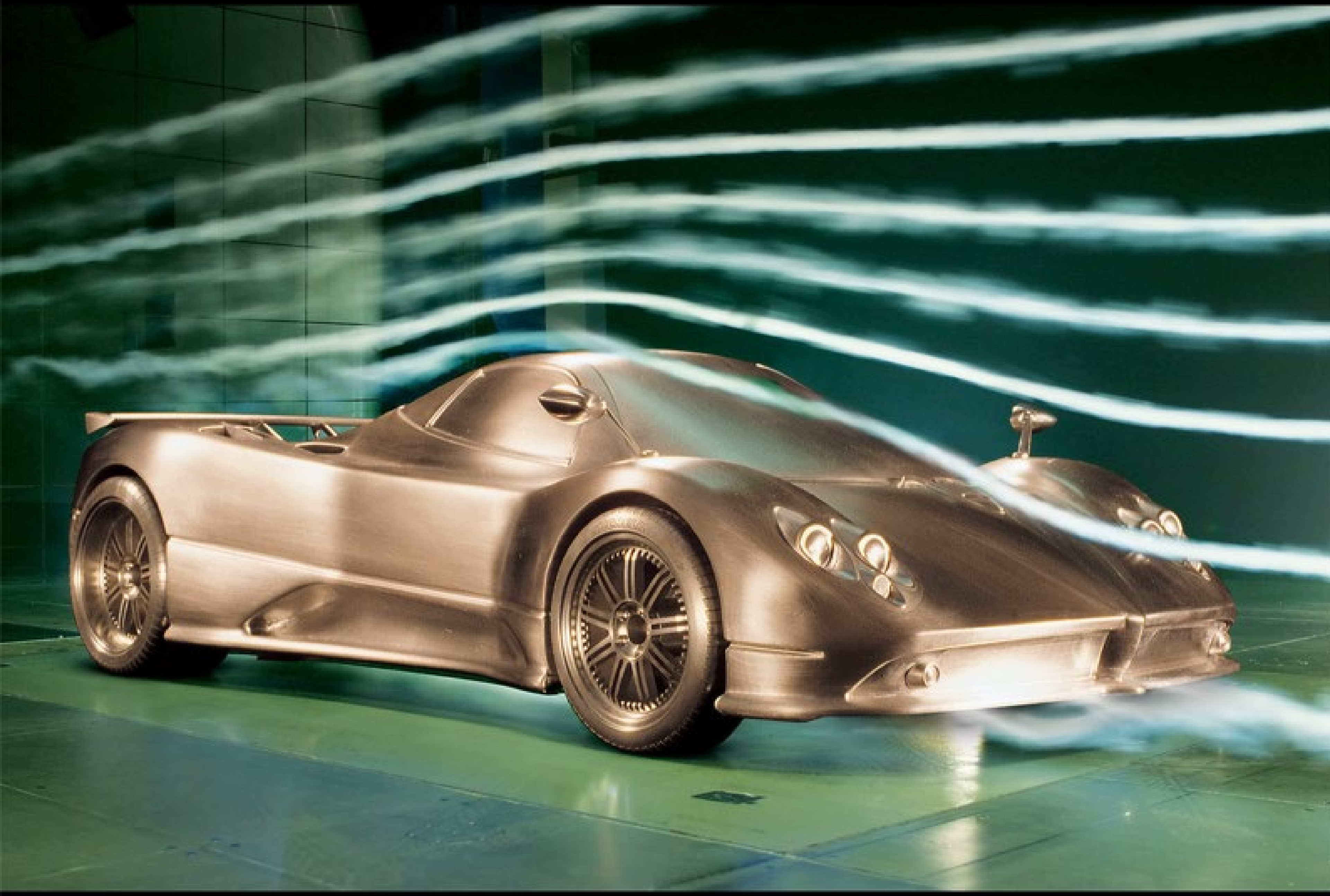
Chapter Five

Dimensional analysis and Similitude

Fourier

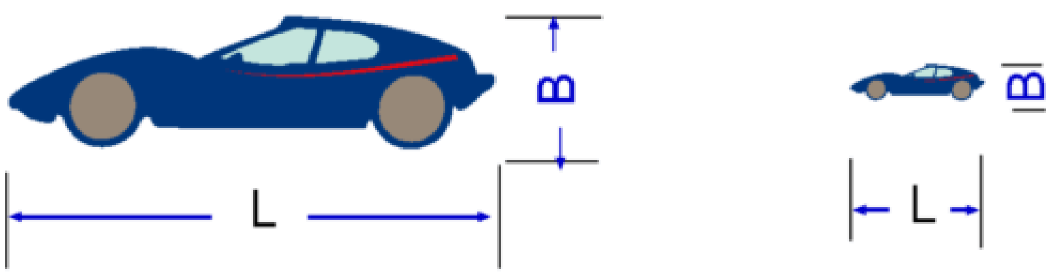
1768-1830



Geometric Similarity

- Geometric similarity – the model is the same shape as the application, usually scaled

Linear scale ratio: $$\delta_l = \frac{l}{l'} = \text{const}$$



Prototype

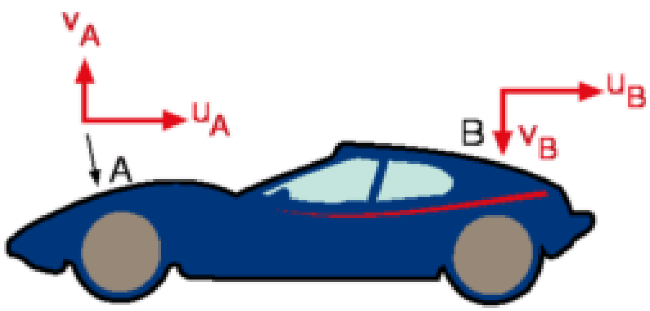
Model

$$\left(\frac{L}{B}\right)_{model} = \left(\frac{L}{B}\right)_{prototype};$$

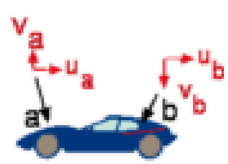
Kinematic Similarity

- Kinematic similarity – fluid flow of both the model and real application must undergo similar time rates of change motions. (fluid streamlines are similar)

Velocity scale ratio: $$\delta_v = \frac{v}{v'} = \text{const}$$



Prototype



Model

