Chapter 15

Dynamic Programming

Chapter 15

The slides for this course are based on the course textbook: Cormen, Leiserson, Rivest, and Stein, *Introduction to Algorithms*, 2nd edition, The MIT Press, McGraw-Hill, 2001.

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Chapter 15 Topics

- Assembly-line scheduling
- Matrix-chain multiplication
- Elements of dynamic programming
- Longest common subsequence
- Optimal binary search trees

Dynamic Programming

- General approach – combine solutions to subproblems to get the solution to an problem
- Unlike Divide and Conquer in that
  - subproblems are dependent rather than independent
  - bottom-up approach
  - save the values of subproblems in a table and use them more than one time

Usual Approach

- Start with the smallest, simplest subproblems
- Combine “appropriate” subproblem solutions to get a solution to the bigger problem
- If you have a Divide and Conquer algorithm that does a lot of duplicate computation, you can often find a Dynamic Programming solution that is more efficient

A Simple Example

- Calculating binomial coefficients

$$\binom{n}{k}$$

- *n choose k* is the number of different combinations of *n* things taken *k* at a time
- These are also the coefficients of the binomial expansion $(x+y)^n$

Two Definitions

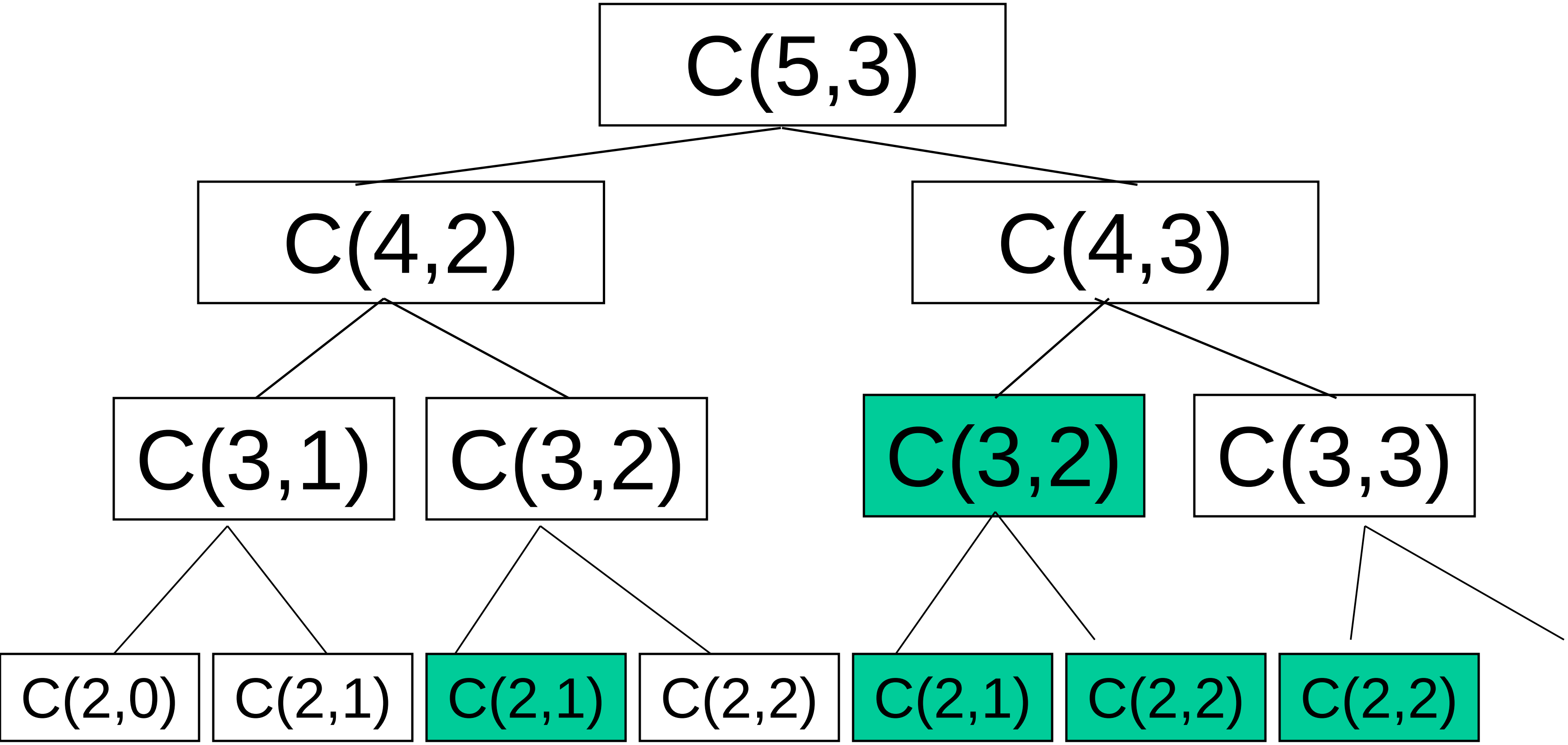
$$\text{Defintion 1: } \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

$$\text{Defintion 2:}$$

$$\binom{\text{n}}{\text{k}} = \begin{cases} \binom{n-1}{k-1} + \binom{n-1}{k} & 0 < k < n \\ 1 & k = 0 \\ 1 & k = n \end{cases}$$

Algorithm for Recursive Definition

```
function C(n,k)
   if k = 0 or k = n then
      return 1
   else
      return C(n-1, k-1) + C(n-1, k)
```



C(5,3)

C(4,2) C(4,3)

C(3,1) C(3,2) C(3,2) C(3,3)

C(2,0) C(2,1) C(2,1) C(2,2) C(2,1) C(2,2) C(2,2)

etc.

Complexity of Divide and Conquer Algorithm

- Time complexity is $\Omega(n!)$
- But we did a lot of duplicate computation
- Dynamic programming approach
  - Store solutions to subproblems in a table
  - Bottom-up approach

$n \geq k$

k

n

	0	1	2	3
1	1	1		
2	1		1	
3	1			1
4	1			
5	1			
6	1			

```
function C(n,k)
  if k = 0 or k = n then
    return 1
  else
    return C(n-1, k-1) + C(n-1, k)
```

k

n

	0	***1***	***2***	***3***
1	1	1		
2	1		1	
3	1			1
4	1			
5	1			

Analysis of Dynamic Programming Version

- Time complexity

  O(n k)

- Storage requirements
  - full table O(n k)
  - less space often possible

Typical Problem

- Dynamic Programming is often used for optimization problems that satisfy the principle of optimality
- Principle of optimality
  – In an optimal sequence of decisions or choices, each subsequence must be optimal

Optimization Problems

- Problem has many possible solutions
- Each solution has a value
- Goal is to find the solution with the optimal value

Steps in Dynamic Programming Solutions to Optimization Problems

1. Characterize the structure of an optimal solution
2. Recursively define the value of an optimal solution
3. Compute the value of an optimal solution in a bottom-up manner
4. Construct an optimal solution from computed information

Assembly-Line Scheduling

- Problem: Determine which stations to choose from line 1 and which to choose from line 2 in order to minimize the total time through the factory for one auto.
- Brute force approach: 2^n possibilities!

