Testing Hypotheses under Lehmann Alternatives with Polya Tree Priors Doctoral Dissertation Proposal

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Outline of Presentation

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   - Review of Testing Lehmann Alternatives
   - Review of Estimation in Cox Proportion Hazards Model
   - Introduction to Polya Tree Processes
3. CURRENT RESULTS
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Lehmann Alternatives

Definition:
Suppose $X_1, \cdots, X_{n_1} \sim F(x)$ and $Y_1, \cdots, Y_{n_2} \sim H(x)$.

$$H_0 : F = H \text{ vs } H_1 : H(x) = 1 - \{1 - F(x)\}^\alpha = 1 - S^\alpha(x)$$

where $S(x) = 1 - F(x)$ is the survival function, $\alpha > 0$ and $\alpha \neq 1$.

Motivations:

- Mathematical simplicity.
- When α is an integer, $H(x)$ is the distribution function of the minimum of α independent random variables each having distribution function F.
- H_1 introduces a very natural stochastic ordering of F and H.
- Related to Cox proportional hazards model

Cox Proportional Hazards Model

Cox proportional hazards model introduced by Cox (1972):

$$h(t \mid x) = h_0(t) exp(x^T \beta) \tag{1}$$

where $h_0(t)$ is the baseline hazard function and β is a vector of coefficients. It follows that the survival function is

$$S(t \mid x) = [S_0(t)]^{exp(x^T \beta)}$$

β is generally estimated by maximizing partial likelihood function

$$PL(\beta) = \prod_{k=1; d_k=1}^{n} \frac{exp(x_k^T \beta)}{\sum_{j=k}^{n} exp(x_j^T \beta)} \tag{2}$$

where d_k is censoring indicator.

Lehmann (1953)

Suppose the ranks of Y's in the combined sample are denoted by $e_1, \cdots, e_{n_2}$. The complete set of rank is determined by the ranks of Y's alone. Lehmann (1953) derived

$$P(E_1 = e_1, \cdots, E_{n_2} = e_{n_2}) = \frac{\alpha^{n_2}}{\binom{n_1+n_2}{n_1}} \prod_{j=1}^{n_2} \frac{\Gamma(e_j + j\alpha - j)}{\Gamma(e_j)} \frac{\Gamma(e_{j+1})}{\Gamma(e_{j+1} + j\alpha - j)} \tag{3}$$

Using Equation (3), one can compute the power of various rank tests against the alternatives.

Savage (1956)

Savage (1956) considered a different but more general question:

$$H_L : F(x) = F_0(x)^{\alpha_1} \text{ and } H(x) = F_0(x)^{\alpha_2}$$

where $\alpha_2 > \alpha_1 > 0$ and $F_0(x)$ is a continuous cumulative distribution function.
Put these two samples together and denote it by $V_1, \cdots, V_{n_1+n_2}$.
Assume:

- WLOG, assume that V_k's are ordered.
- There are no ties.

Then, $V_1 < V_2 < \cdots < V_{n_1+n_2}$.

Savage (1956)

Define indicator functions Z_k, $k = 1, \cdots, n_1 + n_2$, as

$$Z_k = \begin{cases} 0, & \textit{if } V_k \in \mathfrak{X} = \{X_1, \cdots, X_{n_1}\} \\ 1, & \textit{if } V_k \in \mathfrak{Y} = \{Y_1, \cdots, Y_{n_2}\} \end{cases} \tag{4}$$

Under H_L, the probability of a rank order $z_1, \cdots, z_{n_1+n_2}$ is given by

$$Prob. = \frac{n_1!n_2!\alpha_1^{n_1}\alpha_2^{n_2}}{\prod_{i=1}^{n_1+n_2}\left(\sum_{j=1}^{i}\left[(1-z_j)\alpha_1 + z_j\alpha_2\right]\right)}.$$

Other Literature

- Davies (1971) showed asymptotic equivalence of the approaches of Lehmann (1953) and Savage (1956).
- Brooks (1974) assigned an F distribution prior to α and conducted a Bayesian test.
- Miura and Tsukahara (1993) discussed the estimation problem in one-sample generalized Lehmann alternative model

Summary:
The above-mentioned tests are based only on the ranks which remain invariant under one-to-one transformation.This considers only the positions, but not the magnitudes of differences of order statistics.

