04 IMU Sensor Fusion

备注：卡尔曼滤波步骤参见《卡尔曼滤波与组合导航原理》P34

Propagation equations

1. We propagate the bias using the discretized form of eq. (135) as

$$\hat{\mathbf{b}}_{k+1|k} = \hat{\mathbf{b}}_{k|k} \tag{193}$$

2. Using the measurement $\boldsymbol{\omega}_{m_{k+1}}$ and $\hat{\mathbf{b}}_{k+1|k}$, we form the estimate of the new turn rate according to eq. (136) as

$$\hat{\boldsymbol{\omega}}_{k+1|k} = \boldsymbol{\omega}_{m_{k+1}} - \hat{\mathbf{b}}_{k+1|k} \tag{194}$$

3. We propagate the quaternion using a first order integrator (cf. section 1.6.2) with $\hat{\boldsymbol{\omega}}_{k|k}$ and $\hat{\boldsymbol{\omega}}_{k+1|k}$ to obtain $\hat{\bar{q}}_{k+1|k}$.
4. From the formulas in sections 2.5.1 and 2.5.2 we compute the state transition matrix $\boldsymbol{\Phi}$ and the discrete time noise covariance matrix $\mathbf{Q}_d$.
5. We compute the state covariance matrix according to the Extended Kalman Filter equation

$$\mathbf{P}_{k+1|k} = \boldsymbol{\Phi}\mathbf{P}_{k|k}\boldsymbol{\Phi}^{\mathrm{T}} + \mathbf{Q}_d \tag{195}$$

补充：

步骤0 状态量取误差项（线性化更准）：

Since the rotation associated with the error quaternion $\delta\bar{q}$ can be assumed to be very small, we can employ the small angle approximation (as seen in section 1.4) and define the attitude error angle vector $\delta\boldsymbol{\theta}$ as follows

$$\delta\bar{q} = \begin{bmatrix}\delta\mathbf{q}\\ \delta q_4\end{bmatrix} \tag{139}$$

$$= \begin{bmatrix}\hat{\mathbf{k}}\sin(\delta\theta/2)\\ \cos(\delta\theta/2)\end{bmatrix} \tag{140}$$

$$\boxed{\approx \begin{bmatrix}\frac{1}{2}\delta\boldsymbol{\theta}\\ 1\end{bmatrix}} \tag{141}$$

This error angle vector $\delta\boldsymbol{\theta}$ is of dimension 3×1 and will be used together with the bias error in the error state vector. The bias error is defined as

$$\Delta\mathbf{b} = \mathbf{b} - \hat{\mathbf{b}} \tag{142}$$

We can now define the error vector as

$$\boxed{\tilde{\mathbf{x}} = \begin{bmatrix}\delta\boldsymbol{\theta}\\ \Delta\mathbf{b}\end{bmatrix}} \tag{143}$$

状态方程（不是指误差）

In direct consequence of the previous analysis, we define a seven-element state vector consisting of the quaternion and the gyro-bias

$$\boxed{\mathbf{x}(t) = \begin{bmatrix}\bar{q}(t)\\ \mathbf{b}(t)\end{bmatrix}} \tag{131}$$

Using the definition of the quaternion derivative (eq. (86)) and the error model (eqs. (111) and (114)), we find the following system of differential equations governing the state

$${}^L_G\dot{\bar{q}}(t) = \frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}_m - \mathbf{b} - \mathbf{n}_{\mathbf{r}})\,{}^L_G\bar{q}(t) \tag{132}$$

$$\dot{\mathbf{b}} = \mathbf{n}_{\mathbf{w}} \tag{133}$$

Taking the expectation of the above yields the prediction equations (cf. [3, p. 422]) for the state within the EKF-framework

$${}^L_G\dot{\hat{\bar{q}}}(t) = \frac{1}{2}\boldsymbol{\Omega}(\hat{\boldsymbol{\omega}})\,{}^L_G\hat{\bar{q}}(t) \tag{134}$$

ekf：预测方程，不包括噪声项

$$\dot{\hat{\mathbf{b}}} = \mathbf{0}_{3\times1} \tag{135}$$

with

$$\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}_m - \hat{\mathbf{b}} \tag{136}$$

Since the bias is constant over the integration interval, we may integrate the quaternion using the zeroth order (cf. section 1.6.1) or first order (cf. section 1.6.2) integrator, using $\hat{\boldsymbol{\omega}}$ instead of $\boldsymbol{\omega}$.

