机器人技术与实践

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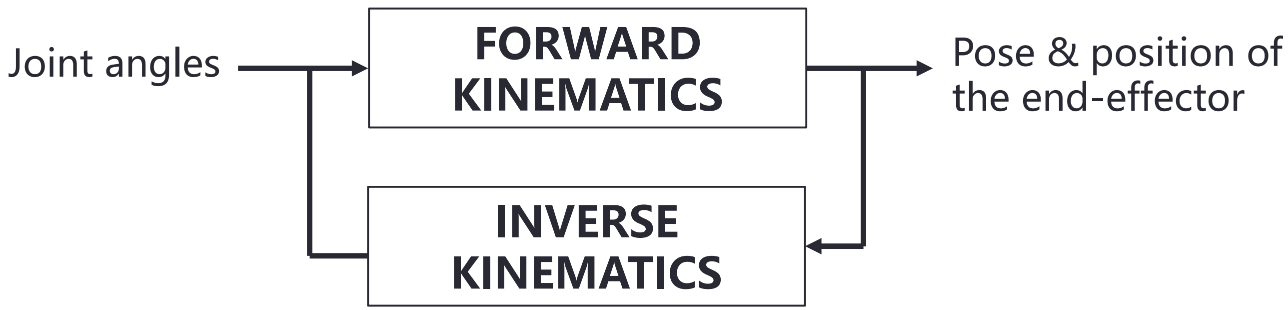
4. INVERSE KINEMATICS I

$$ {}^{0}T_{6} = {}^{0}T_{3}\,{}^{3}T_{6} = \begin{bmatrix} {}^{0}R_{3} & {}^{0}\mathbf{d}_{3} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} {}^{3}R_{6} & {}^{3}\mathbf{d}_{6} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} $$

4.1 Inverse Kinematics

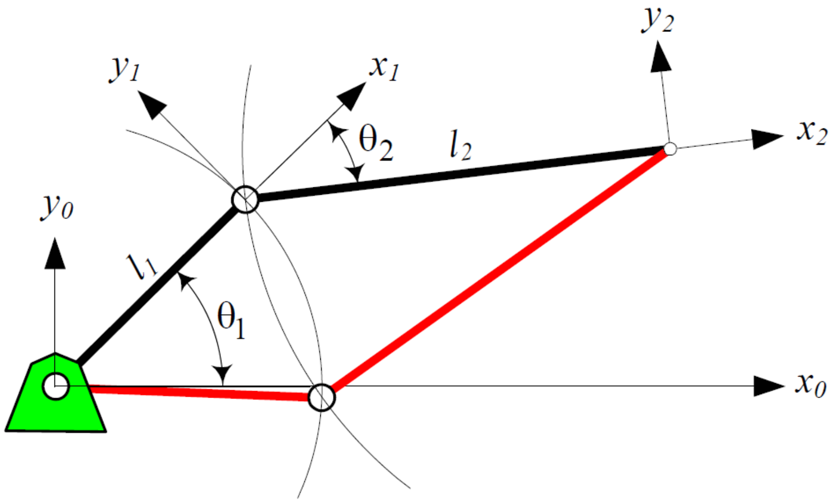
✓ Solving the inverse kinematics problem is a process to find all joint angles of a robot if the position and orientation of the end-effector are known.



$$ {}^{0}T_{6} = \begin{bmatrix} n_x & s_x & a_x & d_X \\ n_y & s_y & a_y & d_Y \\ n_z & s_z & a_z & d_Z \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{?}{\Rightarrow} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{bmatrix} $$

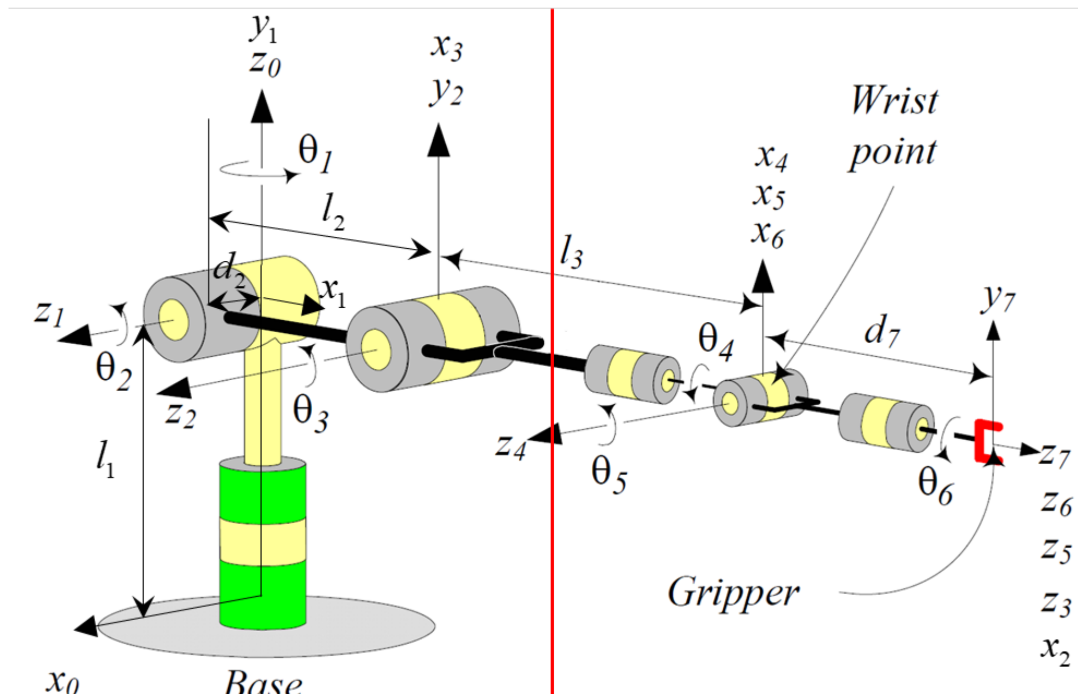
Notes

- ✓ Finding solution to inverse kinematics problem is more complicated than direct kinematics problem.
- ✓ It may suffer from various illnesses such as no analytical solutions, multiple solutions, or even no solution issues, etc.
- ✓ There is no standard and generally applicable method to solve the inverse kinematic problem.
  - Decoupling technique
  - Generic method
  - Numeric solution