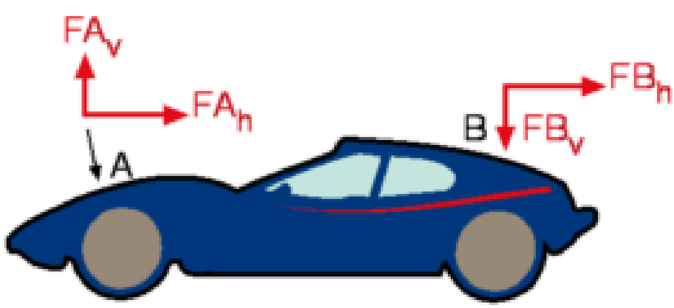
$$\left(\frac{v_A}{u_A}\right)_{prototype} = \left(\frac{v_a}{u_a}\right)_{model}$$

$$\left(\frac{v_B}{u_B}\right)_{prototype} = \left(\frac{v_b}{u_b}\right)_{model}$$

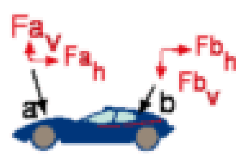
Dynamic Similarity

- Dynamic similarity – Ratios of all forces acting on corresponding fluid particles and boundary surfaces are constant

Density scale ratio: $$\delta_\rho = \frac{\rho}{\rho'} = 常数$$



Prototype



Model

$$\left(\frac{FA_v}{FA_h}\right)_{prototype} = \left(\frac{Fa_v}{Fa_h}\right)_{model}$$

$$\left(\frac{FB_v}{FB_h}\right)_{prototype} = \left(\frac{Fb_v}{Fb_h}\right)_{model}$$

Dynamic Similarity

In a system involving flow of fluid, different forces due to different causes may act on a fluid element.

