Signal Weighting

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Active strategies face the challenge of combining information from different sources in order to maximize that information's after-cost effectiveness.[1] This challenge is evident for investors who follow a systematic, structured investment process. It is also an issue, although often unrecognized, for more traditional managers who must balance top-down views and the views of in-house and sell-side analysts. In any case, a decision is made either through default, analysis, or whimsy. We favor analysis and to that end this article presents a simple and structured approach to the task. This task is commonly called signal weighting, although risk budgeting seems to be a better description. The approach is based on standard methods of portfolio analysis and their extension to dynamic portfolio management.

We can consider an active investment product on three levels: strategic, tactical, and operational. At the strategic level are the target clients, fee structure, level of risk, amount of gearing, level of turnover, marketing channel, decision-making structure, and so forth. At the operational level, we run the product: maintaining and replenishing data, executing trades, tracking positions and performance, informing clients, and so on. At the tactical level, we allocate resources to improve the product and its operations, and do some fine-tuning on such parameters as risk level and turnover level. The signal-weighting decision belongs at the tactical level. Every strategy has an internal risk budget that shows how the aggregate risk is being allocated among the various sources/themes that we anticipate can add value. We should periodically, say, every six months, re-examine and possibly change that allocation. A similar reconsideration of the internal risk budget should take place if there is a significant change in the signal mix (e.g., adding a new theme) or if there is a major change in the market environment.

The method we present in this article gives substance to the signal-weighting process by isolating the important aspects of the decision and providing a structure that links those features to a decision. The procedure we outline mimics the investment process. Each signal is viewed as a sub, or source, portfolio that is a potential place to allocate risk. We allocate risk to the source portfolios in a manner that makes the most sense at the aggregate level. In this effort we are trying to be comprehensive enough to depict the essentials and simple enough to easily capture the link between inputs and outputs. Thus, we compromise between the desire to be reasonable and robust, on the one hand, and to provide a clear box rather than a black box, on the other.

To preserve transparency we should err on the side of simplicity rather than on excessive elaboration. There is a temptation to more and more closely mimic the actual investment process (e.g., backtesting). Our unadorned approach is to assume the portfolio is run with the straightforward and easy-to-analyze method A, although we realize that we actually use the complex and impossible-to-analyze method B. We have to trade off the benefits of getting sensible, comprehensible answers against the costs of departing from the operational reality. Any effect we want to consider has to argue for a place at the table. The watchword is simplicity and the burden of proof is against making it complicated. This focus on simplicity also argues against a profusion of sources. In our view, the number of themes should be more than two and certainly less than, say, seven. If you believe there are a larger number of significant alpha drivers of a strategy you should 1) look up "significant" and 2) aggregate the signals into broader themes.

One benefit of the proposed approach is that it is based on a theory. There is a logical consistency that lets us track cause and effect. The assumptions are few and transparent. It is not a sequence of ad hoc decisions and jury-rigged improvisations where the often hidden assumptions are many and their implications are obscure. This clarity makes it easy to link cause and effect. In particular, it facilitates the use of sensitivity analysis to study the links between important inputs and outputs.

Signal weighting is a forward-looking exercise. We are planning for the future not over-fitting the past. Past performance of signals can, of course, inform us about each signal's presumed future strength, how fast the information dissipates, and how the signals relate to each other. However, we will usually be in a situation with old signals that are part of the implemented strategy and new signals that have only seen hypothetical implementation. In addition, we may feel strongly that some signals will lose strength because they are being arbitraged away. What we see in a backtest is one sample from a nonstationary probability distribution. By all means consider past performance, but add a large pinch of salt.

The analysis is designed for an unconstrained implementation. There may be some idiosyncratic features of a signal that will make it unsuitable for say a long only implementation. These are difficult to predict, so the analysis here should still provide a starting point of any study of the interactions of signals and constraints.

PREVIEW

This article has the following structure. We initially consider a cost-free world with one signal, and then a cost-free world with multiple signals. This exercise allows us to introduce many of the terms and concepts we will use in the more difficult setting with transaction costs.

The next step is to introduce transaction costs. Costs are difficult to handle in an analytic way. Our approach is traditional; we make an assumption. We assume that the portfolio manager acts as if he is following a simple rebalance rule. The rebalance rule has three parts:

- For each signal we construct a source portfolio that captures that signal in an efficient (i.e., most signal per unit of risk) manner.
- We construct a model portfolio that is a mixture of the source portfolios.
- We stipulate a rebalance rule to fix the rate at which the portfolio manager tries to close the gap between the portfolio the manager actually holds and the model portfolio.

In Appendix A, we show that this type of rebalance rule is optimal under certain conditions.[2] We feel that a rule, that is optimal under one set of conditions, will be reasonable in a much wider group of situations. In Appendix B, we study the consequences of using such a decision rule. The signal-weighting model and results are illustrated using a three-signal example.

OTHER WORK

This article stems from the author's previous work on dynamic portfolio analysis, Grinold [2006, 2007]. The most directly related paper is Sneddon [2008]. Similar techniques to those described in Appendix A are used by Garleanu and Pedersen [2009].

