Chapter 6
Heapsort

The slides for this course are based on the course textbook: Cormen, Leiserson, Rivest, and Stein, *Introduction to Algorithms*, 2nd edition, The MIT Press, McGraw-Hill, 2001.

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Chapter 6 Topics

- Heaps
- Maintaining the heap property
- Building a heap
- The heapsort algorithm
- Priority queues

Heapsort

- Running time of heapsort is $O(n \log_2 n)$
- It sorts in place
- It uses a data structure called a *heap*
- The heap data structure is also used to implement a priority queue efficiently

Full and Complete Binary Trees

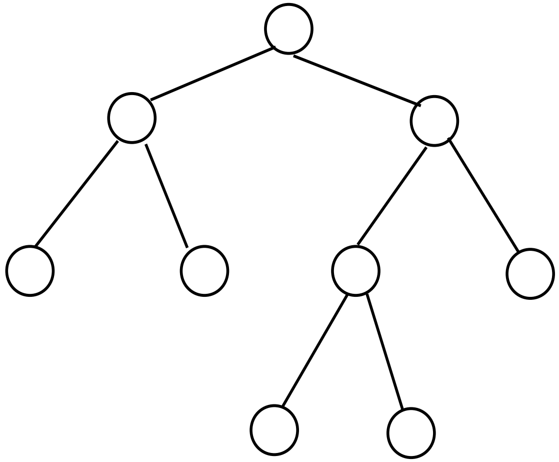
A **full binary tree** is a binary tree in which each node is either a leaf node or has degree 2 (i.e., has exactly 2 children).

A **complete binary tree** is a full binary tree in which all leaves have the same depth.

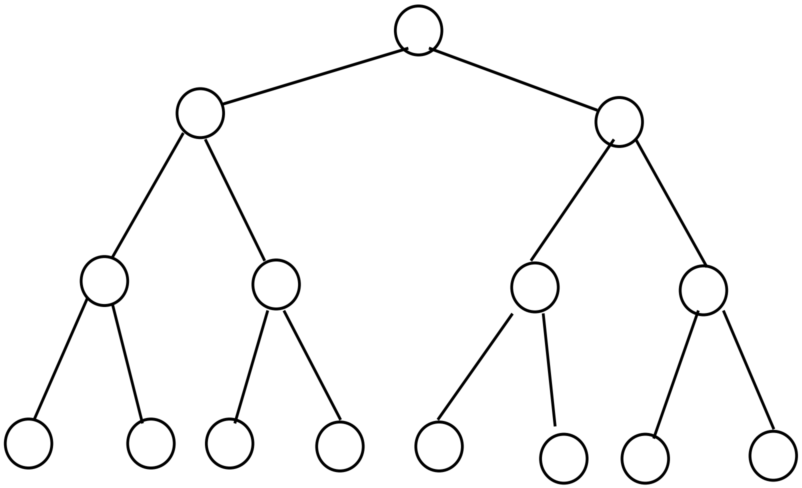
A **nearly complete binary tree** is completely filled on all levels except possibly the lowest, which is filled *from the left* up to a point.

Examples

Full binary tree:



Complete binary tree:



Representation of Nearly Complete Binary Tree

A nearly complete binary tree may be represented as an array (i.e., no pointers):

Number the nodes, beginning with the root node and moving from level to level, left to right within a level.

The number assigned to a node is its index in the array.

Additional Properties of Nearly Complete Binary Trees

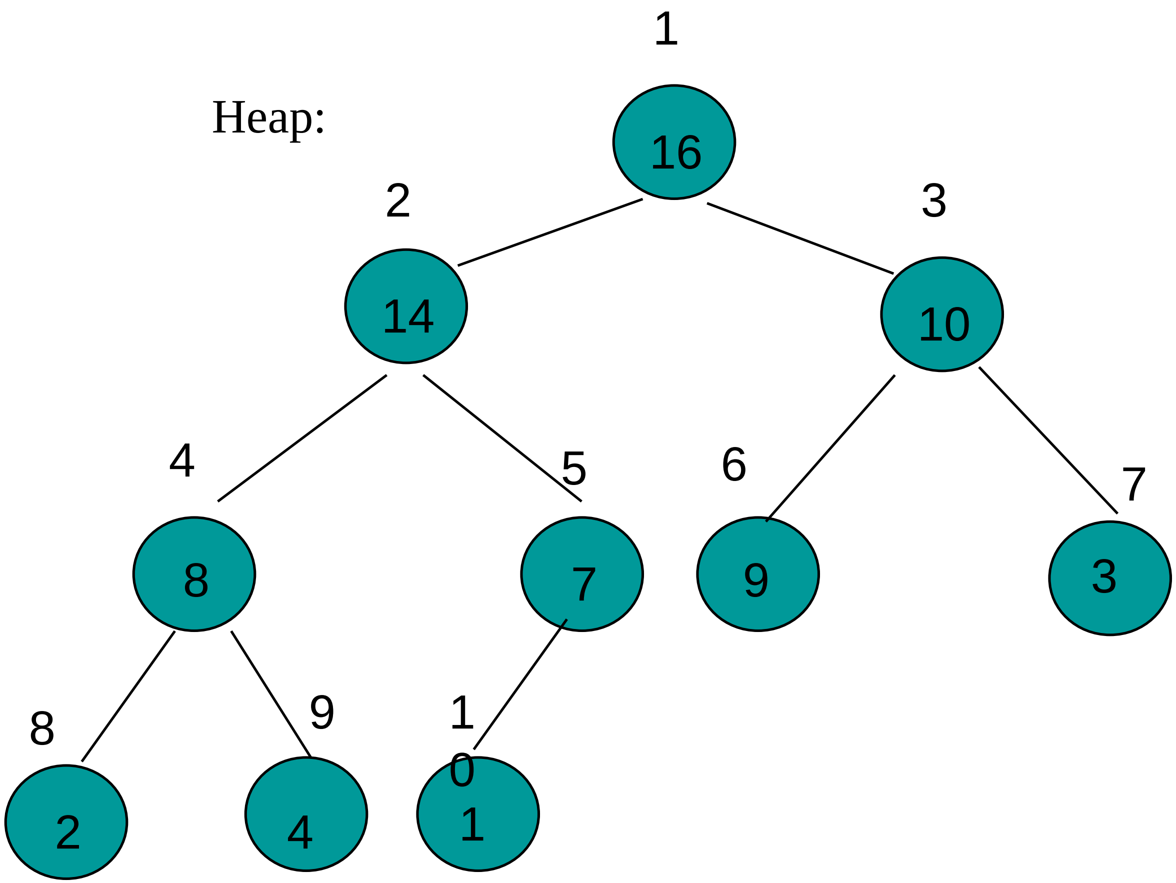
- The root of the tree is A[1].
- If a node has index i, we can easily compute the indices of its:
  - parent $\lfloor i/2 \rfloor$
  - left child $2i$
  - right child $2i + 1$

Numbering

Array:

16	14	10	8	7	9	3	2	4	1

Heap:



Heap

- Implemented as an array object, A[ ]
- Array A that implements the heap has two attributes
  - length(A)
  - heap-size(A)

Heap

A binary tree with n nodes and of height *h* is **almost complete** iff its nodes correspond to the nodes which are numbered 1 to *n* in the complete binary tree of height *h*.

A **heap** is an *almost complete binary tree* that satisfies the **heap property**:

max-heap: For every node *i* other than the root:

$$A[\text{Parent}(i)] \geq A[i]$$

min-heap: For every node i other than the root:

$$A[\text{Parent}(i)] \leq A[i]$$

Max-Heap

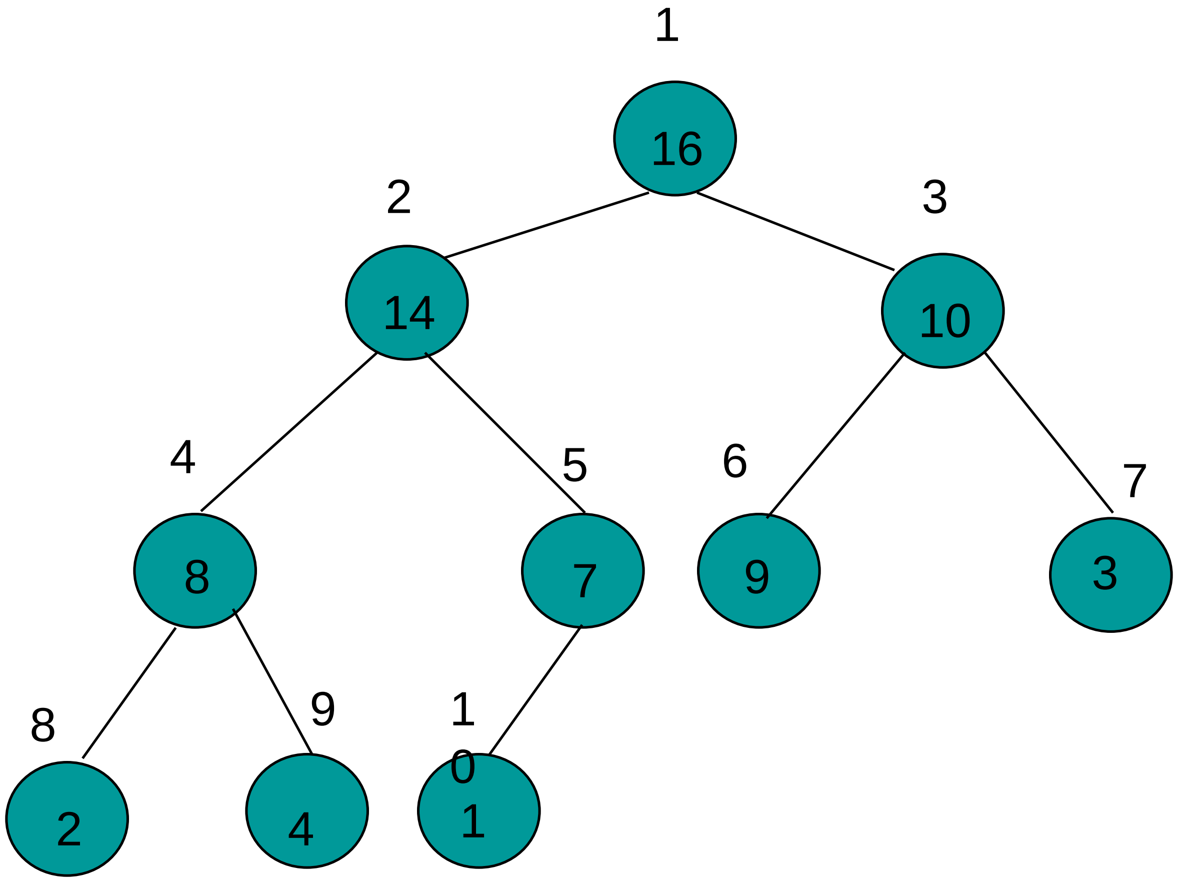
A **max-heap** is an *almost complete binary tree* that satisfies the **heap property**:

For every node i other than the root,

$$A[\text{PARENT}(i)] \geq A[i]$$

What does this mean?

- the value of a node is at most the value of its parent
- the largest element in the heap is stored in the root
- subtrees rooted at a node contain smaller values than the node itself



16	14	10	8	7	9	3	2	4	1
1	2	3	4	5	6	7	8	9	10

Height of a node in a heap

The ***height*** of a node in a heap is the number of edges on the longest simple downward path from the node to a leaf.

The height of a heap is the height of its root.

Since a heap of n elements is based on a complete binary tree, its height is $\Theta(\lg n)$.

Heaps have 5 basic procedures

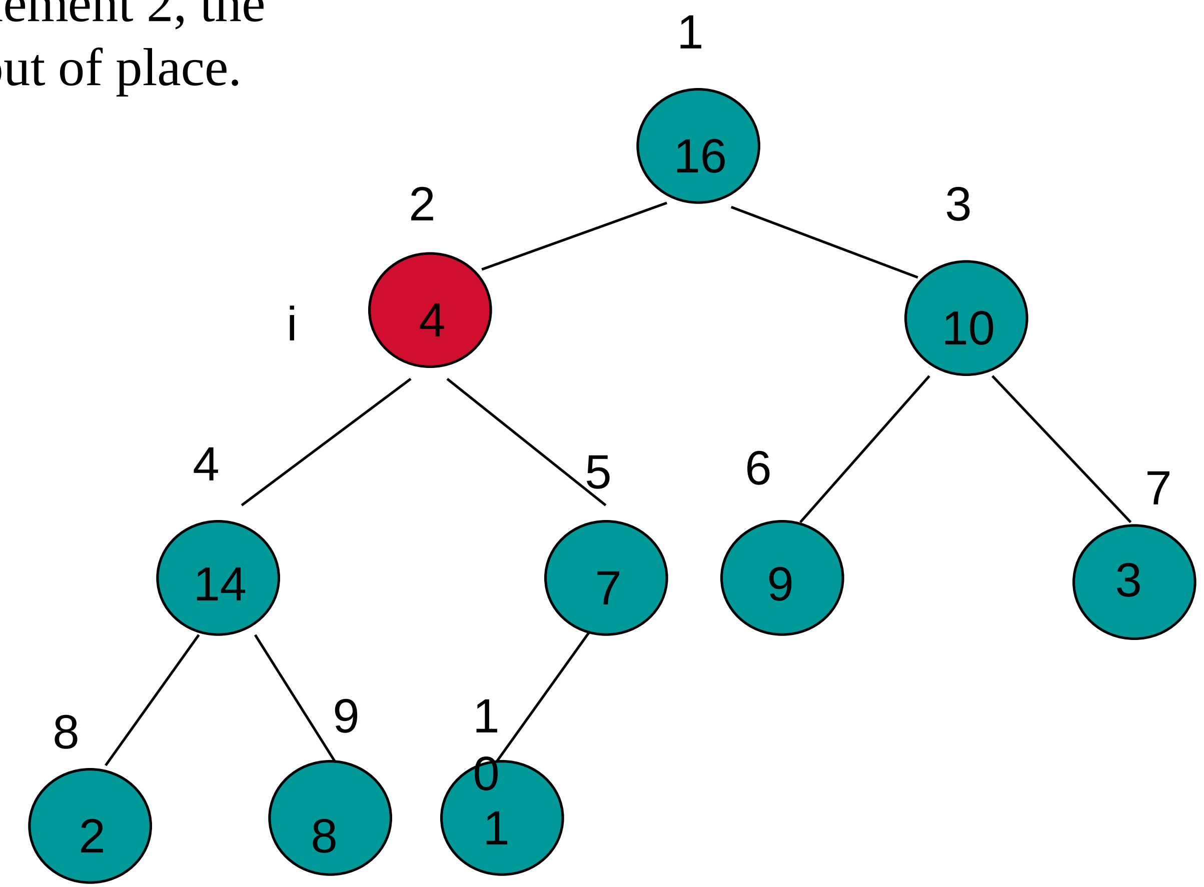
- HEAPIFY: maintains the heap property
- BUILD-HEAP: builds a heap from an unordered array
- HEAPSORT: sorts an array in place
- EXTRACT-MAX: selects max element
- INSERT: inserts a new element

We'll work with MAX heaps

MAX-HEAPIFY(A,i)

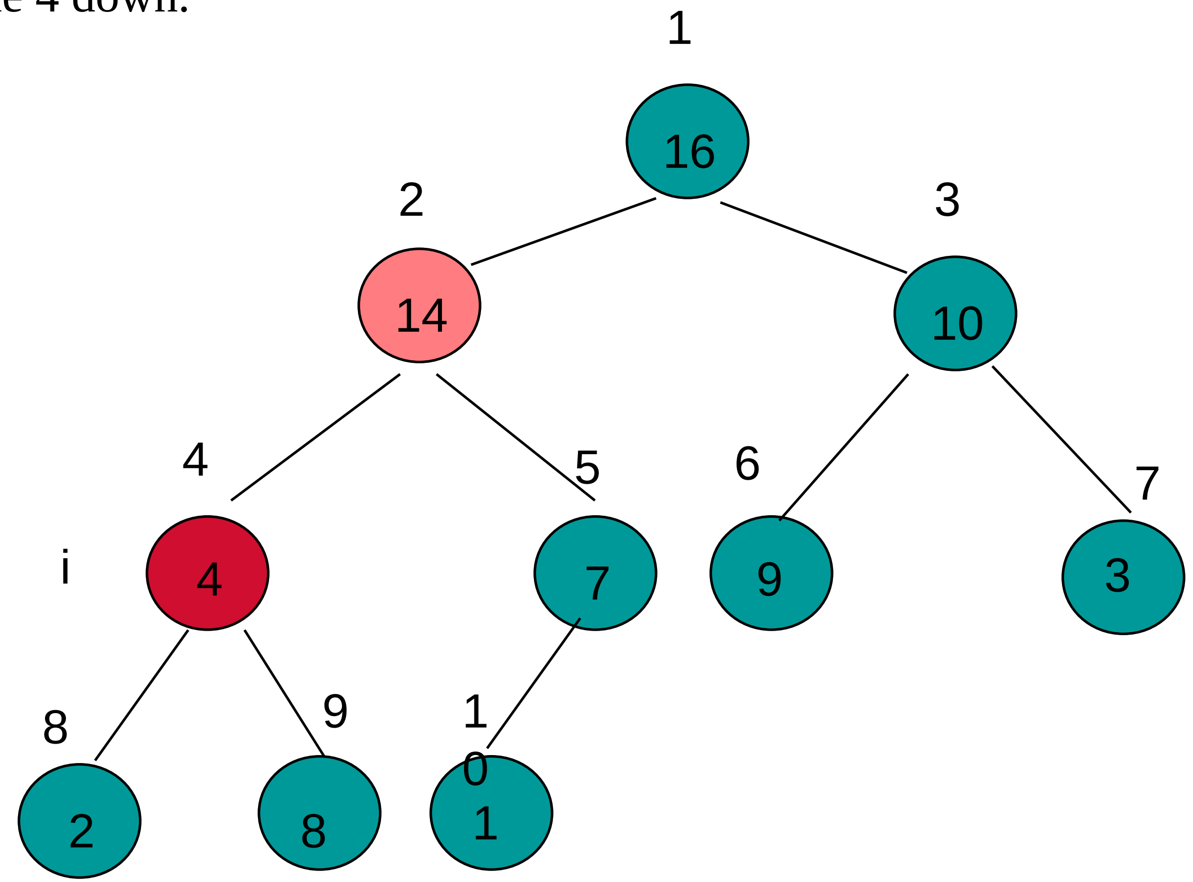
- Goal is to put the i^{th} element in the correct place in a portion of the array that "almost" has the heap property.
- The only element with index of *i* or greater that is out of place is A[*i*].
- Assume that left and right subtrees of A[*i*] have the heap property.
- "Sift" A[*i*] down to the right position.

Array element 2, the "4", is out of place.



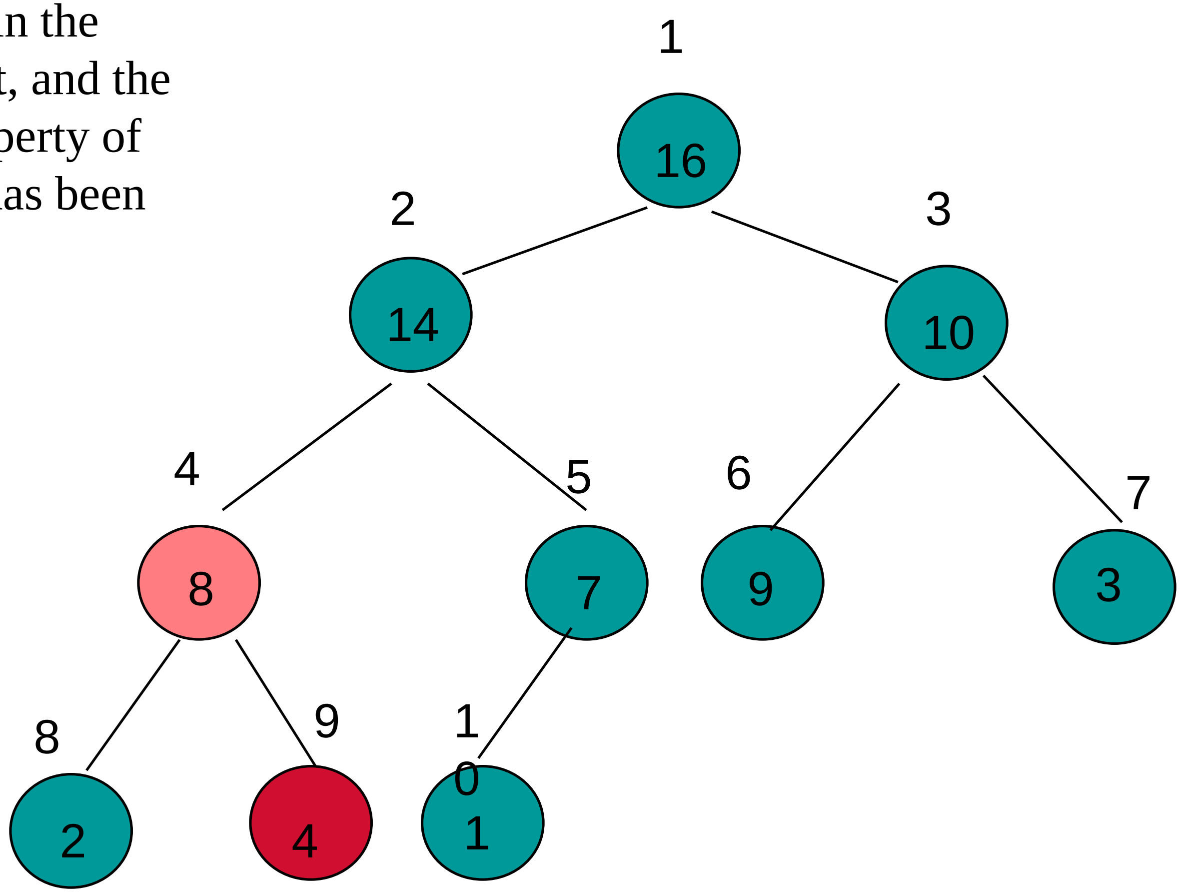
MAX-HEAPIFY(A,2) heap-size[A] = 10

Moving the 4 down.



