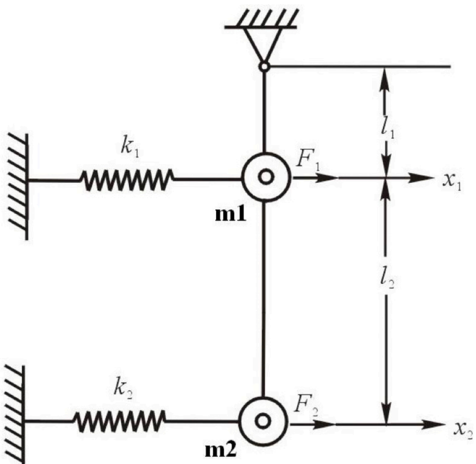
作业1：下图是一个带有附有质量 和 上的约束弹簧双摆，采用质量的微小水平平动 和 为坐标，写出系统运动的作用力方程（参考第4讲内容）

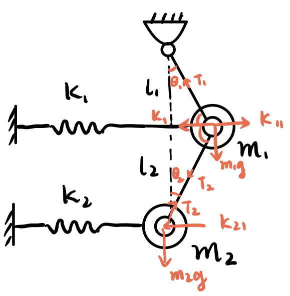


分析：系统自由度为2

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix}$$

① 求刚度矩阵 K

设 $x_1=1$，$x_2=0$



对 m_1 而言，

$\Sigma F_x=0$，$k_1+T_1\sin\theta_1+T_2\sin\theta_2-k_{11}=0$

$\Sigma F_y=0$，$m_1g+T_2\cos\theta_2-T_1\cos\theta_1=0$

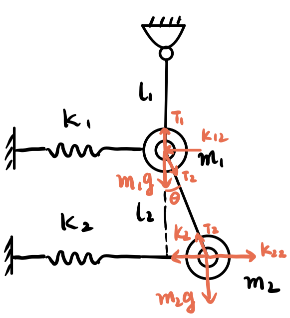
对 m_2 而言，

$\Sigma F_x=0$，$k_{21}+T_2\sin\theta_2=0$

$\Sigma F_y=0$，$m_2g-T_2\cos\theta_2=0$

$$\boxed{\begin{aligned} &\Rightarrow T_2=m_2g/\cos\theta_2 \quad T_1=\frac{m_1g}{\cos\theta_1}+T_2\frac{\cos\theta_2}{\cos\theta_1} \\ &\begin{bmatrix} k_{11} \\ k_{21} \end{bmatrix}=\begin{bmatrix} k_1+T_1\sin\theta_1+T_2\sin\theta_2 \\ -T_2\sin\theta_2 \end{bmatrix} \end{aligned}}$$

设 $x_1=0$，$x_2=1$



对 m_1 而言，

$\Sigma F_x=0$，$k_{12}+T_2\sin\theta=0$

$\Sigma F_y=0$，$m_1g+T_2\cos\theta-T_1=0$

对 m_2 而言，

$\Sigma F_x=0$，$k_2+T_2\sin\theta-k_{22}=0$

$\Sigma F_y=0$，$m_2g-T_2\cos\theta=0$

$$\boxed{\begin{aligned} &\Rightarrow T_2=\frac{m_2g}{\cos\theta} \\ &\begin{bmatrix} k_{12} \\ k_{22} \end{bmatrix}=\begin{bmatrix} -T_2\sin\theta \\ k_2+T_2\sin\theta \end{bmatrix} \end{aligned}}$$

因为 x_1 与 x_2 为微小水平平动，

所以 θ_1、θ_2、$\theta \approx 0 \Rightarrow$ $\sin\theta_1 \approx \tan\theta_1 = \frac{1}{l_1}$

$\sin\theta_2 \approx \tan\theta_2 = \frac{1}{l_2}$ $\quad \cos\theta_1 \approx \cos\theta_2 = 1$

$\sin\theta \approx \tan\theta = \frac{1}{l_2}$ $\quad \cos\theta \approx 1$

$$\Rightarrow \quad k_{11} = k_1 + \frac{(m_1+m_2)g}{l_1} + \frac{m_2 g}{l_2}$$

$$k_{21} = -\frac{m_2 g}{l_2}$$

$$k_{12} = -\frac{m_2 g}{l_2}$$

$$k_{22} = k_2 + \frac{m_2 g}{l_2}$$

② 求质量矩阵 M

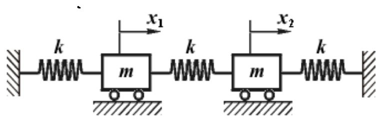
设 $\ddot{x}_1 = 1 \quad \ddot{x}_2 = 0 \Rightarrow m_{11} = m_1 \quad m_{21} = 0$

设 $\ddot{x}_1 = 0 \quad \ddot{x}_2 = 1 \Rightarrow m_{12} = 0 \quad m_{22} = m_2$