ONE SIGNAL AND NO TRANSACTION COSTS

We start with a quick description of the framework with only one signal and then expand the analysis to cover multiple signals. There are N assets whose return covariance is described by the N by N covariance matrix $\mathbf{V}$. The basic material in the weighting exercise is an N element vector $\mathbf{a}$, called a raw alpha. It is scaled to have, on average, an information ratio of one,

$$\mathbf{a}' \cdot \mathbf{V}^{-1} \cdot \mathbf{a} = 1 \tag{1}$$

To obtain a weight, we need an assessment of the strength of the signal. Let *IR* be the information ratio of the signal, then

$$\boldsymbol{\alpha} = IR \cdot \mathbf{a} \tag{2}$$

We are done. With one signal we simply weight the raw alpha to have the desired information ratio.

We can get to the same place by a longer route that approaches the problem from a portfolio perspective. The portfolio route is worth the extra effort because it generalizes in a natural way when there are multiple signals and transaction costs.

We associate a source portfolio **s** with our signal,

$$\mathbf{s} = \mathbf{V}^{-1} \cdot \mathbf{a} \Rightarrow \omega_S^2 = \mathbf{s}' \cdot \mathbf{V} \cdot \mathbf{s} = 1 \tag{3}$$

The source portfolio is an efficient implementation of the signal, that is, among all positions **x**, it maximizes the ratio

$$\mathbf{a} \cdot \mathbf{x} \Big/ \sqrt{\mathbf{x}' \cdot \mathbf{V} \cdot \mathbf{x}}$$

As noted in Equation (3), the source portfolio has an excessive risk level of 100%. Suppose we choose a more realistic risk level ω. This is akin to scaling the source portfolio. If ω is the risk level, then the portfolio is $\mathbf{q} = \omega \cdot \mathbf{s}$. This portfolio has an alpha of $IR \cdot \omega$ and a variance of ω^2. If we have a risk penalty, λ, and we do the standard mean-variance optimization, we choose the risk level ω to maximize,

$$IR \cdot \omega - \frac{\lambda}{2} \cdot \omega^2 \tag{4}$$

The solution is $\omega = {}^{IR}\!/_{\lambda}$. For example, if $IR = 0.75$, and we have a risk penalty of $\lambda = 18.75$, then $\omega = 4\%$.

The alpha resulting from this is

$$\boldsymbol{\alpha} = \lambda \cdot \mathbf{V} \cdot \mathbf{q} = \lambda \cdot \mathbf{V} \cdot \mathbf{s} \cdot \omega = \{\lambda \cdot \omega\} \cdot \mathbf{a} \tag{5}$$

which gets us to the same place as the straightforward approach in Equation (2) since $\lambda \cdot \omega = IR$.

The reasons for the portfolio optimization approach used in Equation (5) are that

- it generalizes nicely when we have multiple signals, and
- it makes things much simpler when we introduce transaction costs.

First, we will consider the multiple-signal case with no transaction costs.

J SIGNALS AND NO TRANSACTION COSTS

We have *J* signals (sub-strategies), each with a *raw* alpha $\mathbf{a}_j$ $j = 1, \ldots, J$. Each $\mathbf{a}_j$ is a vector with elements for each of the *N* assets. In addition, we start with a level playing field; each $\mathbf{a}_j$ is standardized so that its information ratio is one,[3]

$$\mathbf{a}_j' \cdot \mathbf{V}^{-1} \cdot \mathbf{a}_j = 1 \tag{6}$$

We can, as in Equation (3), associate a position with each raw alpha. These positions have, on average, 100% risk and will be the building blocks of our investment strategy,

$$\begin{aligned} &\mathbf{a}_j = \mathbf{V} \cdot \mathbf{s}_j, \quad \mathbf{s}_j = \mathbf{V}^{-1} \cdot \mathbf{a}_j \\ &\mathbf{s}_j' \cdot \mathbf{V} \cdot \mathbf{s}_j = 1 \end{aligned} \tag{7}$$

Because we have multiple signals, we have to take into consideration any correlation between the information sources. We define that correlation in terms of the expected return covariance between any pair of positions,[4]

$$\rho_{i,j} \equiv \mathbf{s}_i' \cdot \mathbf{V} \cdot \mathbf{s}_j \tag{8}$$

For example, if *i* is a value signal and *j* is a momentum signal, they will tend to be negatively correlated. Recall that signal weighting is a forward-looking exercise; the correlations $\rho_{i,j}$ are predictions. Nonetheless, a historical analysis of past correlations should be informative;[5] often, a fairly stable relationship exists. In other cases, the relationship appears to depend on the caprice of the market. As a practical matter, if any of these correlations are very large (say, above 0.5), it might be wise to combine the signals. If a correlation is large and negative, then either you have found a very good thing or, more likely, you have uncovered something that is too good to be true.

We build a strategy by assigning risk levels, ω_j, to each of *J* sub-strategies. The vector representation is $\boldsymbol{\omega} = \{\omega_j\}_{j=1,J}$. The result will be portfolio

$$\mathbf{q}(\boldsymbol{\omega}) = \sum_{j=1,J} \mathbf{s}_j \cdot \omega_j \tag{9}$$

The information ratios of the J sub-strategies are given by the J element vector $\mathbf{IR} = \{IR_j\}_{j=1,J}$. The J by J correlation matrix with elements $\rho_{i,j}$ is denoted as $\mathbf{R}$.

