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An Asymptotic Expansion with Push-Down of Malliavin Weights*

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Abstract. This paper derives asymptotic expansion formulas for option prices and implied volatilities as well as the density of the underlying asset price in multidimensional stochastic volatility models. In particular, the integration-by-parts formula in Malliavin calculus and the push-down of Malliavin weights are effectively applied. We provide an expansion formula for generalized Wiener functionals and closed-form approximation formulas in the stochastic volatility environment. In addition, we present applications of the general formula to expansions of option prices for the shifted log-normal model with stochastic volatility. Moreover, with some results of Malliavin calculus in jump-type models, we derive an approximation formula for the jump-diffusion model in the stochastic volatility environment. Some numerical examples are also shown.

Key words. Malliavin calculus, asymptotic expansion, stochastic volatility, implied volatility, shifted log-normal model, jump-diffusion model, integration-by-parts, Malliavin weight, push-down, Malliavin calculus for Poisson processes

AMS subject classifications. 60H07, 91G20, 91G60

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1. Introduction.

This paper develops an asymptotic expansion method for generalized Wiener functionals by applying *Malliavin weight* (divergence given by the integration-by-parts formula) and *push-down* (the conditional expectation in Malliavin [14] and Malliavin and Thalmaier [15]). As applications, we propose a concrete approximation formula of option prices as well as the density of the underlying asset price in multidimensional stochastic volatility models and then derive a new Taylor expansion formula of the implied volatilities. Moreover, we present applications of the general formula to expansions of option prices for the shifted log-normal model in the stochastic volatility environment. Also, combining some results of Malliavin calculus in jump-type models by Bavouzet and Messaoud [2] with our method, we derive an approximation formula for option prices in the jump-diffusion model with stochastic volatility. To the best of our knowledge, it is the first study with push-down of Malliavin weights for deriving analytical approximation formulas for option prices and implied volatilities in those models. A companion paper by Shiraya, Takahashi, and Yamada [18] applies the method to deriving a concrete approximation formula for valuation of barrier options with discrete monitoring under stochastic volatility models.

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Various stochastic volatility models has been proposed for calibration to market prices of options with so-called volatility skews and smiles. However, closed-form solutions for option prices in the stochastic volatility environment are rarely found, and hence a large number of studies have been made in order to obtain analytical approximations and efficient numerical schemes for option prices and implied volatilities with stochastic volatility models. Takahashi [20, 21] proposed approximation formulas based on the asymptotic expansion method in Watanabe theory (Watanabe [25, 26, 27] and Yoshida [28, 29]) for valuation of various derivatives including options in the stochastic volatility environment. Fournié, Lebuchoux, and Touzi [6] provided an expansion result of the second-order partial differential equation which satisfies the uniform ellipticity condition and showed its application to Monte Carlo simulations. Fouque, Papanicolaou, and Sircar [4] derived a closed-form approximation formula for the fast mean reverting stochastic volatility model using a singular perturbation method and then discussed the calibration problem. Hagan et al. [9] introduced the SABR stochastic volatility model and obtained an approximation implied volatility formula. Labordère [12] generalized the SABR model to the λ-SABR model and derived an approximation of implied volatilities by applying the heat kernel expansion. Moreover, Gatheral et al. [8] and Ben Arous and Laurence [3] presented novel results for this direction. Recently, Antonelli and Scarlatti [1] gave a Taylor series expansions of option prices with respect to a correlation parameter in a stochastic volatility model. Also, we will provide more detailed comments on some related works in section 3.3.

The organization of the paper is as follows: the next section derives an asymptotic expansion formula for generalized Wiener functionals after a brief summary of Malliavin calculus necessary for the remainder of the paper. Section 3 applies the general formula to pricing options in the stochastic volatility environment and then obtains implied volatility expansions. Section 4 presents numerical examples for expansions including calibrations of implied volatilities. As simple applications of our method, section 5 provides approximation formulas for option prices (or/and implied volatilities) in the shifted log-normal and jump-diffusion models under stochastic volatilities. Section 6 concludes. Appendicies A and B gives explicit calculations of push-down of Malliavin weights (coefficients in the expansions) as well as the proof of Proposition 5.2.

2. Asymptotic expansion.

2.1. Malliavin calculus.

This subsection summarizes basic facts on the Malliavin calculus which are necessary for the following discussion. We use the notations and definitions given below.

Let $(\mathcal{W}, \mu)$ be the Wiener space, i.e.,

$$\mathcal{W} = \mathcal{W}^d = C_0([0,T] : \mathbf{R}^d) = \{w : [0,T] \to \mathbf{R}^d;\ \text{continuous},\ w(0) = 0\},$$

and let μ be the Wiener measure. Next, let H be a Hilbert space such that

$$H = \left\{ h \in \mathcal{W}; h_i(t)(i = 1, \dots, d) \text{ is absolute continuous with respect to } t \right.$$
$$\left. \text{and } \sum_{i=1}^{d} \int_0^T \left| \frac{dh_i(t)}{dt} \right|^2 dt < \infty \right\}$$

with an inner product $\langle h, h' \rangle_H = \sum_{i=1}^d \int_0^T \frac{dh_i(t)}{dt} \frac{dh'_i(t)}{dt} dt$. Then, H is called the Cameron–Martin space.

Define $L^{\infty-}(\mathcal{W})$ as $L^{\infty-}(\mathcal{W}) = \cap_{p<+\infty} L^p(\mathcal{W})$ and a distance on $L^{\infty-}(\mathcal{W})$ as

$$d_{L^{\infty-}(\mathcal{W})}(F_1, F_2) = \sum_{j=1}^{\infty} 2^{-j}(\min\{\|F_1 - F_2\|_{L^j}, 1\}),$$

where $\|\cdot\|_{L^p}$ denotes the L^p-norm in $(\mathcal{W}, \mu)$. Given a separable Hilbert space G, let $L^p(\mathcal{W} : G)$ denote the space of measurable maps from $\mathcal{W}$ to G such that $\|f\|_G \in L^p(\mathcal{W})$. The same definition is made for $L^{\infty-}(\mathcal{W} : G)$.

Then, consider the space

$$\begin{aligned}
&\mathbf{D}_1^p(\mathcal{W} : G) \\
&= \Big\{ F \in L^p(\mathcal{W} : G) : \text{there exists } DF \in L^p(\mathcal{W} : H \otimes G) \text{ such that for } h \in H, \\
&\lim_{\epsilon \to 0} \frac{1}{\epsilon}[F(w + \epsilon h) - F(w)] = (DF(w))(h) \Big\}.
\end{aligned}$$

Here, DF is called the (Malliavin) derivative of F. Note also that the tensor product $H \otimes G$ of two separable Hilbert spaces H and G is a Hilbert space formed by all linear operators $A : H \to G$ of Hilbert–Schmidt type endowed with the Hilbert–Schmidt norm

$$\|A\|_{HS} = \left(\sum_{i,j=1}^{\infty} \langle Ae_i, e'_j \rangle_G^2 \right)^{1/2},$$

for some ONB (orthonormal basis)'s $\{e_i\}$ in H and ONB's $\{e'_j\}$ in G, and

$$\|A\|_{HS}^2 = \|A\|_{H \otimes G}^2.$$

Then, the random variable DF takes values in $H \otimes G$.

Due to a canonical identification of the Hilbert spaces $L^2(\mathcal{W} : H)$ and $L^2([0,T] \times \mathcal{W})$, the Malliavin derivative DF may be considered as a stochastic process $\{D_t F = (D_{t,1}F, \ldots, D_{t,d}F) : t \in [0,T]\}$ such that

$$(DF(w))(h) = \langle DF, h \rangle_H = \sum_{i=1}^d \int_0^T (D_{t,i}F) \left(\frac{dh_i(t)}{dt} \right) dt.$$

A norm $\mathbf{D}_1^p(\mathcal{W} : G)$ is given by $\|F\|_{\mathbf{D}_1^p(\mathcal{W}:G)} = \|F\|_{L^p(\mathcal{W}:G)} + \|DF\|_{L^p(\mathcal{W}:H\otimes G)}$. Also, $\mathbf{D}_1^{\infty-}(\mathcal{W} : G)$ is defined by $\mathbf{D}_1^{\infty-}(\mathcal{W} : G) := \cap_{1 \le p < +\infty} \mathbf{D}_1^p(\mathcal{W} : G)$, and a distance on $\mathbf{D}_1^{\infty-}(\mathcal{W} : G)$ is given by

$$d_{\mathbf{D}_1^{\infty-}(\mathcal{W}:G)}(F_1, F_2) = \sum_{j=1}^{\infty} 2^{-j}(\min\{\|F_1 - F_2\|_{\mathbf{D}_1^j(\mathcal{W}:G)}, 1\}).$$

For $r \geq 2$ ($r \in \mathbf{N}$), we introduce the spaces

$$\mathbf{D}^p_r(\mathcal{W} : G) = \{F \in \mathbf{D}^p_{r-1}(\mathcal{W} : G) : DF \in \mathbf{D}^p_{r-1}(\mathcal{W} : H \otimes G)\}$$

with $\|F\|_{\mathbf{D}^p_r(\mathcal{W}:G)} = \|F\|_{\mathbf{D}^p_{r-1}(\mathcal{W}:G)} + \|D^{r-1}F\|_{\mathbf{D}^p_1(H^{\otimes(r-1)}\otimes G)}$. We also define $\mathbf{D}^p_0(\mathcal{W} : G)$ as $\mathbf{D}^p_0(\mathcal{W} : G) = L^p(\mathcal{W} : G)$.

If $G = \mathbf{R}^n$, then we write $\mathbf{D}^p_r(\mathcal{W})$ for $\mathbf{D}^p_r(\mathcal{W} : G)$.

Some properties of these spaces are the following: $\mathbf{D}^{p'}_{r'}(\mathcal{W}) \subset \mathbf{D}^p_r(\mathcal{W})$, $r' \leq r$, and $p' \leq p$. The dual space of $(\mathbf{D}^q_r(\mathcal{W}))$, $(\mathbf{D}^q_r(\mathcal{W}))^*$ is given by $(\mathbf{D}^q_r(\mathcal{W}))^* = \mathbf{D}^p_{-r}(\mathcal{W})$, with $p^{-1} + q^{-1} = 1$, $r \geq 0$.

Furthermore, define the space $\mathbf{D}_\infty(\mathcal{W}) = \cap_{p\geq 1, r\geq 0}\mathbf{D}^p_r(\mathcal{W})$. Then, $\mathbf{D}_\infty(\mathcal{W})$ is a complete metric space under a metric,

$$d_{\mathbf{D}^\infty(\mathcal{W})}(F_1, F_2) = \sum_{p,r=1}^{\infty} 2^{-p-r}(\min\{\|F_1 - F_2\|_{\mathbf{D}^p_r}, 1\}).$$

We call $F \in \mathbf{D}_\infty(\mathcal{W})$ the smooth functional in the sense of Malliavin.

Given $Z = (Z_1(w), \ldots, Z_d(w)) \in \mathbf{D}^p_1(\mathcal{W} : H)$, there exists $D^*_i(Z_i) \in L^p(\mathcal{W})$, $i = 1, \ldots, d$, such that

$$E\left[\int_0^T D_{t,i}F(w)Z_i(w)dt\right] = E[F(w)D^*_i(Z_i(w))]$$

for all $F \in \mathbf{D}^{\infty-}_1(\mathcal{W})$. Then, define $D^*(Z) := \sum_{i=1}^d D^*_i(Z_i(w))$. So, there exists $C_p > 0$ such that

$$\|D^*(Z)\|_{L^p} \leq C_p\|Z\|_{\mathbf{D}^p_1(\mathcal{W}:H)}.$$

We call $D^*(Z)$ the divergence of Z.

We also introduce the notation $D_Z F$ such that

$$D_Z F := \sum_{i=1}^d \int_0^T D_{t,i}F(w)Z_i(w)dt.$$

Then, we have $E[D_Z F] = E[FD^*(Z)]$. Thus,

$$D^*(FZ) = FD^*(Z) - D_Z F$$

is obtained (see, e.g., Proposition 1.16 in Malliavin and Thalmaier [15]).

Definition 2.1. *Let $F = (F_1, \ldots, F_n) \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ be the n-dimensional smooth functional; we call F a nondegenerate functional in the sense of Malliavin if the Malliavin covariance matrix $\{\sigma^{ij}_F\}_{1\leq i,j\leq n}$*

$$\sigma^{ij}_F := \langle DF_i, DF_j\rangle_H = \sum_{k=1}^d \int_0^T (D_{t,k}F_i(w))(D_{t,k}F_j(w))dt \tag{1}$$

is invertible a.s. and

$$(\det \sigma_F)^{-1} \in L^{\infty-}(\mathcal{W}).$$

Theorem 2.2. *Let $F \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ be an n-dimensional nondegenerate functional in the sense of Malliavin and $G \in \mathbf{D}_\infty(\mathcal{W})$. Then, for $\varphi \in C_b^1(\mathbf{R}^n)$,*

$$E[\partial_i \varphi(F)G] = E\left[\varphi(F) D^* \left(\sum_{j=1}^n G\gamma_{ij}^F DF^j\right)\right], \tag{2}$$

where $(\gamma_{ij}^F)_{1\le i,j\le n}$ is the inverse matrix of Malliavin covariance of F.

Proof. See Lemma III.5.2 of Malliavin [14]. ∎

Theorem 2.3. *Let $F \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ be a nondegenerate functional. F has a smooth density $p^F \in \mathcal{S}(\mathbf{R}^n)$, where $\mathcal{S}(\mathbf{R}^n)$ denotes the space of all infinitely differentiable functions $f : \mathbf{R}^n \mapsto \mathbf{R}$ such that for any $k \ge 1$, and for any multi-index $\beta \in \{1, \dots, n\}^j$, one has $\sup_{x\in\mathbf{R}^n} |x|^k |\partial_\beta f(x)| < \infty$ (i.e., $\mathcal{S}(\mathbf{R}^n)$ is the Schwartz space and $\mathcal{S}'(\mathbf{R}^n)$ is its dual).*

Proof. See Theorem III.5.1 of Malliavin [14]. ∎

Definition 2.4. *Consider the space $\mathbf{D}_{-\infty}(\mathcal{W}) = \cup_{p\ge1,r\ge0}\mathbf{D}^p_{-r}(\mathcal{W})$, that is, the dual of $\mathbf{D}_\infty$. We call $F \in \mathbf{D}_{-\infty}(\mathcal{W})$ a distribution on the Wiener space. We define the duality form on $\mathbf{D}_{-\infty}\times\mathbf{D}_\infty$, $(F,G) \mapsto \langle F, G\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty} = E[FG] \in \mathbf{R}$. We call this duality form the generalized expectation.*

2.2. Asymptotic expansion for expectation of generalized Wiener functionals. Let $F \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ be a nondegenerate functional, and let ν and p^F be the law and the smooth density of F, respectively; that is, $\nu(dx) = \mu \circ F^{-1}(dx) = p^F(x)dx$. Also, we define the range O as $O := \{x : p^F(x) > 0\} \subset \mathbf{R}^n$.