Arm with a Spherical Wrist

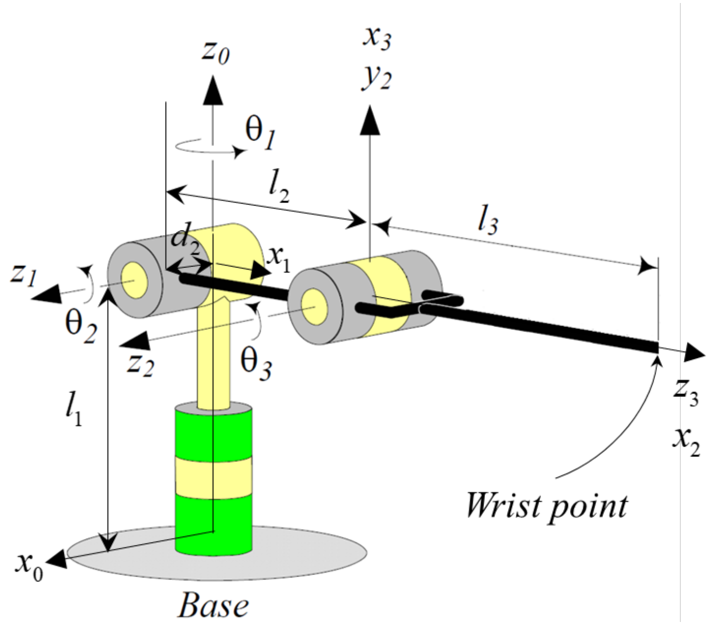
It is the most common design that an industrial arm contains an **anthropomorphic (elbow) manipulator** serially-connected with a **spherical wrist**



Decomposition of the Structure

It is the most common design that an industrial arm contains an anthropomorphic (elbow) manipulator serially-connected with a spherical wrist

3 rotation axes intersects at the same point



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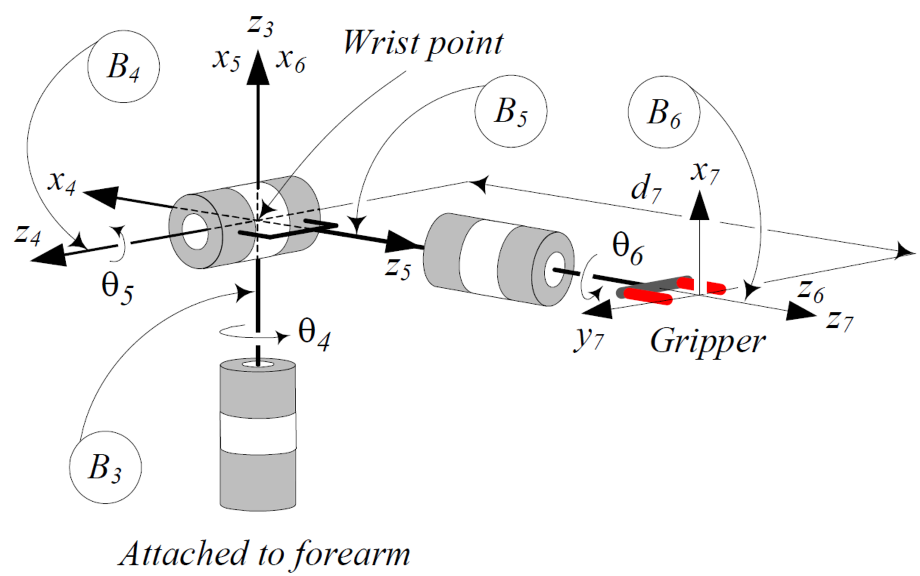


Determine the **position**

Determine the **orientation**

Global Pos. and Orient. of an End-effector

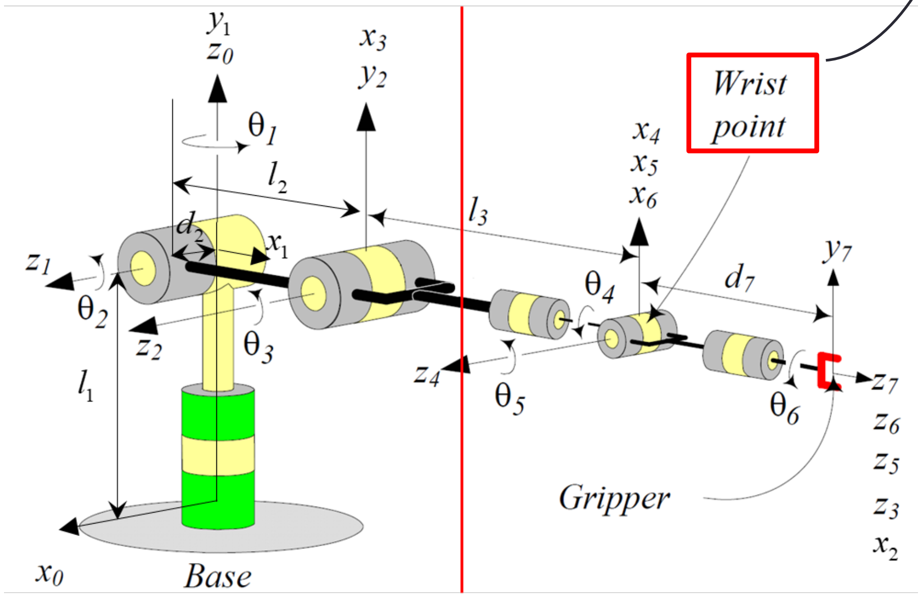
✓ A transformation ${}^0T_6$ is given as a function of the joint variables

$$ {}^0T_6 = {}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4\,{}^4T_5\,{}^5T_6 = {}^0T_3\,{}^3T_6 $$

$$ = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0R_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

$$ {}^0\mathbf{d}_6 = \begin{bmatrix} r_{14} \\ r_{24} \\ r_{34} \end{bmatrix} = \begin{bmatrix} d_X \\ d_Y \\ d_Z \end{bmatrix} $$



Decoupling

- ✓ It is possible to decouple the inverse kinematics problem into two subproblems, known as **inverse position** and **inverse orientation** kinematics.
- ✓ Following the decoupling principle, the overall transformation matrix of a robot can be decomposed to a translation and a rotation.

