

机器人技术与实践

A/P ZHOU, Chunlin (周春琳)

Institute of Cyber-system and Control

College of Control Science and Engineering, Zhejiang University

Email: c_zhou@zju.edu.cn

4. INVERSE KINEMATICS II

$${}^1T_6 = {}^0T_1^{-1} {}^0T_6$$

$${}^2T_6 = {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6$$

$${}^3T_6 = {}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6$$

$${}^4T_6 = {}^3T_4^{-1} {}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6$$

$${}^5T_6 = {}^4T_5^{-1} {}^3T_4^{-1} {}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6$$

$$\mathbf{I} = {}^5T_6^{-1} {}^4T_5^{-1} {}^3T_4^{-1} {}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6$$

4.2 A Generic Method

- ✓ The decomposition of inverse position and inverse orientation kinematics is not always available if a robot has no spherical wrist structure, i.e there is no property that 3 rotation axes intersects at the same point.
- ✓ A way to find analytical solutions can be derived through **the Pieper's technique**:

$${}^0T_6 = {}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5 {}^5T_6 \quad \Rightarrow$$

$${}^1T_6 = ({}^0T_1)^{-1} {}^0T_6 \quad \longrightarrow \quad \text{contains only } \theta_1$$

$${}^2T_6 = ({}^1T_2)^{-1} ({}^0T_1)^{-1} {}^0T_6 \quad \longrightarrow \quad \text{contains } \theta_1, \theta_2$$

$${}^3T_6 = ({}^2T_3)^{-1} ({}^1T_2)^{-1} ({}^0T_1)^{-1} {}^0T_6 \quad \longrightarrow \quad \text{contains } \theta_1, \theta_2, \theta_3$$

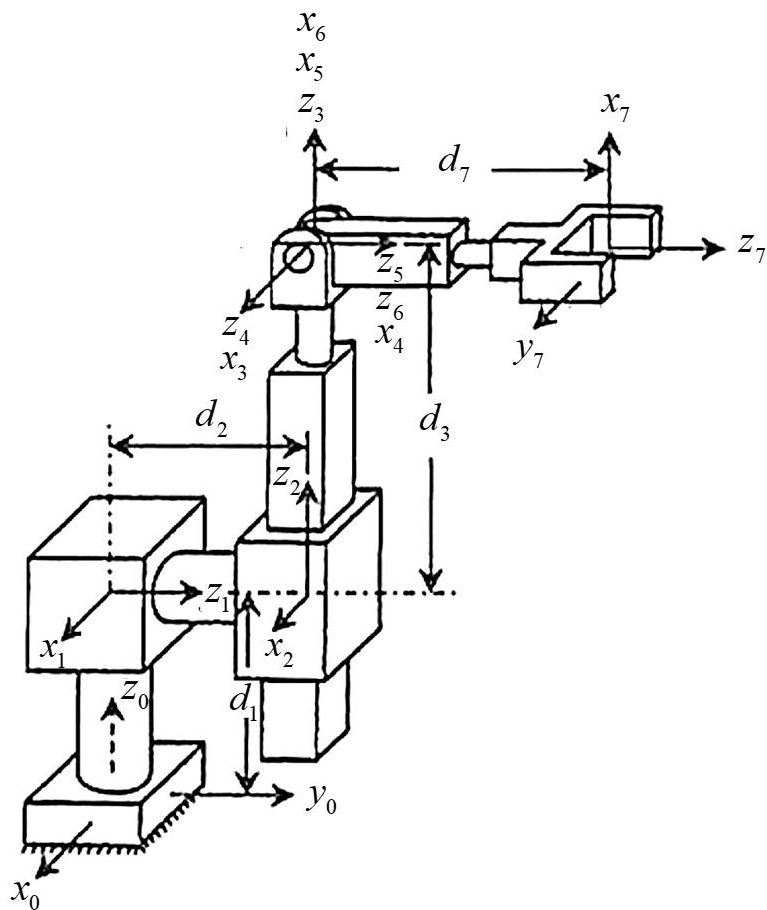
$${}^4T_6 = ({}^3T_4)^{-1} ({}^2T_3)^{-1} ({}^1T_2)^{-1} ({}^0T_1)^{-1} {}^0T_6 \quad \longrightarrow \quad \text{contains } \theta_1, \theta_2, \theta_3, \theta_4$$