步骤1+2 状态预测b w：

$${}^L_G\dot{\hat{\bar{q}}}(t) = \frac{1}{2}\boldsymbol{\Omega}(\hat{\boldsymbol{\omega}})\,{}^L_G\hat{\bar{q}}(t) \tag{134}$$

ekf：预测方程，不包括噪声项

$$\dot{\hat{\mathbf{b}}} = \mathbf{0}_{3\times1} \tag{135}$$

$$\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}_m - \hat{\mathbf{b}} \tag{136}$$

步骤3 状态预测q：

1.6.2 First Order Quaternion Integrator

The first order quaternion integrator makes the assumption of a linear evolution of $\boldsymbol{\omega}$ during the integration interval Δt. In this case, we have to modify the matrix $\boldsymbol{\Theta}(t_{k+1}, t_k)$ from eq. (96). For that purpose, we introduce the average turn rate $\bar{\boldsymbol{\omega}}$, defined as

$$\boxed{\bar{\boldsymbol{\omega}} = \frac{\boldsymbol{\omega}(t_{k+1}) + \boldsymbol{\omega}(t_k)}{2}} \tag{104}$$

Recognizing the first term as the Taylor series expansion of the matrix exponential, and after replacing $\boldsymbol{\Omega}(\dot{\boldsymbol{\omega}})$ with its definition (eq. (105)), we obtain the final formula

$${}^L_G\bar{q}(t_{k+1}) = \left(\exp\left(\frac{1}{2}\boldsymbol{\Omega}(\bar{\boldsymbol{\omega}})\Delta t\right) + \frac{1}{48}\Big(\boldsymbol{\Omega}(\boldsymbol{\omega}(t_{k+1}))\boldsymbol{\Omega}(\boldsymbol{\omega}(t_k)) - \boldsymbol{\Omega}(\boldsymbol{\omega}(t_k))\boldsymbol{\Omega}(\boldsymbol{\omega}(t_{k+1}))\Big)\Delta t^2\right){}^L_G\bar{q}(t_k) \tag{110}$$

Update

3.2 Kalman Filter Update

Given the propagated state estimates $\hat{\bar{q}}_{k+1|k}$ and $\hat{\mathbf{b}}_{k+1|k}$, as well as their covariance matrix $\mathbf{P}_{k+1|k}$, the current measurement $\mathbf{z}(k+1)$, and the measurement matrix $\mathbf{H}$, we can update our estimate in the following way:

1. Compute the measurement matrix $\mathbf{H}(k)$ according to eq. (211)
2. Compute residual $\mathbf{r}$ according to

$$\boxed{\mathbf{r} = \mathbf{z} - \hat{\mathbf{z}}} \tag{212}$$

(213)

3. Compute the covariance of the residual $\mathbf{S}$ as

$$\mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^{\mathrm{T}} + \mathbf{R} \tag{214}$$

4. Compute the Kalman gain $\mathbf{K}$

$$\mathbf{K} = \mathbf{P}\mathbf{H}^{\mathrm{T}}\mathbf{S}^{-1} \tag{215}$$

5. Compute the correction $\Delta\hat{\mathbf{x}}(+)$

$$\boxed{\Delta\hat{\mathbf{x}}(+) = \begin{bmatrix}\delta\hat{\boldsymbol{\theta}}(+)\\ \Delta\hat{\mathbf{b}}(+)\end{bmatrix} = \begin{bmatrix}2\cdot\delta\hat{\mathbf{q}}(+)\\ \Delta\hat{\mathbf{b}}(+)\end{bmatrix} = \mathbf{K}\mathbf{r}} \tag{216}$$