MAX-HEAPIFY(A,4) heap-size[A] = 10

The 4 is in the right spot, and the heap property of the tree has been restored.



MAX-HEAPIFY(A,9) heap-size[A] = 10

MAX-HEAPIFY

```
MAX-HEAPIFY(A, i)
l ← LEFT(i)
r ← RIGHT(i)
if l ≤ heap-size[A] and A[l] > A[i]
    then largest ← l
    else largest ← i
if r ≤ heap-size[A] and A[r] > A[largest]
    then largest ← r
if largest ≠ i
    then exchange A[i] ↔ A[largest]
         MAX-HEAPIFY(A, largest)
```

Running time of MAX-HEAPIFY

- Run time of MAX-HEAPIFY(A,i)
  - Look at lines 1 –9
  - Is there a loop? No.
  - Does the number of steps depend upon *n*? No.
  - So the running time so far is $\Theta(1)$
  - How about line 10? We don't know yet.

Running time of MAX-HEAPIFY

The recursive call to MAX-HEAPIFY in line 10 implies a recurrence relation.

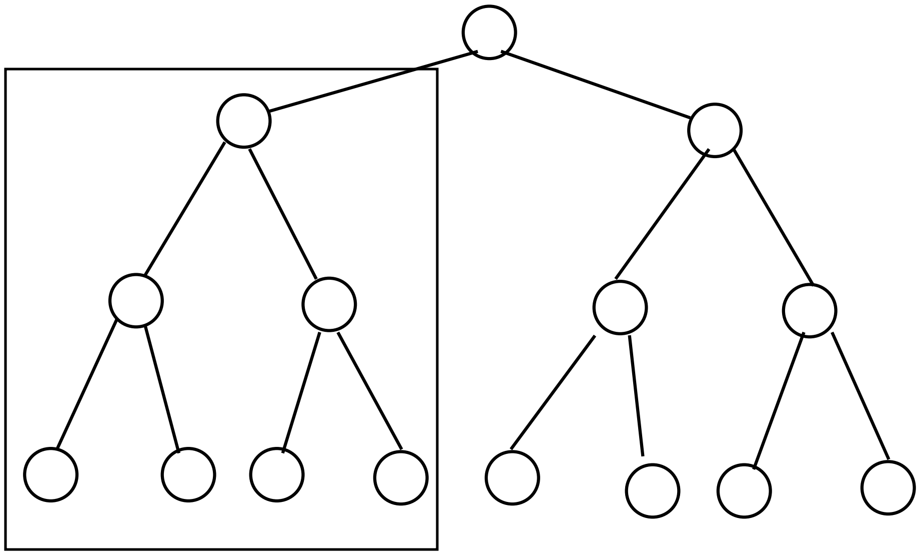
When we call MAX_HEAPIFY again, we already know that lines 1-9 cost $\Theta(1)$ steps.

But we may need to call MAX-HEAPIFY on a subtree rooted at one of the children of the current node, so we have to add the cost of doing that.

Running time of MAX-HEAPIFY

How many nodes might be involved?

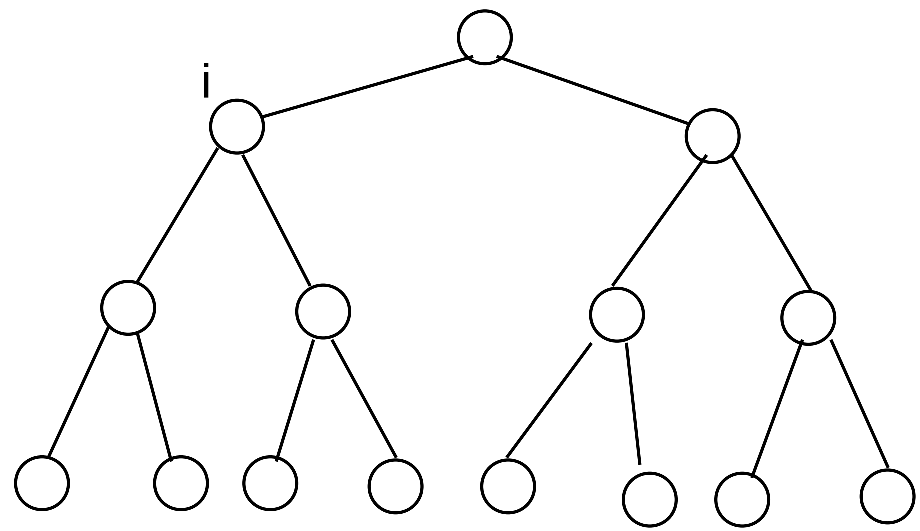
In the case of a <u>full</u> binary tree, about half of the tree might be involved.



Running time of MAX-HEAPIFY

In a complete binary tree with 15 nodes, 8 of those nodes are leaves at the bottom level.

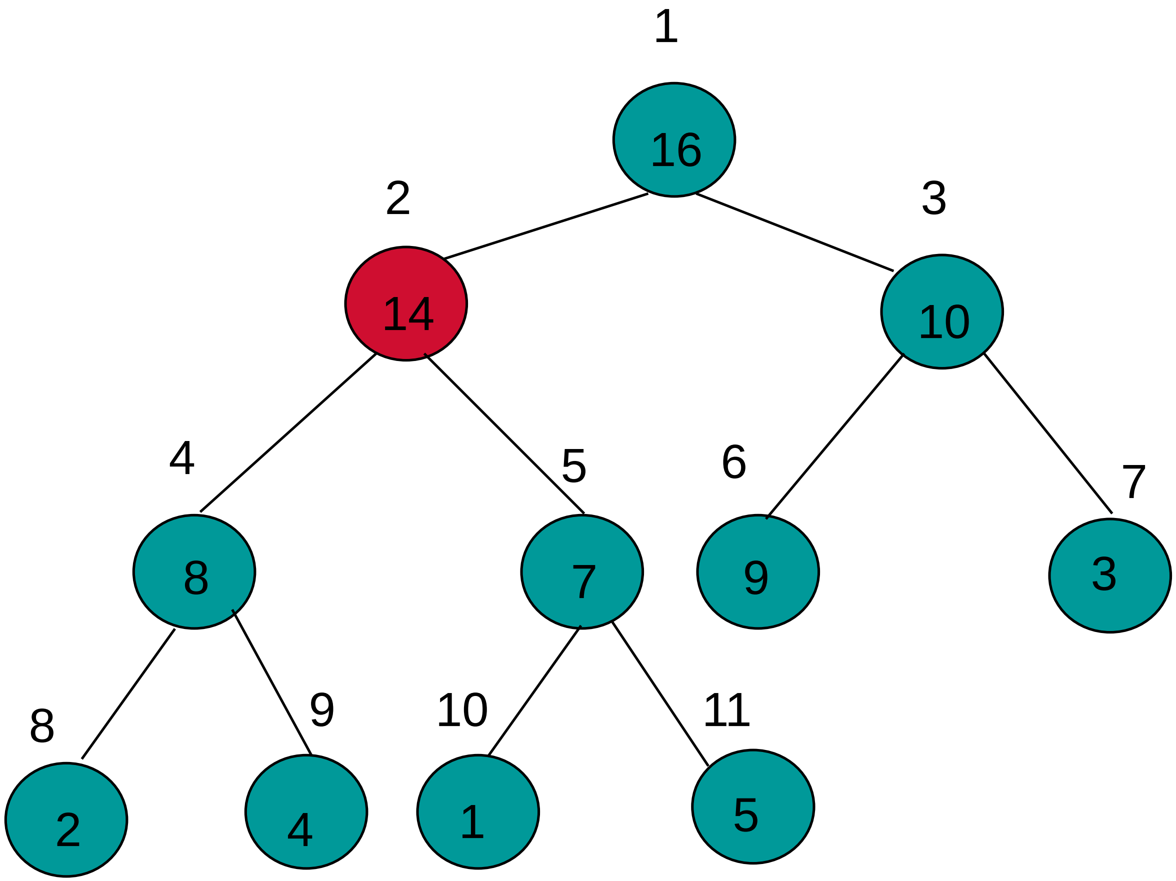
If we perform MAX-HEAPIFY on node i, 7 of the 15 nodes will be involved – about ½ of the nodes.



Running time of MAX-HEAPIFY

What is the worst case?

When the last row of the tree is half full.



Here 7 out of 11 nodes are involved.

In general, $\leq 2/3^{rds}$ of the tree might be involved in the worst case.

Running time of MAX-HEAPIFY

Remember that, in a complete binary tree, *more than half* of the nodes in the entire tree are the leaf nodes on the bottom level of the tree.

But the only nodes involved in MAX-HEAPIFY are the descendants of A[i], which must be in A[i]'s half of the tree.

So worst case is when the last row of the tree is half full on the left side and A[i] is their ancestor.

Running time of MAX-HEAPIFY

The subtrees of the children of our current node have size at most $2n/3$.

The running time of MAX_HEAPIFY can be described by the recurrence:

$$T(n) \le T(2n/3) + \Theta(1)$$

This is Case 2 by the master method, so:

$$T(n) = O(\lg n)$$

Running time of MAX-HEAPIFY

We could also describe the running time of MAX-HEAPIFY for a node of height h as $O(h)$. (This is useful only if we know the height of a specific node.)

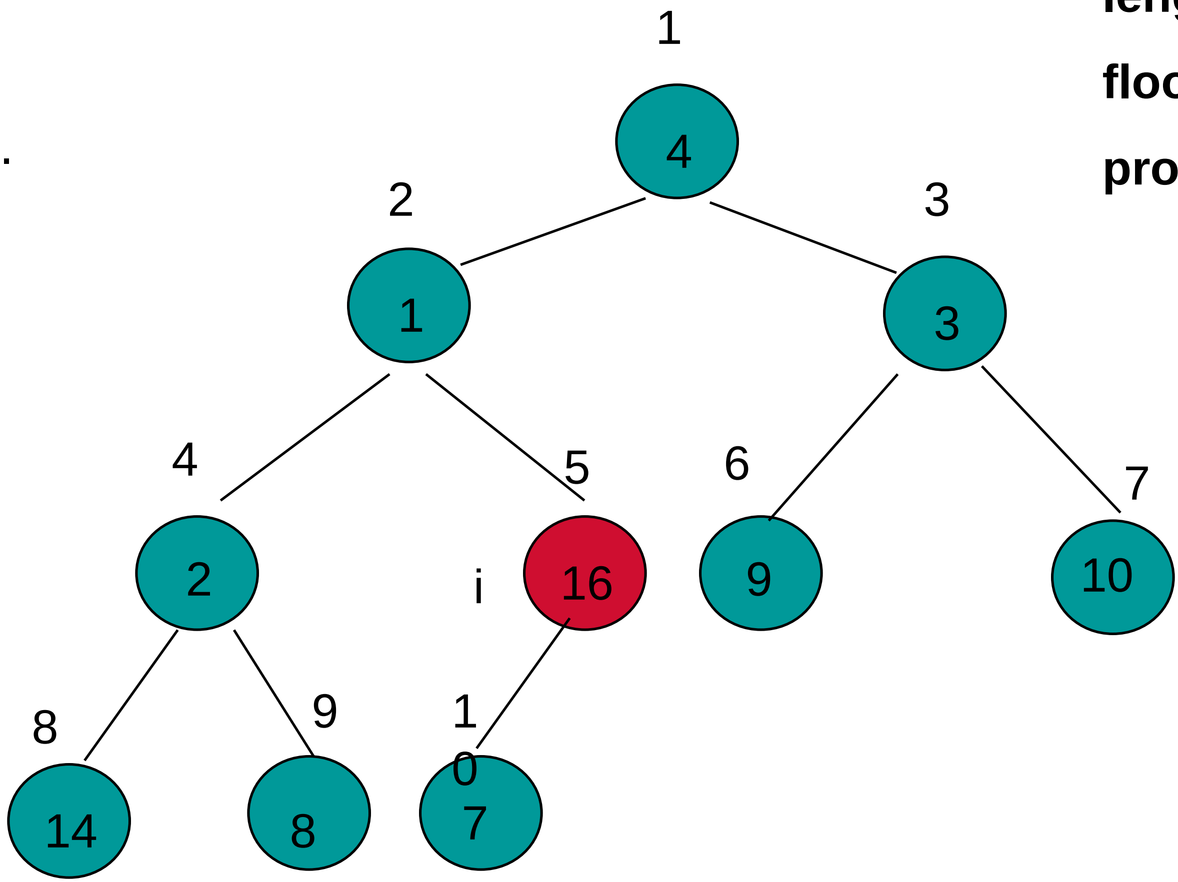
BUILD-MAX-HEAP

- Use MAX-HEAPIFY in a bottom-up manner to convert an array A[1..n] into a heap.
- Each leaf is initially a one-element heap. Elements A[$\lfloor n/2 \rfloor + 1..n$] are leaves.
- MAX-HEAPIFY is called on all interior nodes.

BUILD-MAX-HEAP

```
BUILD-MAX-HEAP(A)
heap-size[A] ← length[A]
for i ← floor(length[A]/2) downto 1 do
    MAX-HEAPIFY(A, i)
```

a.



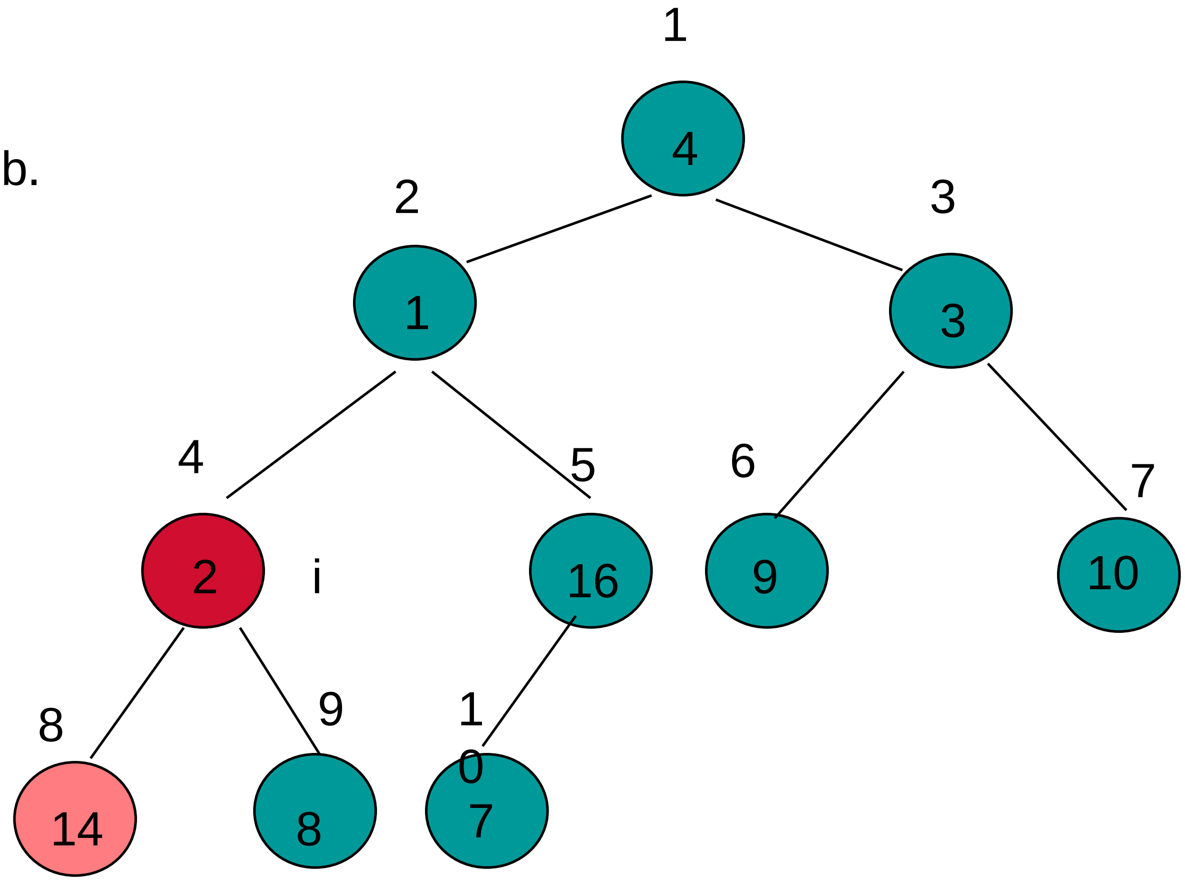
length(A) = 10

floor(length(A)/2) = 5

process from 5 to 1

4	1	3	2	16	9	10	14	8	7
1	2	3	4	5	6	7	8	9	10

b.



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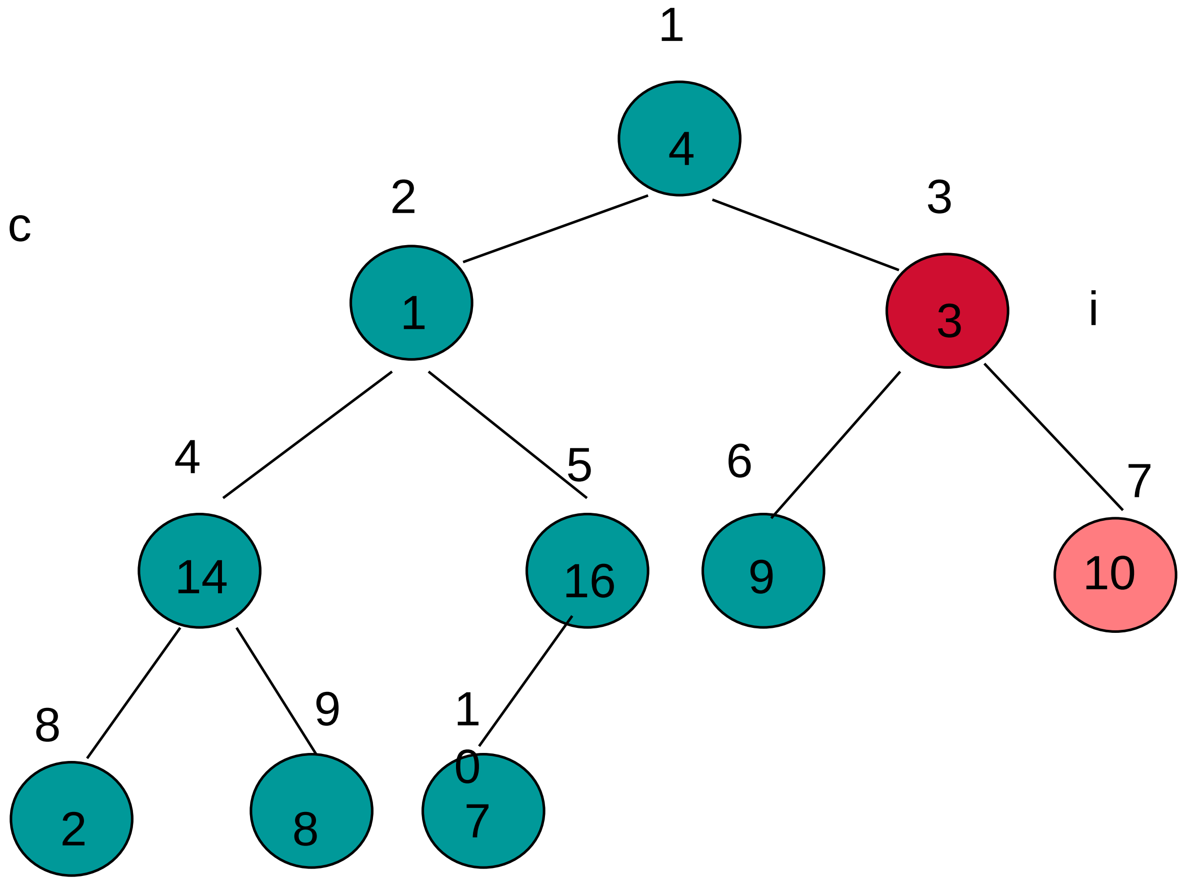
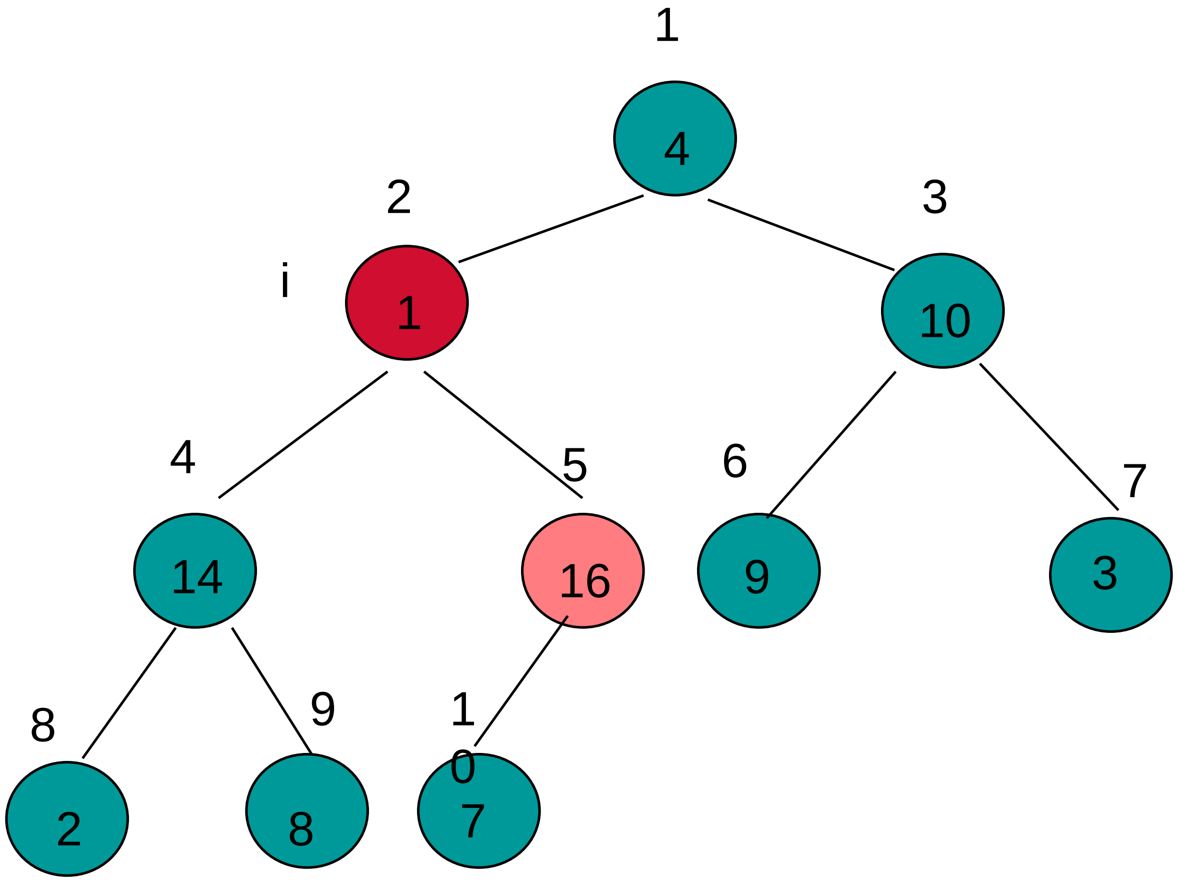
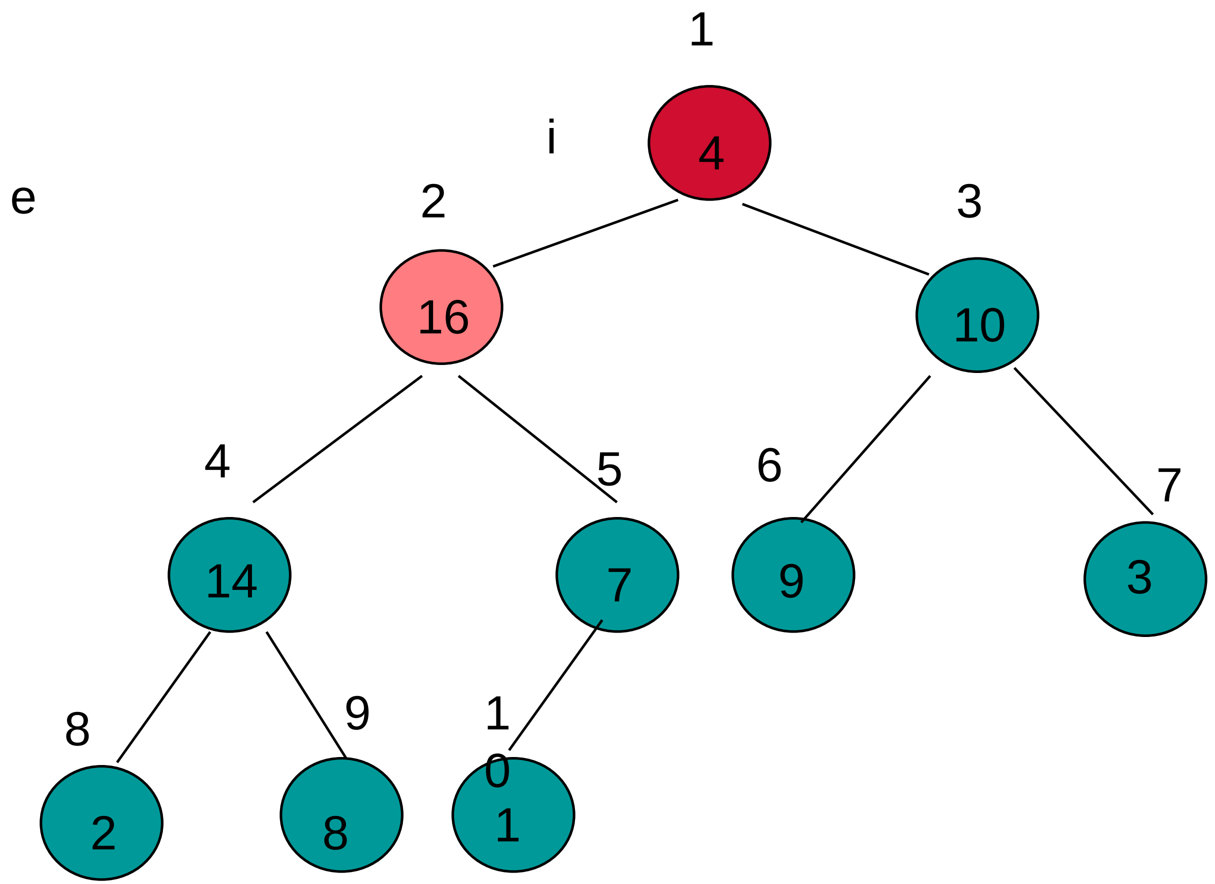


