COMMUNICATIONS IN STATISTICS—THEORY AND METHODS
2017, VOL. 46, NO. 20, 10036–10050
https://doi.org/10.1080/03610926.2016.1228965

Robust test for independence in high dimensions

Guangyu Mao

School of Economics and Management, Beijing Jiaotong University, Beijing, China

ABSTRACT
This article develops a new test based on Spearman's rank correlation coefficients for total independence in high dimensions. The test is robust to the non normality and heavy tails of the data, which is a merit that is not shared by the existing tests in literature. Simulation results suggest that the new test performs well under several typical null and alternative hypotheses. Besides, we employ a real data set to illustrate the use of the new test.

ARTICLE HISTORY
Received 7 July 2015
Accepted 15 August 2016

KEYWORDS
High-dimensional data; higher-order moments; independence test; rank correlation.

MATHEMATICS SUBJECT CLASSIFICATION
62H15

1. Introduction

Statistical tests for high-dimensional data have drawn increasing attention in a variety of fields in recent years. Typically, a high-dimensional data set of sample size N containing m variables has the feature that $m \gg N$. For a specific test problem pertaining to this kind of data set, it has been well understood that the traditional tests may perform poorly since the methods used to approximate the exact distributions of the related statistics usually rely heavily on the assumption that N tends to infinity while m is fixed. To address the $m \gg N$ issue, it is almost inevitable that the approximations underlying the tests should let m approach infinity as N does, which is fundamental for statistical tests in high dimensions. For a summary and bibliography of some high-dimensional tests recently developed, interested reader may refer to Fujikoshi et al. (2010) for examples.

This article is concerned with the test for total independence among an m -dimensional random vector $(Y_1, \ldots, Y_m)$ using N independent realizations in high dimensions. Let $F(y_1, \ldots, y_m)$ be the joint CDF (cumulative distribution function) of $(Y_1, \ldots, Y_m)$ and $F_p(y_p)$ $(p = 1, \ldots, m)$ be the marginal CDFs. The null hypothesis of interest is

$$H_0 : F(y_1, \ldots, y_m) = \prod_{p=1}^{m} F_p(y_p)$$

To test H_0, we assume $(m, N) \to \infty$, i.e., $m \to \infty$ and $N \to \infty$ simultaneously.[1] Besides, we impose an additional restriction on m and N below: the limit of m/N exists and is a finite positive number. This restriction is not rare in the literature related to high-dimensional analysis; see, e.g., Schott (2005). Note that our aim is to test the total independence of m random

CONTACT Guangyu Mao gymao@bjtu.edu.cn School of Economics and Management, Beijing Jiaotong University, Science and Technology Building #926, Shang Yuan Cun #3, Beijing 100044, P. R. China.

[1] The simultaneous asymptotic regime is different from the sequential asymptotic regimes, i.e., $m \to \infty$ first and then $N \to \infty$, or $N \to \infty$ first and then $m \to \infty$.

variables, and m is finite in practice. Here $m \to \infty$ is only used to approximate the finite-sample distribution of the test statistic developed below, which does not mean that we are trying to test the total independence of an infinite number of random variables.

When the data are jointly normal, H_0 is equivalent to

$$\tilde{H}_0 : r_{pq} = 0 \quad \text{for } 2 \le p \le m \quad \text{and } 1 \le q \le p-1$$

where r_{pq} is the Pearson's correlation coefficient between Y_p and Y_q. As a consequence, any method that can be employed to test $\tilde{H}_0$, such as those in Schott (2005), Srivastava (2005), Cai and Jiang (2011) , Srivastava et al. (2011), Qiu and Chen (2012), and Mao (2014), is helpful to check the total independence.[2] However, the normality assumption does not always hold in practice; see, e.g., a real example below. Even though the existing methods are effective to test $\tilde{H}_0$, it is generally difficult to justify that they are also useful for testing independence when dealing with non normal data. Moreover, the non normal data may not have finite variances. For example, heavy-tailed data from financial markets are believed to be of infinite variance under many circumstances. In this case, $\tilde{H}_0$ is definitely inappropriate to the independence test since the parameters r_{pq} are not well defined. Thus, we should be very cautious in making any conclusion after conducting tests for $\tilde{H}_0$ if our goal is to check whether H_0 is true or not.

To circumvent the drawbacks of the existing tests just mentioned, this article develops a non parametric method for high-dimensional independence test, which is robust to possible non normality and heavy tails (even infinite variance) of the data. In literature pertaining to high-dimensional tests, methods that are robust to the non normality have been widely discussed. Nevertheless, there are a relatively small number of articles that concentrate on the robustness to heavy tails. A recent example can be found in Zou et al. (2014), which proposed a sphericity test using multivariate signs for possibly heavy-tailed data in high dimensions.

The rest of this article is organized as follows. Next section introduces the new test and discusses the theoretical properties about it. Section 3 evaluates the new test by simulation. Section 4 is a real example that shows the use of the new test. Section 5 is a short conclusion. All technical proofs are contained in the Appendix.

2. Statistics and related theories

Our new test is largely motivated by the one proposed by Schott (2005). Before presenting our statistic and related theories, we briefly review Schott's test first. Denote $\hat{r}_{pq}$ as the sample (Pearson's) correlation coefficient corresponding to r_{pq}. Because $r_{pq} = 0$ implies independence between Y_p and Y_q under the normality assumption, based on the pairwise sample correlation coefficients, Schott employed $\sum_{p=2}^{m} \sum_{q=1}^{p-1} \hat{r}_{pq}^2$ to construct his test statistic. Since $E(\hat{r}_{pq}^2) = \frac{1}{N-1}$, $var(\hat{r}_{pq}^2) = \frac{N-2}{2(N-1)^2(N+1)}$, and $\hat{r}_{pq}$ are pairwise independent according to Geisser and Mantel (1962) under H_0, Schott formulated the following standardized $\sum_{p=2}^{m} \sum_{q=1}^{p-1} \hat{r}_{pq}^2$:

$$t_{Nm} = \sqrt{\frac{(N-1)^2(N+1)}{m(m-1)(N-2)}} \left(\sum_{p=2}^{m} \sum_{q=1}^{p-1} \hat{r}_{pq}^2 - \frac{m(m-1)}{2(N-1)} \right) \tag{1}$$

[2] Testing $\tilde{H}_0$ is equivalent to testing the diagonality of the covariance matrix of $(Y_1, \ldots, Y_m)$. Besides the diagonality, identity and sphericity are two other important structures for a covariance matrix. Examples that are concerned with testing the two structures in high dimensions include Ledoit and Wolf (2002), Chen et al. (2010), Srivastava et al. (2014), and Ahmad (2016), to name a few.

He proved that $t_{Nm} \xrightarrow{d} N(0,1)$ as long as $(m, N) \to \infty$ and $\lim_{(m,N)\to\infty} \frac{m}{N} = \gamma \in (0, \infty)$, where $\xrightarrow{d}$ denotes convergence in distribution.

That $\hat{r}_{pq}$, as a measurement of dependence, can be used to test independence depends heavily on the normality assumption. To deal with the possibly non normal case, in this article we employ the Spearman's rank correlation coefficient (Spearman, 1904), another one of the oldest measurements of dependence. For Y_p and Y_q $(1 \le p \ne q \le m)$, the Spearman's rank correlation between them is defined by

$$\rho_{pq} = \frac{\sum_{i=1}^{N}(R_{pi} - \bar{R}_p)(R_{qi} - \bar{R}_q)}{\sqrt{\sum_{i=1}^{N}(R_{pi} - \bar{R}_p)^2 \sum_{i=1}^{N}(R_{qi} - \bar{R}_q)}} \tag{2}$$

where $(R_{p1}, \ldots, R_{pN})$ and $(R_{q1}, \ldots, R_{qN})$ denote the corresponding ranks, respectively, $\bar{R}_p = N^{-1}\sum_{i=1}^{N} R_{pi}$ and $\bar{R}_q = N^{-1}\sum_{i=1}^{N} R_{qi}$. Note that ρ_{pq} is just the sample (Pearson's) correlation coefficient of the ranks. Like $\hat{r}_{pq}$, ρ_{pq} can be effectively employed to test independence but without making strong distributional assumption such as normality; see, e.g., Hájek et al. (1999). Due to the similarity of $\hat{r}_{pq}$ and ρ_{pq}, following the idea of Schott (2005), we consider standardizing $\sum_{p=2}^{m}\sum_{q=1}^{p-1}\rho_{pq}^2$ to form our test statistic.