By Malliavin [14] and Malliavin and Thalmaier [15], the conditional expectation of $g \in L^p(\mathcal{W}, \mu)$ conditioned by a set $\{w : F(w) = x\}$ in σ-field $\sigma(F)$, $E[g|F = x]$ gives a map,

$$E^F : L^p(\mathcal{W}, \mu) \ni g \mapsto E[g|F = x] \in L^p(O, \nu). \tag{3}$$

For multi-index $\alpha^{(k)} = (\alpha_1, \dots, \alpha_k)$, we define the iterated Malliavin weight. The Malliavin weight $H_{\alpha^{(k)}}$ is recursively defined as follows: for $G \in \mathbf{D}_\infty$,

$$H_{\alpha^{(k)}}(F, G) = H_{(\alpha_k)}(F, H_{\alpha^{(k-1)}}(F, G)), \tag{4}$$

where

$$H_{(l)}(F, G) = D^*\left(\sum_{i=1}^n G\gamma_{li}^F DF_i\right). \tag{5}$$

Here, $\gamma^F = \{\gamma_{ij}^F\}_{1\le i,j\le n}$ denotes the inverse matrix of the Malliavin covariance matrix of F.

Watanabe [25, 26] introduced the distribution on Wiener space as composition of a nondegenerate map F by a Schwartz distribution T. The next theorem restates the result of Watanabe [26] in terms of Malliavin [14] and Malliavin and Thalmaier [15].

Theorem 2.5.

1. *Let $\mathcal{S}'$ be the Schwartz distributions. There exists a map*

$$(E^F)^* : \mathcal{S}' \ni T \mapsto T \circ F \in \tilde{\mathbf{D}}_{-\infty} := \bigcup_{s\ge0}\bigcap_{q\ge1} \mathbf{D}^q_{-s} \subset \mathbf{D}_{-\infty}. \tag{6}$$

   $(E^F)^*$ is called the lifting up of T.

2. *The conditional expectation defines a map*

$$E^F : \mathbf{D}_\infty \ni G \mapsto E^F[G] \in \mathcal{S}(O), \tag{7}$$

where $\mathcal{S}(O)$ *stands for the Schwartz space of the rapidly decreasing functions on* $O = \{x : p^F(x) > 0\} \subset \mathbf{R}^n$*. We call this map the push-down of* G*.*

3. *The following duality formula is obtained:*

$$\langle (E^F)^* T, G\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty} = \langle T, E^F[G]\rangle_{p^F(x)dx}, \tag{8}$$

where $\langle\cdot,\cdot\rangle_{p^F(x)dx}$ *is defined as follows:*

$$\langle T, E^F[G]\rangle_{p^F(x)dx} = {}_{\mathcal{S}'}\langle T, E^F[G]p^F\rangle_{\mathcal{S}}. \tag{9}$$

Proof. In this proof, we apply the discussions of Watanabe [26], Malliavin [14], Malliavin and Thalmaier [15], and Nualart [16].

1. Given $T \in \mathcal{S}'$, there exists $T_n \in \mathcal{S}$ such that $T_n \to T$ *in* $\mathcal{S}'$, i.e., for $m \geq 1$,

$$\|A^{-m}T_n - A^{-m}T\|_\infty \to 0, \quad n \to \infty, \tag{10}$$

where $\|f\|_\infty = \sup_{x\in\mathbf{R}^n} |f(x)|$, $A = 1 + |x|^2 - \Delta$, $\Delta = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. By the Malliavin integration-by-parts formula, we can estimate as follows: for $p^{-1} + q^{-1} = 1$,

$$\begin{aligned}
&\|T_n(F) - T_{n'}(F)\|_{\mathbf{D}^q_{-2m}}\\
&= \sup_{G\in\mathbf{D}^p_{2m}, \|G\|_{\mathbf{D}^p_{2m}}\leq 1} |E[T_n(F)G] - E[T_{n'}(F)G]|\\
&= \sup_{G\in\mathbf{D}^p_{2m}, \|G\|_{\mathbf{D}^p_{2m}}\leq 1} \left|E[(A^m\{A^{-m}T_n\})(F)G] - E[(A^m\{A^{-m}T_{n'}\})(F)G]\right|\\
&= \sup_{G\in\mathbf{D}^p_{2m}, \|G\|_{\mathbf{D}^p_{2m}}\leq 1} \left|E[\{A^{-m}T_n\}(F)\pi^F_{(2m)}(G)] - E[\{A^{-m}T_{n'}\}(F)\pi^F_{(2m)}(G)]\right|\\
&\leq \sup_{G\in\mathbf{D}^p_{2m}, \|G\|_{\mathbf{D}^p_{2m}}\leq 1} \|A^{-m}T_n - A^{-m}T_{n'}\|_\infty \|\pi^F_{(2m)}(G)\|_{L^1}\\
&= C\|A^{-m}T_n - A^{-m}T_{n'}\|_\infty \to 0
\end{aligned}$$

as $n, n' \to \infty$, where $\pi^F_{(2m)}(G) \in \mathbf{D}_\infty$ and

$$C := \sup_{G\in\mathbf{D}^p_{2m}, \|G\|_{\mathbf{D}^p_{2m}}\leq 1} \|\pi^F_{(2m)}(G)\|_{L^1} < \infty.$$

Here, as in Ikeda and Watanabe [10, p. 379] or Theorem 3.2.1 with the equations (3.2.1) and (3.2.2) of Sakamoto and Yoshida [17], we have used the relation

$$E[(A^m\{A^{-m}T_n\})(F)G] = E[\{A^{-m}T_n\}(F)\pi^F_{(2m)}(G)], \tag{11}$$

where $\pi^F_{(2m)}(G)$ is recursively obtained by

$$\pi^F_{(2)}(G) = (1+|F|^2)G - 1/2\sum_{i=1}^n H_{(i,i)}(F,G),$$
$$\pi^F_{(2(m+1))}(G) = \pi^F_{(2)}(\pi^F_{(2m)}(G)), \quad m \geq 1.$$

Then $(T_n(F))_{n\in\mathbf{N}}$ is a Cauchy sequence in $\mathbf{D}_{-\infty}$, and thus there exists $(E^F)^*T = T(F) \in \mathbf{D}_{-\infty}$: a composite functional $T(F)$ is uniquely determined.

2. Given $G \in \mathbf{D}_\infty$, for any multi-index $s = (s_1,\ldots,s_k)$, and for any $\varphi \in C_b^{|s|}(\mathbf{R}^n)$, by *push-down* and then the integration-by-parts formula on $\mathbf{R}^n$ or the integration-by-parts formula on $\mathcal{W}$ and then *push-down*, we obtain

$$\begin{aligned} E[\partial_s^{|s|}\varphi(F)G] &= (-1)^{|s|}\int_{\mathbf{R}^n}\varphi(x)\partial_s^{|s|}\{E^{F=x}[G]p^F(x)\}dx \\ &= \int_{\mathbf{R}^n}\varphi(x)E^{F=x}[\pi_s^F]p^F(x)dx, \end{aligned} \tag{12}$$

where $\pi_s^F = H_s(F,G) \in \mathbf{D}_\infty$. It implies that

$$(-1)^{|s|}\partial_s^{|s|}\{E^{F=x}[G]p^F(x)\} = E^{F=x}[\pi_s^F]p^F(x), \tag{13}$$

where

$$E^{F=x}[\pi_s^F] \in L^p(O,\nu).$$

We define O_ϵ as $O_\epsilon = \{x \in \mathbf{R}^n : p^F(x) > \epsilon\}$. Therefore,

$$(-1)^{|s|}\partial_s^{|s|}\{E^{F=x}[G]p^F(x)\} \in L^p(O_\epsilon,dx) \tag{14}$$

for all s, which implies that $E^{F=x}[G]p^F(x) \in C^\infty(O)$. As $p^F(x) \in C^\infty(O)$, where $C^\infty(O)$ stands for the set of real-valued C^∞-functions on O,

$$(p^F(x))^{-1}\{E^{F=x}[G]p^F(x)\} = E^{F=x}[G] \in C^\infty(O). \tag{15}$$

Note also that the conditional expectation has the following expression:

$$E^{F=x}[G]p^F(x) = E[\mathbf{1}_{\{F>x\}}H_{(1,\ldots,n)}(F,G)]. \tag{16}$$

Thus, for all $k \in \mathbf{N}$ and for all $j = 1,\ldots,n$, if $x_j > 0$,

$$\begin{aligned} &\sup_{x\in\mathbf{R}^d, x_j>0} x_j^{2k}\partial_s^{|s|}\{E^{F=x}[G]p^F(x)\} \\ &= \sup_{x\in\mathbf{R}^d, x_j>0} x_j^{2k}|E[\mathbf{1}_{F>x}H_s(F,H_{(1,\ldots,n)}(F,G))| \\ &\leq E[|F_j|^{2k}|H_s(F,H_{(1,\ldots,n)}(F,G))|] \\ &< \infty, \end{aligned} \tag{17}$$

and if $x_j < 0$, we can derive a similar estimate. These facts imply that

$$E^{F=x}[G] \in \mathcal{S}(O).$$

3. Hence, because there exist $T_n \in \mathcal{S}$, $n \in \mathbf{N}$ for $T \in \mathcal{S}'$ such that $T_n \to T$ *in* $\mathcal{S}'$, we have

$$\langle T_n, E^F[G]\rangle_{p^F(x)dx} = E[T_n(F)G] \to \langle (E^F)^* T, G\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty} = \langle T, E^F[G]\rangle_{p^F(x)dx}$$

as $n \to \infty$. ■

Hereafter, we use the notation $\int T(x)p(x)dx$ for $T \in \mathcal{S}'(\mathbf{R}^n)$ and $p \in \mathcal{S}(\mathbf{R}^n)$, meaning that ${}_{\mathcal{S}'}\langle T, p\rangle_{\mathcal{S}}$.

The next theorem presents an asymptotic expansion formula for the expectation of generalized Wiener functionals.

Theorem 2.6. *Consider a family of smooth nondegenerate Wiener functionals* $F^\epsilon = (F_1^\epsilon, \dots, F_n^\epsilon) \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$, $\epsilon \in (0,1]$, *such that* F^ϵ *has an asymptotic expansion in* $\mathbf{D}_\infty$ *and satisfies the uniformly nondegenerate condition*

$$\limsup_{\epsilon\downarrow 0} \|(\det \sigma_{F^\epsilon})^{-1}\|_{L^p} < \infty \quad \textit{for all } p < \infty. \tag{18}$$

Then, for a Schwartz distribution $T \in S'(\mathbf{R}^n)$, *we have an asymptotic expansion in* $\mathbf{R}$:

$$\Bigg| E[T(F^\epsilon)] - \Bigg\{ \int_{\mathbf{R}^n} T(x) p^{F^0}(x) dx + \sum_{j=1}^{N} \epsilon^j \int_{\mathbf{R}^n} T(x) E\left[\sum_k^{(j)} H_{\alpha^{(k)}}\left(F^0, \prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}\right) | F^0 = x \right] p^{F^0}(x) dx \Bigg\} \Bigg| = O(\epsilon^{N+1}),$$

where p^{F^0} *is the density of* F^0, *and* $F_i^{0,k} := \frac{1}{k!}\frac{d^k}{d\epsilon^k} F_i^\epsilon|_{\epsilon=0}$, $k \in \mathbf{N}$, $i = 1, \dots, n$. *Also,* $\alpha^{(k)}$ *denotes a multi-index,* $\alpha^{(k)} = (\alpha_1, \dots, \alpha_k)$, *and*

$$\sum_k^{(j)} \equiv \sum_{k=1}^{j} \sum_{\beta_1+\cdots+\beta_k = j, \beta_i \geq 1} \sum_{\alpha^{(k)} \in \{1,\dots,n\}^k} \frac{1}{k!}.$$

Moreover, Malliavin weight $H_{\alpha^{(k)}}$ *is recursively defined as follows:*

$$H_{\alpha^{(k)}}(F, G) = H_{(\alpha_k)}(F, H_{\alpha^{(k-1)}}(F, G)),$$

where

$$H_{(l)}(F, G) = D^*\left(\sum_{i=1}^n G \gamma_{li}^F DF_i \right).$$

Here, $\gamma^F = \{\gamma_{ij}^F\}_{1\leq i,j\leq n}$ *denotes the inverse matrix of the Malliavin covariance matrix of* F.

Proof. We use α as an abbreviation of $\alpha^{(k)}$ in the proof. Under the uniformly nondegenerate condition of $F^\epsilon \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$, the lifting up of $T \in \mathcal{S}'(\mathbf{R}^n)$, $(E^{F^\epsilon})^* T$, has the asymptotic expansion in distributions on the Wiener space $\mathbf{D}_{-\infty}$; i.e., for $N \in \mathbf{N}$, there exists $s \in \mathbf{N}$ such that

$$\left\| (E^{F^\epsilon})^* T - \left\{ T \circ F^0 + \sum_{j=1}^N \epsilon^j \sum_k^{(j)} (\partial_\alpha^k T) \circ F^0 \prod_{l=1}^k F_{\alpha_l}^{0,\beta_l} \right\} \right\|_{\mathbf{D}_{-s}^q} = O(\epsilon^{N+1}), \quad \epsilon \in (0,1], \ q < \infty.$$

Then, there exists an asymptotic expansion of $\langle (E^{F^\epsilon})^* T, \mathbf{1}\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty}$. The push-down of the Malliavin weights are computed as follows:

$$\begin{aligned}\left\langle \partial_\alpha^k T(F^0), \prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}\right\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty} &= \left\langle T(F^0), H_\alpha\left(F^0, \prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}\right)\right\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty} \\ &= \left\langle T, E^{F^0}\left[H_\alpha\left(F^0, \prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}\right)\right]\right\rangle_{p^{F^0}(x)dx} \\ &= \int_{\mathbf{R}^n} T(x) E\left[H_\alpha\left(F^0, \prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}\right)|F^0 = x\right] p^{F^0}(x)dx. \quad \blacksquare\end{aligned}$$

Corollary 2.7. *The density $p^{F^\epsilon}(y)$ is expressed as the following asymptotic expansion with the push-down of Malliavin weights:*

$$p^{F^\epsilon}(y) = p^{F^0}(y) + \sum_{j=1}^N \epsilon^j E\left[\sum_k^{(j)} H_{\alpha^{(k)}}\left(F^0, \prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}\right)|F^0 = y\right] p^{F^0}(y) + O(\epsilon^{N+1}),$$

where $p^{F^0}(y)$ is the density of F^0.

Proof. Take a delta function $\delta_y \in \mathcal{S}'$ in the theorem above. $\blacksquare$

3. Asymptotic expansion in multidimensional stochastic volatility model.

This section applies the general formula in the previous section to pricing options in a stochastic volatility model and then obtains a new implied volatility expansion formula.

3.1. Asymptotic expansion of option prices.

This subsection proves a basic result on an asymptotic expansion for a stochastic volatility model.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space and $W = \{(W_{1,t},\ldots,W_{d,t}) : 0 \le t \le T\}$ be a d-dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. We consider the following stochastic volatility model (n-dimensional volatility factor):

$$\begin{aligned} dS_t^{(\epsilon)} &= rS_t^{(\epsilon)}dt + V(\sigma_t^{(\epsilon)})S_t^{(\epsilon)}dW_{1,t}, \\ d\sigma_t^{(\epsilon)} &= A_0(\sigma_t^{(\epsilon)})dt + \epsilon A(\sigma_t^{(\epsilon)})dW_t, \\ S_0^{(\epsilon)} &= S_0^{(0)} = s > 0, \quad \sigma_0^{(\epsilon)} = \sigma_0^{(0)} = \sigma \in \mathbf{R}_+^n, \end{aligned} \tag{19}$$

where $V \in C_b^\infty(\mathbf{R}^n \to \mathbf{R})$, $A_0 \in C_b^\infty(\mathbf{R}^n \to \mathbf{R}^n)$, $A \in C_b^\infty(\mathbf{R}^n \to \mathbf{R}^{n\times d})$, $r > 0$, and $\epsilon \in [0,1]$. Note that ϵ is the volatility of volatility parameter. $(\sigma_t^{(0)})_{t\in[0,T]}$ is a deterministic process and satisfies an ordinary differential equation,

$$d\sigma_t^{(0)} = A_0(\sigma_t^{(0)})dt. \tag{20}$$

Next, we impose the following condition, which gives the nondegeneracy of the Malliavin covariance of $S_T^{(\epsilon)}$ at $\epsilon = 0$.

