机器人技术与实践

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5. TRAJECTORY PLANNING

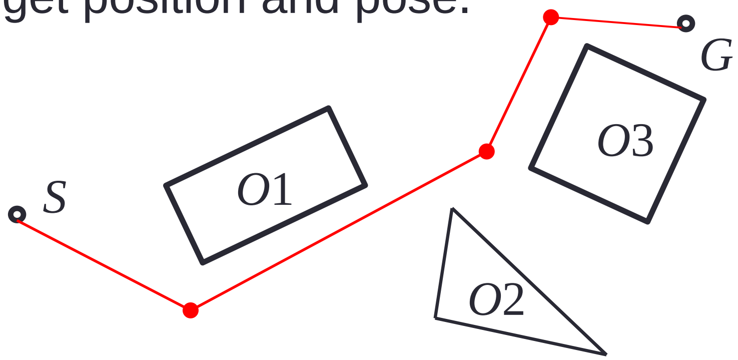
$$q(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

$$\dot{\mathbf{q}} = \mathbf{J}^{-1}\dot{\mathbf{X}}$$

Notes

Path Planning

- ✓ A ***path*** denotes the locus of points in the joint space, or in the operational space, which the manipulator has to follow in the execution of the assigned motion.
- ✓ Task of path planning is to find a feasible way connecting a start position and pose to a target position and pose.

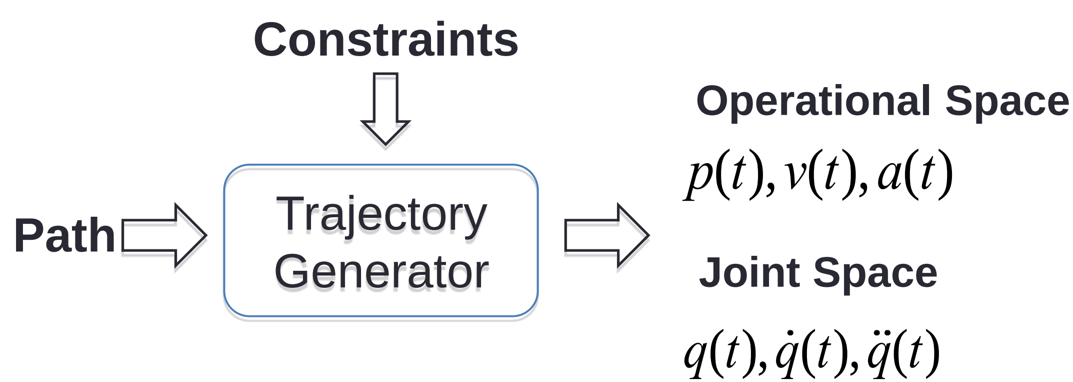


- ✓ A path is a pure geometric description of motion. Path planning does not answer the question that how a robot joint moves to a target angle.

Notes

Trajectory Planning

- ✓ A ***trajectory*** is a path on which a timing law is specified, for instance in terms of velocities and/or accelerations at each point.
- ✓ The task of the trajectory planning is to generate a time sequence of variables that describe end-effector position and orientation over time in respect of the imposed constraints:
  - Path
  - Boundary
  - No collision
  - Kinematics
  - Continuity
  - Smoothness
  - *etc.*

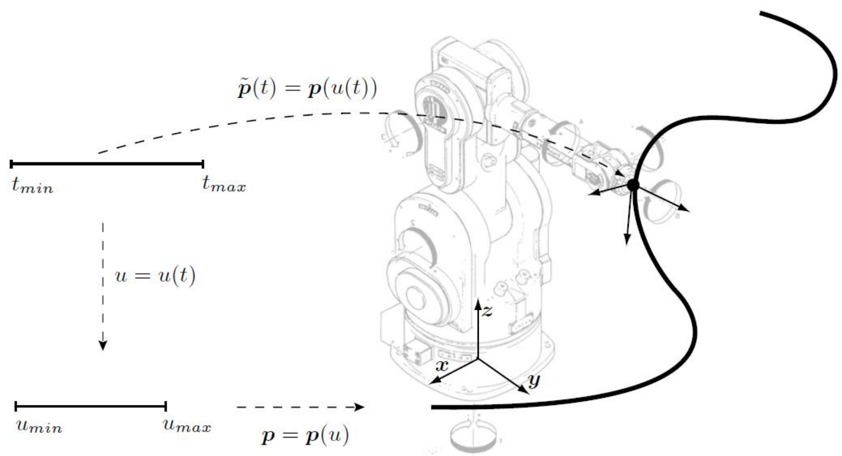


5.1 Planning in Joint Space

Planning in Operational Space v.s. Joint Space

- ✓ The trajectory of the end-effector's motion can be planned in the operational space (*Cartesian Space*) or in the joint space. Usually, the task is given in the operational space.
- ✓ If target angles of joints of a manipulator are available, trajectory functions for all joints can be obtained by explicit functions of time in the joint space.

$q(t), \dot{q}(t), \ddot{q}(t)$

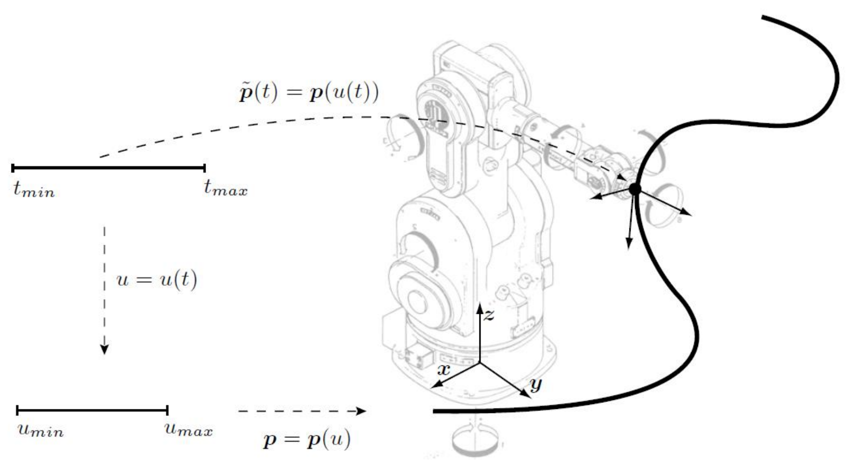


$p(t), v(t), a(t)$

There are pros. & cons. by using two types of planning method.

- ✓ Because a manipulator's motion is typically assigned in the operational space, it is natural to planning the trajectory in the operational space. Then joint control angles can be obtained by solving inverse kinematics problem for a manipulator.
- ✓ This method may suffer from discontinuity, singularity, or even no-solution issues. Therefore, in most cases the planning is made in the Joint Space.

$$q(t), \dot{q}(t), \ddot{q}(t)$$



$$p(t), v(t), a(t)$$

Models of a Trajectory

- ✓ Problem definition: for a given joint, initial joint angle at time t_0 is $\theta_i(t_0)$. The final joint angle at time t_f is $\theta_i(t_f)$. The time-dependent function that transfers $\theta_i(t_0)$ to $\theta_i(t_f)$ is called a trajectory.
- ✓ The trajectory from a start point to the target point is not unique. Theoretically, any-time dependent function can be used to model a trajectory as long as it satisfies constraints in the trajectory. Commonly used functions include polynomial function, harmonic, exponential or cycloid functions.

$$q(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

$$q(t) = a_0 + a_1 \cos a_2 t + a_3 \sin a_2 t$$

$$q(t) = a_0 + a_1 t - a_2 \sin a_3 t$$

Cubic Polynomial Trajectory

The cubic polynomial can be chosen to determine a joint motion as

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

resulting into a parabolic velocity profile and a linear acceleration profile

$$\dot{q}(t) = a_1 + 2a_2 t + 3a_3 t^2 \qquad \ddot{q}(t) = 2a_2 + 6a_3 t$$

A cubic polynomial has four coefficients. Therefore, it can satisfy the position and velocity constraints at the initial and final points.

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ q_0' \\ q_f \\ q_f' \end{bmatrix}$$

where

$$q(t_0) = q_0 \qquad \dot{q}(t_0) = q_0'$$
$$q(t_f) = q_f \qquad \dot{q}(t_f) = q_f'$$

✓ Coefficients of the cubic polynomial can be found by solving the boundary conditions equations:

$$a_0 = -\frac{q_1 t_0^2 (t_0 - 3t_f) + q_0 t_f^2 (3t_0 - t_f)}{(t_f - t_0)^3} - t_0 t_f \frac{q_0' t_f + q_1' t_0}{(t_f - t_0)^2}$$

$$a_1 = 6 t_0 t_f \frac{q_0 - q_1}{(t_f - t_0)^3} + \frac{q_0' t_f \left(t_f^2 + t_0 t_f - 2t_0^2\right) + q_1' t_0 \left(2t_f^2 - t_0^2 - t_0 t_f\right)}{(t_f - t_0)^3}$$

$$a_2 = -\frac{q_0 (3t_0 + 3t_f) + q_1 (-3t_0 - 3t_f)}{(t_f - t_0)^3} - \frac{q_1' \left(t_0 t_f - 2t_0^2 + t_f^2\right) + q_0' \left(2t_f^2 - t_0^2 - t_0 t_f\right)}{(t_f - t_0)^3}$$

$$a_3 = \frac{2q_0 - 2q_1 + q_0' (t_f - t_0) + q_1' (t_f - t_0)}{(t_f - t_0)^3}$$

✓ In case that $t_0 = 0$, the coefficients simplify to

$$a_0 = q_0, \quad a_1 = \dot{q}_0, \quad a_3 = \frac{3(q_f - q_0) - (2q_0' + q_f')t_f}{t_f^2}, \quad a_4 = \frac{(q_0' + q_f')t_f - 2(q_f - q_0)}{t_f^3}$$

✓ It is possible to employ a time shift and search for a cubic polynomial of the form

$$q(t) = a_0 + a_1 (t - t_0) + a_2 t (t - t_0)^2 + a_3 (t - t_0)^3$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & (t_f - t_0) & (t_f - t_0)^2 & (t_f - t_0)^3 \\ 0 & 1 & 2(t_f - t_0) & 3(t_f - t_0)^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ q'_0 \\ q_f \\ q'_f \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ q'_0 \\ -(t_f - t_0)^{-2} \left(3q_0 - 3q_f - 2t_0 q'_0 - t_0 q'_f + 2t_f q'_0 + t_f q'_f\right) \\ (t_f - t_0)^{-3} \left(2q_0 - 2q_f - t_0 q'_0 - t_0 q'_f + t_f q'_0 + t_f q'_f\right) \end{bmatrix}$$