6. Update the quaternion according to

$$\delta\hat{\bar{q}} = \begin{bmatrix}\delta\hat{\mathbf{q}}(+)\\ \sqrt{1 - \delta\hat{\mathbf{q}}(+)^{\mathrm{T}}\delta\hat{\mathbf{q}}(+)}\end{bmatrix} \tag{217}$$

or, if $\delta\hat{\mathbf{q}}(+)^{\mathrm{T}}\delta\hat{\mathbf{q}}(+) > 1$, using

$$\delta\hat{\bar{q}} = \frac{1}{\sqrt{1 + \delta\hat{\mathbf{q}}(+)^{\mathrm{T}}\delta\hat{\mathbf{q}}(+)}}\cdot\begin{bmatrix}\delta\hat{\mathbf{q}}(+)\\ 1\end{bmatrix} \tag{218}$$

$$\hat{\bar{q}}_{k+1|k+1} = \delta\hat{\bar{q}} \otimes \hat{\bar{q}}_{k+1|k} \tag{219}$$

[1] Note here the difference to the expression obtained if we were using an *additive* instead of *multiplicative* error model, where $\bar{q} = \hat{\bar{q}} + \Delta\bar{q}$ with $\Delta\bar{q} = [\boldsymbol{\Delta}\mathbf{q}^{\mathrm{T}} \quad \Delta q_4]^{\mathrm{T}}$:

$$\boxed{{}^L_G\mathbf{C}(\hat{\bar{q}} + \Delta\bar{q}) - {}^{\hat{L}}_G\mathbf{C}(\hat{\bar{q}}) = 4\hat{q}_4\Delta q_4\mathbf{I}_{3\times3} - 2\hat{q}_4\lfloor\boldsymbol{\Delta}\mathbf{q}\times\rfloor - 2\Delta q_4\lfloor\hat{\mathbf{q}}\times\rfloor + 2\boldsymbol{\Delta}\mathbf{q}\hat{\mathbf{q}}^{\mathrm{T}} + 2\hat{\mathbf{q}}\boldsymbol{\Delta}\mathbf{q}^{\mathrm{T}}} \tag{205}$$

7. Update the bias

$$\hat{\mathbf{b}}_{k+1|k+1} = \hat{\mathbf{b}}_{k+1|k} + \Delta\hat{\mathbf{b}}(+) \tag{220}$$

8. Update the estimated turn rate using the new estimate for the bias

$$\hat{\boldsymbol{\omega}}_{k+1|k+1} = \boldsymbol{\omega}_{m_{k+1}} - \hat{\mathbf{b}}_{k+1|k+1} \tag{221}$$

9. Compute the new updated Covariance matrix

$$\mathbf{P}_{k+1|k+1} = (\mathbf{I}_{6\times6} - \mathbf{K}\mathbf{H})\mathbf{P}_{k+1|k}(\mathbf{I}_{6\times6} - \mathbf{K}\mathbf{H})^{\mathrm{T}} + \mathbf{K}\mathbf{R}\mathbf{K}^{\mathrm{T}} \tag{222}$$

补充：

步骤1 计算H矩阵

测量量z也构造成了误差的形式，用测量值– 估计值（等式右端也一样减去了这个估计值，构造出了括号里的误差项）。

$$\boxed{\tilde{\mathbf{z}} = \mathbf{z} - \hat{\mathbf{z}} = \boldsymbol{\Pi}^S_L\mathbf{C}\left({}^L_G\mathbf{C}(\bar{q}) - {}^{\hat{L}}_G\mathbf{C}(\hat{\bar{q}})\right)\cdot{}^G\mathbf{r}_\odot + \mathbf{n}_\mathbf{m}} \tag{201}$$