$${}^5T_6 = ({}^4T_5)^{-1} ({}^3T_4)^{-1} ({}^2T_3)^{-1} ({}^1T_2)^{-1} ({}^0T_1)^{-1} {}^0T_6 \quad \longrightarrow \quad \text{contains } \theta_1, \theta_2, \theta_3, \theta_4, \theta_5$$

$$I_{6 \times 6} = ({}^5T_6)^{-1} ({}^4T_5)^{-1} ({}^3T_4)^{-1} ({}^2T_3)^{-1} ({}^1T_2)^{-1} ({}^0T_1)^{-1} {}^0T_6 \quad \longrightarrow \quad \text{contains } \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$$

Ex 4-2-1

Solve the inverse kinematics problem of the
Stanford manipulator



No.	a_i	α_i	d_i	θ_i
0	0	-90°	d_1	θ_1
1	0	90°	d_2	θ_2
2	0	0	$d_3(+d_0)$	0
3	0	90°	0	$\theta_4(90^\circ)$
4	0	90°	0	$\theta_5(90^\circ)$
5	0	0	0	θ_6

Ex 4-2-1

$${}^0T_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1T_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2T_3 = \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^3T_4 = \begin{bmatrix} c_4 & 0 & s_4 & 0 \\ s_4 & 0 & -c_4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^4T_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^5T_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_6 = {}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5 {}^5T_6 = \begin{bmatrix} n_x & o_x & a_x & d_x \\ n_y & o_y & a_y & d_y \\ n_z & o_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1T_6 = {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5 {}^5T_6 = \begin{bmatrix} X & X & X & d_3 s_2 \\ X & X & X & -d_3 c_2 \\ X & X & X & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1T_6 = ({}^0T_1)^{-1} {}^0T_6 = \begin{bmatrix} X & X & X & d_x c_1 + d_y s_1 \\ X & X & X & d_1 - d_z \\ X & X & X & d_y c_1 - d_x s_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$d_y c_1 - d_x s_1 = d_2 \Rightarrow$$

$$\theta_1 = \text{atan2}(d_y, d_x) - \text{atan2}(d_2, \pm \sqrt{d_x^2 + d_y^2 - d_2^2})$$

$$\left. \begin{aligned} d_3 s_2 &= d_x c_1 + d_y s_1 \\ -d_3 c_2 &= d_1 - d_z \end{aligned} \right\} \Rightarrow \theta_2 = \text{atan2}(d_x c_1 + d_y s_1, d_z - d_1)$$

$$\Rightarrow d_3 = \frac{d_z - d_1}{\cos \theta_2}$$

$${}^4T_6 = ({}^3T_4)^{-1} ({}^2T_3)^{-1} ({}^1T_2)^{-1} ({}^0T_1)^{-1} {}^0T_6 = \begin{bmatrix} X & X & r_{13} & X \\ X & X & r_{23} & X \\ X & X & r_{33} & X \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^4T_6 = {}^4T_5 {}^5T_6 = \begin{bmatrix} c_5 c_6 & -c c_5 s_6 & s_5 & 0 \\ s_5 c_6 & -s_5 s_6 & -c_5 & 0 \\ s_6 & c_6 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow r_{33} = (a_x s_1 - a_y c_1) c_4 + (a_x c_1 c_2 + a_y s_1 c_2 - a_z s_2) s_4 = 0$$

$$\Rightarrow \theta_4 = \text{atan2}(a_x c_1 c_2 + a_y s_1 c_2 - a_z s_2, a_y c_1 - a_x s_1)$$