$$ {}^0T_6 = \begin{bmatrix} {}^0R_6 & {}^0\mathbf{d}_6 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} = \begin{bmatrix} I & {}^0\mathbf{d}_6 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} {}^0R_6 & 0 \\ \mathbf{0} & 1 \end{bmatrix} $$

- ✓ ${}^0T_6$ can also be decomposed as

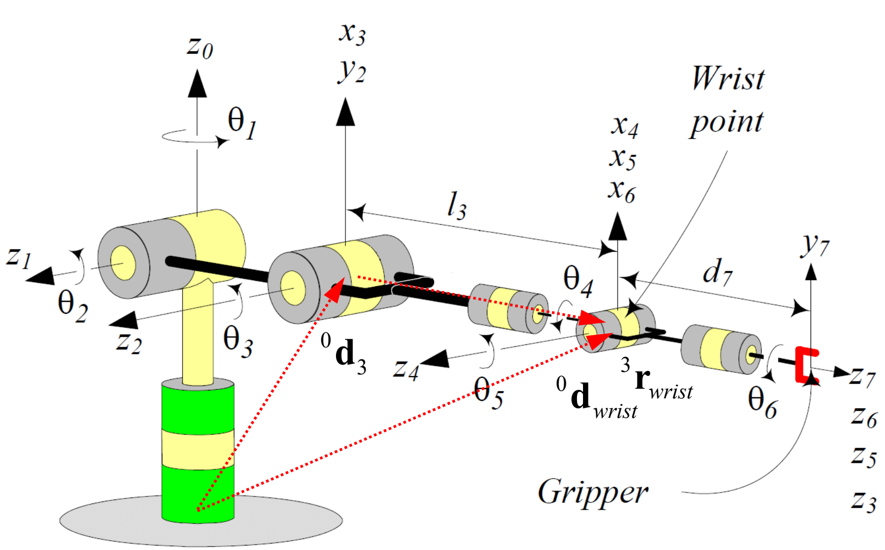
$$ {}^0T_6 = {}^0T_3\,{}^3T_6 = \underbrace{\begin{bmatrix} {}^0R_3 & {}^0\mathbf{d}_3 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}}_{\substack{\text{3 unknowns} \\ \theta_1,\ \theta_2,\ \theta_3}} \underbrace{\begin{bmatrix} {}^3R_6 & {}^3\mathbf{d}_6 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}}_{\substack{\text{3 unknowns} \\ \theta_4,\ \theta_5,\ \theta_6}} $$

$$ {}^0T_6 = {}^0T_3\,{}^3T_6 = \begin{bmatrix} {}^0R_3 & {}^0\mathbf{d}_3 \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix} \begin{bmatrix} {}^3R_6 & {}^3\mathbf{d}_6 \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix} $$

$$ {}^0\mathbf{d}_6 \Rightarrow {}^0R_3 \Rightarrow \theta_1, \theta_2, \theta_3 \Rightarrow \left.\begin{matrix} {}^0R_3 \\ {}^0R_6 \end{matrix}\right\} \Rightarrow {}^3R_6 \Rightarrow \theta_4, \theta_5, \theta_6 $$

Known as position of the wrist point

Known as orientation of the end-effector



$$ {}^3\mathbf{r}_{wrist} = \begin{bmatrix} 0 & 0 & l_3 \end{bmatrix}^T $$

✓ The position of the wrist point is

$$ {}^0\mathbf{d}_{wrist} = {}^0\mathbf{d}_3 + {}^0R_3\,{}^3\mathbf{r}_{wrist} = {}^0\mathbf{d}_6 \qquad (4\text{-}1) $$

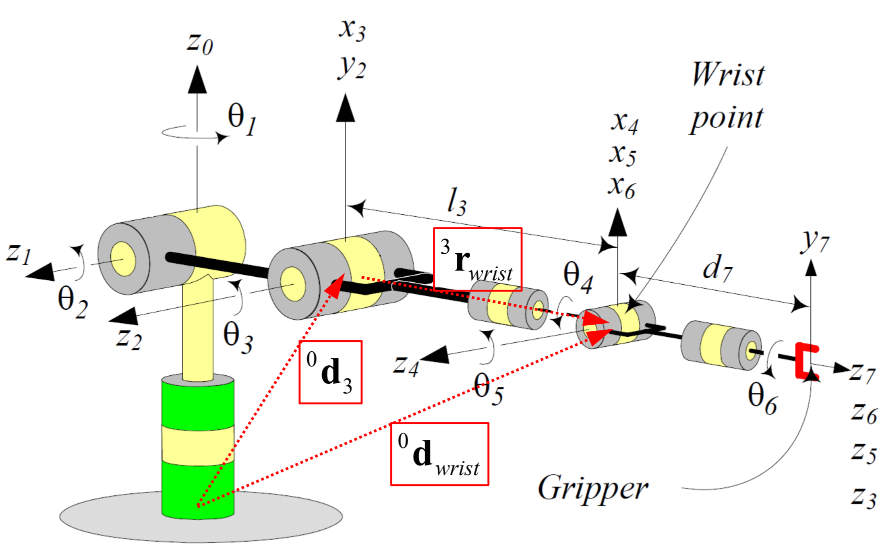
✓ Chain rule implies that

$$ {}^0R_6 = {}^0R_3\,{}^3R_6 \Rightarrow {}^3R_6 = {}^0R_3^{\,T}\,{}^0R_6 \qquad (4\text{-}2) $$

Step 1: find θ_1, θ_2, θ_3 by equating corresponding elements of two sides of Eq. (4-1)

Step 2: substitute θ_1, θ_2, θ_3 into ${}^0R_3 = {}^0R_1{}^1R_2{}^2R_3$ to obtain ${}^0R_3$ and find ${}^3R_6$ using Eq. (4-2)