✓ A cubic trajectory satisfies constraints of position and speed at start and target point

$$\begin{aligned} q(t_0) &= q_0 \qquad \dot{q}(t_0) = q'_0 \\ q(t_f) &= q_f \qquad \dot{q}(t_f) = q'_f \end{aligned}$$

Ex 5-1

Assume $q(0) = 10$, $q(1) = 45$, and $q'(0) = q'(1) = 0$. Find the **rest-to-rest** cubic trajectory.

$$a_0 = q_0$$

$$a_1 = \dot{q}_0$$

$$a_3 = \frac{3(q_f - q_0) - (2q_0' + q_f')t_f}{t_f^2}$$

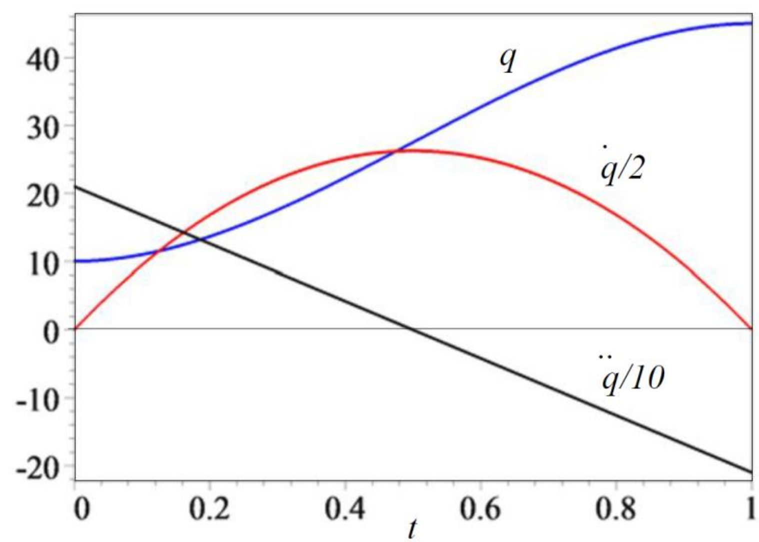
$$a_4 = \frac{(q_0' + q_f')t_f - 2(q_f - q_0)}{t_f^3}$$

The coefficients of the cubic polynomial are

$$a_0 = 10 \qquad a_1 = 0 \qquad a_2 = 105 \qquad a_3 = -70$$

which generates a trajectory as

$$q(t) = 10 + 105t^2 - 70t^3$$



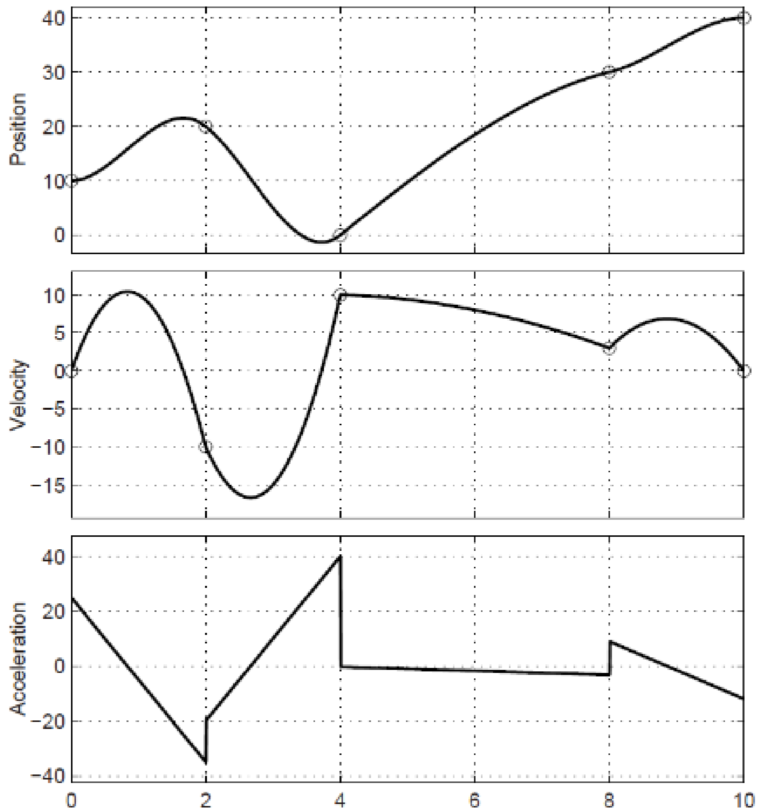
- ✓ By using multiple cubic polynomials, a trajectory passing multiple points can be established.
- ✓ A disadvantage of cubic paths is the acceleration jump at boundaries that introduces **infinite jerks.**

$$\ddot{q}(t) = 2a_2 + 6a_3 t$$

Position is smooth

Velocity is continuous but not smooth

Acceleration is not continuous



Quintic Polynomial Trajectory

- ✓ If it is desired to assign also the initial and final values of acceleration, six constraints have to be satisfied and then a polynomial of at least fifth order is needed.

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

- ✓ Specific position, velocity, and acceleration at boundaries introduces six conditions

$$\begin{aligned} q(t_0) &= q_0 \qquad \dot{q}(t_0) = q_0' \qquad \ddot{q}(t_0) = q_0'' \\ q(t_f) &= q_f \qquad \dot{q}(t_f) = q_f' \qquad \ddot{q}(t_f) = q_f'' \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 & t_0^5 \\ 0 & 1 & 2t_0 & 3t_0^2 & 4t_0^3 & 5t_0^4 \\ 0 & 0 & 2 & 6t_0 & 12t_0^2 & 20t_0^3 \\ 1 & t_f & t_f^2 & t_f^3 & t_f^4 & t_f^5 \\ 0 & 1 & 2t_f & 3t_f^2 & 4t_f^3 & 5t_f^4 \\ 0 & 0 & 2 & 6t_f & 12t_f^2 & 20t_f^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} q_0 \\ q_0' \\ q_0'' \\ q_f \\ q_f' \\ q_f'' \end{bmatrix}$$

✓ Employ a time shift and search for a qntic polynomial of the form

Position $$q(t) = a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + a_3(t - t_0)^3 + a_4(t - t_0)^4 + a_5(t - t_0)^5$$

Speed $$\dot{q}(t) = a_1 + 2a_2(t - t_0) + 3a_3(t - t_0)^2 + 4a_4(t - t_0)^3 + 5a_5(t - t_0)^4$$

Acceleration $$\ddot{q}(t) = 2a_2 + 6a_3(t - t_0) + 12a_4(t - t_0)^2 + 20a_5(t - t_0)^3$$

where

$$\begin{cases} a_0 = q_0 \\ a_1 = \dot{q}_0 \\ a_2 = \dfrac{\ddot{q}_0}{2} \\ a_3 = \dfrac{20q_f - 20q_0 - (8\dot{q}_f + 12\dot{q}_0)(t_f - t_0) - (3\ddot{q}_0 - \ddot{q}_f)(t_f - t_0)^2}{2(t_f - t_0)^3} \\ a_4 = \dfrac{-30q_f + 30q_0 + (14\dot{q}_f + 16\dot{q}_0)(t_f - t_0) + (3\ddot{q}_0 - 2\ddot{q}_f)(t_f - t_0)^2}{2(t_f - t_0)^4} \\ a_5 = \dfrac{12q_f - 12q_0 - (6\dot{q}_f + 6\dot{q}_0)(t_f - t_0) - (\ddot{q}_0 - \ddot{q}_f)(t_f - t_0)^2}{2(t_f - t_0)^5} \end{cases}$$

Ex 5-2

Find a rest-to-rest trajectory with zero acceleration at the rest positions with the following conditions:

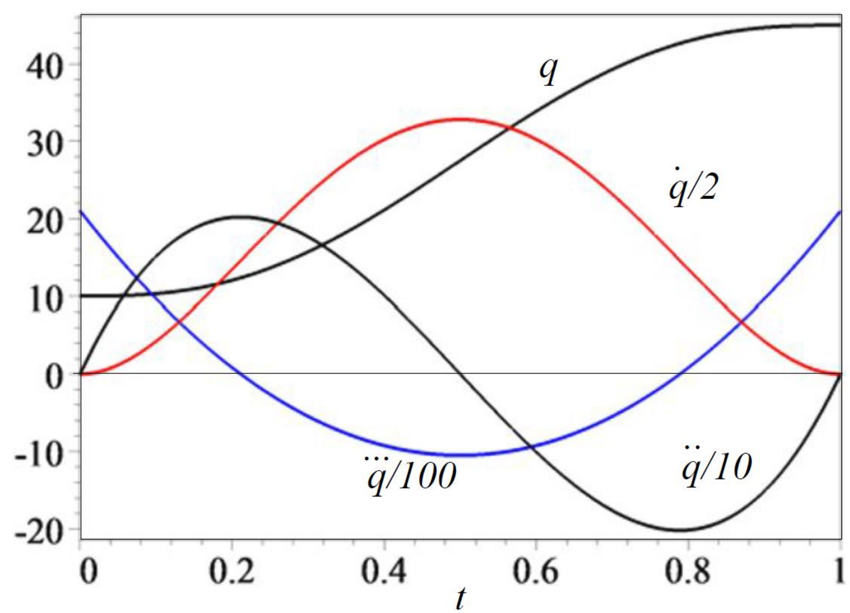
$$\begin{aligned} q(0) &= 10\,\text{deg} \qquad \dot{q}(0) = 0 \qquad \ddot{q}(0) = 0 \\ q(1) &= 45\,\text{deg} \qquad \dot{q}(1) = 0 \qquad \ddot{q}(1) = 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 6 & 12 & 20 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 45 \\ 0 \\ 0 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 350 \\ -525 \\ 210 \end{bmatrix}$$

$$q(t) = 10 + 350t^3 - 525t^4 + 210t^5$$



Zero Jerk Trajectory

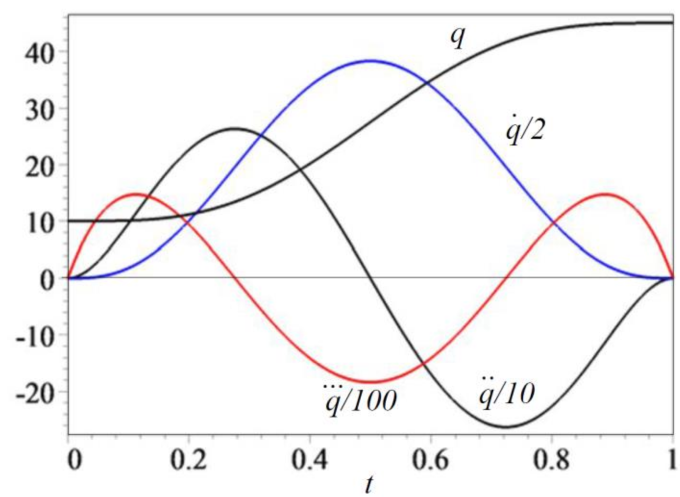
- ✓ To make a path start and stop with zero jerk, a seven degree polynomial and eight boundary conditions can be employed.

$$\begin{array}{lllll} q(0) = q_0 & \dot{q}(0) = 0 & \ddot{q}(0) = 0 & \dddot{q}(0) = 0 \\ q(1) = q_f & \dot{q}(1) = 0 & \ddot{q}(1) = 0 & \dddot{q}(1) = 0 \end{array}$$

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7$$

$$for\ q(0) = 10\deg\ \ and\ q(1) = 45\deg \qquad q(t) = 10 + 1225t^4 - 2940t^5 + 2450t^6 - 700t^7$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 2 & 6 & 12 & 20 & 30 & 42 \\ 0 & 0 & 0 & 6 & 24 & 60 & 120 & 210 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 45 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



Parabolic Trajectory

- ✓ A parabola (2-degree polynomials) yields constant accelerations in motion which has two segments with positive and negative accelerations.