$$\vec{F}_p + \vec{F}_\mu + \vec{F}_g + \vec{F}_c + \vec{F}_K = m\vec{a}$$

Pressure Force (due to different in pressure)	F_p
Viscous Force (due to viscosity)	F_μ
Gravity Force (due to gravitational attraction)	F_g
Capillary Force (due to surface tension)	F_c
Compressibility Force (due to elasticity)	F_k

$$\delta_{F_p} = \frac{(F_p)}{(F_p)'} = \frac{pA}{p'A'} = \delta_p \delta_l^2$$

$$\delta_{F_\mu} = \frac{(F_\mu)}{(F_\mu)'} = \frac{\mu A \dfrac{dv}{dl}}{\mu' A' \dfrac{dv'}{dl'}} = \delta_\mu \delta_l \delta_v$$

$$\delta_{F_g} = \frac{(F_g)}{(F_g)'} = \frac{\rho g V}{\rho' g' V'} = \delta_\rho \delta_g \delta_l^3$$

$$\delta_{F_c} = \frac{(F_c)}{(F_c)'} = \frac{\sigma l}{\sigma' l'} = \delta_\sigma \delta_l$$

$$\delta_{F_K} = \frac{(F_K)}{(F_K)'} = \frac{K l^2}{K' l'^2} = \delta_K \delta_l^2$$

$$\delta_{F_L} = \frac{(ma)}{(ma)'} = \frac{\rho l^3 \dfrac{\partial v}{\partial t}}{\rho' l'^3 \dfrac{\partial v'}{\partial t'}} = \delta_\rho \delta_l^2 \delta_v^2$$

Dynamic Similarity

$$\delta_p \delta_l^2 = \delta_\mu \delta_l \delta_v = \delta_\rho \delta_g \delta_l^3 = \delta_\sigma \delta_l = \delta_K \delta_l^2 = \delta_\rho \delta_l^2 \delta_v^2$$

If similarity:

$$\frac{\delta_\rho \delta_v^2}{\delta_p} = \frac{\delta_\rho \delta_l \delta_v}{\delta_\mu} = \frac{\delta_v^2}{\delta_l \delta_g} = \frac{\delta_\rho \delta_v^2 \delta_l}{\delta_\sigma} = \frac{\delta_\rho \delta_v^2}{\delta_K} = 1$$

- Impossible to have all force ratios the same unless the model is the same size as the prototype
- Need to determine which forces are important and attempt to keep those force ratios the same

Dynamic Similarity

$$\frac{\delta_\rho \delta_v^2}{\delta_p} = 1 \qquad \frac{\rho v^2}{p} = \frac{\rho' v'^2}{p'} = \frac{1}{Eu}$$

- *Eu denotes pressure force/inertia force*

$$\frac{\delta_\rho \delta_l \delta_v}{\delta_\mu} = 1 \qquad \frac{\rho l v}{\mu} = \frac{\rho' l' v'}{\mu'} = Re$$

- *Re denotes inertia force/viscous force*

$$\frac{\delta_v^2}{\delta_l \delta_g} = 1 \qquad \frac{v^2}{gl} = \frac{v'^2}{g'l'} = Fr$$

- *Fr denotes inertia force/gravity force*

$$\frac{\delta_\rho \delta_v^2 \delta_l}{\delta_\sigma} = 1 \qquad \frac{\rho v^2 l}{\sigma} = \frac{\rho' v'^2 l'}{\sigma'} = Bo$$

- *Bo denotes inertia force/surface tension force*

$$\frac{\delta_\rho \delta_v^2}{\delta_K} = 1 \qquad \sqrt{\frac{\rho v^2}{K}} = \sqrt{\frac{\rho' v'^2}{K'}} = Ma$$

- *M denotes inertia force/elastic force*

Example

- Consider a submarine modeled at 1/40 th scale. The application operates in sea water at 0.5 ° C, moving at 5 m/s. The model will be tested in fresh water at 20 ° C. Find the power required for the submarine to operate at the stated speed.