Step 3: find θ_4, θ_5, θ_6 from ${}^3R_6$



$${}^3\mathbf{r}_{wrist} = \begin{bmatrix} 0 & 0 & l_3 \end{bmatrix}^T$$

✓ The position of the wrist point is

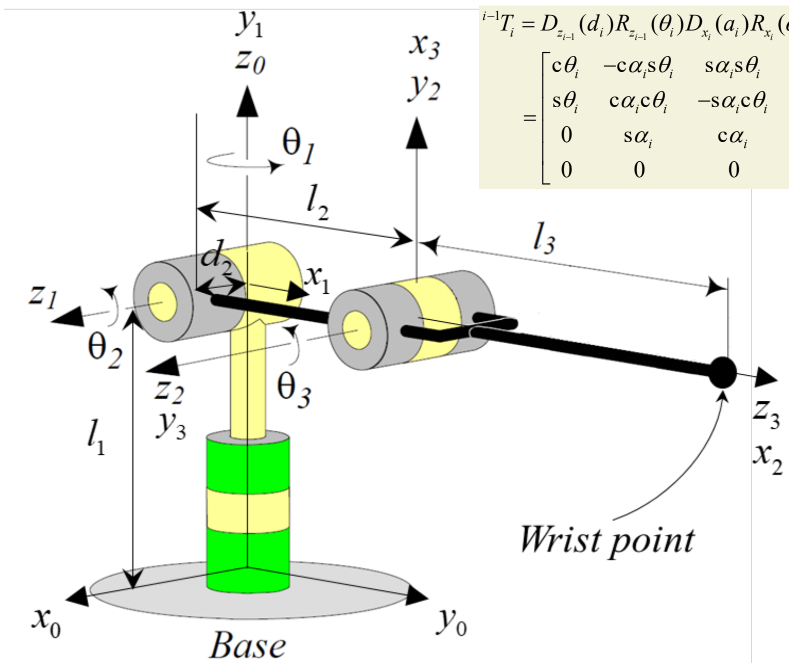
$$\boxed{{}^0\mathbf{d}_{wrist} = {}^0\mathbf{d}_3 + {}^0R_3\,{}^3\mathbf{r}_{wrist} = {}^0\mathbf{d}_6} \qquad (4\text{-}1)$$

✓ Chain rule implies that

$${}^0R_6 = {}^0R_3\,{}^3R_6 \Rightarrow \boxed{{}^3R_6 = {}^0R_3^{T}\,{}^0R_6} \qquad (4\text{-}2)$$

Anthropomorphic Manipulator

- ✓ Point $P=(d_x, d_y, d_z)^T$ at the end of the last link of an anthropomorphic manipulator is supposed to be the point where a spherical wrist will be attached.



$$
{}^{i-1}T_i = D_{z_{i-1}}(d_i)R_{z_{i-1}}(\theta_i)D_{x_i}(a_i)R_{x_i}(\alpha_i) = \begin{bmatrix} c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i & a_i c\theta_i \\ s\theta_i & c\alpha_i c\theta_i & -s\alpha_i c\theta_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

No.	a_i	α_i	d_i	θ_i
1	0	90°	l_1	θ_1（90°）
2	l_2	0	d_2	θ_2
3	0	90°	0	θ_3（90°）

$$
{}^0T_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad
{}^1T_2 = \begin{bmatrix} c_2 & -s_2 & 0 & l_2c_2 \\ s_2 & c_2 & 0 & l_2s_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad
{}^2T_3 = \begin{bmatrix} c_3 & 0 & s_3 & 0 \\ s_3 & 0 & -c_3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$\Downarrow$$

$$
{}^0T_3 = {}^0T_1\,{}^1T_2\,{}^2T_3 = \begin{bmatrix} c_1c_{23} & s_1 & c_1s_{23} & d_2s_1 + l_2c_1c_2 \\ s_1c_{23} & -c_1 & s_1s_{23} & l_2s_1c_2 - d_2c_1 \\ s_{23} & 0 & -c_{23} & l_1 + l_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$\Downarrow$$

$$
{}^0R_3 = \begin{bmatrix} c_1c_{23} & s_1 & c_1s_{23} \\ s_1c_{23} & -c_1 & s_1s_{23} \\ s_{23} & 0 & -c_{23} \end{bmatrix} \qquad
{}^0\mathbf{d}_3 = \begin{pmatrix} d_2s_1 + l_2c_1c_2 \\ l_2s_1c_2 - d_2c_1 \\ l_1 + l_2s_2 \end{pmatrix}
$$

$$ {}^{0}\mathbf{d}_{wrist} = {}^{0}\mathbf{d}_{3} + {}^{0}R_{3}\,{}^{3}\mathbf{r}_{wrist} = {}^{0}\mathbf{d}_{6} \Rightarrow $$

$$ \begin{pmatrix} d_2 s_1 + l_2 c_1 c_2 \\ l_2 s_1 c_2 - d_2 c_1 \\ l_1 + l_2 s_2 \end{pmatrix} + \begin{bmatrix} c_1 c_{23} & s_1 & c_1 s_{23} \\ s_1 c_{23} & -c_1 & s_1 s_{23} \\ s_{23} & 0 & -c_{23} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ l_3 \end{bmatrix} = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} \Rightarrow \begin{cases} d_x = d_2 s_1 + l_2 c_1 c_2 + l_3 c_1 s_{23} & (1) \\ d_y = l_2 s_1 c_2 - d_2 c_1 + l_3 s_1 s_{23} & (2) \\ d_z = l_1 + l_2 s_2 - l_3 c_{23} & (3) \end{cases} $$

$$ \left. \begin{array}{l} (1)\ \&\ (2) \Rightarrow \quad d_x s_1 - d_y c_1 = d_2 \\ \text{Let} \quad d_y = r\sin\alpha,\ d_x = r\cos\alpha \end{array} \right\} \Rightarrow \sin(\theta_1 - \alpha) = \frac{d_2}{r} \Rightarrow \cos(\theta_1 - \alpha) = \pm\frac{1}{r}\sqrt{r^2 - d_2^2} \Rightarrow $$

$$ \Rightarrow \quad \theta_1 = \operatorname{atan2}(d_2, \pm\sqrt{r^2 - d_2^2}) + \alpha \qquad \text{where} \qquad r = \sqrt{d_x^2 + d_y^2}, \quad \alpha = \operatorname{atan2}(d_y, d_x) $$

$$ d_x c_1 + d_y s_1 = (d_2 s_1 + l_2 c_1 c_2 + l_3 c_1 s_{23}) c_1 + (l_2 s_1 c_2 - d_2 c_1 + l_3 s_1 s_{23}) s_1 = l_2 c_2 + l_3 s_{23} \qquad (4) $$

