CSE 303: Introduction to the Theory of Computation

(Finite Automata)

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Deterministic Finite Automata (DFA)

Electric bulb

Problem

- Design the logic behind an electric bulb.

Electric bulb

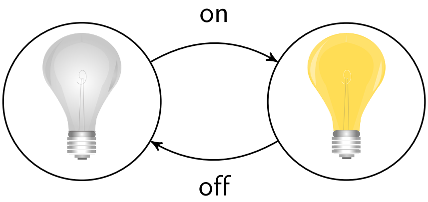
Problem

- Design the logic behind an electric bulb.

Solution

- Diagram.

- Analysis.
  States $= \{\text{nolight}, \text{light}\}$, Input $= \{\text{off}, \text{on}\}$
- Finite Automaton.



Multispeed fan

Problem

- Design the logic behind a multispeed fan.

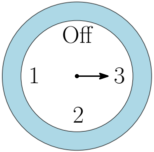
Multispeed fan

Problem

- Design the logic behind a multispeed fan.

Solution

- Diagram.

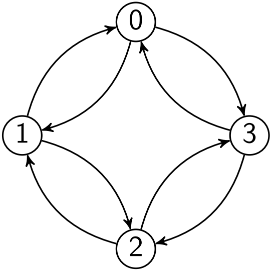


- Analysis.

  States $= \{0, 1, 2, 3\}$

  Input $= \{\circlearrowright, \circlearrowleft\}$

- Finite Automaton.



Automatic doors

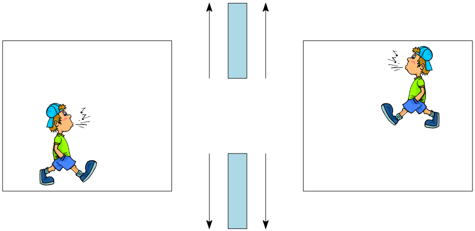
Problem

- Design the logic behind automatic doors in Walmart.

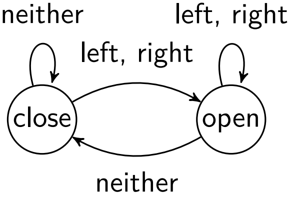
Automatic doors

Solution

- Diagram.



- Analysis.
  States $= \{\text{close}, \text{open}\}$, Input $= \{\text{left}, \text{right}, \text{neither}\}$
- Finite Automaton.



Basic features of finite automata

- A finite automaton is a simple computer with extremely limited memory
- A finite automaton has a finite set of states
- Current state of a finite automaton changes when it reads an input symbol
- A finite automaton acts as a language acceptor i.e., outputs "yes" or "no"

Why should you care?

Deterministic Finite Automata (DFA) are everywhere.

- ATMs
- Ticket machines
- Vending machines
- Traffic signal systems
- Calculators
- Digital watches
- Automatic doors
- Elevators
- Washing machines
- Dishwashing machines
- Thermostats
- Train switches
- (CS) Compilers
- (CS) Search engines
- (CS) Regular expressions

Why should you care?

Probabilistic Finite Automata (PFA) are everywhere, too.

- Speech recognition
- Optical character recognition
- Thermodynamics
- Statistical mechanics
- Chemical reactions
- Information theory
- Queueing theory
- PageRank algorithm
- Statistics
- Reinforcement learning
- Price changes in finance
- Genetics
- Algorithmic music composition
- Bioinformatics
- Probabilistic forecasting

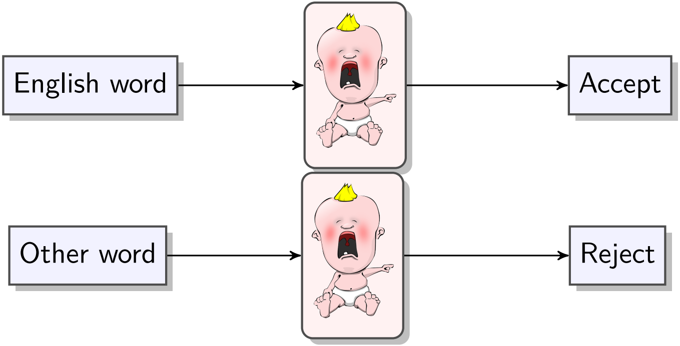
What is a decision problem?

Definition

- A decision problem is a computational problem with a 'yes' or 'no' answer.
- A computer that solves a decision problem is a decider.
  Input to a decider: A string w
  Output of a decider: Accept (w is in the language) or Reject (w is not in the language)

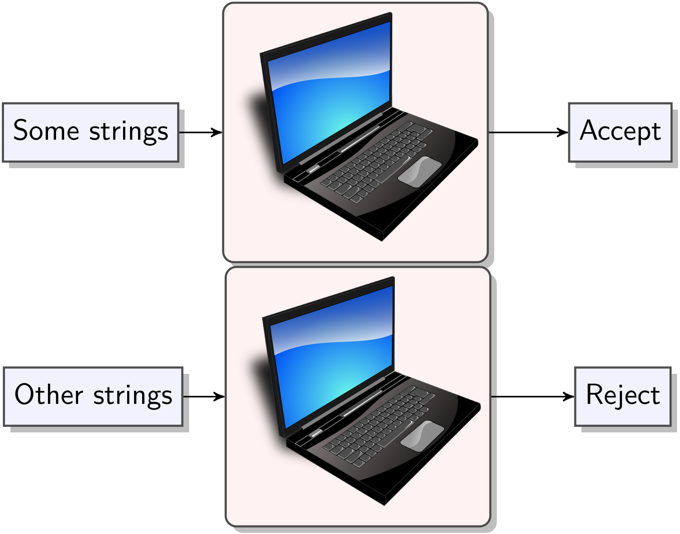
w → Decider → yes/no

What is a decision problem?



- Language = English language = $\{\text{milk}, \text{food}, \text{sleep}, \ldots\}$ $\triangleright$ Accept
- Not in language = $\{\text{zffgb}, \text{cdcapqw}, \ldots\}$ $\triangleright$ Reject

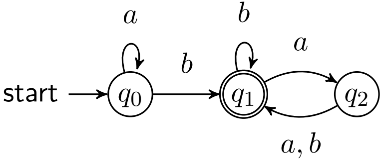
What is a decision problem?



How does a DFA work?

Problem

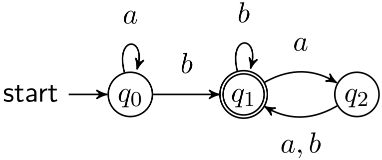
- Does the DFA accept the string $bbab$?



How does a DFA work?

Problem

- Does the DFA accept the string $bbab$?



Solution

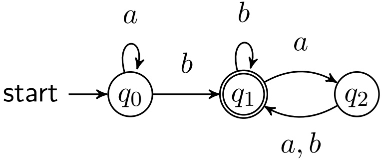
The DFA accepts the string $bbab$. The computation is:

1. Start in state q_0
2. Read b, follow transition from q_0 to q_1.
3. Read b, follow transition from q_1 to q_1.
4. Read a, follow transition from q_1 to q_2.
5. Read b, follow transition from q_2 to q_1.
6. Accept because the DFA is in an accept state q_1 at the end of the input.

How does a DFA work?

Problem

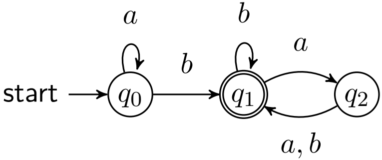
- Does the DFA accept the string $aaba$?



How does a DFA work?

Problem

- Does the DFA accept the string $aaba$?

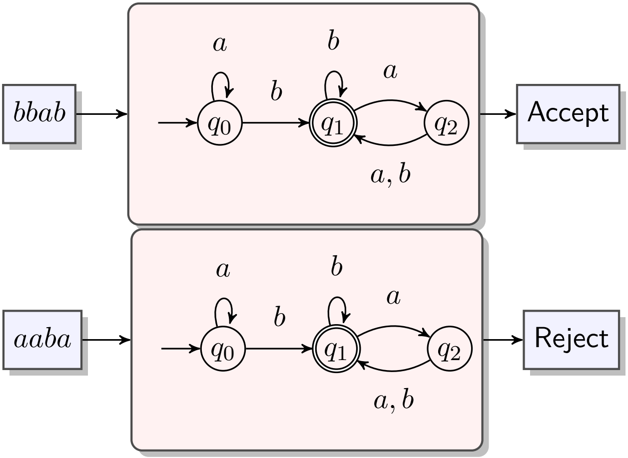


Solution

The DFA rejects the string $aaba$. The computation is:

1. Start in state q_0
2. Read a, follow transition from q_0 to q_0.
3. Read a, follow transition from q_0 to q_0.
4. Read b, follow transition from q_0 to q_1.
5. Read a, follow transition from q_1 to q_2.
6. Reject because the DFA is in a reject state q_2 at the end of the input.