Constraints: $\boxed{|\ddot{q}(t_0)| = a_c}$ $\quad q_1(0) = q_0 \quad \dot{q}_1(0) = 0$

- ✓ The first half of the motion has a positive acceleration

$$\begin{aligned}\dot{q}_1(t) &= a_c t \\ q_1(t) &= \frac{1}{2}a_c t^2 + q_0\end{aligned} \qquad 0 < t < \frac{1}{2}t_f$$

A known parameter

- ✓ For the second half of the path, we may start with a second degree polynomial

$$q_2(t) = a_0 + a_1 t + a_2 t^2 \qquad \frac{1}{2}t_f < t < t_f$$

There are 3 unknowns in q_2

Boundary constraints: $q_2(t_f) = q_f \qquad \dot{q}_2(t_f) = 0$

At the middle time of the whole journey：

$$q_1\left(\frac{t_f}{2}\right) = q_2\left(\frac{t_f}{2}\right) = \frac{1}{8}a_c t_f^2 + q_0$$

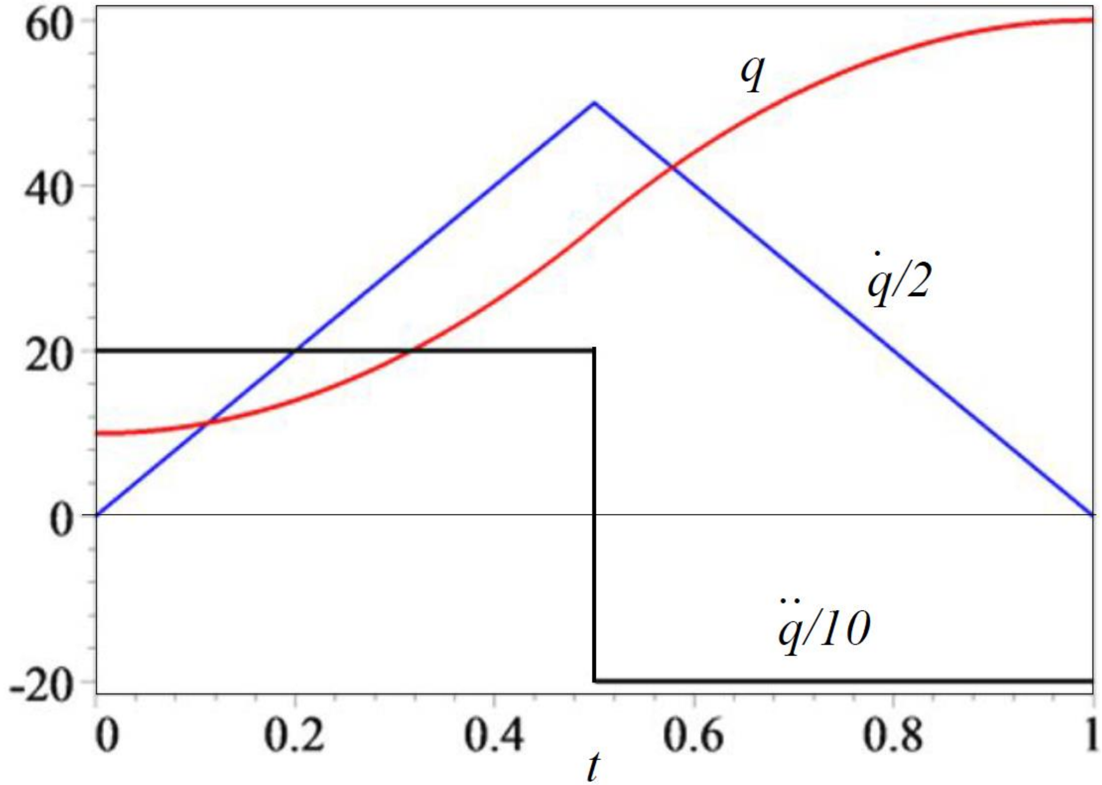
$$\Downarrow$$

$$\begin{bmatrix} 1 & t_f & t_f^2 \\ 0 & 1 & 2t_f \\ 0 & 1 & t_f \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} q_f \\ 0 \\ \frac{1}{8}a_c t_f^2 + q_0 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} q_f - t_f\left(q_0 + \frac{1}{8}a_c t_f^2\right) \\ 2q_0 + \frac{1}{4}a_c t_f^2 \\ -\frac{1}{t_f}\left(q_0 + \frac{1}{8}a_c t_f^2\right) \end{bmatrix}$$

$$q_1(t) = \frac{1}{2}a_c t^2 + q_0$$

$$q_2(t) = a_0 + a_1 t + a_2 t^2$$



$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} q_f - t_f\left(q_0 + \frac{1}{8}a_c t_f^2\right) \\ 2q_0 + \frac{1}{4}a_c t_f^2 \\ -\frac{1}{t_f}\left(q_0 + \frac{1}{8}a_c t_f^2\right) \end{bmatrix}$$

$$q_0 = 10\,\mathrm{deg},\ q_f = 45\,\mathrm{deg}$$

$$t_f = 1,\ and\ a_c = 200\ \mathrm{deg}\,/\,\mathrm{s}^2$$

Point Sequence Trajectory

- ✓ A path can be assigned via a series of points that the variable must attain at specific times. The points may also be defined to approximate a trajectory.
- ✓ Consider an example path specified by four points q_0, q_1, q_2, and q_3, such that the points are reached at times t_0, t_1, t_2, and t_3 respectively.
- ✓ In addition to positions, constraints on initial and final velocities and accelerations are usually imposed.

Constraints:

$$\begin{aligned} q(t_0) &= q_0 \qquad \dot{q}(t_0) = 0 \qquad \ddot{q}(t_0) = 0 \\ q(t_3) &= q_3 \qquad \dot{q}(t_3) = 0 \qquad \ddot{q}(t_3) = 0 \end{aligned}$$

$$\begin{aligned} q(t_1) &= q_1 \\ q(t_2) &= q_2 \end{aligned}$$

A seven degree polynomial can satisfy eight conditions.

The set of equations for the unknown coefficients of the seventh-order polynomial:

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 & t_0^5 & t_0^6 & t_0^7 \\ 0 & 1 & 2t_0 & 3t_0^2 & 4t_0^3 & 5t_0^4 & 6t_0^5 & 7t_0^6 \\ 0 & 0 & 2 & 6t_0 & 12t_0^2 & 20t_0^3 & 30t_0^4 & 42t_0^5 \\ 1 & t_1 & t_1^2 & t_1^3 & t_1^4 & t_1^5 & t_1^6 & t_1^7 \\ 1 & t_2 & t_2^2 & t_2^3 & t_2^4 & t_2^5 & t_2^6 & t_2^7 \\ 1 & t_3 & t_3^2 & t_3^3 & t_3^4 & t_3^5 & t_3^6 & t_3^7 \\ 0 & 1 & 2t_3 & 3t_3^2 & 4t_3^3 & 5t_3^4 & 6t_3^5 & 7t_3^6 \\ 0 & 0 & 2 & 6t_3 & 12t_3^2 & 20t_3^3 & 30t_3^4 & 42t_3^5 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} q_0 \\ 0 \\ 0 \\ q_1 \\ q_2 \\ q_3 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} q(0) &= 10\,\mathrm{deg} \\ q(0.4) &= 20\,\mathrm{deg} \\ q(0.7) &= 30\,\mathrm{deg} \\ q(1) &= 45\,\mathrm{deg} \end{aligned}$$