With this notation the alpha expected by using the weights $\boldsymbol{\omega}$ is

$$\alpha(\boldsymbol{\omega}) = \sum_{j=1,J} IR_j \cdot \omega_j = \mathbf{IR}' \cdot \boldsymbol{\omega} \tag{10}$$

The variance of the portfolio, $\mathbf{q}(\boldsymbol{\omega})$, with weights $\boldsymbol{\omega}$ is

$$\sigma^2(\boldsymbol{\omega}) = \sum_{i=1,J} \omega_i \cdot \sum_{j=1,J} \rho_{i,j} \cdot \omega_j = \boldsymbol{\omega}' \cdot \mathbf{R} \cdot \boldsymbol{\omega} \tag{11}$$

We choose $\boldsymbol{\omega}$ to maximize,

$$\alpha(\boldsymbol{\omega}) - \frac{\lambda}{2} \cdot \sigma^2(\boldsymbol{\omega}) \tag{12}$$

The solution is obtained by solving the first-order equations,[6]

$$IR_i = \lambda \cdot \sum_{j=1,J} \rho_{i,j} \cdot \omega_j$$

or

$$\mathbf{IR} = \lambda \cdot \mathbf{R} \cdot \boldsymbol{\omega} \tag{13}$$

The maximizing vector of weights is given by

$$\boldsymbol{\omega} = \frac{1}{\lambda} \cdot \mathbf{R}^{-1} \cdot \mathbf{IR} \tag{14}$$

We will refer to the optimal portfolio (a.k.a., the *ideal* portfolio) as Q with positions

$$\mathbf{q}(\boldsymbol{\omega}) = \sum_{j=1,J} \mathbf{s}_j \cdot \omega_j \tag{15}$$

The information ratio and risk of the ideal portfolio Q are

$$IR_Q \equiv \sqrt{\mathbf{IR}' \cdot \mathbf{R}^{-1} \cdot \mathbf{IR}}, \quad \omega_Q = \sqrt{\boldsymbol{\omega}' \cdot \mathbf{R} \cdot \boldsymbol{\omega}},\ IR_Q = \lambda \cdot \omega_Q \tag{16}$$

Note that IR_Q is independent of the risk penalty λ, and thus we can choose λ to get the desired level of active risk, ω_Q.

EXHIBIT 1

Three-Signal Example

Source	IR	SLOW	INTERMEDIATE	FAST	Risk
SLOW	0.4	1.0	-0.15	0.1	1.96%
INTERMEDIATE	0.5	-0.15	1.0	-0.3	3.78%
FAST	0.7	0.1	-0.3	1.0	4.10%

Note: The input data are the information ratio, IR, and the correlation. The optimal result is in the column labeled "Risk."

EXAMPLE

The following is an example with three sub-strategies. In anticipation of the next section, we have classified the signal by the rate of information turnover. The sources are SLOW, INTERMEDIATE, and FAST. The goal is to mix these three sources to get an ideal portfolio with 5% risk.

The data in Exhibit 1 represent our best estimates for a reasonably long planning horizon, say, a year or more. Referring to Equation (16), the net effect of these data is an aggregate information ratio of $IR_Q = 1.11$. If we want an overall risk of $\omega_Q = 5.00\%$, we can set the risk penalty as $\lambda = IR_Q/\omega_Q = 1.11/0.05 = 22.17$ and achieve the desired result. The final column is the optimal risk level derived from Equation (14). We can interpret this as running the INTERMEDIATE sub-strategy at 3.78% risk, the SLOW sub-strategy at 1.96% risk, and the FAST sub-strategy at 4.10% risk.

RISK BUDGETING

Signal weighting, like portfolio management, is an exercise in risk budgeting. We can express the result in terms of the resulting risk budget. The variance of the ideal portfolio Q is

$$i.\ \omega_Q^2 = \sum_{j=1,J} \omega_j \cdot \sum_{k=1,J} \rho_{j,k} \cdot \omega_k$$

or

$$ii.\ 1 = \sum_{j=1,J} \left\{\frac{\omega_j}{\omega_Q}\right\} \cdot \sum_{k=1,J} \rho_{j,k} \cdot \left\{\frac{\omega_k}{\omega_Q}\right\} \tag{17}$$

If we allocate a fraction $\{\frac{\omega_j}{\omega_Q}\} \cdot \Sigma_{k=1,J} \rho_{j,k} \cdot \{\frac{\omega_k}{\omega_Q}\}$ of the risk to alpha source j, we are in effect taking the two covariance terms for sources j and k, $\omega_j \cdot \rho_{j,k} \cdot \omega_k + \omega_k \cdot \rho_{k,j} \cdot \omega_j$, and splitting the baby, thus attributing $\omega_j \cdot \rho_{j,k} \cdot \omega_k$ to signal j and $\omega_k \cdot \rho_{k,j} \cdot \omega_j$ to signal k.

In our example, over 52% of the risk comes from the FAST signal, 34% from the INTERMEDIATE signal, and only 14% from the SLOW signal.