$$\left. \begin{aligned} s_5 &= r_{13} \\ c_5 &= -r_{23} \end{aligned} \right\} \Rightarrow \theta_5 = \text{atan2}(r_{13}, -r_{23}) \quad \text{where}$$

$$r_{13} = a_x (c_1 c_2 c_4 - s_1 s_4) + a_y (c_1 s_4 + s_1 c_2 c_4) - a_z s_2 c_4 \quad r_{23} = a_x c_1 s_2 + a_y s_1 s_2 + a_z c_2$$

$${}^5T_6 = ({}^4T_5)^{-1}({}^3T_4)^{-1}({}^2T_3)^{-1}({}^1T_2)^{-1}({}^0T_1)^{-1}{}^0T_6 = \begin{bmatrix} r_{11} & X & X & X \\ r_{21} & X & X & X \\ X & X & X & X \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^5T_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left. \begin{array}{l} s_6 = r_{21} \\ c_6 = r_{11} \end{array} \right\} \Rightarrow \theta_6 = \text{atan2}(r_{21}, r_{11}) \quad \text{where}$$

$$r_{11} = (s_2 s_5 + c_2 c_4 c_5)(n_x c_1 + n_y s_1) + s_4 c_5 (n_y c_1 - n_x s_1) + n_z (c_2 s_5 - s_2 c_4 c_5)$$

$$r_{21} = c_2 s_4 (n_x c_1 + n_y s_1) + c_4 (n_x s_1 - n_y c_1) - n_z s_2 s_4$$

4.3 Numerical Method

- ✓ Analytical solution may not exist or it can hardly be obtained due to multiplicity or singularity issues.
- ✓ Finding a numerical solution can be interpreted as searching for the solution q_k of a set of nonlinear algebraic equations.

$$\begin{aligned}
 {}^0T_n &= \mathbf{T}(\mathbf{q}) \\
 &= {}^0T_1(q_1) {}^1T_2(q_2) {}^2T_3(q_3) {}^3T_4(q_4) \cdots {}^{n-1}T_n(q_n) \\
 &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{T}_d
 \end{aligned}$$

- ✓ The most common method, known as the Newton-Raphson method can be used to find the zeros of the following equation:

$$T = T(\mathbf{q}) - {}^0T_n = 0$$

Newton-Raphson Method

Step 1: start with an initial guess $\mathbf{q}^* = \mathbf{q} + \delta\mathbf{q}$

Step 2: Use the forward kinematics to determine the configuration of the end-effector frame for the guessed joint variables.

$$T^* = T(\mathbf{q}^*)$$

Step 3: Evaluate the difference between the desired orientation & position T_d and the present T^* . By using **first order Taylor expansion**,

$$T_d = T(\mathbf{q} + \delta\mathbf{q}) = T(\mathbf{q}) + \frac{\partial T}{\partial \mathbf{q}} \delta\mathbf{q} + O(\delta\mathbf{q}^2)$$



$$\delta T = T_d - T(\mathbf{q}) \approx \frac{\partial T(\mathbf{q})}{\partial \mathbf{q}} \delta\mathbf{q} = J \delta\mathbf{q} \quad \Rightarrow \quad \delta\mathbf{q} \approx J^{-1} \delta T$$

J is called the **Jacobian Matrix**

$$J = \frac{\partial T(\mathbf{q})}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial T_1}{\partial q_1} & \frac{\partial T_1}{\partial q_2} & \cdots & \frac{\partial T_1}{\partial q_n} \\ \frac{\partial T_2}{\partial q_1} & \frac{\partial T_2}{\partial q_2} & \cdots & \frac{\partial T_2}{\partial q_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial T_6}{\partial q_1} & \frac{\partial T_6}{\partial q_2} & \cdots & \frac{\partial T_6}{\partial q_n} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & \cdots & J_{1n} \\ J_{21} & J_{22} & \cdots & J_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ J_{61} & J_{62} & \cdots & J_{6n} \end{bmatrix}$$