Assumption 1. For some $s \in [0,T]$, $V(\sigma_s^{(0)}) \neq 0$.

We define the logarithmic process of $(S_t^{(\epsilon)})_{t\in[0,T]}$ as

$$X_t^{(\epsilon)} = \log\left(\frac{S_t^{(\epsilon)}}{s}\right).$$

Let $p^{SV}(y)$ be the density of the underlying asset of the stochastic volatility model and $C^{SV}(T,K)$ and $P^{SV}(T,K)$ be the call and the put option prices under the stochastic volatility with maturity T and strike price K.

Also let $p^{BS}(y)$ be the log-normal density of the Black–Scholes model, i.e.,

$$p^{BS}(y) := \frac{1}{y\sqrt{2\pi\int_0^T V(\sigma_t^{(0)})^2 dt}} \exp\left(-\frac{1}{2\int_0^T V(\sigma_t^{(0)})^2 dt}\left(\log\left(\frac{y}{s}\right) - rT + \int_0^T V(\sigma_t^{(0)})^2 dt\right)^2\right).$$

$C^{BS}(T,K,\bar{\sigma})$ and $P^{BS}(T,K,\bar{\sigma})$ denote the Black–Scholes formula of the call and the put options with maturity T and strike price K, i.e.,

$$C^{BS}(T,K,\bar{\sigma}) := sN(d_1) - Ke^{-rT}N(d_2), \tag{21}$$

$$P^{BS}(T,K,\bar{\sigma}) := Ke^{-rT}N(-d_2) - sN(-d_1), \tag{22}$$

where

$$\begin{aligned} n(x) &:= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \\ N(x) &:= \int_{-\infty}^{x} n(y)dy, \\ d_1 &:= \frac{\log\left(\frac{s}{K}\right) + rT}{\bar{\sigma}\sqrt{T}} + \frac{1}{2}\bar{\sigma}\sqrt{T}, \\ d_2 &:= \frac{\log\left(\frac{s}{K}\right) + rT}{\bar{\sigma}\sqrt{T}} - \frac{1}{2}\bar{\sigma}\sqrt{T}, \end{aligned}$$

with

$$\bar{\sigma} := \left(\frac{1}{T}\int_0^T V(\sigma_t^{(0)})^2 dt\right)^{1/2}. \tag{23}$$

Let $\sigma_{S_T^{(0)}}$ be the Malliavin (co)variance of $S_T^{(\epsilon)}$ at $\epsilon = 0$, i.e.,

$$\sigma_{S_T^{(0)}} = \|D_1 S_T^{(0)}\|_H^2 = (S_T^{(0)})^2 \int_0^T V(\sigma_s^{(0)})^2 ds. \tag{24}$$

We introduce the expressions

$$\begin{aligned} Z(t) &:= \frac{D_{t,1}S_T^{(0)}}{\|D_1 S_T^{(0)}\|_H^2} = \frac{S_T^{(0)} V(\sigma_t^{(0)})}{(S_T^{(0)})^2 \int_0^T V(\sigma_u^{(0)})^2 du}, \\ S_t^{(k)} &= \frac{1}{k!}\frac{\partial^k}{\partial \epsilon^k} S_t^{(\epsilon)}|_{\epsilon=0}, \\ \Psi^{\beta_1,\ldots,\beta_k}(t) &= \prod_{i=1}^{k} S_t^{(\beta_i)}, \end{aligned}$$

where $\beta_l \geq 1$ satisfy

$$\sum_{l=1}^{k} \beta_l = j, \quad j \in \mathbf{N}, \quad 1 \leq k \leq j.$$

Given Z above, for $\Psi \in \mathbf{D}_\infty$, $D^*(Z) - D_Z : \mathbf{D}_\infty \to \mathbf{D}_\infty$ is expressed as

$$(D^*(Z) - D_Z) \circ \Psi = \Psi \int_0^T Z(t) dW_{1,t} - \int_0^T D_{t,1} \Psi Z(t) dt. \tag{25}$$

Then, we have the following result.

Theorem 3.1. *Under the stochastic volatility model* (19)*, we have asymptotic expansions of the density and the option prices as follows:*

$$\begin{aligned}
p^{SV}(y) &= p^{BS}(y) + \sum_{j=1}^{N} \epsilon^j E[\pi_j | S_T^{(0)} = y] p^{BS}(y) + O(\epsilon^{N+1}), \\
C^{SV}(T,K) &= C^{BS}(T,K,\bar{\sigma}) + \sum_{j=1}^{N} \epsilon^j \int_{\mathbf{R}_+} e^{-rT} (y-K)^+ E[\pi_j | S_T^{(0)} = y] p^{BS}(y) dy \\
&\quad + O(\epsilon^{N+1}), \\
P^{SV}(T,K) &= P^{BS}(T,K,\bar{\sigma}) + \sum_{j=1}^{N} \epsilon^j \int_{\mathbf{R}_+} e^{-rT} (K-y)^+ E[\pi_j | S_T^{(0)} = y] p^{BS}(y) dy \\
&\quad + O(\epsilon^{N+1}),
\end{aligned}$$

where π_j is the jth-order Malliavin weight, i.e.,

$$\pi_j = \sum_{k=1}^{j} \sum_{\beta_1 + \cdots + \beta_k = j, \beta_i \geq 1} \frac{1}{k!} H_k(S_T^{(0)}, \Psi^{\beta_1,\ldots,\beta_k}(T)) \in \mathbf{D}_\infty,$$

with

$$\begin{aligned}
H_1(S_T^{(0)}, \Psi^{\beta_1,\ldots,\beta_k}(T)) &= \Psi^{\beta_1,\ldots,\beta_k}(T) \int_0^T Z(s) dW_{1,s} - \int_0^T D_{s,1} \Psi^{\beta_1,\ldots,\beta_k}(T) Z(s) ds, \\
H_k(S_T^{(0)}, \Psi^{\beta_1,\ldots,\beta_k}(T)) &= (D^*(Z) - D_Z)^k \circ \Psi^{\beta_1,\ldots,\beta_k}.
\end{aligned}$$

Proof. Let $(Y_t)_t$ be the solution of the following stochastic differential equation:

$$\begin{aligned}
dY_j^i(t) &= \sum_{l=1}^{d} \sum_{k=1}^{n} a_{k,l}^i(s) Y_j^k(s) dW_t^l + \sum_{k=1}^{n} b_k^i(s) Y_j^k(s) ds, \\
Y_j^i(0) &= \delta_j^i,
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
a_{k,l}^i(s) &= \partial_k A_l^i(\sigma_s^{(\epsilon)}), \\
b_k^i(s) &= \partial_k A_0^i(\sigma_s^{(\epsilon)}),
\end{aligned}$$

and δ^i_j is the Kronecker delta. Let $D_{s,k}$, $k=1,\dots,d$, be the Malliavin derivative acting on the Brownian motion $W_{k,t}$. Then, the Malliavin covariance of $S_T^{(\epsilon)}$ is given by

$$\sigma_{S_T^{(\epsilon)}} = \sum_{i=1}^{d}\int_0^T (D_{s,i}S_T^{(\epsilon)})^2 ds = (S_T^{(\epsilon)})^2\sum_{i=1}^{d}\int_0^T (D_{s,i}X_T^{(\epsilon)})^2 ds, \tag{27}$$

where

$$D_{s,1}X_T^{(\epsilon)} = V(\sigma_s^{(\epsilon)}) - \epsilon\sum_{i=1}^{n}\int_s^T V(\sigma_u^{(\epsilon)})\partial_i V(\sigma_u^{(\epsilon)})D_{s,1}\sigma_{i,u}^{(\epsilon)}du + \epsilon\sum_{i=1}^{n}\int_s^T \partial_i V(\sigma_u^{(\epsilon)})D_{s,1}\sigma_{i,u}^{(\epsilon)}dW_{1,u}, \tag{28}$$

$$D_{s,k}X_T^{(\epsilon)} = -\epsilon\sum_{i=1}^{n}\int_s^T V(\sigma_u^{(\epsilon)})\partial_i V(\sigma_u^{(\epsilon)})D_{s,k}\sigma_{i,u}^{(\epsilon)}du + \epsilon\sum_{i=1}^{n}\int_s^T \partial_i V(\sigma_u^{(\epsilon)})D_{s,k}\sigma_{i,u}^{(\epsilon)}dW_{1,u}. \tag{29}$$

Here, $\partial_i V(\sigma_u^{(\epsilon)}) = \frac{\partial V(x)}{\partial x_i}|_{x=\sigma_u^{(\epsilon)}}$ and for $s\le u$,

$$D_{s,k}\sigma_{i,u}^{(\epsilon)} = \sum_{l,j=1}^{n} Y_l^i(u)Y^{-1}(s)_j^l A_k^j(\sigma_s^{(\epsilon)}). \tag{30}$$

Assumption 1 yields the nondegeneracy of $S_T^{(0)}$:

$$\|\sigma^{-1}_{S_T^{(0)}}\|_{L^p} < \infty \quad \text{for all } p<\infty. \tag{31}$$

Then, with an argument similar to that in Takahashi and Yoshida [23], we have an asymptotic expansion in $\mathbf{D}_{-\infty}$:

$$\delta_y(S_T^{(\epsilon)}) = \delta_y(S_T^{(0)}) + \sum_{i=1}^{N}\epsilon^j\sum_{k=1}^{j}\sum_{\beta_1+\cdots+\beta_k=j,\beta_i\ge 1}\frac{1}{k!}\partial^k\delta_y(S_T^{(0)})\Psi^{\beta_1,\dots,\beta_k}(T) + O(\epsilon^{N+1}), \tag{32}$$

$$e^{-rT}(S_T^{(\epsilon)}-K)^+ = e^{-rT}(S_T^{(0)}-K)^+ + \sum_{i=1}^{N}\epsilon^j\sum_{k=1}^{j}\sum_{\beta_1+\cdots+\beta_k=j,\beta_i\ge 1}\frac{1}{k!}e^{-rT}\partial^k(S_T^{(0)}-K)^+\Psi^{\beta_1,\dots,\beta_k}(T) + O(\epsilon^{N+1}). \tag{33}$$

By the integration-by-parts formula,

$$
\begin{aligned}
&\left\langle \partial^k \delta_y(S_T^{(0)}), \Psi^{\beta_1,\dots,\beta_k}(T)\right\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty} \\
&= \left\langle \delta_y(S_T^{(0)}), H_k(S_T^{(0)}, \Psi^{\beta_1,\dots,\beta_k}(T))\right\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty} \\
&= \left\langle \delta_y, E^{S_T^{(0)}}[H_k(S_T^{(0)}, \Psi^{\beta_1,\dots,\beta_k}(T))]\right\rangle_{p^{BS}(x)dx} \\
&= E[H_k(S_T^{(0)}, \Psi^{\beta_1,\dots,\beta_k}(T))|S_T^{(0)} = y]p^{BS}(y),
\end{aligned} \tag{34}
$$

where

$$
\partial^k \delta_y(S_T^{(0)}) = \frac{\partial^k}{\partial x^k}\delta_y(x)|_{x=S_T^{(0)}}.
$$

Similarly, we have

$$
\begin{aligned}
&\left\langle e^{-rT}\partial^k (S_T^{(0)} - K)^+, \Psi^{\beta_1,\cdots,\beta_k}(T)\right\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty} \\
&= \int_{\mathbf{R}_+} e^{-rT}(y-K)^+ E[H_k(S_T^{(0)}, \Psi^{\beta_1,\dots,\beta_k}(T))|S_T^{(0)} = y]p^{BS}(y)dy,
\end{aligned} \tag{35}
$$

where

$$
\partial^k \left(S_T^{(0)} - K\right)^+ = \frac{\partial^k}{\partial x^k}(x-K)^+|_{x=S_T^{(0)}}.
$$

Then, we obtain asymptotic expansion formulas of the density and the option prices:

$$
p^{SV}(y) = p^{BS}(y) + \sum_{j=1}^{N} \epsilon^j E[\pi_j|S_T^{(0)} = y]p^{BS}(y) + O(\epsilon^{N+1}), \tag{36}
$$

$$
\begin{aligned}
&C^{SV}(T,K) = C^{BS}(T,K,\bar{\sigma}) \\
&+\sum_{j=1}^{N} \epsilon^j \int_{\mathbf{R}_+} e^{-rT}(y-K)^+ E[\pi_j|S_T^{(0)} = y]p^{BS}(y)dy + O(\epsilon^{N+1}),
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
&P^{SV}(T,K) = P^{BS}(T,K,\bar{\sigma}) \\
&+\sum_{j=1}^{N} \epsilon^j \int_{\mathbf{R}_+} e^{-rT}(K-y)^+ E[\pi_j|S_T^{(0)} = y]p^{BS}(y)dy + O(\epsilon^{N+1}).
\end{aligned} \tag{38}
$$

The Malliavin weights are computed by the iterated Skorohod integrals

$$
\begin{aligned}
&H_1\left(S_T^{(0)}, \Psi^{\beta_1,\dots,\beta_k}\right) = D^*\left(\Psi^{\beta_1,\dots,\beta_k}\sigma^{-1}_{S_T^{(0)}} DS_T^{(0)}\right) \\
&= (D^*(Z) - D_Z)\circ \Psi^{\beta_1,\dots,\beta_k} \\
&= \Psi^{\beta_1,\dots,\beta_k}(T)\int_0^T Z(s)dW_{1,s} - \int_0^T D_{s,1}\Psi^{\beta_1,\dots,\beta_k}(T)Z(s)ds,
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
&H_k(S_T^{(0)}, \Psi^{\beta_1,\dots,\beta_k}(T)) = H_1(S_T^{(0)}, H_{k-1}(S_T^{(0)}, \Psi^{\beta_1,\dots,\beta_k}(T))) \\
&= (D^*(Z) - D_Z)^k \circ \Psi^{\beta_1,\dots,\beta_k}. \qquad \blacksquare
\end{aligned} \tag{40}
$$

3.2. Implied volatility expansion. This subsection derives an asymptotic expansion formula for the implied volatility in the stochastic volatility model considered in the previous subsection, where we obtained an approximation formula of a call option:

$$C^{SV}(T,K) = C^{BS}(T,K,\bar{\sigma}) + \epsilon C_1 + \epsilon^2 C_2 + \epsilon^3 C_3 + O(\epsilon^4), \tag{41}$$

where

$$C_i = \int_{\mathbf{R}} e^{-rT}\left(se^{y+rT-\frac{1}{2}\Sigma} - K\right)^+ E\left[\pi_i \middle| \int_0^T V(\sigma_t^{(0)})dW_{1,t} = y\right]\frac{1}{\sqrt{2\pi\Sigma}}e^{-\frac{1}{2\Sigma}y^2}dy, \quad i = 1,2,3.$$