THE ALPHA PERSPECTIVE

We can view the result in terms of alphas as well as of portfolios. According to Equation (15), our ideal portfolio is

$$\mathbf{q}(\boldsymbol{\omega}) = \sum_{j=1,J} \mathbf{s}_j \cdot \omega_j$$

The alphas that lead to this ideal are[7]

$$\boldsymbol{\alpha}_Q = \lambda \cdot \mathbf{V} \cdot \mathbf{q} = \sum_{j=1,J} \mathbf{a}_j \cdot (\lambda \cdot \omega_j) \tag{18}$$

WEIGHTS

Equation (18) can be interpreted in terms of weights (i.e., numbers that sum to one), as follows:

$$\begin{aligned} &i.\ \boldsymbol{\alpha}_Q = scale \cdot \left\{ \sum_{j=1,J} \mathrm{a}_j \cdot weight_j \right\}, \quad \text{where} \\ &ii.\ scale = \lambda \cdot \sum_{j=1,J} \omega_j \\ &iii.\ weight_j = \frac{\omega_j}{\sum_{i=1,J} \omega_i} \end{aligned} \tag{19}$$

The scale factor is to ensure that the resulting alphas have the correct information ratio, for example,

$$\sqrt{\boldsymbol{\alpha}'_Q \cdot \mathbf{V}^{-1} \cdot \boldsymbol{\alpha}_Q} = IR_Q = 1.11 \tag{20}$$

The weights in our examples are 38% for INTERMEDIATE, 20% for SLOW, and 42% for FAST. Note that these are similar, but not equal, to the risk budgeting numbers mentioned previously.

The following three ways of presenting the results are illustrated in Exhibit 2:

- risk of each source
- risk budget
- weights that sum to one

Now we turn to the more challenging and realistic situation where we consider transaction costs.

EXHIBIT 2

Three Views of the Solution to the Signal/Source-Weighting Problem with No Transaction Costs

Ideal Portfolio	Risk Q	Risk Budget Q	Weight Q
SLOW	1.96%	14.16%	19.95%
INTERMEDIATE	3.78%	34.10%	38.42%
FAST	4.10%	51.74%	41.64%

TRANSACTION COSTS

The previous analysis showed how to successfully approach the source/signal-weighting problem when there are no transaction costs. But the presence of transaction costs introduces three new obstacles. These obstacles are

- Transaction costs are paid either directly through commissions, spread, taxes, and so forth, or indirectly by demanding liquidity and shelling out too much for purchases and getting too little for sales (a.k.a., market impact).
- Transaction costs are intimidating. They keep you from fully exploiting your information, which creates an opportunity loss.
- Transaction costs are levied on changes in positions, so the initial position is a crucial part of the analysis.

The last point implies a multi-period perspective is required. Our approach will be to look for strategies or decision rules that can, in some sense, be considered optimal in a multi-period setting. To do this, we make assumptions that allow us to abstract from the reality of the day-to-day portfolio management environment and still capture something of its essence. This is, after all, both the power and the Achilles' heel of any economic analysis. As Solow [1956] said, "All theory depends on assumptions that are not quite true."

THE PORTFOLIO'S LAW OF MOTION

An active portfolio is an object in motion. Call it portfolio *P*. If we look at portfolio *P* periodically, say, every Δt years,[8] then we see a sequence of positions. The changes in these positions, defined as

$$\Delta\mathbf{p}(t) \equiv \mathbf{p}(t) - \mathbf{p}(t - \Delta t) \tag{21}$$

determine the cost.

The challenge is to capture these changes in a useful way. To do this we will specify a rule, called the law of motion, to show how portfolio P changes in response to changes in the source portfolios. We will do this in two steps, as follows:

1. Combine the source portfolios into a *model* portfolio M.
2. Show how the portfolio P attempts to track the model portfolio.

THE MODEL PORTFOLIO

The model portfolio is a mixture of the source portfolios. Its holdings at time t are

$$\mathbf{m}(t) \equiv \sum_{j=1,J} \mathbf{s}_j(t) \cdot \omega_j \cdot \psi_j \tag{22}$$

The risk levels ω_j are the same levels we used to define the ideal portfolio; see Equations (13) and (14). The parameters ψ_j, where $0 < \psi_j < 1$, are explained in detail in the next section.

THE TRACKING RULE

The changes in portfolio P are governed by a linear decision rule of the form

$$i.\ \ \mathbf{p}(t) = \delta \cdot \mathbf{p}(t - \Delta t) + (1 - \delta) \cdot \mathbf{m}(t)$$

or

$$ii.\ \ \frac{\Delta \mathbf{p}(t)}{\Delta t} = \frac{(1-\delta)}{\Delta t} \cdot \{\mathbf{m}(t) - \mathbf{p}(t - \Delta t)\} \tag{23}$$

The parameter *delta*, $0 < \delta < 1$, is discussed later. In Appendix A, we demonstrate that the rule described in Equations (22) and (23) is optimal under special circumstances. The leap of faith is to assume that the rule is at least "reasonable" under more general conditions.

The parameters δ, ψ_j mentioned in Equations (22) and (23) depend on the rate of change of the signals as represented by either the raw alphas $\mathbf{a}_j(t)$ or the source portfolios $\mathbf{s}_j(t)$ and on the change in the portfolio $\mathbf{p}(t)$ itself. We now turn our attention to measuring those changes and determining values for these parameters.