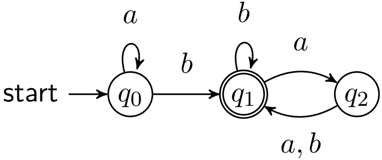
How does a DFA work?



How does a DFA work?

Problem

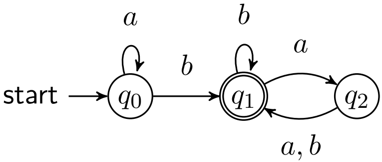
- What language does the DFA accept?



How does a DFA work?

Problem

- What language does the DFA accept?



Examples

- The DFA accepts the following strings:
  $b, ab, bb, aabbbb, abababab, \ldots$ ▷ ends with b
  $baa, abaa, ababaaaaa, \ldots$ ▷ ends with b followed by even a's
- The DFA rejects the following strings:
  $a, ba, babaaa, \ldots$
- What language does the DFA accept?

Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\epsilon, a, aa, aaa, aaaa, \ldots\}$

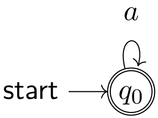
Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\epsilon, a, aa, aaa, aaaa, \dots\}$

Solution

- Language L: $\Sigma^* = \{\epsilon, a, aa, aaa, aaaa, \dots\}$
- Expression: a^*
- Deterministic Finite Automaton (DFA) M:



Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\}$

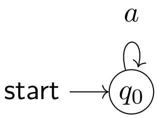
Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\}$

Solution

- Language L: $\phi = \{\}$ $\triangleright$ Empty language
- Expression: ϕ
- DFA M:



Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{a, aa, aaa, aaaa, \ldots\}$

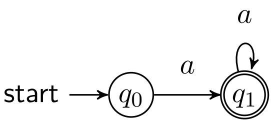
Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{a, aa, aaa, aaaa, \ldots\}$

Solution

- Language L: $\Sigma^* - \{\epsilon\} = \{a, aa, aaa, aaaa, \ldots\}$
- Expression: a^+
- DFA M:



Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\epsilon\}$

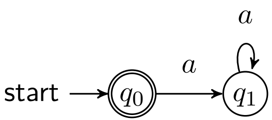
Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\epsilon\}$

Solution

- Language L: $= \{\epsilon\}$
- Expression: ϵ
- DFA M:



Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{aaa\}$

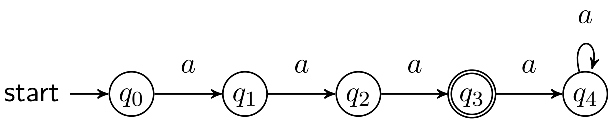
Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{aaa\}$

Solution

- Language L: $\{aaa\}$
- Expression: aaa
- DFA M:



Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\text{strings with even size}\}$

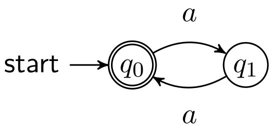
Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\text{strings with even size}\}$

Solution

- Language L: $\{\epsilon, aa, aaaa, aaaaaa, \ldots\}$
- Expression: $(aa)^*$
- DFA M:



Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\text{strings with odd size}\}$

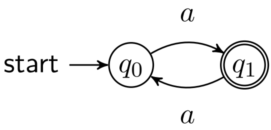
Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\text{strings with odd size}\}$

Solution

- Language L: $\{a, aaa, aaaaa, \ldots\}$
- Expression: $a(aa)^*$
- DFA M:



Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{$strings of size divisible by 3$\}$

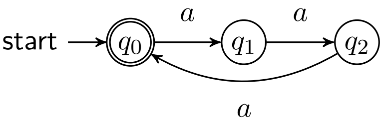
Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{$strings of size divisible by 3$\}$

Solution

- Language L: $\{\epsilon, aaa, aaaaaa, aaaaaaaaa, \ldots\}$
- Expression: $(aaa)^*$
- DFA M:



Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{$strings of size not divisible by 3$\}$

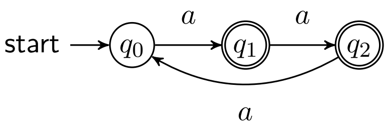
Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{$strings of size not divisible by 3$\}$

Solution

- Language L: $\{a, aa, aaaa, aaaaa, \ldots\}$
- Expression: $(a \cup aa)(aaa)^*$
- DFA M:



Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{$strings of size divisible by 6$\}$

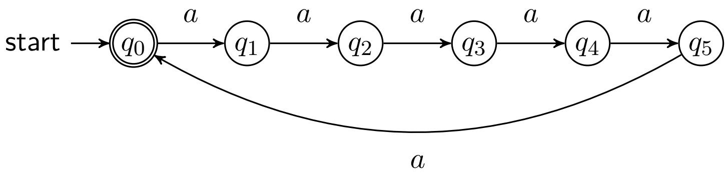
Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\text{strings of size divisible by 6}\}$

Solution

- Language L: $\{\epsilon, aaaaaa, aaaaaaaaaaaa, \ldots\}$
- Expression: $(aaaaaa)^*$
- DFA M:



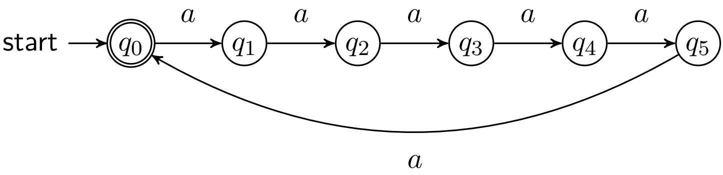
Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{$strings of size divisible by 6$\}$

Solution

- Language L: $\{\epsilon, aaaaaa, aaaaaaaaaaaa, \ldots\}$
- Expression: $(aaaaaa)^*$
- DFA M:



- Can you think of another approach?

Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{$strings of size divisible by 6$\}$

Solution

- Let $n =$ string size
- Observation
  $n \bmod 6 = 0 \iff n \bmod 2 = 0$ and $n \bmod 3 = 0$
- Idea
  Build DFA M_1 for $n \bmod 2 = 0$.
  Build DFA M_2 for $n \bmod 3 = 0$.
  Run M_1 and M_2 in parallel.
  Accept a string if both DFAs M_1 and M_2 accept the string.
  Reject a string if at least one of the DFAs M_1 and M_2 reject the string.
- It is possible to build complicated DFAs from simpler DFAs

Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\text{strings with size } n \text{ where } n \bmod 4 = 2\}$

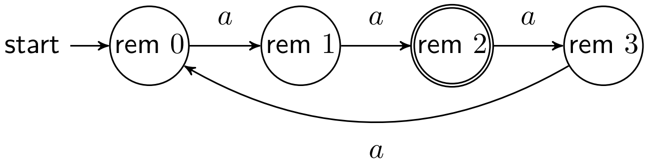
Construct DFA for $\Sigma = \{a\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{$strings with size n where $n \bmod 4 = 2\}$

Solution

- Language L: $\{aaa, aaaaaaa, aaaaaaaaaaa, \ldots\}$
- Expression: $aa(aaaa)^*$
- DFA M:



- What about strings with size n where $n \bmod k = i$?

Construct DFA for $\Sigma = \{a\}$

More Problems

Construct a DFA that accepts all strings from the language $L = \{\text{strings with size } n\}$ such that

- $n^2 - 5n + 6 = 0$
- $n \in [4, 37]$
- n is a perfect cube
- n is a prime number
- n satisfies a mathematical function $f(n)$

Specifying a DFA

The specification of DFA consists of:

- A (finite) alphabet
- A (finite) set of states
- Which state is the start state?
- Which states are the final states?
- What is the transition from each state, on each input character?

What is a deterministic finite automaton (DFA)?

- Deterministic = Events can be determined precisely
- Finite = Finite and small amount of space used
- Automaton = Computing machine

What is a deterministic finite automaton (DFA)?

- Deterministic = Events can be determined precisely
- Finite = Finite and small amount of space used
- Automaton = Computing machine

Definition

A deterministic finite automaton (DFA) M is a 5-tuple
$M = (Q, \Sigma, \delta, q_0, F)$, where,

1. Q: A finite set (set of states). ▷ Space (computer memory)
2. Σ: A finite set (alphabet).
3. $\delta : Q \times \Sigma \to Q$ is the transition function. ▷ Time (computation)
4. q_0: The start state (belongs to Q).
5. F: The set of accepting/final states, where $F \in Q$.

Acceptance and rejection of strings

Definition

- A DFA accepts a string $w = w_1 w_2 \ldots w_k$ iff there exists a sequence of states $r_0, r_1, \ldots, r_k$ such that the current state starts from the start state and ends at a final state when all the symbols of w have been read.
- A DFA rejects a string iff it does not accept it.

What is a regular language?

Definition

- We say that a DFA M accepts a language L if $L = \{w \mid M \text{ accepts } w\}$.
- A language is called a regular language if some DFA accepts or recognizes it.