To compute $E(\sum_{p=2}^{m}\sum_{q=1}^{p-1}\rho_{pq}^2)$ and $\sigma_{Nm}^2 = var(\sum_{p=2}^{m}\sum_{q=1}^{p-1}\rho_{pq}^2)$, which are useful to the standardization, and develop the related theories, throughout the rest of the article, we postulate that the following assumption holds.

Assumption 1. $(Y_1, \ldots, Y_m)$ is both a jointly and marginally continuous random vector whose joint density is $f(y_1, \ldots, y_m)$ and marginal densities are $f_p(y_p)$ $(p = 1, \ldots, m)$.

This assumption excludes the cases of non continuous distributions to avoid the occurrence of rank ties, but it does not require that the data are jointly normal or of finite variance. In fact, this assumption is quite weak and applies to lots of data sets in practice. Based on it, the targeted null hypothesis becomes

$$\bar{H}_0 : f(y_1, \ldots, y_m) = \prod_{p=1}^{m} f_p(y_p)$$

It is easy to show that $\tilde{H}_0$ is only a special case of $\bar{H}_0$ when normality holds.

Now, we are in a position to calculate $E(\sum_{p=2}^{m}\sum_{q=1}^{p-1}\rho_{pq}^2)$ and σ_{Nm}^2. It has been known that $E(\rho_{pq}^2) = (N-1)^{-1}$ under $\bar{H}_0$; see, e.g., page 124 in Hájek et al. (1999). Thus, $E(\sum_{p=2}^{m}\sum_{q=1}^{p-1}\rho_{pq}^2) = 0.5(N-1)^{-1}m(m-1)$. Besides, since ρ_{pq} is pairwise independent under $\bar{H}_0$ according to a lemma in page 153 in Buckley and Eagleson (1986), which is very similar to the pairwise independence of the sample (Pearson's) correlation coefficients (Geisser and Mantel, 1962), we have $\sigma_{Nm}^2 = \sum_{p=2}^{m}\sum_{q=1}^{p-1} var(\rho_{pq}^2)$ under $\bar{H}_0$. To compute σ_{Nm}^2, the following proposition is helpful.

Proposition 1. *Under $\bar{H}_0$, it holds for ρ_{pq} $(1 \le p \ne q \le m)$ that*

$$(i)\ E\left(\rho_{pq}^2 - \frac{1}{N-1}\right)^2 = \frac{2(25N^3 - 57N^2 - 40N + 108)}{25(N-1)^3N(N+1)}$$

$$(ii)\ E\left(\rho_{pq}^2 - \frac{1}{N-1}\right)^4 = \frac{60N^{13} + O(N^{12})}{(N-1)^7N^5(N+1)^5}$$

Remark 1. In the proof of this proposition below, the exact higher even-order moments of ρ_{pq}, $E(\rho_{pq}^4)$, $E(\rho_{pq}^6)$, and $E(\rho_{pq}^8)$ are reported. To the best of our knowledge, these exact moments have not been seen in literature. We believe that they may be of independent interest.

In terms of (i), we obtain our standardized test statistic:

$$s_{Nm} = (\sigma_{Nm}^2)^{-\frac{1}{2}} \left(\sum_{p=2}^{m} \sum_{q=1}^{p-1} \rho_{pq}^2 - \frac{m(m-1)}{2(N-1)} \right) \tag{3}$$

where $\sigma_{Nm}^2 = \frac{m(m-1)(25N^3-57N^2-40N+108)}{25(N-1)^3N(N+1)}$. Note that $\sigma_{Nm}^2 \to \gamma^2$ as $(m, N) \to \infty$ and $\lim_{(m,N)\to\infty} \frac{m}{N} = \gamma \in (0, \infty)$.

By virtue of (ii), we can further prove our main theoretical result.

Theorem 1. *Under $\bar{H}_0$, $s_{Nm} \xrightarrow{d} N(0, 1)$ as $(m, N) \to \infty$ and $\lim_{(m,N)\to\infty} \frac{m}{N} = \gamma \in (0, \infty)$.*

Using the result above, the total independence of an m-dimensional random vector of possible non normality can be tested based on s_{Nm}. Like the test procedure based on t_{nm} suggested by Schott (2005), $\bar{H}_0$ will be rejected if $s_{Nm} > c_{1-\alpha}$ and not rejected otherwise, where α is the significance level, and $c_{1-\alpha}$ is the $1-\alpha$ quantile of the $N(0, 1)$ distribution.

Remark 2. When preparing this article, we found that rank-based tests for $\bar{H}_0$ were also studied by Leung and Drton (2015). They showed that under $\bar{H}_0$,

$$T_\tau = \sum_{p=2}^{m} \sum_{q=1}^{p-1} \tau_{pq}^2 - \frac{m(m-1)(2N+5)}{9N(N-1)} \xrightarrow{d} N\left(0, \frac{16}{81}\gamma^2\right)$$

$$T_\rho = \sum_{p=2}^{m} \sum_{q=1}^{p-1} \rho_{pq}^2 - \frac{m(m-1)}{2(N-1)} \xrightarrow{d} N(0, \gamma^2) \tag{4}$$

where τ_{pq} denotes Kendall's rank correlation between Y_p and Y_q. Since $s_{Nm} = \sigma_{Nm}^{-1} T_\rho$, the second result is a direct corollary of our Theorem 1. It is worth noting that to prove the results above, Leung and Drton (2015) employed the theories concerning U-statistics. Contrastingly, to prove $s_{Nm} \xrightarrow{d} N(0, 1)$ under $\bar{H}_0$, unlike them, we still follow a procedure similar to that formulated by Schott (2005), which is largely due to the fact that we have exact higher even-order moments of ρ_{pq}. To implement tests, Leung and Drton (2015) used $\frac{m}{N}$, the estimator of γ, to standardize T_ρ, whereas we adopt σ_{Nm}, the exact standard deviation of T_ρ.

3. Simulation studies

To illustrate the robustness of the test based on s_{Nm} (s_{Nm}-test) in (3), we take the tests based on t_{Nm} (t_{Nm}-test) in (1) proposed by Schott (2005) and another statistic T_3 (T_3-test) developed by Srivastava (2005) as two competitors. Since the two statistics are constructed under the normality assumption and useful for testing $\tilde{H}_0$, they are directly related to the total independence test. In Srivastava (2005), T_3 is defined as

$$T_3 = \frac{(N-1)(\hat{\lambda}_3 - 1)}{2\sqrt{1 - \hat{a}_{40}/(m\hat{a}_{20}^2)}}$$

Table 1. Size of tests under independence (normal distribution).