A MEASURE OF SIGNAL SPEED

We can gauge the rate of change in the signal j between times t and $t - \Delta t$ by measuring the correlation of the positions $\mathbf{s}_j(t)$ and $\mathbf{s}_j(t - \Delta t)$ using the asset covariance, $\mathbf{V}(t)$, at time t. To do the calculation in a sensible way, we should tailor the time interval, calling it Δt_j, on a signal-by-signal basis so that a reasonable amount of change takes place resulting in a value for $\gamma_j(t)$ in Equation (24) in the range of 0.8 to 0.975. This will avoid spurious results if Δt is either too long or too short. Even if we keep daily data we might want to calculate the correlation over one week, two weeks, or one month so that we can see significant changes.

This past correlation is given by

$$\gamma_j(t) = \frac{\mathbf{s}'_j(t) \cdot \mathbf{V}(t) \cdot \mathbf{s}_j(t - \Delta t_j)}{\sqrt{\mathbf{s}'_j(t) \cdot \mathbf{V}(t) \cdot \mathbf{s}_j(t)} \cdot \sqrt{\mathbf{s}'_j(t - \Delta t_j) \cdot \mathbf{V}(t) \cdot \mathbf{s}_j(t - \Delta t_j)}} \tag{24}$$

We can eliminate the effects of the tailored time interval selection, Δt_j, by calculating a rate of change,

$$g_j(t) = -\frac{\ln(\gamma_j(t))}{\Delta t_j} \quad \text{or } e^{-g_j(t)\cdot \Delta t_j} = \gamma_j(t) \tag{25}$$

The data will show how the signal has moved in the past; for example, seasonality (e.g., quarterly or semi-annual release of earnings) or a trend (e.g., getting faster or slower) may be present. The challenge is to use this analysis to choose a point estimate that we will call g_j, that will determine how fast, on average, we believe the signal will move in the future. One way to consider this question is in terms

EXHIBIT 3
Values of g_j for Various Half-Lives

Half-Life	g
One week	36.0
One month	8.3
Two months	4.2
One quarter	2.8
Six months	1.4
Nine months	0.9
One year	0.7

of a half-life, that is, the time it will take for the correlation, as calculated by Equation (24), to drop to 0.5. Exhibit 3 shows how g_j depends on the half-life.

In our example, we will use nine months as the half-life for the SLOW signal, one month for the FAST signal, and six months for the INTERMEDIATE signal.

THE PACE OF PORTFOLIO CHANGE

The parameter *delta* in Equation (23) governs the rate of change of portfolio *P*. We will express *delta* as $\delta = e^{-d\cdot\Delta t}$, where *d* measures how fast we attempt to close the gap between where we start, $\mathbf{p}(t-\Delta t)$, and where we would like to be, $\mathbf{m}(t)$. The choice of *d*, whether it is made explicitly or implicitly, is an attempt to balance two costs:

1. Transaction costs: A larger *d* (thus lower δ) leads to more trading and higher costs.
2. Opportunity loss: A smaller *d* (thus higher δ) implies a less efficient implementation; the fraction of portfolio *P*'s risk budget directed toward alpha declines and the amount of uncompensated risk increases.

In Appendix A and in Grinold [2007], we show how, under special circumstances, the optimal choice of *d* is possible. In what follows, we first discuss the interpretation of *d*, which should isolate a reasonable range of values, and then we consider procedures that will help us discover an implied value of *d* by examining the past behavior of the portfolio.

INTERPRETATION OF *d*

Portfolio *P* chases the model *M*. It is always behind the curve. Indeed, $1/d$ is a measure of just how much *P* lags *M*. Both *P* and *M* are based on information that has flowed in over previous periods. The age-weighted exposure of *P* to this information is about $1/d$ years longer than the age-weighted exposure of *M* to the same information. Slow trading leaves *P* under exposed to recent information and relatively over exposed to older stale information. Exhibit 4 shows lags of one to six months and the corresponding values for *d*.

EXHIBIT 4

Relationship between Trading Rate *d* and Lag between Average Age of Information in Model Portfolio *M* and Tracking Portfolio *P*

Months Lag	*d*
One month	12.0
Two months	6.0
Three months	4.0
Four months	3.0
Five months	2.4
Six months	2.0

ESTIMATION OF *d*; REVEALED PREFERENCE

It is possible to get a rough estimate of *d* by examining the history of the portfolio and the source portfolios. As we indicate in Appendix B, we can use a regression, or portfolio optimization, approach to estimate parameters $\hat{\delta}(t)$ and $\hat{\beta}_j(t)$, so that

$$\mathbf{p}(t) = \mathbf{p}(t-\Delta t_0)\cdot\hat{\delta}(t) + \sum_{j=1,J}\mathbf{s}_j(t)\cdot\hat{\beta}_j(t) + \hat{\varepsilon}(t) \quad (26)$$

and the unexplained variance, $\hat{\varepsilon}'(t)\cdot\mathbf{V}(t)\cdot\hat{\varepsilon}(t)$, is minimized. If things are going along relatively smoothly (i.e., no radical changes of policy) and the time interval Δt_0 is selected in a reasonable manner,[9] then estimates $\hat{\delta}(t)$ will tend to be positive and less than one. In such cases, we get an estimate of *d* as

$$\hat{d}(t) = -\frac{\ln(\hat{\delta}(t))}{\Delta t_0}, \quad \hat{\delta}(t) = e^{-\hat{d}(t)\cdot\Delta t_0} \quad (27)$$