Variable	Application	Scaled mode	Units
L (diameter of submarine)	**1**	**1/40**	**(m)**
V (speed)	**5**	***calculate***	**(m/s)**
(density)	**1028**	**998**	**(kg/m³)**
(dynamic viscosity)	**1.88×10^{-3}**	**1.00×10^{-3}**	**Pa·s (N s/m²)**
F (force)	***calculate***	***to be measured***	**N (kg m/s²)**

$$R_e = \left(\frac{\rho V L}{\mu}\right)$$

$$C_p = \left(\frac{2\Delta p}{\rho V^2}\right), F = \Delta p L^2$$

Dimensional Analysis

- In physics and all science, dimensional analysis is a tool to find or check relations among physical quantities by using their dimensions.
- The dimension of a physical quantity is the combination of the basic physical dimensions (usually length, mass, time, electric current, temperature, amount of substance and luminous intensity) which describe it.
- Any meaningful equation (and any inequality and inequation) must have the same dimensions in the left and right sides. Checking this is the basic way of performing dimensional analysis.

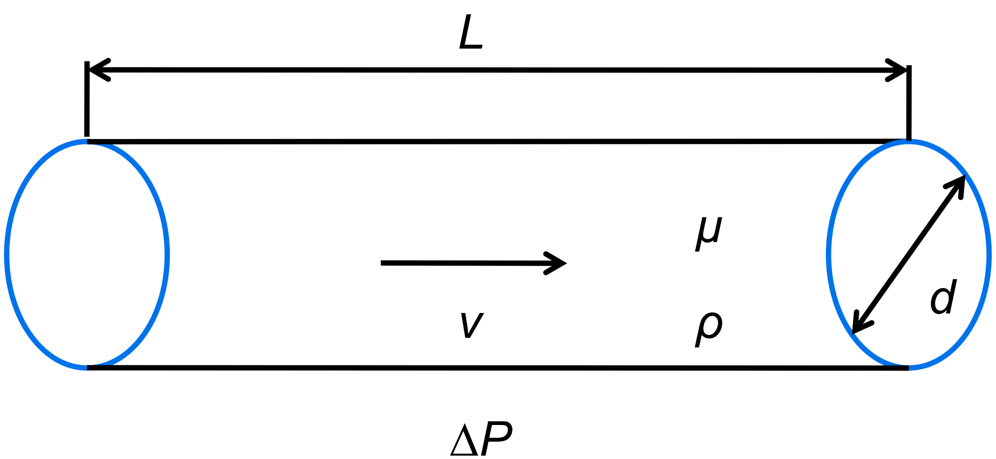
Dimensional Analysis

Dimensions of physical quantities

Physical quantity	Dimension	Physical quantity	Dimension
Area A	L^2	Pressure p	$ML^{-1}T^{-2}$
Volume V	L^3	Stress τ	$ML^{-1}T^{-2}$
Velocity v	LT^{-1}	Force F	MLT^{-2}
Acceleration a	LT^{-2}	Dynamic vis μ	$ML^{-1}T^{-1}$
Rotation speed n	T^{-1}	Kinematic vis ν	L^2T^{-1}
Heat quantity Q_H	H	Flow rate Q	L^3T^{-1}
Density ρ	ML^{-3}	Energy E	ML^2T^{-2}

Example

- A typical fluid mechanics problem in which experimentation is required, consider the steady flow of an incompressible Newtonian fluid through a long, smoothwalled, horizontal, circular pipe.



$$\Delta P = f(L, d, v, \mu, \rho)$$

Pressure drop depends on five variables:

Length(L);Diameter (D); speed (V); fluid density (ρ); fluid viscosity(μ)

Buckingham Pi Theorem

- If an equation involving k variables is dimensionally homogeneous, it can be reduced to a relationship among k-r independent dimensionless products, where r is the minimum number of reference dimensions required to describe the variables.

Pi terms

- Given a physical problem in which the dependent variable is a function of k-1 independent variables.