4	1	3	14	16	9	10	2	8	7
1	2	3	4	5	6	7	8	9	10

d

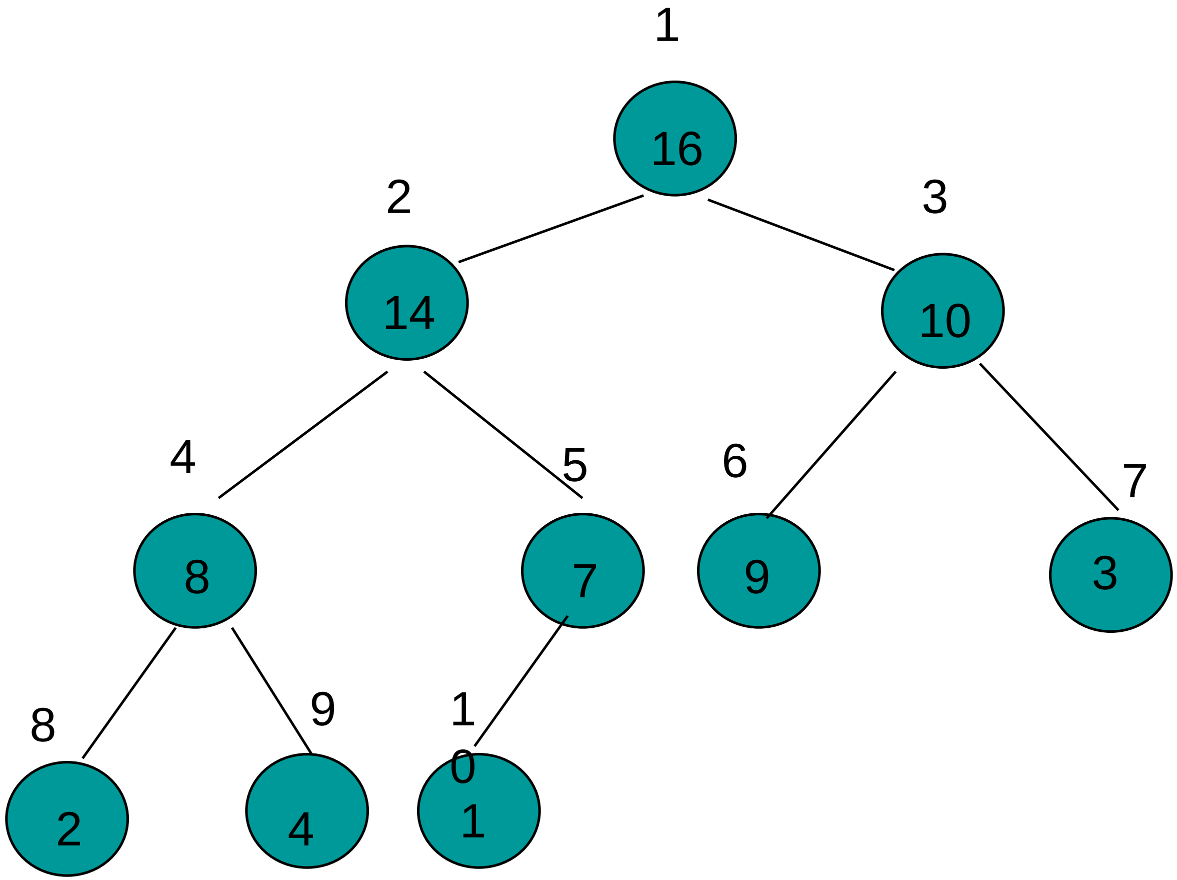


4	1	10	14	16	9	3	2	8	7
1	2	3	4	5	6	7	8	9	10



4	16	10	14	7	9	3	2	8	1
1	2	3	4	5	6	7	8	9	10

f



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Running Time of BUILD-MAX-HEAP

- Simple upper bound:
  - each call to MAX-HEAPIFY costs O(lg n)
  - O(n) such calls
  - running time at most O(n lg n)
- Previous bound is not tight:
  - lots of the elements are leaves
  - most elements are near leaves (small height)

Tighter Bound for BUILD-MAX-HEAP

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right)$$

By substituting x = ½ in the formula for differentiating infinite geometric series, we have:

$$\sum_{h=0}^{\infty} \frac{h}{2^h} = \frac{1/2}{(1-1/2)^2} = 2$$

Tighter Bound for BUILD-MAX-HEAP (continued)

Thus the running time is bounded by:

$$O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right) = O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right) = O(n)$$

Therefore, we can build a heap from an unordered array in linear time.

Heapsort

- First build a heap.
- Then successively remove the biggest element from the heap and move it to the first position in the sorted array.
- The element currently in that position is then placed at the top of the heap and sifted to the proper position.