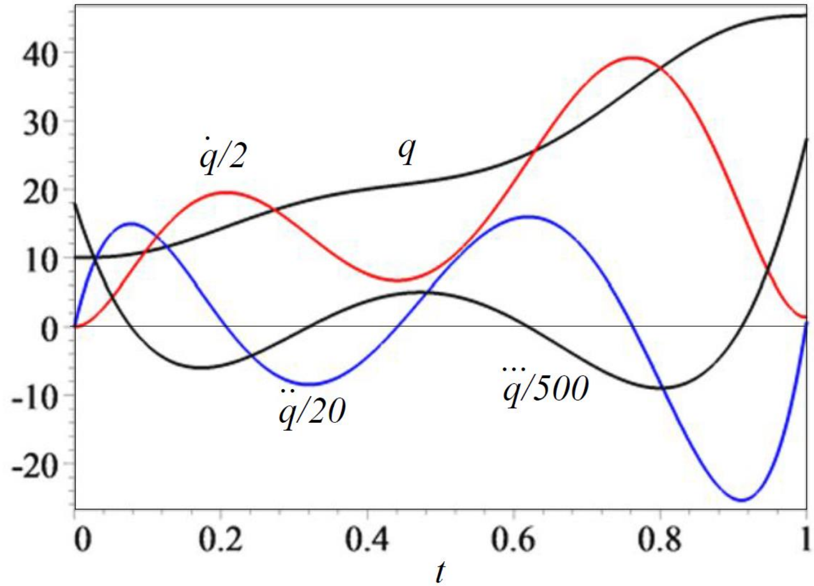
$$\dot{q}(0) = 0$$

$$\ddot{q}(0) = 0$$

$$\dot{q}(1) = 0$$

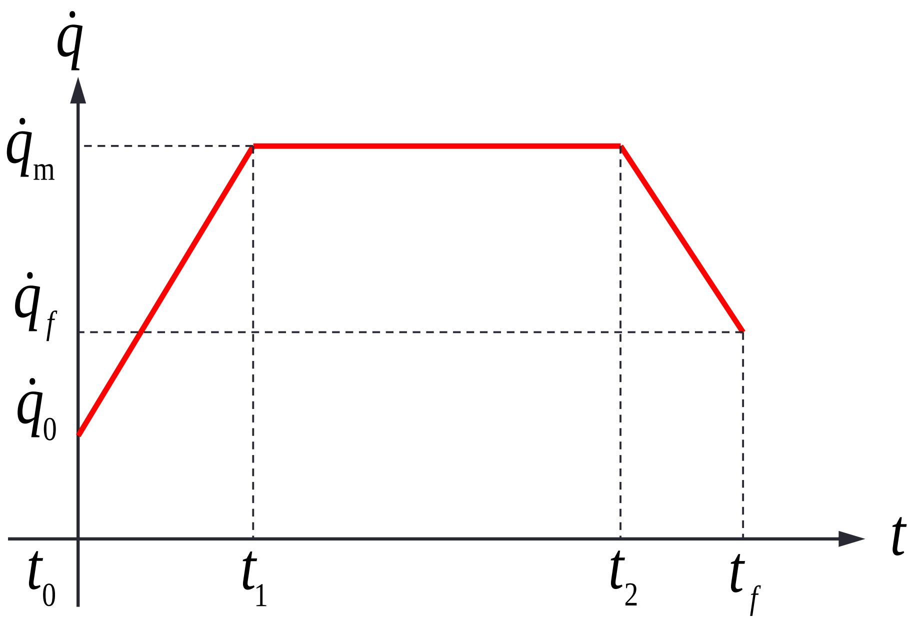
$$\ddot{q}(1) = 0$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 1500.5 \\ -7053 \\ 12891 \\ -10380 \\ 3076.9 \end{bmatrix}$$



5.2 Trapezoidal Velocity Trajectory

- ✓ A high order polynomial trajectory can be replaced by several lower order polynomials by setting specific velocity and acceleration boundaries.
- ✓ A time-optimal trajectory is to let the joint work at its maximum speed while maintain all constraints.



$\boldsymbol{t_0 - t_1}$

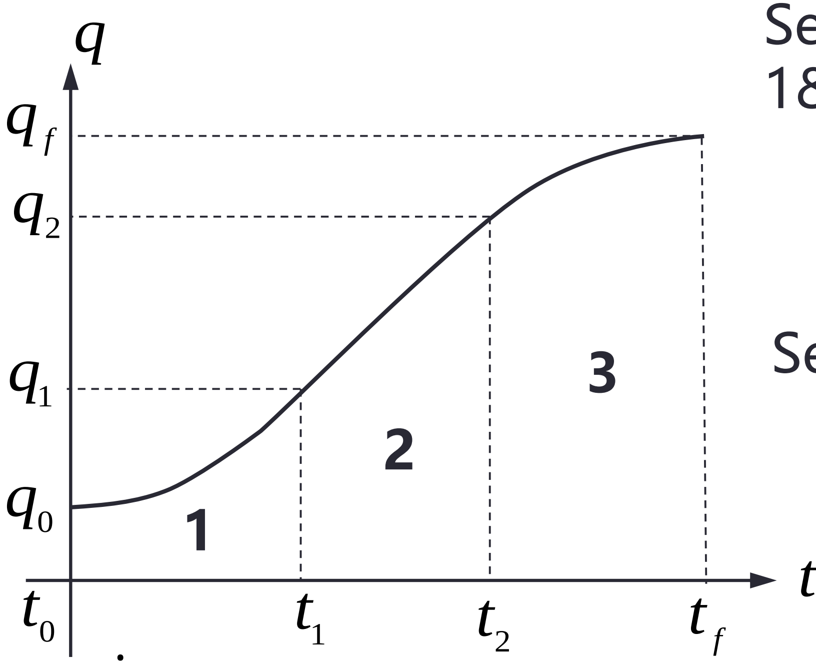
accelerates - *quadratic* $q(t)$

$\boldsymbol{t_1 - t_2}$

uniform moving – *linear* $q(t)$

$\boldsymbol{t_2 - t_f}$

decelerates – *quadratic* $q(t)$

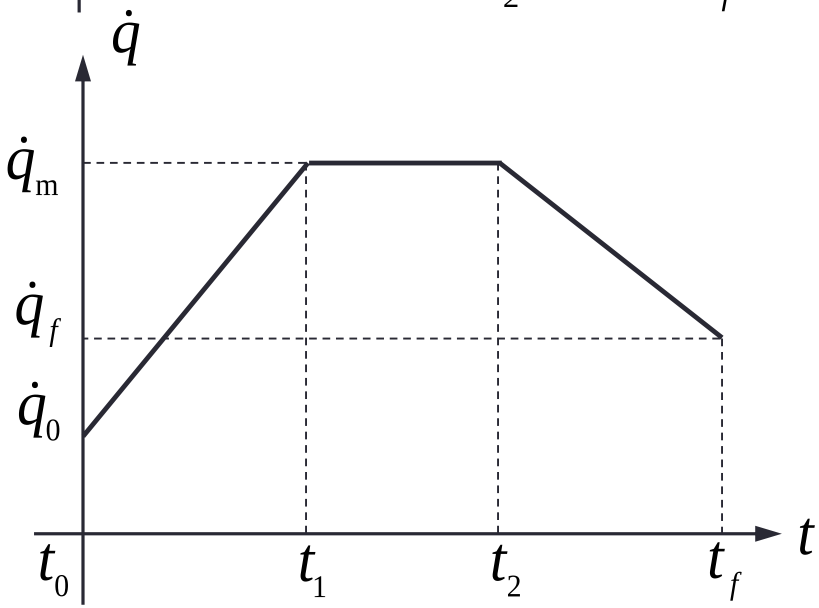


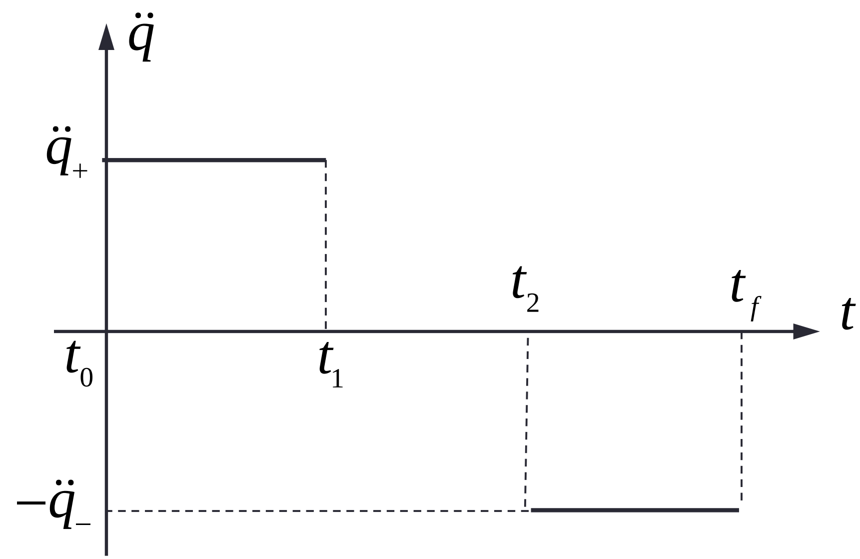
Segment 1&3

$$q(t) = a_0 + a_1 t + a_2 t^2$$
$$\dot{q}(t) = a_1 + 2a_2 t \qquad (5\text{-}1)$$
$$\ddot{q}(t) = 2a_2$$

Segment 2

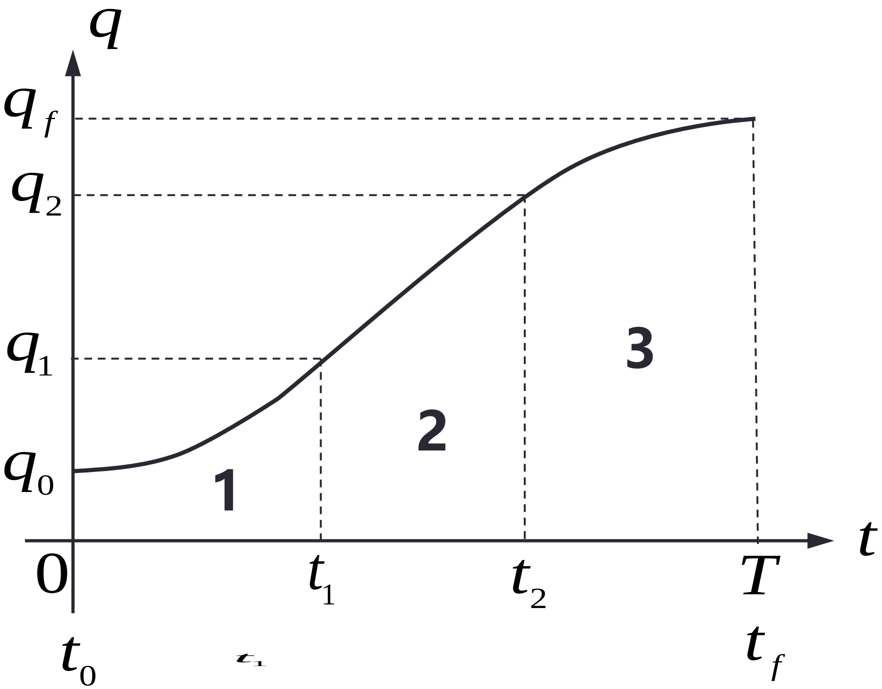
$$q(t) = b_0 + b_1 t$$
$$\dot{q}(t) = b_1 \qquad (5\text{-}2)$$
$$\ddot{q}(t) = 0$$