The analysis will also produce implied risk levels, $\hat{\beta}_j(t)/(1-\hat{\delta}(t))$, and a measure of the fitting error, $\{\hat{\varepsilon}'(t)\cdot\mathbf{V}(t)\cdot\hat{\varepsilon}(t)\}/\{\mathbf{p}'(t)\cdot\mathbf{V}(t)\cdot\mathbf{p}(t)\}$.

From these estimates of implied past policy in addition to the intuition that might be gained from Exhibit 4 and future policy plans, we settle on values for the parameters *d* and g_j, $j = 1, \ldots, J$. Then, for any rebalance interval Δt, we can calculate the risk adjustments ψ_j promised in Equation (22),[10]

$$\begin{aligned} &i.\ \delta = e^{-d\cdot\Delta t}, \quad \gamma_j = e^{-g_j\cdot\Delta t} \\ &ii.\ \psi_j \equiv \frac{1-\delta}{1-\delta\cdot\gamma_j} \end{aligned} \quad (28)$$

Now let's examine how this would work with the example introduced earlier. The data in Exhibit 5 are

EXHIBIT 5

Assumed Half-lives of the Three Signals and Corresponding Values of g_j, γ_j, and ψ_j based on $\Delta t = 1/52$ and $\delta = 0.926$

Source	HL Months	g_j	γ_j	Ψ_j	Risk Q	Risk M
SLOW	Nine	0.92	0.982	0.819	1.96%	1.61%
INTERMEDIATE	Six	1.39	0.974	0.752	3.78%	2.84%
FAST	One	8.32	0.852	0.351	4.10%	1.44%

based on a value of $d = 4$ and a rebalance interval of one week, $\Delta t = 1/52$, thus $\delta = 0.926$.

In Exhibit 5, the g_j are based on the assumed half-lives. If HL_j is the half-life in months, then $g_j = 12 \cdot \{\ln(2)/HL_j\}$. The γ_j are calculated using item (*i*) of Equation (28) and the ψ_j are derived from item (*ii*) of Equation (28). The column "Risk *Q*" repeats the optimal risk levels, ω_j, of the ideal portfolio previously reported in Exhibits 1 and 2. The final column, "Risk *M*," contains the risk levels $\psi_j \cdot \omega_j$ associated with the model portfolio, as in Equation (22). The risk levels of all the signals fall, although the decrease, evident in the "*psi*" column, is more dramatic for the FAST signal. The risk[11] of model portfolio *M* is 3.06% compared to the 5% risk level for the ideal portfolio *Q*. To raise the risk level of the model portfolio to, say, 5%, merely decreases the risk penalty lambda from 22.17 to $\{3.06/5\} \cdot 22.17 = 13.57$ and repeats the calculation. This will scale up the risk, but keep the risk budget allocation and weights the same.

THE ALPHA PERSPECTIVE REVISITED

The adjustment for transaction costs works for alphas as well as for portfolios. According to Equation (22), our model portfolio is

$$\mathbf{m} = \sum_{j=1,J} \mathbf{s}_j \cdot \{\psi_j \cdot \omega_j\}$$

The alphas that would lead to this portfolio are

$$\alpha_M = \lambda \cdot \mathbf{V} \cdot \mathbf{m} = \lambda \cdot \sum_{j=1,J} \mathbf{a}_j \cdot (\psi_j \cdot \omega_j) \tag{29}$$

WEIGHTS

Equation (29) can, as before, be interpreted in terms of weights (i.e., numbers that sum to one), as follows:

EXHIBIT 6

Risk Budgets and Weights before (*Q*) and after (*M*) Adjustment for Trading

Source	Risk Budget *Q*	Risk Budget *M*	Weight *Q*	Weight *M*
SLOW	14.16%	22.72%	19.95%	27.30%
INTERMEDIATE	34.10%	65.84%	38.42%	48.28%
FAST	51.74%	11.43%	41.64%	24.41%

$$\begin{aligned} &i.\ \boldsymbol{\alpha}_M = scale \cdot \left\{ \sum_{j=1,J} \boldsymbol{\alpha}_j \cdot weight_j \right\}, \quad \text{where} \\ &ii.\ scale = \lambda \cdot \sum_{j=1,J} \{\psi_j \cdot \omega_j\} \\ &iii.\ weight_j = \frac{\{\psi_j \cdot \omega_j\}}{\sum_{i=1,J} \{\psi_i \cdot \omega_i\}} \end{aligned} \tag{30}$$

Exhibit 6 compares the results pre-adjustment (ideal *Q*) and post-adjustment (model *M*) for transaction costs.

OPPORTUNITY LOSS

The adjustment from ideal portfolio *Q* to model portfolio *M* effectively lowers our sights from one unobtainable goal to another slightly less unobtainable goal. In the process, we have left some potential value-added on the table.

