推导过程

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- **推导：**

$$\Pr(y \in B \mid x, r=0) = \Pr(y \in B \mid x, r=1) \times \frac{\Pr(r=0 \mid x, y \in B)/\Pr(r=1 \mid x, y \in B)}{\Pr(r=0 \mid x)/\Pr(r=1 \mid x)}.$$

  过程：

$$\begin{aligned}
\Pr(y \in B \mid x, r=0) &= \frac{\Pr(y \in B, x, r=0)}{\Pr(x, r=0)} = \frac{\Pr(r=0 \mid x, y \in B)\Pr(x, y \in B)}{\Pr(r=0 \mid x) f(x)}, \\
\Pr(y \in B \mid x, r=1) &= \frac{\Pr(r=1 \mid x, y \in B)\Pr(x, y \in B)}{\Pr(r=1 \mid x) f(x)}.
\end{aligned}$$

  两式相除：

$$\frac{\Pr(y \in B \mid x, r=0)}{\Pr(y \in B \mid x, r=1)} = \frac{\Pr(r=0 \mid x, y \in B)/\Pr(r=1 \mid x, y \in B)}{\Pr(r=0 \mid x)/\Pr(r=1 \mid x)},$$

  即

$$\Pr(y \in B \mid x, r=0) = \Pr(y \in B \mid x, r=1) \times \frac{\Pr(r=0 \mid x, y \in B)/\Pr(r=1 \mid x, y \in B)}{\Pr(r=0 \mid x)/\Pr(r=1 \mid x)}.$$

- **推导：**

$$\mathrm{E}[Y \mid X, r=0] = \frac{\mathrm{E}[(1-\pi(x,y))y \mid x]}{\mathrm{E}[1-\pi(x,y) \mid x]}.$$

过程：

$$
\begin{aligned}
\mathrm{E}[Y \mid X, r=0] &= \int y f(y \mid x, r=0) \mathrm{d}y \\
&= \int y f(y, x, r=0)/f(x, r=0) \mathrm{d}y \\
&= \frac{\int y \mathrm{Pr}(r=0 \mid x, y) f(x, y) \mathrm{d}y}{\mathrm{Pr}(r=0 \mid x) f(x)} \\
&= \frac{\int y(1-\pi(x, y)) f(y \mid x) \mathrm{d}y}{1-\mathrm{Pr}(r=1 \mid x)} \\
&= \frac{\mathrm{E}[y(1-\pi(x, y)) \mid x]}{1-\mathrm{Pr}(r=1 \mid x)}, \\
1-\mathrm{Pr}(r=1 \mid x) &= \mathrm{E}[(1-r) \mid x] \\
&= \mathrm{E}[\mathrm{E}[1-r \mid x, y] \mid x] \\
&= \mathrm{E}[(1-\pi(x, y)) \mid x].
\end{aligned}
$$

- **推导：**

$$
\mathrm{E}[ry + (1-r)\mathrm{E}(y \mid x, r=0)] = \mathrm{E}y.
$$

过程：

$$
\begin{aligned}
\mathrm{E}[ry] = \mathrm{E}[\mathrm{E}[ry \mid x, y]] &= \mathrm{E}[y\mathrm{E}(r \mid x, y)] = \mathrm{E}[y\pi(x, y)], \\
\mathrm{E}[(1-r)\mathrm{E}(y \mid x, r=0)] &= \mathrm{E}\left[(1-r)\frac{\mathrm{E}[y(1-\pi(x, y)) \mid x]}{\mathrm{E}[1-r \mid x]}\right] \\
&= \mathrm{E}[y(1-\pi(x, y))].
\end{aligned}
$$

故：$\mathrm{E}[ry + (1-r)\mathrm{E}(y \mid x, r=0)] = \mathrm{E}y.$

- **（3）式的推导：**

$$
f_0(y \mid x) = f_1(y \mid x) \times \frac{O(x, y)}{\mathrm{E}[O(x, y) \mid x, r=1]},
$$

其中

$$
O(x, y) = \frac{\mathrm{Pr}(r=0 \mid x, y)}{\mathrm{Pr}(r=1 \mid x, y)}.
$$

过程：由已得结论：

$$
\mathrm{Pr}(y \in B \mid x, r=0) = \mathrm{Pr}(y \in B \mid x, r=1)\frac{\mathrm{Pr}(r=0 \mid x, y \in B)/\mathrm{Pr}(r=1 \mid x, y \in B)}{\mathrm{Pr}(r=0 \mid x)/\mathrm{Pr}(r=1 \mid x)},
$$

可得：

$$
f_0(y \mid x) = f_1(y \mid x)\frac{\mathrm{Pr}(r=0 \mid x, y)/\mathrm{Pr}(r=1 \mid x, y)}{\mathrm{Pr}(r=0 \mid x)/\mathrm{Pr}(r=1 \mid x)}.
$$

下面验证：$\mathrm{E}[O(x,y) \mid x, r=1] = \Pr(r=0 \mid x)/\Pr(r=1 \mid x)$.