We can now write

$$\boxed{\tilde{\mathbf{z}}} = \boldsymbol{\Pi}^S_L\mathbf{C}\left({}^L_{\hat{L}}\mathbf{C}(\delta\bar{q}) - \mathbf{I}\right){}^{\hat{L}}_G\mathbf{C}(\hat{\bar{q}})\cdot{}^G\mathbf{r}_\odot + \mathbf{n}_\mathbf{m} \tag{207}$$

$$\approx \boldsymbol{\Pi}^S_L\mathbf{C}\left(-\lfloor\delta\boldsymbol{\theta}\times\rfloor\right){}^{\hat{L}}_G\mathbf{C}(\hat{\bar{q}})\cdot{}^G\mathbf{r}_\odot + \mathbf{n}_\mathbf{m} \tag{208}$$

$$= \boldsymbol{\Pi}^S_L\mathbf{C}\lfloor{}^{\hat{L}}_G\mathbf{C}(\hat{\bar{q}}){}^G\mathbf{r}_\odot\times\rfloor\cdot\delta\boldsymbol{\theta} + \mathbf{n}_\mathbf{m} \tag{209}$$

$$\boxed{= \begin{bmatrix}\boldsymbol{\Pi}^S_L\mathbf{C}\lfloor{}^{\hat{L}}_G\mathbf{C}(\hat{\bar{q}}){}^G\mathbf{r}_\odot\times\rfloor & \mathbf{0}\end{bmatrix}\cdot\begin{bmatrix}\delta\boldsymbol{\theta}\\ \tilde{\mathbf{b}}\end{bmatrix} + \mathbf{n}_\mathbf{m}} \tag{210}$$

so that the measurement matrix $\mathbf{H}$ corresponds to

$$\boxed{\mathbf{H} = \begin{bmatrix}\boldsymbol{\Pi}^S_L\mathbf{C}\lfloor{}^{\hat{L}}_G\mathbf{C}(\hat{\bar{q}}){}^G\mathbf{r}_\odot\times\rfloor & \mathbf{0}\end{bmatrix}} \tag{211}$$

步骤2 计算测量残差

$$\boxed{\mathbf{r} = \mathbf{z} - \hat{\mathbf{z}}}$$

步骤3 计算残差方差矩阵S（为计算滤波增益K做准备）

$$\mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^{\mathrm{T}} + \mathbf{R}$$

步骤4 计算滤波增益K

$$\mathbf{K} = \mathbf{P}\mathbf{H}^{\mathrm{T}}\mathbf{S}^{-1}$$

此处直接用

q(k+1|k) = q(k)x[1,1/2*dt*w] 比较方便，直接四元数乘法计算，精度也差不多够了。

步骤4 计算状态转移矩阵（为计算一步预测均方误差P(k+1/k)做准备）：

2.5.1 The State Transition Matrix

Since the continuous time system matrix $\mathbf{F}_c$ is constant over the integration time step, we may write the state transition matrix as [4, eq. (2-58a)]

$$\mathbf{\Phi}(t+\Delta t, t) = \exp(\mathbf{F}_c \Delta t) \tag{163}$$

$$= \boxed{\mathbf{I}_{6\times 6} + \mathbf{F}_c \Delta t} + \frac{1}{2!}\mathbf{F}_c^2 \Delta t^2 + \dots \tag{164}$$

Straightforward calculation produces the powers of $\mathbf{F}_c$ as

$$\mathbf{F}_c = \begin{bmatrix} -\lfloor\hat{\omega}\times\rfloor & -\mathbf{I}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{bmatrix}, \quad \mathbf{F}_c^2 = \begin{bmatrix} \lfloor\hat{\omega}\times\rfloor^2 & \lfloor\hat{\omega}\times\rfloor \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{bmatrix}$$
$$\mathbf{F}_c^3 = \begin{bmatrix} -\lfloor\hat{\omega}\times\rfloor^3 & -\lfloor\hat{\omega}\times\rfloor^2 \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{bmatrix}, \quad \mathbf{F}_c^4 = \begin{bmatrix} \lfloor\hat{\omega}\times\rfloor^4 & \lfloor\hat{\omega}\times\rfloor^3 \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{bmatrix} \tag{165}$$