Construct DFA for $\Sigma = \{a, b\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\text{strings with odd number of } b\text{'s}\}$

Construct DFA for $\Sigma = \{a, b\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\text{strings with odd number of } b\text{'s}\}$

Solution

States

- q_{odd}: DFA is in this state if it has read odd b's.
- q_{even}: DFA is in this state if it has read even b's.

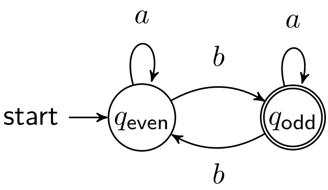
Construct DFA for $\Sigma = \{a, b\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\text{strings with odd number of } b\text{'s}\}$

Solution

- Language L: $\{\text{strings with odd number of } b\text{'s}\}$
- Expression: $a^*b(a \cup ba^*b)^*$ or $a^*ba^*(ba^*ba^*)^*$
- DFA M:



Construct DFA for $\Sigma = \{a, b\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\text{strings with odd number of } b\text{'s}\}$

Solution (continued)

- DFA M is specified as
  Set of states is $Q = \{q_{\text{even}}, q_{\text{odd}}\}$
  Set of symbols is $\Sigma = \{a, b\}$
  Start state is q_{even}
  Set of accept states is $F = \{q_{\text{odd}}\}$
  Transition function δ is:

δ	a	b
q_{even}	q_{even}	q_{odd}
q_{odd}	q_{odd}	q_{even}

Construct DFA for $\Sigma = \{a, b\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{$strings containing $bab\}$

Construct DFA for $\Sigma = \{a, b\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\text{strings containing } bab\}$

Solution

States

- q_b: DFA is in this state if the last symbol read was b, but the substring bab has not been read.
- q_{ba}: DFA is in this state if the last two symbols read were ba, but the substring bab has not been read.
- q_{bab}: DFA is in this state if the substring bab has been read in the input string.
- q: In all other cases, the DFA is in this state.

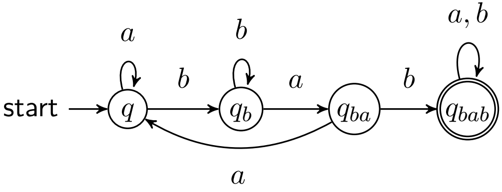
Construct DFA for $\Sigma = \{a, b\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{$strings containing $bab\}$

Solution (continued)

- Language L: $\{$strings containing $bab\}$
- Expression: $(a^*b^+aa)^*bab(a \cup b)^*$
- DFA M:



Construct DFA for $\Sigma = \{a, b\}$

Problem

- Construct a DFA that accepts all strings from the language $L = \{\text{strings containing } bab\}$

Solution (continued)

- DFA M is specified as
  Set of states is $Q = \{q, q_b, q_{ba}, q_{bab}\}$
  Set of symbols is $\Sigma = \{a, b\}$
  Start state is q
  Set of accept states is $F = \{q_{bab}\}$
  Transition function δ is:

δ	a	b
q	q	q_b
q_b	q_{ba}	q_b
q_{ba}	q	q_{bab}
q_{bab}	q_{bab}	q_{bab}

Closure properties of regular languages

Properties

Let L_1 and L_2 be regular languages.

Then, the following languages are regular.

- Complement. $\overline{L_1} = \{x \mid x \in \Sigma^* \text{ and } x \notin L_1\}$.
- Union. $L_1 \cup L_2 = \{x \mid x \in L_1 \text{ or } x \in L_2\}$.
- Intersection. $L_1 \cap L_2 = \{x \mid x \in L_1 \text{ and } x \in L_2\}$.
- Concatenation. $L_1 \cdot L_2 = \{xy \mid x \in L_1 \text{ and } y \in L_2\}$.
- Star. $L_1^* = \{x_1 x_2 \ldots x_k \mid k \geq 0 \text{ and each } x_i \in L_1\}$.

Closure properties for languages

Language	Operation						
	$L_1 \cup L_2$	$L_1 \cap L_2$	L'	$L_1 L_2$	L^*	L^R	L^T
Regular	✓	✓	✓	✓	✓	✓	✓
DCFL	✗	✗	✓	✗	✗	✗	✗
CFL	✓	✗	✗	✓	✓	✓	✓
Recursive	✓	✓	✓	✓	✓	✓	✗
R.E.	✓	✓	✗	✓	✓	✓	✓

- $L_1 \cup L_2 =$ Union of L_1 and L_2
- $L_1 \cap L_2 =$ Intersection of L_1 and L_2
- $L' =$ Complement of L
- $L_1 L_2 =$ Concatenation of L_1 and L_2
- $L^* =$ Powers of L
- $L^R =$ Reverse of L
- $L^T =$ Finite transduction of L (may include: intersection/shuffle/perfect-shuffle/quotient with arbitrary regular languages)

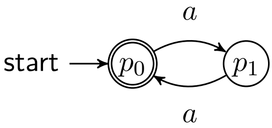
Construct DFA for $L_1 \cup L_2$

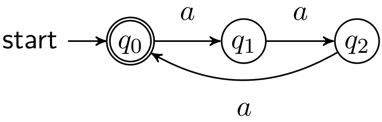
Problem

- Construct a DFA that accepts all strings from the language $L = \{$strings with size multiples of 2 or 3$\}$ where $\Sigma = \{a\}$

Solution

- Language $L_1 = \{$strings with size multiples of 2$\}$
- Language $L_2 = \{$strings with size multiples of 3$\}$

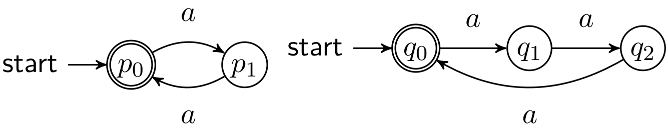


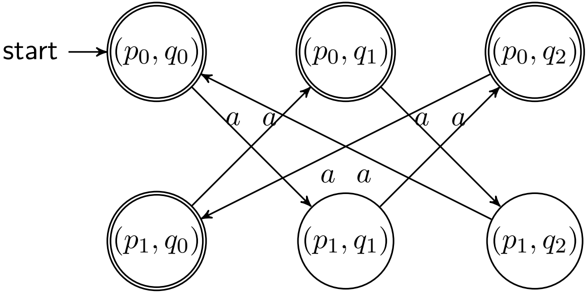


Construct DFA for $L_1 \cup L_2$

Solution (continued)

- Language $L_1 \cup L_2 = \{$strings with size multiples of 2 and 3$\}$





Construct DFA for $L_1 \cup L_2$

Union

- Let M_1 accept L_1, where $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$
  Let M_2 accept L_2, where $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$
- Let M accept $L_1 \cup L_2$, where $M = (Q, \Sigma, \delta, q, F)$. Then
  $Q = \{(r_1, r_2) \mid r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$ $\triangleright$ Cartesian product
  $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a)) \ \ \forall (r_1, r_2) \in Q, a \in \Sigma$
  $q_0 = (q_1, q_2)$
  $F = \{(r_1, r_2) \mid r_1 \in F_1 \text{ or } r_2 \in F_2\}$

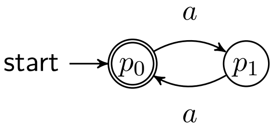
Construct DFA for $L_1 \cap L_2$

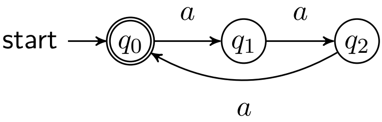
Problem

- Construct a DFA that accepts all strings from the language $L = \{\text{strings with size multiples of 2 and 3}\}$ where $\Sigma = \{a\}$

Solution

- Language $L_1 = \{\text{strings with size multiples of 2}\}$
- Language $L_2 = \{\text{strings with size multiples of 3}\}$

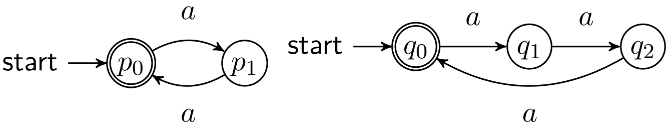


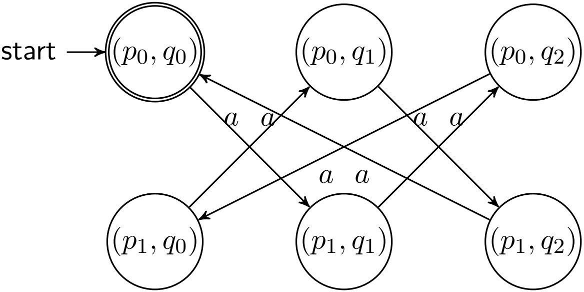


Construct DFA for $L_1 \cap L_2$

Solution (continued)