HEAPSORT

```
HEAPSORT(A)
BUILD-MAX-HEAP(A)
for i ← length[A] downto 2 do
    exchange A[1] ↔ A[i]
    heap-size[A] ← heap-size[A] – 1
    MAX-HEAPIFY(A, 1)
```

```
BUILD-MAX-HEAP(A)
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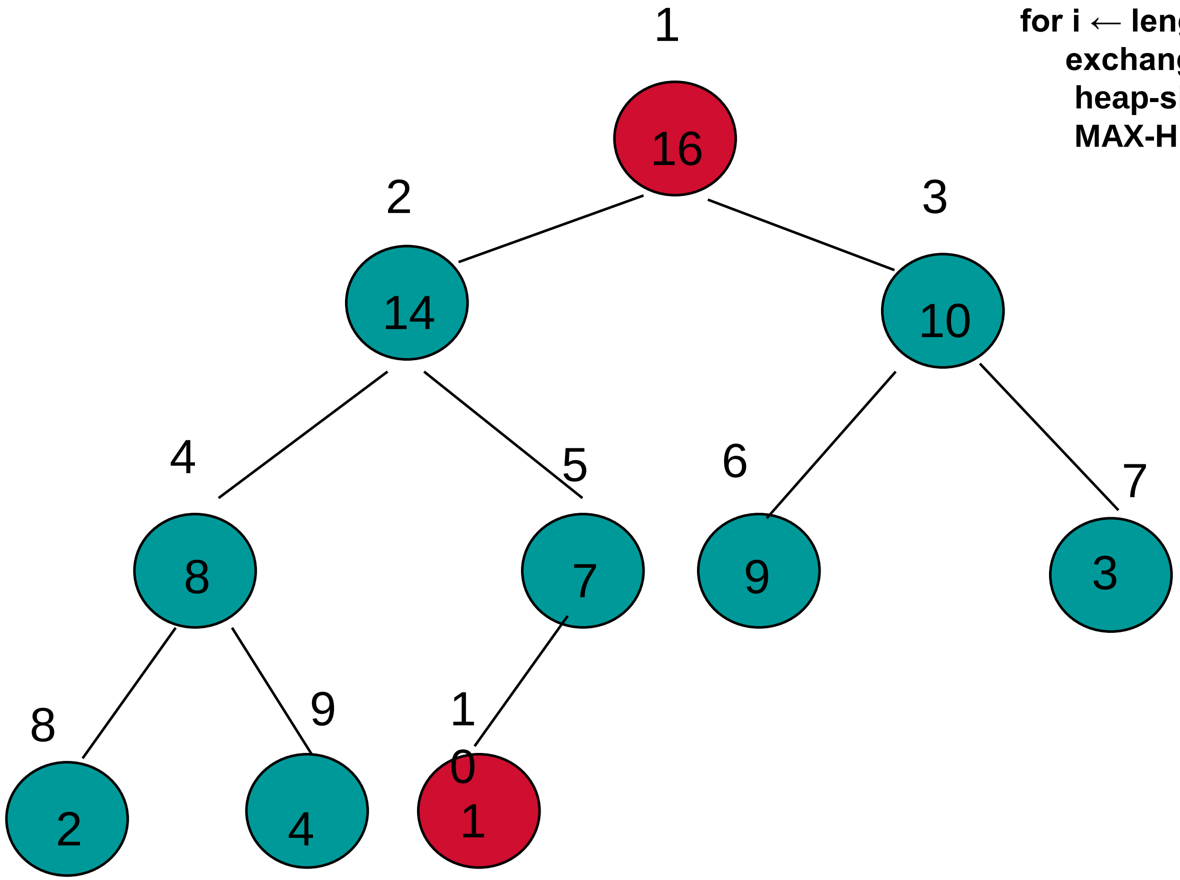

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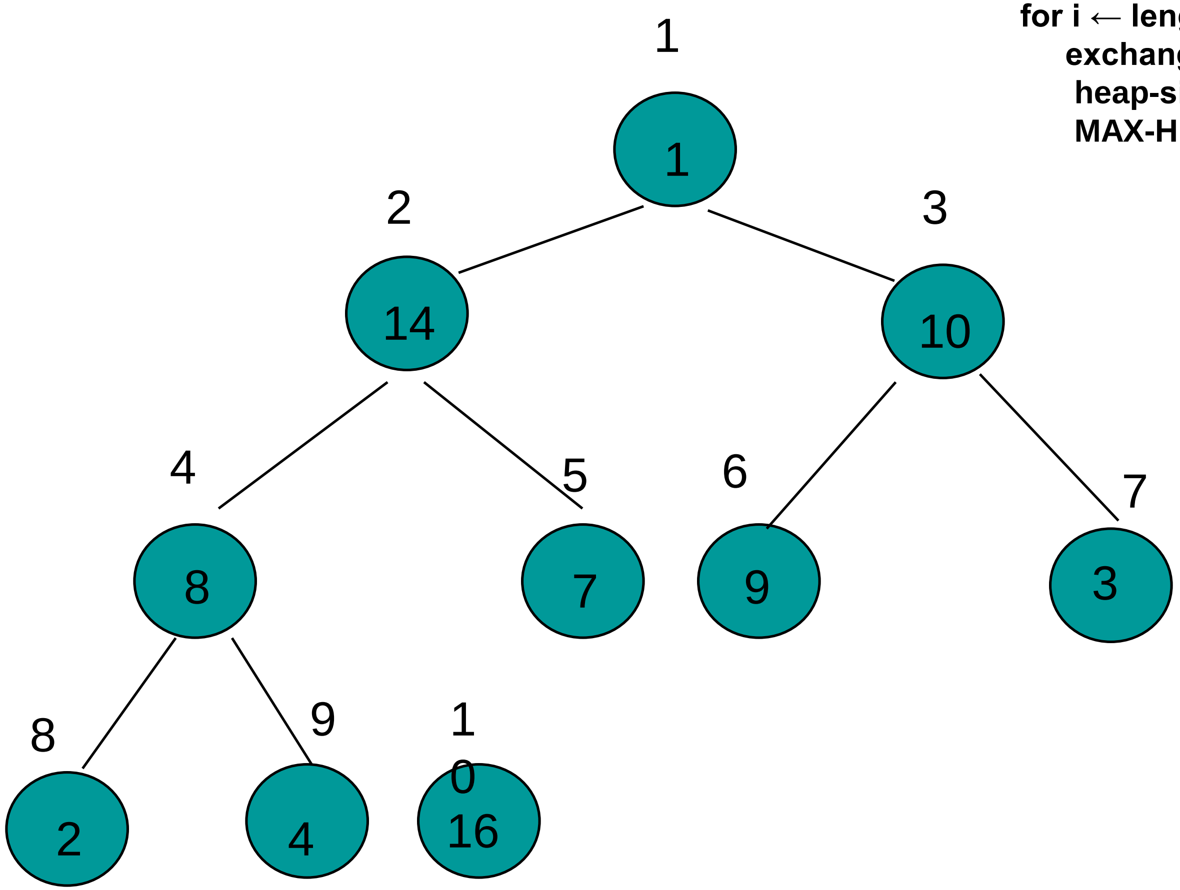

1	14	10	8	7	9	3	2	4	16
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BUILD-MAX-HEAP(A)
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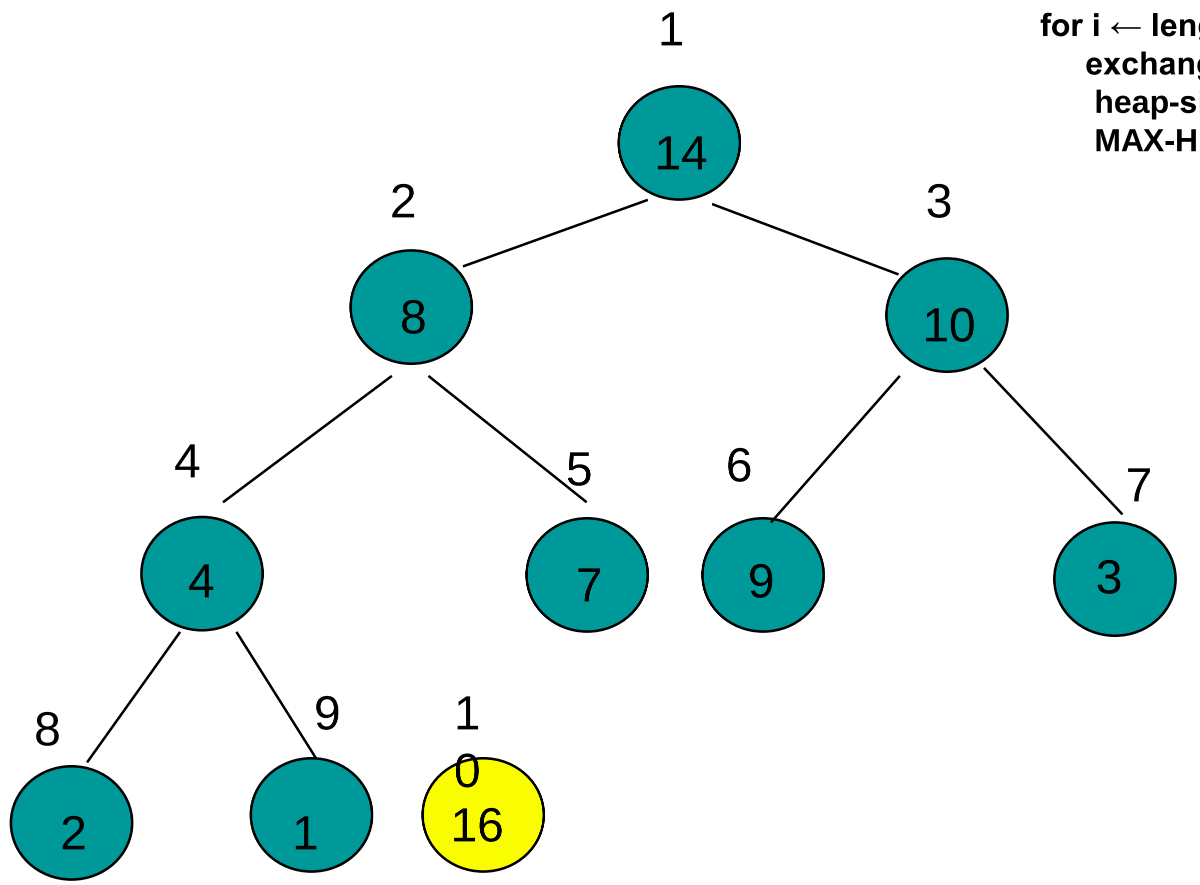

14	8	10	4	7	9	3	2	1	16
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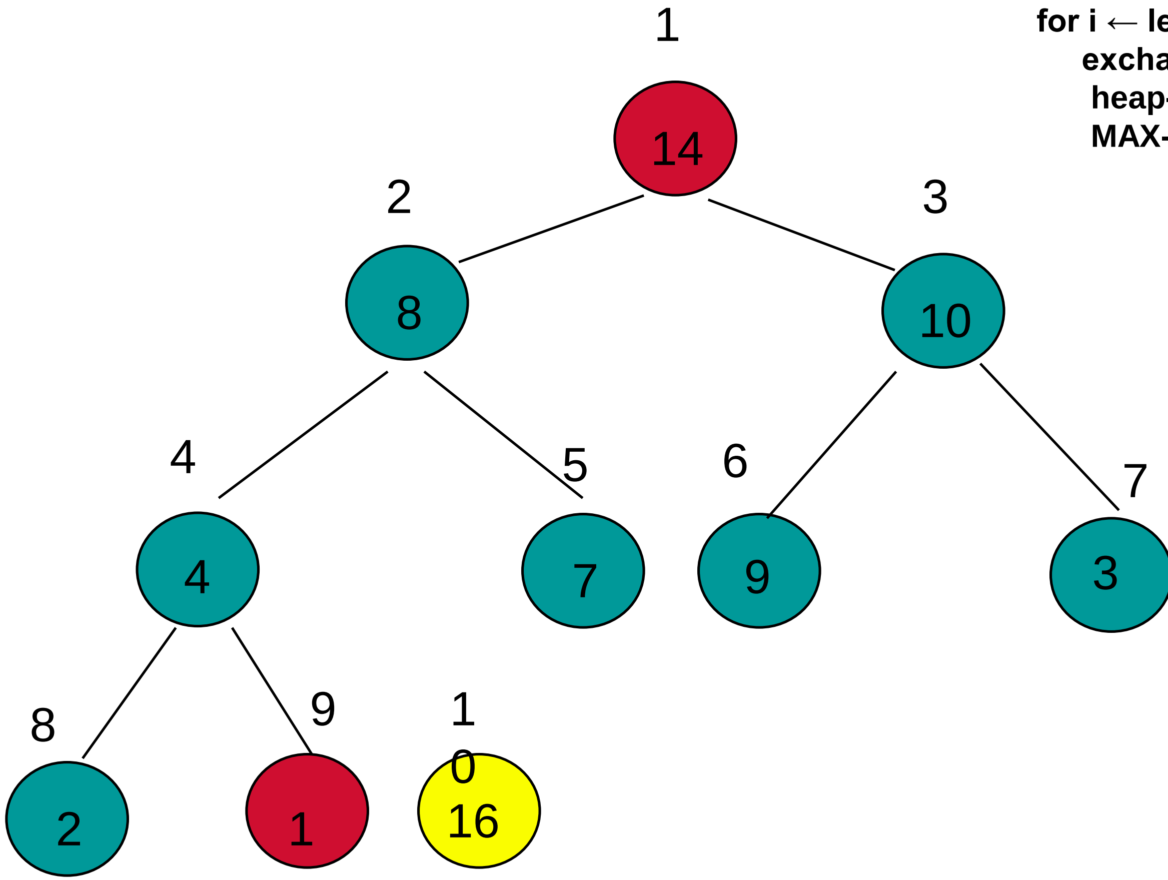

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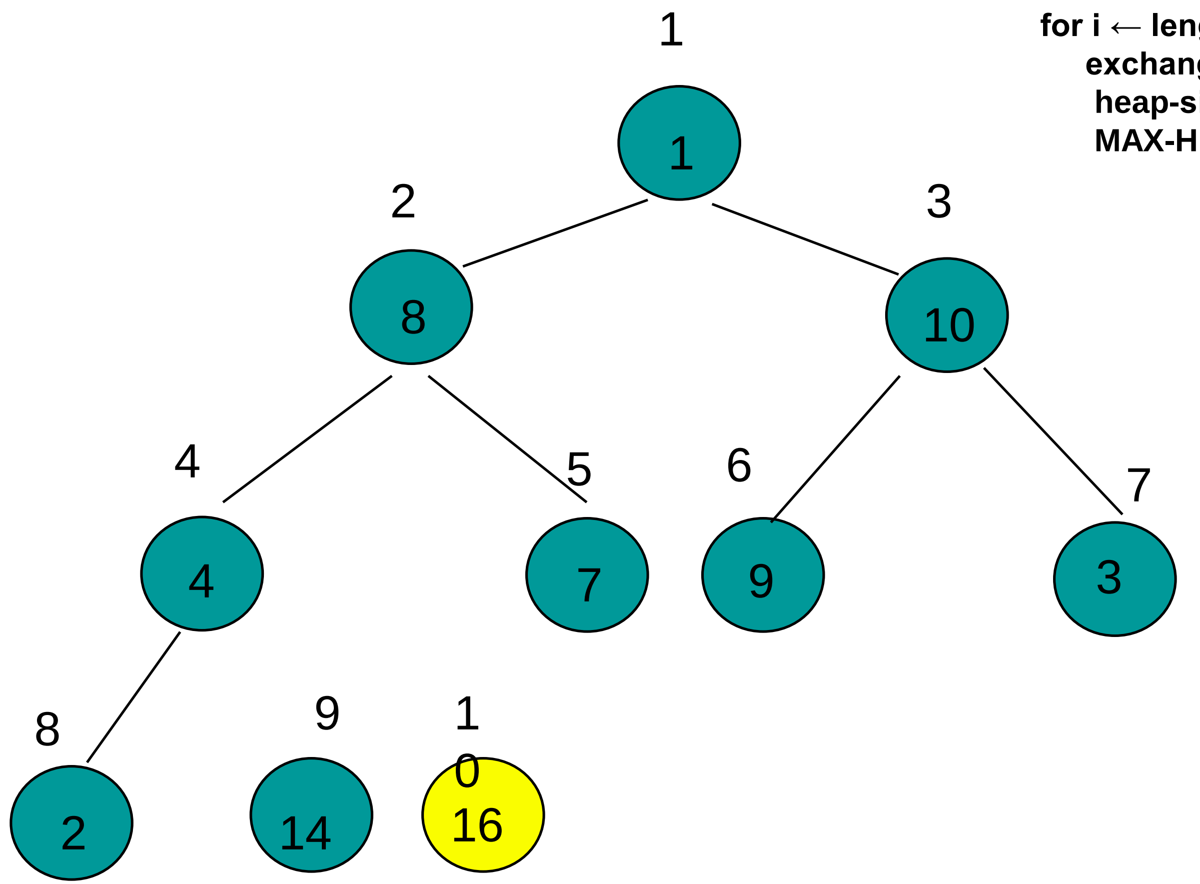

1	8	10	4	7	9	3	2	14	16
1	2	3	4	5	6	7	8	9	10

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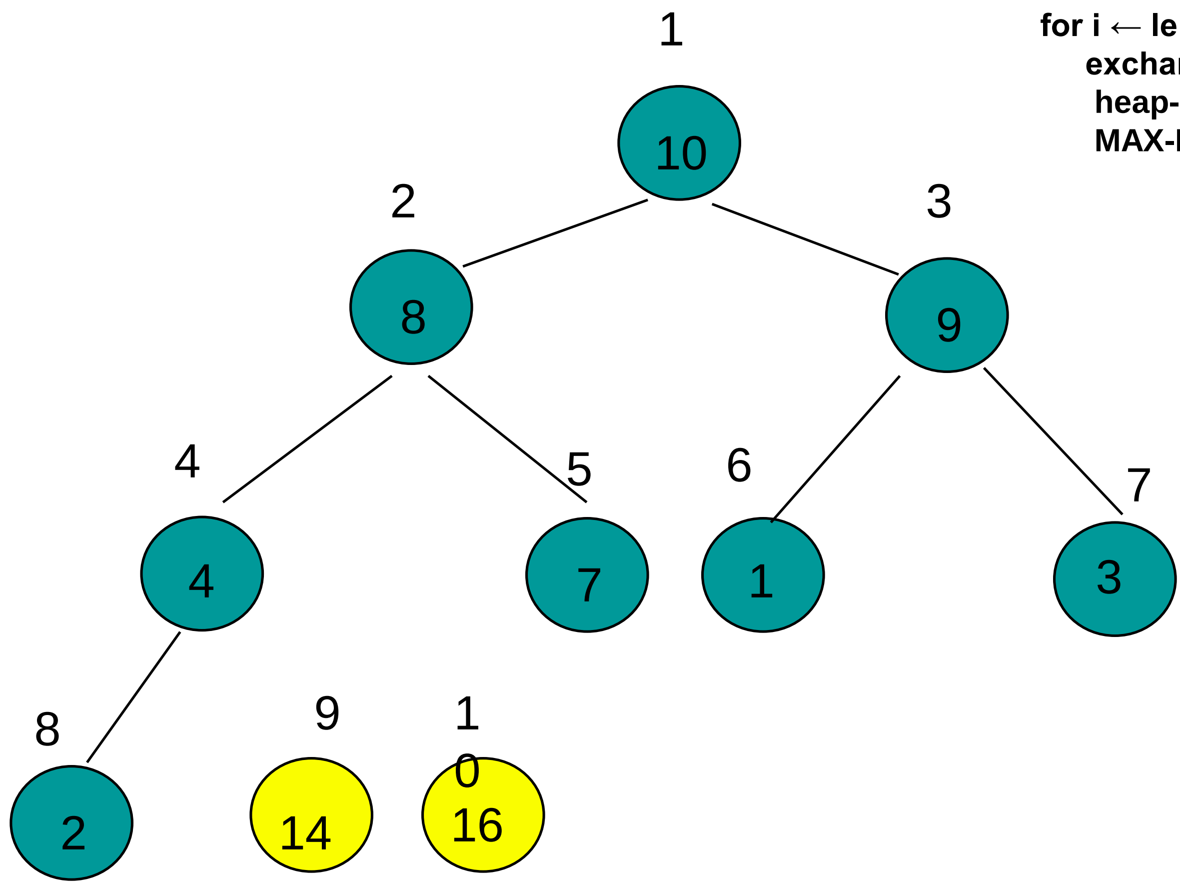

10	8	9	4	7	1	3	2	14	16
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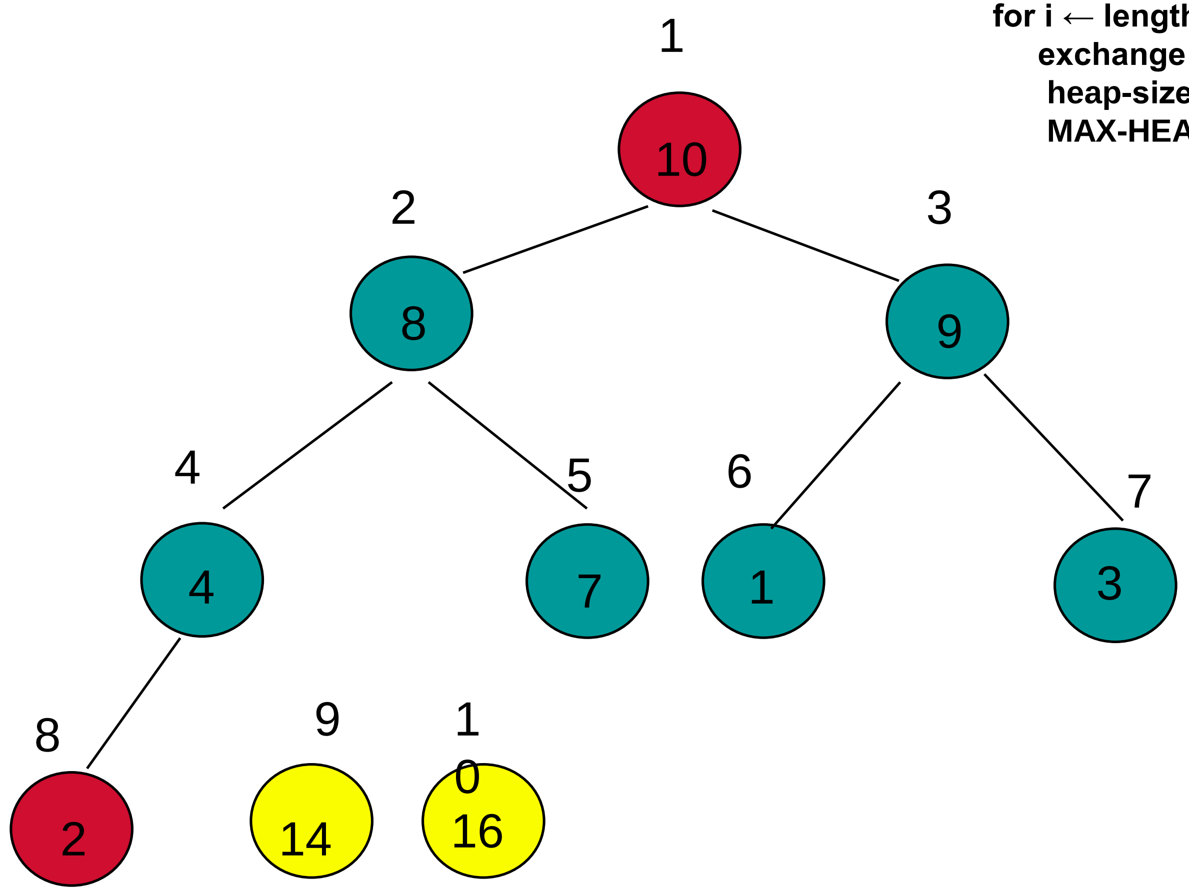

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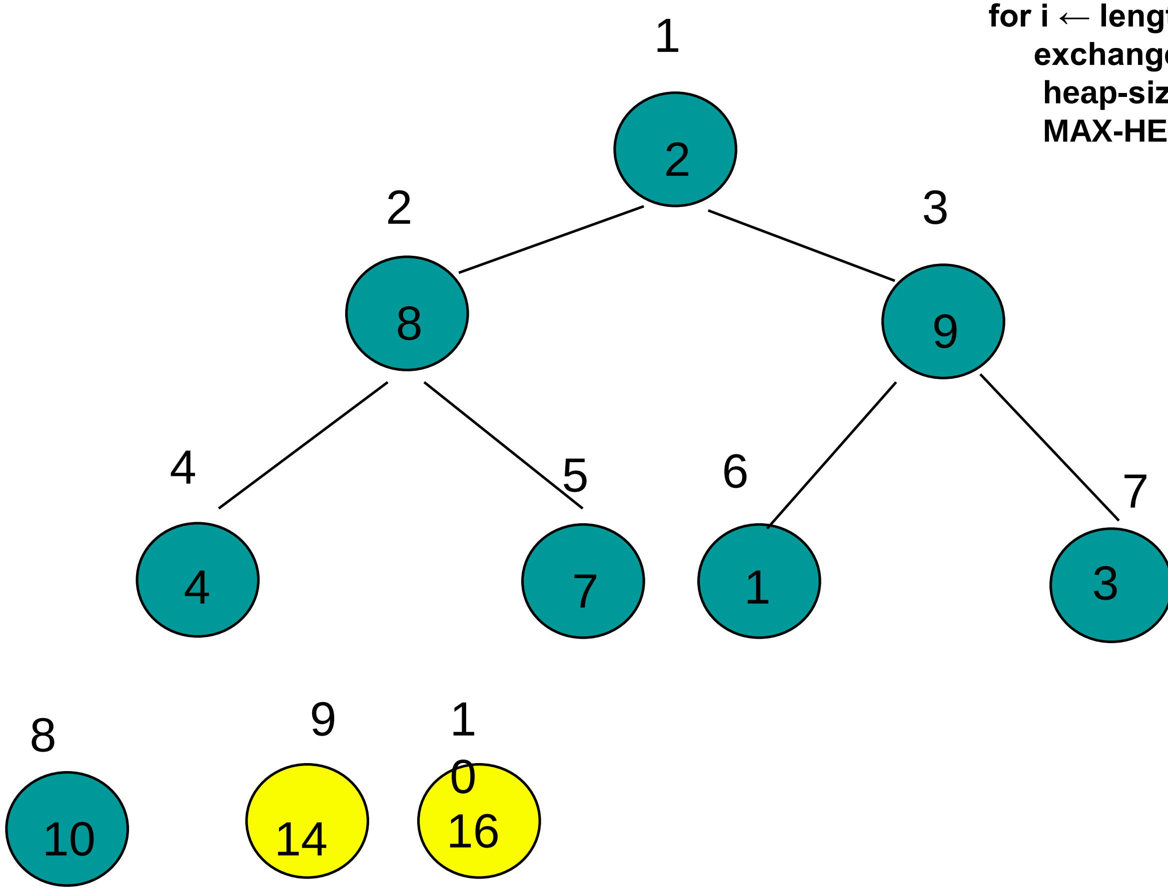

2	8	9	4	7	1	3	10	14	16
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```
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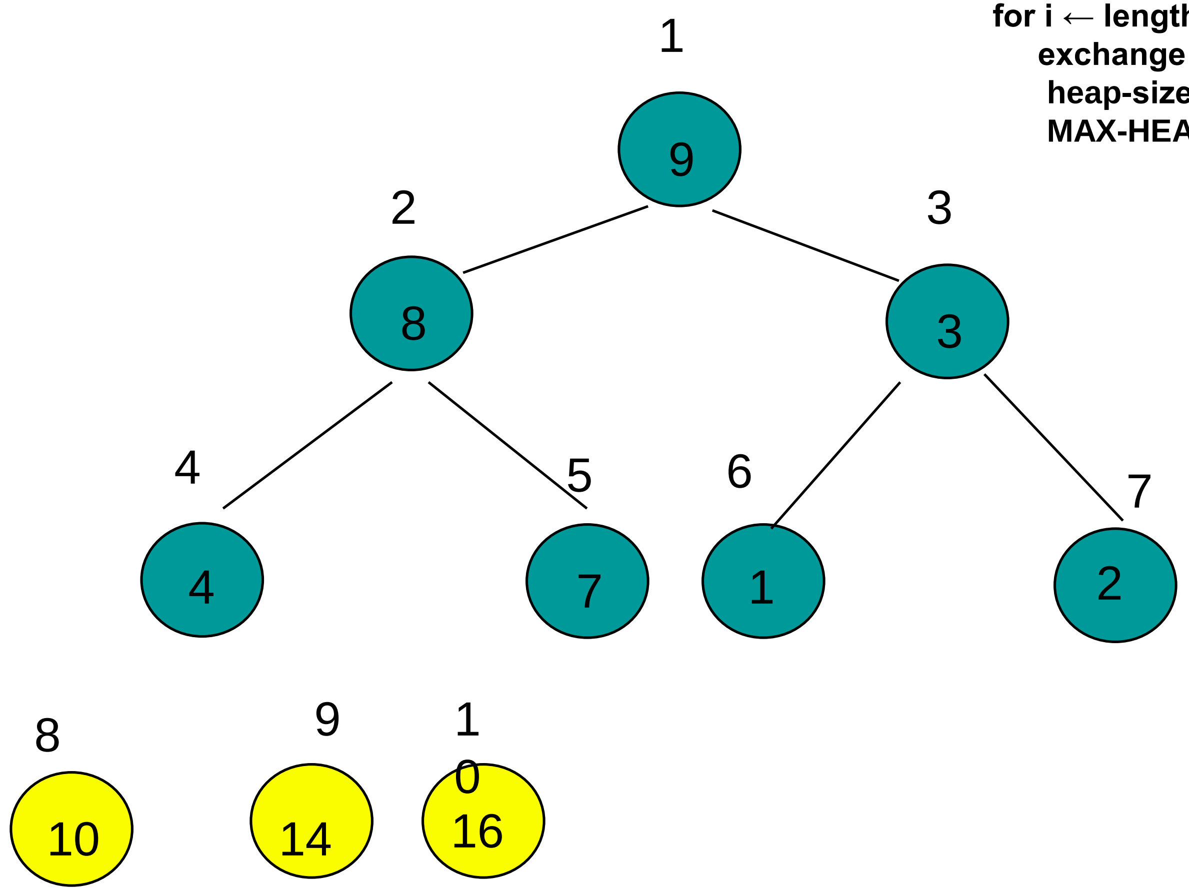

9	8	3	4	7	1	2	10	14	16
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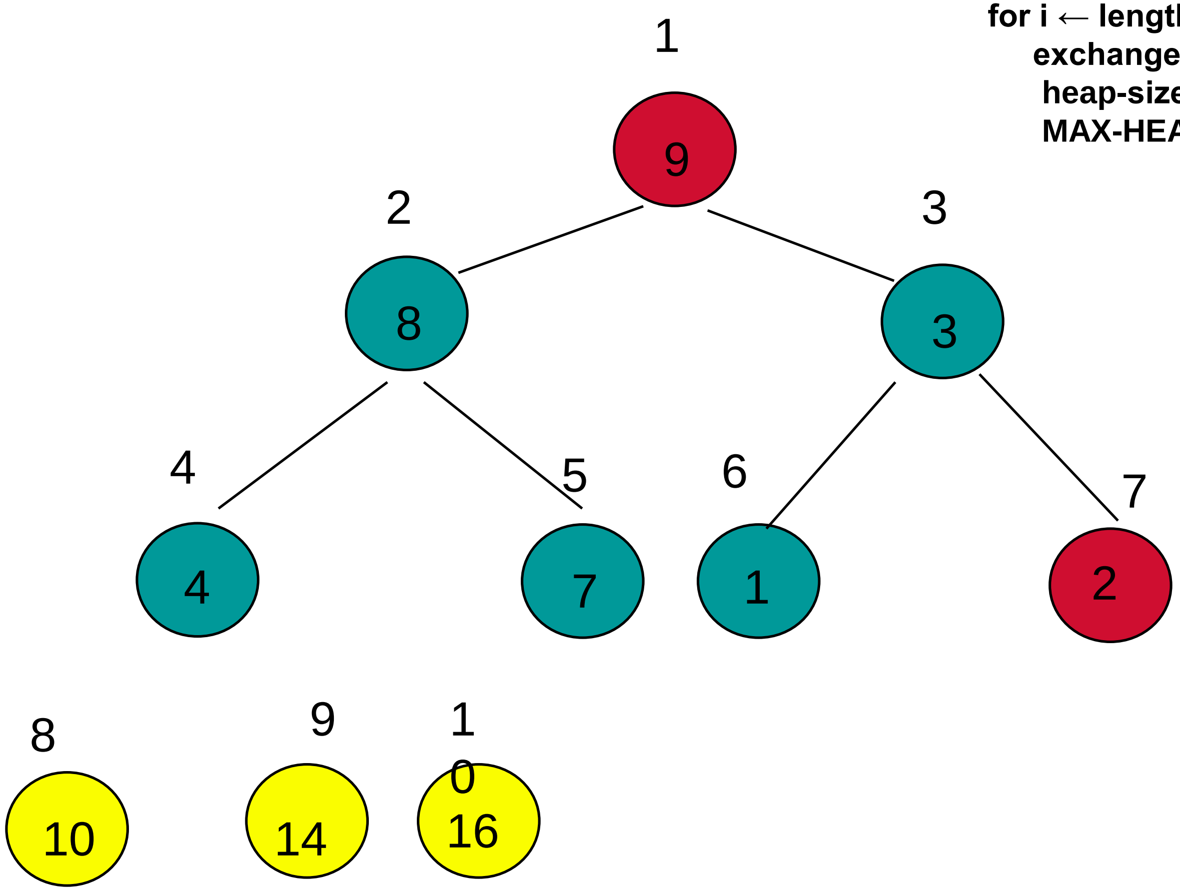

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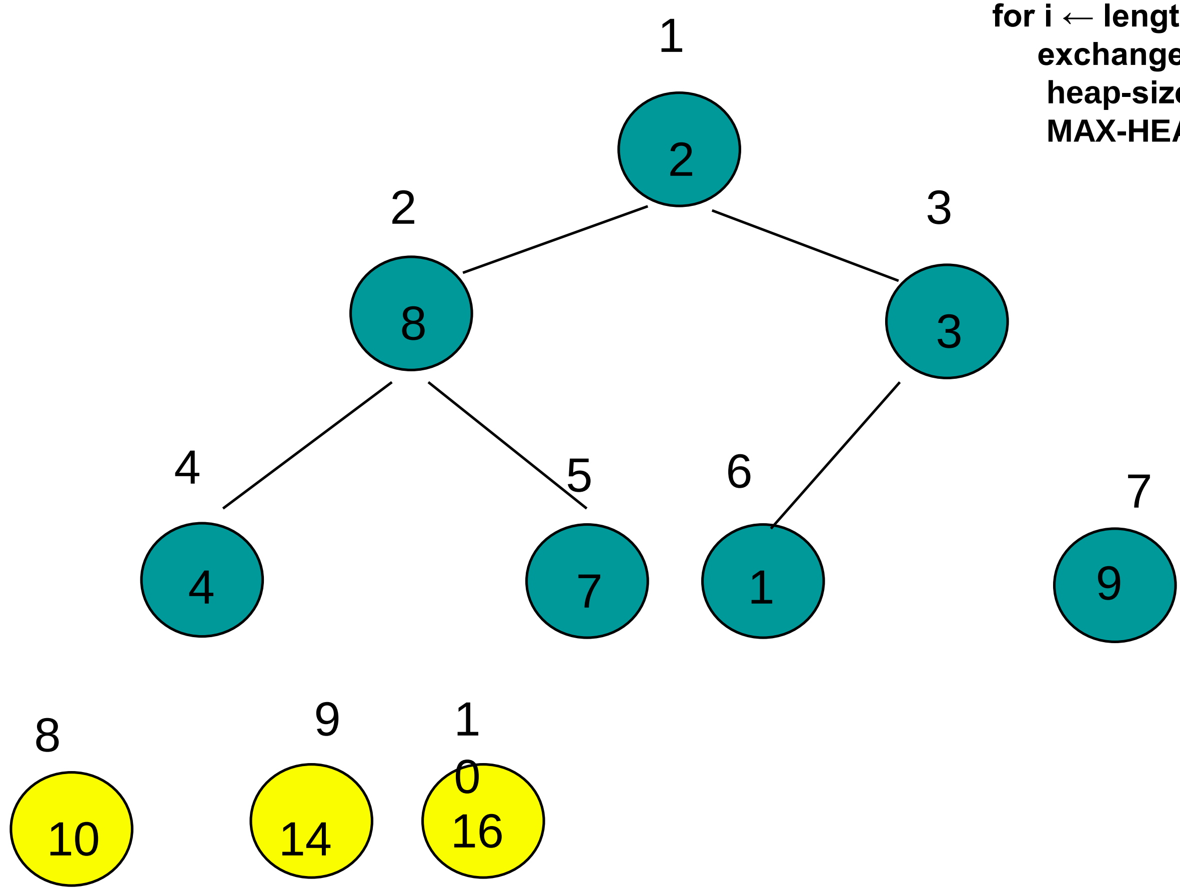

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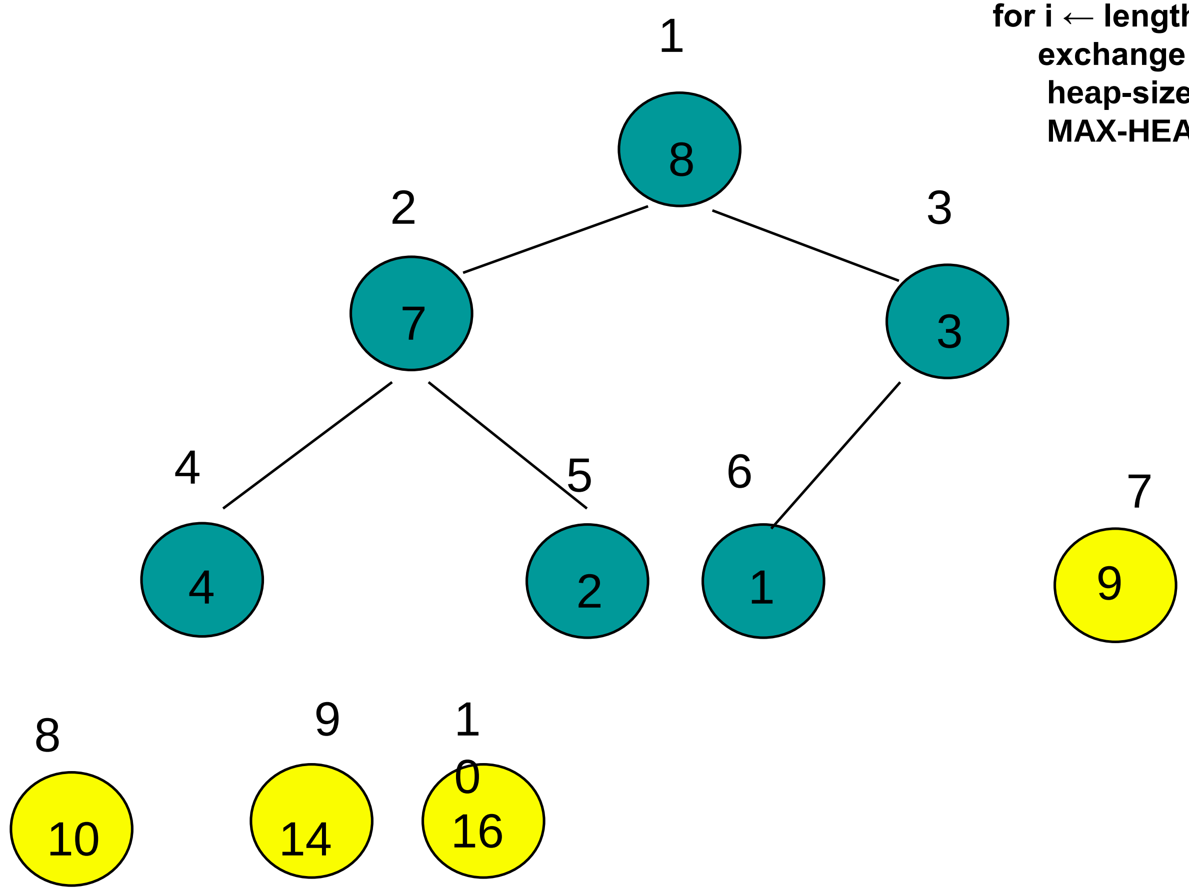

8	7	3	4	2	1	9	10	14	16
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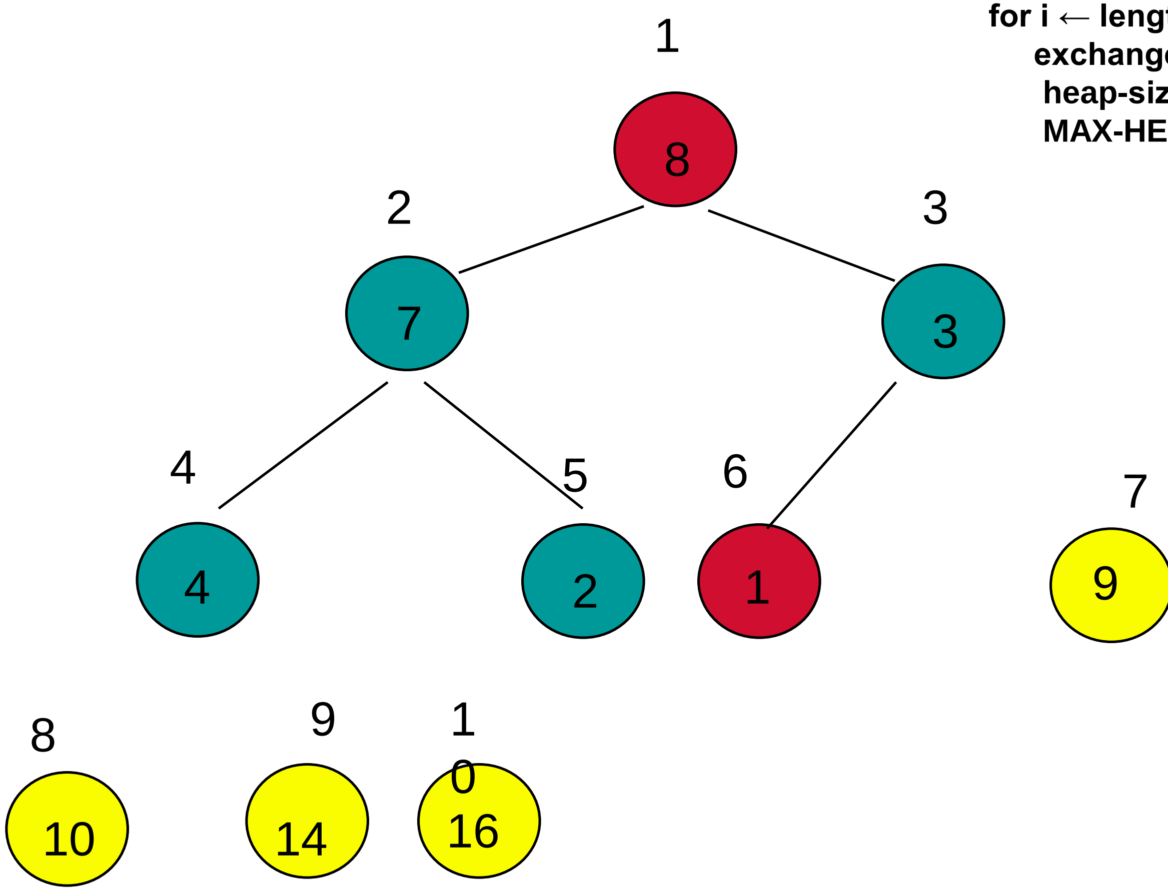

8	7	3	4	2	1	9	10	14	16
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```
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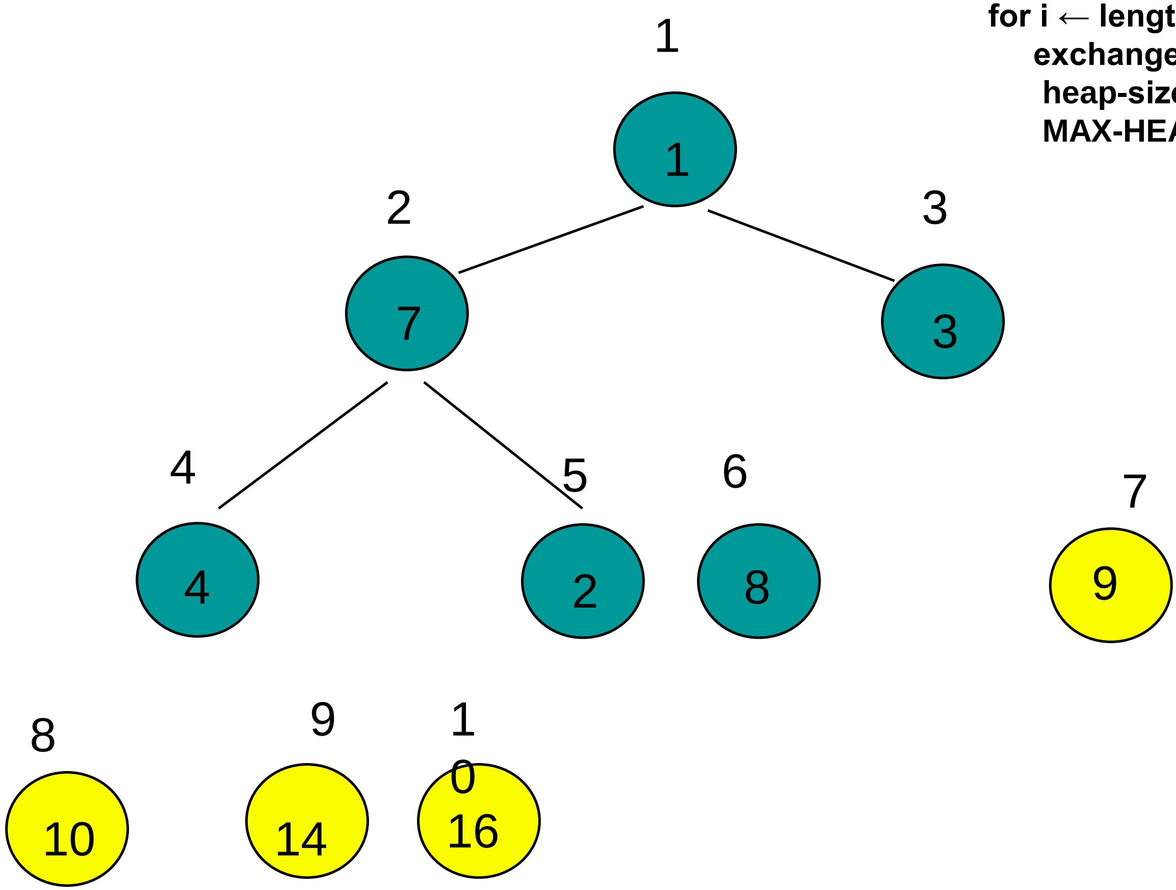

1	7	3	4	2	8	9	10	14	16
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BUILD-MAX-HEAP(A)
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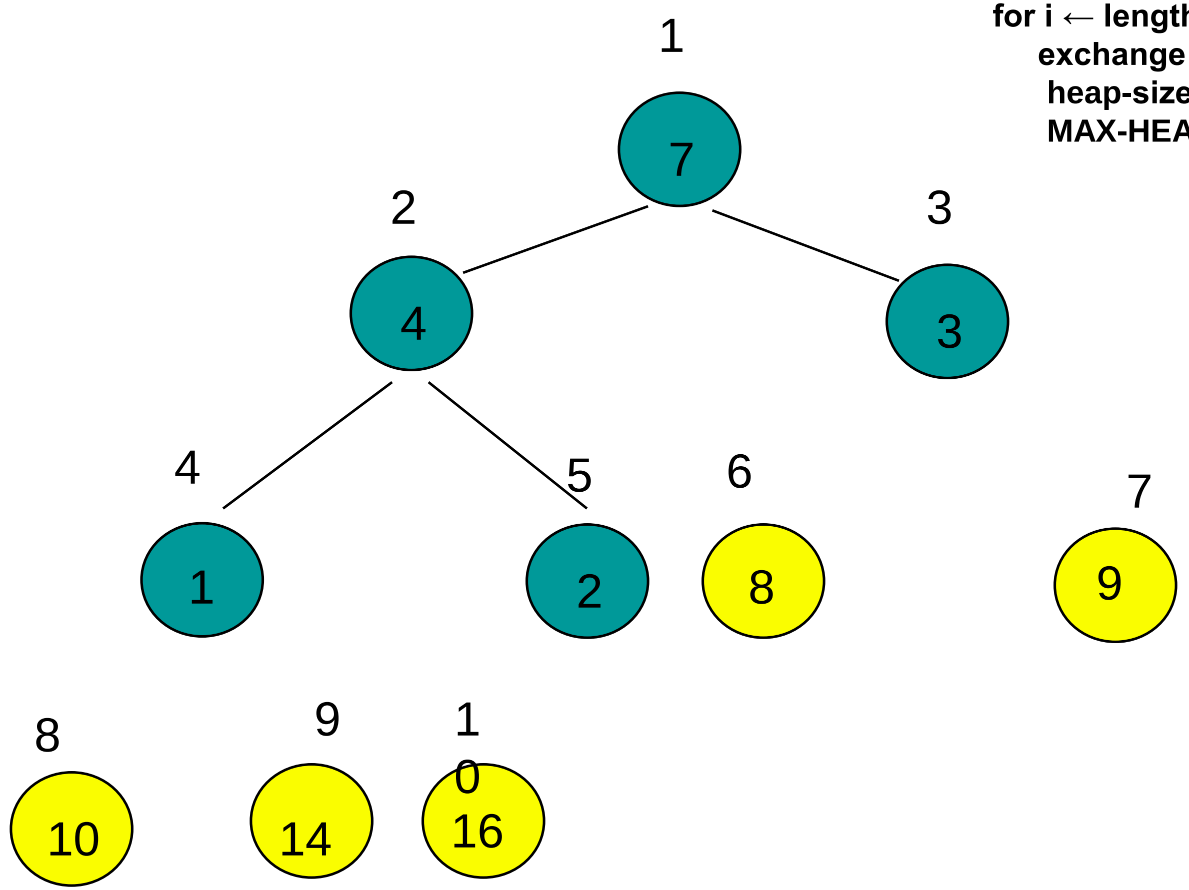

7	4	3	1	2	8	9	10	14	16
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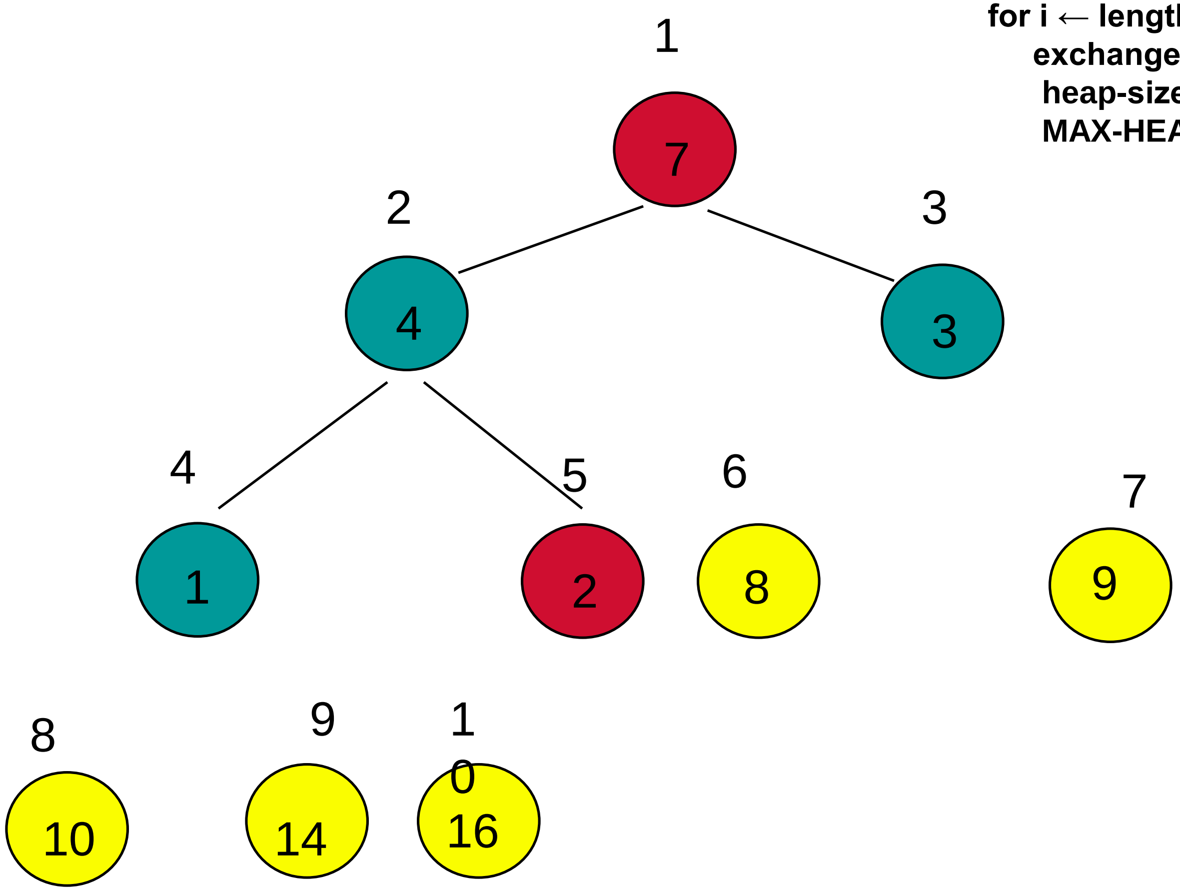

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     MAX-HEAPIFY(A, 1)
```
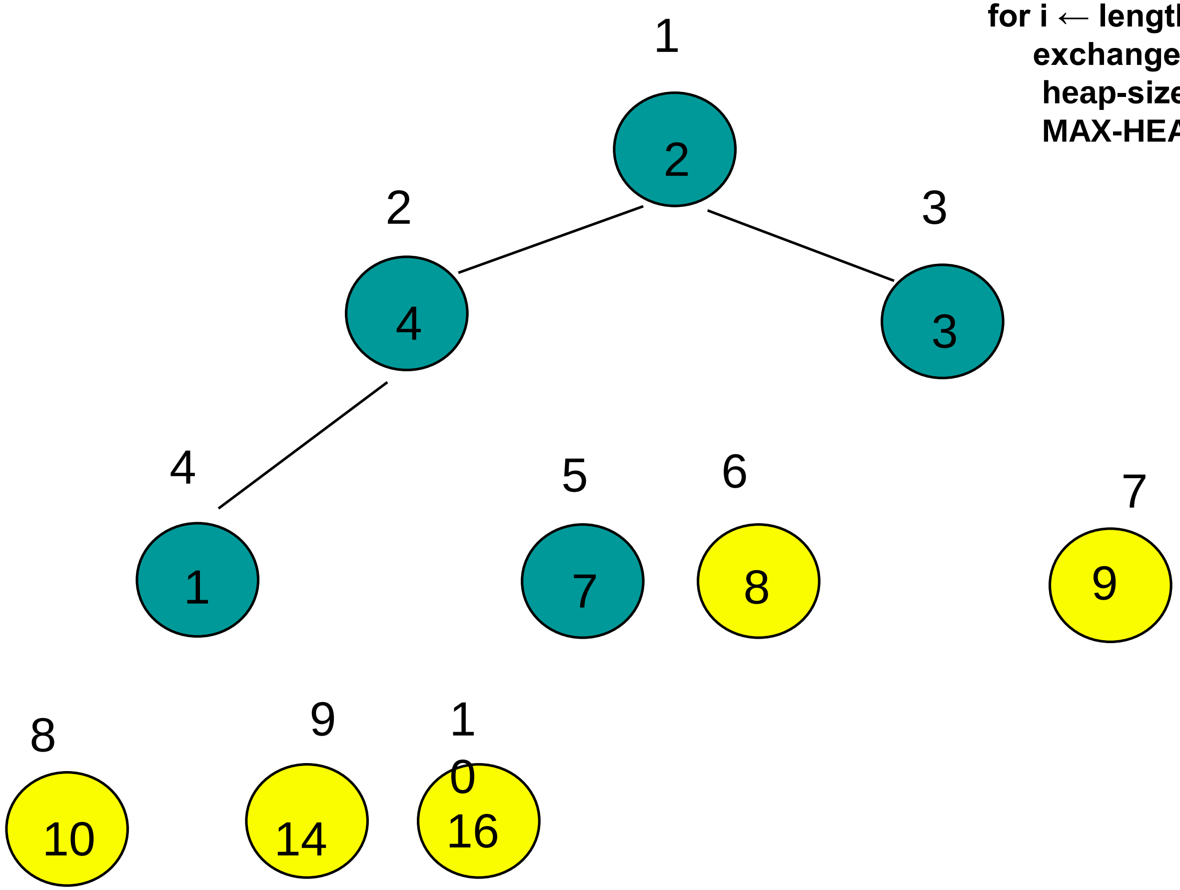
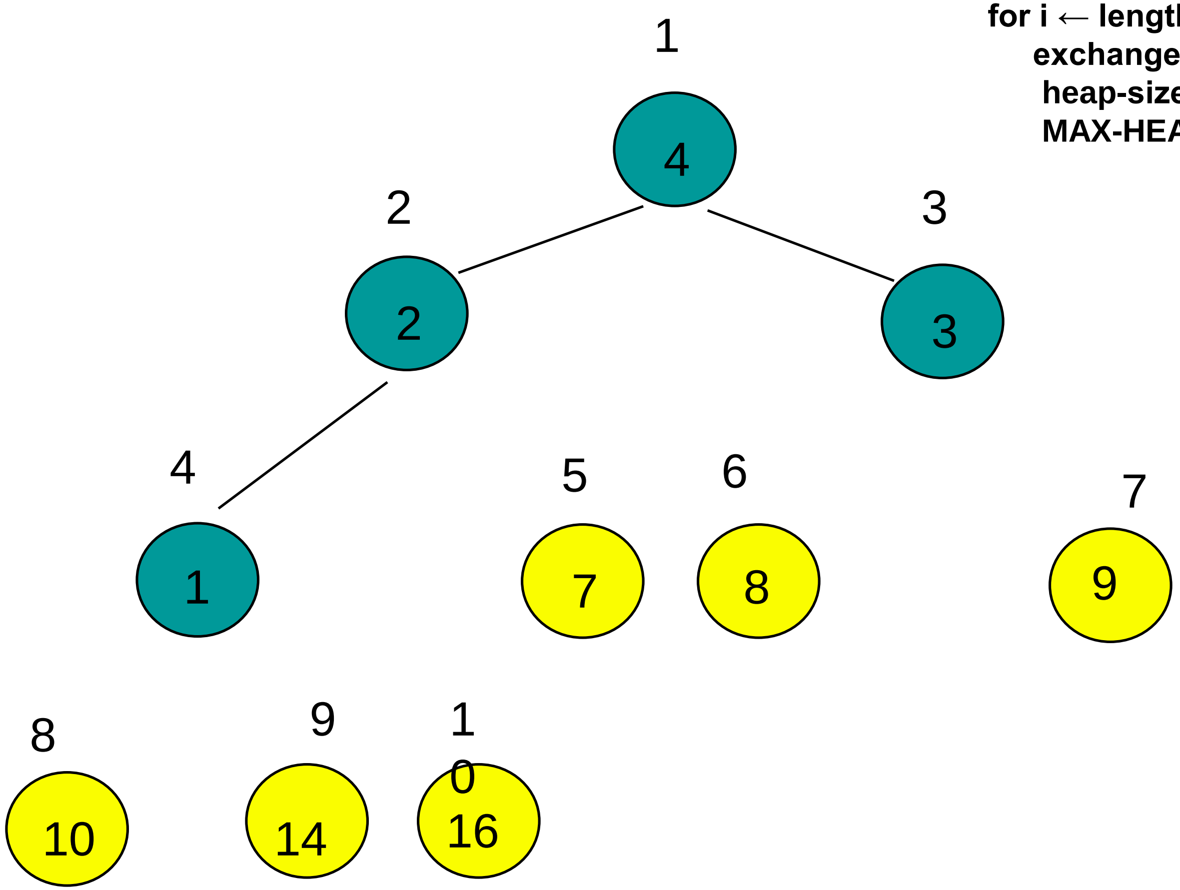