Goal: Establish a testing method such that not only the ranks, but also the spacings of order statistics are taken into account.

Cox proportional hazards model

Bayesian nonparametric methods:

- Kalbfleisch (1978) — Gamma process
- Hjort (1990) — Beta process
- Muliere and Walker (1997) — Polya tree process (without covariates)
- Hanson (2006) — Mixture of finite Polya tree model
- Hanson and Jara (2012) — Mixture of Polya tree process

What is Polya tree process?

Notations:

- Let $E = \{0, 1\}$, $E^0 = \emptyset$. Let $E^m = E \times E \times E \cdots \times E$ be the m-fold product and $E^* = \bigcup_0^\infty E^m$.
- let Ω be a separable measurable space. Define a separating binary tree of partition of Ω, $\Pi = \{\pi_m, m = 0, 1, 2, ...\}$, such that :
    - $\pi_0 = \Omega$.
    - $\pi_0, \pi_1, ...$ form a sequence of partitions such that $\bigcup_0^\infty \pi_m$ generates the measurable sets.
    - Every $B \in \pi_{m+1}$ is obtained by splitting some $B' \in \pi_m$ into two sets. Degenerate splits are permitted, i.e. some $B \in \pi_m$ can be split into $B \bigcup \emptyset$.

What is Polya tree process?

Definition: For each m, $\pi_m = \{B_{\vec{\epsilon_m}} : \vec{\epsilon_m} = \epsilon_1, ..., \epsilon_m \in E^m\}$ is a partition of Ω such that for all $\vec{\epsilon_m} \in E^*$, $B_{\vec{\epsilon_m},0}$, $B_{\vec{\epsilon_m},1}$ is a partition of $B_{\vec{\epsilon_m}}$. Let $A = \{a_{\vec{\epsilon_m}} : \vec{\epsilon_m} \in E^*\}$ be a set of nonnegative real numbers and $\mathfrak{y} = \{W_{\vec{\epsilon_m}} : \vec{\epsilon_m} \in E^*\}$ be a collection of random variables. Then we say a random probability measure P on Ω have a Polya tree distribution with parameter (Π, A), written $P \sim PT(\Pi, A)$, if the following hold:

What is Polya tree process?

1. all the random variables in $\mathfrak{y}$ with subscripts ending with 0 are independent, i.e. $W_{\vec{\epsilon_m},0}$, for all $\vec{\epsilon_m} \in E^*$, are independent; $W_{\vec{\epsilon_m},1} = 1 - W_{\vec{\epsilon_m},0}$
2. for every $\vec{\epsilon_m} \in E^*$, $W_{\vec{\epsilon_m},0} \sim Beta(a_{\vec{\epsilon_m},0}, a_{\vec{\epsilon_m},1})$;
3. for every m=1,2,... and every $\vec{\epsilon_m} \in E^*$,

$$P(B_{\epsilon_1,\ldots,\epsilon_m}) = \left(\prod_{j=1;\epsilon_j=0}^{m} W_{\epsilon_1,\ldots,\epsilon_j}\right) \prod_{j=1;\epsilon_j=1}^{m} (1 - W_{\epsilon_1,\ldots,\epsilon_{j-1},0})$$

$$= \prod_{j=1}^{m} W_{\epsilon_1,\ldots,\epsilon_j} \tag{5}$$

Properties of Polya tree processes

Some established properties of Polya tree process that we need to use by Lavein (1992 & 1994), Mauldin et al. (1992) and Ghosh and Ramamoorthi (2003):

- Conjugacy: update when observe $X = x$
- Continuity: Polya trees can assign probability 1 to the set of continuous distributions. A sufficient condition is, for example, $a_{\vec{\epsilon}_m} = m^2$.

