The 3rd Report of Assignments of Advanced Mathematics II

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1 Selected Exercises of §11.1 Fundamental Concept of Number Series

Theorem 1 (Cauchy Convergence Test). *A series* $\sum_{n=1}^{\infty} u_n$ *is convergent if and only if for any* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that*

$$|u_{n+1} + u_{n+2} + \cdots + a_{n+p}| < \varepsilon$$

holds for all $n > N$ *and* $p \geq 1$*.*

A Wrong Solution of 7-(1)
Let $u_n = (-1)^{n+1}/n$. For any $p \in \mathbb{Z}^+$, we have

$$\begin{aligned}
&|u_{n+1} + u_{n+2} + \cdots + a_{n+p}| \\
=&\left|\frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{n+3}}{n+2} + \cdots + \frac{(-1)^{n+p+1}}{n+p}\right| \\
<&\left|\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p}\right| \\
<&\frac{p}{n} \ .
\end{aligned}$$

For any $\varepsilon > 0$, we choose any $N \geq \lceil \frac{p}{\varepsilon} \rceil$ such that for all $n > N$,

$$|u_{n+1} + u_{n+2} + \cdots + a_{n+p}| < \frac{p}{n} < \frac{p}{\lceil \frac{p}{\varepsilon} \rceil} \leq \varepsilon \ .$$

Therefore, the series $\sum_{n=1}^{\infty} u_n$ is convergent.
A General Wrong Example
Let $u_n = \xi(n)$ be any function of n. Furthermore, $|\xi(n)|$ is monotone decreasing and tending to 0 by n. For any $p \in \mathbb{Z}^+$, we have

$$\begin{aligned}
&|u_{n+1} + u_{n+2} + \cdots + a_{n+p}| \\
=&|\xi(n+1) + \xi(n+2) + \cdots + \xi(n+p)| \\
<&\sum_{i=1}^{p} |\xi(n+i)| \\
=&\varphi(n,p) \ .
\end{aligned}$$

Obviously, $\varphi(n,p)$ also decreases monotonically and tends to zero by n. Thus, for any $\varepsilon > 0$, we can always find an N such that $\varphi(N,p) < \varepsilon$. Therefore, the series $\sum_{n=1}^{\infty} u_n$ is convergent.

Error: If N is dependent on p, then it doesn't holds for all $p \geq 1$. Or if N is selected independent on p, then it doesn't holds when p is sufficient large, since $\varphi(n,p)$ increases by p.

A Solution of 7-(1)

$$\begin{aligned}
&|u_{n+1} + u_{n+2} + \cdots + a_{n+p}| \\
=&\Big|\frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{n+3}}{n+2} + \cdots + \frac{(-1)^{n+p+1}}{n+p}\Big| \\
<&\frac{1}{n+1} - (\frac{1}{n+2} - \frac{1}{n+3}) - (\frac{1}{n+4} - \frac{1}{n+5}) - \cdots \\
<&\frac{1}{n} \ .
\end{aligned}$$

Then we can choose any $N \geq \lceil \frac{1}{\varepsilon} \rceil$.

2 Exercises of §11.6 Fourier Series

Fourier Series is used to decompose a *periodic* function into the sum of simple oscillating functions (viz. sines and cosines).

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}) \ ,$$

where

$$\begin{aligned}
a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \qquad (n \in \mathbb{N}) \ , \\
b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx \qquad (n \in \mathbb{Z}^+) \ .
\end{aligned}$$

1-(1)
$f(x) = |x|$ is an even function. Thus $b_n = 0$ and

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|dx = \pi \ , \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)dx = \frac{2}{n^2\pi}(\cos(n\pi) - 1) \qquad (n \in \mathbb{Z}^+) \ .
\end{aligned}$$

1-(2)
$f(x) = \sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$ has already satisfied the form of Fourier series.
1-(3)

$f(x) = 3x^2 + 1$ is an even function. Thus $b_n = 0$ and

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (3x^2+1)dx = 2\pi^2 + 2 \ ,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (3x^2+1)\cos(nx)dx = \frac{12}{n^2}\cos(n\pi) \qquad (n \in \mathbb{Z}^+) \ .$$