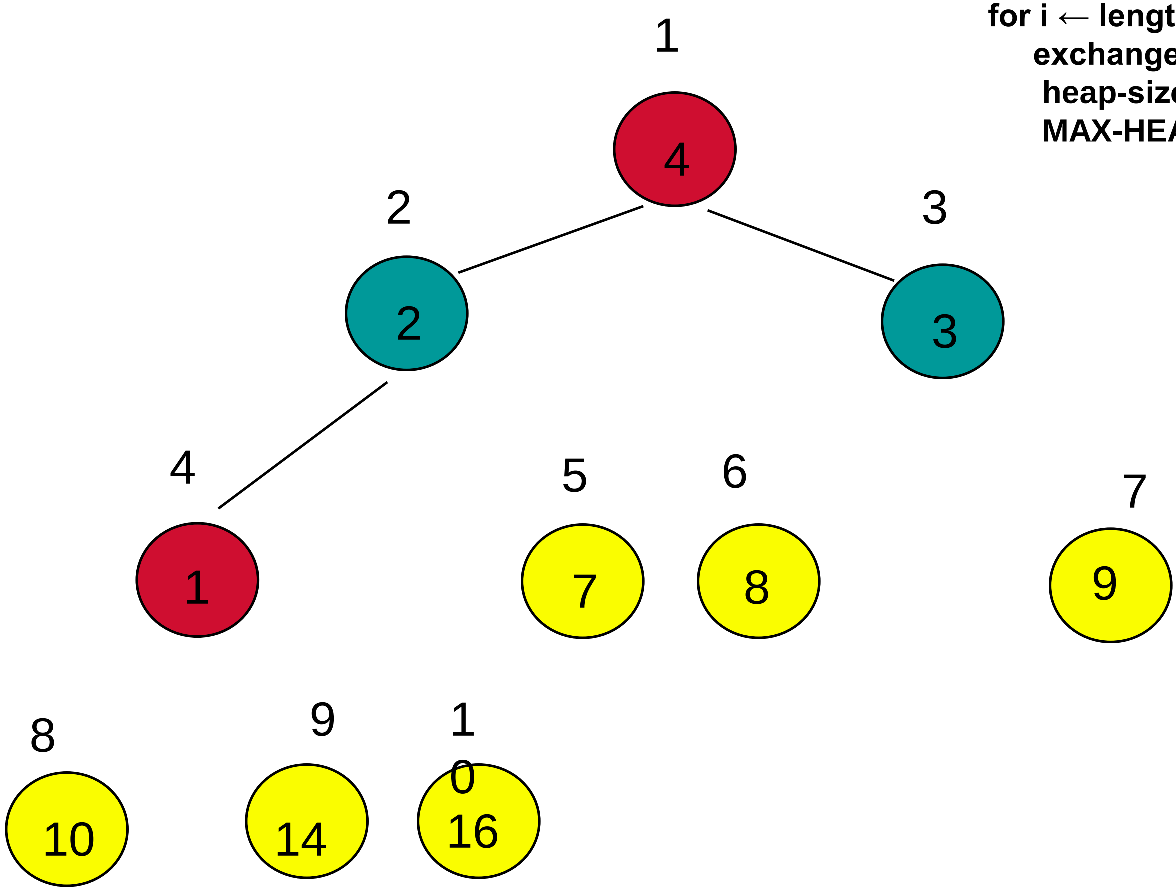
2	4	3	1	7	8	9	10	14	16
1	2	3	4	5	6	7	8	9	10