Properties of Polya tree processes

- Density: a Polya tree with partitions $\{B_{\vec{\epsilon_m}} : \vec{\epsilon_m} \in E^*\}$ and parameters A has predictive density at $x \in B_{\vec{\epsilon_m}}$ is

$$f(x) = \lim_{m \to +\infty} \frac{Pr(B_{\vec{\epsilon_m}})}{\lambda(B_{\vec{\epsilon_m}})} = \lim_{m \to +\infty} \frac{\prod_{i=1}^{m} \frac{a_{\epsilon_1,\ldots,\epsilon_j}}{a_{\epsilon_1,\ldots,\epsilon_{j-1},0} + a_{\epsilon_1,\ldots,\epsilon_{j-1},1}}}{\lambda(B_{\vec{\epsilon_m}})} \tag{6}$$

where $\lambda(\cdot)$ is the Lebesgue measure.

Properties of Polya tree processes

- Centering: A Polya tree can be constructed centering at an arbitrary distribution. Suppose $\Omega = \mathbb{R}$, and any pre-specified distribution function G. Two ways to do this:
  - M1 let the partition be such that the elements of π_m are taken as the intervals $[G^{-1}(k/2^m), G^{-1}((k+1)/2^m))$ for $k = 0, 1, ..., 2^m - 1$, with the obvious interpretation for $G^{-1}(0)$ and $G^{-1}(1)$, and $a_{\vec{\epsilon_m}} = m^2$.
  - M2 let the partition be data-dependent, i.e. $B_1 = [x_1, +\infty)$, $B_{11} = [x_2, +\infty)$,...,$B_{\underbrace{1,...,1}_{n}} = [x_n, +\infty)$. Parameters $a_{\vec{\epsilon_m}}$ need to satisfy

$$\frac{a_{\epsilon_1,\ldots,\epsilon_{j-1},0}}{a_{\epsilon_1,\ldots,\epsilon_{j-1},1}} = \frac{G(B_{\epsilon_1,\ldots,\epsilon_{j-1},0})}{G(B_{\epsilon_1,\ldots,\epsilon_{j-1},1})}$$

    and $a_{\vec{\epsilon_m}}$ should grow quickly enough to ensure the continuity property.

Bayes Factor

Setup:

- $X_1, \cdots, X_{n_1} \sim F(x)$ and $Y_1, \cdots, Y_{n_2} \sim H(x)$.
- Suppose $V_1 < V_2 < \cdots < V_{n_1+n_2}$ is the order statistic of combined sample when no ties occur. Indicator functions Z_k, $k = 1, \cdots, n_1 + n_2$ are defined as usual.
- Let $d_1, \cdots, d_{n_1+n_2}$ be the censoring indicators corresponding to $V_1, \cdots, V_{n_1+n_2}$.

By definition, the Bayes factor is

$$BF_{01} = \frac{posterior \quad odds}{prior \quad odds} = \frac{\frac{P(H_0|V_1,\ldots,V_{n_1+n_2})}{P(H_1|V_1,\ldots,V_{n_1+n_2})}}{\frac{\pi(H_0)}{\pi(H_1)}}$$

Let $\pi(H_0) = p_0 = 1 - \pi(H_1)$, where $0 < p_0 < 1$. Suppose the joint pdf of $V_1, \ldots, V_{n_1+n_2}$ is f_0 under H_0 and f_1 under H_1.

Bayes Factor— Two-sample case

Then by simple calculation,

$$BF_{01} = \frac{\int f_0(v_1, ..., v_{n_1+n_2} \mid P)dPT(P)}{\int f_1(v_1, ..., v_{n_1+n_2} \mid P)dPT(P)} \tag{7}$$

Thus the Bayes factor is given by the ratio of the marginal distributions of $V_1, ..., V_{n_1+n_2}$ under H_0 to that under H_1.