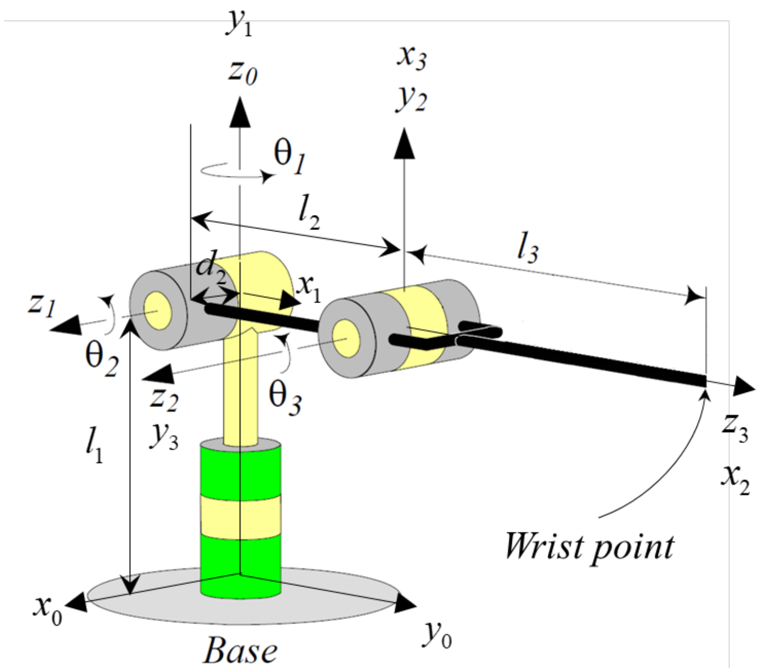
$$ (3)\ \&\ (4) \Rightarrow (l_3 s_{23})^2 + (l_3 c_{23})^2 = (d_x c_1 + d_y s_1 - l_2 c_2)^2 + (l_1 + l_2 s_2 - d_z)^2 $$

$$ \Rightarrow \quad m s_2 - n c_2 = s \Rightarrow \theta_2 = \operatorname{atan2}(s, \pm\sqrt{t^2 - s^2}) + \beta $$

$$ \text{where} \qquad t = \sqrt{m^2 + n^2}, \quad \beta = \operatorname{atan2}(m, n) $$

$$ m = 2l_2(l_1 - d_z) \qquad n = 2l_2(d_x c_1 + d_y s_1) \qquad s = l_3^2 - l_1^2 - l_2^2 - d_z^2 - (d_x c_1 + d_y s_1)^2 + 2l_1 d_z $$

(3) & (4) $\Rightarrow$ $l_3 s_{23} = d_x c_1 + d_y s_1 - l_2 c_2$, $l_3 c_{23} = l_1 + l_2 s_2 - d_z$ $\Rightarrow$ $\theta_3 = \text{atan2}(d_x c_1 + d_y s_1 - l_2 c_2,\quad l_1 + l_2 s_2 - d_z) - \theta_2$



If the position of the end-effector is known

$$ {}^{0}\mathbf{d}_{wrist} = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} $$

✓ There are four sets of solutions to the inverse kinematics problem

$$\theta_1 = \text{atan2}(d_2, \pm\sqrt{d_x^2 + d_y^2 - d_2^2}) + \text{atan2}(d_y, d_x)$$

$$\theta_2 = \text{atan2}(s, \pm\sqrt{m^2 + n^2 - s^2}) + \text{atan2}(m, n)$$

$$m = 2l_2(d_x c_1 + d_y s_1)$$

$$n = 2l_2(l_1 - d_z)$$

$$s = l_3^2 - l_1^2 - l_2^2 - d_z^2 - (d_x c_1 + d_y s_1)^2 + 2l_1 d_z$$

$$\theta_3 = \text{atan2}(d_x c_1 + d_y s_1 - l_2 c_2,\quad l_1 + l_2 s_2 - d_z) - \theta_2$$

Ex 4-1-1

For an anthropomorphic arm, find solution to its forward kinematics problem given $L_1 = L_2 = L_3 = 1$, $d_2 = 0.1$, $(q_1, q_2, q_3) = (0, \pi/4, -\pi/2)$. Prove that $(0, -\pi/4, \pi/2)$ and $(-3.2828, -3\pi/4, -\pi/2)$ are also solutions.

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1. $(q_1, q_2, q_3) = (0, \pi/4, -\pi/2)$

$\Rightarrow$ $\theta_1 = \pi/2$, $\theta_2 = \pi/4$, $\theta_3 = 0$ $\Rightarrow$

$$d_x = d_2 s_1 + l_2 c_1 c_2 + l_3 c_1 s_{23} = 0.1$$
$$d_y = l_2 s_1 c_2 - d_2 c_1 + l_3 s_1 s_{23} = 1.4142$$
$$d_z = l_1 + l_2 s_2 - l_3 c_{23} = 1$$

2. $(0, -\pi/4, \pi/2)$

$\Rightarrow$ $\theta_1 = \pi/2$, $\theta_2 = -\pi/4$, $\theta_3 = \pi$ $\Rightarrow$

$$d_x = d_2 s_1 + l_2 c_1 c_2 + l_3 c_1 s_{23} = 0.1$$
$$d_y = l_2 s_1 c_2 - d_2 c_1 + l_3 s_1 s_{23} = 1.4142$$
$$d_z = l_1 + l_2 s_2 - l_3 c_{23} = 1$$