BUILD-MAX-HEAP(A)
for i ← length[A] downto 2 do
exchange A[1] ↔ A[i]
heap-size[A] ← heap-size[A] – 1
MAX-HEAPIFY(A, 1)



4	2	3	1	7	8	9	10	14	16
1	2	3	4	5	6	7	8	9	10

```
BUILD-MAX-HEAP(A)
for i ← length[A] downto 2 do
    exchange A[1] ↔ A[i]
    heap-size[A] ← heap-size[A] – 1
    MAX-HEAPIFY(A, 1)
```



1	2	3	4	5	6	7	8	9	10
4	2	3	1	7	8	9	10	14	16

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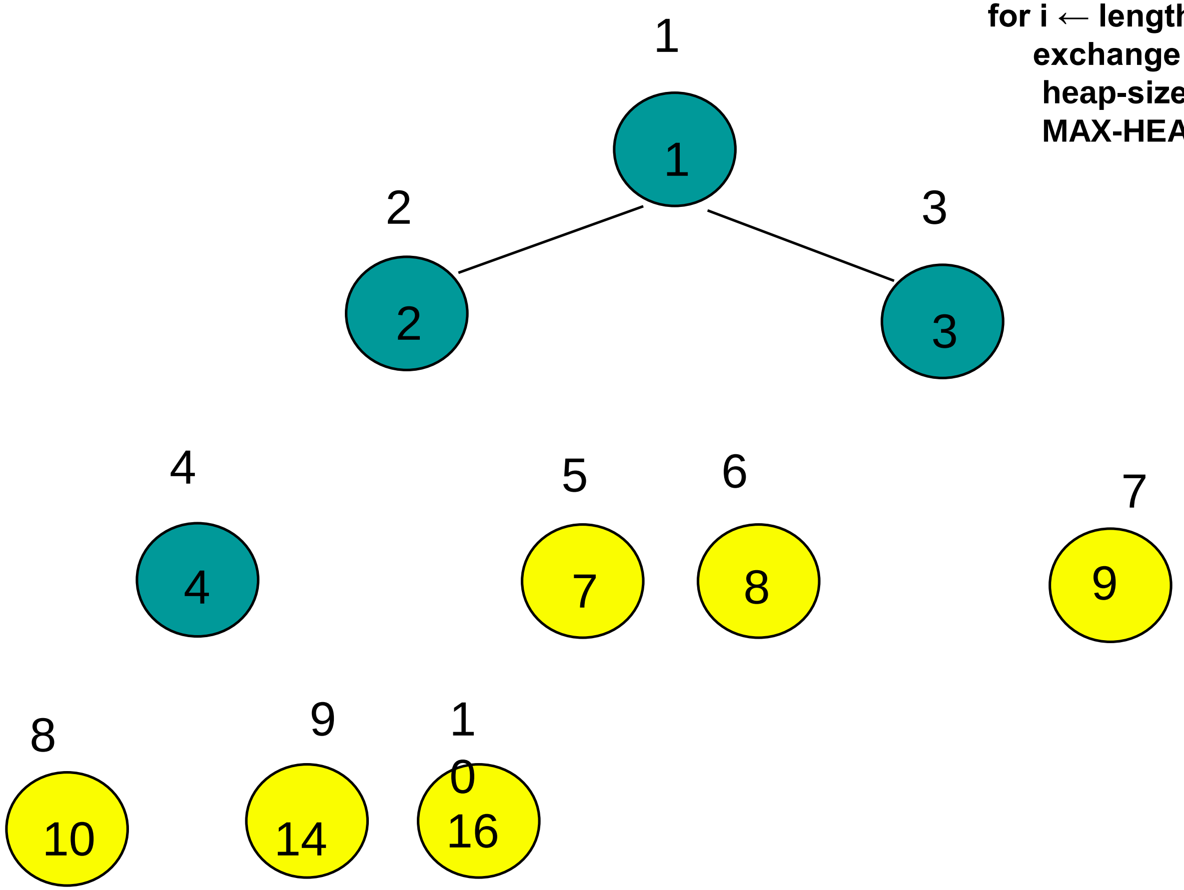
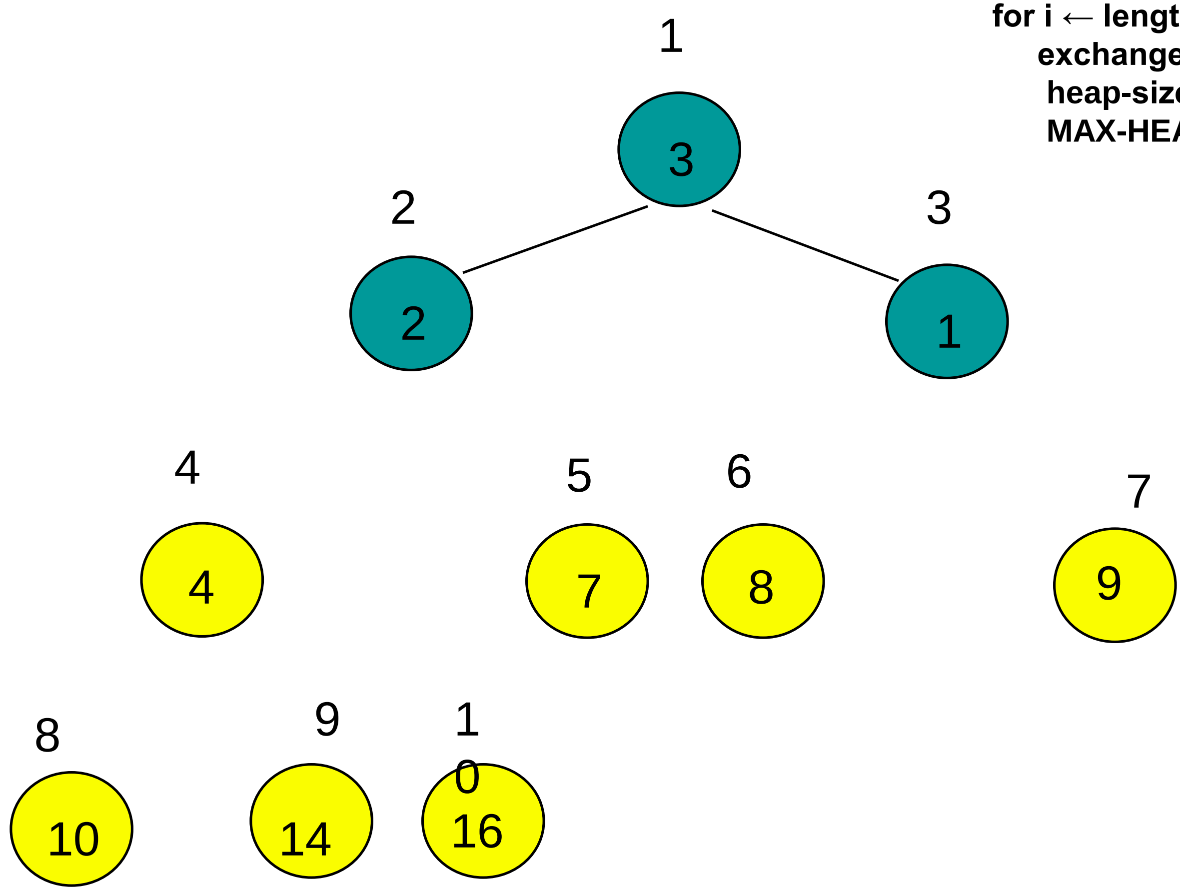


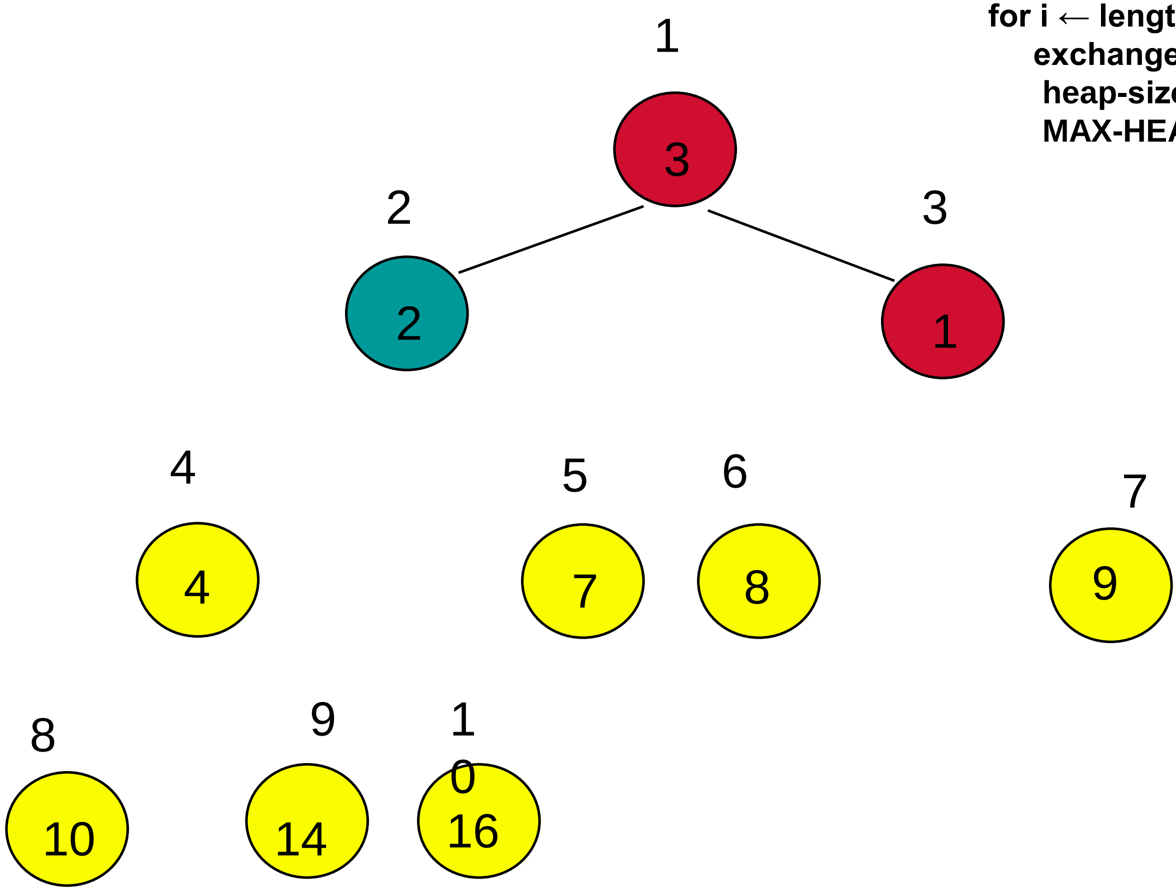
1	2	3	4	7	8	9	10	14	16
1	2	3	4	5	6	7	8	9	10

BUILD-MAX-HEAP(A)
for i ← length[A] downto 2 do
exchange A[1] ↔ A[i]
heap-size[A] ← heap-size[A] – 1
MAX-HEAPIFY(A, 1)



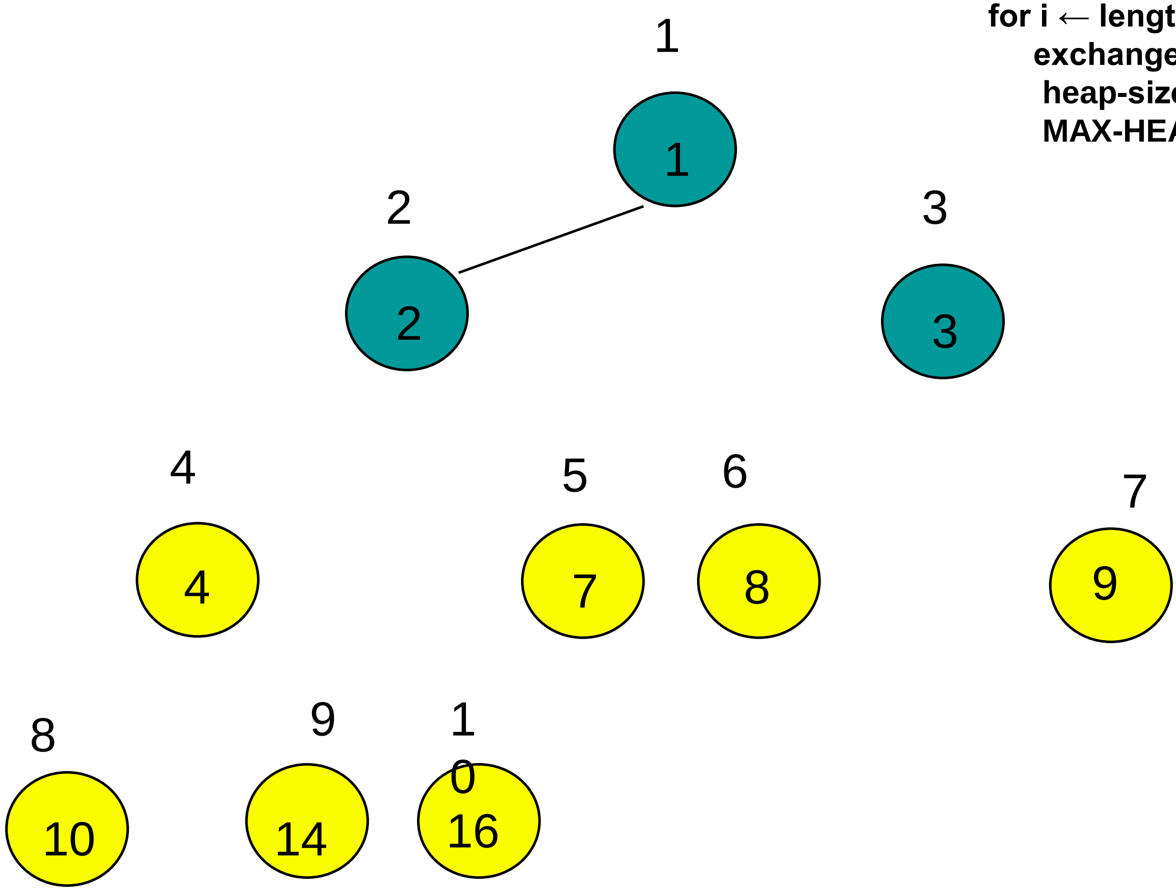
3	2	1	4	7	8	9	10	14	16
1	2	3	4	5	6	7	8	9	10

BUILD-MAX-HEAP(A)
for i ← length[A] downto 2 do
exchange A[1] ↔ A[i]
heap-size[A] ← heap-size[A] – 1
MAX-HEAPIFY(A, 1)



3	2	1	4	7	8	9	10	14	16
1	2	3	4	5	6	7	8	9	10

BUILD-MAX-HEAP(A)
for i ← length[A] downto 2 do
    exchange A[1] ↔ A[i]
    heap-size[A] ← heap-size[A] – 1
    MAX-HEAPIFY(A, 1)