Test	$n\backslash m$	4	8	16	32	64	128	256	512
s_{Nm}	4	0.072	0.060	0.062	0.054	0.057	0.052	0.053	0.054
	8	0.065	0.066	0.060	0.060	0.055	0.055	0.053	0.055
	16	0.058	0.061	0.055	0.059	0.052	0.053	0.055	0.052
	32	0.056	0.061	0.056	0.060	0.053	0.055	0.053	0.051
	64	0.060	0.059	0.053	0.055	0.053	0.048	0.052	0.054
	128	0.054	0.057	0.055	0.052	0.048	0.053	0.048	0.048
T_ρ	4	0.035	0.041	0.044	0.050	0.049	0.048	0.045	0.046
	8	0.040	0.053	0.054	0.056	0.051	0.051	0.057	0.051
	16	0.038	0.051	0.051	0.054	0.049	0.055	0.053	0.050
	32	0.042	0.054	0.056	0.056	0.050	0.053	0.052	0.050
	64	0.051	0.057	0.049	0.053	0.051	0.048	0.051	0.053
	128	0.046	0.056	0.049	0.050	0.046	0.052	0.047	0.048
t_{Nm}	4	0.069	0.071	0.076	0.073	0.072	0.065	0.068	0.066
	8	0.070	0.062	0.060	0.069	0.060	0.057	0.056	0.058
	16	0.061	0.061	0.064	0.064	0.051	0.064	0.057	0.053
	32	0.069	0.064	0.065	0.058	0.056	0.053	0.053	0.054
	64	0.075	0.063	0.059	0.057	0.049	0.048	0.047	0.055
	128	0.073	0.065	0.055	0.053	0.045	0.054	0.052	0.050
T_3	4	0.100	0.130	0.115	0.089	0.078	0.069	0.067	0.062
	8	0.106	0.076	0.061	0.061	0.055	0.056	0.053	0.052
	16	0.071	0.060	0.059	0.063	0.051	0.059	0.055	0.045
	32	0.073	0.065	0.063	0.058	0.058	0.053	0.054	0.054
	64	0.077	0.063	0.057	0.054	0.051	0.048	0.046	0.053
	128	0.073	0.066	0.059	0.054	0.047	0.055	0.050	0.049

where

$$\hat{\lambda}_3 = \frac{(N-1)}{(N-1)} \frac{[tr(S^2) - \frac{1}{N-1}(trS)^2]}{\sum_{i=1}^m s_{ii}^2}, \qquad \hat{a}_{20} = \frac{(N-1)\sum_{i=1}^m s_{ii}^2}{m(N+1)}, \qquad \hat{a}_{40} = \frac{\sum_{i=1}^m s_{ii}^4}{m}$$

in which S is the $m \times m$ sample covariance matrix, and s_{ii} is the ith diagonal element of S. Under different assumptions, Srivastava (2005) showed that $T_3 \xrightarrow{d} N(0,1)$ under $\tilde{H}_0$ as $(m, N) \to \infty$ and $N = O(m^\delta)$ for some $\delta \in (0, 1]$.[3] Additionally, we also consider the comparison between our s_{Nm}-test with those rank-based tests developed by Leung and Drton (2015). Since the test based on T_τ (T_τ-test) significantly suffers from size distortion in finite samples and is outperformed by that based on T_ρ (T_ρ -test) in (4) according to the simulation results in Leung and Drton (2015) , to save the space, we only include T_ρ-test in our simulation.

We evaluate the four tests by comparing their empirical size and power. Our simulation considers different combinations of $m \in \{4, 8, 16, 32, 64, 128, 256, 512\}$ and $N \in \{4, 8, 16, 32, 64, 128\}$. For each combination, 5000 independent repetitions are implemented. All the tests are studied at 5% significance level.

First, we investigate the performance of the three tests under $\bar{H}_0$. Three cases are considered: (a)$Y_i \overset{i.i.d.}{\sim} N(0, 1) (i = 1, \ldots, m)$; (b)$Y_i \overset{i.i.d.}{\sim} Cauchy(0, 1) (i = 1, \ldots, m)$; (c)$Y_i \overset{i.i.d.}{\sim} t(2) (i = 1, \ldots, m)$. The first case is normal and the other two are of non normality and infinite variance. Simulation results are collected in Tables 1–3. As we can see, when the data are normal, the empirical sizes of the four tests are quite close to 5%. However, when

[3] In T_3, $1 - \hat{a}_{40}/m\hat{a}_{20}^2$ may be smaller than zero in terms of the definition of $\hat{a}_{20}$ and $\hat{a}_{40}$. In our simulation, if $1 - \hat{a}_{40}/m\hat{a}_{20}^2 < 0$ we will compute T_3 by letting $\hat{a}_{20} = \sum_{i=1}^p s_{ii}^2/m$, under which T_3 is always well defined in terms of the Cauchy–Schwarz inequality.

Table 2. Size of tests under independence (Cauchy distribution).

Test	$n\backslash m$	4	8	16	32	64	128	256	512
s_{Nm}	4	0.073	0.065	0.055	0.061	0.054	0.055	0.057	0.055
	8	0.064	0.062	0.053	0.058	0.057	0.057	0.062	0.052
	16	0.056	0.058	0.056	0.055	0.053	0.056	0.057	0.053
	32	0.055	0.056	0.052	0.055	0.052	0.057	0.054	0.053
	64	0.051	0.059	0.056	0.053	0.053	0.051	0.052	0.049
	128	0.050	0.056	0.054	0.055	0.054	0.055	0.046	0.051
T_ρ	4	0.034	0.039	0.041	0.040	0.047	0.048	0.049	0.051
	8	0.040	0.045	0.052	0.049	0.050	0.051	0.058	0.056
	16	0.044	0.055	0.056	0.057	0.049	0.054	0.055	0.057
	32	0.047	0.052	0.048	0.051	0.050	0.056	0.053	0.052
	64	0.050	0.050	0.051	0.053	0.054	0.060	0.052	0.048
	128	0.048	0.051	0.056	0.052	0.053	0.054	0.046	0.051
t_{Nm}	4	0.068	0.074	0.070	0.077	0.077	0.072	0.074	0.073
	8	0.091	0.085	0.096	0.096	0.092	0.100	0.099	0.093
	16	0.134	0.132	0.140	0.139	0.142	0.152	0.139	0.142
	32	0.152	0.182	0.196	0.205	0.203	0.214	0.216	0.206
	64	0.140	0.228	0.249	0.260	0.260	0.266	0.274	0.270
	128	0.108	0.205	0.291	0.304	0.309	0.319	0.330	0.331
T_3	4	0.107	0.122	0.135	0.143	0.145	0.151	0.151	0.150
	8	0.135	0.139	0.155	0.162	0.159	0.165	0.168	0.169
	16	0.118	0.125	0.139	0.145	0.149	0.155	0.152	0.159
	32	0.102	0.106	0.120	0.133	0.127	0.124	0.131	0.126
	64	0.084	0.096	0.101	0.103	0.098	0.104	0.103	0.108
	128	0.062	0.075	0.082	0.079	0.075	0.086	0.082	0.087

Table 3. Size of tests under independence (t distribution).

Test	$n\backslash m$	4	8	16	32	64	128	256	512
s_{Nm}	4	0.081	0.067	0.060	0.050	0.053	0.054	0.053	0.056
	8	0.066	0.062	0.061	0.060	0.057	0.055	0.055	0.054
	16	0.063	0.059	0.055	0.059	0.056	0.056	0.054	0.057
	32	0.059	0.061	0.057	0.054	0.050	0.055	0.057	0.056
	64	0.055	0.054	0.055	0.056	0.056	0.053	0.052	0.053
	128	0.056	0.056	0.054	0.051	0.055	0.055	0.050	0.049
T_ρ	4	0.037	0.046	0.040	0.043	0.050	0.049	0.045	0.050
	8	0.042	0.048	0.051	0.056	0.059	0.056	0.058	0.056
	16	0.048	0.050	0.052	0.055	0.053	0.054	0.051	0.056
	32	0.048	0.050	0.051	0.049	0.052	0.054	0.056	0.055
	64	0.046	0.050	0.050	0.054	0.043	0.052	0.051	0.052
	128	0.047	0.055	0.053	0.048	0.054	0.055	0.049	0.049
t_{Nm}	4	0.065	0.074	0.080	0.066	0.071	0.070	0.068	0.072
	8	0.076	0.066	0.073	0.068	0.072	0.065	0.070	0.068
	16	0.087	0.085	0.090	0.085	0.082	0.081	0.080	0.083
	32	0.095	0.087	0.106	0.102	0.105	0.100	0.103	0.102
	64	0.097	0.117	0.125	0.123	0.128	0.126	0.125	0.124
	128	0.094	0.126	0.139	0.157	0.160	0.160	0.164	0.164
T_3	4	0.082	0.107	0.120	0.120	0.117	0.131	0.128	0.135
	8	0.116	0.114	0.125	0.129	0.142	0.162	0.171	0.182
	16	0.108	0.116	0.135	0.141	0.160	0.188	0.199	0.200
	32	0.099	0.110	0.130	0.157	0.174	0.205	0.204	0.206
	64	0.087	0.108	0.135	0.143	0.178	0.198	0.211	0.210
	128	0.082	0.107	0.127	0.149	0.170	0.191	0.204	0.212

Table 4. Power of tests under dependence.