- Language $L_1 \cap L_2 = \{$strings with size multiples of 2 and 3$\}$





Construct DFA for $L_1 \cap L_2$

Intersection

- Let M_1 accept L_1, where $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$
  Let M_2 accept L_2, where $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$
- Let M accept $L_1 \cap L_2$, where $M = (Q, \Sigma, \delta, q, F)$. Then
  $Q = \{(r_1, r_2) \mid r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$ $\quad \triangleright$ Cartesian product
  $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a)) \ \ \forall (r_1, r_2) \in Q, a \in \Sigma$
  $q_0 = (q_1, q_2)$
  $F = \{(r_1, r_2) \mid r_1 \in F_1 \text{ and } r_2 \in F_2\}$

Problems for practice

Problems

Assume $\Sigma = \{a, b\}$ unless otherwise mentioned.
Construct DFA's for the following languages and generalize:

- $L = \{w \mid |w| = 2\}$
- $L = \{w \mid |w| \leq 2\}$
- $L = \{w \mid |w| \geq 2\}$
- $L = \{w \mid n_a(w) = 2\}$
- $L = \{w \mid n_a(w) \leq 2\}$
- $L = \{w \mid n_a(w) \geq 2\}$
- $L = \{w \mid n_a(w) \bmod 3 = 1\}$
- $L = \{w \mid n_a(w) \bmod 2 = 0$ and $n_b(w) \bmod 2 = 0\}$
- $L = \{w \mid n_a(w) \bmod 3 = 2$ and $n_b(w) \bmod 2 = 1\}$
- $L = \{w \mid n_a(w) \bmod 5 = 3, n_b(w) \bmod 3 = 2,$ and $n_c(w) \bmod 2 = 1\}$ for $\Sigma = \{a, b, c\}$
- $L = \{w \mid n_a(w) \bmod 3 \geq n_b(w) \bmod 2\}$

Problems for practice

Problems (continued)

- $L = \{b \mid \text{binary number } b \bmod 3 = 1\}$ for $\Sigma = \{0, 1\}$
- $L = \{t \mid \text{ternary number } t \bmod 4 = 3\}$ for $\Sigma = \{0, 1, 2\}$
- $L = \{w \mid w \text{ starts with } a\}$
- $L = \{w \mid w \text{ contains } a\}$
- $L = \{w \mid w \text{ ends with } a\}$
- $L = \{w \mid w \text{ starts with } ab\}$
- $L = \{w \mid w \text{ contains } ab\}$
- $L = \{w \mid w \text{ ends with } ab\}$
- $L = \{w \mid w \text{ starts with } a \text{ and ends with } b\}$
- $L = \{w \mid w \text{ starts and ends with different symbols}\}$
- $L = \{w \mid w \text{ starts and ends with the same symbol}\}$
- $L = \{w \mid \text{every } a \text{ in } w \text{ is followed by a } b\}$
- $L = \{w \mid \text{every } a \text{ in } w \text{ is never followed by a } b\}$

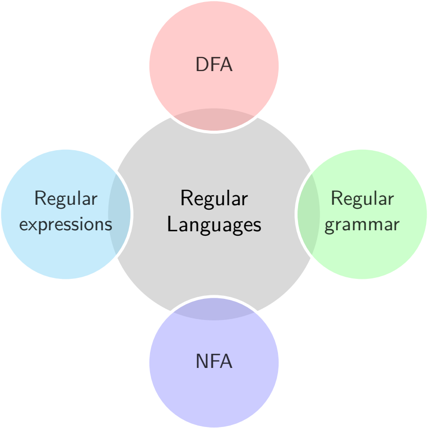
Problems for practice

Problems (continued)

- $L = \{w \mid \text{every } a \text{ in } w \text{ is followed by } bb\}$
- $L = \{w \mid \text{every } a \text{ in } w \text{ is never followed by } bb\}$
- $L = \{w \mid w = a^m b^n \text{ for } m, n \geq 1\}$
- $L = \{w \mid w = a^m b^n \text{ for } m, n \geq 0\}$
- $L = \{w \mid w = a^m b^n c^\ell \text{ for } m, n, \ell \geq 1\}$ for $\Sigma = \{a, b, c\}$
- $L = \{w \mid w = a^m b^n c^\ell \text{ for } m, n, \ell \geq 0\}$ for $\Sigma = \{a, b, c\}$
- $L = \{w \mid \text{second symbol from left end of } w \text{ is } a\}$
- $L = \{w \mid \text{second symbol from right end of } w \text{ is } a\}$
- $L = \{w \mid w = a^3 b x a^3 \text{ such that } x \in \{a, b\}^*\}$

Equivalence of different computation models

- Two machines or computational models are computationally equivalent if they accept/recognize the same language.
- The following models are computationally equivalent:
  DFA, regular expressions, NFA, and regular grammars.



Closure properties for languages

Language	Operation				
	$L_1 \cup L_2$	$L_1 \cap L_2$	$\bar{L}$	$L_1 \circ L_2$	L^*
DFA	Easy	Easy	Easy	Hard	Hard
Regex	Easy	Hard	Hard	Easy	Easy
NFA	Easy	Hard	Hard	Easy	Easy

- $L_1 \cup L_2 =$ Union of L_1 and L_2
- $L_1 \cap L_2 =$ Intersection of L_1 and L_2
- $\bar{L} =$ Complement of L
- $L_1 \circ L_2 =$ Concatenation of L_1 and L_2
- $L^* =$ Powers of L

Regular Expressions

Example

Example

- Arithmetic expression.
  $(5 + 3) \times 4 = 32 =$ Number
- Regular expression.
  $(a \cup b)a^* = \{a, b, aa, ba, aaa, baa, \dots\} =$ Regular language

Application

- Regular expressions in Linux.
  Used to find patterns in filenames, file content etc.
  Used in Linux tools such as awk, grep, and Perl.
  Google search: `http://www.googleguide.com/advanced_operators_reference.html`

What is a regular expression?

Definition

- The following are regular expressions.
  $\epsilon, \phi, a \in \Sigma$.
- If R_1 and R_2 are regular expressions, then the following are regular expressions.
  (Union.) $R_1 \cup R_2$
  (Concatenation.) $R_1 \circ R_2$
  (Kleene star.) R_1^*

Examples

Regular language	Regular expression
$\{\}$	ϕ
$\{\epsilon\}$	ϵ
$\{a\}$	a
$\{a,b\}$	$a \cup b$
$\{a\}\{b\}$	ab
$\{a\}^* = \{\epsilon, a, aa, aaa, \ldots\}$	a^*
$\{aab\}^*\{a, ab\}$	$(aab)^*(a \cup ab)$
$(\{aa, bb\} \cup \{a,b\}\{aa\}^*\{ab, ba\})^*$	$(aa \cup bb \cup (a \cup b)(aa)^*(ab \cup ba))^*$

Equality

- Two regular expressions are equal if they describe the same regular language. E.g.:
  $(a^*b^*)^* = (a \cup b)^*ab(a \cup b)^* \cup b^*a^* = (a \cup b)^* = \Sigma^*$

Examples

Examples

Let $\Sigma = a \cup b$, $R^{+} = RR^{*}$, and $R^{k} = \underbrace{R \cdots R}_{k \text{ times}}$

- $L = \{w \mid |w| = 2\}$
  $R = \Sigma\Sigma$
- $L = \{w \mid |w| \leq 2\}$
  $R = \epsilon \cup \Sigma \cup \Sigma\Sigma$
- $L = \{w \mid |w| \geq 2\}$
  $R = \Sigma\Sigma\Sigma^{*}$
- $L = \{w \mid n_a(w) = 2\}$
  $R = b^{*}ab^{*}ab^{*}$
- $L = \{w \mid n_a(w) \leq 2\}$
  $R = b^{*} \cup b^{*}ab^{*} \cup b^{*}ab^{*}ab^{*}$
- $L = \{w \mid n_a(w) \geq 2\}$
  $R = b^{*}ab^{*}ab^{*}(ab^{*})^{*}$

Rules

Beware of ϕ and ϵ

Suppose R is a regular expression.

- $R \cup \phi = R$
- $R \circ \epsilon = R$
- $\boxed{R \cup \epsilon \text{ may not equal } R}$
  (e.g.: $R = 0$, $L(R) = \{0\}$, $L(R \cup \epsilon) = \{0, \epsilon\}$)
- $\boxed{R \circ \phi \text{ may not equal } R}$
  (e.g.: $R = 0$, $L(R) = \{0\}$, $L(R \circ \phi) = \phi$)

Rules

Rules

Suppose R_1, R_2, R_3 are regular expressions. Then

- $R_1\phi = \phi R_1 = \phi$
- $R_1\epsilon = \epsilon R_1 = R_1 \cup \phi = \phi \cup R_1 = R_1$
- $R_1 \cup R_1 = R_1$
- $R_1 \cup R_2 = R_2 \cup R_1$
- $R_1(R_2 \cup R_3) = R_1R_2 \cup R_1R_3$
- $(R_1 \cup R_2)R_3 = R_1R_3 \cup R_2R_3$
- $R_1(R_2R_3) = (R_1R_2)R_3$
- $\phi^* = \epsilon$
- $(\epsilon \cup R_1)^* = (\epsilon \cup R_1)^+ = R_1^*$
- $R_1^*(\epsilon \cup R_1) = (\epsilon \cup R_1)R_1^* = R_1^*$
- $R_1^*R_2 \cup R_2 = R_1^*R_2$
- $R_1(R_2R_1)^* = (R_1R_2)^*R_1$
- $(R_1 \cup R_2)^* = (R_1 * R_2)^*R_1^* = (R_2^*R_1)^*R_2^*$

Construct a regex for $\Sigma = \{a, b\}$

Problem

- Construct a regular expression to describe the language $L = \{w \mid n_a(w) \text{ is odd}\}$

Construct a regex for $\Sigma = \{a, b\}$

Problem

- Construct a regular expression to describe the language $L = \{w \mid n_a(w) \text{ is odd}\}$

Solution

- Incorrect expressions.