Assembly-Line Scheduling

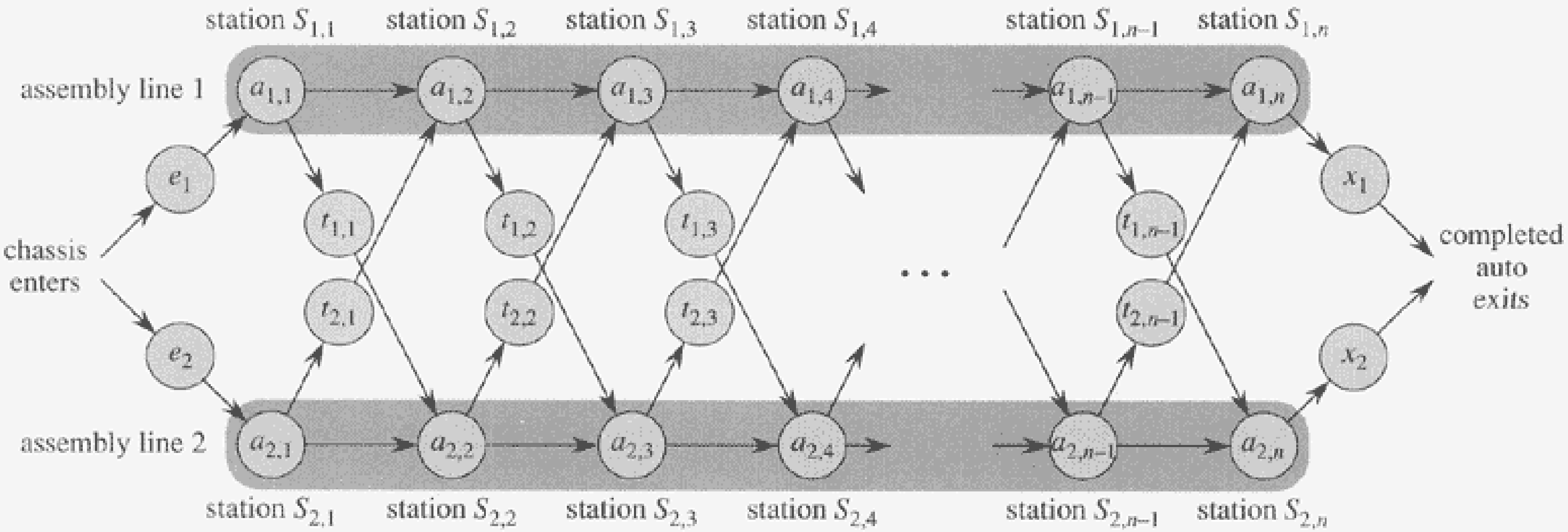


Figure 15.1 A manufacturing problem to find the fastest way through a factory. There are two assembly lines, each with n stations; the jth station on line i is denoted $S_{i,j}$ and the assembly time at that station is $a_{i,j}$. An automobile chassis enters the factory, and goes onto line i (where $i = 1$ or 2), taking e_i time. After going through the jth station on a line, the chassis goes on to the $(j+1)$st station on either line. There is no transfer cost if it stays on the same line, but it takes time $t_{i,j}$ to transfer to the other line after station $S_{i,j}$. After exiting the nth station on a line, it takes x_i time for the completed auto to exit the factory. The problem is to determine which stations to choose from line 1 and which to choose from line 2 in order to minimize the total time through the factory for one auto.

Step 1: Structure of Optimal Solution

- Fastest way through station $S_{1,j}$ is either:
  - the fastest way through $S_{1,j-1}$ and then directly through $S_{1,j}$, or
  - the fastest way through $S_{2,j-1}$, a transfer from line 2 to line 1, and then through $S_{1,j}$.
- There is a symmetric argument for fastest way through $S_{2,j}$.

Step 2: Recursive Solution

$$f_1[j] = \begin{cases} e_1 + a_{1,1} & \text{if } j = 1 \\ \min(f_1[j-1]+a_{1,j},\ f_2[j-1]+t_{2,j-1}+a_{1,j}) & \text{if } j \geq 2 \end{cases}$$

$$f_2[j] = \begin{cases} e_2 + a_{2,1} & \text{if } j = 1 \\ \min(f_2[j-1]+a_{2,j},\ f_1[j-1]+t_{1,j-1}+a_{2,j}) & \text{if } j \geq 2 \end{cases}$$

Step 3: Computing Fastest Times

- Recursive algorithm based on equations 15.1, 15.6, and 15.7 will have running time of Θ(2n).
- Better way: Compute the $f_i[j]$ values in order of increasing station numbers; time will be Θ(n).

FASTEST-WAY

Think about it this way:

For both assembly lines, simultaneously, work your way from the entrance to the exit. As you encounter each station:

1. Stand on the station and ask yourself, “What was the fastest way for me to get to and through this station?”

   A. For the first station on either assembly line, just add the cost of the station you are standing on to the entrance cost to the line you are in.

FASTEST-WAY

B. For station 2 through *n*, the cost is computed this way:

1) Add the cost of the station you are standing on to the cost of the previous station in your line.

2) Add the cost of the station you are standing on to the cost of the previous station in the *other* assembly line, plus the transit cost of moving from one line to another.

3) The min of these 2 values is your cost.

FASTEST-WAY

2. Record the cost of the station in Table 1. In Table 2, keep track of the previous station you passed through on your fastest way to this station.
3. If you are on one of the two end stations, add the cost of the station to the exit cost for that station. The station with the *min* total value is the station from which you want to exit the assembly process.

FASTEST-WAY

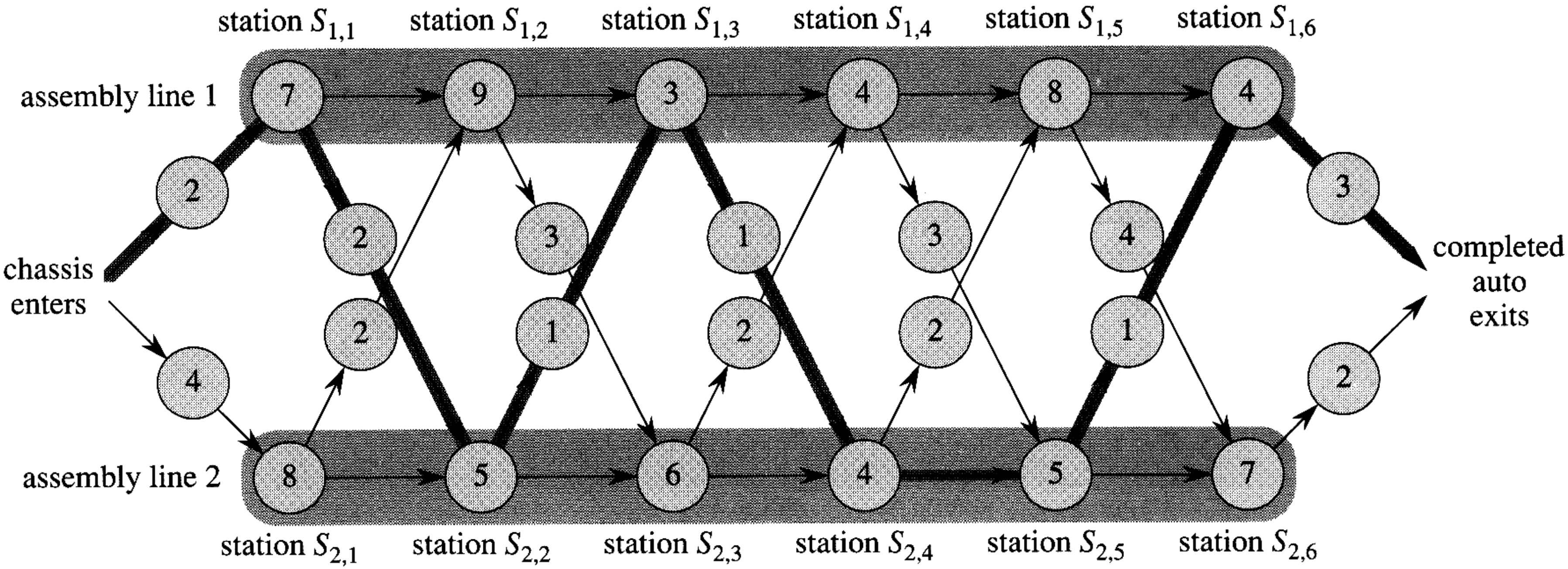
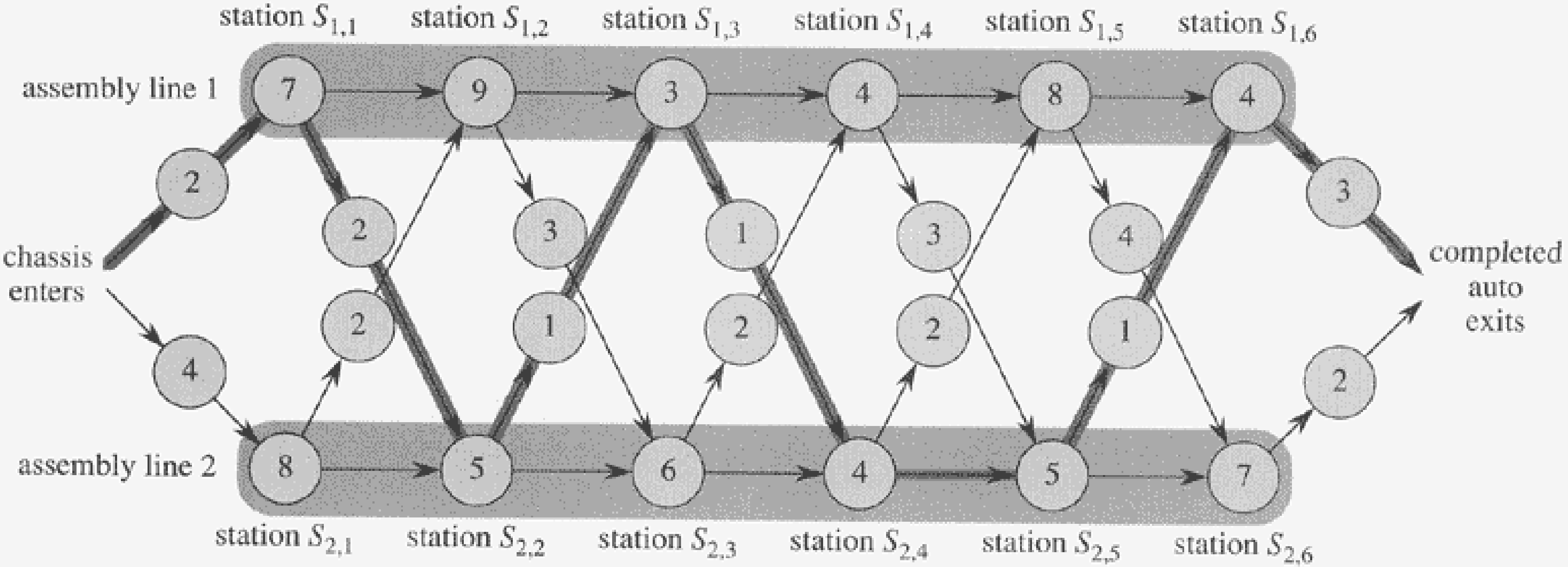