Step 4: Update variables

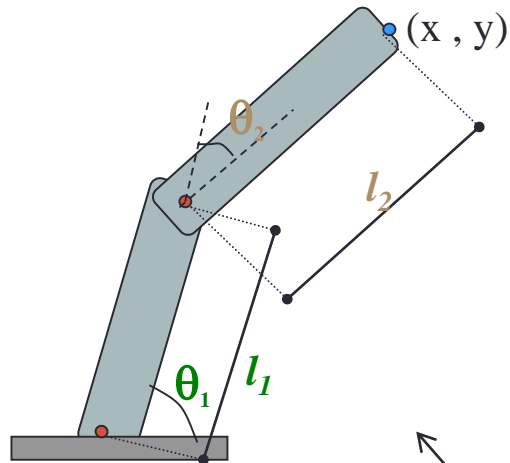
$$\delta \mathbf{q} = J^{-1} \delta T$$

$$\mathbf{q}^{(i+1)} = \mathbf{q}^{(i)} + J^{-1}(\mathbf{q}^{(i)}) \delta T(\mathbf{q}^{(i)})$$

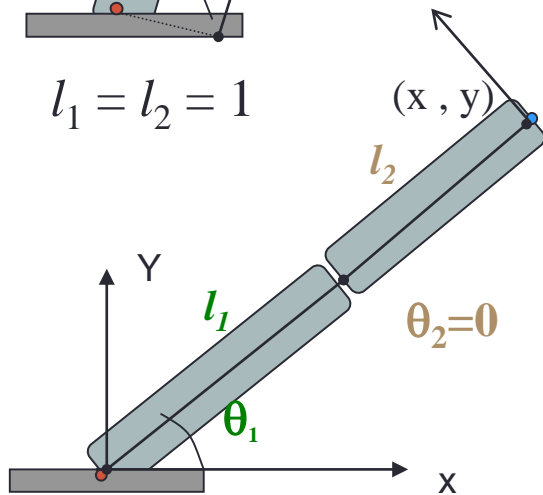
Step 5: Repeat the above process from step 2. The iteration can be terminated if every elements of $T(\mathbf{q}^{(i)})$ or its norm is less than a give tolerance, $\|\delta T(\mathbf{q}^{(i)})\| < \varepsilon$, or $\|J\| < \varepsilon$, or $\|\mathbf{q}^{(i+1)} - \mathbf{q}^{(i)}\| < \varepsilon$

EX 4-3-1

Find the solution of inverse kinematics problem of the 2R Planar Arm using Newton-Raphson method: $[x \ y]^T = [1 \ 1]^T$



$$l_1 = l_2 = 1$$



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{bmatrix} = \begin{bmatrix} T_1(\theta_1, \theta_2) \\ T_2(\theta_1, \theta_2) \end{bmatrix} \quad T_d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{\partial T_1}{\partial \theta_1} & \frac{\partial T_1}{\partial \theta_2} \\ \frac{\partial T_2}{\partial \theta_1} & \frac{\partial T_2}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

$$J^{-1} = \frac{1}{l_1 l_2 s_2} \begin{bmatrix} l_2 c_{12} & l_2 s_{12} \\ -l_1 c_1 - l_2 c_{12} & -l_1 s_1 - l_2 s_{12} \end{bmatrix}$$