  $b^*ab^*(ab^*a)^*b^*$ ▷ Why?

  $b^*a(b^*ab^*ab^*)^*$ ▷ Why?

- Correct expressions.

  $b^*ab^*(b^*ab^*ab^*)^*$ ▷ Why?

  $b^*ab^*(ab^*ab^*)^*$ ▷ Why?

  $b^*a(b^*ab^*a)^*b^*$ ▷ Why?

  $b^*a(b \cup ab^*a)^*$ ▷ Why?

  $(b \cup ab^*a)^*ab^*$ ▷ Why?

Construct a regex for $\Sigma = \{a, b\}$

Problem

- Construct a regular expression to describe the language $L = \{w \mid w \text{ ends with } b \text{ and does not contain } aa\}$

Construct a regex for $\Sigma = \{a, b\}$

Problem

- Construct a regular expression to describe the language $L = \{w \mid w \text{ ends with } b \text{ and does not contain } aa\}$

Solution

- A string not containing aa means that every a in the string:
  - is immediately followed by b, or
  - is the last symbol of the string
- Each string in the language has to end with b.
- Hence, every a in each string of the language is immediately followed by b
- Regular expression is: $(b \cup ab)^+$

Construct a regex to recognize identifiers in C

Problem

- Identifiers are the names you supply for variables, types, functions, and labels.
- Construct a regular expression to recognize the identifiers in the C programming language i.e., $L = \{\text{identifiers in C}\}$

Construct a regex to recognize identifiers in C

Problem

- Identifiers are the names you supply for variables, types, functions, and labels.
- Construct a regular expression to recognize the identifiers in the C programming language i.e., $L = \{\text{identifiers in C}\}$

Solution

- C identifier = FirstLetter OtherLetters
  FirstLetter = English letter or underscore
  OtherLetters = Alphanumeric letters or underscore
- Let $L = \{a, \ldots, z, A, \ldots, Z\}$ and $D = \{0, 1, \ldots, 9\}$
- Regular expression is:
  $R =$ FirstLetter $\circ$ OtherLetters
  FirstLetter $= (L \cup \_)$
  OtherLetters $= (L \cup D \cup \_)$

Construct a regex to recognize decimals in C

Problem

- Construct a regular expression to recognize the decimal numbers in the C programming language i.e., $L = \{\text{decimal numbers in C}\}$
- Examples: $14, +1, -12, 14.3, -.99, 16., 3E14, -1.00E2, 4.1E-1$, and $.3E+2$

Construct a regex to recognize decimals in C

Problem

- Construct a regular expression to recognize the decimal numbers in the C programming language i.e.,
  $L = \{\text{decimal numbers in C}\}$
- Examples: $14, +1, -12, 14.3, -.99, 16., 3E14, -1.00E2, 4.1E-1$, and $.3E+2$

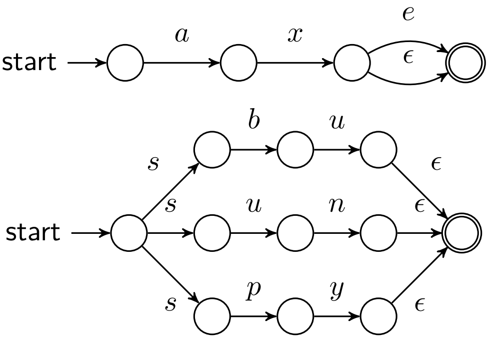
Solution

- C decimal number = Sign Decimals Exponent
- Let $D = \{0, 1, \ldots, 9\}$
- Regular expression is:
  $R = \text{Sign} \circ \text{Decimals} \circ \text{Exponent}$
  $\text{Sign} = (+ \cup - \cup \epsilon)$
  $\text{Decimals} = (D^+ \cup D^+.D^* \cup D^*.D^+)$
  $\text{Exponent} = (\epsilon \cup E\ \text{Sign}\ D^+)$

Nondeterministic Finite Automata (NFA)

Example NFA's

Examples



Difference	DFA	NFA
Multiple transitions	1 exiting arrow	≥ 0 exiting arrows
Epsilon transitions	✗	✓
Missing transitions	No missing transitions	Missing transitions mean transitions to sink/reject state

What is the intuition behind nondeterminism?

Intuition

Nondeterministic computation = Parallel computation
(NFA searches all possible paths in a graph to the accept state)

- When NFA has multiple choices for the same input symbol, think of it as a process forking multiple processes for parallel computation.
- A string is accepted if any of the parallel processes accepts the string.

Nondeterministic computation = Tree of possibilities
(NFA magically guesses a right path to the accept state)

- Root of the tree is the start of the computation.
- Every branching point is the decision-making point consisting of multiple choices.
- Machine accepts a string if any of the paths from the root of the tree to a leaf leads to an accept state.

Why care for NFA's?

Uses of NFA's

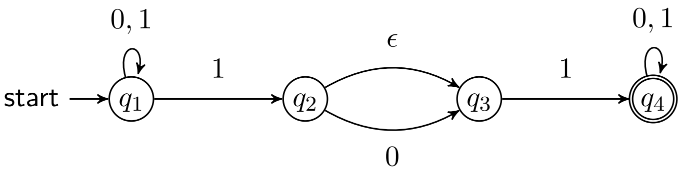
- Constructing NFA's is easier than directly constructing DFA's for many problems.
  Hence, construct NFA's and then convert them to DFA's.
- NFA's are easier to understand than DFA's.

Construct NFA for $\Sigma = \{0, 1\}$

Problem

- Construct a NFA that accepts all strings from the language $L = \{$strings containing 11 or 101$\}$

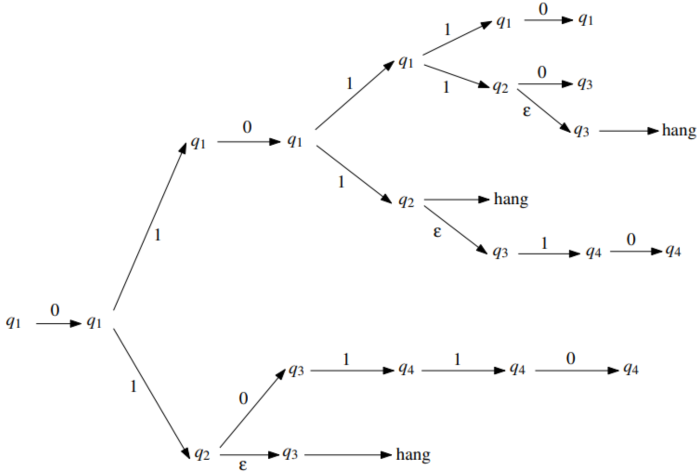
Solution



- How does the machine work for the input 010110?
- What is the equivalent DFA for solving the problem?

Construct NFA for $\Sigma = \{0, 1\}$

Solution (continued)



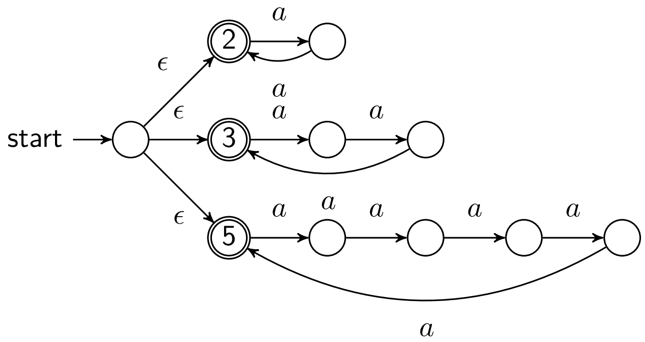
Source: Anil Maheshwari and Michiel Smid's Theory of Computation

Construct NFA for $\Sigma = \{a\}$

Problem

- Construct a NFA that accepts all strings from the language $L = \{$strings of size multiples of 2 or 3 or 5$\}$

Solution



- What is the equivalent DFA for solving the problem?

What is a nondeterministic finite automaton (NFA)?