We obtain an asymptotic expansion formula of the implied volatility around

$$\bar{\sigma} = \left(\frac{1}{T}\int_0^T V(\sigma_t^{(0)})^2 dt\right)^{1/2}. \tag{42}$$

Theorem 3.2. *Under the stochastic volatility model (19), an asymptotic expansion of the implied volatility is given by*

$$\begin{aligned}
\sigma^{IV}(T,K) &= \bar{\sigma} + \epsilon\frac{C_1}{C_\sigma^{BS}(\bar{\sigma})} + \epsilon^2\left\{\frac{C_2}{C_\sigma^{BS}(\bar{\sigma})} - \frac{1}{2}\frac{C_1^2}{C_\sigma^{BS}(\bar{\sigma})^3}C_{\sigma\sigma}^{BS}(\bar{\sigma})\right\}\\
&+ \epsilon^3\left\{\frac{C_3}{C_\sigma^{BS}(\bar{\sigma})} - \left(\frac{C_1}{C_\sigma^{BS}(\bar{\sigma})^2}\right)\left\{\frac{C_2}{C_\sigma^{BS}(\bar{\sigma})} - \frac{1}{2}\frac{C_1^2}{C_\sigma^{BS}(\bar{\sigma})^3}C_{\sigma\sigma}^{BS}(\bar{\sigma})\right\}C_{\sigma\sigma}^{BS}(\bar{\sigma}) - \frac{1}{3!}\frac{C_1^3}{C_\sigma^{BS}(\bar{\sigma})^4}C_{\sigma\sigma\sigma}^{BS}(\bar{\sigma})\right\}\\
&+ O(\epsilon^4),
\end{aligned}$$

where $C_\sigma^{BS}(\bar{\sigma})$, $C_{\sigma\sigma}^{BS}(\bar{\sigma})$, $C_{\sigma\sigma\sigma}^{BS}(\bar{\sigma})$ are derivatives of the Black–Scholes formula with respect to the volatility, i.e.,

$$\begin{aligned}
C_\sigma^{BS}(\bar{\sigma}) &:= \frac{\partial}{\partial\sigma}C^{BS}|_{\sigma=\bar{\sigma}} = s\sqrt{T}n(d_1),\\
C_{\sigma\sigma}^{BS}(\bar{\sigma}) &:= \frac{\partial^2}{\partial\sigma^2}C^{BS}|_{\sigma=\bar{\sigma}} = \frac{s\sqrt{T}}{\bar{\sigma}}n(d_1)d_1d_2,\\
C_{\sigma\sigma\sigma}^{BS}(\bar{\sigma}) &:= \frac{\partial^3}{\partial\sigma^3}C^{BS}|_{\sigma=\bar{\sigma}} = \frac{s\sqrt{T}}{\bar{\sigma}^2}n(d_1)\{d_1^2d_2^2 - d_1d_2 - d_1^2 - d_2^2\}.
\end{aligned}$$

Proof. Suppose that an implied volatility is expanded as

$$\sigma^{IV}(T,K) = \bar{\sigma} + \epsilon\sigma_1 + \epsilon^2\sigma_2 + \epsilon^3\sigma_3 + O(\epsilon^4). \tag{43}$$

Then we have

$$\begin{aligned}
C^{BS}(T,K,\sigma^{IV}(T,K)) &= C^{BS}(T,K,\bar{\sigma}) + \epsilon C_\sigma^{BS}(\bar{\sigma})\sigma_1\\
&+ \epsilon^2\left\{C_\sigma^{BS}(\bar{\sigma})\sigma_2 + \frac{1}{2}C_{\sigma\sigma}^{BS}(\bar{\sigma})(\sigma_1)^2\right\}\\
&+ \epsilon^3\left\{\sigma_3C_\sigma^{BS}(\bar{\sigma}) + \sigma_1\sigma_2C_{\sigma\sigma}^{BS}(\bar{\sigma}) + \frac{1}{3!}\sigma_1^3C_{\sigma\sigma\sigma}^{BS}(\bar{\sigma})\right\} + O(\epsilon^4).
\end{aligned} \tag{44}$$

By the definition of the implied volatility in the stochastic volatility, i.e., $C^{SV}(T,K) = C^{BS}(T,K,\sigma^{IV}(T,K))$, the approximation terms of the implied volatility are given by

$$\sigma_1 = \frac{C_1}{C^{BS}_\sigma(\bar\sigma)},$$
$$\sigma_2 = \left\{ \frac{C_2}{C^{BS}_\sigma(\bar\sigma)} - \frac{1}{2}\frac{C_1^2}{C^{BS}_\sigma(\bar\sigma)^3} C^{BS}_{\sigma\sigma}(\bar\sigma) \right\},$$
$$\sigma_3 = \frac{C_3}{C^{BS}_\sigma(\bar\sigma)} - \left(\frac{C_1}{C^{BS}_\sigma(\bar\sigma)^2} \right) \left\{ \frac{C_2}{C^{BS}_\sigma(\bar\sigma)} - \frac{1}{2}\frac{C_1^2}{C^{BS}_\sigma(\bar\sigma)^3} C^{BS}_{\sigma\sigma}(\bar\sigma) \right\} C^{BS}_{\sigma\sigma}(\bar\sigma)$$
$$-\frac{1}{3!}\frac{C_1^3}{C^{BS}_\sigma(\bar\sigma)^4} C^{BS}_{\sigma\sigma\sigma}(\bar\sigma). \quad \blacksquare$$

3.3. Comments on related works. Fournié et al. [5] applied Malliavin calculus to efficient Monte Carlo estimators for computing Greeks of options in the Black–Scholes framework; e.g., for $dS_t = rS_t dt + \sigma^{(\epsilon)} S_t dW_t$ and $\sigma^{(\epsilon)} = \sigma + \epsilon$,

$$\frac{\partial}{\partial\sigma} E[e^{-rT}(S_T - K)^+] = \frac{\partial}{\partial\epsilon} E[e^{-rT}(S_T^{(\epsilon)} - K)^+]|_{\epsilon=0} = E[e^{-rT}(S_T^{(0)} - K)^+ \pi]. \tag{45}$$

The estimator π is the so-called *Malliavin weight*. Subsequently, a number of papers extended the method. In particular, related to our work, Siopacha and Teichmann [19] developed strong and weak Taylor methods for a system of stochastic differential equations (SDEs) with perturbations. Especially, the weak Taylor method is based on the integration by parts on the Wiener space, which is a powerful tool for efficient Monte Carlo simulations and is general enough to be applied to multidimensional SDEs. As an example, they applied the method to swaptions under a market model of interest rates with one-dimensional Heston-type stochastic volatility and obtained an approximation of the option prices including the expectation of the divergence on the Wiener space. In the last step, they used Monte Carlo simulations for computing the option prices and demonstrated the efficiency of their method comparing with well-known existing Monte Carlo schemes.

Lewis [13] developed a new method for approximations of European option prices and implied volatilities. In particular, he proposed an expansion with respect to a stochastic volatility parameter, “*vol-of-vol* ” (ξ below), under a general one-dimensional time-homogeneous diffusion process of the stochastic volatility, which is considered as a special case of our setup. However, he took an approach different from ours to obtain an approximation: for example, he considered the following generalized Heston model:

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t} S_t dW_{1t}, \\ dv_t &= b(v_t)dt + \xi\eta(v_t)(\rho(v)dW_{1t} + \sqrt{1-\rho(v)^2} dW_{2t}), \end{aligned} \tag{46}$$

where r and ξ are constants. Also, $W = (W_{1t}, W_{2t})$ is a two-dimensional Brownian motion and $\rho(v)$ stands for the instantaneous correlation between S and v. For valuation of call options, he used the following formula:

$$C(T,K) = S_0 - \frac{Ke^{-rT}}{2\pi} \int_{\frac{i}{2}-\infty}^{\frac{i}{2}+\infty} e^{-ikx} \frac{H(k,v,T)}{k^2 - ik} dk, \tag{47}$$

where $x = \log\left(S/Ke^{-rT}\right)$, and H (called "*the fundamental transform*") is the solution to

$$\frac{\partial H}{\partial T} = \frac{1}{2}\xi^2\eta^2(v)\frac{\partial^2 H}{\partial v^2} + (b(v) - ik\xi\rho(v)\eta(v)\sqrt{v})\frac{\partial H}{\partial v} - \{(k^2 - ik)/2\}vH, \tag{48}$$

with the initial condition $H(k, V, 0) = 1$. Then, he gave an approximation for a call price showing a scheme for the expansion of H with respect to ξ:

$$H(k, v, T) = H^{(0)}(k, v, T) + \xi H^{(1)}(k, v, T) + \xi^2 H^{(2)}(k, v, T) + \cdots . \tag{49}$$

He also presented an explicit result of the expansion for the following model:

$$dv_t = (\omega - \theta v_t)dt + \xi v_t^{\beta}(\rho dW_{1t} + \sqrt{1-\rho^2}dW_{2t}),$$

where ω, θ, β, and ρ are some appropriate constants.

This paper applies an asymptotic expansion approach to stochastic volatility models and derives the approximation terms including the Malliavin weights. Then, these weights are transformed into a finite-dimensional integration through the *duality formula* of Malliavin [14]; that is, for $T \in \mathcal{S}'$,

$$\begin{aligned} E[T(S_T)\pi] &= \langle (E^{S_T})^* T, \pi\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_{\infty}} \\ &= \langle T, E^{S_T}[\pi]\rangle_{p(x)dx} = \int_{\mathbf{R}} T(x)E[\pi | S_T = x]p(x)dx, \end{aligned} \tag{50}$$

where $p(x)$ is the density of $S_T \in \mathbf{D}_\infty$. This formula suggests that an element of the distributions on the Wiener space $\mathbf{D}_{-\infty}$ is the adjoint operator of the conditional expectation. It also shows that the push-down of the Malliavin weights (the conditional expectation of the divergence on the Wiener space) is an element of the Schwartz space $\mathcal{S}$. Thus, applying both the integration-by-parts and the duality formulas, we can obtain analytical approximations for density functions, option prices, and implied volatilities.

A simple example, for $dS_t = rS_t dt + \sigma_t^{(\epsilon)} S_t dW_{1t}$ and $d\sigma_t^{(\epsilon)} = A_0(\sigma_t^{(\epsilon)})dt + \epsilon A_1(\sigma^{(\epsilon)})(\rho dW_{1t} + \sqrt{1-\rho^2}dW_{2t})$ for some real-valued functions $A_0(x)$ and $A_1(x)$, is when we can obtain an approximation for a call option as follows:

$$\begin{aligned} C(T,K) &= E[e^{-rT}(S_T^{(\epsilon)} - K)^+] \\ &= E[e^{-rT}(S_T^{(0)} - K)^+] + \epsilon\frac{\partial}{\partial\epsilon}E[e^{-rT}(S_T^{(\epsilon)} - K)^+]|_{\epsilon=0} + O(\epsilon^2) \\ &= E[e^{-rT}(S_T^{(0)} - K)^+] + \epsilon E[e^{-rT}(S_T^{(0)} - K)^+\pi] + O(\epsilon^2) \\ &= C^{BS}(T,K) + \epsilon\int_{\mathbf{R}}\left(se^{x-\frac{1}{2}\Sigma} - K\right)^+ \vartheta(x)\frac{1}{\sqrt{2\pi\Sigma}}e^{-\frac{1}{2\Sigma}x^2}dx + O(\epsilon^2), \end{aligned} \tag{51}$$

where $C^{BS}(T,K)$ represents a Black–Scholes call price and

$$\vartheta(x) = E\left[\pi | \int_0^T \sigma_t^{(0)}dW_{1,t} = x\right] = \zeta\left(\frac{1}{\Sigma^3}x^3 - \frac{1}{\Sigma^2}x^2 - \frac{1}{\Sigma^2}3x + \frac{1}{\Sigma}\right).$$

Here, ζ and Σ are some constants. (See Corollary A.2 in Appendix A for the details.) Further, the integral in the last line of (51) can be given explicitly.

Consequently, our method is a natural extension of Fournié et al. [5] and Siopacha and Teichmann [19] in the sense that we obtain analytical approximations that do not rely on Monte Carlo simulations by making use of push-down of Malliavin weights. Although Lewis [13] took a different approach, our method can be regarded as its extension because his models are included in our framework: in fact, Theorems 3.1 and 3.2 show that our method and formulas can be applied to multidimensional models including time-inhomogeneous ones in a unified way. That is, we can readily deal with multidimensional stochastic volatility models by the same procedure as in one-dimensional ones. A concrete example will be shown in section 4.2. In this regard, it does not seem an easy task for other analytical approximations such as Lewis [13], Hagan et al. [9], Labordère [12], and Antonelli and Scarlatti [1] to apply the methods to higher-dimensional models.

4. Numerical examples.

4.1. One-dimensional stochastic volatility models.

This subsection shows numerical examples for the following stochastic volatility model:

$$\begin{aligned} dS_t^{(\epsilon)} &= (\sigma_t^{(\epsilon)})^{\delta} S_t^{(\epsilon)} dW_{1,t}, \\ d\sigma_t^{(\epsilon)} &= \kappa(\theta - \sigma_t^{(\epsilon)})dt + \epsilon(\sigma_t^{(\epsilon)})^{\gamma}(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}), \end{aligned} \tag{52}$$

where all the parameters will be specified later. We set the risk-free interest rate to be zero (i.e., $r = 0$).

Remark. In section 4, we deal with various stochastic volatility models including the Heston-type square-root volatility (i.e., (52) with $\gamma = 1/2$). Although the square-root volatility model does not satisfy the regularity conditions stated in section 3 since the coefficient function of the Heston-type model $A(\sigma) = \sqrt{\sigma}$ has an unbounded derivative at $\sigma = 0$, our method is formally applicable.