0

$$\mathbf{q}^0 = \begin{bmatrix} 2\pi/3 \\ -2\pi/3 \end{bmatrix} \quad T^0 = \begin{bmatrix} 0.5 \\ 0.866 \end{bmatrix} \quad \delta T^0 = \begin{bmatrix} 0.5 \\ 0.134 \end{bmatrix}$$

$$\text{norm}(\delta T) = 0.5176 > 0.0001$$

1

$$J^{-1}(\mathbf{q}^0) = \begin{bmatrix} -1.1547 & 0 \\ 0.5774 & 1 \end{bmatrix}$$

$$\mathbf{q}^1 = \mathbf{q}^0 + J^{-1}(\mathbf{q}^0)\delta T(\mathbf{q}^0) = \begin{bmatrix} 1.517 \\ -1.6717 \end{bmatrix}$$

$$T^1 = \begin{bmatrix} 1.0418 \\ 0.8445 \end{bmatrix} \quad \delta T^1 = \begin{bmatrix} -0.0418 \\ 0.1555 \end{bmatrix} \quad \text{norm}(\delta T) = 0.161 > 0.0001$$

2

$$J^{-1}(\mathbf{q}^1) = \begin{bmatrix} -0.9931 & 0.1549 \\ 1.0471 & 0.8488 \end{bmatrix}$$

$$\mathbf{q}^2 = \mathbf{q}^1 + J^{-1}(\mathbf{q}^1)\delta T(\mathbf{q}^1) = \begin{bmatrix} 1.5826 \\ -1.5835 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} 0.9882 \\ 0.9991 \end{bmatrix} \quad \delta T^2 = \begin{bmatrix} 0.0118 \\ 0.0009 \end{bmatrix} \quad \text{norm}(\delta T) = 0.0119 > 0.0001$$

3

$$J^{-1}(\mathbf{q}^2) = \begin{bmatrix} -1.0001 & 0.0009 \\ 0.9882 & 0.9992 \end{bmatrix}$$

$$\mathbf{q}^3 = \mathbf{q}^2 + J^{-1}(\mathbf{q}^2)\delta T(\mathbf{q}^2) = \begin{bmatrix} 1.5708 \\ -1.5709 \end{bmatrix} \approx \begin{bmatrix} \pi / 2 \\ -\pi / 2 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} 1 \\ 0.9999 \end{bmatrix} \quad \delta T^2 = \begin{bmatrix} 0 \\ 0.0001 \end{bmatrix} \quad \text{norm}(\delta T) < 0.0001$$

Notes on Numerical Method

- ✓ Effectiveness of the procedure depends on the number of iterations to be performed, which depends on the initial estimate of \mathbf{q} and on the dimension of the Jacobian matrix.
- ✓ Since the solution to nonlinear equations is not unique, it may generate different sets of solutions depending on the initial guess.
- ✓ Convergence may not occur if the initial estimate of the solution falls outside the convergence domain of the algorithm.
- ✓ Convergence speed can be improved by using better termination conditions.

Different Cases

- ✓ Consider the m nonlinear equations with n unknown variables.

$$\mathbf{y} = [y_i]_{m \times 1} = [f_i(\mathbf{q})] \quad \mathbf{q} = [q_1, \dots, q_j, \dots, q_n]^T \quad i = 1, \dots, m$$

- ✓ Assume that \mathbf{q} is the exact solution of these equations and \mathbf{q}^* is an approximate solution. By using the first-order Taylor expansion

$$\mathbf{y} = \mathbf{f}(\mathbf{q}^*) + \left[\sum_{j=1}^n \frac{\partial f_i}{\partial q_j} \delta q_j + O(\delta q_j^2) \right]_{m \times 1} = \mathbf{f}(\mathbf{q}^*) + J \delta \mathbf{q} + O(\delta \mathbf{q}^2) \quad (\delta \mathbf{q} \triangleq \mathbf{q} - \mathbf{q}^*, \quad J = \left[\frac{\partial f_i}{\partial q_j} \right])$$

- ✓ The difference between the exact solution and the estimated solution can be approximately given by

$$\mathbf{r} = \mathbf{y} - \mathbf{f}(\mathbf{q}^*) \approx J \delta \mathbf{q}$$

-
- ✓ **Case 1: $m = n$** , the linearized residual equation has a unique solution and the Newton-Raphson technique can be utilized.