1	2	3	4	7	8	9	10	14	16
1	2	3	4	5	6	7	8	9	10

```
BUILD-MAX-HEAP(A)
for i ← length[A] downto 2 do
    exchange A[1] ↔ A[i]
     heap-size[A] ← heap-size[A] – 1
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```

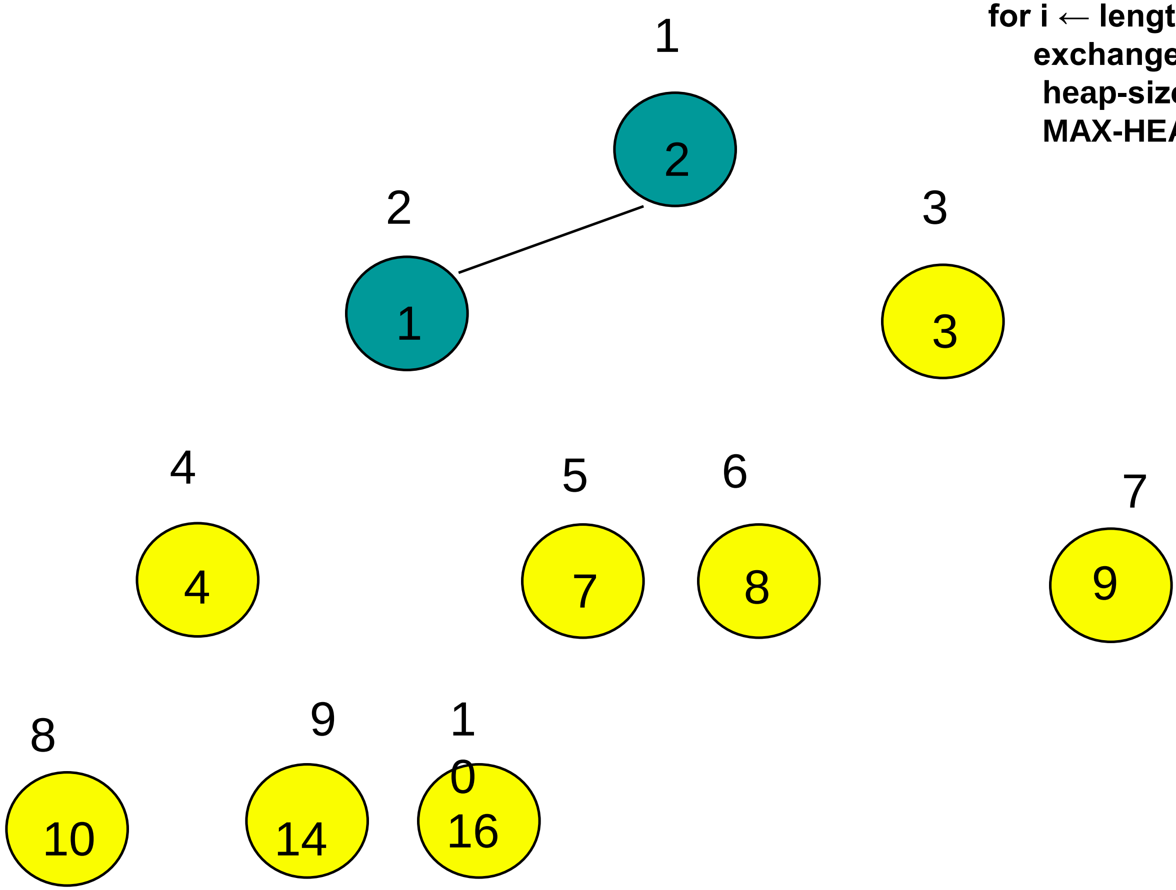

2	1	3	4	7	8	9	10	14	16
1	2	3	4	5	6	7	8	9	10

```
BUILD-MAX-HEAP(A)
for i ← length[A] downto 2 do
    exchange A[1] ↔ A[i]
     heap-size[A] ← heap-size[A] – 1
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```
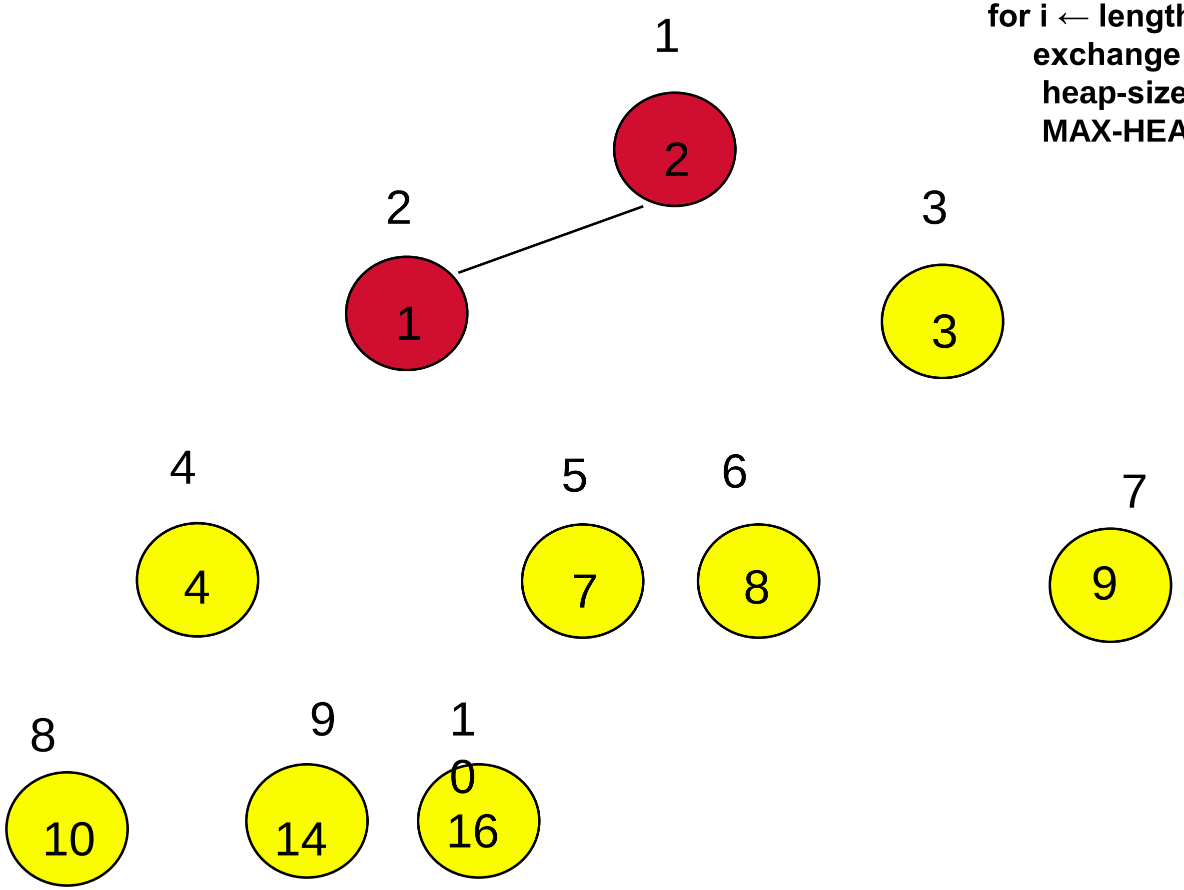
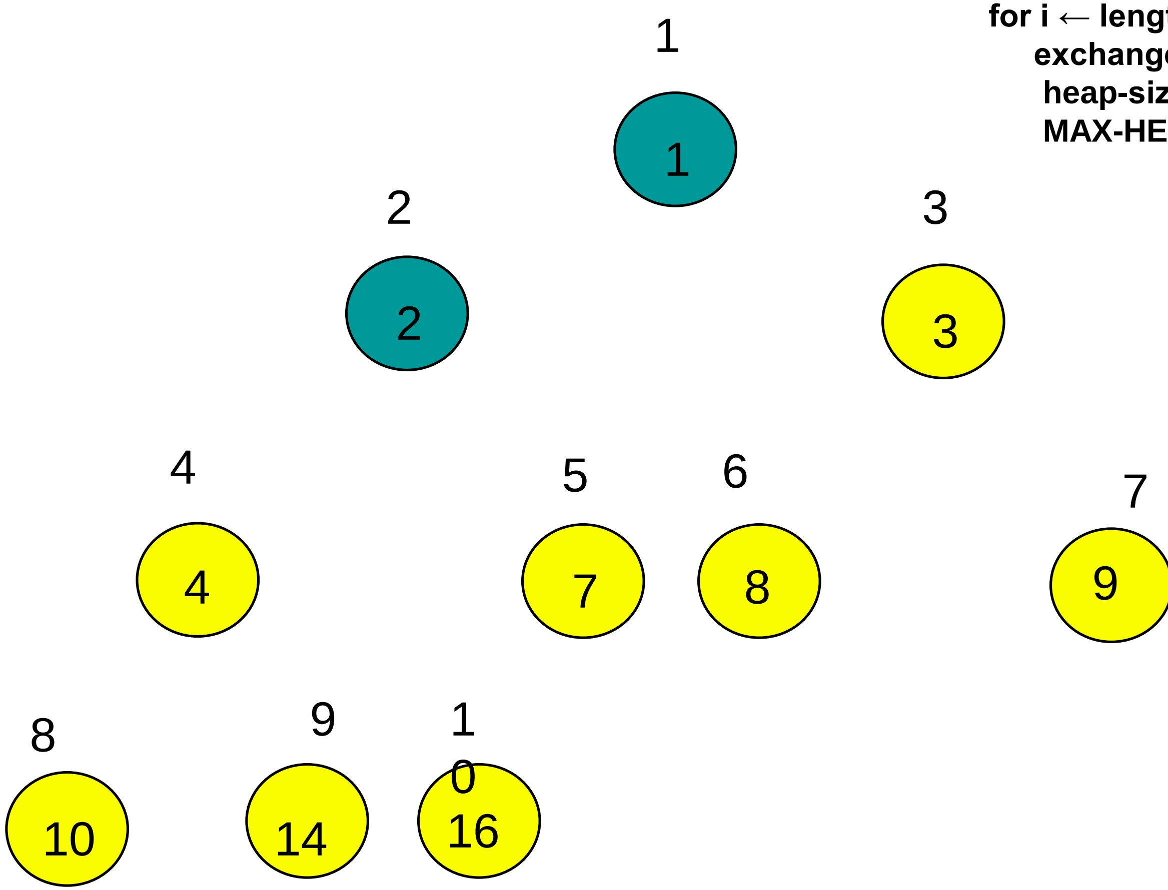



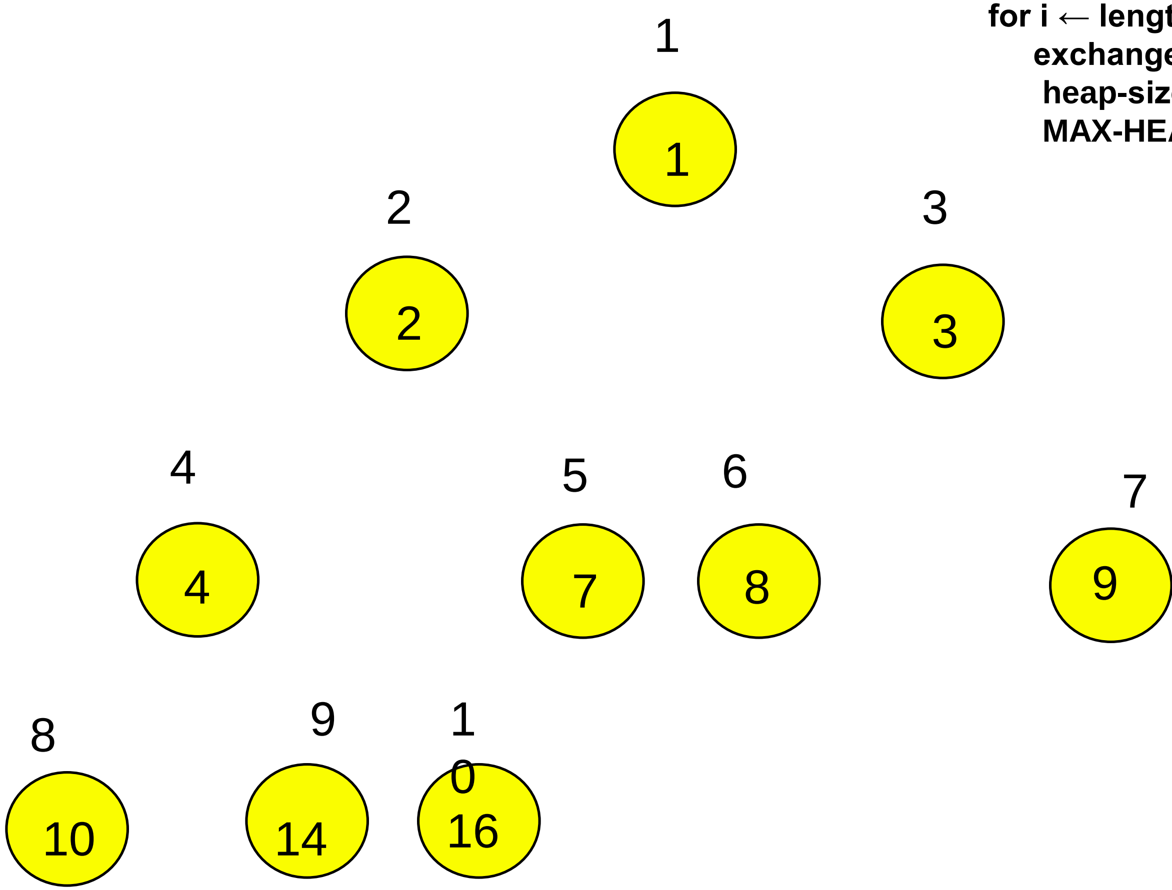
2	1	3	4	7	8	9	10	14	16
1	2	3	4	5	6	7	8	9	10

BUILD-MAX-HEAP(A)
for i ← length[A] downto 2 do
exchange A[1] ↔ A[i]
heap-size[A] ← heap-size[A] – 1
MAX-HEAPIFY(A, 1)



1	2	3	4	7	8	9	10	14	16
1	2	3	4	5	6	7	8	9	10

BUILD-MAX-HEAP(A)
for i ← length[A] downto 2 do
exchange A[1] ↔ A[i]
heap-size[A] ← heap-size[A] – 1
MAX-HEAPIFY(A, 1)



1	2	3	4	7	8	9	10	14	16
1	2	3	4	5	6	7	8	9	10

Running time of Heapsort

```
HEAPSORT(A)
BUILD-MAX-HEAP(A)
for i ← length[A] downto 2 do
  exchange A[1] ↔ A[i]
  heap-size[A] ← heap-size[A] – 1
  MAX-HEAPIFY(A, 1)
```

Is there a loop? If so, how many times will it execute? What is the cost of one iteration of the loop?

Running time of Heapsort

```
HEAPSORT(A)
1   BUILD-MAX-HEAP(A)                         O(n)
2   for i ← length[A] downto 2 do             O(n-1)
3    exchange A[1] ↔ A[i]                     O(1)
4    heap-size[A] ← heap-size[A] – 1          O(1)
5    MAX-HEAPIFY(A, 1)                        O(lg n)
```

Total time is:

$$O(n) + O(n\text{-}1) * [\ O(1) + O(1) + O(\lg n)\ ]$$

which is approximately

$$O(n) + O(n \lg n)$$

or just $O(n \lg n)$

Running time of Heapsort

- BUILD-MAX-HEAP takes O(n).
- We have a loop. Each of the n-1 calls to MAX-HEAPIFY takes O(lg n) time.
- Total time is O(n lg n).
- Will heap sort always take O(n lg n) time? Is there a best-case scenario? Is there a worst-case scenario? Why or why not?

Space requirements of Heapsort

- Heapsort uses an array as its data structure.
- Heapsort sorts “in place”.
- Any extra storage needed?
- Only a negligible amount – one extra storage location is needed as temporary storage when swapping two array elements.

Priority Queues

- A priority queue is a data structure for maintaining a set S of elements, each with an associated value called a key.
- Applications include
  - scheduling jobs on a shared computer (max-priority queue)
  - event-driven simulators (min-priority queue)

Handles

- Elements of priority queue correspond to objects in application.
- We must be able to determine which application object corresponds to a given priority-queue element.
- We store a **handle** (pointer, integer, etc.) to the corresponding application object in each heap element.
- We also store a **handle** (array index) to the corresponding heap element in each application object.

Max-Priority Queue Operations

- INSERT(S,x): insert element x into set S
- MAXIMUM(S): return element of S with the largest key
- EXTRACT-MAX(S): remove and return element of S with the largest key
- INCREASE-KEY(S, x, k): increase value of x's key to k, where k is at least as large as x's current key value

Min-Priority Queue Operations

- INSERT(S,x): insert element x into set S
- MINIMUM(S): return element of S with the smallest key
- EXTRACT-MIN(S): remove and return element of S with the smallest key
- DECREASE-KEY(S, x, k): decrease value of x’s key to k, where k is at least as small as x’s current key value

Priority Queue Operations

- All operations can be done on a set of size n in O(lg n) time

HEAP-MAXIMUM

```
HEAP-MAXIMUM(A)
1  return A[1]
```

- Returns the item at the top of the heap
- Runs in $\Theta(1)$ time

HEAP-EXTRACT-MAX

```
HEAP-EXTRACT-MAX(A)
if heap-size[A] < 1
    then error “heap underflow”
max ← A[1]
A[1] ← A[heap-size[A]]
heap-size[A] ← heap-size[A] - 1
MAX-HEAPIFY(A,1)
return max
```

Running time of HEAP-EXTRACT-MAX

```
HEAP-EXTRACT-MAX(A)
if heap-size[A] < 1                               O(1)
    then error “heap underflow”                   O(1)
max ← A[1]                                        O(1)
A[1] ← A[heap-size[A]]                            O(1)
heap-size[A] ← heap-size[A] – 1                   O(1)
MAX-HEAPIFY(A,1)                                  O(lg n)
return max                                        O(1)
```

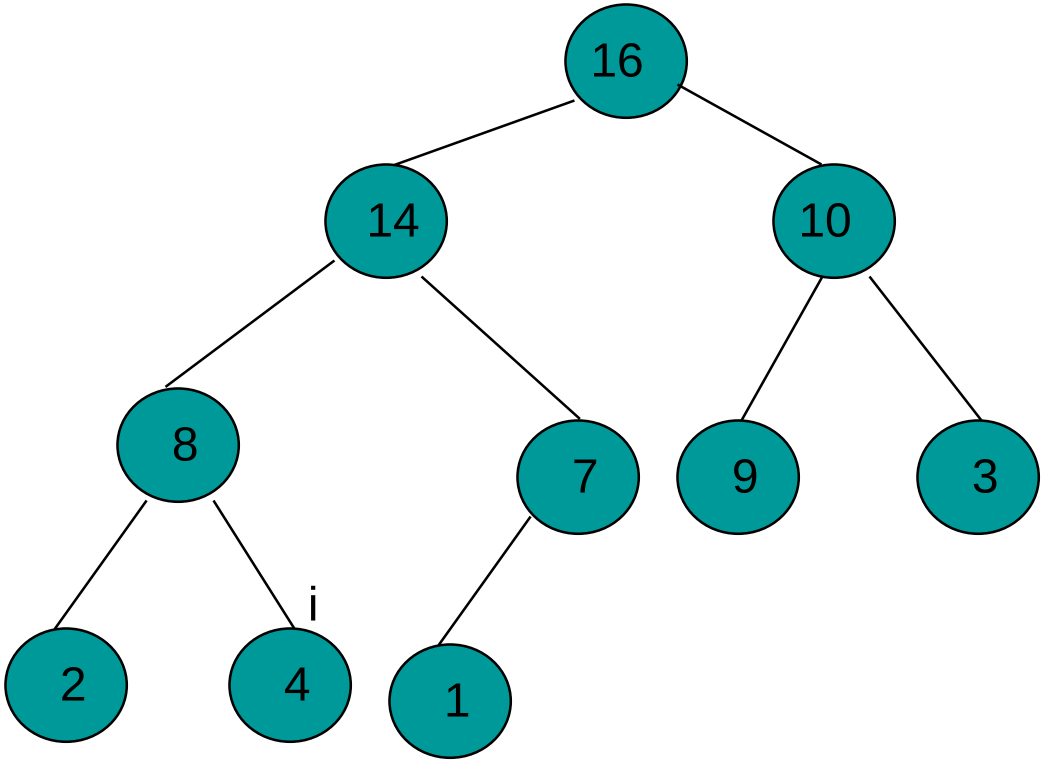
Any loops? No. So just sum up the times: O(6) + O(lg *n*)

The dominant term is O(lg *n*).