1-(6)

$$f(x) = \begin{cases} bx, & -\pi \le x < 0 \ , \\ ax, & 0 \le x < \pi \ . \end{cases} \qquad (a > b > 0) \ .$$

We have

$$a_0 = \frac{1}{\pi}\Big(\int_{-\pi}^{0} (bx)dx + \int_{0}^{\pi} (ax)dx\Big) = \frac{\pi}{2}(a-b) \ ,$$

$$a_n = \frac{1}{\pi}\Big(\int_{-\pi}^{0} (bx\cos(nx))dx + \int_{0}^{\pi} (ax\cos(nx))dx\Big) = \frac{a-b}{n^2\pi}(\cos(n\pi)-1) \qquad (n \in \mathbb{Z}^+) \ ,$$

$$b_n = \frac{1}{\pi}\Big(\int_{-\pi}^{0} (bx\sin(nx))dx + \int_{0}^{\pi} (ax\sin(nx))dx\Big) = -\frac{a+b}{n}\cos(n\pi) \qquad (n \in \mathbb{Z}^+) \ .$$

3

$f(x) = |sin(x)|$ is an even function. Thus $b_n = 0$ and

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (|sin(x)|)dx = \frac{4}{\pi} \ ,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (|sin(x)|)\cos(nx)dx = -\frac{2(\cos(n\pi)+1)}{\pi(n^2-1)} \qquad (n \in \mathbb{Z}^+) \ .$$

Thus

$$\begin{aligned} f(x) &\sim \frac{2}{\pi} - \frac{4}{\pi}\sum_{n\in 2\mathbb{Z}^+} \frac{1}{n^2-1}\cos(nx) \\ &= \frac{2}{\pi} - \frac{4}{\pi}\sum_{N\in \mathbb{Z}^+} \frac{1}{(2N)^2-1}\cos(2Nx) \ . \end{aligned}$$

It is clear that

$$\begin{aligned} f(0) &= \frac{2}{\pi} - \frac{4}{\pi}\sum_{N\in \mathbb{Z}^+} \frac{1}{(2N)^2-1} \\ \Leftrightarrow \sum_{N\in \mathbb{Z}^+} \frac{1}{4N^2-1} &= \frac{1}{2} \ . \end{aligned}$$

5-(2)

Similarly to 1-(1), $f(x) = |x|$ is an even function. Thus $b_n = 0$ and

$$a_0 = \frac{1}{1}\int_{-1}^{1} |x|dx = 1 \ ,$$

$$a_n = \frac{1}{1}\int_{-1}^{1} |x|\cos(n\pi x)dx = \frac{2}{n^2\pi^2}(\cos(n\pi)-1) \qquad (n \in \mathbb{Z}^+) \ .$$

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \ , \\ 1, & 1 < x < 2 \ . \end{cases} \ .$$

(1)After even extension, it become an even function.

$$\begin{aligned} a_0 &= 2 \cdot \frac{1}{2} \int_0^2 f(x) dx = \frac{3}{2} \ , \\ a_n &= 2 \cdot \frac{1}{2} \int_0^2 f(x) \cos \frac{n\pi x}{2} dx = \frac{4}{n^2\pi^2} (\cos \frac{n\pi}{2} - 1) \qquad (n \in \mathbb{Z}^+) \ . \end{aligned}$$

(2)After odd extension, it become an odd function.

$$b_n = 2 \cdot \frac{1}{2} \int_0^2 f(x) \sin \frac{n\pi x}{2} dx = -\frac{2}{n\pi} \cos(n\pi) \qquad (n \in \mathbb{Z}^+) \ .$$

3 Exercises of §11

1-(3)
$u_n = n^s/a^n$ where $a > 1$ and $s > 1$. The series is convergent since

$$\lim_{n\to\infty} \frac{(n+1)^s}{a^{n+1}} \cdot \frac{a^n}{n^s} = \lim_{n\to\infty} \frac{1}{a} (1 + \frac{1}{n})^s = a^{-1} < 1 \ .$$