- Nondeterministic = Event paths cannot be determined precisely
- Finite = Finite and small amount of space used
- Automaton = Computing machine

Definition

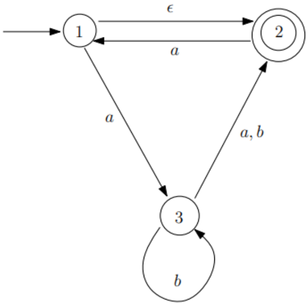
A nondeterministic finite automaton (NFA) M is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$, where,

1. Q: A finite set (set of states). ▷ Space (computer memory)
2. Σ: A finite set (alphabet).
3. $\boxed{\delta : Q \times (\Sigma \cup \epsilon) \rightarrow P(Q)}$ is the transition function, where $P(Q)$ is the power set of Q. ▷ Time (computation)
4. q_0: The start state (belongs to Q).
5. F: The set of accepting/final states, where $F \in Q$.

Convert NFA to DFA

Problem

- Convert the NFA to a DFA.



Source: Anil Maheshwari and Michiel Smid's Theory of Computation

Construct DFA for the given NFA

Solution

- NFA M is specified as
  Set of states is $Q = \{1, 2, 3\}$
  Set of symbols is $\Sigma = \{a, b\}$
  Start state is 1
  Set of accept states is $F = \{1\}$
  Transition function δ is:

δ	a	b	ϵ
1	$\{3\}$	ϕ	$\{2\}$
2	$\{1\}$	ϕ	ϕ
3	$\{2\}$	$\{2, 3\}$	ϕ

- How do you convert this NFA to DFA?

Construct DFA for the given NFA

Solution

- NFA M is specified as
  Set of states is $Q = \{1, 2, 3\}$
  Set of symbols is $\Sigma = \{a, b\}$
  Start state is 1
  Set of accept states is $F = \{1\}$
  Transition function δ is:

δ	a	b	ϵ
1	$\{3\}$	ϕ	$\{2\}$
2	$\{1\}$	ϕ	ϕ
3	$\{2\}$	$\{2, 3\}$	ϕ

- How do you convert this NFA to DFA?
  If NFA has states Q, then construct a DFA with states $P(Q)$.

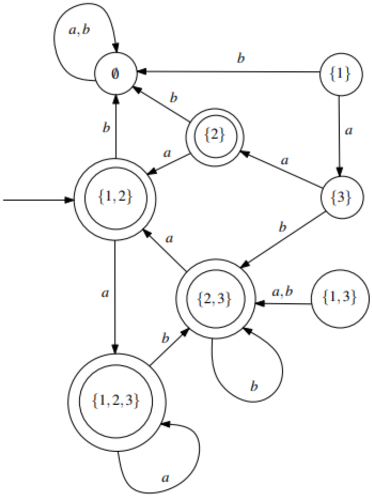
Construct DFA for the given NFA

Solution (continued)

- $\phi \xrightarrow{a} \phi$
- $\phi \xrightarrow{b} \phi$
- $\{1\} \xrightarrow{a} \{3\}$
- $\{1\} \xrightarrow{b} \phi$
- $\{2\} \xrightarrow{a} \{1,2\}$
- $\{2\} \xrightarrow{b} \phi$
- $\{3\} \xrightarrow{a} \{2\}$
- $\{3\} \xrightarrow{b} \{2,3\}$
- $\{1,2\} \xrightarrow{a}$?
- $\{1,2\} \xrightarrow{b}$?
- $\{1,3\} \xrightarrow{a}$?
- $\{1,3\} \xrightarrow{b}$?
- $\{2,3\} \xrightarrow{a}$?
- $\{2,3\} \xrightarrow{b}$?
- $\{1,2,3\} \xrightarrow{a}$?
- $\{1,2,3\} \xrightarrow{b}$?

Construct DFA for the given NFA

Solution (continued)



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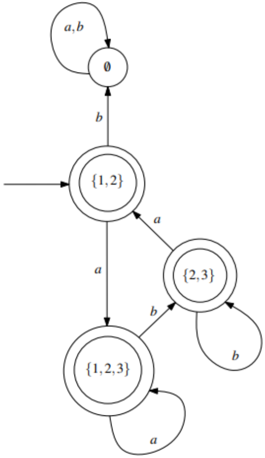
Construct DFA for the given NFA

Solution (continued)



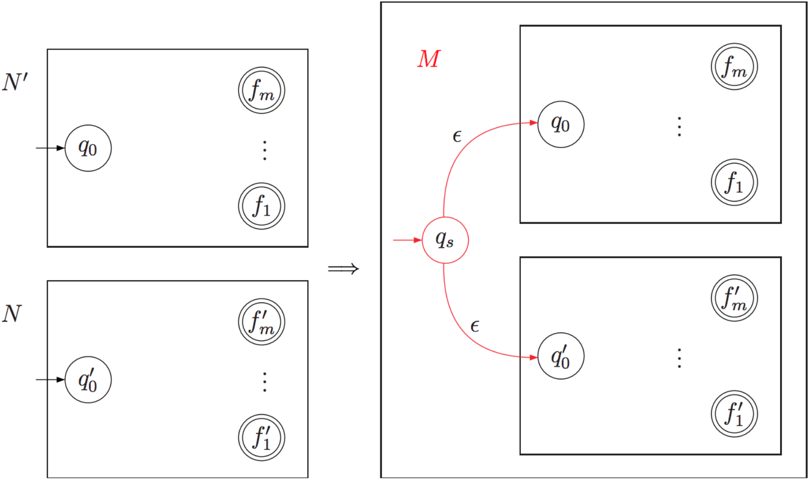
Source: Anil Maheshwari and Michiel Smid's Theory of Computation

Construct DFA for the given NFA

Convert NFA to DFA

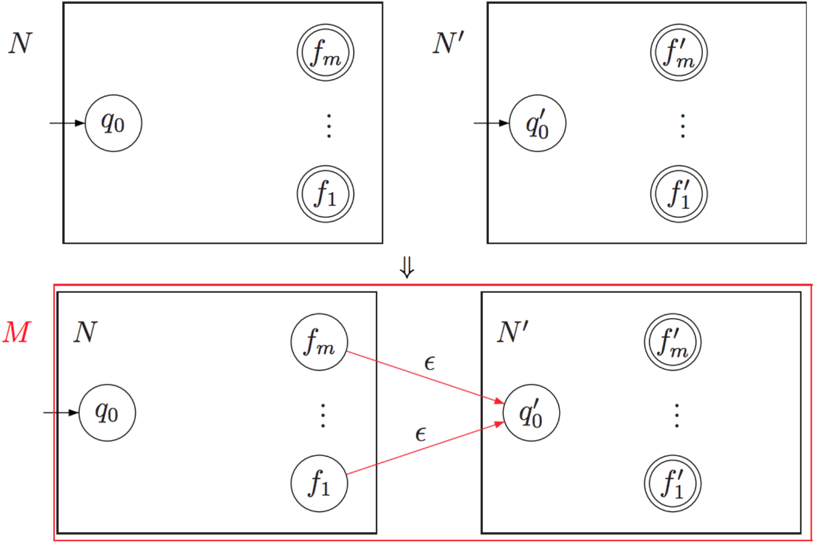
- Let $N = (Q, \Sigma, \delta, q, F)$ be the NFA.
  Let $M = (Q', \Sigma, \delta', q', F')$ be the DFA. Then
- $Q' = P(Q)$ $\triangleright$ Power set of Q
  $q' = C_\epsilon(\{q\})$ $\triangleright$ ϵ-closure of the start state
  $F' = \{S \in Q' \mid S \cap F \neq \phi\}$ $\triangleright$ $S \cap F \neq \phi$ means that S contains at least one accept state of N
  $\delta' : Q' \times \Sigma \rightarrow Q'$ is defined as follows:
  For all state $S \in Q'$ and for all letter $a \in \Sigma$,
  $\boxed{\delta'(S, a) = \bigcup_{s \in S} C_\epsilon(\delta(s, a))}$

Union of NFA



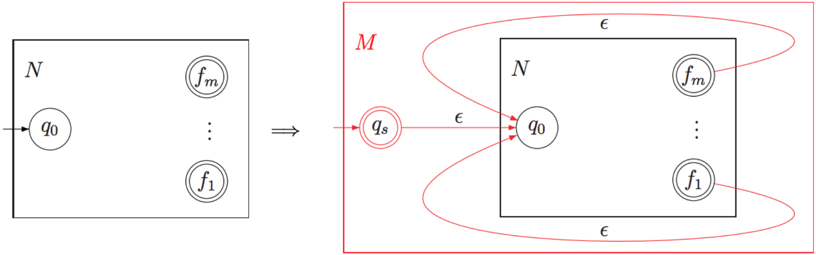
Source: Margaret Fleck and Sariel Har-Peled's Notes on Theory of Computation

Concatenation of NFA



Source: Margaret Fleck and Sariel Har-Peled's Notes on Theory of Computation

Star of NFA



Source: Margaret Fleck and Sariel Har-Peled's Notes on Theory of Computation

Construct a NFA for $(aa \cup aab)^*b$

Problem

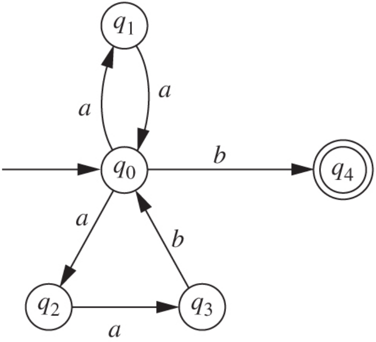
- Construct a NFA for the regular expression $(aa \cup aab)^*b$.