综上，作用力方程为

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + \frac{(m_1+m_2)g}{l_1} + \frac{m_2 g}{l_2} & -\frac{m_2 g}{l_2} \\ -\frac{m_2 g}{l_2} & k_2 + \frac{m_2 g}{l_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix}$$

作业2：如图所示2自由度系统。（1）求系统固有频率和模态矩阵，并画出各阶主振型图形；（2）当系统存在初始条件 和 时，试采用模态叠加法求解系统响应



(1) ① 求解运动状态方程： $M\ddot{X} + KX = 0$ （二自由度）

存在刚性耦合： $x_1 = 1,\ x_2 = 0$ 时 $\Rightarrow \begin{bmatrix} K_{11} \\ K_{21} \end{bmatrix} = \begin{bmatrix} 2K \\ -K \end{bmatrix}$

$x_1 = 0,\ x_2 = 1$ 时 $\Rightarrow \begin{bmatrix} K_{12} \\ K_{22} \end{bmatrix} = \begin{bmatrix} -K \\ 2K \end{bmatrix}$

不存在惯性耦合： $M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2K & -K \\ -K & 2K \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

② 求解固有频率 w_i

$$(K - w^2 M)\phi = 0 \Rightarrow \begin{bmatrix} 2K - w^2 m & -K \\ -K & 2K - w^2 m \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$|K - w^2 M| = 0 \Rightarrow (2K - w^2 m)^2 - K^2 = 0 \Rightarrow w_1 = \sqrt{K/m} \quad w_2 = \sqrt{3K/m}$$

③ 求解模态矩阵 $\phi^{(i)}$

$w_1 = \sqrt{K/m}$ 代回方程，得

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \phi_1 - \phi_2 = 0 \\ -\phi_1 + \phi_2 = 0 \end{cases} \quad \text{取 } \phi_2 = 1 \text{ 则 } \phi_1 = 1 \quad \phi^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

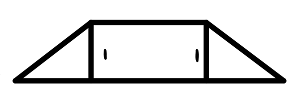
$W_2 = 2\sqrt{K\text{-}m}$ 代回方程，得

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -\phi_1 - \phi_2 = 0 \\ -\phi_1 - \phi_2 = 0 \end{cases} \quad \text{取 } \phi_2 = 1 \text{ 则 } \phi_1 = -1 \quad \phi^2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

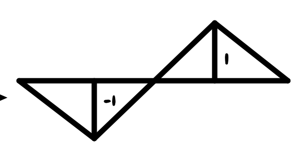
综上①②③，系统固有频率 $w_1 = \sqrt{K/m}$ $\quad w_2 = \sqrt{3K/m}$

系统模态矩阵 $\phi = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

当 $w_1 = \sqrt{K/m}$ 时的主振型 $\longrightarrow$



当 $w_2 = \sqrt{3K/M}$ 时的主振型 $\longrightarrow$



(2) 初始条件 $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ x_0 \end{bmatrix}$ 和 $\begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Phi = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \Phi^T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \Phi^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$M_p = \Phi^T M \Phi = \begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix}$$

$$K_p = \Phi^T K \Phi = \begin{bmatrix} 2k & 0 \\ 0 & 6k \end{bmatrix}$$

$$X = \Phi X_p \Rightarrow M_p \ddot{X}_p + K_p X_p = 0$$

$$\Rightarrow \begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix} \begin{bmatrix} \ddot{x}_{p1} \\ \ddot{x}_{p2} \end{bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & 6k \end{bmatrix} \begin{bmatrix} x_{p1} \\ x_{p2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_{pi} = x_{pi}(0) \cos w_i t + \frac{\dot{x}_{pi}}{w_2} \sin w_i t$$

$$X_p(0) = \Phi^{-1} X(0) = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ x_0 \end{bmatrix} = \begin{bmatrix} x_0/2 \\ x_0/2 \end{bmatrix}$$

$$\dot{X}_p(0) = \Phi^{-1} \dot{X}(0) = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$

$$x_{p1} = \frac{x_0}{2} \cos\left(\sqrt{\frac{k}{m}}\, t\right) \qquad X_p = \begin{bmatrix} \frac{x_0}{2} \cos\left(\sqrt{\frac{k}{m}}\, t\right) \\ \frac{x_0}{2} \cos\left(\sqrt{\frac{3k}{m}}\, t\right) \end{bmatrix}$$

$$x_{p2} = \frac{x_0}{2} \cos\left(\sqrt{\frac{3k}{m}}\, t\right)$$

$$X = \Phi X_p = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{x_0}{2} \cos\left(\sqrt{\frac{k}{m}}\, t\right) \\ \frac{x_0}{2} \cos\left(\sqrt{\frac{3k}{m}}\, t\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x_0}{2} \left[\cos\left(\sqrt{\frac{k}{m}}\, t\right) - \cos\left(\sqrt{\frac{3k}{m}}\, t\right) \right] \\ \frac{x_0}{2} \left[\cos\left(\sqrt{\frac{k}{m}}\, t\right) + \cos\left(\sqrt{\frac{3k}{m}}\, t\right) \right] \end{bmatrix}$$