Test	$n\backslash m$	4	8	16	32	64	128	256	512
s_{Nm}	4	0.100	0.083	0.088	0.112	0.160	0.239	0.346	0.482
	8	0.081	0.098	0.132	0.216	0.357	0.552	0.736	0.868
	16	0.103	0.155	0.263	0.475	0.720	0.905	0.979	0.996
	32	0.158	0.290	0.519	0.827	0.973	1.000	1.000	1.000
	64	0.278	0.546	0.864	0.994	1.000	1.000	1.000	1.000
	128	0.502	0.878	0.998	1.000	1.000	1.000	1.000	1.000
T_ρ	4	0.046	0.051	0.063	0.087	0.128	0.210	0.311	0.448
	8	0.056	0.079	0.115	0.204	0.341	0.541	0.729	0.867
	16	0.081	0.136	0.243	0.460	0.713	0.904	0.979	0.996
	32	0.127	0.261	0.502	0.819	0.972	1.000	1.000	1.000
	64	0.234	0.522	0.868	0.994	1.000	1.000	1.000	1.000
	128	0.451	0.857	0.997	1.000	1.000	1.000	1.000	1.000
t_{Nm}	4	0.075	0.078	0.094	0.130	0.193	0.281	0.406	0.546
	8	0.081	0.107	0.150	0.257	0.417	0.609	0.786	0.902
	16	0.116	0.165	0.293	0.529	0.771	0.928	0.985	0.998
	32	0.173	0.324	0.575	0.863	0.984	0.999	1.000	1.000
	64	0.301	0.590	0.907	0.996	1.000	1.000	1.000	1.000
	128	0.543	0.901	0.998	1.000	1.000	1.000	1.000	1.000
T_3	4	0.102	0.158	0.150	0.170	0.239	0.344	0.479	0.618
	8	0.134	0.128	0.171	0.283	0.449	0.651	0.820	0.922
	16	0.133	0.187	0.309	0.544	0.789	0.935	0.987	0.998
	32	0.183	0.329	0.582	0.870	0.983	0.999	1.000	1.000
	64	0.306	0.592	0.910	0.997	1.000	1.000	1.000	1.000
	128	0.545	0.901	0.999	1.000	1.000	1.000	1.000	1.000

it comes to the other two cases, the size distortion of t_{Nm}-test and T_3-test is significant. In contrast, s_{Nm}-test and T_ρ-test still perform well. Next, we employed a simulation scheme adopted by Schott (2005) to illustrate the performance of the s_{Nm}-test when $\bar{H}_0$ does not hold. The scheme generates the data according to $(Y_1, \ldots, Y_m)' \overset{i.i.d.}{\sim} N_m(0, \Sigma_\rho)$, where $N_m(0, \Sigma_\rho)$ denotes m-dimensional normal distribution with zero means and variance matrix $\Sigma_\rho = (1-\rho)I_m + \rho ee'$, in which $\rho = 0.1$, I_m is the identity matrix of dimension m, and e is the m-dimensional column vector of ones. Table 4 reports the related results. It can be found that the four tests are consistent and have similar performance. According to size and power results, s_{Nm}-test and T_ρ-test are comparable in performance.

In summary, our simulation results suggest that the newly developed s_{Nm}-test is more robust than the existing t_{Nm}-test and T_3-test. When the data are heavy-tailed and may be of infinite variance, s_{Nm}-test may be a better choice.

4. A real example

This section illustrates the use of the test proposed by this article via a real data set. In financial markets, a necessary condition for constructing an asset portfolio, which may be used to, i.e., make risk hedging, is that the targeted assets are correlated. Therefore, it is natural to test the total independence of the targeted assets before employing them to making an asset portfolio. Here we consider a data set which consists of daily returns of 96 public companies in China's financial markets.[4] These companies belong to the category "Transportation, Warehousing, and Postal Industry" proposed by China Securities Regulatory Commission. It has been widely recognized that stock markets are highly volatile. Therefore, even though there

[4] The data set comes from CSMAR Solution.

are abundant data across the time dimension, usually only a small proportion of the data is employed in practice for analysis. In this article, we use the data concerning the 96 companies from October 8 to December 31, 2015. There are 61 trading days in this sample period and hence $n = 61$ in the present case. Of these companies, 27 companies have the experience of suspending trading during the period, and hence there is a problem of missing data. For simplicity, we exclude the 27 companies in our analysis. Therefore, our sample only comprises daily returns of 69 companies, i.e., $m = 69$.

Before proceeding to the independence test, we first employ Jarque–Bera test (Jarque and Bera, 1980, 1987) to detect the normality for each return series. It is found that the p values of the 69 tests that are smaller than 0.01 (0.05 or 0.1) account for 37.7% (49.3% or 60.0%), which suggests that it may be inappropriate to assume that the daily returns in our sample are normally distributed. Thus, to test the total independence of the corresponding assets, it may be more reasonable to apply the methods that are robust to non normality. By implementing the test developed in this article and the T_ρ-test proposed by Leung and Drton (2015), we find $s_{Nm} = 523$ and $T_\rho = 518$, which is strong evidence that the assets considered here are correlated.

5. Concluding remarks

This article proposes a new statistic based on pairwise Spearman's rank correlation coefficients, which has been shown to be helpful to test total independence among an high-dimensional random vector. The new statistic is closely related to T_ρ in Leung and Drton (2015), and is identical to $\sigma_{Nm}^{-1} T_\rho$, where σ_{Nm} is the exact standard deviation of T_ρ. Intuitively, we may analogously standardize T_τ in Leung and Drton (2015) using its exact standard deviation to construct new statistic. Given the fact that T_τ-test has poor finite-sample performance, this standardization may bring some improvement. To achieve the aim, we need to find the exact higher even-order moments of τ_{pq}. However, the method in this article cannot be trivially extended to this case since τ_{pq}, unlike ρ_{pq}, is not a linear rank statistic (see, e.g., page 126 in Hájek et al. (1999)). As a result, special treatment for τ_{pq} is required. Since this topic is largely beyond the scope of this article, we do not make further discussion about it here and may study it in the future.

Acknowledgment

The author would like to thank the referees for providing valuable comments, which substantially improved this article. The author is also grateful to the editor Prof. N. Balakrishnan for his encouragement.

Funding

Financial support from Beijing Natural Science Foundation (Grant No. 1174018) and Special Foundation for Young Researchers of Beijing Jiaotong University (Grant No. 2014RC023) are gratefully acknowledged.

References

Ahmad, M.R. (2016). On testing sphericity and identity of a covariance matrix with large dimensions. *Math. Methods Stat.* 25:121–132.