Construct a NFA for $(aa \cup aab)^*b$

Problem

- Construct a NFA for the regular expression $(aa \cup aab)^*b$.

Solution



Source: John Martin's Introduction to Languages and the Theory of Computation.

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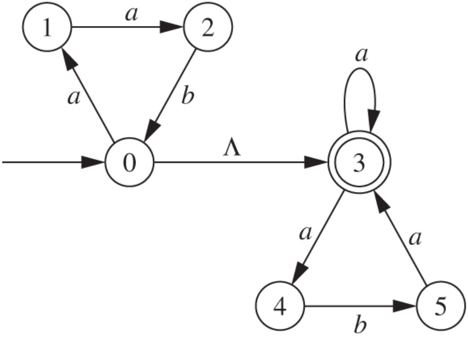

Source: John Martin's Introduction to Languages and the Theory of Computation.

Non-Regular Languages

Regular or non-regular languages

Problems

Let $\Sigma = \{a, b\}$ unless mentioned otherwise. Check if the languages are regular or non-regular (✗):

- $L = \{w \mid w = a^n \text{ and } n \leq 10^{100}\}$
- $L = \{w \mid w = a^n \text{ and } n \geq 1\}$
- $L = \{w \mid w = a^m b^n \text{ and } m, n \geq 1\}$
- $L = \{w \mid w = a^* b^*\}$
- $L = \{w \mid w = a^n b^n \text{ and } n \geq 1\}$
- $L = \{ww^R \mid |w| = 3\}$
- $L = \{ww^R \mid |w| \geq 1\}$
- $L = \{w \mid w = w^R \text{ and } |w| \geq 1\}$
- $L = \{w \mid w = a^{2i+1} b^{3j+2} \text{ and } i, j \geq 1\}$
- $L = \{w \mid w = a^n \text{ and } n \text{ is a square}\}$
- $L = \{w \mid w = a^n \text{ and } n \text{ is a prime}\}$
- $L = \{w \mid w = a^i b^{j^2} \text{ and } i, j \geq 1\}$

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- $L = \{w \mid w = a^* b^*\}$
- $L = \{w \mid w = a^n b^n \text{ and } n \geq 1\}$ .... ✗
- $L = \{ww^R \mid |w| = 3\}$
- $L = \{ww^R \mid |w| \geq 1\}$ .... ✗
- $L = \{w \mid w = w^R \text{ and } |w| \geq 1\}$ .... ✗
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- $L = \{w \mid w = a^n \text{ and } n \text{ is a square}\}$ .... ✗
- $L = \{w \mid w = a^n \text{ and } n \text{ is a prime}\}$ .... ✗
- $L = \{w \mid w = a^i b^{j^2} \text{ and } i, j \geq 1\}$ .... ✗

Regular or non-regular languages

Problems (continued)

- $L = \{w \mid n_a(w) = n_b(w)\}$
- $L = \{w \mid n_a(w) \bmod 3 \geq n_b(w) \bmod 5\}$
- $L = \{w \mid w = a^i b^j \text{ and } j > i \geq 1\}$
- $L = \{wxw^R \mid x \in \Sigma^*, |w|, |x| \geq 1, \text{ and } |x| \leq 5\}$
- $L = \{wxw^R \mid x \in \Sigma^* \text{ and } |w|, |x| \geq 1\}$
- $L = \{xww^Ry \mid x, y \in \Sigma^* \text{ and } |w|, |x|, |y| \geq 1\}$
- $L = \{xww^R \mid x \in \Sigma^* \text{ and } |w|, |x| \geq 1\}$
- $L = \{ww^Ry \mid y \in \Sigma^* \text{ and } |w|, |y| \geq 1\}$

Regular or non-regular languages

Problems (continued)

- $L = \{w \mid n_a(w) = n_b(w)\}$ .... ✗
- $L = \{w \mid n_a(w) \bmod 3 \geq n_b(w) \bmod 5\}$
- $L = \{w \mid w = a^i b^j$ and $j > i \geq 1\}$ .... ✗
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- $L = \{xww^R \mid x \in \Sigma^*$ and $|w|, |x| \geq 1\}$ .... ✗
- $L = \{ww^R y \mid y \in \Sigma^*$ and $|w|, |y| \geq 1\}$ .... ✗

How to prove that certain languages are not regular?

Pumping lemma

- Many languages are not regular.
- Pumping lemma is a method to prove that certain languages are not regular.

Pumping property

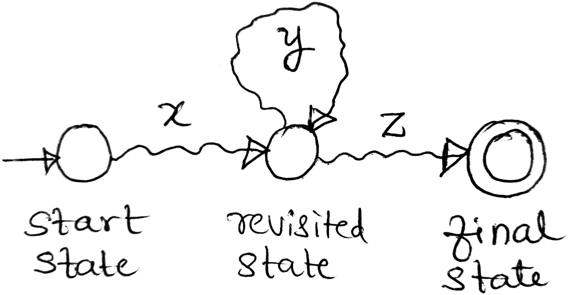
- If a language is regular, then it must have the pumping property.
- If a language does not have the pumping property, then the language is not regular. $\triangleright$ Proof by contraposition

How to prove languages non-regular using pumping lemma?

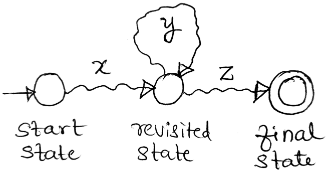
- Proof by contradiction.
  Assume that the language is regular.
  Show that the language does not have the pumping property.
  Contradiction! Hence, the language has to be non-regular.

Pumping property of regular languages

- Suppose a DFA M with s number of states accepts a very long string w such that $|w| \geq s$ from a language L.
- From pigeonhole principle, at least one state is visited twice.
- This implies that the string went through a loop.



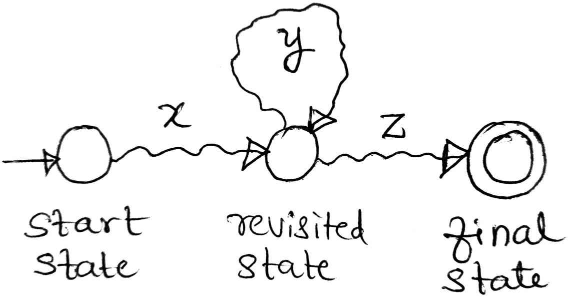
Pumping property of regular languages



Observations

- Suppose string w has more characters than the number of states in the DFA, i.e., $|w| \geq s$
- String w can be split into three parts i.e., $w = xyz$ where
  x: string before the first loop
  y: string of the first loop
  z: string after the first loop (might contain loops)
- Loop must appear i.e., $|y| \geq 1$
  (x and z can be empty)
- Loop must appear in the first s characters of w i.e., $|xy| \leq s$

Pumping property of regular languages



Idea

- An infinite number of strings can be pumped with loop length and they must also be in the language.
- Formally, for all $i \geq 0$, xy^iz must be in the language.
- xz, xyz, $xyyz$, $xyyyz$, etc must also belong to the language.

Pumping lemma for regular languages

Theorem

Suppose L is a language over alphabet Σ. Suppose L is accepted by a finite automaton M having s states. Then, every long string $w \in L$ satisfying $|w| \geq s$ can be split into three strings $w = xyz$ such that the following three conditions are true.

- $|xy| \leq s$.
- $|y| \geq 1$.
- For every $i \geq 0$, the string xy^iz also belongs to L.

$L = \{a^n b^n \mid n \geq 0\}$ is non-regular

Problem

- Prove that $L = \{a^n b^n \mid n \geq 0\}$ is not a regular language.

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Solution

- Suppose L is regular. Then it must satisfy pumping property.
- Suppose $w = a^s b^s$.
- Let $w = xyz = \boxed{a^p}\boxed{a^q}\boxed{a^r b^s}$
  where $|xy| \leq s$, $|y| \geq 1$, and $p + q + r = s$.
- Also, $xy^i z$ must belong to L for all $i \geq 0$.
- But, $xyyz$ is not in L.
  Reason: $xyyz = a^p a^q a^q a^r b^s = a^{s+q} b^s \notin L$.
  $xyyz$ has more a's than b's.
- Contradiction! Hence, L is not regular.