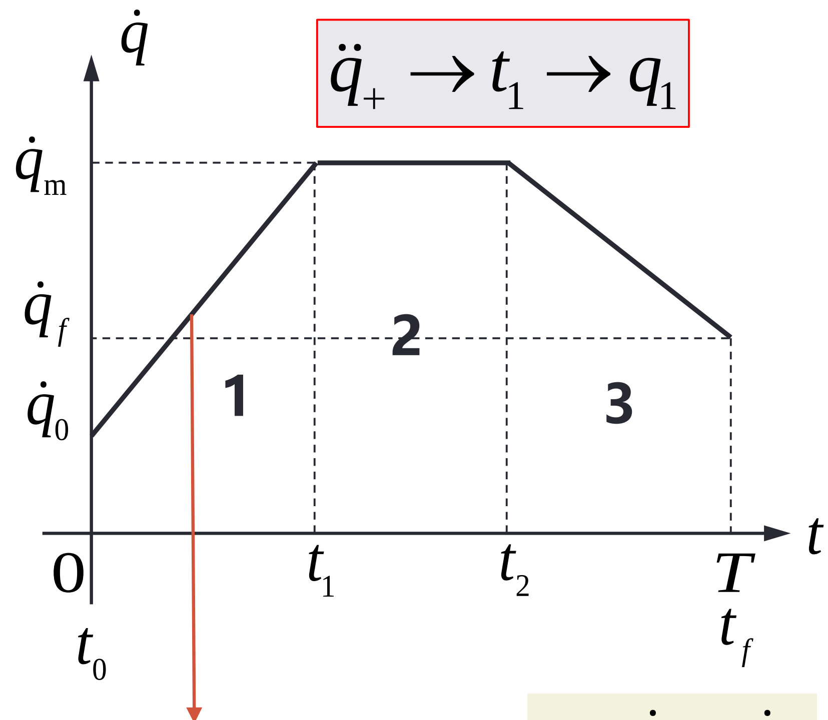




Boundary Conditions of trapezoidal velocity trajectory

1) $q(t_0) = q_0$
2) $\dot{q}(t_0) = \dot{q}_0$ (*usually* 0)
3) $\ddot{q}(t_0) = \ddot{q}_0$ (*usually* 0)
4) t_1 (*or* $\ddot{q}_+$)
5) pos. continuity at t_1
6) vel. continuity at t_1
7) acc. continuity at t_1
8) t_2 (*or* $-\ddot{q}_-$)
9) pos. continuity at t_2
10) vel. continuity at t_2
11) acc. continuity at t_2
12) $q(t_f) = q_f$
13) $\dot{q}(t_f) = \dot{q}_f$ (*usually* 0)
14) $\ddot{q}(t_f) = \ddot{q}_f$ (*usually* 0)





$$\ddot{q}_+ \rightarrow t_1 \rightarrow q_1$$

Segment 1: $0 \le t < t_1$

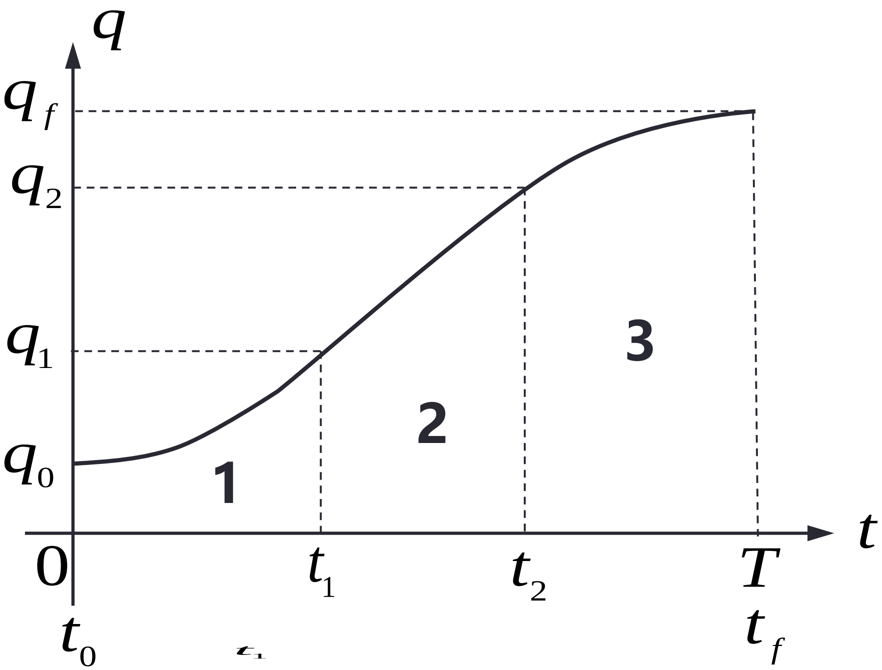
$$\dot{q}_m = \dot{q}_0 + \ddot{q}_+ t_1 \;\Rightarrow\; t_1 = \frac{\dot{q}_m - \dot{q}_0}{\ddot{q}_+}$$

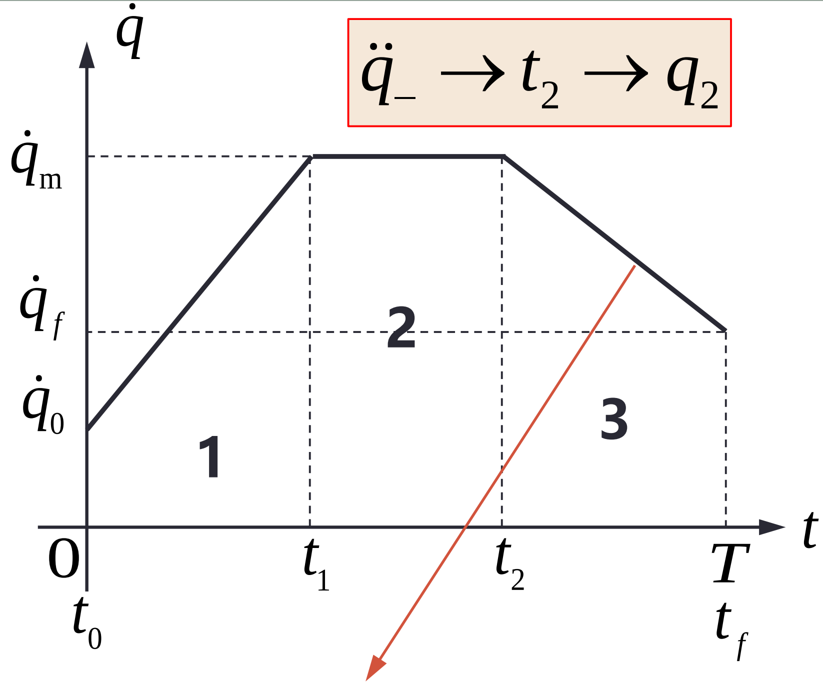
$$q(t) = a_0 + a_1 t + a_2 t^2$$
$$\dot{q}(t) = a_1 + 2a_2 t$$
$$\ddot{q}(t) = 2a_2$$

$$\xRightarrow{t=0} \begin{cases} a_0 = q_0 \\ a_1 = \dot{q}_0 \\ a_2 = \ddot{q}_+ / 2 \end{cases}$$

$$q_1 = q(t_1) = q_0 + \boxed{\frac{1}{2}(\dot{q}_0 + \dot{q}_m)t_1} = q_0 + \frac{\dot{q}_m^2 - \dot{q}_0^2}{2\ddot{q}_+}$$

area below $\dot{q}$ *profile* 1





$$\ddot{q}_- \rightarrow t_2 \rightarrow q_2$$

Segment 2: $t_1 \leq t < t_2$

$$\dot{q}_m - \dot{q}_f = \ddot{q}_-(T - t_2)$$

$$q_1 = q_0 + \frac{\dot{q}_m^2 - \dot{q}_0^2}{2\ddot{q}_+}$$

$$t_2 = T - \frac{\dot{q}_m - \dot{q}_f}{\ddot{q}_-}$$

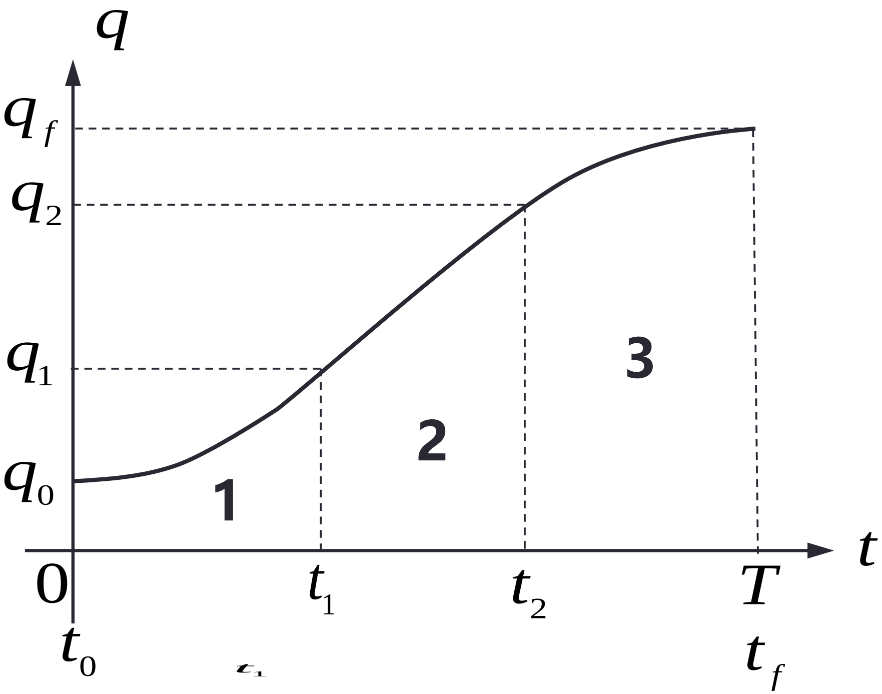
$$q(t) = b_0 + b_1 t$$
$$\dot{q}(t) = b_1$$
$$\ddot{q}(t) = 0$$