Buckley, M.J., Eagleson, G.K. (1986). Assessing large sets of rank correlations. *Biometrika* 73:151–157.
Cai, T.T., Jiang, T. (2011). Limiting laws of coherence of random matrices with applications to testing covariance structure and construction of compressed sensing matrices. *Ann. Stat.* 39:1496–1525.
Chen, S.X., Zhang, L., Zhong, P. (2010). Tests for high-dimensional covariance matrices. *J. Am. Stat. Assoc.* 105:810–819.
Fujikoshi, Y., Ulyanov, V.V., Shimizu, R. (2010). Multivariate statistics: High-dimensional and large-sample approximations. Hoboken, NJ: Wiley.
Geisser, S., Mantel, N. (1962). Pairwise independence of jointly dependent variables. *Ann. Math. Stat.* 33:290–291.
Hájek, J., Šidák, Z., Sen, P.K. (1999). *Theory of Rank Tests*. San Diego, CA: Academic Press.
Jarque, C.M., Bera, A.K. (1980). Efficient tests for normality, homoscedasticity and serial independence of regression residuals. *Econ. Lett.* 6:255–259.
Jarque, C.M., Bera, A.K. (1987). A test for normality of observations and regression residuals. *Int. Stat. Rev.* 55:163–172.
Ledoit, O., Wolf, M. (2002). Some hypothesis tests for the covariance matrix when the dimension is large compared to the sample size. *Ann. Stat.* 30:1081–1102.
Leung, D., Drton, M. (2015). Testing mutual independence in high dimensions with sums of squares of rank correlations. Available at: *http://arxiv.org/abs/1501.01732*.
Mao, G. (2014). A new test of independence for high-dimensional data. *Stat. Probab. Lett.* 93:14–18.
McLeish, D.L. (1974). Dependent central limit theorems and invariance principles. *Ann. Probab.* 2:620–628.
Qiu, Y., Chen, S.X. (2012). Test for bandedness of high-dimensional covariance matrices and bandwidth estimation. *Ann. Stat.* 40:1285–1314.
Schott, J.R. (2005). Testing for complete independence in high dimensions. *Biometrika* 92:951–956.
Spearman, C. (1904). The proof and measurement of association between two things. *Am. J. Psychol.* 15:72–101.
Srivastava, M.S. (2005). Some tests concerning the covariance matrix in high dimensional data. *J. Jpn. Stat. Soc.* 35:251–272.
Srivastava, M.S., Kollo, T., von Rosen, D. (2005). Some tests for the covariance matrix with fewer observations than the dimension under non-normality. *J. Multivar. Anal.* 102:1090–1103.
Srivastava, M.S., Yanagihara, H., Kubokawa, T. (2014). Tests for covariance matrices in high dimension with less sample size. *J. Multivar. Anal.* 130:289–309.
Zou, C., Peng, L., Feng, L., Wang, Z. (2014). Multivariate sign-based high-dimensional tests for sphericity. *Biometrika* 101:229–236.

Appendix: Proofs

Let $(D_{p1}, \ldots, D_{pN})$ be the antiranks of $(R_{p1}, \ldots, R_{pN})$, i.e., the statistics satisfying $D_{pR_{pi}} = R_{pD_{pi}} = i$ for $i \in \{1, \ldots, N\}$. Based on the antiranks, ρ_{pq} defined in (2) can be equivalently written as

$$\rho_{pq} = \frac{12}{N^3 - N} \sum_{i=1}^{N} (i - c_N)(Q_{qpi} - c_N)$$

where $Q_{qpi} = R_{qD_{pi}}$ and $c_N = \frac{N+1}{2}$; see, e.g., page 124 in Hájek et al. (1999). Note that if Y_p is independent of Y_q, $(Q_{pq1}, \ldots, Q_{pqN})$ is uniformly distributed on the space of all permutations of $\{1, \ldots, N\}$.

First, we compute $E(\rho_{pq}^k)$ for $i \in \{1, \ldots, 8\}$. To do so, it suffices to calculate the moments of $\sum_{i=1}^N iQ_{qpi}$ by noting that $\sum_{i=1}^N (i - c_N)(Q_{qpi} - c_N) = \sum_{i=1}^N iQ_{qpi} - Nc_N^2$. For simplicity, we abbreviate Q_{qpi} to Q_i below unless there is a risk of misunderstanding, and denote $A_N = \sum_{i=1}^N iQ_i$. Before proceeding to the calculation of $E(A_N^k)$ for $k = 1, \ldots, 8$, we first give a useful lemma.

Lemma A.1. *Define* $L_J^{k_1\ldots k_J}$ *as* $E(\prod_{j=1}^J Q_{i_j}^{k_j})$*, where* $k_1, \ldots, k_J$ *are positive integers, and* $i_1, \ldots, i_J$ *are mutually distinct positive numbers in* $\{1, \ldots, N\}$*. Then under* $\bar{H}_0$*,* $L_J^{k_1\ldots k_J}$ *satisfies* $L_1^{k_1} = \frac{S_{k_1}}{N}$ *and*

$$L_J^{k_1\ldots k_J} = \frac{NL_1^{k_1}L_{J-1}^{k_2\ldots k_J} - L_{J-1}^{(k_1+k_2)k_3\ldots k_J} - L_{J-1}^{k_2(k_1+k_3)\ldots k_J} \cdots - L_{J-1}^{k_2k_3\ldots(k_1+k_J)}}{N-J+1} \quad \text{for } J > 1 \tag{5}$$

where $S_{k_1} = \sum_{i=1}^N i^{k_1}$.

The definition of $L_J^{k_1\ldots k_J}$ implies that $E(\prod_{j=1}^J Q_{i_j}^{k_j})$ are always the same for mutually distinct $i_1, \ldots, i_J$ as long as $k_1, \ldots, k_J$, and J are fixed. For example, $E(Q_2^{k_1}Q_4^{k_2}Q_5^{k_3}) = E(Q_1^{k_1}Q_3^{k_2}Q_5^{k_3}) = E(Q_6^{k_1}Q_7^{k_2}Q_8^{k_3})$. This is due to the fact that $(Q_1, \ldots, Q_N)$ is uniformly distributed on the space of all permutations of $\{1, \ldots, N\}$ as mentioned above. Since $A_N = \sum_{i=1}^N iQ_i$, $L_J^{k_1\ldots k_J}$ for different $k_1, \ldots, k_J$, and J are the ingredients for us to compute $E(A_N^k)$.

Proof. Because of the uniformity of $(Q_1, \ldots, Q_N)$ on the permutation space of $\{1, \ldots, N\}$, we have $L_J^{k_1\ldots k_J} = E(\prod_{j=1}^J Q_{i_j}^{k_j}) = E(\prod_{j=1}^J Q_j^{k_j})$. It is easy to verify $L_1^{k_1} = E(Q_1^{k_1}) = \frac{S_{k_1}}{N}$. Besides,

$$L_J^{k_1\ldots k_J} = E\left(\prod_{j=1}^J Q_j^{k_j}\right) = E\left[\prod_{j=2}^J Q_j^{k_j} E\left(Q_1^{k_1}|Q_2, \ldots, Q_J\right)\right]$$

Since $(Q_1, \ldots, Q_N)$ is a permutation of $\{1, \ldots, N\}$, note that $\sum_{i=1}^N Q_i^{k_1}$ is always identical to S_{k_1}, and Q_1 can only take value in $\{1, \ldots, N\}\backslash\{Q_2, \ldots, Q_J\}$ with equal probability $(N-J+1)^{-1}$ provided that the values of $Q_2, \ldots, Q_J$ are given. Thus, we further have

$$\begin{aligned}
L_J^{k_1\ldots k_J} &= \frac{E\left[\prod_{j=2}^J Q_j^{k_j}(S_{k_1} - Q_2^{k_1} - \cdots - Q_J^{k_1})\right]}{N-J+1} \\
&= \frac{S_{k_1}E\left(\prod_{j=1}^{J-1} Q_j^{k_{j+1}}\right) - E\left[\prod_{j=2}^J Q_j^{k_j}(Q_2^{k_1} - \cdots - Q_J^{k_1})\right]}{N-J+1} \\
&= \frac{NL_1^{k_1}L_{J-1}^{k_2\ldots k_J} - L_{J-1}^{(k_1+k_2)k_3\ldots k_J} - L_{J-1}^{k_2(k_1+k_3)\ldots k_J} \cdots - L_{J-1}^{k_2k_3\ldots(k_1+k_J)}}{N-J+1}
\end{aligned}$$

□

Lemma A.1 provides a useful iteration formula to compute $L_J^{k_1\ldots k_J}$ for any $k_1, \ldots k_J$, and J based on $L_1^{k_1}$ $(k_1 \geq 1)$. Since S_{k_1} can be expressed by a polynomial of N of (k_1+1)th degree according to the Faulhaber's formula

$$S_{k_1} = \frac{1}{k_1+1}\sum_{j=0}^{k_1}(-1)^j\binom{k_1+1}{j}B_jN^{k_1+1-j}$$

where B_j are the Bernoulli numbers with $B_1 = -\frac{1}{2}$, we can find the closed-form expression of $L_J^{k_1\ldots k_J}$, i.e., the functional relationship between $L_J^{k_1\ldots k_J}$ and N. This can be easily done by virtue of any package that can conduct symbolic computation. The results in the following Lemma are obtained with the help of the MATLAB Symbolic Math Toolbox.