$$\begin{aligned}
\mathrm{E}[O(x,y) \mid x, r=1] &= \mathrm{E}\left[\frac{\Pr(r=0 \mid x,y)}{\Pr(r=1 \mid x,y)} \mid x, r=1\right] \\
&= \mathrm{E}\left[\frac{1-\Pr(r=1 \mid x,y)}{\Pr(r=1 \mid x,y)} \mid x, r=1\right] \\
&= \mathrm{E}\left[\frac{1}{\Pr(r=1 \mid x,y)} - 1 \mid x, r=1\right] \\
&= \mathrm{E}\left[\frac{1}{\pi(x,y)} | x, r=1\right] - 1.
\end{aligned} \tag{\# 1}$$

下面考虑 $\mathrm{E}\left[\frac{1}{\pi(x,y)} \mid x, r=1\right]$：

$$\begin{aligned}
&\mathrm{E}\left[\frac{1}{\pi(x,y)} \mid x, r=1\right] = \int \frac{1}{\pi(x,y)} f(y \mid x, r=1) \mathrm{d}y \\
&= \int \frac{1}{\pi(x,y)} \frac{f(y,x,r=1)}{f(x,r=1)} \mathrm{d}y \\
&= \int \frac{1}{\pi(x,y)} \frac{\Pr(r=1 \mid x,y) f(x,y)}{\Pr(r=1 \mid x) f(x)} \mathrm{d}y \\
&= \frac{1}{\Pr(r=1 \mid x)} \int f(y \mid x) \mathrm{d}y = \frac{1}{\Pr(r=1 \mid x)}.
\end{aligned} \tag{\# 2}$$

由公式 (# 1) 和 (# 2) 可得 $\mathrm{E}[O(x,y) \mid x, r=1] = \Pr(r=0 \mid x)/\Pr(r=1 \mid x)$.

- **推导：**

$$\frac{\mathrm{E}[r(y-m(x))^2 \mid x]}{\mathrm{E}[r \mid x]} = \mathrm{E}\left[(y-m(x))^2 \mid x, r=1\right]. \tag{*}$$

首先，

$$\begin{aligned}
\mathrm{E}\left[r(y-m(x))^2 \mid x\right] &= \mathrm{E}\left[\mathrm{E}\left[r(y-m(x))^2 \mid x, y\right] \mid x\right] \\
&= \mathrm{E}\left[(y-m(x))^2 \mathrm{E}(r \mid x, y) \mid x\right] \\
&= \mathrm{E}\left[\pi(x,y)(y-m(x))^2 \mid x\right].
\end{aligned}$$

接下来有：

$$\begin{aligned}
\mathrm{E}\left[(y-m(x))^2 \mid x, r=1\right] &= \int (y-m(x))^2 f(y \mid x, r=1) \mathrm{d}y \\
&= \frac{\int (y-m(x))^2 f(y, x, r=1) \mathrm{d}y}{f(x, r=1)} \\
&= \frac{\int (y-m(x))^2 \operatorname{Pr}(r=1 \mid x, y) f(x, y) \mathrm{d}y}{\operatorname{Pr}(r=1 \mid x) f(x)} \\
&= \frac{\int (y-m(x))^2 \pi(x, y) f(y \mid x) \mathrm{d}y}{\operatorname{Pr}(r=1 \mid x)} \\
&= \frac{\mathrm{E}\left[(y-m(x))^2 \pi(x, y) \mid x\right]}{\mathrm{E}(r \mid x)}
\end{aligned}$$

这样，公式 (*) 得证. 文中 (7) 式是

$$\frac{\mathrm{E}[r(y-m(x))^2 \mid x]}{\mathrm{E}(r \mid x)}$$

的相合估计，求解使 (7) 式达到最小的 $m(x)$，得到估计量：

$$\hat{m}_1(x) = \sum_{i=1}^{n} w_{i1}(x) y_i,$$

即文中 (8) 式，其中

$$w_{i1}(x) = \frac{r_i K_h(x_i, x)}{\sum_{j=1}^{n} r_j K_h(x_j, x)}.$$

- **(9) 式的推导：**

$$p \lim_{n\to\infty} \sum_{i=1}^{n} w_{i1}(x) y_i = \frac{\mathrm{E}(ry \mid x)}{\mathrm{E}(r \mid x)} = \mathrm{E}(y \mid x, r=1).$$

过程： 首先推导第一个等号，即要证明

$$\lim_{n\to\infty} \sum_{i=1}^{n} w_{i1}(x) y_i \xrightarrow{P} \frac{\mathrm{E}(ry \mid x)}{\mathrm{E}(r \mid x)}.$$

注意到

$$\begin{aligned}
\sum_{i=1}^{n} w_{i1}(x) y_i &= \frac{\sum_{i=1}^{n} r_i y_i K_h\left(x_i, x\right)}{\sum_{i=1}^{n} r_i K_h\left(x_i, x\right)} \\
&= \frac{\sum_{i=1}^{n} r_i y_i K_h(x_i x)}{\sum_{i=1}^{n} K_h(x_i, x)} \Big/ \frac{\sum_{i=1}^{n} r_i K_h(x_i, x)}{\sum_{i=1}^{n} K_h(x_i, x)} \\
&\xrightarrow{P} \mathrm{E}(ry \mid x)/\mathrm{E}(r \mid x) \qquad (\text{slutsky 定理})
\end{aligned}$$