Table 1

Table 2



(a)

j	1	2	3	4	5	6
$f_1[j]$	9	18	20	24	32	35
$f_2[j]$	12	16	22	25	30	37

$f^* = 38$

j	2	3	4	5	6
$l_1[j]$	1	2	1	1	2
$l_2[j]$	1	2	1	2	2

$l^* = 1$

(b)

Figure 15.2 **(a)** An instance of the assembly-line problem with costs e_i, $a_{i,j}$, $t_{i,j}$, and x_i indicated. The heavily shaded path indicates the fastest way through the factory. **(b)** The values of $f_i[j]$, f^*, $l_i[j]$, and l^* for the instance in part (a).

FASTEST-WAY

```
FASTEST-WAY (a, t, e, x, n)
    $f_1[1] \leftarrow e_1 + a_{1,1}$
    $f_2[1] \leftarrow e_2 + a_{2,1}$
    for $j \leftarrow 2$ to $n$ do
        if $f_1[j-1] + a_{1,j} \leq f_2[j-1] + t_{2,j-1} + a_{1,j}$
            then $f_1[j] \leftarrow f_1[j-1] + a_{1,j}$
                 $l_1[j] \leftarrow 1$
            else $f_1[j] \leftarrow f_2[j-1] + t_{2,j-1} + a_{1,j}$
                 $l_1[j] \leftarrow 2$
        if $f_2[j-1] + a_{2,j} \leq f_1[j-1] + t_{1,j-1} + a_{2,j}$
            then $f_2[j] \leftarrow f_2[j-1] + a_{2,j}$
                 $l_2[j] \leftarrow 2$
            else $f_2[j] \leftarrow f_1[j-1] + t_{1,j-1} + a_{2,j}$
                 $l_2[j] \leftarrow 1$
    if $f_1[n] + x_1 \leq f_2[n] + x_2$
        then $f^* = f_1[n] + x_1$
             $l^* = 1$
        else $f^* = f_2[n] + x_2$
             $l^* = 2$
```

FASTEST-WAY

Why is this the fastest way?

1. Because we are not re-doing any of our steps. We start at the beginning and work our way to the end, saving the work that we have done so far at each stage. The recursive solution makes us recalculate solutions we already calculated previously.
2. Because we skip some of the possible paths. The fastest path to $a_{1,4}$ must include the fastest path to $a_{1,3}$ or $a_{2,3}$. This algorithm ignores all other possible paths to $a_{1,4}$.

Step 4: Construct an Optimal Solution

```
PRINT-STATIONS
i ← $l^*$
print “line ” i “, station ” n
for j ← n downto 2 do
    i ← $l_i[j]$
    print “line” i “,station” j-1
```

Matrix-Chain Multiplication

Matrix multiplication is another example of a problem for which the dynamic programming approach can save us a lot of work.

We can multiply two matrices only if they are compatible; the number of rows of one must equal the number of columns of the other.

Matrix-Chain Multiplication

We know how to multiply 2 matrices:

```
MATRIX-MULTIPLY (A, B)
if columns[A] ≠ rows[B]
   then error “incompatble dimensions”
   else for i ← 1 to rows[A] do
           for j ← 1 to columns[B] do
              C[i, j] ← 0
              for k ← 1 to columns[A] do
                 C[i, j] ← C[i, j] + A[i, k] * B[k, j]
return C
```

Matrix-Chain Multiplication

But what if we want to multiply a sequence of matrices, such as:

A_1 = 20 x 100, A_2 = 100 x 40, A_3 = 40 x 20

and return the product of this “chain” of matrix multiplications?

We could multiply A_1 x A_2 to get B_1 = 20 x 40, and then multiply B x A_3 to get C = 20 x 20, or

We could multiply A_2 x A_3 to get B_2 = 100 x 20, and then multiply B_2 x A_1 to get C = 100 x 100

Matrix-Chain Multiplication

We can enclose the components of the sequence in parentheses to show in what order we would carry out the matrix multiplications:

$(A_1 \text{ x } A_2) \text{ x } A_3$ or

$A_1 \text{ x } (A_2 \text{ x } A_3)$

Matrix Chain Multiplication

$$A_1 \times A_2 \times A_3 \times .... \times A_n$$

- Matrix multiplication is associative
- We can determine all ways that a sequence can be parenthesized give the same answer
- But some are much less expensive to compute

Matrix-Chain Multiplication

With a sequence of several matrices, the order in which we multiply them may make a difference in the number of multiplication operations required.

The matrix chain multiplication problem asks:

Given a sequence $< A_1, A_2, \ldots A_n>$ of matrices to be multiplied, how do we parenthesize the sequence so to require the fewest multiplication operations?

Matrix Chain Multiplication Problem

- Given a chain $< A_1, A_2, . . ., A_n >$ of n matrices, where i = 1, 2, . . ., n and matrix A_i has dimension $p_{i-1} \times p_i$, fully parenthesize the product $A_1 \times A_2 \times .... \times A_n$ in a way that minimizes the number of scalar multiplications

Matrix-Chain Multiplication

Suppose we take as our example:

A_1 = 20 x 100, A_2 = 100 x 40,

A_3 = 40 x 30, A_4 = 30 x 10

Do we parenthesize the sequence as, e.g., $((A_1 x A_2) x (A_3 x A_4))$ or $((A_1 x (A_2 x A_3)) x A_4)$?

Which sequence will require the fewest multiplications?

Example

Matrix	Dimensions
A	13 x 5
B	5 X 89
C	89 X 3
D	3 X 34
M = A B C D	13 x 34

Matrix	*Dimensions*
A	13 x 5
B	5 X 89
C	89 X 3
D	3 X 34

Multiply:	*# of operations*	*Resulting matrix*
(A B)	13 x 5 x 89	13 x 89
((AB)C)	$\underline{\text{13 x 89}}$ x 3	13 x 3
(((AB)C)D)	$\underline{\text{13 x 3}}$ x 34	13 x 34

Comparison of results

Parenthesization	Scalar multiplications
1 ((A B) C) D	10,582
2 (A B) (C D)	54,201
3 (A (B C)) D	2,856
4 A ((B C) D)	4,055
5 A (B (C D))	26,418

Number of Parenthesizations

$T(n) =$ number of different ways to parenthesize

Make first cut between position i and $(i+1)$ where

$(1 \leq \mathrm{i} < \mathrm{n})$

$$M = M_1 M_{2\ldots} M_i M_{i+1} M_{i+2} \ldots M_n$$

$$T(n) = \sum_{i=1}^{n-1} T(i)T(n-i)$$

T(n) ways to parenthesize

n	1	2	3	4	5	10	15
T(n)	1	1	2	5	14	4,862	2,674,440

$$T(n) = \frac{1}{n}\binom{2n-2}{n-1}$$

Catalan numbers

Very fast growing exponenti

$$T(n) = \Omega\left(\frac{4^n}{n}\right)$$

Steps in DP Solutions to Optimization Problems

1 <u>Characterize the structure of an optimal solution</u>

2 Recursively define the value of an optimal solution

3 Compute the value of an optimal solution in a bottom-up manner

4 Construct an optimal solution from computed information

Step 1

- Show that the principle of optimality applies
  - An optimal solution to the problem contains within it optimal solutions to subproblems
  - Let $A_{i..j}$ be the optimal way to parenthesize $A_iA_{i+1}...A_j$
  - Suppose the optimal solution has the first split at position k