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Solution

- Suppose L is regular. Then it must satisfy pumping property.
- Suppose $w = (ab)^s$.
- Let $w = xyz = \boxed{\epsilon}\boxed{(ab)^1}\boxed{(ab)^{s-1}}$
- We have $|xy| \leq s$ and $|y| \geq 1$.
- Also, xy^iz must belong to L for all $i \geq 0$.
- xy^iz belongs to L for all $i \geq 0$.
- No contradiction! Hence, L is regular.

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- xy^iz belongs to L for all $i \geq 0$.
- No contradiction! Hence, L is regular.

Mistakes

Incorrect solution! Why? Multiple reasons:

1. If we cannot find a contradiction, that does not prove anything.
2. We must try for all possible values of x, y such that $|xy| \leq s$.
3. The chosen string $(ab)^s$ is a bad string to work on.

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Takeaway

1. Reduction! Reduce a problem to another. Reuse its solution.

Superset of a non-regular language

Problem

- $\{a^n b^n\}$ is a subset of $\{w \mid n_a(w) = n_b(w)\}$.
  We used the fact that $\{a^n b^n\}$ is non-regular to prove that $\{w \mid n_a(w) = n_b(w)\}$ is non-regular.
  Is a superset of a non-regular language non-regular?

Superset of a non-regular language

Problem

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  We used the fact that $\{a^n b^n\}$ is non-regular to prove that $\{w \mid n_a(w) = n_b(w)\}$ is non-regular.
  Is a superset of a non-regular language non-regular?

Solution

- No!
  Σ^* is a superset of every non-regular language.
  But, it is regular.

$L = \{w \mid n_a(w) = n_b(w)\}$ is non-regular

Problem

- Prove that $L = \{w \mid n_a(w) = n_b(w)\}$ is not a regular language.

Solution (without using pumping lemma)

- Suppose L is regular.
- We know that $L' = \{w \mid w = a^i b^j, i, j \geq 0\}$ is regular.
- As regular languages are closed under intersection, $L \cap L'$ must also be regular.
- We see that $L \cap L' = \{w \mid w = a^n b^n$ and $n \geq 0\}$.
- But, this language was earlier proved to be non-regular.
- Contradiction! Hence, L is not regular.

$L = \{ww\}$ is non-regular

Problem

- Prove that $L = \{ww\}$ is not a regular language.

$L = \{ww\}$ is non-regular

Problem

- Prove that $L = \{ww\}$ is not a regular language.

Solution

- Suppose L is regular. Then it must satisfy pumping property.
- Suppose $ww = a^s a^s$.
- Let $ww = xyz = \boxed{a^p}\boxed{a^1}\boxed{a^{s-p-1}a^s}$
- We have $|xy| \leq s$ and $|y| \geq 1$.
- Also, xy^iz must belong to L for all $i \geq 0$.
- But, $xyyz$ is not in L.
  Reason: $xyyz = a^p a^1 a^1 a^{s-p-1} a^p = a^{s+1} a^s \notin L$.
  $xyyz$ has odd number of a's.
- Contradiction! Hence, L is not regular.

$L = \{ww\}$ is non-regular

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- Prove that $L = \{ww\}$ is not a regular language.

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  $xyyz$ has odd number of a's.
- Contradiction! Hence, L is not regular.

Mistakes

Incorrect solution! Why?

1. We must try all possible values of x, y such that $|xy| \leq s$.
2. Try pumping with $y \in \{a^2, a^4, \ldots\}$ such that $|y| \leq s$.

$L = \{ww\}$ is non-regular

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$L = \{ww\}$ is non-regular

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- Prove that $L = \{ww\}$ is not a regular language.

Solution

- Suppose L is regular. Then it must satisfy pumping property.
- Suppose $ww = a^s b^s a^s b^s$.
- Let $ww = xyz = \boxed{a^p}\boxed{a^q}\boxed{a^r b^s a^s b^s}$
  where $|xy| \leq s$, $|y| \geq 1$, and $p + q + r = s$.
- Also, $xy^i z$ must belong to L for all $i \geq 0$.
- But, $xyyz$ is not in L.
  Reason: $xyyz = a^p a^q a^q a^r b^s a^s b^s = a^{s+q} b^s a^s b^s \notin L$.
- Contradiction! Hence, L is not regular.

$L = \{w \mid w = a^n, n \geq 0, n$ is a square$\}$ is non-regular

Problem

- Prove that $L = \{w \mid w = a^{n^2}, n \geq 0\}$ is not a regular language.

$L = \{w \mid w = a^n, n \geq 0, n$ is a square$\}$ is non-regular

Problem

- Prove that $L = \{w \mid w = a^{n^2}, n \geq 0\}$ is not a regular language.

Solution

- Suppose L is regular. Then it must satisfy pumping property.
- Suppose $w = a^{s^2}$.
- Let $w = xyz = \boxed{a^p}\boxed{a^q}\boxed{a^r a^{s^2-s}}$
  where $|xy| \leq s$, $|y| \geq 1$, and $p + q + r = s$.
- Also, xy^iz must belong to L for all $i \geq 0$.
- But, $xyyz$ is not in L.
  Reason: $xyyz = a^p a^q a^q a^r a^{s^2-s} = a^{s^2+q} \notin L$.
  Because, $a^{s^2} < a^{s^2+q} < a^{(s+1)^2}$.
- Contradiction! Hence, L is not regular.

$L = \{w \mid w = a^n, n$ is prime$\}$ is non-regular

Problem

- Prove that $L = \{w \mid w = a^n, n \text{ is prime}\}$ is not regular.

$L = \{w \mid w = a^n, n$ is prime$\}$ is non-regular

Problem

- Prove that $L = \{w \mid w = a^n, n \text{ is prime}\}$ is not regular.

Solution

- Suppose L is regular. Then it must satisfy pumping property.
- Suppose $w = a^m$, where m is prime and $m \geq s$.
- Let $w = xyz = \boxed{a^p}\boxed{a^q}\boxed{a^r}$
  where $|xy| \leq s$, $|y| \geq 1$, and $p + q + r = m$.
- Also, xy^iz must belong to L for all $i \geq 0$.
- But, $xy^{m+1}z$ is not in L.
  Reason: $xy^{m+1}z = a^p a^{q(m+1)} a^r = a^{m(q+1)} \notin L$.
- Contradiction! Hence, L is not regular.

$L = \{w \mid w = a^m b^n, m > n\}$ is non-regular

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- Prove that $L = \{w \mid w = a^m b^n, m > n\}$ is not regular.

Solution

- Suppose L is regular. Then it must satisfy pumping property.
- Suppose $w = a^{s+1} b^s$.
- Let $w = xyz = \boxed{a^p}\boxed{a^q}\boxed{a^r b^s}$
  where $|xy| \leq s$, $|y| \geq 1$, and $p + q + r = s + 1$.
- Also, $xy^i z$ must belong to L for all $i \geq 0$.
- But, xz is not in L. $\triangleright$ Pumping down
  Reason: $xz = a^p a^r b^s = a^{p+r} b^s \notin L$.
  Because, $p + r \leq s$ i.e., $\#a$'s is not greater than $\#b$'s.
- Contradiction! Hence, L is not regular.

$L = \{w \mid w = a^m b^n, m \neq n\}$ is non-regular

Problem

- Prove that $L = \{w \mid w = a^m b^n, m \neq n\}$ is not regular.

$L = \{w \mid w = a^m b^n, m \neq n\}$ is non-regular

Problem

- Prove that $L = \{w \mid w = a^m b^n, m \neq n\}$ is not regular.

Solution

- Suppose L is regular. Then it must satisfy pumping property.
- Suppose $w = a^s b^{s+s!}$.
- Let $w = xyz = \boxed{a^p}\boxed{a^q}\boxed{a^r b^{s+s!}}$
  where $|xy| \leq s$, $|y| \geq 1$, and $p + q + r = s$.
- Also, $xy^i z$ must belong to L for all $i \geq 0$.
- But, $xy^i z$ is not in L for some i.
  We pump a^q to get $a^{s+s!} b^{s+s!}$.
  Reason: $xy^i z = a^p a^{qi} a^r b^{s+s!} = a^{s+(i-1)q} b^{s+s!} \notin L$.
  This means $(i-1)q = s! \implies i = s!/q + 1$.
- Contradiction! Hence, L is not regular.

$L = \{w \mid w = a^m b^n, m \neq n\}$ is non-regular

Problem

- Prove that $L = \{w \mid w = a^m b^n, m \neq n\}$ is not regular.

Solution (without using pumping lemma)

- Suppose L is regular.
- We know that $L' = \{w \mid w = a^i b^j, i, j \geq 0\}$ is regular.
- Let $L'' = \{w \mid w = a^n b^n, n \geq 0\}$.
- As regular languages are closed under intersection and complementation, $L = L' - L'' = L' \cap \bar{L}''$ is regular.
  This implies that L'' is regular.
- But, the language L'' was earlier proved to be non-regular.
- Contradiction! Hence, L is not regular.