A rigorous treatment for the application of our method to this case could be made in a manner similar to that in Takahashi and Yoshida [24] that utilizes a smooth modification technique: In fact, we take a modified process $(\tilde{\sigma}_t)_{t\in[0,T]}$ of square-root process $(\sigma_t)_{t\in[0,T]}$:

$$\begin{aligned} dS_t^{(\epsilon)} &= (\sigma_t^{(\epsilon)})^{\delta} S_t^{(\epsilon)} dW_{1,t}, \\ d\sigma_t^{(\epsilon)} &= \kappa(\theta - \sigma_t^{(\epsilon)})dt + \epsilon\sqrt{\sigma_t^{(\epsilon)}}(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}), \\ d\tilde{S}_t^{(\epsilon)} &= (\sigma_t^{(\epsilon)})^{\delta} \tilde{S}_t^{(\epsilon)} dW_{1,t}, \\ d\tilde{\sigma}_t^{(\epsilon)} &= \kappa(\theta - \tilde{\sigma}_t^{(\epsilon)})dt + \epsilon g(\tilde{\sigma}_t^{(\epsilon)})(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}). \end{aligned}$$

Here, $g(x)$ is a smooth modification of $\sqrt{x}$ such that $g(x) = \sqrt{x}$ when $x \geq \epsilon_1$ for some small $\epsilon_1 > 0$, and $g(x) = 0$ when $x \leq \epsilon_2$ for some $\epsilon_2 \in (0, \epsilon_1)$. Specifically, we can set $g(x)$ as follows:

$$\begin{aligned} g(x) &= h(x)\sqrt{x}, \\ h(x) &= \frac{\psi(x-\epsilon_2)}{\psi(x-\epsilon_2) + \psi(\epsilon_1 - x)}, \quad 0 < \epsilon_2 < \epsilon_1, \\ \psi(x) &= e^{-1/x} \text{ for } x > 0, \quad \psi(x) = 0 \text{ for } x \leq 0. \end{aligned}$$

Let $X_T^{(\epsilon)} = \log(S_T^{(\epsilon)}/s)$ and $\tilde{X}_T^{(\epsilon)} = \log(\tilde{S}_T^{(\epsilon)}/s)$. Then,

$$\begin{aligned} X_T^{(\epsilon)} &= \frac{-1}{2}\int_0^T (\sigma_t^{(\epsilon)})^{2\delta} dt + \int_0^T (\sigma_t^{(\epsilon)})^{\delta} dW_{1,t}, \\ \tilde{X}_T^{(\epsilon)} &= \frac{-1}{2}\int_0^T (\tilde{\sigma}_t^{(\epsilon)})^{2\delta} dt + \int_0^T (\tilde{\sigma}_t^{(\epsilon)})^{\delta} dW_{1,t}. \end{aligned}$$

Suppose that, for an $\mathbf{R}$-valued function f which, for example, stands for an option payoff, $f(x) = (e^x - K)^+, (K - e^x)^+$, $E[|f(X_T^{(\epsilon)})|^2] < \infty$, and $E[|f(\tilde{X}_T^{(\epsilon)})|^2] < \infty$. Then, we have

$$E\left[\left|f(X_T^{(\epsilon)}) - f(\tilde{X}_T^{(\epsilon)})\right| 1_{\{\sigma^{(\epsilon)} \neq \tilde{\sigma}^{(\epsilon)}\}}\right] \leq \left(E\left[|f(X_T^{(\epsilon)})|^2\right]^{\frac{1}{2}} + E\left[|f(\tilde{X}_T^{(\epsilon)})|^2\right]^{\frac{1}{2}}\right) P(\{\sigma^{(\epsilon)} \neq \tilde{\sigma}^{(\epsilon)}\})^{\frac{1}{2}}.$$

It also holds that

$$\begin{aligned} &P(\{\sigma^{(\epsilon)} \neq \tilde{\sigma}^{(\epsilon)}\}) = P(\{\sigma_t^{(\epsilon)} \leq \epsilon_1 \text{ for some } t \in [0,T]\}) \\ &\leq P\left(\left\{\sup_{0\leq t\leq T} |\sigma_t^{(\epsilon)} - \sigma_t^{(0)}| > a\right\}\right) \\ &+ P\left(\{\sigma_t^{(\epsilon)} \leq \epsilon_1 \text{ for some } t \in [0,T]\} \cap \left\{\sup_{0\leq t\leq T} |\sigma_t^{(\epsilon)} - \sigma_t^{(0)}| \leq a\right\}\right). \end{aligned}$$

We can easily see that the second term after the last inequality is 0. The first term is smaller than any ϵ^n for $n = 1, 2, \ldots$ by the following lemma of a large deviation inequality.

Lemma 4.1. *Suppose that $Z_t^{(\epsilon)}$, $t \in [0,T]$, follows an SDE:*

$$dZ_t^{(\epsilon)} = a(Z_t^{(\epsilon)})dt + \epsilon b(Z_t^{(\epsilon)})dW_t,$$

where $a(z)$ satisfies Lipschitz and linear growth conditions, and $b(z)$ satisfies the linear growth condition. We assume that the unique strong solution exists. Then, there exist positive constants c_1 and c_2 independent of ϵ such that

$$P\left(\left\{\sup_{0\leq s\leq T} |Z_s^{(\epsilon)} - Z_s^{(0)}| > c\right\}\right) \leq c_1 \exp(-c_2\epsilon^{-2}) \tag{53}$$

for all $c > 0$.

The lemma can be proved by slight modification of Lemma 7.1 in Kunitomo and Takahashi [11]. Note also that $\sigma^{(\epsilon)}$ and $\tilde{\sigma}^{(\epsilon)}$ satisfy the conditions in the lemma above.

Hence,

$$E\left[\left|f(X_T^{(\epsilon)}) - f(\tilde{X}_T^{(\epsilon)})\right|\right] = o(\epsilon^n), \quad n = 1, 2, \ldots. \tag{54}$$

Therefore, the difference between $f(X_T^{(\epsilon)})$ and $f(\tilde{X}_T^{(\epsilon)})$ is negligible in the *small disturbance asymptotic theory*. Then, the mathematical conditions in section 3 are met and our expansion method is rigorously applicable.

Hereafter, we use the following notations:

$$
\begin{aligned}
n(x) &:= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \\
N(x) &:= \int_{-\infty}^{x} n(y)dy, \\
\zeta &:= \rho \int_0^T \delta(\sigma_t^{(0)})^{(\delta-1)} e^{-\kappa t} (\sigma_t^{(0)})^{\delta} \int_0^t e^{\kappa s} (\sigma_s^{(0)})^{\gamma+\delta} ds dt, \\
\Sigma &:= \int_0^T \left((\sigma_t^{(0)})^{\delta} \right)^2 dt, \\
\vartheta(x) &:= E\Big[\frac{\partial}{\partial \epsilon} S_T^{(\epsilon)}|_{\epsilon=0} \int_0^T \frac{D_{t,1} S_T^{(0)}}{\int_0^T (D_{s,1} S_T^{(0)})^2 ds} dW_{1,t} \\
&\qquad - \int_0^T D_{s,1} \frac{\partial}{\partial \epsilon} S_T^{(\epsilon)}|_{\epsilon=0} \frac{D_{t,1} S_T^{(0)}}{\int_0^T (D_{s,1} S_T^{(0)})^2 ds} dt \Big| \int_0^T (\sigma_t^{(0)})^{\delta} dW_{1,t} = x \Big] \\
&= \zeta \left(\frac{1}{\Sigma^3} x^3 - \frac{1}{\Sigma^2} x^2 - \frac{1}{\Sigma^2} 3x + \frac{1}{\Sigma} \right), \\
\bar{\sigma} &:= \left(\frac{1}{T} \int_0^T \left((\sigma_t^{(0)})^{\delta} \right)^2 dt \right)^{1/2}, \\
d_1 &:= \frac{\log\left(\frac{s}{K}\right)}{\bar{\sigma}\sqrt{T}} + \frac{1}{2} \bar{\sigma} \sqrt{T}, \\
d_2 &:= \frac{\log\left(\frac{s}{K}\right)}{\bar{\sigma}\sqrt{T}} - \frac{1}{2} \bar{\sigma} \sqrt{T}, \\
C^{BS} &:= sN(d_1) - KN(d_2), \\
C_1 &:= \int_{\mathbf{R}} \left(s e^{x - \frac{1}{2}\bar{\sigma}^2 T} - K \right)^+ \vartheta(x) \frac{1}{\sqrt{2\pi \bar{\sigma}^2 T}} \exp\left(-\frac{1}{2\bar{\sigma}^2 T} x^2 \right) dx, \\
vega &:= s\sqrt{T} n(d_1).
\end{aligned}
$$

By using Corollary A.2 in Appendix A, the first-order approximations of the asymptotic expansions of the option price and the implied volatility are given by

$$\textbf{Malliavin AE} := C^{BS} + \epsilon C_1$$

and

$$\textbf{Malliavin IV} := \bar{\sigma} + \epsilon \frac{C_1}{vega}.$$

First, we give the numerical results on the accuracy of our approximation formula for call option prices.

$T = 1,\ \delta = 1/2,\ \gamma = 1/2,\ s = 100,\ (\sigma_0)^{\frac{1}{2}} = 0.3,\ \kappa = 2.0,\ \theta = 0.09,\ \epsilon = 0.1,\ \rho = -0.5$

Strike price	**Benchmark**	**Malliavin AE**	Relative error
70	31.5478	31.5496	0.00%
80	23.6382	23.6471	0.04%
90	17.0487	17.0631	0.08%
100	11.8647	11.8816	0.14%
110	7.9947	8.0105	0.20%
120	5.2356	5.2474	0.23%

Benchmark: Heston's Fourier transform solutions.

$T = 5,\ \delta = 1,\ \gamma = 1,\ s = 100,\ \sigma_0 = 0.4,\ \kappa = 0.1,\ \theta = 0.4,\ \epsilon = 0.2,\ \rho = -0.5$

Strike price	**Benchmark**	**Malliavin AE**	Relative error
70	46.4585	46.3591	−0.21%
80	41.5476	41.4427	−0.25%
90	37.1770	37.0591	−0.31%
100	33.2892	33.1521	−0.41%
110	29.8338	29.6697	−0.55%
120	26.7639	26.5645	−0.74%

Benchmark: Monte Carlo simulation (1,000,000 trials, 500 time steps).

$T = 10,\ \delta = 1,\ \gamma = 1,\ s = 100,\ \sigma_0 = 0.4,\ \kappa = 0.1,\ \theta = 0.4,\ \epsilon = 0.2,\ \rho = -0.5$

Strike price	**Benchmark**	**Malliavin AE**	Relative error
70	55.3842	55.2118	−0.31%
80	51.4141	51.2058	−0.10%
90	47.8183	47.5671	−0.53%
100	44.5521	44.2513	−0.68%
110	41.5785	41.2207	−0.86%
120	38.8648	38.4432	−1.10%

Benchmark: Monte Carlo simulation (3,000,000 trials, 1000 time steps).

Next, we show the numerical results for implied volatilities.

$T = 0.25,\ \delta = 1/2,\ \gamma = 1/2,\ s = 100,\ (\sigma_0)^{\frac{1}{2}} = 0.15,\ \kappa = 4.0,\ \theta = 0.0225,\ \epsilon = 0.1,\ \rho = -0.5$

Strike price	**Exact IV**	**Malliavin IV**	Relative error
70	17.25	17.17	−0.46%
80	16.39	16.35	−0.24%
90	15.62	15.63	0.06%
100	14.95	14.98	0.20%
110	14.39	14.40	0.07%
120	13.96	13.86	−0.72%

$T = 0.5,\ \delta = 1/2,\ \gamma = 1/2,\ s = 100,\ (\sigma_0)^{\frac{1}{2}} = 0.4,\ \kappa = 2.5,\ \theta = 0.16,\ \epsilon = 0.1,\ \rho = -0.5$

Strike price	**Exact IV**	**Malliavin IV**	Relative error
70	40.63	40.34	−0.71%
80	40.30	40.20	−0.26%
90	40.02	40.07	0.13%
100	39.78	39.96	0.45%
110	39.58	39.85	0.69%
120	39.41	39.76	0.89%

$T = 1,\ \delta = 1/2,\ \gamma = 1/2,\ s = 100,\ (\sigma_0)^{1/2} = 0.2,\ \kappa = 2.0,\ \theta = 0.04,\ \epsilon = 0.1,\ \rho = -0.25$

Strike price	**Exact IV**	**Malliavin IV**	Relative error
70	20.68	20.60	−0.40%
80	20.36	20.36	0.00%
90	20.10	20.15	0.26%
100	19.90	19.96	0.32%
110	19.75	19.80	0.23%
120	19.63	19.64	0.06%

4.2. Double Heston model. Our method is general enough to be applied to multidimensional models. For instance, Gatheral [7] introduced the following double stochastic volatility model:

$$
\begin{aligned}
dS_t^{(\epsilon)} &= rS_t^{(\epsilon)}dt + \sqrt{v_t^{(\epsilon)}}S_t^{(\epsilon)}dW_{1,t},\\
dv_t^{(\epsilon)} &= \kappa(\bar{v}_t^{(\epsilon)} - v_t^{(\epsilon)})dt + \epsilon\sqrt{v_t^{(\epsilon)}}\left(\rho_1 dW_{1,t} + \sqrt{1-\rho_1^2}dW_{2,t}\right),\\
d\bar{v}_t^{(\epsilon)} &= \bar{\kappa}(\theta - \bar{v}_t^{(\epsilon)})dt + \epsilon\sigma_2\sqrt{\bar{v}_t^{(\epsilon)}}\left(\rho_2 dW_{1,t} + \sqrt{1-\rho_2^2}dW_{3,t}\right),
\end{aligned}
$$

with $\rho_1, \rho_2 \in [-1, 1]$.

We derive an approximation formula for the double Heston model and show some numerical examples. Let $C^{Double.SV}$, and let $\sigma^{IV}(T,K)$ be the call option and the implied volatility under the double Heston model:

$$
\begin{aligned}
C^{Double.SV}(T,K) &= e^{-rT}E[(S_T^{(\epsilon)} - K)^+],\\
C^{BS}(T,K,\sigma^{IV}(T,K)) &= C^{Double.SV}(T,K).
\end{aligned}
$$

We have the following approximation formulas for $C^{Double.SV}$ and $\sigma^{IV}(T,K)$.

Proposition 4.2.

$$
\begin{aligned}
C^{Double.SV}(T,K) &= C^{BS}(T,K,\bar{\sigma}) + \epsilon_1 C_{11} + \epsilon_2 C_{12} + O(\epsilon^2),\\
\sigma^{IV}(T,K) &= \bar{\sigma} + \epsilon_1\frac{1}{vega}C_{11} + \epsilon_2\frac{1}{vega}C_{12} + O(\epsilon^2),
\end{aligned}
$$

where $\epsilon_1 = \epsilon$, $\epsilon_2 = \epsilon\sigma_2$,

$$
\begin{aligned}
C_{1i} &= e^{-rT}\int_{\mathbf{R}}\left(se^{x+rT-\frac{1}{2}\Sigma} - K\right)^+ \eta_i\left(\frac{1}{\Sigma^3}x^3 - \frac{1}{\Sigma^2}x^2 - \frac{1}{\Sigma^2}3x + \frac{1}{\Sigma}\right)\frac{1}{\sqrt{2\pi\Sigma}}e^{-\frac{1}{2\Sigma}x^2}dx,\\
&i = 1, 2,\\
\eta_1 &= \frac{1}{2}\rho_1\int_0^t e^{-\kappa s}\int_0^s e^{\kappa u}v_u^{(0)}duds,\\
\eta_2 &= \frac{1}{2}\rho_2\int_0^t \kappa e^{-\kappa s}\int_0^s e^{\bar{\kappa}u}\sqrt{\bar{v}_u^{(0)}}\int_u^s e^{(\kappa-\bar{\kappa})v}dv\sqrt{v_u^{(0)}}duds,\\
\bar{\sigma} &= \left(\theta + (v_0^{(0)} - \theta)\frac{(1-e^{-\kappa T})}{\kappa T} + (\bar{v}_0 - \theta)\frac{\kappa}{\kappa - \bar{\kappa}}\left(\frac{(1-e^{-\bar{\kappa}T})}{\bar{\kappa}T} - \frac{(1-e^{-\kappa T})}{\kappa T}\right)\right)^{1/2},\\
vega &= s\sqrt{T}n(d_1),
\end{aligned}
$$

with $\kappa \neq \bar{\kappa}$.