Lemma A.2. *Under $\bar{H}_0$,*

$$E(A_N) = \frac{N(N+1)^2}{4} \qquad E\left(A_N^2\right) = \frac{N^2(N+1)^2(9N^2+19N+8)}{144}$$

$$E\left(A_N^3\right) = \frac{N^3(N+1)^4(N+2)(3N+1)}{192} \qquad E\left(A_N^4\right) = \frac{N^3(N+1)^3(\sum_{k=0}^{6} a_{4k}N^k)}{172800}$$

$$E\left(A_N^5\right) = \frac{N^4(N+1)^5(N+2)(\sum_{k=0}^{5} a_{5k}N^k)}{138240} \qquad E\left(A_N^6\right) = \frac{N^3(N+1)^3(\sum_{k=0}^{12} a_{6k}N^k)}{243855360}$$

$$E\left(A_N^7\right) = \frac{N^4(N+1)^5(N+2)(\sum_{k=0}^{11} a_{7k}N^k)}{139345920} \qquad E\left(A_N^8\right) = \frac{N^3(N+1)^3(\sum_{k=0}^{18} a_{8k}N^k)}{125411328000}$$

where the coefficients of the polynomials in the formulae for $E(A_N^j)(j=4,\dots,8)$ can be found in the proof.

Proof.

(i) $E(A_N) = \sum_{i=1}^N iE(Q_i) = E(Q_1)\sum_{i=1}^N i = L_1^1 S_1 = N(L_1^1)^2 = \frac{N(N+1)^2}{4}$

(ii)

$$E\left(A_N^2\right) = E\left(\sum_{i_1=1}^N\sum_{i_2=1}^N i_1 i_2 Q_{i_1}Q_{i_2}\right) = E\left(\sum_{i_1=1}^N i_1^2 Q_{i_1}^2\right) + E\left(\sum_{i_1=1}^N\sum_{i_2\neq i_1}^N i_1 i_2 Q_{i_1}Q_{i_2}\right)$$
$$= E\left(Q_1^2\right)\sum_{i_1=1}^N i_1^2 + E(Q_1Q_2)\sum_{i_1=1}^N\sum_{i_2\neq i_1}^N i_1 i_2$$

Recall that $(Q_1, Q_2, \dots, Q_N)$ is uniformly distributed on the permutation space of $\{1,\dots,N\}$. Thus, $E(Q_1Q_2) = \frac{\sum_{i_1=1}^N\sum_{i_2\neq i_1}^N i_1 i_2}{N(N-1)}$, which suggests

$$E\left(A_N^2\right) = L_1^2 S_2 + N(N-1)[E(Q_1Q_2)]^2$$
$$= N\left(L_1^2\right)^2 + N(N-1)\left(L_2^{11}\right)^2 = \frac{N^2(N+1)^2(9N^2+19N+8)}{144}$$

(iii) Similarly,

$$E(A_N^3) = E\left(\sum_{i_1=1}^N\sum_{i_2=1}^N\sum_{i_3=1}^N i_1 i_2 i_3 Q_{i_1}Q_{i_2}Q_{i_3}\right)$$
$$= E\left(\sum_{i_1=1}^N i_1^3 Q_{i_1}^3\right) + 3E\left(\sum_{i_1=1}^N\sum_{i_2\neq i_1}^N i_1^2 i_2 Q_{i_1}^2 Q_{i_2}\right)$$
$$+E\left(\sum_{i_1=1}^N\sum_{i_2\neq i_1}^N\sum_{i_3\neq i_1, i_3\neq i_2}^N i_1 i_2 i_3 Q_{i_1}Q_{i_2}Q_{i_3}\right)$$
$$= N\left(L_1^3\right)^2 + 3N(N-1)\left(L_2^{21}\right)^2 + N(N-1)(N-2)\left(L_3^{111}\right)^2$$
$$= \frac{N^3(N+1)^4(N+2)(3N+1)}{192}$$

(iv) Following analysis is entirely analogous, so we only give a sketch. For simplicity, below we use N_J to denote $\prod_{j=0}^{J}(N-j)$.

$$\begin{aligned} E\left(A_N^4\right) &= E\left(\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\sum_{i_3=1}^{N}\sum_{i_4=1}^{N} i_1 i_2 i_3 i_4 Q_{i_1} Q_{i_2} Q_{i_3} Q_{i_4}\right) \\ &= N_0\left(L_1^4\right)^2 + 4N_1\left(L_2^{31}\right)^2 + 3N_1\left(L_2^{22}\right)^2 + 6N_2\left(L_3^{211}\right)^2 + N_3\left(L_4^{1111}\right)^2 \\ &= \frac{N^3(N+1)^3\left(\sum_{k=0}^{6} a_{4,k}N^k\right)}{172800} \end{aligned}$$

where $a_{4,0}=-72$, $a_{4,1}=332$, $a_{4,2}=2478$, $a_{4,3}=6687$, $a_{4,4}=7675$, $a_{4,5}=3825$, and $a_{4,6}=675$.

(v)

$$\begin{aligned} E\left(A_N^5\right) &= N_0\left(L_1^5\right)^2 + 5N_1\left(L_2^{41}\right)^2 + 10N_1\left(L_2^{32}\right)^2 + 10N_2\left(L_3^{311}\right)^2 \\ &\quad +15N_2\left(L_3^{221}\right)^2 + 10N_3\left(L_4^{2111}\right)^2 + N_4\left(L_5^{11111}\right)^2 \\ &= \frac{N^4(N+1)^5(N+2)\left(\sum_{k=0}^{5} a_{5,k}N^k\right)}{138240} \end{aligned}$$

where $a_{5,0}=-36$, $a_{5,1}=64$, $a_{5,2}=157$, $a_{5,3}=565$, $a_{5,4}=555$, and $a_{5,5}=135$.

(vi)

$$\begin{aligned} E(A_N^6) &= N_0\left(L_1^6\right)^2 + 6N_1\left(L_2^{51}\right)^2 + 15N_1\left(L_2^{42}\right)^2 + 10N_1\left(L_2^{33}\right)^2 + 15N_2\left(L_3^{411}\right)^2 \\ &\quad +60N_2\left(L_3^{321}\right)^2 + 15N_2\left(L_3^{222}\right)^2 + 20N_3\left(L_4^{3111}\right)^2 + 45N_3\left(L_4^{2211}\right)^2 \\ &\quad +15N_4\left(L_5^{21111}\right)^2 + N_5\left(L_6^{111111}\right)^2 \\ &= \frac{N^3\,(N+1)^3\left(\sum_{k=0}^{12} a_{6,k}N^k\right)}{243855360} \end{aligned}$$

where $a_{6,0}=28800$, $a_{6,1}=-72960$, $a_{6,2}=-105376$, $a_{6,3}=-174792$, $a_{6,4}=-88156$, $a_{6,5}=1108378$, $a_{6,6}=3830267$, $a_{6,7}=7295236$, $a_{6,8}=8753997$, $a_{6,9}=6440266$, $a_{6,10}=2771685$, $a_{6,11}=635040$, and $a_{6,12}=59535$.