其中用到的w的估计，是指前一时刻k的估计值。

其实因为采样频率足够高，delta t很小，所以应该只保留一阶精度就够了。

计算噪声方差矩阵

The resulting matrix $\mathbf{Q}_d$ has the following structure

$$\mathbf{Q}_d = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{12}^{\mathrm{T}} & \mathbf{Q}_{22} \end{bmatrix} \tag{187}$$

and the elements follow after considerable algebra as

$$\mathbf{Q}_{11} = \underline{\sigma_r^2 \Delta t \cdot \mathbf{I}_{3\times3}} + \sigma_w^2 \cdot \left(\mathbf{I}_{3\times3}\frac{\Delta t^3}{3} + \frac{\frac{(|\hat{\omega}|\Delta t)^3}{3} + 2\sin(|\hat{\omega}|\Delta t) - 2|\hat{\omega}|\Delta t}{|\hat{\omega}|^5} \cdot \lfloor\hat{\omega}\times\rfloor^2 \right) \tag{188}$$

$$\mathbf{Q}_{12} = -\sigma_w^2 \cdot \left(\mathbf{I}_{3\times3}\frac{\Delta t^2}{2} - \frac{|\hat{\omega}|\Delta t - \sin(|\hat{\omega}|\Delta t)}{|\hat{\omega}|^3} \cdot \lfloor\hat{\omega}\times\rfloor + \frac{\frac{(|\hat{\omega}|\Delta t)^2}{2} + \cos(|\hat{\omega}|\Delta t) - 1}{|\hat{\omega}|^4} \cdot \lfloor\hat{\omega}\times\rfloor^2 \right) \tag{189}$$

$$\mathbf{Q}_{22} = \underline{\sigma_w^2 \Delta t \cdot \mathbf{I}_{3\times3}} \tag{190}$$

此处，应该可以不考虑噪声的高阶项和耦合项。（此处留个坑，关于噪声方差矩阵数值怎么填的问题）

步骤5 计算一步预测均方误差矩阵P(k+1/k)

$$\mathbf{P}_{k+1|k} = \mathbf{\Phi}\mathbf{P}_{k|k}\mathbf{\Phi}^{\mathrm{T}} + \mathbf{Q}_d$$

步骤5+6+7 更新四元数姿态、更新bias，根据新的bias更新w

$$\delta\hat{\bar{q}} = \begin{bmatrix} \delta\hat{\mathbf{q}}(+) \\ \sqrt{1 - \delta\hat{\mathbf{q}}(+)^{\mathrm{T}}\delta\hat{\mathbf{q}}(+)} \end{bmatrix}$$

or, if $\delta\hat{\mathbf{q}}(+)^{\mathrm{T}}\delta\hat{\mathbf{q}}(+) > 1$, using

$$\delta\hat{\bar{q}} = \frac{1}{\sqrt{1 + \delta\hat{\mathbf{q}}(+)^{\mathrm{T}}\delta\hat{\mathbf{q}}(+)}} \cdot \begin{bmatrix} \delta\hat{\mathbf{q}}(+) \\ 1 \end{bmatrix}$$

$$\underline{\hat{\bar{q}}_{k+1|k+1} = \delta\hat{\bar{q}} \otimes \hat{\bar{q}}_{k+1|k}}$$

步骤8 更新均方差矩阵P(k+1/k+1)，其余量均用最后更新的最新值

$$\mathbf{P}_{k+1|k+1} = (\mathbf{I}_{6\times6} - \mathbf{K}\mathbf{H})\mathbf{P}_{k+1|k}(\mathbf{I}_{6\times6} - \mathbf{K}\mathbf{H})^{\mathrm{T}} + \mathbf{K}\mathbf{R}\mathbf{K}^{\mathrm{T}}$$