Proof. By the asymptotic expansion with push-down of the Malliavin weights, we have

$$
\begin{aligned}
&C^{Double.SV}(T,K) = e^{-rT}E[(S_T^{(\epsilon)}-K)^+]\\
&= e^{-rT}E[(S_T^{(0)}-K)^+] + \epsilon e^{-rT}E\left[(S_T^{(0)}-K)^+ H\left(S_T^{(0)}, \frac{\partial}{\partial\epsilon}S_t^{(\epsilon)}|_{\epsilon=0}\right)\right] + O(\epsilon^2)\\
&= C^{BS}(T,K,\bar{\sigma}) + \epsilon e^{-rT}\int_{\mathbf{R}} (se^{x+rT-\frac{1}{2}\Sigma}-K)^+ \vartheta(x)\frac{1}{\sqrt{2\pi\Sigma}}e^{-\frac{x^2}{2\Sigma}}dx + O(\epsilon^2),
\end{aligned}
$$

where

$$
\vartheta(x) = E\left[H\left(S_T^{(0)}, \frac{\partial}{\partial\epsilon}S_t^{(\epsilon)}|_{\epsilon=0}\right) \mid \int_0^T \sqrt{v_s^{(0)}}dW_{1,s} = x\right].
$$

$\frac{\partial}{\partial\epsilon}S_t^{(\epsilon)}|_{\epsilon=0}$ is given by

$$
\frac{\partial}{\partial\epsilon}S_t^{(\epsilon)}|_{\epsilon=0} = S_t^{(0)}X_t,
$$

where

$$
X_t = \left(\int_0^t \frac{1}{2\sqrt{v_s^{(0)}}}\frac{\partial}{\partial\epsilon}v_s^{(\epsilon)}|_{\epsilon=0}dW_{1,s} - \frac{1}{2}\int_0^t \frac{\partial}{\partial\epsilon}v_s^{(\epsilon)}|_{\epsilon=0}ds\right).
$$

Then ϑ, the push-down of Malliavin weight, is computed as follows:

$$
\begin{aligned}
\vartheta(x) &= E\left[(D^*(Z)-D_Z)\circ\frac{\partial}{\partial\epsilon}S_t^{(\epsilon)}|_{\epsilon=0}| \int_0^T \sqrt{v_s^{(0)}}dW_{1,s} = x\right]\\
&= \left(\frac{x}{\Sigma}-\frac{\partial}{\partial x}\right)\circ E\left[X_T|\int_0^T \sqrt{v_s^{(0)}}dW_{1,s} = x\right]\\
&= (\zeta_1+\zeta_2)\left(\frac{1}{\Sigma^3}x^3 - \frac{1}{\Sigma^2}x^2 - \frac{1}{\Sigma^2}3x + \frac{1}{\Sigma}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
Z &= \frac{D_1S_T^{(0)}}{\|D_1S_T^{(0)}\|_H^2},\\
\zeta_1 &= \frac{1}{2}\rho_1\int_0^t e^{-\kappa s}\int_0^s e^{\kappa u}v_u^{(0)}duds,\\
\zeta_2 &= \frac{1}{2}\rho_2\sigma_2\int_0^t \kappa e^{-\kappa s}\int_0^s e^{\bar{\kappa}u}\sqrt{\bar{v}_u^{(0)}}\int_u^s e^{(\kappa-\bar{\kappa})v}dv\sqrt{v_u^{(0)}}duds.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&C^{Double.SV}(T,K)\\
&= C^{BS}(T,K,\bar{\sigma})\\
&\quad + \epsilon e^{-rT}\int_{\mathbf{R}}\left(se^{x+rT-\frac{1}{2}\Sigma}-K\right)^+(\zeta_1+\zeta_2)\left(\frac{1}{\Sigma^3}x^3-\frac{1}{\Sigma^2}x^2-\frac{1}{\Sigma^2}3x+\frac{1}{\Sigma}\right)\frac{1}{\sqrt{2\pi\Sigma}}e^{-\frac{1}{2\Sigma}x^2}dx\\
&\quad + O(\epsilon^2).
\end{aligned}
$$

By an argument similar to that in section 3.2, the implied volatility is expanded as follows:

$$\begin{aligned}&\sigma^{IV}(T,K)\\&=\bar{\sigma}+\epsilon\frac{1}{vega}e^{-rT}\int_{\mathbf{R}}\left(se^{x+rT-\frac{1}{2}\Sigma}-K\right)^{+}(\zeta_1+\zeta_2)\\&\times\left(\frac{1}{\Sigma^3}x^3-\frac{1}{\Sigma^2}x^2-\frac{1}{\Sigma^2}3x+\frac{1}{\Sigma}\right)\frac{1}{\sqrt{2\pi\Sigma}}e^{-\frac{1}{2\Sigma}x^2}dx\\&+O(\epsilon^2).\qquad\blacksquare\end{aligned}$$

Next, we show the numerical results on the accuracy of our approximation formula for the call option prices under the double Heston model. Also, in the tables below we define **Malliavin AE** and **Malliavin IV** as follows:

$$\begin{aligned}\textbf{Malliavin AE}&:=C^{BS}+\epsilon_1C_{11}+\epsilon_2C_{12},\\\textbf{Malliavin IV}&:=\bar{\sigma}+\epsilon_1\frac{C_{11}}{vega}+\epsilon_2\frac{C_{12}}{vega}.\end{aligned}$$

$T=1$, $s=100$, $\sigma_0=v_0^{1/2}=0.4$, $\kappa=0.1$, $\epsilon_1=0.1$, $\rho_1=-0.5$, $\bar{\kappa}=1.0$, $\theta=0.16$, $\epsilon_2=0.1$, $\rho_2=-0.25$

Strike price	**Benchmark**	**Malliavin AE**	Relative error
70	33.6198	33.5304	0.27%
80	26.5663	26.4797	0.33%
90	20.6084	20.5439	0.31%
100	15.7241	15.6937	0.19%
110	11.8225	11.8301	−0.06%
120	8.7741	8.8198	−0.52%

Benchmark: Monte Carlo simulation (1,000,000 trials, 500 time steps).

$T=2.5$, $s=100$, $\sigma_0=v_0^{1/2}=0.4$, $\kappa=0.1$, $\epsilon_1=0.1$, $\rho_1=-0.5$, $\bar{\kappa}=1.0$, $\theta=0.16$, $\epsilon_2=0.1$, $\rho_2=-0.25$

Strike price	**Benchmark**	**Malliavin AE**	Relative error
70	39.3315	39.4345	0.26%
80	33.5871	33.6975	0.33%
90	28.6053	28.7170	0.39%
100	24.3161	24.4228	0.44%
110	20.6430	20.7397	0.47%
120	17.5111	17.5926	0.47%

Benchmark: Monte Carlo simulation (1,000,000 trials, 500 time steps).

4.3. Parameter identification. It is useful to see if the model parameter can be identified by our implied volatility expansion. For this purpose, we first compute the implied volatility for a stochastic volatility model with some fixed parameters by a numerical method such as Monte Carlo simulation. Next, we estimate the model parameters using our implied volatility expansion. As an example, we use the Heston model

$$\begin{aligned}dX_t&=-\frac{v_t}{2}dt+\sqrt{v_t}dW_{1,t},\\dv_t&=\kappa(\theta-v_t)dt+\epsilon\sqrt{v_t}(\rho dW_{1,t}+\sqrt{1-\rho^2}dW_{2,t}),\end{aligned}\tag{55}$$

where $\rho \in [-1,1]$. Here, κ and θ are known to be the speed and the level of the mean-reversion parameters, respectively. Also, the parameter $\nu = \epsilon\rho$ is regarded as a skewness parameter.

We compute the implied volatility with $S_0 = e^{X_0} = 100$, $v_0 = 0.04$, $\kappa = 1.5$, $\theta = 0.09$, and $\nu = -0.05$. Then, we obtain data set $\{\sigma^{MC}(T_i, K_{ij})\}_{i,j}$ of the implied volatilities (**Exact IV**). Using our implied volatility expansion and the data set $\{\sigma^{MC}(T_i, K_{ij})\}_{i,j}$, we solve the following minimization problem to estimate model parameters as if they were unknown:

$$\min_{v_0,\kappa,\theta,\nu} \sum_{i=1}^{n}\sum_{j=1}^{m} |\sigma^{MC}(T_i,K_{ij}) - \textbf{Malliavin IV}(T_i,K_{ij},v_0,\kappa,\theta,\nu)|^2, \tag{56}$$

where

$$\textbf{Malliavin IV}(T,K,v_0,\kappa,\theta,\nu) = \bar{\sigma} + \nu\frac{C_1}{vega}, \tag{57}$$

with

$$\bar{\sigma} = \left(\theta + (v_0^{(0)} - \theta)\frac{(1-e^{-\kappa T})}{\kappa T}\right)^{1/2},$$

$$C_1 = e^{-rT}\int_{\mathbf{R}}\left(se^{x-\frac{1}{2}\Sigma} - K\right)^{+}\eta_1\left(\frac{1}{\Sigma^3}x^3 - \frac{1}{\Sigma^2}x^2 - \frac{1}{\Sigma^2}3x + \frac{1}{\Sigma}\right)\frac{1}{\sqrt{2\pi\Sigma}}e^{-\frac{1}{2\Sigma}x^2}dx,$$

$$\eta_1 = \frac{1}{2}\int_0^t e^{-\kappa s}\int_0^s e^{\kappa u}v_u^{(0)}duds.$$

As a result, we obtain the optimal parameters $v_0^* = 0.0404$, $\kappa^* = 1.5022$, $\theta^* = 0.0863$, and $\nu^* = -0.0336$, which are close to the true ones given above. Hence, the model parameters seem well identified by our implied volatility expansion for this case.

The fitting results are shown in the following tables, where **Malliavin IV** are computed by these optimal parameters $(v_0^*, \kappa^*, \theta^*, \nu^*) = (0.0404, 1.5022, 0.0863, -0.0336)$.

Monte Carlo parameter versus optimal parameter

Parameters	**Exact IV**	**Malliavin IV**
v_0	0.0400	0.0404
κ	1.5000	1.5022
θ	0.0900	0.0863
ν	−0.0500	−0.0336

$T = 0.4$:

Strike price	**Exact IV**	**Malliavin IV**	Relative error
70	24.31	24.23	−0.32%
80	23.68	23.66	−0.05%
90	23.17	23.17	−0.01%
100	22.67	22.72	0.23%
110	22.30	22.32	0.09%
120	21.92	21.96	0.17%

$T = 0.5$:

Strike price	**Exact IV**	**Malliavin IV**	Relative error
70	24.71	24.74	0.13%
80	24.13	24.17	0.14%
90	23.64	23.66	0.06%
100	23.18	23.20	0.10%
110	22.83	22.79	−0.20%
120	22.48	22.41	−0.33%

4.4. Calibration to market data. In this subsection, we show calibration examples using an implied volatility data of Nikkei-225 option as of January 31, 2011, where the strikes are the closest to ATM with maturities 0.20, 0.28, 0.37, 0.62, 0.87, 1.37, and 1.87 years. Moreover, the data for 10 different strikes other than ATM with maturity 0.87 year are also used since this kind of data is most available for the maturity. We compare the fitting result of our expansion in the single Heston model with that in the double Heston model. The single Heston model is expressed as

$$\begin{aligned} dX_t &= -\frac{v_t}{2}dt + \sqrt{v_t}dW_{1,t}, \\ dv_t &= \kappa(\theta - v_t)dt + \epsilon_1\sqrt{v_t}\left(\rho_1 dW_{1,t} + \sqrt{1-\rho_1^2}dW_{2,t}\right). \end{aligned} \tag{58}$$

The double Heston model is expressed as

$$\begin{aligned} dX_t &= -\frac{v_t}{2}dt + \sqrt{v_t}dW_{1,t}, \\ dv_t &= \kappa_1(\bar{v}_t - v_t)dt + \epsilon_1\sqrt{v_t}\left(\rho_1 dW_{1,t} + \sqrt{1-\rho_1^2}dW_{2,t}\right), \\ d\bar{v}_t &= \kappa_2(\theta - \bar{v}_t)dt + \epsilon_2\sqrt{\bar{v}_t}\left(\rho_2 dW_{1,t} + \sqrt{1-\rho_2^2}dW_{3,t}\right), \end{aligned} \tag{59}$$

where $W = (W_1, W_2, W_3)$ is a 3-dimensional Brownian motion. We define the skewness parameters ν_1 and ν_2 as $\nu_1 = \epsilon_1\rho_1$ and $\nu_2 = \epsilon_2\rho_2$. We also define **Malliavin IV 1** and **Malliavin IV 2** as

$$\textbf{Malliavin IV 1}\ (T, K, v_0, \kappa_1, \theta, \nu_1) := \bar{\sigma}_1 + \nu_1\frac{C_{11}}{vega}$$

and

$$\textbf{Malliavin IV 2}\ (T, K, v_0, \bar{v}_0, \kappa_1, \kappa_2, \theta, \nu_1, \nu_2) := \bar{\sigma}_2 + \nu_1\frac{C_{11}}{vega} + \nu_2\frac{C_{12}}{vega},$$

where C_{1i} $(i = 1, 2)$ are previously given, and

$$\begin{aligned} \bar{\sigma}_1 &= \left(\theta + (v_0^{(0)} - \theta)\frac{(1-e^{-\kappa T})}{\kappa T}\right)^{1/2}, \\ \bar{\sigma}_2 &= \left(\theta + (v_0^{(0)} - \theta)\frac{(1-e^{-\kappa T})}{\kappa T} + (\bar{v}_0 - \theta)\frac{\kappa}{\kappa - \bar{\kappa}}\left(\frac{(1-e^{-\bar{\kappa}T})}{\bar{\kappa}T} - \frac{(1-e^{-\kappa T})}{\kappa T}\right)\right)^{1/2}. \end{aligned}$$

To reduce the degree of freedom in **Malliavin IV 2**, we give the condition $v_0 = \bar{v}_0 = \theta$. For the Nikkei-225 data $\{\sigma^{Data}(T_i, K_{ij})\}_{i,j}$, we solve the following minimization problems using our implied volatility expansions.

Optimization for **Malliavin IV 1** (single Heston model):

$$\textbf{Error 1} = \min_{v_0,\kappa_1,\theta,\nu_1} \sum_{i=1}^{n}\sum_{j=1}^{m} |\sigma^{Data}(T_i, K_{ij}) - \textbf{Malliavin IV 1}\ (T_i, K_{ij}, v_0, \kappa_1, \theta, \nu_1)|^2.$$

Optimization for **Malliavin IV 2** (double Heston model):

$$\textbf{Error 2} = \min_{v_0,\kappa_1,\kappa_2,\theta,\nu_1,\nu_2} \sum_{i=1}^{n}\sum_{j=1}^{m} |\sigma^{Data}(T_i, K_{ij}) - \textbf{Malliavin IV 2}\ (T_i, K_{ij}, v_0, \kappa_1, \kappa_2, \theta, \nu_1, \nu_2)|^2.$$

Then, we obtain the optimal parameters for the single and double Heston models, as shown in the following two tables.

Parameters	**Malliavin IV 1**
v_0	0.024
κ_1	2.805
θ	0.0444
ν_1	−0.136

Parameters	**Malliavin IV 2**
v_0	0.038
κ_1	9.355
θ	0.038
ν_1	−0.195
κ_2	9.070
ν_2	0.580

The total differences between the calculated volatilities and the actual data are given as $\sqrt{\textbf{Error 1}}$= 2.956% for **Malliavin IV 1** and $\sqrt{\textbf{Error 2}}$=2.663% for **Malliavin IV 2**. Hence, the fitting result of the expansion of the double Heston is better than that of the single Heston. Also, the figures of the result for the double Heston model are shown in Figures 1 and 2.

5. Applications.

This section provides approximation formulas for option prices under the shifted log-normal and jump-diffusion models with stochastic volatilities; an expansion of the implied volatility is also given for the jump-diffusion model. Also, for simplicity we set the risk-free interest rate $r = 0$ in this section.