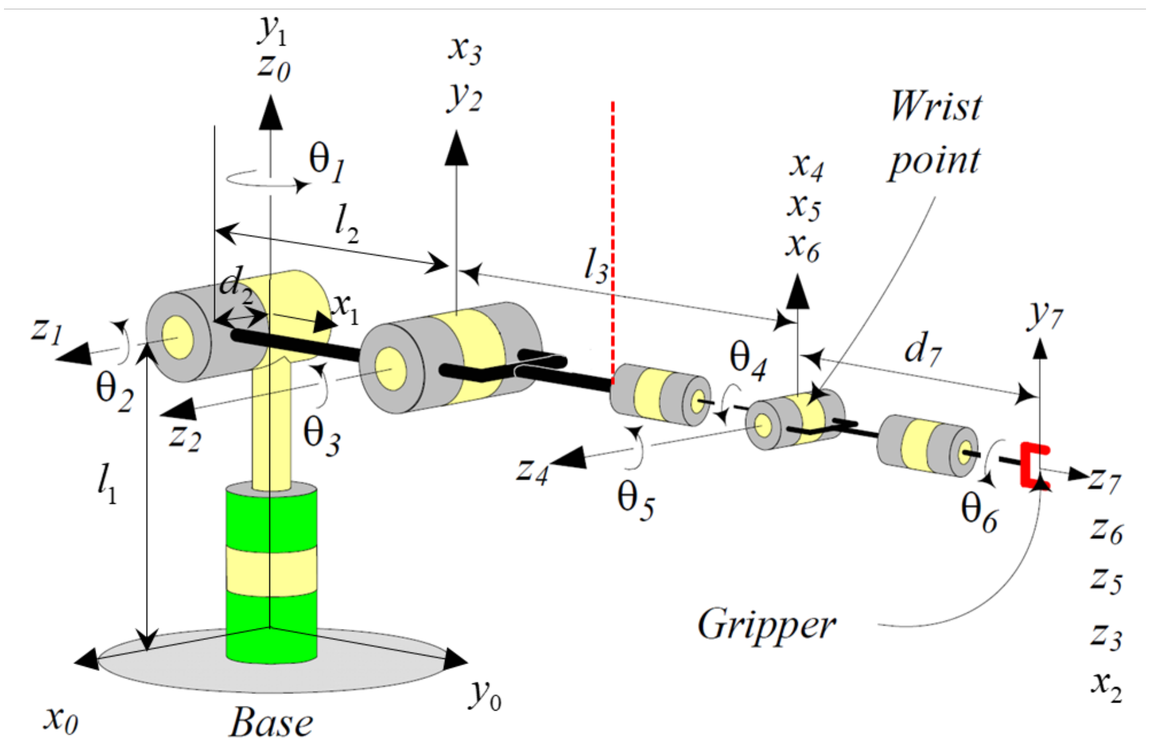
3. $(-3.2828, -3\pi/4, -\pi/2)$

$\Rightarrow$ $\theta_1 = -1.712$, $\theta_2 = -3\pi/4$, $\theta_3 = 0$ $\Rightarrow$

$$d_x = d_2 s_1 + l_2 c_1 c_2 + l_3 c_1 s_{23} = 0.1$$
$$d_y = l_2 s_1 c_2 - d_2 c_1 + l_3 s_1 s_{23} = 1.4142$$
$$d_z = l_1 + l_2 s_2 - l_3 c_{23} = 1$$

Anthropomorphic Arm + Spherical Wrist

✓ Orientation and position of the end-effector is supposed to be represented by a homogeneous rotation transformation matrix ${}^0T_6$



$$
{}^0T_6 = \begin{bmatrix} n_x & o_x & a_x & d_x \\ n_y & o_y & a_y & d_y \\ n_z & o_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

No.	a_i	α_i	d_i	θ_i
0	0	90°	l_1	$\theta_1(90°)$
1	l_2	0	d_2	θ_2
2	0	90°	0	$\theta_3(90°)$
3	0	-90°	l_3	θ_4
4	0	90°	0	θ_5
5	0	0	0	θ_6

Anthropomorphic Arm + Spherical Wrist

$$ {}^0T_6 = \begin{bmatrix} n_x & o_x & a_x & d_x \\ n_y & o_y & a_y & d_y \\ n_z & o_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^3T_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^4T_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^5T_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^3T_6 = {}^3T_4\,{}^4T_5\,{}^5T_6 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -s_4c_6 - c_4c_5s_6 & c_4s_5 & 0 \\ c_4s_6 + s_4c_5c_6 & c_4c_6 - s_4c_5s_6 & s_4s_5 & 0 \\ -s_5c_6 & s_5s_6 & c_5 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \leftarrow {}^3R_6 $$

Alternatively, $ {}^3R_6 = {}^0R_3^{T}\,{}^0R_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $

By comparing both ${}^3R_6$, wrist joint angles can be obtained.

$$
{}^{3}R_6 = {}^{0}R_3^{T}\,{}^{0}R_6 = \begin{bmatrix} c_1c_{23} & s_1c_{23} & s_{23} \\ s_1 & -c_1 & 0 \\ c_1s_{23} & s_1s_{23} & -c_{23} \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix}
$$

$$
= \begin{bmatrix} n_xc_1c_{23} + n_ys_1c_{23} + n_zs_{23} & o_xc_1c_{23} + o_ys_1c_{23} + o_zs_{23} & a_xc_1c_{23} + a_ys_1c_{23} + a_zs_{23} \\ n_xs_1 - n_yc_1 & o_xs_1 - o_yc_1 & a_xs_1 - a_yc_1 \\ n_xc_1s_{23} + n_ys_1s_{23} - n_zc_{23} & o_xc_1s_{23} + o_ys_1s_{23} - o_zc_{23} & a_xc_1s_{23} + a_ys_1s_{23} - a_zc_{23} \end{bmatrix}
$$

$$
= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}
\qquad
{}^{3}R_6 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -s_4c_6 - c_4c_5s_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & c_4c_6 - s_4c_5s_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix}
$$

If $\theta_5 \neq k\pi$

$$\theta_5 = \operatorname{atan2}(\pm\sqrt{1 - r_{33}{}^2}, r_{33})$$

$$\theta_4 = \operatorname{atan2}\left(\frac{r_{23}}{s_5}, \frac{r_{13}}{s_5}\right)$$

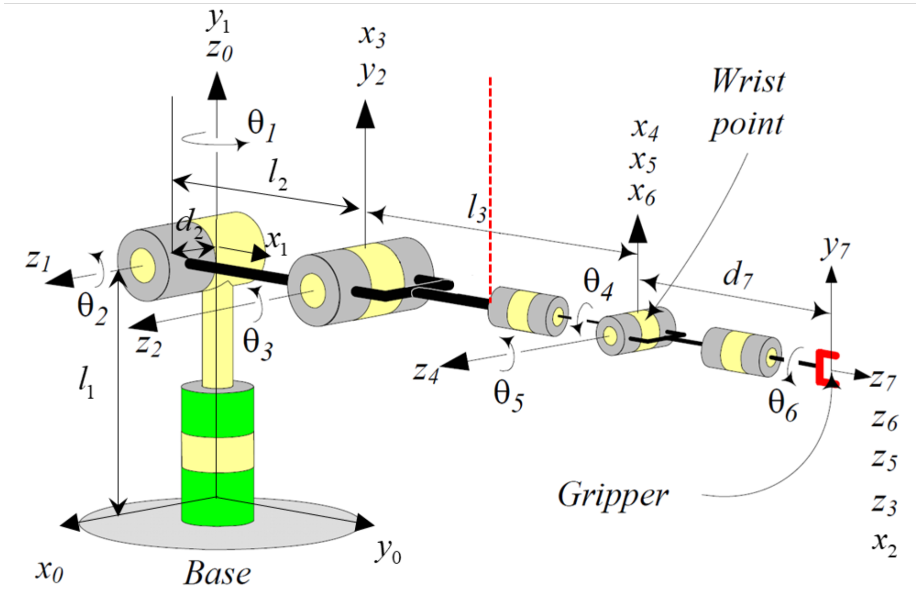
$$\theta_6 = \operatorname{atan2}\left(\frac{r_{32}}{s_5}, -\frac{r_{31}}{s_5}\right)$$