(vii)

$$\begin{aligned} E\left(A_N^7\right) &= N_0\left(L_1^7\right)^2 + 7N_1\left(L_2^{61}\right)^2 + 21N_1\left(L_2^{52}\right)^2 + 35N_1\left(L_2^{43}\right)^2 + 21N_2\left(L_3^{511}\right)^2 \\ &\quad +105N_2\left(L_3^{421}\right)^2 + 70N_2\left(L_3^{331}\right)^2 + 105N_2\left(L_3^{322}\right)^2 + 35N_3\left(L_4^{4111}\right)^2 \\ &\quad +210N_3\left(L_4^{3211}\right)^2 + 105N_3\left(L_4^{2221}\right)^2 + 35N_4\left(L_5^{31111}\right)^2 \\ &\quad +105N_4\left(L_5^{22111}\right)^2 + 21N_5\left(L_6^{211111}\right)^2 + N_6\left(L_7^{1111111}\right)^2 \\ &= \frac{N^4\,(N+1)^5\,(N+2)\,\left(\sum_{k=0}^{11} a_{7,k}N^k\right)}{139345920} \end{aligned}$$

where $a_{7,0}=14400$, $a_{7,1}=-43680$, $a_{7,2}=904$, $a_{7,3}=6148$, $a_{7,4}=-38206$, $a_{7,5}=76789$, $a_{7,6}=224312$, $a_{7,7}=390691$, $a_{7,8}=454762$, $a_{7,9}=277515$, $a_{7,10}=79380$, and $a_{7,11}=8505$.

(viii)

$$\begin{aligned} E\left(A_N^8\right) &= N_0\left(L_1^8\right)^2 + 8N_1\left(L_2^{71}\right)^2 + 28N_1\left(L_2^{62}\right)^2 + 56N_1\left(L_2^{53}\right)^2 + 35N_1\left(L_2^{44}\right)^2 \\ &\quad +28N_2\left(L_3^{611}\right)^2 + 168N_2\left(L_3^{521}\right)^2 + 280N_2\left(L_3^{431}\right)^2 + 280N_2\left(L_3^{332}\right)^2 \\ &\quad +210N_2\left(L_3^{422}\right)^2 + 56N_3\left(L_4^{5111}\right)^2 + 420N_3\left(L_4^{4211}\right)^2 + 280N_3\left(L_4^{3311}\right)^2 \end{aligned}$$

$$+840N_3\left(L_4^{3221}\right)^2+105N_3\left(L_4^{2222}\right)^2+70N_4\left(L_5^{41111}\right)^2+560N_4\left(L_5^{32111}\right)^2$$
$$+420N_4\left(L_5^{22211}\right)^2+56N_5\left(L_6^{311111}\right)^2+210N_5\left(L_6^{221111}\right)^2$$
$$+28N_6\left(L_7^{2111111}\right)^2+N_7\left(L_8^{11111111}\right)^2$$
$$=\frac{N^3(N+1)^3\left(\sum_{k=0}^{18}a_{8,k}N^k\right)}{125411328000}$$

where $a_{8,0}=-40642560$, $a_{8,1}=92510208$, $a_{8,2}=48773376$, $a_{8,3}=-147593280$, $a_{8,4}=-91515776$, $a_{8,5}=-178362512$, $a_{8,6}=-256876896$, $a_{8,7}=-62518156$, $a_{8,8}=104223488$, $a_{8,9}=637694109$, $a_{8,10}=1886379965$, $a_{8,11}=3248699216$, $a_{8,12}=3900623568$, $a_{8,13}=3388945450$, $a_{8,14}=2062031650$, $a_{8,15}=837540900$, $a_{8,16}=213759000$, $a_{8,17}=30830625$, and $a_{8,18}=1913625$. □

Based on the exact moments of A_N, we now prove Proposition 1.

Proof (Proposition 1). Using $\rho_{pq}=\frac{12}{N^3-N}\sum_{i=1}^N(i-c_N)(Q_i-c_N)$ and the results in Lemma A.2, it is straightforward to verify that $E(\rho_{pq}^k)=0$ for $k=1,3,5,7$, and

$$E(\rho_{pq}^2)=\frac{1}{N-1}\qquad E(\rho_{pq}^4)=\frac{3\left(25N^3-38N^2-35N+72\right)}{25(N-1)^3N(N+1)}$$
$$E(\rho_{pq}^6)=\frac{3\left(\sum_{k=0}^{8}b_{6,k}N^k\right)}{245(N-1)^5N^3(N+1)^3}\qquad E(\rho_{pq}^8)=\frac{3\left(\sum_{k=0}^{13}b_{8,k}N^k\right)}{875(N-1)^7N^5(N+1)^5}$$

where $b_{6,0}=-28800$, $b_{6,1}=44160$, $b_{6,2}=54280$, $b_{6,3}=-50081$, $b_{6,4}=-22783$, $b_{6,5}=23818$, $b_{6,6}=-178$, $b_{6,7}=-4361$, and $b_{6,8}=1225$; $b_{8,0}=40642560$, $b_{8,1}=-51867648$, $b_{8,2}=-74721024$, $b_{8,3}=110888256$, $b_{8,4}=71871632$, $b_{8,5}=-79689881$, $b_{8,6}=-27330110$, $b_{8,7}=30402746$, $b_{8,8}=2142858$, $b_{8,9}=-6275976$, $b_{8,10}=1090534$, $b_{8,11}=451718$, $b_{8,12}=-218050$, and $b_{8,13}=30625$. As a result, (i) and (ii) in this proposition can be directly verified. □

Finally, we prove our main theoretical result by virtue of the proposition above.

Proof (Theorem 1). Let $\pi_{pq}=\rho_{pq}^2-(N-1)^{-1}$, $X_{Np}=\sigma_{Nm}^{-1}\sum_{q=1}^{p-1}\pi_{pq}$, and $T_{Np}^*=X_{Np}+T_{N,p-1}^*$ for $p=2,\dots,m$, where $T_{N,1}^*=0$. Besides, denote $\mathcal{F}_{Np}$ as the σ-field generated by $\{Y_1,\dots,Y_p\}$. Note that $\mathcal{F}_{N1}\subseteq\cdots\subseteq\mathcal{F}_{Nm}$, $T_{Np}^*=\sum_{k=2}^{p}X_{Nk}$, and $T_{N,p-1}^*$ is measurable with respect to $\mathcal{F}_{N,p-1}$.

First, we show that $\{T_{Np}^*,\mathcal{F}_{Np},1\le p\le m,N\ge 1\}$ is a martingale array with differences X_{Np}. To do so, it is sufficient to prove $E(X_{Np}|\mathcal{F}_{N,p-1})=0$ since $E(T_{Np}^*|\mathcal{F}_{N,p-1})=T_{N,p-1}^*+E(X_{Np}|\mathcal{F}_{N,p-1})$. Note that ρ_{pq} defined in (2) is identical to $12(N^3-N)^{-1}\sum_{i=1}^N(R_{pi}-c_N)(R_{qi}-c_N)$; see, e.g., page 124 in Hájek et al. (1999). Thus, for $1\le q<p$, under $\bar{H}_0$, according to the independence between Y_p and $\{Y_1,\dots,Y_{p-1}\}$, we have

$$E\left(\rho_{pq}^2|\mathcal{F}_{N,p-1}\right)=\frac{144}{(N^3-N)^2}\sum_{i=1}^N\sum_{j=1}^N(R_{qi}-c_N)(R_{qj}-c_N)E[(R_{pi}-c_N)(R_{pj}-c_N)]$$
$$=\frac{144}{(N^3-N)^2}\Bigg[E(R_{pi}-c_N)^2\sum_{i=1}^N(R_{qi}-c_N)^2$$

$$+ E[(R_{pi}-c_N)(R_{pj}-c_N)]\sum_{i=1}^{N}\sum_{j\neq i}^{N}(R_{qi}-c_N)(R_{qj}-c_N)\Bigg]$$
$$= \frac{144}{(N^3-N)^2}(I_1+I_2)$$