接下来推导第二个等号，首先：

$$\begin{aligned}\mathrm{E}(ry \mid x) &= \mathrm{E}[\mathrm{E}(ry \mid x, y) \mid x] \\ &= \mathrm{E}[y\mathrm{E}(r \mid x, y) \mid x] \\ &= \mathrm{E}[y\pi(x, y) \mid x].\end{aligned}$$

接下来考虑：

$$\begin{aligned}\mathrm{E}(y \mid x, r=1) &= \int yf(y \mid x, r=1)\mathrm{d}y \\ &= \int yf(y, x, r=1)/f(x, r=1)\mathrm{d}y \\ &= \frac{\int pr(r=1 \mid x, y)f(x, y)\mathrm{d}y}{\Pr(r=1 \mid x)f(x)} \\ &= \frac{(y\pi(x, y)f(y \mid x)\mathrm{d}y}{\mathrm{E}(r \mid x)} \\ &= \frac{\mathrm{E}(y\pi(x, y) \mid x)}{\mathrm{E}(r \mid x)} = \frac{\mathrm{E}(ry \mid x)}{\mathrm{E}(r \mid x)}.\end{aligned}$$

- $m_0(x) = \mathrm{E}(y \mid x, r=0)$，下面考虑 $m_0(x)$ 的估计，在 (10) 中给出.

  推导：

$$\begin{aligned}\mathrm{E}(y \mid x, r=0) &= \frac{\mathrm{E}[(1-\pi(x, y))y \mid x]}{\mathrm{E}[1-\pi(x, y) \mid x]} = \frac{\mathrm{E}\left[\pi(x, y)y\exp\left(\gamma^* y\right) \mid x\right]}{\mathrm{E}\left[\pi(x, y)\exp\left(\gamma^* y\right) \mid x\right]} \\ &= \frac{\mathrm{E}\left[ry\exp\left(\gamma^* y\right) \mid x\right]}{\mathrm{E}\left[r\exp\left(\gamma^* y\right) \mid x\right]}\end{aligned}$$

  过程：首先推导第一个等号：

$$\begin{aligned}\mathrm{E}[y|x, r=0] &= \int yf(y \mid x, r=0)\mathrm{d}y \\ &= \int \frac{y(y \mid x, r=0)}{f(x, r=0)}\mathrm{d}y \\ &= \int y\frac{\Pr(r=0 \mid x, y)f(x, y)}{f(r=0 \mid x)f(x)}\mathrm{d}y \\ &= \int \frac{y(1-\pi(x, y))f(y \mid x)}{1-\mathrm{E}(r \mid x)}\mathrm{d}y \\ &= \frac{\mathrm{E}[y(1-\pi(x, y)) \mid x]}{1-\mathrm{E}(r \mid x)}.\end{aligned}$$

  下面证明

$$1-\mathrm{E}(r \mid x) = \mathrm{E}[1-\pi(x, y) \mid x]. \tag{*}$$

$$\mathrm{E}(r \mid x) = \mathrm{E}[\mathrm{E}(r \mid x, y) \mid x] = \mathrm{E}[\pi(x, y) \mid x],$$

带入上式即证公式 (*).

因为

$$\pi(x,y)=\frac{\exp(g(x)+\phi y)}{1+\exp(g(x)+\phi y)}$$

且 $\gamma=-\phi$，通过简单的计算可证得第二个等号和第三个等号.

因为

$$\mathrm{E}[y\mid x,r=0]=\frac{\mathrm{E}[ry\exp(\gamma^* y)\mid x]}{\mathrm{E}[r\exp(\gamma^* y)\mid x]},$$

故可估计 $\mathrm{E}[y\mid x,r=0]$ 如下:

$$\hat{m}_0\left(x;\gamma^*\right)=\sum_{i=1}^n w_{i0}\left(x;\gamma^*\right)y_i,$$

即文中的 (10) 式，其中

$$\begin{aligned} w_{i0}\left(x;\gamma^*\right) &= \frac{r_i K_h\left(x,x_i\right)\exp\left(\gamma^* y_i\right)}{\sum_{j=1}^n r_j K_h\left(x,x_j\right)\exp\left(\gamma^* y_j\right)} \\ &= \frac{w_{i1}(x)\exp\left(\gamma^* y_i\right)}{\sum_{j=1}^n w_{j1}(x)\exp\left(\gamma^* y_j\right)}. \end{aligned}$$

- 因为 $\mathrm{E}(y)=\mathrm{E}[ry+(1-r)\mathrm{E}(y\mid x,r=0)]$，故 $\mathrm{E}(y)$ 的估计可以定义如下:

$$\hat{\theta}_{\mathrm{NP}}=\frac{1}{n}\sum_{i=1}^n\left\{r_i y_i+\left(1-r_i\right)\hat{m}_0\left(x_i;\gamma^*\right)\right\},$$

即文中的 (11) 式。