Different Cases

- ✓ **Case 2: $m > n$** , the manipulator is under-actuated and not all degrees-of-freedom can be controlled. Solutions do not exist. We may only find a solution to minimize the end-effector's location error.

$$\min D = \frac{1}{2} \sum_{i=1}^m w_i [y_i - f_i(\mathbf{q})]^2 \quad \Rightarrow \quad \min D = \frac{1}{2} [\mathbf{y} - f(\mathbf{q})]^T W [\mathbf{y} - f(\mathbf{q})]$$

$(W = \text{diag}(w_1, \dots, w_n))$

- ✓ The error is minimum if

$$\frac{\partial D}{\partial \mathbf{q}} = - \sum_{i=1}^m \frac{\partial f_i}{\partial \mathbf{q}} w_i [y_i - f_i(\mathbf{q})] = 0 \quad \Rightarrow \quad \left. \begin{aligned} J^T W [\mathbf{y} - f(\mathbf{q}^*)] &= J^T W \mathbf{r} = 0 \\ J^T W [\mathbf{y} - f(\mathbf{q}^*)] &\approx J^T W J \delta \mathbf{q} \end{aligned} \right\} \Rightarrow$$

$$J^T W J \delta \mathbf{q} = J^T W \mathbf{r} \quad \Rightarrow \quad \delta \mathbf{q} = (J^T W J)^{-1} J^T W \mathbf{r}$$

Different Cases

- ✓ **Case 3: $m < n$** , the manipulator is **redundant** and infinite number of solutions may exist.
- ✓ Selection of an appropriate solution can be made under the condition that it is optimal in some sense. For example, let us find a solution which minimizes the deviation from a given reference configuration \mathbf{q}_d . The problem may then be formulated as that of finding the minimum of a constrained function

$$\begin{aligned} \min \quad & D = \frac{1}{2}(\mathbf{q}_d - \mathbf{q})^T W(\mathbf{q}_d - \mathbf{q}) \\ \text{s.t.} \quad & \mathbf{y} - f(\mathbf{q}) = 0 \end{aligned}$$

Using the technique of Lagrangian multipliers

$$L = \frac{1}{2}(\mathbf{q}_d - \mathbf{q})^T W(\mathbf{q}_d - \mathbf{q}) + \boldsymbol{\lambda}^T [\mathbf{y} - f(\mathbf{q})] \quad \frac{\partial L}{\partial \mathbf{q}} = W(\mathbf{q}_d - \mathbf{q}) - J^T \boldsymbol{\lambda} = 0 \quad \Rightarrow \quad W \delta \mathbf{q} = J^T \boldsymbol{\lambda}$$

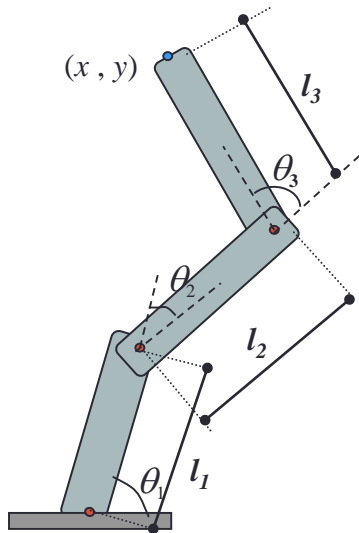
$$\left. \begin{aligned} \Rightarrow \quad \delta \mathbf{q} &= W^{-1} J^T \boldsymbol{\lambda} \\ \mathbf{r} &= J \delta \mathbf{q} \end{aligned} \right\} \Rightarrow \mathbf{r} = J W^{-1} J^T \boldsymbol{\lambda} \Rightarrow \boldsymbol{\lambda} = (J W^{-1} J^T)^{-1} \mathbf{r} \Rightarrow \delta \mathbf{q} = J^\dagger \mathbf{r}$$

$J^\dagger = W^{-1} J^T (J W^{-1} J^T)^{-1}$ is called a pseudo-inverse to the singular Jacobian matrix .