If we measure the value-added from any portfolio, say *F* with positions **f**, as

$$U_F = \boldsymbol{\alpha}'_Q \cdot \mathbf{f} - \frac{\lambda}{2} \cdot \boldsymbol{\omega}_F^2 \tag{31}$$

then the loss we incur by lowering our sights from *Q* to *M* is

$$U_Q - U_M = \frac{\lambda}{2} \cdot \omega_{Q-M}^2 \tag{32}$$

where ω_{Q-M} is the risk of a position that is long the ideal portfolio Q and short the model portfolio M.[12] In our example that risk is $\omega_{Q-M} = 2.58\%$ and the corresponding loss in value-added is $U_Q - U_M = 0.74\%$.

THE END RESULT

In Appendix B, we examine the results of following the rebalancing strategy portrayed in Equation (23). If we let $\omega_{i,M} \equiv \Sigma_{j=1,J} \rho_{i,j} \cdot \psi_j \cdot \omega_j$ be the covariance of source portfolio i with the model portfolio M, then we can easily determine the alpha, risk, and implementation efficiency of the implemented strategy P, as follows:

$$\begin{aligned} &i.\ \alpha_P = \lambda \cdot \omega_M^2 = \lambda \cdot \sum_{i=1,J} \psi_i \cdot \omega_i \cdot \omega_{i,M} \\ &ii.\ \omega_P^2 = \sum_{i=1,J} \psi_i^2 \cdot \omega_i \cdot \omega_{i,M} \\ &iii.\ \omega_{M-P}^2 = \sum_{i=1,J} \psi_i \cdot (1-\psi_i) \cdot \omega_i \cdot \omega_{i,M} \end{aligned} \tag{33}$$

ADDITIONAL OPPORTUNITY LOST

The opportunity loss just described was due to dropping our sights from portfolio Q to portfolio M. Because of trading costs, the portfolio we actually hold, P, will not capture all the potential value added to the model portfolio M. It is possible to take the analysis one step further (see Appendix B) and look at the efficiency of portfolio P, the portfolio that results from using the updating rule described in Equations (22) and (23). In our example, portfolio P has 2.58% risk. Its correlation (transfer coefficient) with Q and M is 0.727 and 0.857, respectively. The loss in potential value-added moving from the ideal portfolio Q to the actual portfolio P is 1.43%. Given that we lost 0.74% by lowering our sights from the ideal Q to the model M, we lose another 0.69% because we cannot exactly track M. That means a certain fraction of portfolio P's risk budget will be residual (i.e., not correlated with any source of alpha) and thus that risk is taken with no expectation of return. Exhibit 7 contains the risk budgets for the three portfolios Q, M, and P.

Exhibit 7 shows the magnitude of the compromises forced upon us by trading frictions. With no trading costs, we would allocate 51.74% of our risk budget to the FAST signal. When costs are taken into consideration, however, we allocate only 2.93% to the FAST signal. This is a huge shift.

EXHIBIT 7
Risk Budgets for the Portfolios: Ideal *Q*, Model *M*, and Actual *P*

Risk Budget	Portfolio *Q* (ideal)	Portfolio *M* (model)	Portfolio *P* (actual)
SLOW	14.16%	22.72%	21.49%
INTERMEDIATE	34.10%	65.84%	50.36%
FAST	51.74%	11.43%	2.93%
Residual	0.00%	0.00%	25.22%

REVERSE ENGINEERING

If the target risk level for portfolio P is, say, 4.00%, then we can alter the analysis to get that outcome. If we change the risk penalty lambda from 22.17 to 22.17 · {2.58/4.00} = 14.30, we will scale up the risks. The risk budget numbers in Exhibit 7 will not change.

ESTIMATE OF COSTS

As previously discussed, opportunity losses are incurred by lowering our sights from portfolio Q to portfolio M, 0.74%, and by, in fact, lagging behind M, 0.69%. We also will incur direct costs in the form of fees and commissions or indirect costs through market impact. We can estimate these costs as well (see Appendix A). Let's call the difference between portfolios M and P the backlog. This is the trade we would make if we had one period free of transaction costs. We can measure the size of this backlog by its variance, ω_{M-P}^2. The estimated annual transaction costs, c_P, for portfolio P are a function of that backlog variance

$$c_P = \frac{\lambda}{2} \cdot \omega_{M-P}^2 \tag{34}$$

Based on the data, this cost is 0.30%. In addition to the 1.43% opportunity loss, transaction costs bring the total bill to 1.73% and the annual value added after costs to 1.04%.

SENSITIVITY ANALYSIS

One of the attractive features of our approach to signal weighting is the ability to do sensitivity analysis. The equations used in this article can be coded in a spreadsheet

EXHIBIT 8

Alpha Weights for the Model Portfolio *M* as Trading Rate *d* Varies and Other Parameters Are Held Constant

Months Lag *d*	0.5 24	1 12	2 6	4 3	6 2	12 1
SLOW	21.66%	23.32%	25.68%	28.52%	30.26%	33.09%
INTERMEDIATE	41.13%	43.55%	46.56%	49.36%	50.55%	51.62%
FAST	37.20%	33.12%	27.76%	22.12%	19.19%	15.30%

in a matter of minutes. As an example, we calculated the weights of the model portfolio *M* for various values of the trading rate parameter *d*, as shown in Equation (30).