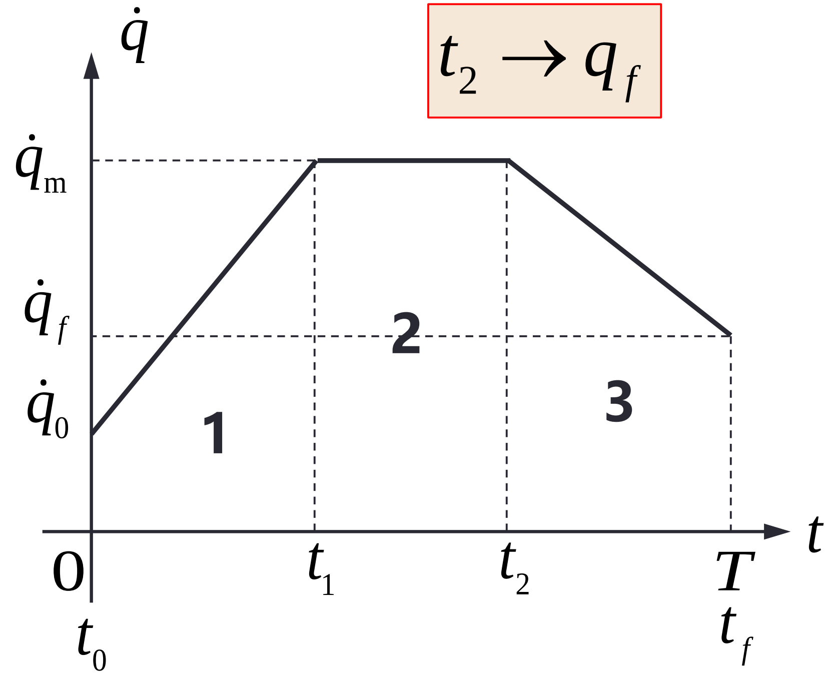
$$\Rightarrow \begin{cases} b_0 = q_1 \\ b_1 = \dot{q}_m \end{cases}$$

area below $\dot{q}$ profile 2

$$q_2 = q(t_2) = q_1 + \boxed{\dot{q}_m(t_2 - t_1)}$$
$$= q_0 + \frac{1}{2}(\dot{q}_0 - \dot{q}_m)t_1 + \dot{q}_m t_2$$

$t_2 \to q_f$





Segment 3: $t_2 \le t \le T$

$$q_f = q(T) = q_2 + \boxed{\frac{1}{2}(\dot{q}_m + \dot{q}_f)(T - t_2)} \longleftarrow \text{area below } \dot{q} \text{ profile } 3$$

$$\begin{aligned} q(t) &= c_0 + c_1 t + c_2 t^2 \\ \dot{q}(t) &= c_1 + 2c_2 t \\ \ddot{q}(t) &= 2c_2 \end{aligned} \quad \xRightarrow{t=T} \quad \left\{ \begin{aligned} c_0 &= q_f - \dot{q}_f T - \ddot{q}\_T^2/2 \\ c_1 &= \dot{q}_f + \ddot{q}\_T \\ c_2 &= -\ddot{q}\_/2 \end{aligned} \right.$$

Trapezoidal velocity trajectory between 2 points:

$$q(t)=\begin{cases} a_0+a_1t+a_2t^2 & 0\le t<t_1 \\ b_0+b_1t & t_1\le t<t_2 \\ c_0+c_1t+c_2t^2 & t_2\le t\le T \end{cases}$$

$$a_0=q_0$$

$$a_1=\dot{q}_0$$

$$t_1=\frac{\dot{q}_m-\dot{q}_0}{\ddot{q}_+}$$

$$a_2=\ddot{q}_+/2$$

$$b_0=q_0+(\dot{q}_0+\dot{q}_m)t_1/2$$

$$b_1=\dot{q}_m$$

$$c_0=q_f-\dot{q}_fT-\ddot{q}_-T^2/2$$

$$c_1=\dot{q}_f+\ddot{q}_-T$$

$$t_2=T-\frac{\dot{q}_\mathrm{m}-\dot{q}_f}{\ddot{q}_-}$$

$$c_2=-\ddot{q}_-/2$$

Case 1: $\dot{q}_m$ and T are planned

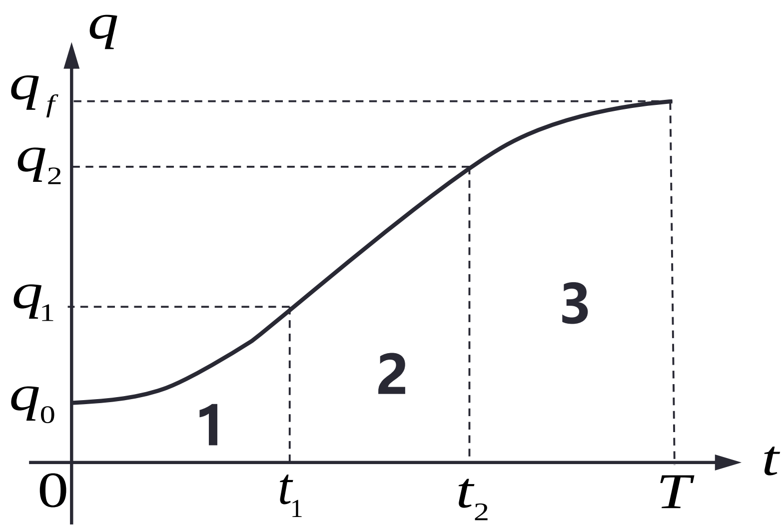
$\ddot{q}_+,\ \ddot{q}_-,\ t_1,\ t_2$

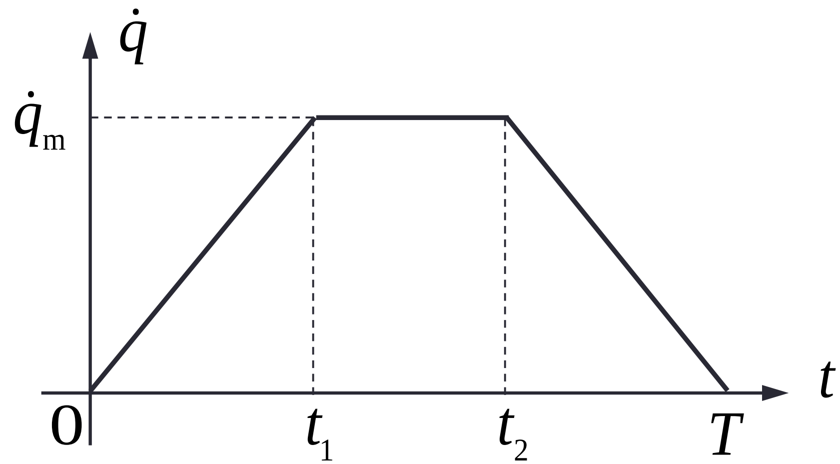
are unknowns to be computed

Case 2: $\dot{q}_m, \ddot{q}_+,\ \ddot{q}_-$ are planned

$T,\ t_1,\ t_2$

are unknowns to be computed





In most cases, trapezoidal velocity profile is used to plan the trajectory of rest-to-rest motion between, i.e. $\dot{q}_0 = \dot{q}_f = 0, \quad \ddot{q}_+ = \ddot{q}_-$

$$q(t) = \begin{cases} q_0 + \ddot{q}_+ t^2 / 2 & 0 \le t < t_1 \\ q_0 + \dot{q}_m^2 / (2\ddot{q}_+) + \dot{q}_m t & t_1 \le t < t_2 \\ q_0 - \ddot{q}_-(t-T)^2 / 2 & t_2 \le t \le T \end{cases}$$

$$t_1 = \frac{\dot{q}_m - \dot{q}_0}{\ddot{q}_+}$$

$$t_2 = T - \frac{\dot{q}_\mathrm{m} - \dot{q}_f}{\ddot{q}_-}$$

Case 1: $\dot{q}_m$ and *T* are planned. $\ddot{q}_+, \ddot{q}_-, t_1, t_2$ are unknowns to be computed

$$\ddot{q}_+ t_1 = \dot{q}_m - \dot{q}_0 \Rightarrow \ddot{q} t_1 = A \quad \boxed{1} \qquad A = \dot{q}_m - \dot{q}_0 \quad (q_+ = q_- = \ddot{q})$$

$$\ddot{q}_-(T - t_2) = \dot{q}_m - \dot{q}_f \Rightarrow \ddot{q}(T - t_2) = B \quad \boxed{2} \qquad B = \dot{q}_m - \dot{q}_f$$

$$q_f = q_0 + \frac{1}{2}(\dot{q}_0 + \dot{q}_m)t_1 + \dot{q}_m(t_2 - t_1) + \frac{1}{2}(\dot{q}_m + \dot{q}_f)(T - t_2) \Rightarrow$$

$$(\dot{q}_m - \dot{q}_0)t_1 - (\dot{q}_m - \dot{q}_f)t_2 + \left[2(q_f - q_0) - T(\dot{q}_m + \dot{q}_f)\right] = 0$$

$$\Downarrow$$

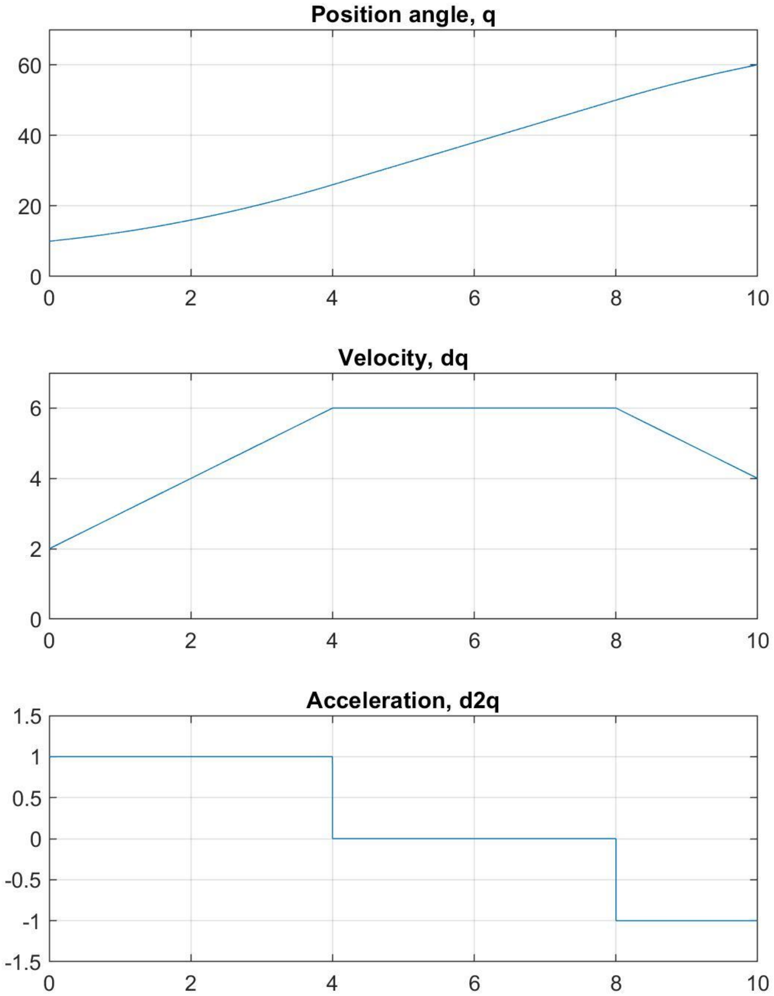
$$At_1 - Bt_2 + C = 0 \quad \boxed{3} \qquad C = 2(q_f - q_0) - T(\dot{q}_m + \dot{q}_f)$$