If $\theta_5 = 0$, the wrist is in a SINGULAR position, θ_4 and θ_6 have infinite number of combinations and they should be specially treated, e.g. manually assigning zero to θ_4 or θ_6,

Location of the Tool

In real applications, the position of the wrist point is usually not explicitly obtained. Instead, we can easily know the position of the tool mounted on the end link of a manipulator (${}^{0}\mathbf{d}_n$). In this case, ${}^{0}\mathbf{d}_6$ should be calculated first.

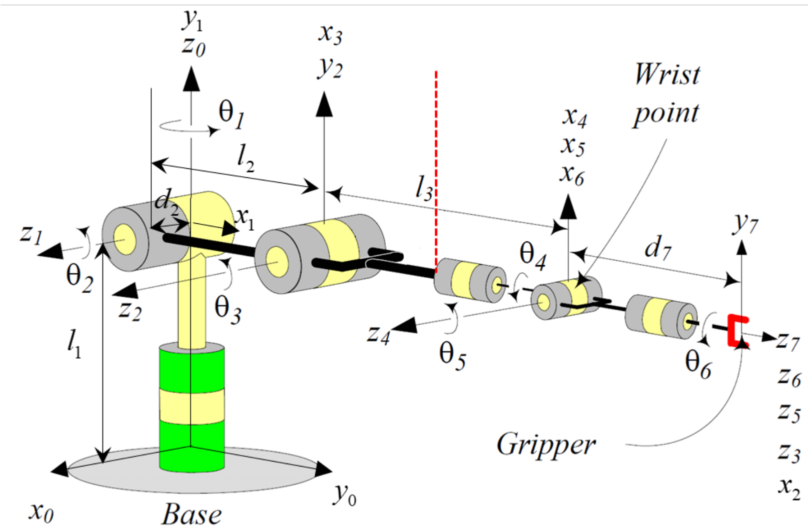
1. Model the tool as a vector in coordinate frame 6, or
2. Set the 7th coordinate frame attached to the tool mounted on the end link



$${}^{6}\mathbf{d}_n = \begin{bmatrix} 0 \\ 0 \\ d_7 \end{bmatrix} \quad \text{or} \quad {}^{6}T_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{cases} {}^{0}R_6 = {}^{0}R_n \\ {}^{0}\mathbf{d}_6 = {}^{0}\mathbf{d}_n - {}^{0}R_6 \, {}^{6}\mathbf{d}_n \end{cases}$$

Summary: 6-DoF Manipulator



$$\theta_1 = \text{atan2}(d_2, \pm\sqrt{d_x^2 + d_y^2 - d_2^2}) + \text{atan2}(d_y, d_x)$$

$$\theta_2 = \text{atan2}(s, \pm\sqrt{m^2 + n^2 - s^2}) + \text{atan2}(m, n)$$

$$\theta_3 = \text{atan2}(d_x c_1 + d_y s_1 - l_2 c_2, l_1 + l_2 s_2 - d_z) - \theta_2$$

$$\theta_5 = \text{atan2}(\pm\sqrt{1 - r_{33}^{\ 2}}, r_{33}) \qquad (\theta_5 \neq k\pi)$$

$$\theta_4 = \text{atan2}(\frac{r_{23}}{s_5}, \frac{r_{13}}{s_5})$$

$$\theta_6 = \text{atan2}(\frac{r_{32}}{s_5}, -\frac{r_{31}}{s_5})$$

$${}^0R_n, \quad {}^0\mathbf{d}_n \quad \Rightarrow$$

$${}^0R_6 = {}^0R_n \qquad {}^0\mathbf{d}_6 = {}^0\mathbf{d}_n - {}^0R_6\,{}^6\mathbf{d}_n$$

$${}^0T_6 = \begin{bmatrix} R_6 & \mathbf{d}_6 \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & d_x \\ n_y & o_y & a_y & d_y \\ n_z & o_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$m = 2l_2(d_x c_1 + d_y s_1)$$

$$n = 2l_2(l_1 - d_z)$$

$$s = l_3^2 - l_1^2 - l_2^2 - d_z^2 - (d_x c_1 + d_y s_1)^2 + 2l_1 d_z$$

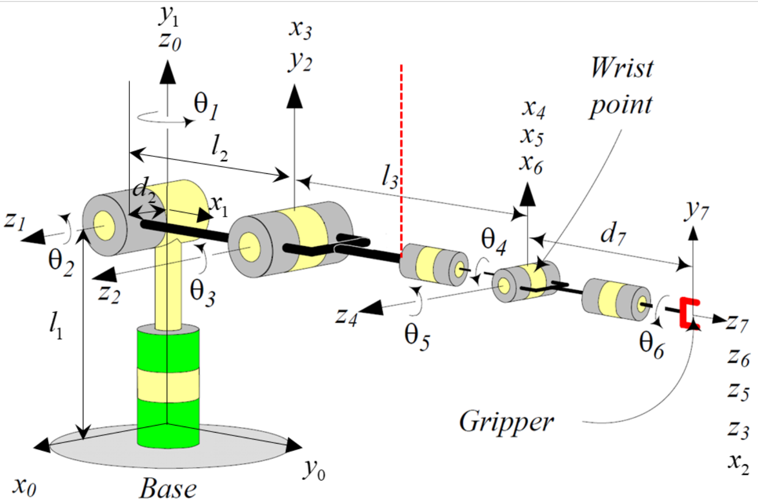
$$r_{13} = a_x c_1 c_{23} + a_y s_1 c_{23} + a_z s_{23} \qquad r_{31} = n_x c_1 s_{23} + n_y s_1 s_{23} - n_z c_{23}$$