The cases in Exhibit 8 bracket the base case with $d = 4$ that is summarized in Exhibit 6. We can also consider the ideal portfolio *Q* as the case with very large *d*. The weight for the ideal portfolio is shown in Exhibit 6 as 19.95%, 38.42%, and 41.64% for SLOW, INTERMEDIATE, and FAST, respectively.

CONCLUSION

We have presented a relatively straightforward approach to the signal-weighting problem in the face of transaction costs. Our approach has several attractive attributes. First, it is portfolio based. It looks at the signal-weighting problem as an investment problem by linking each signal to an investment strategy and then looking for the optimal mix of strategies. Second, the approach is an extension of the solution of the signal-weighting problem in the absence of costs. And third, it is forward looking and depends on three traits of signals: predicted information ratio, IR_j; predicted return correlation, $\rho_{i,j}$; and rate of change of each signal, g_j. In addition, our approach depends on two investment strategy parameters: 1) trading rate of the portfolio, *d,* a surrogate for turnover, and 2) a risk penalty, λ, that controls the strategy's level of risk.

Additional attributes include the fact that it is easy to perform sensitivity analysis with our approach, a capability that is vitally important in establishing a firm understanding of the model and its results and in testing the robustness of the answers to reasonable adjustments in the inputs. Our approach also lends itself to reverse engineering in that it is relatively easy to adjust the risk penalty and trading rate to attain the desired level of turnover and active variance. Furthermore, the approach predicts some of the characteristics of the resulting strategy, including level of risk, risk budget, opportunity loss, and annual trading cost.

Finally, our approach can change perspective by viewing each signal as a sub-strategy and the weights as active risk levels for the sub-strategy. These risk levels, in turn, allow the calculation of risk budgets that are a better measure of the importance of each signal. It changes perspective by focusing on the dynamics of the signals and the portfolio. The exercise of trying to measure the rate of change of the signals, g_j, and the trading rate, *d*, of the portfolio requires a novel, dynamic look at the signals and the portfolio.

ENDNOTES

[1]In this article, the following terms—sources, themes, and signals—are used as synonyms.

[2]The technical appendices, A (Optimization) and B (Policy Analysis), are available by writing to the author at rcgrinold@hotmail.com.

[3]It is best practice to have this standardization work in a time-average fashion. This allows a natural emphasis on a theme when more information is available and, in particular, avoids trying to make something out of nothing when less information is on hand.

[4]Note this also implies that $\rho_{i,j} = \mathbf{a}_i' \cdot \mathbf{V}^{-1} \cdot \mathbf{a}_j = \mathbf{a}_i' \cdot \mathbf{s}_j = \mathbf{a}_j' \cdot \mathbf{s}_i$.

[5]If there is a history of the past source portfolios $s_j(t)$, and dated covariance matrices, $\mathbf{V}(t)$, then $\rho_{i,j}(t) \equiv \mathbf{s}_i'(t) \cdot \mathbf{V}(t) \cdot \mathbf{s}_j(t) / \{\sqrt{\mathbf{s}_i'(t) \cdot \mathbf{V}(t) \cdot \mathbf{s}_i(t)} \cdot \sqrt{\mathbf{s}_j'(t) \cdot \mathbf{V}(t) \cdot \mathbf{s}_j(t)}\}$ gives a measure of the correlation at time *t*.

[6]If one or more of the ω_j turn out to be negative, it just means that we are willing to short that particular sub-strategy or replace $\mathbf{s}_j$ with $-\mathbf{s}_j$. This is not of any technical concern although it definitely is a danger sign.

[7]Another way to say this is $\alpha_Q = \Sigma_{j=1,J}\{IR_j \cdot a_j\} \cdot \{\frac{\lambda \cdot \omega_j}{IR_j}\}$. In this case we have scaled each signal *j* to have the correct information ratio, IR_j. If the signals are not correlated, $\rho_{i,j} = 0$ for $i \neq j$, then the expression $\lambda \cdot \omega_j / IR_j$ equals one for all signals. In the more general case, we see $\lambda \cdot \omega_j / IR_j = \omega_j / \Sigma_{k=1,J} \rho_{j,k} \cdot \omega_k$ as an adjustment for the correlations among the signals.

[8]In other words, monthly $\Delta t = 1/12$, weekly $\Delta t = 1/52$, and so forth.

[9]As mentioned earlier, even if the portfolio were to change on a daily basis, we would likely want Δt to represent a longer, say, one- or two-week timeframe, so that more substantial changes take place.

[10]As Δt goes to zero $\psi_j \to d/\{d + g_j\}$.

[11]If $\hat{\omega}_j = \psi_j \cdot \omega_j$, then $\omega_M^2 = \Sigma_i \hat{\omega}_i \{\Sigma_j \rho_{i,j} \cdot \hat{\omega}_j\}$.

[12]The portfolio $Q - M$ has positions $\mathbf{q} - \mathbf{m} = \Sigma_{j=1,J}$, $\mathbf{s}_j \cdot \omega_j \cdot (1 - \psi_j)$.

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