    $A_{1..k}$ $A_{k+1..n}$
  - Each of these subproblems must be optimally parenthesized

Step 1 continued

- Example: $A_1A_2A_3A_4A_5A_6$

- Suppose this is the optimal solution $(A_1A_2)(A_3A_4A_5A_6)$, where

  $i = 1, k = 2, j = 6$
- The subproblem $(A_3A_4A_5A_6)$ must be optimally parenthesized

Step 2: Recursive Solution

$$m[i, j] = \begin{cases} 0 & \text{if i = j} \\ \min\limits_{\text{i} \leq \text{k} < \text{j}} \{m[i, k] + m[k+1, j] + p_{i-1} p_k p_j\} & \text{if i < j} \end{cases}$$

The running time of this recurrence is: $\Omega(2^n)$

Step 2: Example

A: 20 x 2

B: 2 x 30

C: 30 x 12

D: 12 x 8

$p_0 = 20$

$p_1 = 2$

$p_2 = 30$

$p_3 = 12$

$p_4 = 8$

Step 2: Example

A(B(CD)): 30x12x8 + 2x30x8 + 20x2x8 = 3,680

(AB)(CD): 20x2x30 + 30x12x8 + 20x30x8 = 8,880

A((BC)D): 2x30x12 + 2x12x8 + 20x2x8 = 1,232

(A(BC))D: 2x30x12 + 20x2x12 + 20x12x8 = 3,120

((AB)C)D: 20x2x30 + 20x30x12 + 20x12x8 = 10,320

We want the MIN of these matrix multiplication operations. That is, we want to know the optimal order for multiplying *n* matrices.

Step 2: Example

A_1: 30 x 35

A_2: 35 x 15

A_3: 15 X 5

A_4: 5 X 10

A_5: 10 X 20

A_6: 20 X 25

$p_0 = 30$

$p_1 = 35$

$p_2 = 15$

$p_3 = 5$

$p_4 = 10$

$p_5 = 20$

$p_6 = 25$

Step 2: Example (continued)

$$m[2, 5] = \min \begin{cases} m[2, 2] + m[3, 5] + p_1p_2p_5 = 0 + 2500 + 35 * 15 * 20 = 13000 \\ m[2, 3] + m[4, 5] + p_1p_3p_5 = 2625 + 1000 + 35 * 5 * 20 = 7125 \\ m[2, 4] + m[5,5] + p_1p_4p_5 = 4375 + 0 + 35 * 10 * 20 = 11375 \end{cases}$$

Step 3: Computing Optimal Costs

```
MATRIX-CHAIN-ORDER (p)
n ← length[p] – 1
for i ← 1 to n do
    m[i, i] ← 0
for L ← 2 to n do    ▷ L is the chain length
    for i ← 1 to n-L+1 do
        j ← i + L – 1
        m[i, j] ← ∞
        for k ← i to j-1 do
            q ← m[i, k] + m[k+1, j] + $p_{i-1}p_kp_j$
            if q < m[i, j]
                then m[i, j] ← q
                     s[i, j] ← k
return m and s
```

Step 4: Constructing Optimal Solution

- Consider two procedures:
  - one that prints optimal parenthesization for $A_{i..j}$
  - one that computes $A_{i..j}$ using optimal parenthesization

Printing Optimal Parenthesization

```
PRINT-OPTIMAL-PARENS (s, i, j)
  if $i = j$
    then print “A”$_i$
    else print “(”
         PRINT-OPTIMAL-PARENS (s, i, s[i, j])
         PRINT-OPTIMAL-PARENS (s, s[i, j] + 1, j)
         print “)”
```

Multiplying Using Optimal Parenthesization

```
MATRIX-CHAIN-MULTIPLY (A, s, i, j)
    if j > i
       then X ← MATRIX-CHAIN-MULTIPLY (A, s, i, s[i, j])
            Y ← MATRIX-CHAIN-MULTIPLY (A, s, s[i, j] + 1, j)
            return MATRIX-MULTIPLY (X, Y)
       else return $A_i$
```

Multiplying Using Optimal Parenthesization

```
MATRIX-CHAIN-MULTIPLY (A, s, i, j)
    if j > i
      then X ← MATRIX-CHAIN-MULTIPLY (A, s, i, s[i, j])
           Y ← MATRIX-CHAIN-MULTIPLY (A, s, s[i, j] + 1, j)
           return MATRIX-MULTIPLY (X, Y)
      else return $A_i$
```

Initial call:

MATRIX-CHAIN-MULTIPLY(A,s,1,n) where **A = < A1,A2, . . .,An>**

(From Exercise 15.2-2)

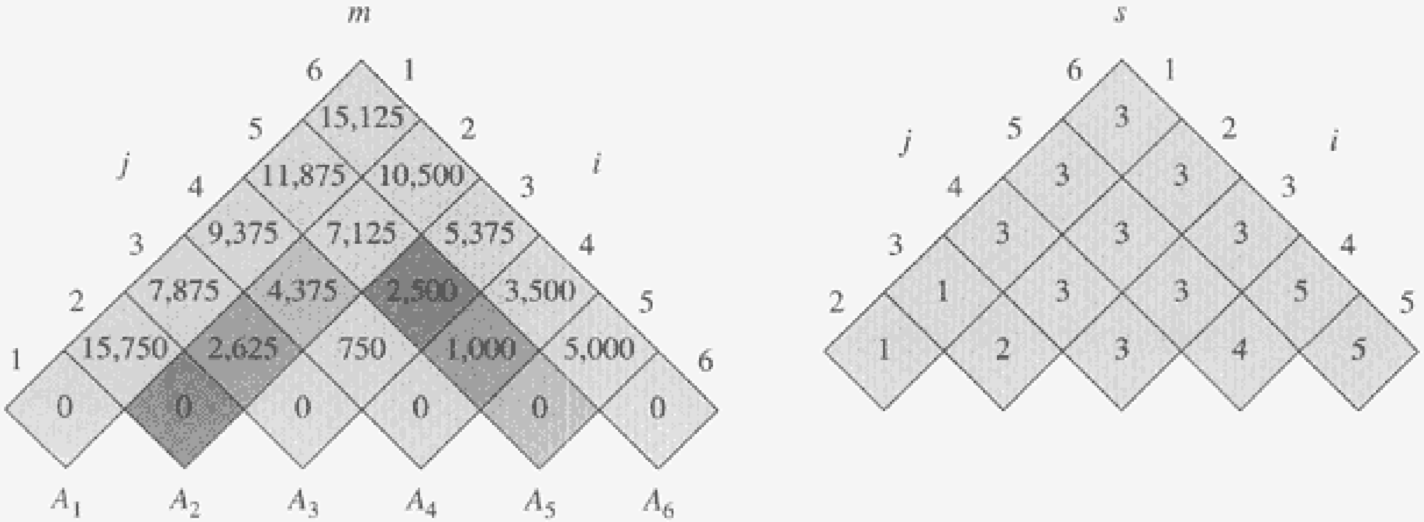


Figure 15.3 The m and s tables computed by MATRIX-CHAIN-ORDER for $n = 6$ and the following matrix dimensions:

matrix	dimension
A_1	30×35
A_2	35×15
A_3	15×5
A_4	5×10
A_5	10×20
A_6	20×25

The tables are rotated so that the main diagonal runs horizontally. Only the main diagonal and upper triangle are used in the m table, and only the upper triangle is used in the s table. The minimum number of scalar multiplications to multiply the 6 matrices is $m[1, 6] = 15{,}125$. Of the darker entries, the pairs that have the same shading are taken together in line 9 when computing

$$m[2,5] = \min \begin{cases} m[2,2] + m[3,5] + p_1 p_2 p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 & = 13000\,, \\ m[2,3] + m[4,5] + p_1 p_3 p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 & = 7125\,, \\ m[2,4] + m[5,5] + p_1 p_4 p_5 = 4375 + 0 + 35 \cdot 10 \cdot 20 & = 11375 \end{cases}$$
$$= 7125\,.$$

Building Table *m*

1	2	3	4	5	6	
0	15,750	7,875	9,375	11,875	15,125	1
	0	2,625	4,375	7,125	10,500	2
		0	750	2,500	5,375	3
			0	1,000	3,500	4
				0	5,000	5
					0	6