Theorem 1:

Suppose that a Polya tree prior as described is applied to $F(x)$ with partitions $B_1 = [v_1, +\infty)$,
$B_{11} = [v_2, +\infty)$,...,$B_{\underbrace{1, ..., 1}_{m+n}} = [v_{n_1+n_2}, +\infty)$, and G is a strictly increasing baseline measure (with respect to Polya tree). Then the Bayes factor of the test is as follows

Properties of Bayes Factor— Two-sample case

- For fixed t_i, BF_{01} is a decreasing function of r_i ($\alpha > 1$) .

$$\textit{small } BF_{01} \leftarrow \textit{large } r_i \rightarrow \textit{data clustered} \rightarrow H_1$$

- Monotonicity on t_i:
  - BF_{01} increase when $\alpha > 1$, then $F(x) < H(x)$. More likely to observe large t_i's under H_0.
  - BF_{01} decrease when $\alpha < 1$, then $F(x) > H(x)$. More likely to observe large t_i's under H_1.

Summary:
Both the rank order statistics and the spacings of order statistics are considered in the test.

Effects of α

- When $\alpha \to \infty$ ($\alpha > 1$), $BF_{01} \to \infty$ if there exists at least one uncencored observations in sample $\mathfrak{X}$ and such that t_{i+1} is nonzero.
- Similar results when $\alpha \to 0^+$ and $\alpha < 1$.

An extreme case: when all uncensored observations in sample $\mathfrak{Y}$ are clustered closely and they are uniformly smaller than any observations in sample $\mathfrak{X}$.

$$\begin{cases} t_1 = n_2,\ t_2 = n_2 - 1, \cdots, t_{n_2} = 1 \\ t_i = 0,\ for\ i = n_2 + 1, \cdots, n_1 + n_2 \end{cases} \tag{8}$$

$BF_{01} \to 0$
As $\alpha \to \infty$, H degenerates at the infimum of the support of F.

Effects of spacings

Suppose $V_{k_0-1} \approx V_{k_0}$ for some $k_0 \in \{2, \cdots, n_1 + n_2\}$ and they come from different samples. Switching the positions of V_{k_0-1} and V_{k_0} results in

$$BF_{01}^{new} = \frac{\sigma_{k_0} + n_1 + n_2 + 1 - k_0 + (\alpha - 1)t_{k_0}^{new}}{\sigma_{k_0} + n_1 + n_2 + 1 - k_0 + (\alpha - 1)t_{k_0}^{old}} BF_{01}^{old}$$
$$= (1 \pm \frac{\alpha - 1}{\sigma_{k_0} + n_1 + n_2 + 1 - k_0 + (\alpha - 1)t_{k_0}^{old}}) BF_{01}^{old}$$

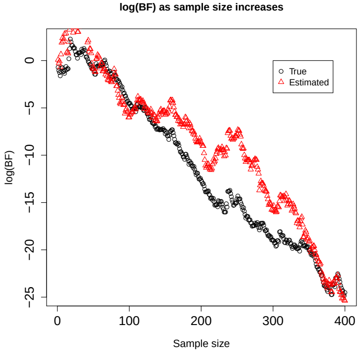
$\left| \frac{\alpha-1}{\sigma_{k_0}+n_1+n_2+1-k_0+(\alpha-1)t_{k_0}^{old}} \right|$ is called the *expansion rate*.

Effects of spacings

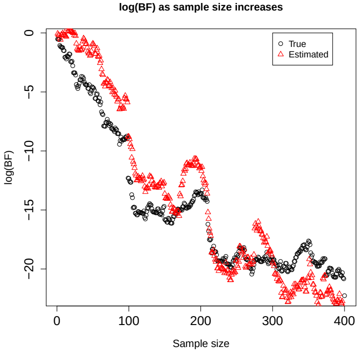
One can choose suitable parameters to meet his needs:

- When $\sigma_1 = \cdots = \sigma_{n_1+n_2}$, the expansion rate goes up as k_0 increases
  $\Rightarrow$ the order statistics play a more important role in the tail than in the beginning.
- When σ_{k_0} grow fast with k_0, the expansion rate could be a decreasing function of k_0
  $\Rightarrow$ a test more sensitive in the beginning.
- The expansion rate grows when α increases if $\alpha > 1$ or $\alpha \to 0^+$ if $\alpha < 1$
  $\Rightarrow$ the effect of order is enlarged when the distance of the alternative from the null is increased.