By combining (1)-(3), we can obtain

$$t_2 = \frac{A^2 T + BC}{A^2 + B^2} \qquad t_1 = \frac{A}{B}(T - t_2) = \frac{A(BT - C)}{A^2 + B^2} \qquad \ddot{q} = q_+ = q_- = \frac{A}{t_1}$$

Ex 5-4

Assume $q(0)=10$, $q(10)=60$, $q'(0)=2$, $q'(10)=4$, $q'_m=6$. Find the trapezoidal trajectory.



$$q(t) = \begin{cases} a_0 + a_1 t + a_2 t^2 & 0 \le t < t_1 \\ b_0 + b_1 t & t_1 \le t < t_2 \\ c_0 + c_1 t + c_2 t^2 & t_2 \le t \le T \end{cases}$$

$$a_0 = 10 \quad b_0 = 2 \quad c_0 = -30$$

$$a_1 = 2 \quad b_1 = 6 \quad c_1 = 14$$

$$a_2 = 0.5 \quad c_2 = -0.5$$

$$t_1 = 4 \quad t_2 = 8 \quad \ddot{q} = 1$$

Case 2: $\dot{q}_m$, $\ddot{q}_+$, $\ddot{q}_-$ are planned . T, t_1, t_2 are unknowns to be computed

$$\ddot{q}_+ t_1 = \dot{q}_m - \dot{q}_0 \Rightarrow t_1 = A / \ddot{q} \quad \boxed{1} \qquad A = \dot{q}_m - \dot{q}_0 \quad (q_+ = q_- = \ddot{q})$$

$$\ddot{q}_-(T - t_2) = \dot{q}_m - \dot{q}_f \Rightarrow T - t_2 = B / \ddot{q} \quad \boxed{2} \qquad B = \dot{q}_m - \dot{q}_f$$

$$q_f = q_0 + \frac{1}{2}(\dot{q}_0 + \dot{q}_m)t_1 + \dot{q}_m(t_2 - t_1) + \frac{1}{2}(\dot{q}_m + \dot{q}_f)(T - t_2) \quad \boxed{3}$$

By combining (1)-(3), we can obtain

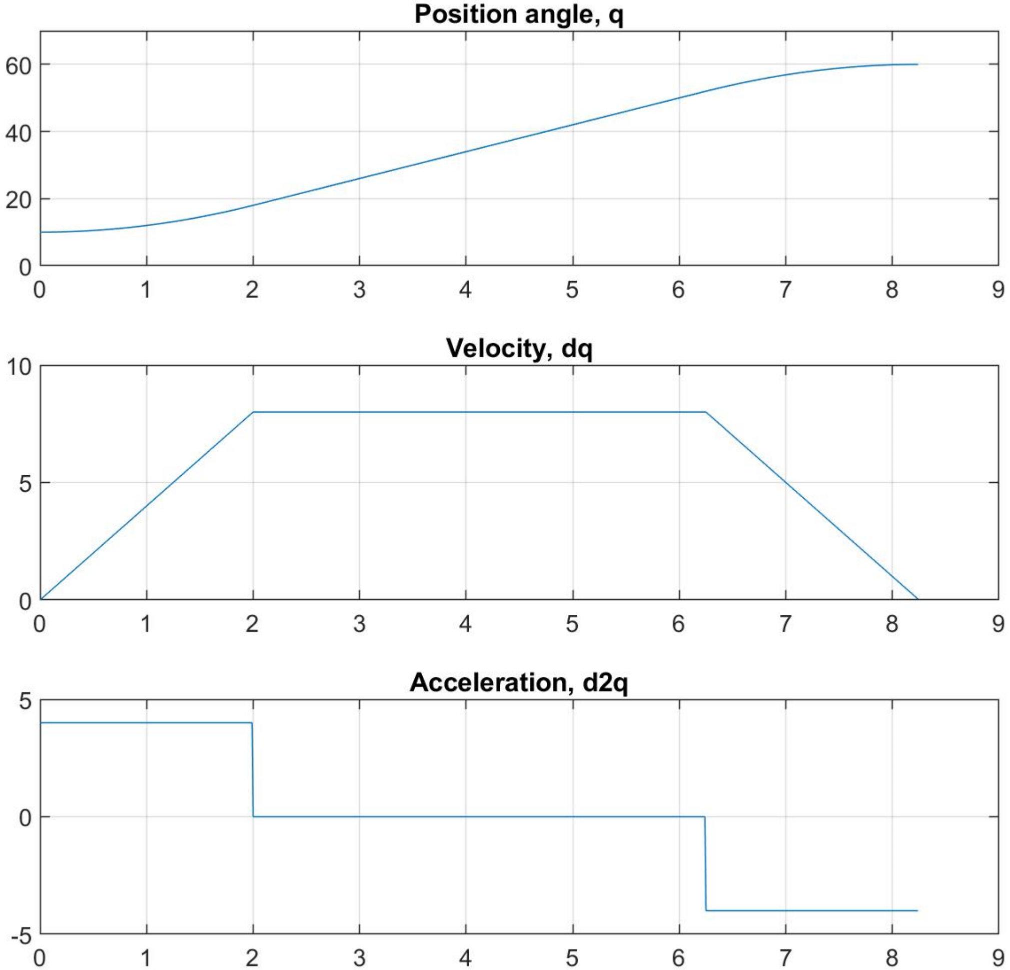
$$t_1 = A / \ddot{q}$$

$$t_2 = \frac{D}{2\dot{q}_m \ddot{q}} \qquad D = A^2 + 2\ddot{q}(q_f - q_0) - B(\dot{q}_m + \dot{q}_f)$$

$$T = B / \ddot{q} + t_2$$

Ex 5-5

Given the boundary conditions of q'_m=8 and q''=4, find a **time-optimal** rest-to-rest trajectory to perform motion from $q(0)$=10 to $q(t_f)$=60.



$$q(t) = \begin{cases} a_0 + a_1 t + a_2 t^2 & 0 \le t < t_1 \\ b_0 + b_1 t & t_1 \le t < t_2 \\ c_0 + c_1 t + c_2 t^2 & t_2 \le t \le T \end{cases}$$

$$a_0 = 10 \quad b_0 = 2 \quad c_0 = -76.125$$

$$a_1 = 0 \quad b_1 = 8 \quad c_1 = 33$$

$$a_2 = 2 \quad c_2 = -2$$

$$t_1 = 4 \quad t_2 = 6.25$$

$$T = 8.25$$

Existence of solution

$t_1 \le t_2$ $\Rightarrow$

Case 1: $AT(B-A)-(A+B)C \le 0$

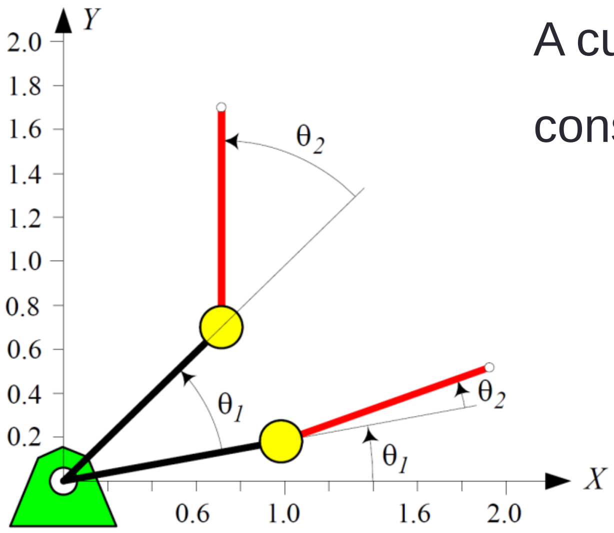
Case 2: $2\dot{q}_m A - D \le 0$

5.3 Planning in Operational Space

- ✓ The planning in joint space results that end-effector's motion is not easily predictable, in view of the nonlinear effects introduced by direct kinematics.
- ✓ Cartesian trajectory planning is the most natural application of planning. Whenever it is desired that the end-effector motion follows a geometrically specified path in the operational space, it is necessary to plan trajectory execution directly in the same space.
- ✓ Planning can be done either by interpolating a sequence of prescribed path points or by generating the analytical motion primitive and the relative trajectory in a punctual way.
- ✓ In both cases, the time sequence of the values attained by the operational space variables is utilized in real time to obtain the corresponding sequence of values of the joint space variables, via an inverse kinematics algorithm.

Ex 5-3

For a given 2R manipulator, consider a rest-to-rest Cartesian trajectory from point (1, 1.5) to point (−1, 1.5) on a straight line Y = 1.5. Find joint control functions for the given trajectory.



A cubic polynomial can satisfy the position and velocity constraints at initial and final points.

$$\begin{aligned} X(0) &= X_0 = 1 \qquad \dot{X}(0) = \dot{X}_0 = 0 \\ X(1) &= X_f = -1 \qquad \dot{X}(1) = \dot{X}_f = 0 \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} X &= 1 - 6t^2 + 4t^3 \\ Y &= 1.5 \end{aligned}$$

By using the inverse kinematics of a 2R planar manipulator,

$$\begin{aligned} \theta_2 &= \pm 2\,\mathrm{atan2}\sqrt{\frac{(l_1+l_2)^2 - (X^2+Y^2)}{(X^2+Y^2) - (l_1-l_2)^2}} \\ \theta_1 &= \mathrm{atan2}\,\frac{X\,(l_1 + l_2\cos\theta_2) + Y\,l_2\sin\theta_2}{Y\,(l_1 + l_2\cos\theta_2) - X l_2 \sin\theta_2} \end{aligned}$$

Ex 5-3

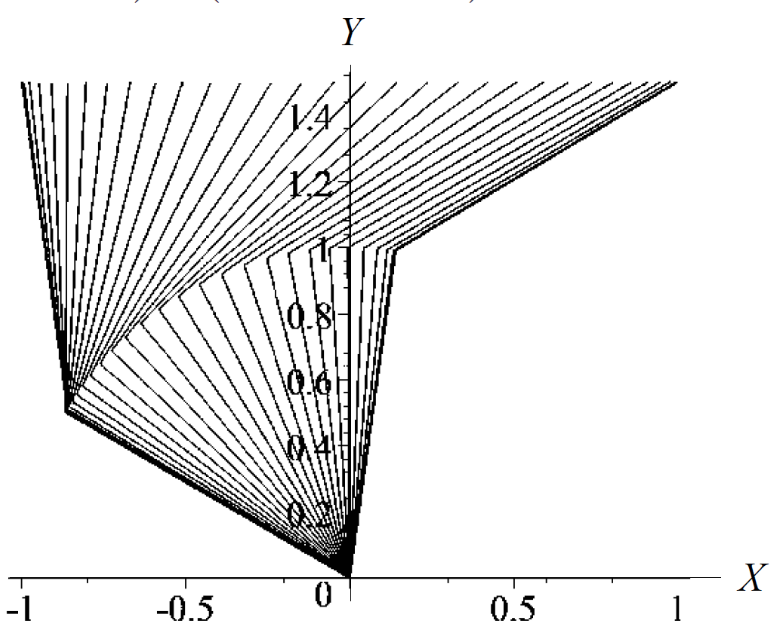
For a given 2R manipulator, consider a rest-to-rest Cartesian trajectory from point (1, 1.5) to point (−1, 1.5) on a straight line Y = 1.5. Find joint control functions for the given trajectory.