5.1. Shifted log-normal model.

This subsection derives an approximation formula of the option price in the shifted log-normal model with stochastic volatility:

$$\begin{aligned} dS_t^{(\epsilon)} &= V(\sigma_t^{(\epsilon)})(S_t^{(\epsilon)} - \beta)dW_{1,t};\ S_0^{(\epsilon)} = s > 0, \\ d\sigma_t^{(\epsilon)} &= A_0(\sigma_t^{(\epsilon)})dt + \epsilon A_1(\sigma_t^{(\epsilon)})(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}), \end{aligned} \tag{60}$$

where β is a constant such that $s > \beta$. At $\epsilon = 0$, the option price is given by

$$C^{BS}(\beta) = \int_{\mathbf{R}} ((s-\beta)e^{x-\frac{1}{2}\Sigma} - (K-\beta))^+ \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{x^2}{2\Sigma}} dx. \tag{61}$$

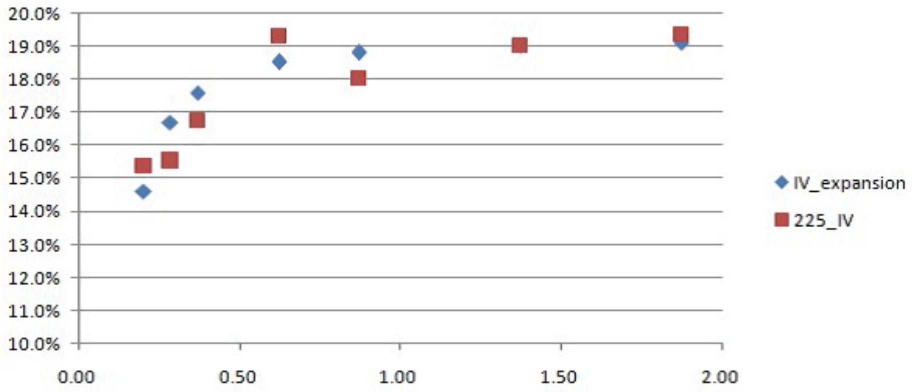

Figure 1. *Term structure.*

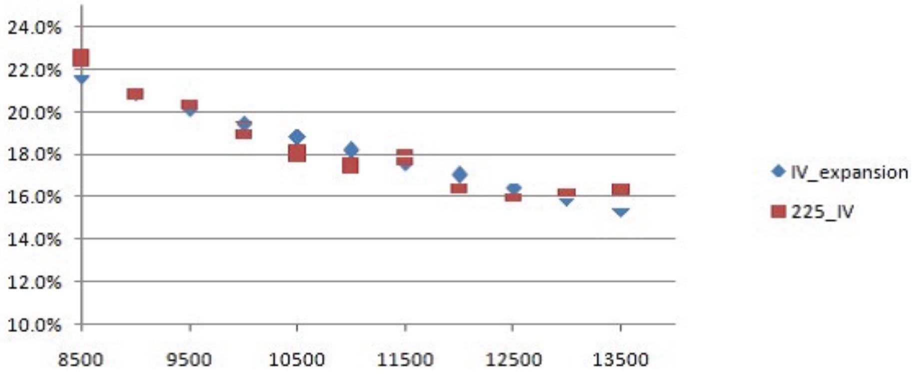

Figure 2. *Skew.*

We define the following deterministic process:

$$\eta_t := \exp\left\{\int_0^t A_0'(\sigma_u^{(0)})du\right\}.$$

Then, the following proposition is obtained.

Proposition 5.1. *An asymptotic expansion formula for the shifted log-normal model* (60) *is given as follows:*

$$C^{SV}(\beta) = C^{BS}(\beta) + \epsilon C_1(\beta) + O(\epsilon^2),$$

where

$$C_1(\beta) = \int_{\mathbf{R}} ((s-\beta)e^{x-\frac{1}{2}\Sigma} - (K-\beta))^+ \vartheta(x) \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{x^2}{2\Sigma}} dx,$$

with

$$\Sigma = \int_0^T V(\sigma_t^{(0)})^2 dt,$$
$$\vartheta(x) = \zeta\left(\frac{1}{\Sigma^3}x^3 - \frac{1}{\Sigma^2}x^2 - \frac{1}{\Sigma^2}3x + \frac{1}{\Sigma}\right),$$
$$\zeta = \rho\int_0^T \partial V(\sigma_t^{(0)})\eta_t V(\sigma_t^{(0)})\int_0^t \eta_s^{-1}A_1(\sigma_s^{(0)})V(\sigma_s^{(0)})dsdt.$$

Proof. We first note that

$$dS_t^{(\epsilon)} = dF_t^{(\epsilon)} = V(\sigma_t^{(\epsilon)})F_t^{(\epsilon)}dW_{1,t};\ F_0^{(\epsilon)} = s - \beta > 0, \tag{62}$$

where $F_t^{(\epsilon)} = S_t^{(\epsilon)} - \beta$. Note also that $(S_T^{(\epsilon)} - K)^+ = (F_T^{(\epsilon)} - (K-\beta))^+$. Hence, instead of the original problem we can consider the option pricing problem with the underlying asset price process $F^{(\epsilon)}$ and strike $K - \beta$. Then, the same argument as in Theorem 3.1 and Corollary A.2 can be applied to computation of the push-down of the Malliavin weight: $\vartheta(x) := E[\pi_1 | \int_0^T V(\sigma_t^{(0)})dW_{1,t} = x]$, where

$$\pi_1 = H_1(F_T^{(0)}, \Psi^{\beta_1}(T)),$$
$$\Psi^{\beta_1}(T) = \frac{\partial}{\partial\epsilon}F_T^{(\epsilon)}|_{\epsilon=0}.$$

Thus, the result is obtained. ∎

5.2. Jump-diffusion with stochastic volatility model. This subsection applies the Malliavin calculus to a jump-diffusion stochastic volatility (SVJ) model. Let $(\Omega, \mathcal{F}, P)$ be a probability space on which we define a Brownian motion $(W_t)_t$, a Poisson process $(N_t)_t$ with intensity λ, and independent and identically distributed random variables $(\Delta_j)_{j\in\mathbf{N}}$ such that $\Delta_j \sim N(0,1)$. We will assume that the σ-algebras generated by $(W_t)_t$, $(N_t)_t$, $(\Delta_j)_j$ are independent.

First, we introduce the perturbed stochastic differential equation: For $\epsilon \in [0,1]$,

$$\begin{aligned} dS_t^{(\epsilon)} &= V(\sigma_t^{(\epsilon)})S_{t-}^{(\epsilon)}dW_{1,t} + S_{t-}^{(\epsilon)}(dJ_t - \lambda m dt), \\ S_0^{(\epsilon)} &= s, \\ d\sigma_t^{(\epsilon)} &= A_0(\sigma_t^{(\epsilon)})dt + \epsilon A_1(\sigma_t^{(\epsilon)})(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}), \\ \sigma_0^{(\epsilon)} &= \sigma, \end{aligned} \tag{63}$$

where J_t is defined by

$$J_t = \sum_{j=1}^{N(t)}(e^{Y_j} - 1), \tag{64}$$

with $Y_j = a + b\Delta_j$ and $m = E[e^{Y_j} - 1]$.

The solution of (63) is given by

$$S_T = s\exp\left\{\int_0^T V(\sigma_t^{(\epsilon)})dW_{1,t} - \frac{1}{2}\int_0^T V(\sigma_t^{(\epsilon)})^2 dt\right\}\prod_{j=1}^{N(T)} e^{Y_j}$$
$$= s\exp\left\{\int_0^T V(\sigma_t^{(\epsilon)})dW_{1,t} - \frac{1}{2}\int_0^T V(\sigma_t^{(\epsilon)})^2 dt + \sum_{j=1}^{N(T)} Y_j\right\}.$$

We define the following notations:

$$X_t^{(\epsilon)} := \log(S_t^{(\epsilon)}/s),$$
$$\sigma^{BS} := \left(\frac{1}{T}\int_0^T V(\sigma_t^{(0)})^2 dt\right)^{1/2},$$
$$\Sigma_n := \left(\int_0^T V(\sigma_t^{(0)})^2 dt + b^2 n\right),$$
$$\sigma_n := \left(\frac{1}{T}\int_0^T V(\sigma_t^{(0)})^2 dt + \frac{b^2 n}{T}\right)^{1/2} = \left((\sigma^{BS})^2 + (b^2 n/T)\right)^{1/2},$$
$$\hat{r} := \frac{-\lambda m T + (a + b^2/2)n}{T},$$
$$d_n := \frac{1}{\sigma_n\sqrt{T}}\left[\log\left(\frac{s}{K}\right) + \left(\hat{r} + \frac{\sigma_n^2}{2}\right)T\right],$$
$$C_n^{BS}(T,K,\sigma_n) := sN(d_n) - e^{-\hat{r}T}KN(d_n - \sigma_n\sqrt{T}),$$
$$C^M(T,K,\sigma^{BS}) := \sum_{n=0}^{\infty}\frac{(\lambda(1+m)T)^n e^{-\lambda(1+m)T}}{n!}C_n^{BS}(T,K,\sigma_n),$$
$$C^{SVJ}(T,K) := E[(S_T^{(\epsilon)} - K)^+].$$

Moreover, let us call $\sigma^{SVJ.IV}(T,K)$ an implied volatility, which satisfies

$$C^{SVJ}(T,K) = C^M(T,K,\sigma^{SVJ.IV}(T,K)). \tag{65}$$

Then, the following proposition is obtained.

Proposition 5.2. *Under the jump-diffusion stochastic volatility (SVJ) model (63), the call option price $C^{SVJ}(T,K)$ and its implied volatility $\sigma^{SVJ.IV}(T,K)$ are expanded up to the ϵ-order as follows:*

$$C^{SVJ}(T,K) = C^M(T,K,\sigma^{BS}) + \epsilon C_1^M + O(\epsilon^2), \tag{66}$$
$$\sigma^{SVJ.IV}(T,K) = \sigma^{BS} + \epsilon\frac{1}{vega^M}C_1^M + O(\epsilon^2), \tag{67}$$

where

$$C_1^M = \sum_{n=0}^{\infty}\frac{(\lambda T)^n e^{-\lambda T}}{n!}\int_{\mathbf{R}}\left(se^{x-\lambda mT - \frac{1}{2}\int_0^T V(\sigma_t^{(0)})^2 dt + na} - K\right)^+ \vartheta_n(x)\frac{1}{\sqrt{2\pi\Sigma_n}}e^{-\frac{1}{2\Sigma_n}x^2}dx,$$
$$vega^M = \sum_{n=0}^{\infty}\frac{(\lambda(m+1)T)^n e^{-\lambda(m+1)T}}{n!}\frac{\sigma^{BS}}{\sigma_n}s\sqrt{T}n(d_n),$$

with

$$\vartheta_n(x) = \zeta\left(\frac{1}{\Sigma_n^3}x^3 - \frac{1}{\Sigma_n^2}x^2 - \frac{1}{\Sigma_n^2}3x + \frac{1}{\Sigma_n}\right),$$
$$\zeta = \rho\int_0^T \partial V(\sigma_t^{(0)})V(\sigma_t^{(0)})\eta_t\int_0^t \eta_s^{-1}A_1(\sigma_s^{(0)})V(\sigma_s^{(0)})dsdt,$$
$$\eta_t = \exp\left\{\int_0^t A_0'(\sigma_u^{(0)})du\right\}.$$

Proof. See Appendix B. ■

6. Conclusion. This paper developed an asymptotic expansion method for generalized Wiener functionals based on the integration-by-parts formula in Malliavin calculus and the push-down of Malliavin weights. As an application, we derived asymptotic expansion formulas for option prices and implied volatilities as well as the density of the underlying asset price in the stochastic volatility environment. We also presented numerical examples for expansions up to the first order (ϵ-order). Moreover, we applied the general formula to expansions of option prices for the shifted log-normal model with stochastic volatility. Further, combining some existing results of Malliavin calculus in jump-type models with our method, in this paper we presented an approximation formula for the jump-diffusion model in the stochastic volatility environment. More detailed numerical experiments and higher-order expansions are our next research topics.

Appendix A. Computation of push-down of Malliavin weights. This section derives the push-down of Malliavin weights for the first- and second-order approximation terms of the asymptotic expansion of option prices.

We consider the following stochastic volatility model:

$$dS_t^{(\epsilon)} = V(\sigma_t^{(\epsilon)})S_t^{(\epsilon)}dW_{1,t},$$
$$d\sigma_t^{(\epsilon)} = A_0(\sigma_t^{(\epsilon)})dt + \epsilon A_1(\sigma_t^{(\epsilon)})(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}),$$
$$S_0^{(\epsilon)} = S_0^{(0)} = s,$$

where $V, A_0, A_1 \in C_b^\infty(\mathbf{R})$, $\rho \in [-1,1]$, and $\epsilon \in [0,1]$.

Also we use the following notations:

$$B_t = \rho W_{1,t} + \sqrt{1-\rho^2}W_{2,t},$$
$$\eta_t = \exp\left\{\int_0^t A'(\sigma_u^{(0)})du\right\},$$
$$\sigma_t^{(\epsilon)} = \sigma_t^{(0)} + \epsilon\sigma_t^{(1)} + \epsilon^2\sigma_t^{(2)} + O(\epsilon^3),$$
$$\sigma_{0t} = \sigma_t^{(0)},$$
$$\sigma_t^{(1)} = \frac{\partial}{\partial\epsilon}\sigma_t^{(\epsilon)}|_{\epsilon=0} = \int_0^t \eta_t\eta_s^{-1}A_1(\sigma_{0s})dB_s,$$
$$\sigma_t^{(2)} = \frac{1}{2}\frac{\partial^2}{\partial\epsilon^2}\sigma_t^{(\epsilon)}|_{\epsilon=0} = \int_0^t \partial A_1(\sigma_{0s})\int_0^s \eta_s\eta_u^{-1}A_1(\sigma_{0u})dB_udB_s,$$

$$\begin{aligned}
v_t^{(0)} &= V(\sigma_t^{(0)}),\\
v_t^{(1)} &= V'(\sigma_{0t})\sigma_t^{(1)},\\
v_t^{(2)} &= 2V'(\sigma_{0t})\sigma_t^{(2)} + V''(\sigma_{0t})(\sigma_t^{(1)})^2,\\
X_T^{(0)} &= \int_0^T V(\sigma_{0t})dW_{1,t} - \frac{1}{2}\int_0^T V(\sigma_{0t})^2 dt,\\
X_T^{(1)} &= \frac{\partial}{\partial\epsilon} X_T^{(\epsilon)}|_{\epsilon=0}\\
&= \int_0^T v_t^{(1)} dW_{1,t} - \int_0^T V(\sigma_{0t})v_t^{(1)} dt,\\
X_T^{(2)} &= \frac{\partial^2}{\partial\epsilon^2} X_T^{(\epsilon)}|_{\epsilon=0}\\
&= \int_0^T v_t^{(2)} dW_{1,t} - \int_0^T (v_t^{(1)})^2 dt - \int_0^T V(\sigma_{0t})v_t^{(2)} dt,\\
\bar{\sigma} &= \left(\frac{1}{T}\int_0^T V(\sigma_t^{(0)})^2 dt\right)^{1/2}.
\end{aligned}$$