Note that $(R_{p1},\dots,R_{pn})$ is uniformly distributed on the permutation space of $\{1,\dots,N\}$. Therefore, $E(R_{pi}-c_N)^2 = E(R_{pi}^2) - 2c_N E(R_{pi}) + c_N^2 = L_1^2 - 2c_N L_1^1 + c_N^2$ according to the definition of $L_J^{k_1\dots k_J}$ in Lemma A.1. Besides, $\sum_{i=1}^N (R_{qi}-c_N)^2 = \sum_{i=1}^N i^2 - 2c_N\sum_{i=1}^N i + Nc_N^2 = S_2 - 2c_N S_1 + Nc_N^2$. Thus, using the result $L_1^{k_1} = S_{k_1}/N$ in Lemma A.1, we have $I_1 = N(L_1^2 - 2c_N L_1^1 + c_N^2)^2$. In the same way, we can show that $I_2 = N(N-1)(L_2^{11} - 2c_N L_1^1 + c_N^2)^2$. Recall that $c_N = \frac{N+1}{2}$. As a result, $E(\rho_{pq}^2|\mathcal{F}_{N,p-1}) = (N-1)^{-1}$, which implies $E(\pi_{pq}|\mathcal{F}_{N,p-1}) = 0$ for $1 \le q < p$ and hence $E(X_{Np}|\mathcal{F}_{Np-1}) = 0$.

Next, note that $s_{Nm} = \sum_{p=2}^m X_{Np}$. Therefore, to prove $s_{Nm} \xrightarrow{d} N(0,1)$ under $\bar{H}_0$, central limit theorems for martingale differences are helpful. In terms of Theorem 2.3 in McLeish (1974) and the related discussion therein, it suffices for us to show

$$\sum_{p=2}^m EX_{Np}^4 \to 0 \qquad \text{and} \sum_{p=2}^m X_{Np}^2 \xrightarrow{p} 1$$
$$\text{as } (m,N)\to\infty \quad \text{and} \lim_{(m,N)\to\infty}\frac{m}{N} = \gamma \in (0,\infty)$$

where the first Liapounov condition can ensure (a) and (b) in McLeish's theorem, and $\xrightarrow{p}$ denotes convergence in probability. According to the lemma in page 153 in Buckley and Eagleson (1986), the rank correlations are pairwise independent under $\bar{H}_0$. As a consequence, it is easy to verify that $E(\pi_{pq_1}\pi_{pq_2}\pi_{pq_3}\pi_{pq_4}) \neq 0$ only if $q_1 = q_2 = q_3 = q_4$, or $q_1 = q_2 \neq q_3 = q_4$, or $q_1 = q_3 \neq q_2 = q_4$, or $q_1 = q_4 \neq q_2 = q_3$. Using Proposition 1 (ii), in this first case we have

$$E(\pi_{pq_1}\pi_{pq_2}\pi_{pq_3}\pi_{pq_4}) = E\left(\pi_{pq_1}^4\right) = \frac{60N^{13}+O(N^{12})}{(N-1)^7N^5(N+1)^5}$$

and by Proposition 1 (i), in the other cases we have

$$E\left(\pi_{pq_1}\pi_{pq_2}\pi_{pq_3}\pi_{pq_4}\right) = E(\pi_{pq_1}^2)E\left(\pi_{pq_2}^2\right) = \frac{4N^6+O(N^5)}{(N-1)^6N^2(N+1)^2}$$

Consequently,

$$EX_{Np}^4 = \sigma_{Nm}^{-4}\sum_{q_1=1}^{p-1}\sum_{q_2=1}^{p-1}\sum_{q_3=1}^{p-1}\sum_{q_4=1}^{p-1}E(\pi_{pq_1}\pi_{pq_2}\pi_{pq_3}\pi_{pq_4})$$
$$= \sigma_{Nm}^{-4}(p-1)O\left(\frac{1}{N^4}\right) + 3\sigma_{Nm}^{-4}(p-1)^2O\left(\frac{1}{N^4}\right)$$

Therefore,

$$\sum_{p=2}^m EX_{Np}^4 = \sigma_{Nm}^{-4}O\left(\frac{1}{N^4}\right)\sum_{p=2}^m(p-1) + 3\sigma_{Nm}^{-4}O\left(\frac{1}{N^4}\right)\sum_{p=2}^m(p-1)^2$$
$$= \sigma_{Nm}^{-4}O\left(\frac{1}{N^4}\right)\frac{m(m-1)}{2} + 3\sigma_{Nm}^{-4}O\left(\frac{1}{N^4}\right)\frac{m(m-1)(2m-1)}{6}$$

As a result, $\sum_{p=2}^{m} EX_{Np}^4 \to 0$ since $(m, N) \to \infty$, $\lim_{(m,N)\to\infty} \frac{m}{N} = \gamma \in (0, \infty)$, and $\sigma_{Nm}^2 \to \gamma^2$.

To verify the second condition, we first note that in terms of the pairwise independence of π_{pq} $(1 \le p \ne q \le m)$, $E(X_{Np}^2) = \sigma_{Nm}^{-2}(p-1)E(\pi_{p1}^2) = \frac{2(p-1)}{m(m-1)}$ since $\sigma_{Nm}^2 = 0.5m(m-1)E(\pi_{pq}^2)$. Thus, $E(\sum_{p=2}^{m} X_{Np}^2) = 1$. Thus, $E(\sum_{p=2}^{m} X_{Np}^2 - 1)^2 = \sum_{p=2}^{m} EX_{Np}^4 + \sum_{p=2}^{m}\sum_{p\ne\tilde{p}}^{m} E(X_{Np}^2 X_{N\tilde{p}}^2) - 1$. We have proved that the first term converges to zero, so it suffices to show the second term converges to 1. To see this, using the property of pairwise independence of π_{pq} $(1 \le p \ne q \le m)$ again, we have

$$\begin{aligned}
E\left(X_{Np}^2 X_{N\tilde{p}}^2\right) &= \sigma_{Nm}^{-4} \sum_{q_1=1}^{p-1}\sum_{q_2=1}^{p-1}\sum_{\tilde{q}_1=1}^{\tilde{p}-1}\sum_{\tilde{q}_2=1}^{\tilde{p}-1} E(\pi_{pq_1}\pi_{pq_2}\pi_{\tilde{p}\tilde{q}_1}\pi_{\tilde{p}\tilde{q}_2}) \\
&= \sigma_{Nm}^{-4} \sum_{q_1=1}^{p-1}\sum_{\tilde{q}_1=1}^{\tilde{p}-1} E\left(\pi_{pq_1}^2 \pi_{\tilde{p}\tilde{q}_1}^2\right) = \sigma_{Nm}^{-4}(p-1)(\tilde{p}-1)E\left(\pi_{p1}^2\right)^2 \\
&= \frac{4(p-1)(\tilde{p}-1)}{m^2(m-1)^2}
\end{aligned}$$

In consequence,

$$\begin{aligned}
\sum_{p=2}^{m}\sum_{p\ne\tilde{p}}^{m} E(X_{Np}^2 X_{N\tilde{p}}^2) &= \frac{4}{m^2(m-1)^2}\left[\sum_{p=1}^{m}\sum_{\tilde{p}=1}^{m}(p-1)(\tilde{p}-1) - \sum_{p=1}^{m}(p-1)^2\right] \\
&= 1 - \frac{2(2m-1)}{3m(m-1)} \to 1 \quad \text{as } m \to \infty
\end{aligned}$$

which implies $\sum_{p=2}^{m} X_{Np}^2 \xrightarrow{p} 1$ as $(m, N) \to \infty$ and $\lim_{(m,N)\to\infty} \frac{m}{N} = \gamma \in (0, \infty)$. □