$$r_{23} = a_x s_1 - a_y c_1 \qquad r_{32} = o_x c_1 s_{23} + o_y s_1 s_{23} - o_z c_{23}$$

$$r_{33} = a_x c_1 s_{23} + a_y s_1 s_{23} - a_z c_{23}$$

Ex 4-1-2

For the 6-DOF manipulator, find solution to its forward kinematics problem given $L_1 = L_2 = L_3 = 1$, $d_2 = 0.1$, $(q_1, q_2, q_3, q_4, q_5, q_6) = (0, \pi/4, -\pi/2, 0, \pi/4, \pi/2)$, and $d_7 = 0.15$. *Prove that* $(0, \pi/4, -\pi/2, -\pi, -\pi/4, \pi/2)$ *is a solution to the inverse kinematics problem.*



$$^0T_3 = {}^0T_1\,{}^1T_2\,{}^2T_3$$

$$= \begin{bmatrix} c_1c_{23} & s_1 & c_1s_{23} & l_2c_1c_2 + d_2s_1 \\ s_1s_{23} & -c_1 & s_1s_{23} & l_2c_1s_2 - d_2c_1 \\ s_{23} & 0 & -c_{23} & l_1 + l_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0.1 \\ 0.7071 & 0 & 0.7071 & 0.7071 \\ 0.7071 & 0 & -0.7071 & 1.7071 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^3T_6 = {}^3T_4\,{}^4T_5\,{}^5T_6$$

$$= \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -s_4c_6 - c_4c_5s_6 & c_4s_5 & 0 \\ c_4s_6 + s_4c_5c_6 & c_4c_6 - s_4c_5s_6 & s_4s_5 & 0 \\ -s_5c_6 & s_5s_6 & c_5 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -0.7071 & 0.7071 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0.7071 & 0.7071 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_7 = {}^0T_3\,{}^3T_6\,{}^6T_7 = {}^0T_6\,{}^6T_7$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0.1 \\ 0 & 0 & 1 & 1.4142 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.15 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0.1 \\ 0 & 0 & 1 & 1.5642 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

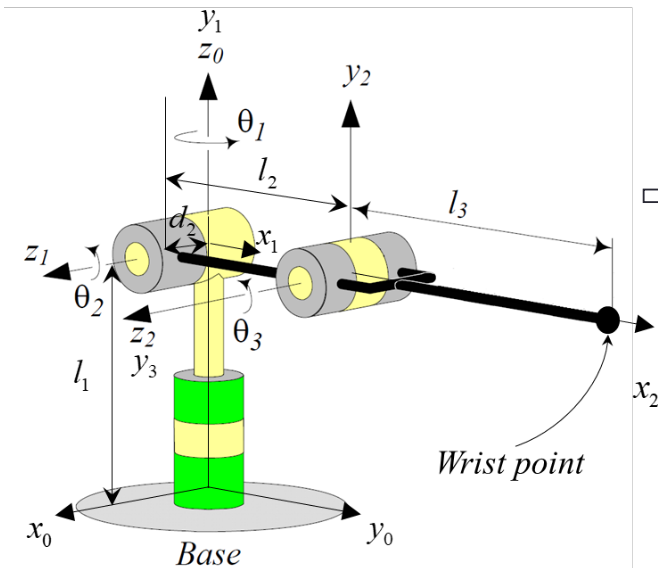
$$\theta_5 = \text{atan2}(\pm\sqrt{1 - r_{33}{}^2}, r_{33}) = \text{atan2}(\pm\sqrt{1 - 0.7071^2}, 0.7071) = \begin{cases} 45^\circ \\ -45^\circ \end{cases}$$

$$\theta_4 = \text{atan2}\left(\frac{r_{23}}{\sin_5}, \frac{r_{13}}{s_5}\right) = \text{atan2}\left(\frac{0}{0.7071}, \pm\frac{0.7071}{0.7707}\right) = \begin{cases} 0^\circ \\ -180^\circ \end{cases}$$

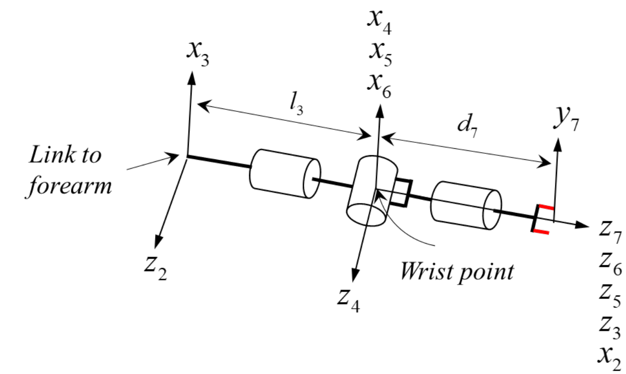
$$\theta_6 = \text{atan2}\left(\frac{r_{32}}{s_5}, -\frac{r_{31}}{s_5}\right) = \text{atan2}\left(\pm\frac{0.7071}{0.7071}, \mp\frac{0}{0.7071}\right) = \begin{cases} 90^\circ \\ 90^\circ \end{cases}$$

Notes on Spherical Wrist

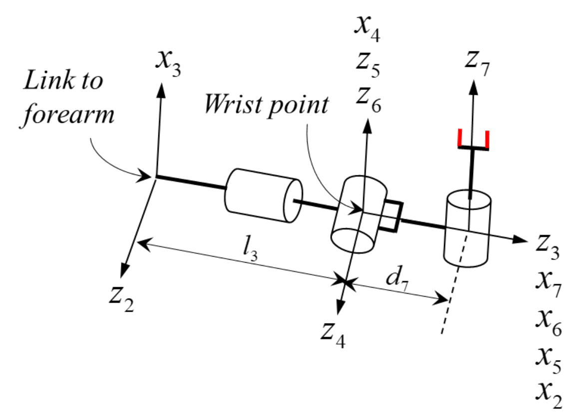
Spherical wrist has 3 different mechanism types, each of them has different coordinate frames and different configurations of DH notations.



Type 1: RPR



Type 2: RPY



Type 3: PYR