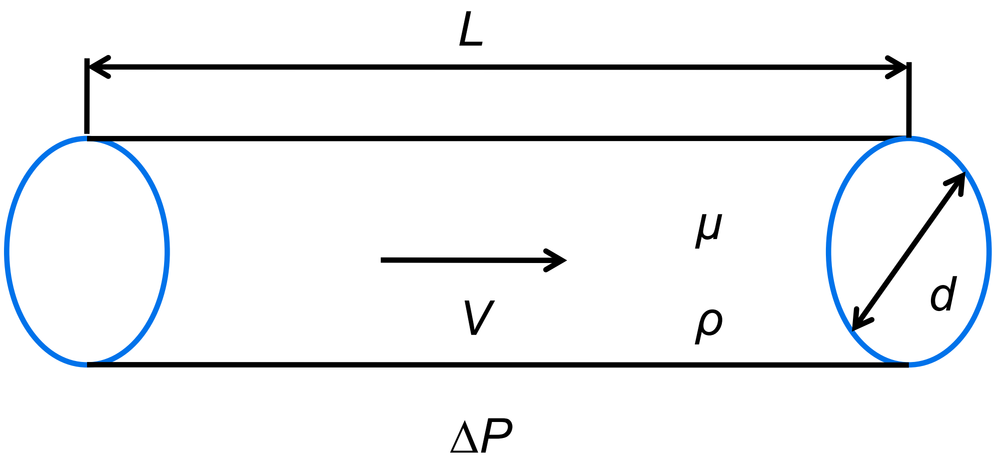
u1 = f (u2 , u3 ,....., u k)

- The k variables may be grouped into k-r independent dimensionless products, or Π terms, expressible in functional form by

$$\Pi_1 = \phi(\Pi_2, \Pi_3 \ldots, \Pi_{k-r})$$

Determination of Pi Terms

- Step 1 List all the variables
  - Let *k* be the number of variables
  -



$$\Delta P = f(L, d, v, \mu, \rho)$$

Determination of Pi Terms

- Step 2 List all the variables
  - Express each of the variables in terms of basic dimensions. Find the number of reference dimensions
  - Example: For pressure drop, we choose MLT

variables	L	d	v	μ	ρ	ΔP
Dimensions	L	L	LT^{-1}	$ML^{-1}T^{-1}$	ML^{-3}	$ML^{-1}T^{-2}$

- Step 3 Determine the required number of pi terms
  - The number of pi terms is k-r
  - Example: For pressure drop, The number of pi terms is 6-3

Determination of Pi Terms

- Step 4 Select a number of repeating variables, where the number required is equal to the number of reference dimensions
  - Example: For pressure drop (r = 3) select ρ, v, d

- Step 5 Form a Pi term by multiplying one of the repeating variables, each raised to an exponent that will make the combination dimensionless.
  - Set up dimensional equations, combining the variables selected in Step 4 with each of the other variables (nonrepeating variables) in turn, to form dimensionless groups or dimensionless product.

Determination of Pi Terms

$$\Pi_3 = \frac{L}{\rho^a v^b d^c} = \mathrm{M}^0\mathrm{L}^0\mathrm{T}^0 \Rightarrow \frac{\mathrm{L}}{(\mathrm{ML}^{-3})^a (\mathrm{LT}^{-1})^b (\mathrm{L})^c} = \mathrm{M}^0\mathrm{L}^0\mathrm{T}^0$$

$$\Rightarrow a = 0, b = 0, c = 1 \Rightarrow \boxed{\Pi_3 = \frac{L}{d} = \mathrm{M}^0\mathrm{L}^0\mathrm{T}^0}$$

Determination of Pi Terms

- Step 6 Check all the resulting Pi terms to make sure they are dimensionless
  - check to see that each group obtained is dimensionless
- Step 7 Express the final form as a relationship among the Pi terms, and think about what is means.
  - Example:

$$\frac{\Delta P}{\rho v^2} = \phi\left(\frac{L}{d}, \frac{\mu}{\rho v D}\right) = \phi\left(\frac{L}{d}, \frac{1}{Re}\right)$$

Pipe pressure loss

Laminar flow

$$\frac{\Delta P}{\rho g} = \frac{64}{Re}\frac{L}{d}\frac{v^2}{2g}$$