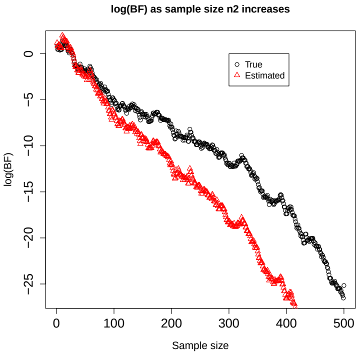
Simulations



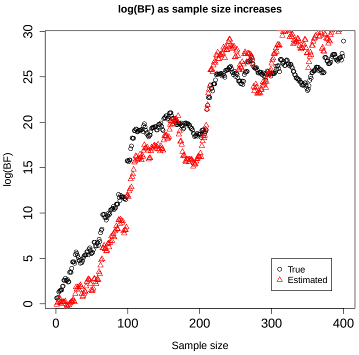
(a) $\alpha = 1.5$



(b) $\alpha = 0.7$



(c) $\alpha = 1.5$



(d) $\alpha = 0.7$

Real data application

Acute myelogenous leukemia (AML)

Table : **Acute Myelogenous Leukemia**

Treatment	Survival Time
Maintained	9, 13, 13+, 18, 23, 28+, 31, 34, 45+, 48, 161+
Nonmaintained	5, 8, 12, 16+, 23, 27, 30, 33, 43, 45

Table : Cox proportional hazards model

	coef	**exp(coef)**	**se(coef)**	**P-value**	**lower .95**	**upper .95**
Nonmaintained	-0.9155	0.4003	0.5119	0.0737	0.1468	1.092

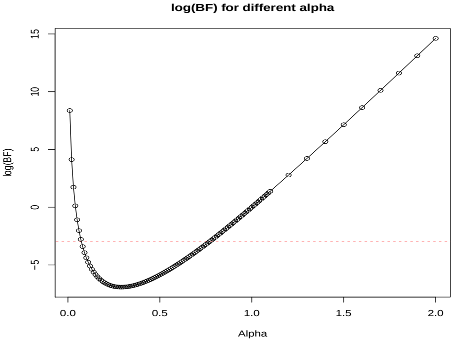
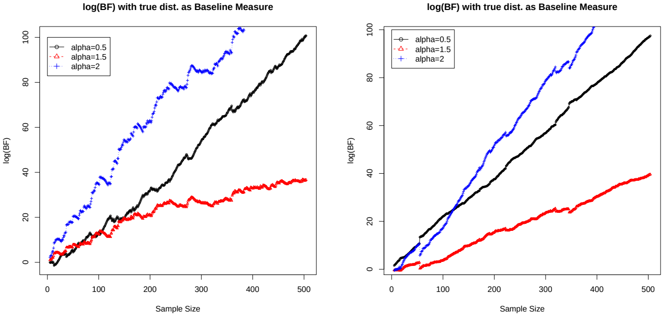


Figure : Leukemia: log(BF) for different α

Strong evidence interval based on Kass and Raftery (1995) criteria: $(0.10, 0.75)$.

Simulations- One-sample Case



(a) Baseline Measure: True Weibull Distribution

(b) Baseline Measure: Estimated Weibull Distribution

Figure : Log(BF) grows drastically as sample size increases under the null hypothesis

Simulations

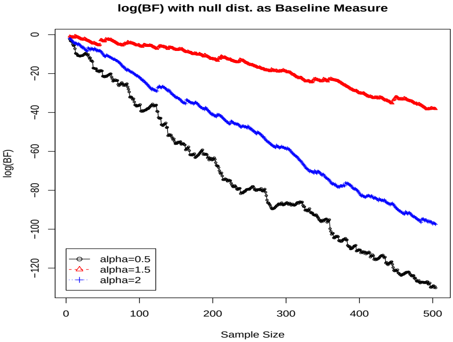


Figure : Log(BF) decreases drastically as sample size increases under the alternative hypothesis

Future Work

- Modification for ties
- Comparison to other Bayesian nonparametric estimators
- Fully Bayesian analysis for Cox model
- Reliability analysis and life testing

THANK YOU!