1-(4)
$u_n = n^n/(n!)^2$. The series is convergent since

$$\begin{aligned} & \lim_{n\to\infty} \frac{(n+1)^{n+1}}{((n+1)!)^2} \cdot \frac{(n!)^2}{n^n} = \lim_{n\to\infty} (n+1) (\frac{n+1}{n})^n (\frac{n!}{(n+1)!})^2 \\ &= \lim_{n\to\infty} \frac{1}{n+1} (1 + \frac{1}{n})^n = 0 \ . \end{aligned}$$

3-(1)
$u_n = (-1)^n/n^p$ is a alternating series. According to Leibnize's rule, u_n is convergent if $p > 0$ since $|u_n| = n^{-p}$ decreases monotonically and tends to zero. Otherwise, u_n is divergent. Furthermore, $|u_n|$ is convergent when $p > 1$. Thus,

$$u_n \text{is} = \begin{cases} \text{divergent}, & p \le 0 \ , \\ \text{conditional convergent}, & 0 < p \le 1 \ , \\ \text{absolute convergent}, & 1 < p \ . \end{cases}$$

3-(4)
$u_n = (-1)^n a_n$ is a alternating series, where $a_n = |u_n| = (n+1)!/n^{n+1}$. It is absolute convergent, because

$$\begin{aligned} \lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+2)!}{(n+1)^{n+2}} \cdot \frac{n^{n+1}}{(n+1)!} \\ &= \lim_{n\to\infty} (1 + \frac{1}{n+1}) \cdot (1 - \frac{1}{n+1})^{n+1} = e^{-1} \ . \end{aligned}$$

4-(1)
$u_n = x^{2n}/(2n-1)!$ is convergent since $\lim_{n\to\infty} u_{n+1}/u_n = 0$. So the convergence domain is $\mathbb{R}$.
4-(2)
By instituting $\xi = x+4$, $u_n = (x+4)^n/n = \xi^n/n$.

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n}{n+1}\xi = \xi \ .$$

u_n is convergent when $\xi \in (-1,1)$. If $\xi = 1$, u_n will be harmonic series and become divergent. If $\xi = -1$, u_n will be alternating series and become convergent. Thus the convergence domain of u_n is $x \in [-5,-3)$.
4-(3)
$u_n = a_n x^n$ where $a_n = (3^n + 5^n)/n$. The radius of convergence

$$\mathcal{R} = (\lim_{n\to\infty} |a_{n+1}/a_n|)^{-1} = \frac{1}{5} \ .$$

When $x = 1/5$, u_n become divergent. When $x = -1/5$, will be alternating series and become convergent by Leibnize's rule. Thus the convergence domain of u_n is $x \in [-1/5, 1/5)$.
4-(4)(5)(6)
The convergence domains are $(9/10, 11/10)$, $(-1/e, 1/e)$ and $(-\sqrt{2}, \sqrt{2})$ respectively.
5-(1)
The convergence domains of $u_n = nx^{n+1}$ is $(-1,1)$. Let $S_n = \sum_{i\in[n]} u_i$. We have

$$\begin{aligned} S_n &= \frac{1}{1-x}(S_n - xS_n) = \frac{1}{1-x}(x^2 + x^3 + \cdots + x^{n+1} - nx^{n+2}) \\ &= \frac{x^2(1-x^n)}{(1-x)^2} - \frac{nx^{n+2}}{1-x} \ . \end{aligned}$$

Since $|x| < 1$, $S = \sum_{i\in\mathbb{Z}^+} u_i = \lim_{n\to\infty} S_n = x^2/(1-x)^2$.
5-(2)
$u_n = ne^{-nx}$. We have

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} (1 + \frac{1}{n})e^{-x} = e^{-x} \ .$$

Clearly, when $x > 0 \Leftrightarrow |e^{-x}| < 1$, u_n is convergent. Namely, The domains of convergence is $(0, +\infty)$.

$$S = \lim_{n\to\infty} \Big(\sum_{i\in[n]} u_i \Big) = \lim_{n\to\infty} \Big(\frac{1}{1-e^{-x}} (S_n - e^{-x} S_n) \Big) = \frac{e^x}{(e^x - 1)^2} \ .$$

5-(3)