A1 = 30x35

A2 = 35x15

A3 = 15x05

A4 = 05x10

A5 = 10x20

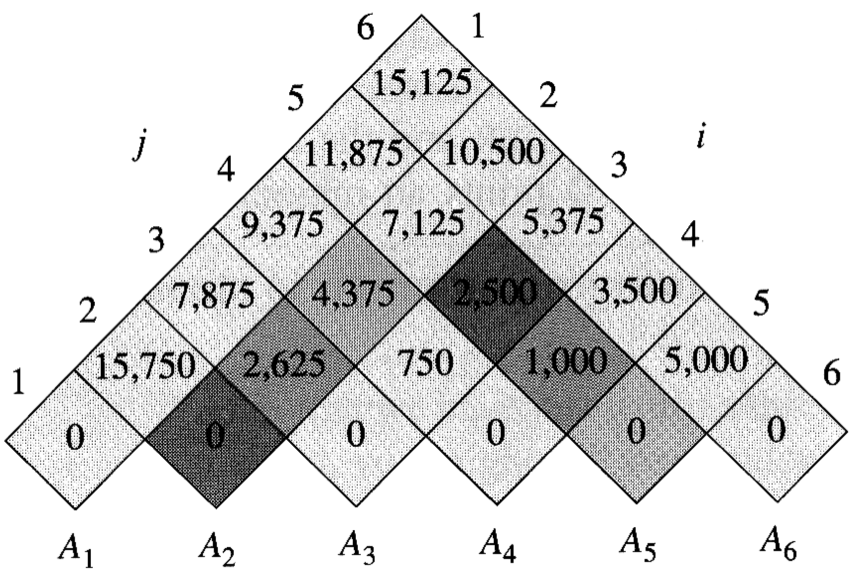
A6 = 20x25

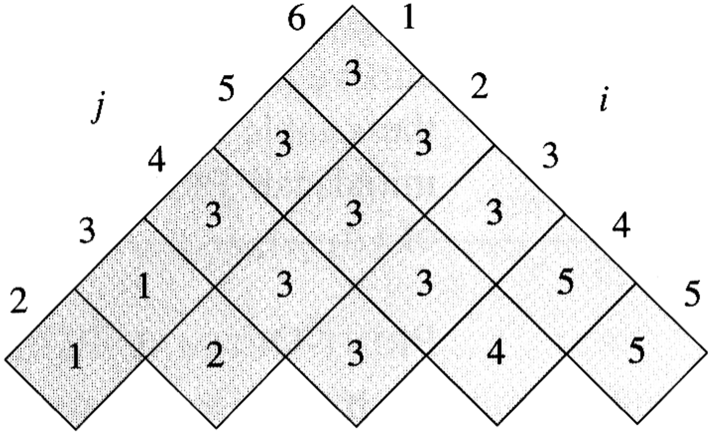
Building Table *s*

2	3	4	5	6	
1	1	3	3	3	1
	2	3	3	3	2
		3	3	3	3
			4	5	4
				5	5

P1 = 30
P2 = 35
P3 = 15
P4 = 05
P5 = 10
P6 = 20
P7 = 25

Building the Tables





Compute k, where k = the position to at which to split the sequence of matrix multiplications in order to minimize the cost to multiply a specific pair of tables. For example,

$$m[2, 5] = \min \begin{cases} m[2, 2] + m[3, 5] + p1p2p5 = 0 + 2500 + 35 * 15 * 20 = 13000 \\ m[2, 3] + m[4, 5] + p1p3p5 = 2625 + 1000 + 35 * 5 * 20 = 7125 \\ m[2, 4] + m[5,5] + p1p4p5 = 4375 + 0 + 35 * 10 * 20 = 11375 \end{cases}$$

$= 7125$

Elements of Dynamic Programming

- Optimal Substructure
- Overlapping subproblems

Optimal Substructure

- An optimal solution to the problem contains within it optimal solutions to subproblems
- Steps in discovering optimal substructure:
  - Show that a solution to the problem involves making a choice which leaves one or more subproblems to be solved
  - You assume you are given a choice that leads to an optimal solution
  - Given this choice, which subproblems ensue and how do you characterize the resulting space of subproblems?
  - Show that the solutions to the subproblems used within an optimal solution to the problem must themselves be optimal

Optimal Substructure (Subtleties)

- Don’t assume that optimal substructure applies where it doesn’t
- Examples:
  - Unweighted shortest path (i.e., finding a simple path in a graph from node u to node v consisting of the fewest edges) does exhibit optimal substructure
  - Unweighted longest path *does not*

Overlapping subproblems

- The space of subproblems must be “small” in the sense that a recursive algorithm for the problem solves the same subproblems over and over, rather than always generating new subproblems.
- DP algorithms typically take advantage of overlapping subproblems by solving each subproblem once and then storing the solution in a table (for constant time lookup)

Step 2

- Define value of the optimal solution recursively in terms of optimal solutions to subproblems.
- Consider the problem $A_{i..j}$
- Let m[i,j] be the minimum number of scalar multiplications needed to compute matrix $A_{i..j}$
- The cost of the cheapest way to compute $A_{1..n}$ is m[1,n]

Step 2 continued

- Define m[i,j]

If i = j the chain has one matrix and the cost m[i,i] = 0 for all i ≤ n

If i < j

Assume the optimal split is at position k where i ≤ k < j

The cost of this optimal parenthesization is the cost of computing the matrix A i..k, plus the cost of computing A k+1..j plus the cost of multiplying them together.

Step 2 continued

- m[i,j] = m[i,k] + m[k+1,j] + $p_{i-1} p_k p_j$
  to compute $A_{i..k}$ and $A_{k+1..j}$
- Problem: we don’t know what the value for k is, but there are only j-i possibilities

$$m[i,j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k < j} \{ m[i,k] + m[k+1,j] + p_{i-1} p_k p_j \} & \text{if } i < j \end{cases}$$

Step 2 continued

- A_1: 30 x 35
- A_2: 35 x 15
- A_3: 15 X 5
- A_4: 5 X 10
- A_5: 10 X 20
- A_6: 20 X 25

- $p_0 = 30$
- $p_1 = 35$
- $p_2 = 15$
- $p_3 = 5$
- $p_4 = 10$
- $p_5 = 20$
- $p_6 = 25$

Step 2 continued

- Consider the subproblem $A_1(A_2A_3A_4A_5)A_6$
- i = 2, j = 5, and three possible k’s:

k = 2, $(A_2)(A_3A_4A_5)$

k = 3, $(A_2A_3)(A_4A_5)$

k = 4, $(A_2A_3A_4)(A_5)$

Step 3

- Compute optimal cost by using a bottom-up approach
- Example: 4 matrix chain

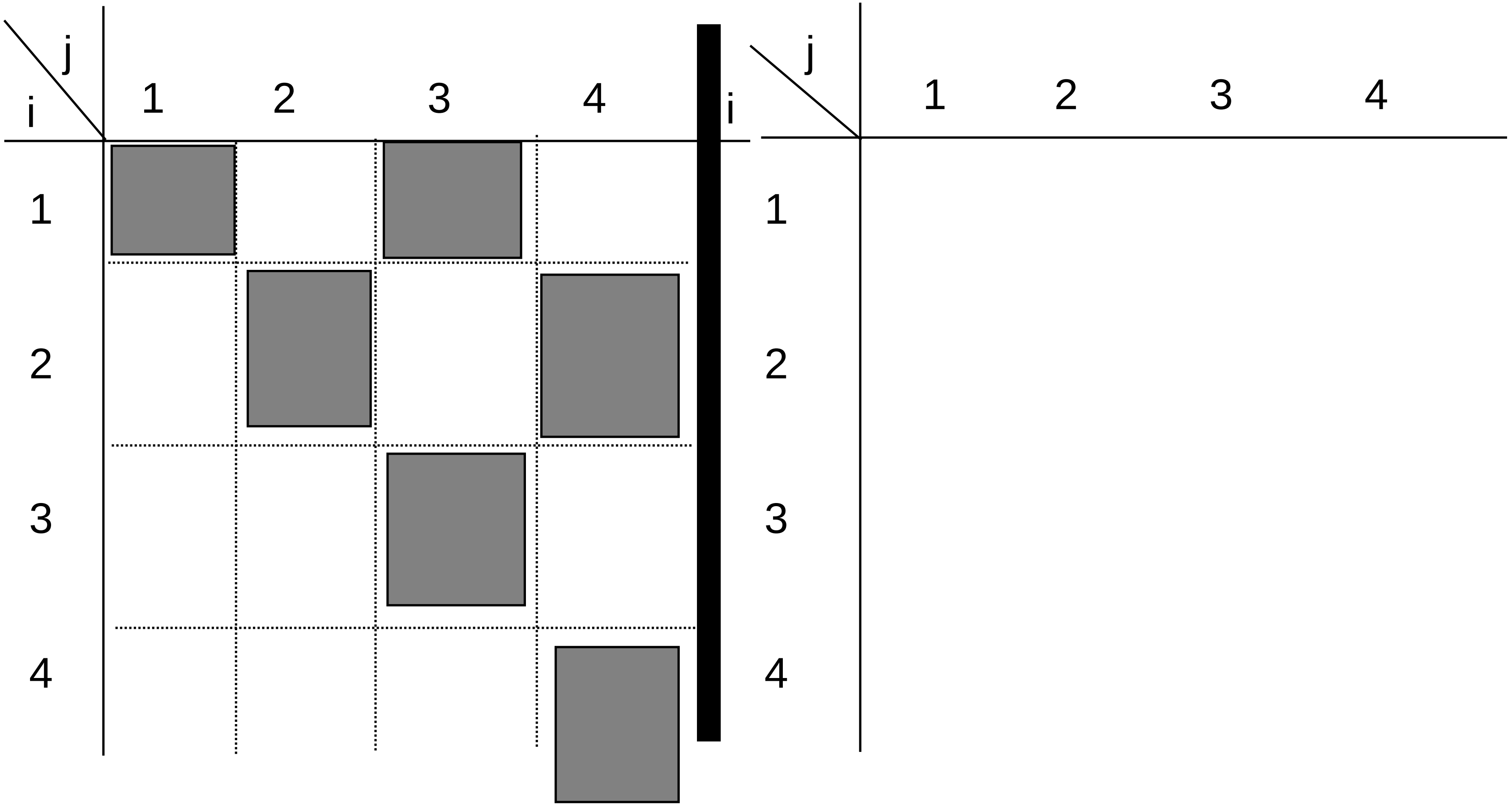
Matrix	Dimensions
A_1	100 x 1
A_2	1 x 100
A_3	100 x 1
A_4	1 x 100

p_o	p_1	p_2	p_3	p_4
100	1	100	1	100

Step 3 continued

- Bottom-up approach

- (A_1A_2), (A_2A_3), (A_3A_4)
- $(A_1A_2)(A_3)$,$(A_1)(A_2A_3)$
- $(A_2A_3)(A_4)$,$(A_2)(A_3A_4)$
- $(A_1)(A_2A_3A_4)$,….



m

s

p_o	p_1	p_2	p_3	p_4
100	1	100	1	100

MATRIX-CHAIN-ORDER

```
MATRIX-CHAIN-ORDER(p)
n ← length[p] - 1
for i ← 1 to n do
    m[i,i] ← 0
for l ← 2 to n do
    for i ← 1 to n - l + 1 do
        j ← i + l - 1
        m[i,j] ← ∞
        for  k ← i to j – 1 do
            q ← m[i,k] + m[k+1,j] + $p_{i-1} p_k p_j$
            if q < m[i,j]
                then m[i,j] = q
                     s[i,j] = k
return m and s
```

Time and Space Complexity

- Time complexity
  - Triply nested loops
  - $O(n^3)$
- Space complexity
  - n x n 2 dimensional table
  - $O(n^2)$

Step 4: Constructing the Optimal Solution

- Matrix-Chain-Order determines the optimal number of scalar multiplications
- Does not directly compute the product
- Step 4 of the dynamic programming paradigm is to construct an optimal solution from computed information.
- The *s* matrix contains the optimal split for every level

MATRIX-CHAIN-MULTIPLY

```
MATRIX-CHAIN-MULTIPLY(A,s,i,j)
if j > i
   then X ← MATRIX-CHAIN-MULTIPLY(A,s,i,s[i,j])
        Y ← MATRIX-CHAIN-MULTIPLY(A,s,s[i,j]+1,j)
        return MATRIX-MULTIPLY(X,Y)
   else return $A_i$
```

Initial call:

`MATRIX-CHAIN-MULTIPLY(A,s,1,n)` where

$A = < A_1,A_2, . . .,A_n>$

Comments on Dynamic Programming

- Useful when the problem can be reduced to several “overlapping” subproblems.
- All possible subproblems are computed
- Computation is done by maintaining a large matrix
- Usually has large space requirement
- Running times are usually at least quadratic

Memoization

- Variation on dynamic programming
- Idea–memoize the natural but inefficient recursive algorithm
- As each subproblem is solved, store values in table
- Initialize table with values that indicate if the value has been computed

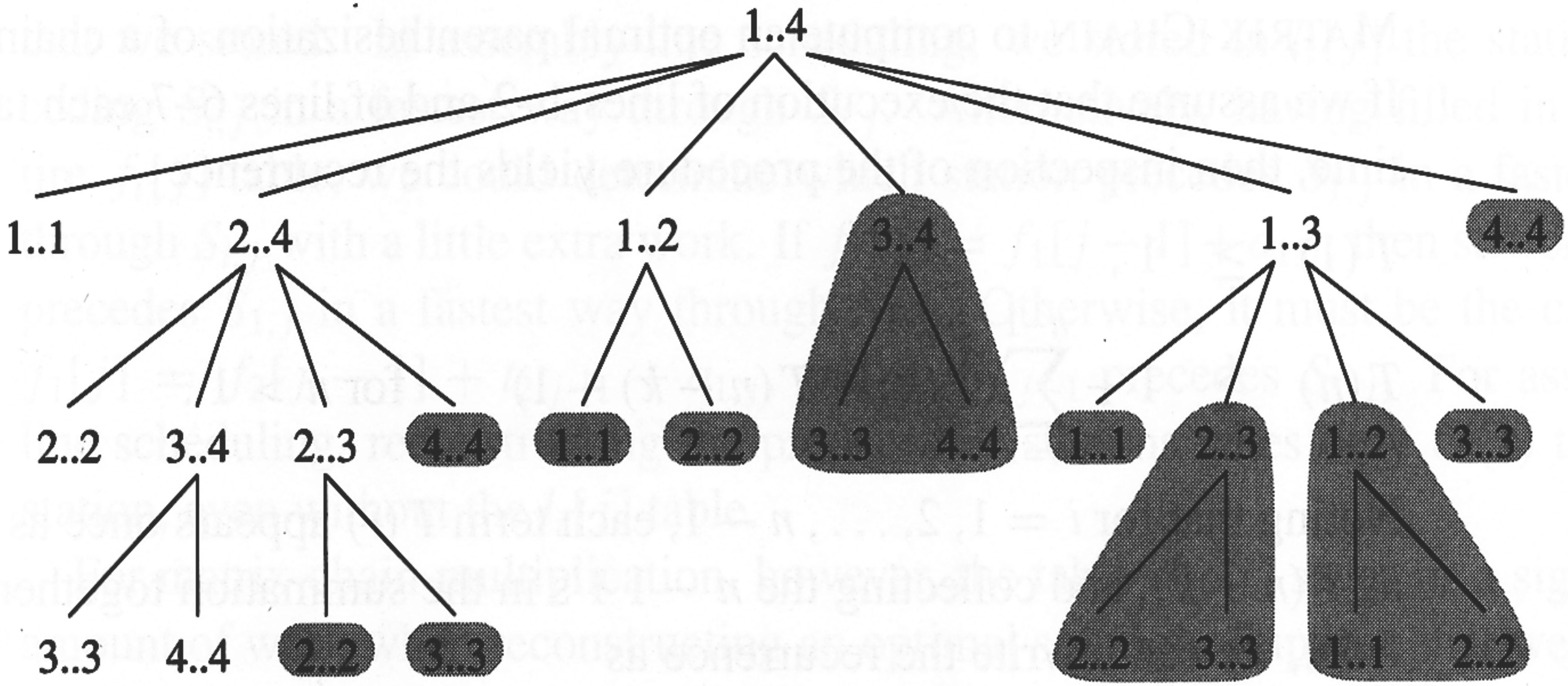
Memoization

- DP approach with the *top-down* strategy of recursion
- Trade space for speed by storing solutions to sub-problems rather than re-computing them.
- As solutions are found for suproblems, they are recorded in a table
- Before any recursive call on a given subproblem, say Q,
  - If no solution has been stored, go ahead with recursive call.
  - If a solution has been stored for Q, retrieve the stored solution, and do not make the recursive call.
- Just before returning the solution store it in the table