EX 4-3-2

Find the solution of inverse kinematics problem of the 3R Planar

Arm: $[x \ y]^T = [2 \ 2]^T$



$$l_1 = l_2 = l_3 = 1$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ l_1 s_1 + l_2 s_{12} + l_3 s_{123} \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \quad T_d = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad W = I$$

$$J = \begin{bmatrix} \frac{\partial T_1}{\partial \theta_1} & \frac{\partial T_1}{\partial \theta_2} & \frac{\partial T_1}{\partial \theta_3} \\ \frac{\partial T_2}{\partial \theta_1} & \frac{\partial T_2}{\partial \theta_2} & \frac{\partial T_2}{\partial \theta_3} \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \end{bmatrix}$$

$$J^\dagger = W^{-1} J^T (J W^{-1} J^T)^{-1}$$

0

$$\mathbf{q}^0 = \begin{bmatrix} \pi/3 \\ -\pi/3 \\ \pi/3 \end{bmatrix}$$

$$T^0 = \begin{bmatrix} 2 \\ 1.732 \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} 0 \\ 0.2679 \end{bmatrix}$$

$$\text{norm}(\mathbf{r}) = 0.2679 > 0.0001$$

1

$$J^\dagger(\mathbf{q}^0) = \begin{bmatrix} -0.3849 & 0 \\ 0.9623 & 1 \\ -1.3472 & -1 \end{bmatrix}$$

$$\mathbf{q}^1 = \mathbf{q}^0 + J^\dagger(\mathbf{q}^0) \mathbf{r} = \begin{bmatrix} 1.0472 \\ -0.7792 \\ 0.7792 \end{bmatrix}$$

$$T^1 = \begin{bmatrix} 1.9643 \\ 1.9968 \end{bmatrix} \quad \mathbf{r}^1 = \begin{bmatrix} 0.0357 \\ 0.0032 \end{bmatrix} \quad \text{norm}(\mathbf{r}) = 0.0358 > 0.0001$$

4

$$J^\dagger(\mathbf{q}^4) = \begin{bmatrix} -0.5207 & -0.1851 \\ 1.3156 & 1.4757 \\ -1.7232 & -1.5514 \end{bmatrix}$$

$$\mathbf{q}^4 = \mathbf{q}^3 + J^\dagger(\mathbf{q}^3) \mathbf{r}^3 = \begin{bmatrix} 1.0299 \\ -0.7311 \\ 0.714 \end{bmatrix} = \begin{bmatrix} 59.01^\circ \\ -41.9^\circ \\ 40.9^\circ \end{bmatrix}$$

$$\text{norm}(\mathbf{r}) = 3.67\text{e}-12 > 0.0001$$

1

$$J^{-1}(\mathbf{q}^0) = \begin{bmatrix} -1.1547 & 0 \\ 0.5774 & 1 \end{bmatrix}$$

$$\mathbf{q}^1 = \mathbf{q}^0 + J^{-1}(\mathbf{q}^0)\delta T(\mathbf{q}^0) = \begin{bmatrix} 1.517 \\ -1.6717 \end{bmatrix}$$

$$T^1 = \begin{bmatrix} 1.0418 \\ 0.8445 \end{bmatrix} \quad \delta T^1 = \begin{bmatrix} -0.0418 \\ 0.1555 \end{bmatrix} \quad \text{norm}(\delta T) = 0.161 > 0.0001$$

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$$J^{-1}(\mathbf{q}^1) = \begin{bmatrix} -0.9931 & 0.1549 \\ 1.0471 & 0.8488 \end{bmatrix}$$

$$\mathbf{q}^2 = \mathbf{q}^1 + J^{-1}(\mathbf{q}^1)\delta T(\mathbf{q}^1) = \begin{bmatrix} 1.5826 \\ -1.5835 \end{bmatrix}$$

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