The sign (±) indicates the elbow-up and elbow-down configurations, depending on the initial configuration at point (1, 1.5).

$$\theta_2 = \pm 2\,\mathrm{atan2}\sqrt{\frac{(l_1+l_2)^2-(4t^3-6t^2+2.5)^2}{(4t^3-6t^2+1)^2-(l_1-l_2)^2}}$$

$$\theta_1 = \mathrm{atan2}\,\frac{(1-6t^2+4t^3)(l_1+l_2\cos\theta_2)+1.5l_2\sin\theta_2}{1.5(l_1+l_2\cos\theta_2)-(1-6t^2+4t^3)\,l_2\sin\theta_2}$$

For an elbow-up configuration



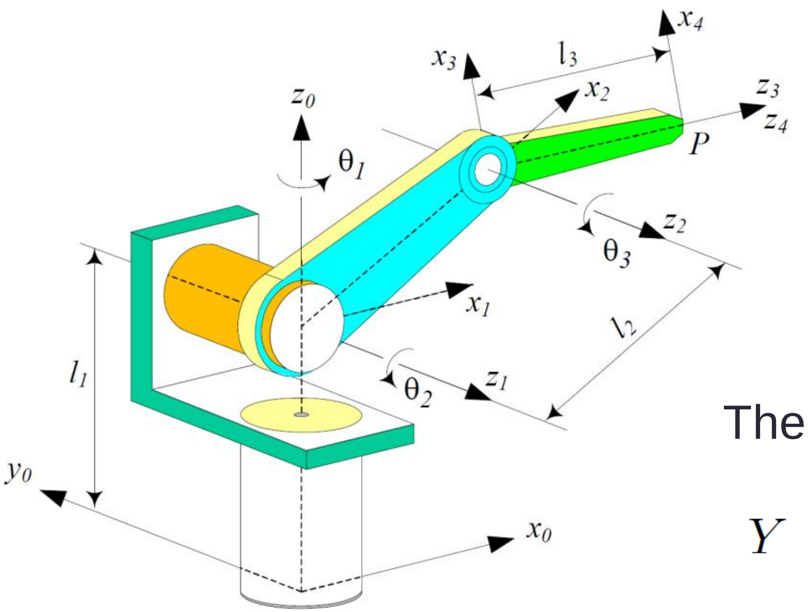
Ex 5-4

The figure illustrates an articulated manipulator. Assume that

$$l_1 = 0.5\,\mathrm{m} \qquad l_2 = 1.0\,\mathrm{m} \qquad l_3 = 1.0\,\mathrm{m}$$

The tip point of the manipulator is supposed to move from point P1 to P2 in 10 sec.

Find the rest-to-rest trajectory for the manipulator on a line.



$$\mathbf{r}_{P_1} = \begin{bmatrix} 1.5 \\ 0.0 \\ 1.0 \end{bmatrix} \quad \mathbf{r}_{P_2} = \begin{bmatrix} -1.0 \\ 1.0 \\ 1.5 \end{bmatrix}$$

Using a quintic trajectory for

$$X = 1.5 - 0.025t^3 + 0.00375t^4 - 0.00015t^5$$

The straight line connecting P1 and P2 is

$$\begin{aligned} Y &= Y_{P_1} + \frac{Y_{P_2} - Y_{P_1}}{X_{P_2} - X_{P_1}}(X - X_{P_1}) \\ &= 0.010t^3 - 0.0015t^4 + 0.00006t^5 \end{aligned}$$

$$Z = Z_{P_1} + \frac{Z_{P_2} - Z_{P_1}}{X_{P_2} - X_{P_1}}(X - X_{P_1}) = 1 + 0.005t^3 - 0.00075t^4 + 0.00003t^5$$

Ex 5-4

The figure illustrates an articulated manipulator. Assume that

$$l_1 = 0.5\,\mathrm{m} \qquad l_2 = 1.0\,\mathrm{m} \qquad l_3 = 1.0\,\mathrm{m}$$

The tip point of the manipulator is supposed to move from point P1 to P2 in 10 sec.

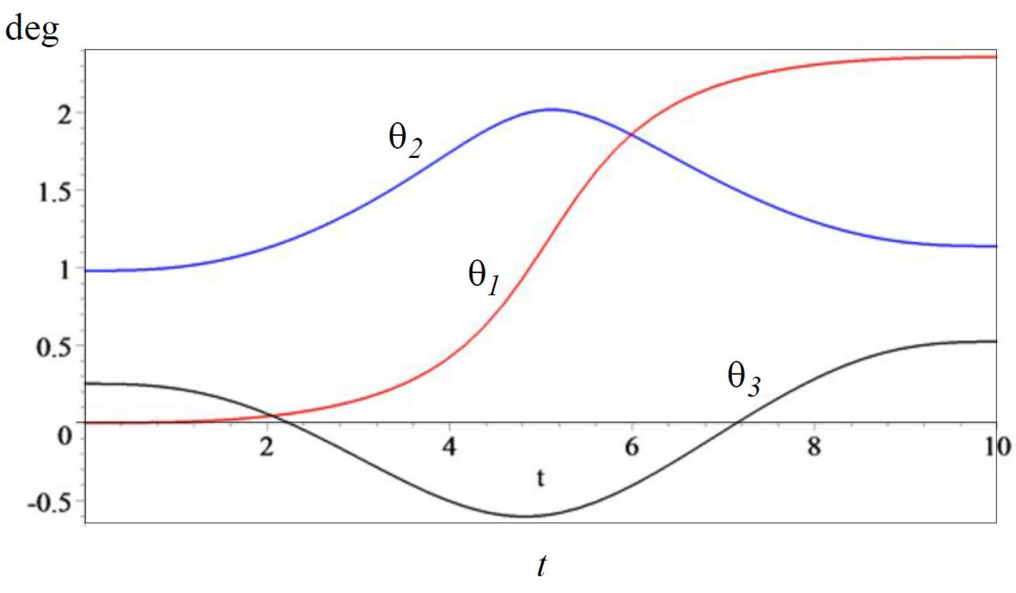
Find the rest-to-rest trajectory for the manipulator on a line.

Using the inverse kinematic equations

$$\theta_1 = \begin{cases} \arctan \dfrac{Y}{X} & X \geq 0 \\ \arctan \dfrac{Y}{X} + \pi & X < 0 \end{cases}$$

$$\theta_2 = 2 \arctan \frac{-C_2 + \sqrt{C_2^2 - C_1 C_3}}{C_1}$$

$$\theta_3 = \arccos \left(\frac{l_1 - Z + l_2 \sin \theta_2}{l_3} \right) - \theta_2$$



$$\begin{aligned} C_1 &= l_1^2 - 2l_1 Z + l_2^2 + \frac{2l_2 X}{\cos \theta_1} - l_3^2 + \frac{X^2}{\cos^2 \theta_1} + Z^2 \\ C_2 &= 2l_1 l_2 - 2l_2 Z \\ C_3 &= l_1^2 - 2l_1 Z + l_2^2 - \frac{2l_2 X}{\cos \theta_1} - l_3^2 + \frac{X^2}{\cos^2 \theta_1} + Z^2 \end{aligned}$$

Inverse Jacobian Matrix

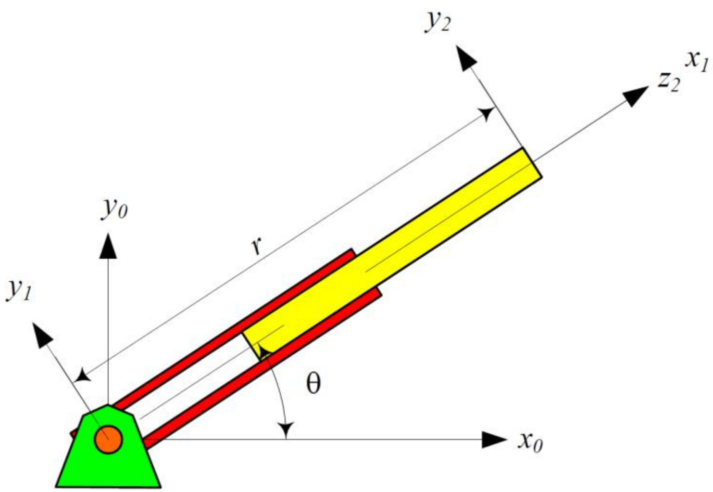
If a trajectory of end-effector is given in the operational space, the inverse Jacobian matrix can be used to find trajectories in joint space by using

$$\dot{\mathbf{q}} = \mathbf{J}^{-1}\dot{\mathbf{X}}$$

It is obvious that once the inverse of the Jacobian matrix is determined, an analytical solution to trajectory planning problem in the joint space can be obtained.

Ex 5-5

Determine the Jacobian matrix for a planar polar manipulator.



The tip point of the manipulator is at

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} \Rightarrow$$

$$\mathbf{J}_D = \begin{bmatrix} \dfrac{\partial X}{\partial r} & \dfrac{\partial X}{\partial \theta} \\ \dfrac{\partial Y}{\partial r} & \dfrac{\partial Y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$${}_0\boldsymbol{\omega}_2 = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \Rightarrow \begin{array}{rcl} \omega_3 & = & \mathbf{J}_R\,\dot{\theta} \\ \mathbf{J}_R & = & 1 \end{array}$$

$$\Rightarrow \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix}$$

$$\dot{\mathbf{q}} = \mathbf{J}^{-1}\dot{\mathbf{X}}$$