Memoization

- See MEMOIZED-MATRIX-CHAIN (p348)
- See figure 15.5 (p345)
- In general if all subproblems must be solved at least once, bottom-up DP approach usually outperforms memoization
- However, memoization solves only those subproblems required

Advantage of Memoized Version



MEMOIZED-MATRIX-CHAIN

```
MEMOIZED-MATRIX-CHAIN(p)
    n ← length(p) - 1
    for i ← 1 to n
        do for j ← i to n
            do m[i,j] ← ∞
    return LOOKUP-CHAIN(p,1,n)
```

LOOKUP-CHAIN Function

```
LOOKUP-CHAIN(p,i,j)
if m[i,j] < $\infty$
      then return m[i,j]
if i = j
      then m[i,j] $\leftarrow$ 0
      else for k $\leftarrow$ i  to j – 1 do
                  q $\leftarrow$ LOOKUP-CHAIN(p,i,k) +
                     LOOKUP-CHAIN(p,k+1,j) + $p_{i-1}p_kp_j$
                  if q < m[i,j]
                     then m[i,j] $\leftarrow$ q
 return m[i,j]
```

Space and Time Requirements of Memoized Version

- Running time $\theta(n^3)$

- Storage $\theta(n^2)$

Longest Common Subsequence

- Definition 1: Subsequence

Given a sequence

$X = < x_1, x_2, . . . , x_m>$

then another sequence

$Z = < z_1, z_2, . . . , z_k>$

is a subsequence of X if there exists a strictly increasing sequence $<i_1, i_2, . . . , i_k>$ of indices of x such that for all j = 1,2,...k we have $x_{i_j}= z_j$

Example

The items in the sequence do not have to be adjacent. For example:

X = <A,B,D,F,M,Q>

Z = <B, F, M>

Z is a subsequence of X with index sequence <2,4,5>

More Definitions

- Definition 2: Common subsequence
  - Given 2 sequences X and Y, we say Z is a common subsequence of X and Y if Z is a subsequence of X and a subsequence of Y
- Defintion 3: Longest common subsequence problem
  - Given X = < x_1, x_2, . . . , x_m> and Y = < y_1, y_2, . . . , y_n> find a maximum length common subsequence of X and Y

Example

X = <A,B,C,B,D,A,B>

Y = <B,D,C,A,B,A>

What is the *longest common subsequence*?

Is it <B, C, A, B> ?

Or maybe <B, D, A, B>?

Or <B, C, B, A>?

Is there a subsequence longer than 4?

Brute Force Algorithm

1. For every subsequence of X
2. Is there a subsequence in Y?
3. If yes, is it longer than the longest subsequence found so far?

What is the complexity of this algorithm?

Brute Force Algorithm

How many subsequences are there in <A, B>?

3: <A>, <B>, and <A, B>

How many subsequences are there in <A, B, C>?

7: <A>, <B>, <C>, <A, B>, <A, C>, <B, C>, and <A, B, C>

How many subsequences are there in <A, B, C, D>?

15: <A>, <B>, <C>, <D>, <A, B>, <A, C>, <A, D>, <B, C>, <B, D> <C, D>, <A, B, C>, <A, B, D>, <A, C, D>, <B, C, D>, and <A, B, C, D>

What is the pattern?

Yet More Definitions

- Definition 4: Prefix of a subsequence

  If $X = < x_1, x_2, . . . , x_m>$, the ith prefix of X for $i = 0,1,...,m$ is $X_i = < x_1, x_2, . . . , x_i>$

- Example
  - if $X = <A,B,C,D,E,F,H,I,J,L>$ then:

    $X_4 = <A,B,C,D>$

    $X_0 = <>$

Optimal Substructure

- Theorem 15.1 Optimal Substructure of LCS

  Let X = < x_1, x_2, . . . , x_m> and Y = < y_1, y_2, . . . , y_n> be sequences and let Z = < z_1, z_2, . . . ,z_k> be any LCS of X and Y

  1. if $x_m = y_n$ then $z_k = x_m = y_n$ and z_{k-1} is an LCS of x_{m-1} and y_{n-1}
  2. if $x_m \neq y_n$ and $z_k \neq x_m$ Z is an LCS of X_{m-1} and Y
  3. if $x_m \neq y_n$ and $z_k \neq y_n$ Z is an LCS of X_m and Y_{n-1}

Subproblem structure

- Case 1
  - if $x_m = y_n$ then there is one subproblem to solve: find a LCS of X_{m-1} and Y_{n-1} and append x_m
- Case 2
  - if $x_m \neq y_n$ then there are two subproblems:
    - find an LCS of X_m and Y_{n-1}
    - find an LCS of X_{m-1} and Y_n
    - pick the longer of the two

Cost of Optimal Solution

- Cost is the length of the common subsequence
- We want to pick the longest one
- Let c[i,j] be the length of an LCS of the sequences X_i and Y_j
- Base case is an empty subsequence:

  in that case, c[i,j] = 0 because there is no LCS

Recurrence for LCS

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & \text{if } i, j > 0 \text{ and } x_i \neq y_j \end{cases}$$

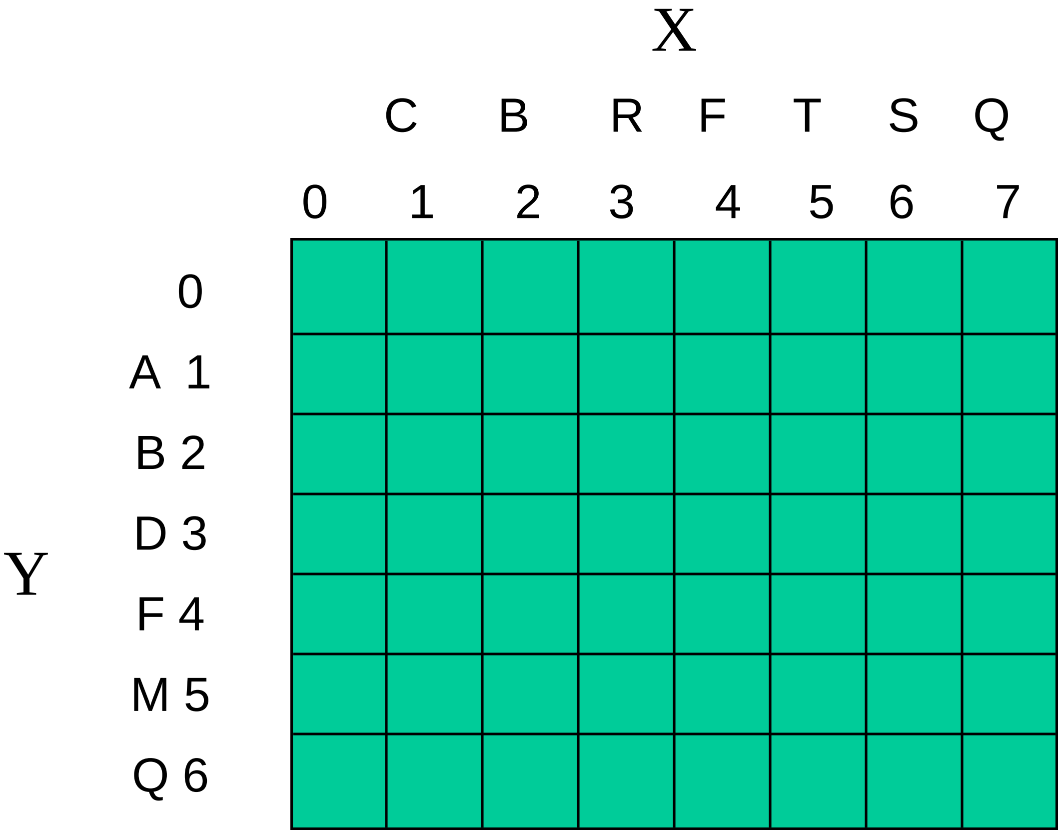
Recurrence for LCS

There are 2^m subsequences, so the running time of this brute force method would be $O(2^m)$.

Using a Dynamic Programming approach, we can compute all O(nm) distinct subproblems in O(nm) time, and use them to construct a LCS in O(m + n) time.