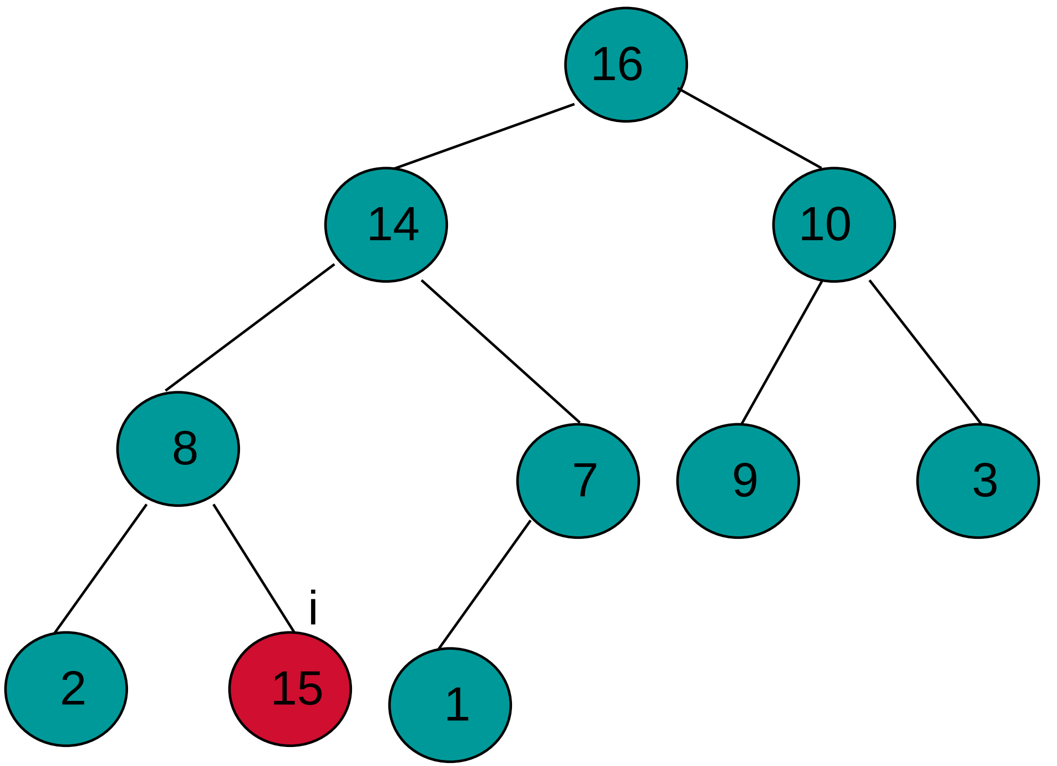
HEAP-INCREASE-KEY

```
HEAP-INCREASE-KEY(A, i, key)
if key < A[i]
    then error “new key is smaller
      than current key”
A[i] ← key
while i > 1 and A[PARENT(i)] < A[i] do
      exchange A[i] ↔ A[PARENT(i)]
      i ← PARENT(i)
```

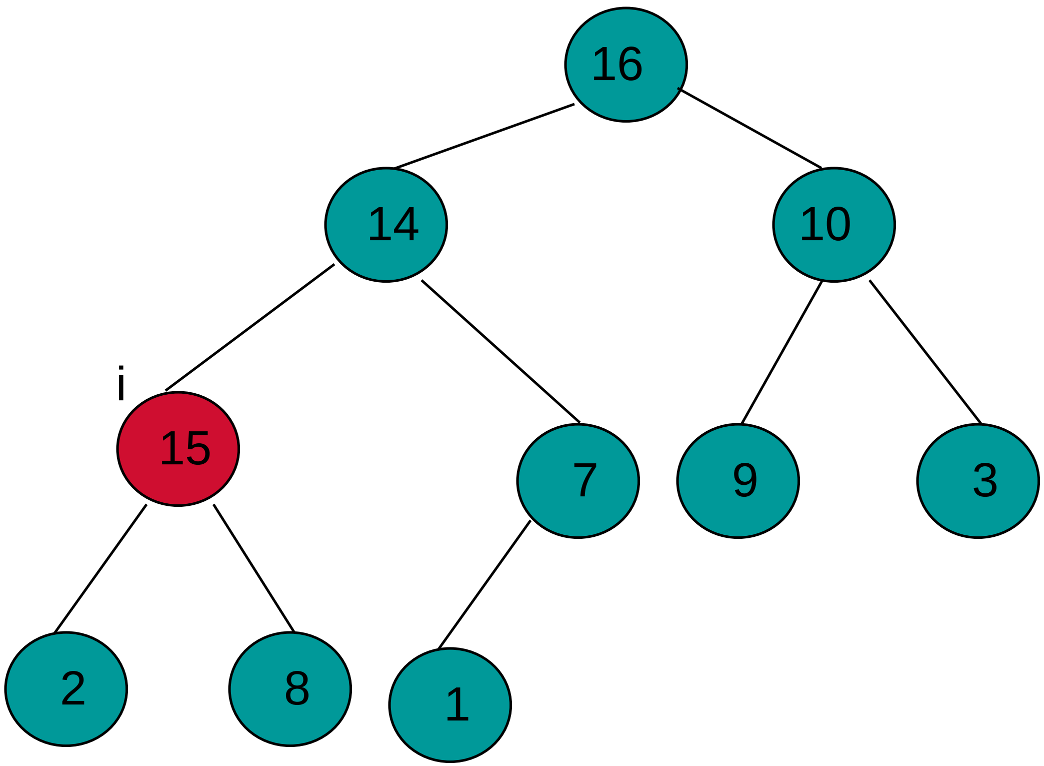
Example of HEAP-INCREASE-KEY



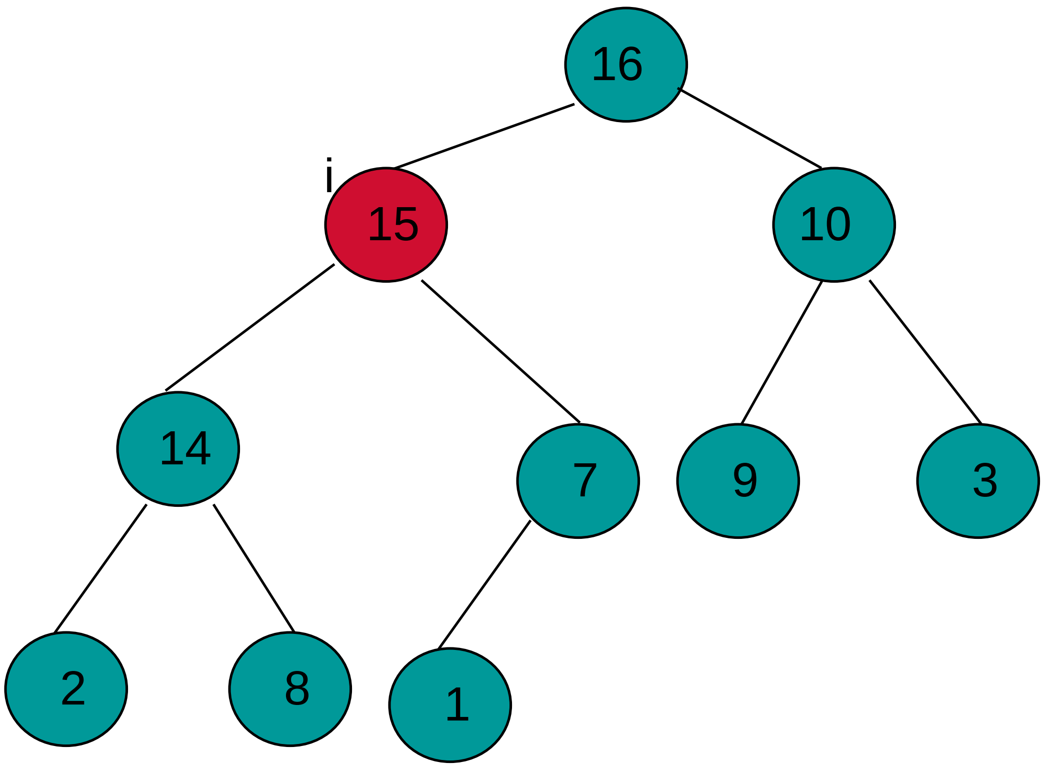
Example of HEAP-INCREASE-KEY (continued)



Example of HEAP-INCREASE-KEY (continued)



Example of HEAP-INCREASE-KEY (continued)



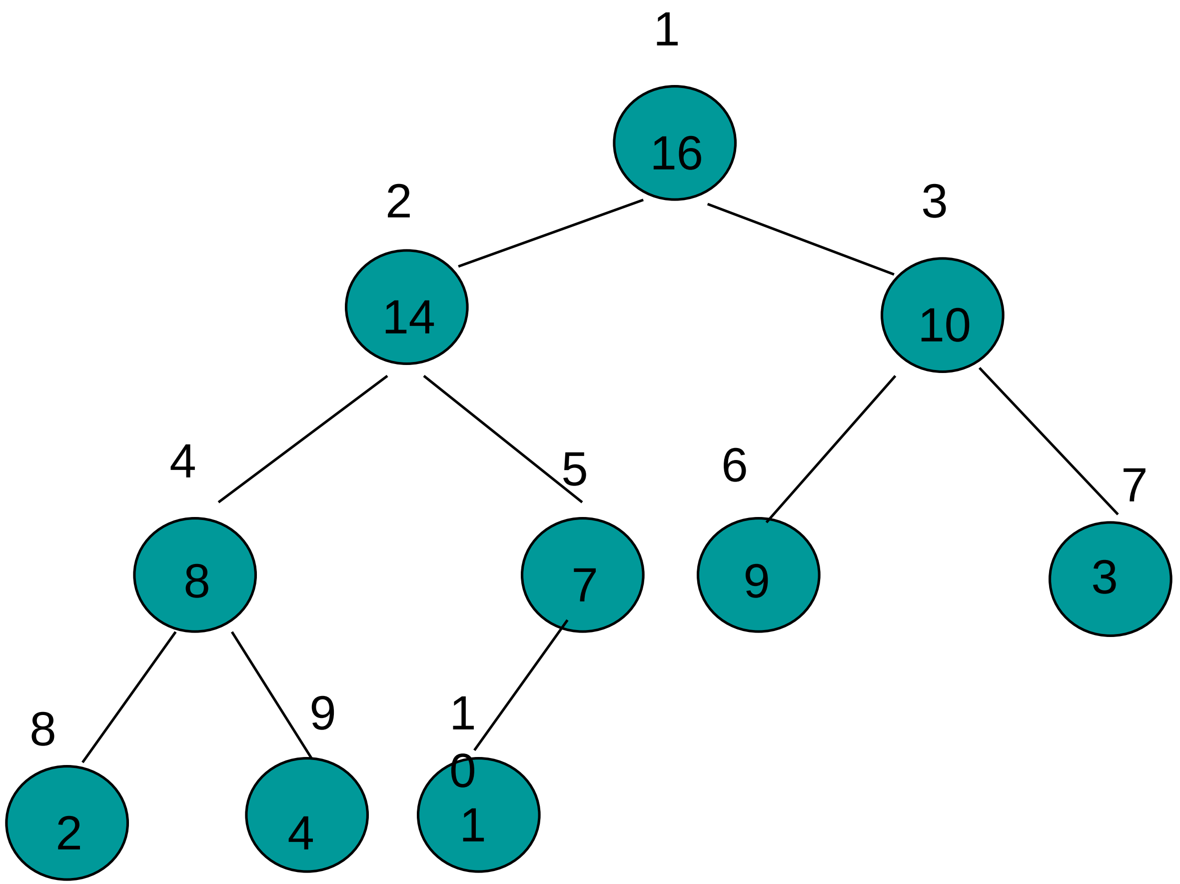
Running time of HEAP-INCREASE-KEY

```
HEAP-INCREASE-KEY(A, i, key)
if key < A[i]                                       O(1)
   then error                                       O(1)
       "new key is smaller than current key"
A[i] ← key                                          O(1)
while i > 1 and A[PARENT(i)] < A[i] do              O(lg n)
     exchange A[i] ↔ A[PARENT(i)]                   O(3)
     i ← PARENT(i)                                  O(1)
```

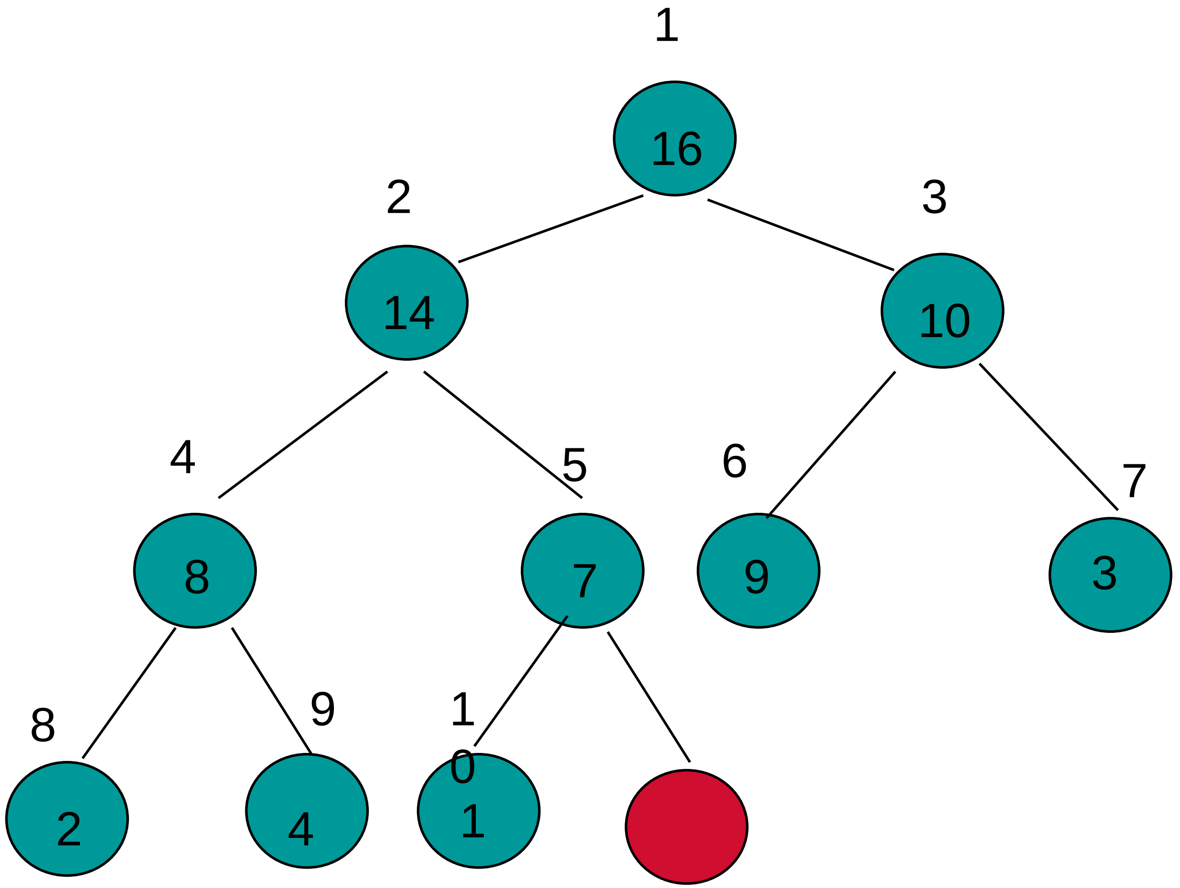
Any loops? Yes. How many times will the loop execute? As many times as node i has ancestors, which = the depth of the tree. The depth of a binary tree is O(lg n). We do a constant amount of work in the loop. Cost is: O(3) + O(4 lg n), or just O(lg n)

MAX-HEAP-INSERT

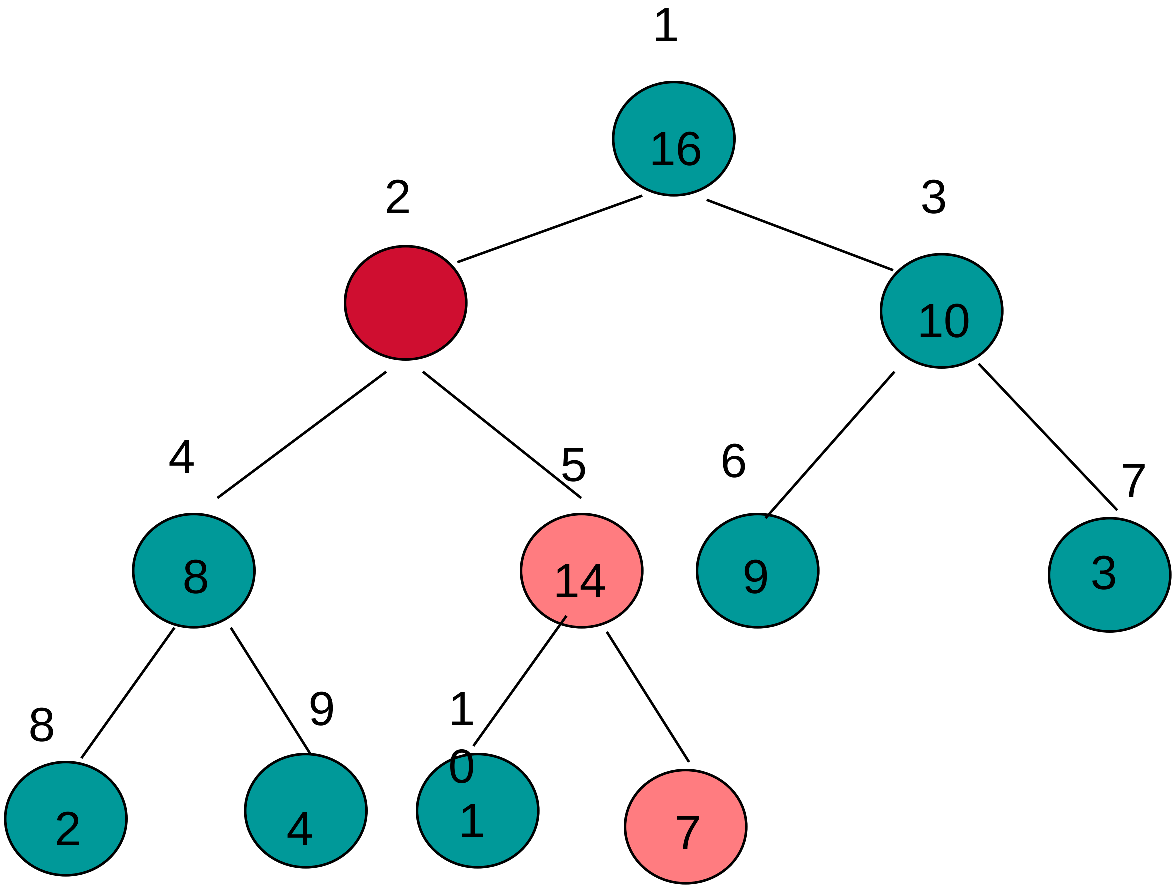
```
MAX-HEAP-INSERT(A,key)
heap-size[A] ← heap-size[A] + 1
A[heap-size] ← -∞
HEAP-INCREASE-KEY(A, heap-size[A], key)
```



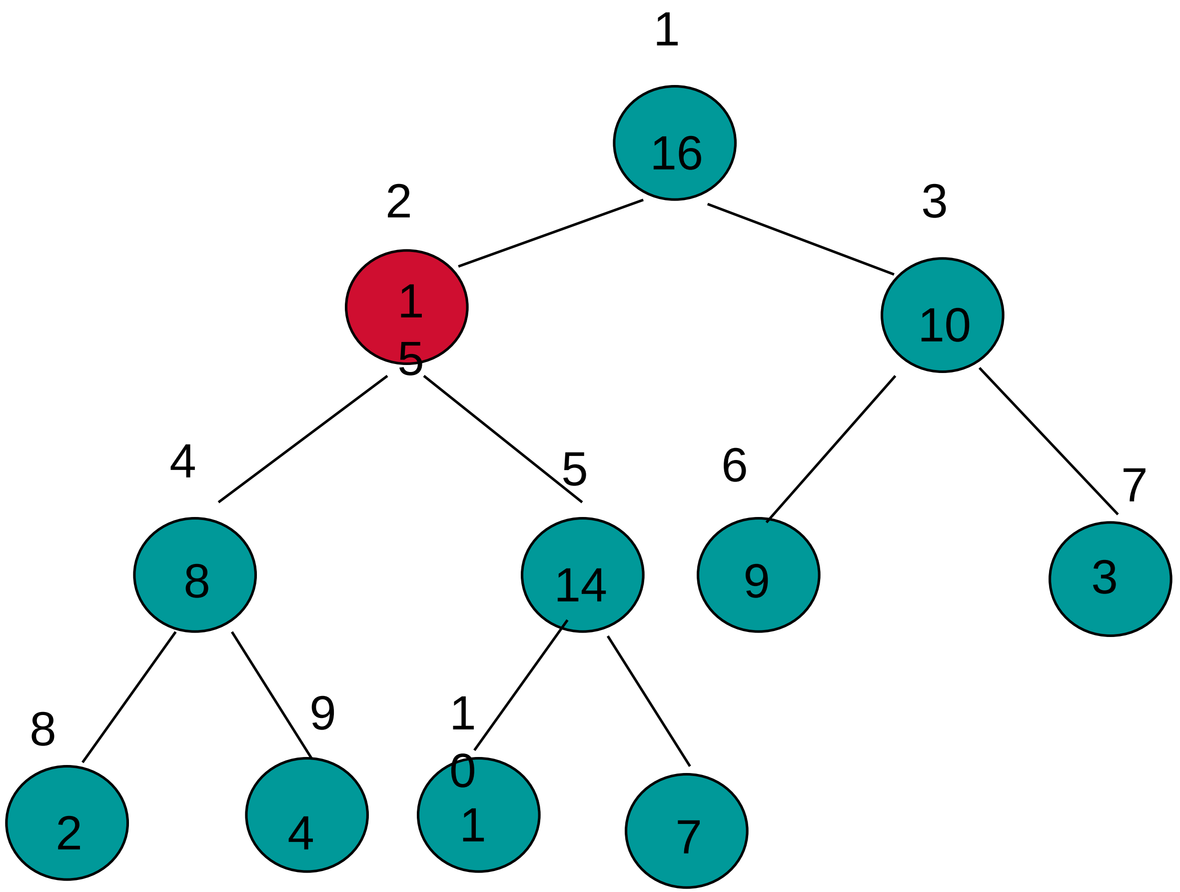
MAX-HEAP-INSERT(A,15)



HEAP-INSERT(A,15)



MAX-HEAP-INSERT(A,15)



MAX-HEAP-INSERT(A,15)

MAX-HEAP-INSERT

```
MAX-HEAP-INSERT(A,key)
heap-size[A] ← heap-size[A] + 1                   O(1)
A[heap-size] ← -∞                                 O(1)
HEAP-INCREASE-KEY(A, heap-size[A], key)           O(lg n)
```

Any loops? No.

Add up the times: O(1) + O(1) + O(lg n) = O(2) + O(lg *n*)

Dominant term is O(lg *n*), so running time is just O(lg *n*).

Conclusion

We have seen:

- what a heap is
- how to build a heap
- how to use a heap for sorting
- how to analyze heapsort’s running time
- how to use a heap for priority queues