The convergence domains of $u_n = \frac{2n-1}{2^n}x^{2(n-1)}$ is $(-\sqrt{2}, \sqrt{2})$. We have

$$\begin{aligned} S &= \sum_{n\in\mathbb{Z}^+} u_n = \sum_{n\in\mathbb{Z}^+} \Big(\frac{x^{2n-1}}{2^n}\Big)' = \Big(\frac{1}{x}\sum_{n\in\mathbb{Z}^+}\Big(\frac{x^2}{2}\Big)^n\Big)' \\ &= \Big(\frac{1}{x}\cdot\frac{\frac{x^2}{2}}{1-\frac{x^2}{2}}\Big)' \qquad \text{since } |x| < \sqrt{2} \\ &= \frac{x^2+2}{(x^2-2)^2} \ . \end{aligned}$$

5-(4)

The convergence domains of $u_n = x^n/(n(n+1))$ is $[-1, 1]$. We have

$$\begin{aligned} S &= \sum_{n\in\mathbb{Z}^+}\frac{x^n}{n} - \sum_{n\in\mathbb{Z}^+}\frac{x^n}{n+1} \\ &= \sum_{n\in\mathbb{Z}^+}\frac{x^n}{n} - \frac{1}{x}\sum_{n\in\mathbb{Z}^+}\frac{x^{n+1}}{n+1} \qquad (x \neq 0,\ \text{otherwise } S\big|_{x=0} = 0) \\ &= \sum_{n\in\mathbb{Z}^+}\int\Big(\frac{x^n}{n}\Big)'dx - \frac{1}{x}\sum_{n\in\mathbb{Z}^+}\int\Big(\frac{x^{n+1}}{n+1}\Big)'dx \\ &= \int\sum_{n\in\mathbb{Z}^+}\Big(\frac{x^n}{n}\Big)'dx - \frac{1}{x}\int\sum_{n\in\mathbb{Z}^+}\Big(\frac{x^{n+1}}{n+1}\Big)'dx \\ &= \int\sum_{n\in\mathbb{Z}^+}x^{n-1}dx - \frac{1}{x}\int\sum_{n\in\mathbb{Z}^+}x^n dx \\ &= \int\frac{1}{1-x}dx - \frac{1}{x}\int\frac{x}{1-x}dx \qquad (x \in (-1,1)) \\ &= -\ln(1-x) + \mathcal{C}_1 - \frac{1}{x}(-\ln(1-x) - x + \mathcal{C}_2) \\ &= (\frac{1}{x} - 1)\ln(1-x) + 1 + (\mathcal{C}_1 + \mathcal{C}_2) \ , \end{aligned}$$

where both $\mathcal{C}_1$ and $\mathcal{C}_2$ are zero, since

$$\begin{aligned} (-\ln(1-x) + \mathcal{C}_1)\big|_{x=0} &= \sum_{n\in\mathbb{Z}^+}\frac{x^n}{n}\Big|_{x=0} \ , \\ (-\ln(1-x) - x + \mathcal{C}_2)\big|_{x=0} &= \sum_{n\in\mathbb{Z}^+}\frac{x^{n+1}}{n+1}\Big|_{x=0} \ . \end{aligned}$$

When $x = 1$, $S = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots = 1$. When $x = -1$,

$$\begin{aligned} S &= -1 + \frac{1}{2} + \frac{1}{2} - \frac{1}{3} - \frac{1}{3} + \cdots \\ &= 1 - 2(1 - \frac{1}{2} + \frac{1}{3} - \cdots) \\ &= 1 - 2\ln 2 \ . \end{aligned}$$

Finally, we obtain that

$$S = \begin{cases} 0 & x = 0 \ , \\ 1 & x = 1 \ , \\ (\frac{1}{x} - 1)\ln(1-x) + 1 & x \in [-1, 1) \ . \end{cases}$$