Dynamic Programming Solution

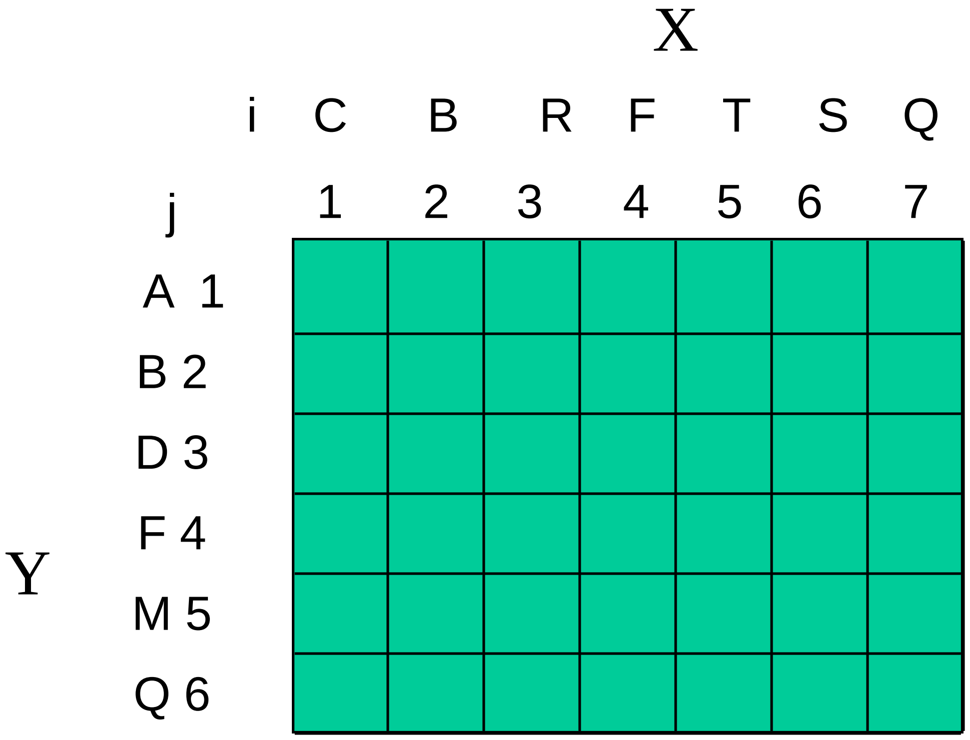
- Table c[0..m, 0..n] stores the length of an LCS of X_i and Y_j c[i,j]
- Table b[1..m, 1..n] stores pointers to optimal subproblem solutions



X

Y		C	B	R	F	T	S	Q
	0	1	2	3	4	5	6	7
0								
A 1								
B 2								
D 3								
F 4								
M 5								
Q 6								

Matrix C



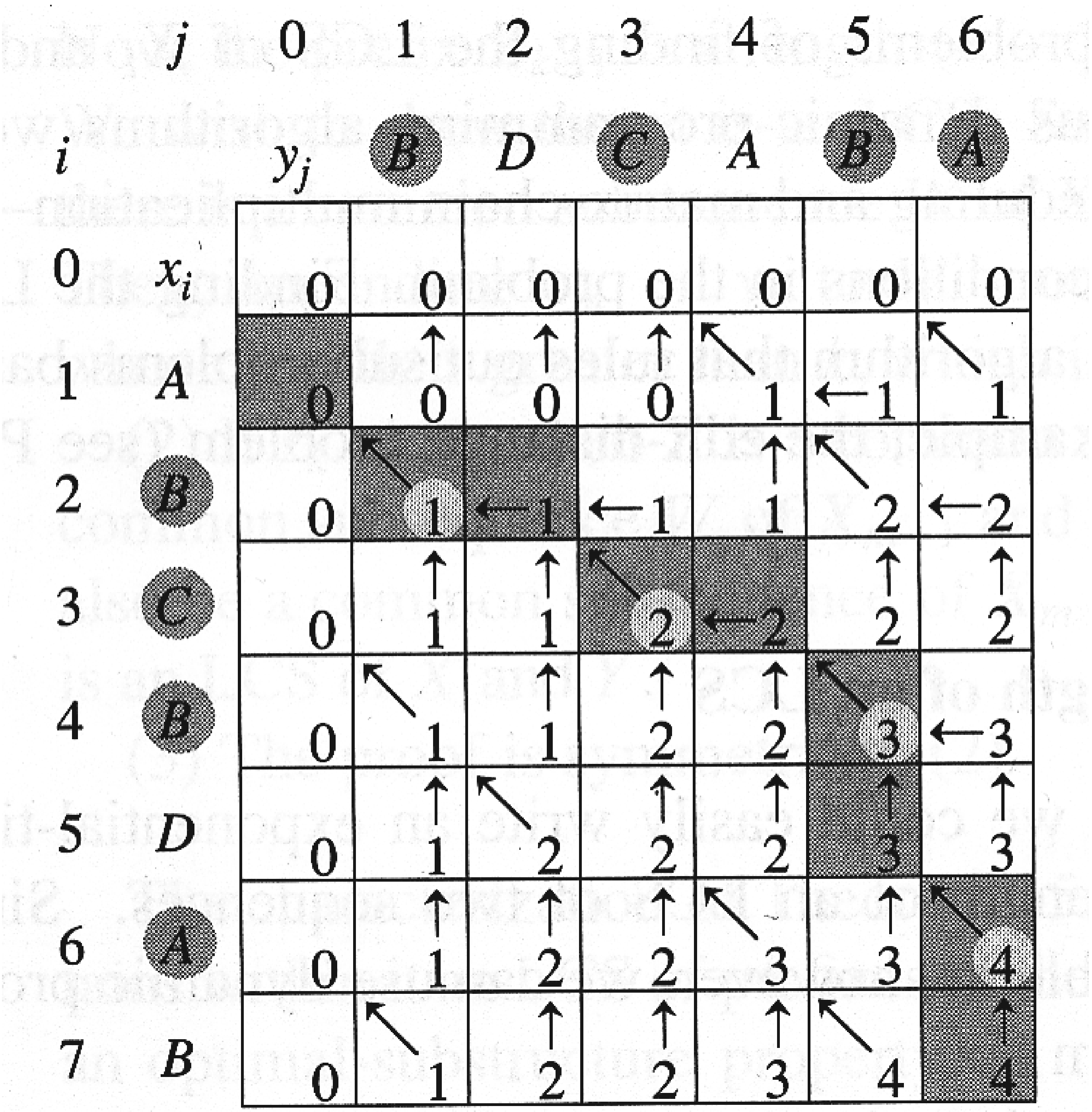
X

Y

	i	C	B	R	F	T	S	Q
j		1	2	3	4	5	6	7
A	1							
B	2							
D	3							
F	4							
M	5							
Q	6							

Matrix B

Tables c and b



	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

```
LCS-LENGTH(X,Y)
m ← length[X]
n ← length[Y]
for i ← 1 to m do
    c[i,0] ← 0
for j ← 0 to n do
    c[0,j] ← 0
for i ← 1 to m do
    for j ← 1 to n do
        if $x_i = y_j$
            then c[i,j] ← c[i-1,j-1] + 1
                 b[i,j] ← "↖"
            else if c[i-1,j] ≥ c[i,j-1]
                    then c[i,j] ← c[i-1,j]
                         b[i,j] ← "↑"
                    else c[i,j] ← c[i,j-1]
                         b[i,j] ← "←"
return c and b
```

PRINT-LCS

```
PRINT-LCS (b, X, i, j)
  if i = 0 or j = 0
     then return
  if b[i,j] = ”↖”
     then PRINT-LCS(b,X,i-1,j-1)
          print $x_i$
     else if b[i,j] = ”↑”
          then PRINT-LCS(b,X,i-1,j)
          else PRINT-LCS(b,X,i,j-1)
```

Code Complexity

- Time complexity?
  - O(mn) to compute tables b and c
  - O(m + n) to print out the LCS
- Space complexity
  - O(mn) for tables b and c
- Improved space complexity?
  - We can eliminate table b, but still need table c
  - Print-LCS can be improved to use only two rows of table C at a time to compute the *length* of an LCS
  - But not if we need to reconstruct the *sequence*

LCS & Matrix-chain multiplication

$$m[i, j] = \begin{cases} 0 & \text{if i = j} \\ \min\limits_{\text{i} \le \text{k} < \text{j}} \{m[i,k] + m[k+1, j] + p_{i-1} p_k p_j\} & \text{if i < j} \end{cases}$$

$$c[i, j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & \text{if } i, j > 0 \text{ and } x_i \neq y_j \end{cases}$$

The recurrences for matrix-chain-order parenthesization and longest common subsequence are very similar; no wonder they can be solved in very similar ways.

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i							
1	A							
2	B							
3	C							
4	B							
5	D							
6	A							
7	B							