Or,

$$\begin{aligned}
S &= \int \sum_{n\in\mathbb{Z}^+} x^{n-1} dx - \frac{1}{x} \int \sum_{n\in\mathbb{Z}^+} x^n dx \\
&= \lim_{n\to\infty} \int \frac{1-x^n}{1-x} dx - \frac{1}{x} \lim_{n\to\infty} \int \frac{x(1-x^n)}{1-x} dx \qquad (x \in [-1, 1)) \\
&= (\frac{1}{x} - 1)\ln(1-x) + 1 + \lim_{n\to\infty} \Big(\frac{x^n}{n(n+1)} - \frac{x^{n-1}}{n} + x^{n-1}(1-x)\Phi(x, 1, n) \Big) \\
&\qquad \text{where } \Phi(x, a, v) = \sum_{n=0}^{\infty} \frac{x^n}{(v+n)^a} \text{ is Lerchi Phi-Function} \\
&= (\frac{1}{x} - 1)\ln(1-x) + 1 + \lim_{n\to\infty} x^{n-1}(1-x) \sum_{n=0}^{\infty} \frac{x^n}{2n} \\
&= (\frac{1}{x} - 1)\ln(1-x) + 1 + \sum_{n=0}^{\infty} \lim_{n\to\infty} \frac{x^{2n-1}(1-x)}{2n} \\
&= (\frac{1}{x} - 1)\ln(1-x) + 1 \ .
\end{aligned}$$

7

$$\begin{aligned}
S = \sum_{n=1}^{\infty} \frac{n^2}{n!} &= \sum_{n=1}^{\infty} \frac{n}{(n-1)!} \\
&= \sum_{n=1}^{\infty} \frac{1}{(n-2)!} + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \\
&= 2e \ .
\end{aligned}$$

8-(1)

$$
\begin{aligned}
f(x) &= \int_0^x \frac{sin(t)}{t} dt \\
&= \int_0^x \frac{1}{t} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^{2n-1}}{(2n-1)!} dt \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)!} \int_0^x x^{2n-2} dt \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^{2n-1}}{(2n-1)\cdot(2n-1)!}\Big|_0^x \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)\cdot(2n-1)!} \quad ,
\end{aligned}
$$

with convergence domain $(-\infty, +\infty)$.

8-(2)

$$
\begin{aligned}
f(x) &= \ln(x + \sqrt{1+x^2}) \\
&= \int d(\ln(x+\sqrt{1+x^2})) \\
&= \int (1+x^2)^{-\frac{1}{2}} dx \quad .
\end{aligned} \tag{1}
$$

We expand $(1+x^2)^{-\frac{1}{2}}$ as

$$
\begin{aligned}
(1+x^2)^{-\frac{1}{2}} &= 1 + \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-(n-1))}{n!} x^{2n} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1\cdot 3\cdot \cdots \cdot (2n-1)}{2^n \cdot n!} x^{2n} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{2^n \cdot n!} \cdot \frac{(2n-1)!}{2\cdot 4\cdot \cdots \cdot (2n-2)} x^{2n} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{2^n \cdot n!} \cdot \frac{(2n-1)!}{2^{n-1}\cdot (n-1)!} x^{2n} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{2^n \cdot n!} \cdot \frac{(2n)!}{2^n \cdot n!} x^{2n} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{(2^n \cdot n!)^2} x^{2n} \quad .
\end{aligned} \tag{2}
$$

Combining (1) and (2), we have

$$\begin{aligned}
f(x) &= \int \Big(1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{(2^n \cdot n!)^2} x^{2n}\Big) dx \\
&= x + \sum_{n=1}^{\infty} \Big((-1)^n \frac{(2n)!}{(2^n \cdot n!)^2} \int x^{2n} dx\Big) \\
&= x + \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{(2n+1)(2^n \cdot n!)^2} x^{2n+1} + \mathcal{C} \ ,
\end{aligned}$$

where $\mathcal{C} = 0$ since $f(0) = 0$. The domain of convergence is $[-1, 1]$.

8-(3)

$$\begin{aligned}
f(x) &= \sqrt{x} \\
&= \sqrt{4} + \sum_{n=0}^{\infty} (-1)^n \frac{1}{2} \frac{1}{(n+1)!} \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{2^n} 4^{-\frac{1}{2}-n} (x-4)^{n+1} \\
&= \sqrt{4} + \sum_{n=0}^{\infty} (-1)^n \frac{1}{2(n+1)} \frac{(2n)!}{(2^n \cdot n!)^2} 4^{-\frac{1}{2}-n} (x-4)^{n+1} \\
&= \sqrt{4} + \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{(n+1)(2^{2n+1} \cdot n!)^2} (x-4)^{n+1} \ ,
\end{aligned}$$

with convergence domain $[0, 8]$.