The closed-form approximation of the density and the call option price at $t = 0$ with strike K and maturity T are given by

$$\begin{aligned}
p^{SV}(y) &= p^{BS}(y) + \epsilon E[\pi_1|S_T^{(0)} = x]p^{BS}(y) + \epsilon^2 E[\pi_2|S_T^{(0)} = x]p^{BS}(y) + O(\epsilon^2),\\
C^{SV}(T,K) &= C^{BS}(T,K,\bar{\sigma})\\
&\quad + \epsilon \int_{\mathbf{R}} (y-K)^+ E[\pi_1|S_T^{(0)} = x]p^{BS}(y)dy\\
&\quad + \epsilon^2 \int_{\mathbf{R}} (y-K)^+ E[\pi_2|S_T^{(0)} = x]p^{BS}(y)dy + O(\epsilon^3)\\
&= C^{BS}(T,K,\bar{\sigma})\\
&\quad + \epsilon \int_{\mathbf{R}} \left(se^{x-\frac{1}{2}\int_0^T V(\sigma_{0s})^2 ds} - K\right)^+ E\left[\pi_1 | \int_0^T V(\sigma_{0t})dW_{1,t} = x\right] n(\Sigma : x)dx\\
&\quad + \epsilon^2 \int_{\mathbf{R}} \left(se^{x-\frac{1}{2}\int_0^T V(\sigma_{0s})^2 ds} - K\right)^+ E\left[\pi_2 | \int_0^T V(\sigma_{0t})dW_{1,t} = x\right] n(\Sigma : x)dx\\
&\quad + O(\epsilon^3),
\end{aligned}$$

where π_1 and π_2 are the Malliavin weights:

$$\begin{aligned}
\pi_1 &= H_1\left(S_T^{(0)}, \frac{\partial}{\partial\epsilon} S_T^{(\epsilon)}|_{\epsilon=0}\right),\\
\pi_2 &= \frac{1}{2}H_1\left(S_T^{(0)}, \frac{\partial^2}{\partial\epsilon^2} S_T^{(\epsilon)}|_{\epsilon=0}\right) + \frac{1}{2}H_2\left(S_T^{(0)}, \left(\frac{\partial}{\partial\epsilon} S_T^{(\epsilon)}|_{\epsilon=0}\right)^2\right).
\end{aligned}$$

Note first that for $t \in [0,T]$, $D_{t,2}X_T^{(0)} = D_{t,2}S_T^{(0)} = 0$, $D_{t,1}X_T^{(0)} = V(\sigma_t^{(0)})$, $D_{t,1}S_T^{(0)} = S_T^{(0)}D_{t,1}X_T^{(0)} = S_T^{(0)}V(\sigma_t^{(0)})$, and $\|DS_T^{(0)}\|_H^2 = \|D_1S_T^{(0)}\|_H^2 = (S_T^{(0)})^2\Sigma$, where $\Sigma = \int_0^T V(\sigma_t^{(0)})^2dt$. Let Z be the function on $L^2[0,T]$ defined by $Z(s) = \frac{D_{s,1}X_T^{(0)}}{\|D_1X_T^{(0)}\|_H^2} = \frac{V(\sigma_s^{(0)})}{\Sigma}$. Recall also that D_1 and D_1^* are the Malliavin derivative and its adjoint operator (Skorohod integral) for the Brownian motion W_1. Then, the second-order Malliavin weight is computed as follows:

$$
\begin{aligned}
&H_1\left(S_T^{(0)}, \frac{\partial^2}{\partial\epsilon^2}S_T^{(\epsilon)}|_{\epsilon=0}\right) = D_1^*\left(\frac{D_1S_T^{(0)}S_T^{(0)}\{\frac{\partial^2}{\partial\epsilon^2}X_T^{(\epsilon)}|_{\epsilon=0} + (\frac{\partial}{\partial\epsilon}X_T^{(\epsilon)}|_{\epsilon=0})^2\}}{\|D_1S_T^{(0)}\|_H^2}\right)\\
&= D_1^*\left(\frac{D_1S_T^{(0)}S_T^{(0)}\frac{\partial^2}{\partial\epsilon^2}X_T^{(\epsilon)}|_{\epsilon=0}}{\|D_1S_T^{(0)}\|_H^2}\right) + D_1^*\left(\frac{D_1S_T^{(0)}S_T^{(0)}(\frac{\partial}{\partial\epsilon}X_T^{(\epsilon)}|_{\epsilon=0})^2}{\|D_1S_T^{(0)}\|_H^2}\right)\\
&= H_1\left(X_T^{(0)}, X_T^{(2)}\right) + H_1\left(X_T^{(0)}, (X_T^{(1)})^2\right)\\
&= X_T^{(2)}\int_0^T Z(s)dW_{1,s} - \int_0^T D_{s,1}X_T^{(2)}Z(s)ds\\
&\quad + (X_T^{(1)})^2\int_0^T Z(s)dW_{1,s} - \int_0^T D_{s,1}(X_T^{(1)})^2Z(s)ds,
\end{aligned}
$$

$$
\begin{aligned}
&H_2\left(S_T^{(0)}, \left(\frac{\partial}{\partial\epsilon}S_T^{(\epsilon)}|_{\epsilon=0}\right)^2\right) = H_1\left(S_T^{(0)}, H_1\left(S_T^{(0)}, \left(\frac{\partial}{\partial\epsilon}S_T^{(\epsilon)}|_{\epsilon=0}\right)^2\right)\right)\\
&= D_1^*\left(\frac{D_1S_T^{(0)}}{\|D_1S_T^{(0)}\|_H^2}D_1^*\left(\frac{D_1S_T^{(0)}(S_T^{(0)}\frac{\partial}{\partial\epsilon}X_T^{(\epsilon)}|_{\epsilon=0})^2}{\|D_1S_T^{(0)}\|_H^2}\right)\right)\\
&= D_1^*\left(\frac{D_1S_T^{(0)}}{\|D_1S_T^{(0)}\|_H^2}D_1^*\left((S_T^{(0)}X_T^{(1)})^2Z\right)\right)\\
&= D_1^*\left(\frac{D_1S_T^{(0)}}{\|D_1S_T^{(0)}\|_H^2}\left\{S_T^{(0)}(X_T^{(1)})^2\int_0^T Z(s)dW_{1,s} - \int_0^T D_{s,1}(S_T^{(0)}(X_T^{(1)})^2)Z(s)ds\right\}\right)\\
&= D_1^*\Bigg(\frac{D_1S_T^{(0)}}{\|D_1S_T^{(0)}\|_H^2}\Bigg\{S_T^{(0)}(X_T^{(1)})^2\int_0^T Z(s)dW_{1,s} - \int_0^T (D_{1,s}S_T^{(0)})(X_T^{(1)})^2Z(s)ds\\
&\quad - \int_0^T S_T^{(0)}(D_{s,1}(X_T^{(1)})^2)Z(s)ds\Bigg\}\Bigg)\\
&= D_1^*\Bigg(\frac{D_1S_T^{(0)}}{\|D_1S_T^{(0)}\|_H^2}\Bigg\{S_T^{(0)}(X_T^{(1)})^2\int_0^T Z(s)dW_{1,s}\\
&\quad -S_T^{(0)}(X_T^{(1)})^2\int_0^T V(\sigma_s^{(0)})Z(s)ds - S_T^{(0)}\int_0^T (D_{s,1}(X_T^{(1)})^2)Z(s)ds\Bigg\}\Bigg)\\
&= D_1^*\left(\frac{D_1X_T^{(0)}}{\|D_1X_T^{(0)}\|_H^2}\left\{(X_T^{(1)})^2\int_0^T Z(s)dW_{1,s} - \int_0^T (D_{s,1}(X_T^{(1)})^2)Z(s)ds - (X_T^{(1)})^2\right\}\right)\\
&= H_2\left(X_T^{(0)}, (X_T^{(1)})^2\right) - H_1\left(X_T^{(0)}, (X_T^{(1)})^2\right).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\pi_2 &= \frac{1}{2}H_1\left(X_T^{(0)}, X_T^{(2)}\right) + \frac{1}{2}H_2\left(X_T^{(0)}, (X_T^{(1)})^2\right) \\
&= \frac{1}{2}H_1\left(X_T^{(0)}, \int_0^T v_t^{(2)} dW_{1,t}\right) \\
&- \frac{1}{2}H_1\left(X_T^{(0)}, \int_0^T (v_t^{(1)})^2 dt\right) - \frac{1}{2}H_1\left(X_T^{(0)}, \int_0^T V(\sigma_{0t})v_t^{(2)} dt\right) \\
&+ \frac{1}{2}H_2\left(X_T^{(0)}, \left(\int_0^T v_t^{(1)} dW_{1,t}\right)^2\right) \\
&- H_2\left(X_T^{(0)}, \int_0^T v_t^{(1)} dW_{1,t} \int_0^T V(\sigma_{0t})v_t^{(1)} dt\right) \\
&+ \frac{1}{2}H_2\left(X_T^{(0)}, \left(\int_0^T V(\sigma_{0t})v_t^{(1)} dt\right)^2\right).
\end{aligned}
\tag{68}
$$

In order to compute the expansion coefficient including the push-down of the Malliavin weights, we give the following formula (69), which is a modified version of that in Malliavin [14] and Malliavin and Thalmaier [15].

Proposition A.1. *For $\Psi \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R})$ and $Z = \frac{h}{\|h\|_H^2}$ with $h \in H$,*

$$
\begin{aligned}
&E[D^*(Z)\Psi - D_Z\Psi | D^*(h) = x] \\
&= \frac{x}{\|h\|_H^2} E[\Psi | D^*(h) = x] - \frac{\partial}{\partial x} E[\Psi | D^*(h) = x].
\end{aligned}
\tag{69}
$$

Proof. We follow Proposition (8.2) on p. 82 of Malliavin [14].

Let g be a nondegenerate Wiener smooth map, and let $g \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ and $z = (z^1, z^2, \ldots, z^n)$ be a vector field on $\mathbf{R}^n$. Suppose also that $Z = (Z^1, Z^2, \ldots, Z^d)$ is a lift of z to the Wiener space (a covering vector field of z)

$$
Z^k(t) = \sum_{1 \le s,l \le n} \gamma^{s,l}(D_{t,k}g^s)z^l,
$$

where $\gamma^{s,l}$ is the (s,l)-element of the inverse matrix of the Malliavin covariance matrix of g.

Then, Proposition (8.2) on p. 82 of Malliavin [14] says that for $\Psi \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R})$ and $k(x) = E^{g=x}[\Psi]$, where $E^{g=x}$ denotes the conditional expectation under $g = x$,

$$
E^{g=x}[D^*(Z)\Psi - D_Z\Psi] = k(x)E^{g=x}[D^*(Z)] - \partial_z k(x),
$$

where $\partial_z k(x) := \langle z, (dk(x)/dx)\rangle$. In our case, set $g = D^*(h)$, and then $g \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R})$ (i.e., $n = 1$). Also, let $z \equiv 1$, and hence

$$
Z = \frac{Dg}{\|Dg\|_H^2} = \frac{h}{\|h\|_H^2}.
$$

Finally, note that $E^{g=x}[D^*(Z)] = \frac{x}{\|h\|_H^2}$ in our case, and thus the result (69) is obtained. ∎

Using (69), we give the first-order approximation formula of the call option price under the stochastic volatility.

Corollary A.2. *The first-order approximation of the call price is given by*

$$C^{SV}(T,K) = C^{BS}(T,K,\bar{\sigma}) + \epsilon \int_{\mathbf{R}} \left(se^{x-\frac{1}{2}\Sigma} - K\right)^{+} \vartheta(x) \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{1}{2\Sigma}x^2} dx,$$

where

$$\vartheta(x) = \zeta \left(\frac{1}{\Sigma^3} x^3 - \frac{1}{\Sigma^2} x^2 - \frac{1}{\Sigma^2} 3x + \frac{1}{\Sigma} \right),$$

$$\Sigma = \int_0^T V(\sigma_s^{(0)})^2 ds,$$

$$\zeta = \rho \int_0^T V'(\sigma_t^{(0)}) \eta_t V(\sigma_t^{(0)}) \int_0^t \eta_s^{-1} A_1(\sigma_s^{(0)}) V(\sigma_s^{(0)}) ds dt,$$

with

$$\eta_t = \exp\left\{ \int_0^t A_0'(\sigma_u^{(0)}) du \right\},$$

where $A_0'(\sigma_u^{(0)}) = \frac{dA_0(x)}{dx}|_{x=\sigma_u^{(0)}}$.

Proof. By (69) and the formula[1]

$$E\left[\int_0^T \int_0^t h_{2u} dW_{1,u} h_{3t} dW_{1,t} | \int_0^T h_{1v} dW_{1,v} = x \right]$$
$$= \left(\int_0^T \int_0^t h_{2u} h_{1u} du h_{3t} h_{1t} dt \right) \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right),$$

for $h_i \in L^2[0,T]$, $i = 1,2,3$. The push-down of the Malliavin weight $H_1(X_T^{(0)}, \int_0^T v_t^{(1)} dW_{1,t})$ is obtained as follows:

$$E\left[H_1\left(X_T^{(0)}, \int_0^T v_t^{(1)} dW_{1,t} \right) | \int_0^T V(\sigma_{0s}) dW_{1,s} = x \right]$$
$$= E\left[(D^*(Z) - D_Z) \circ \int_0^T v_t^{(1)} dW_{1,t} | \int_0^T V(\sigma_{0s}) dW_{1,s} = x \right]$$
$$= \left(\frac{x}{\Sigma} - \frac{\partial}{\partial x} \right) \circ E\left[\int_0^T V'(\sigma_{0t}) \int_0^t \eta_t \eta_s^{-1} A_1(\sigma_{0s}) dB_s dW_{1,t} | \int_0^T V(\sigma_{0s}) dW_{1,s} = x \right]$$
$$= \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2} \right) \zeta,$$

where $\zeta = \rho \int_0^T V'(\sigma_{0t}) \eta_t V(\sigma_{0t}) \int_0^t \eta_s^{-1} A_1(\sigma_{0s}) V(\sigma_{0s}) ds dt$.

[1] For the derivation and more general results, see section 3 in Takahashi, Takehara, and Toda [22].

By (69) and the formula

$$E\left[\int_0^T h_{2t}dW_{1,t}|\int_0^T h_{1v}dW_{1,v}=x\right]=\left(\int_0^T h_{2t}h_{1t}dt\right)\frac{x}{\Sigma},$$

the push-down of the Malliavin weight $H_1(X_T^{(0)},\int_0^T V(\sigma_{0t})v_t^{(1)}dt)$ is obtained as

$$\begin{aligned}
&E\left[H_1\left(X_T^{(0)},\int_0^T V(\sigma_{0t})v_t^{(1)}dt\right)|\int_0^T V(\sigma_{0s})dW_{1,s}=x\right]\\
&=E\left[(D^*(Z)-D_Z)\circ\int_0^T V(\sigma_{0t})V'(\sigma_{0t})\int_0^t \eta_t\eta_s^{-1}A_1(\sigma_{0s})dB_sdt|\int_0^T V(\sigma_{0s})dW_{1,s}=x\right]\\
&=E\left[(D^*(Z)-D_Z)\circ\int_0^T \eta_s^{-1}A_1(\sigma_{0s})\int_t^T V(\sigma_{0t})V'(\sigma_{0t})\eta_t dtdB_s|\int_0^T V(\sigma_{0s})dW_{1,s}=x\right]\\
&=\left(\frac{x}{\Sigma}-\frac{\partial}{\partial x}\right)\circ E\left[\int_0^T \eta_s^{-1}A_1(\sigma_{0s})\int_t^T V(\sigma_{0t})V'(\sigma_{0t})\eta_t dtdB_s|\int_0^T V(\sigma_{0s})dW_{1,s}=x\right]\\
&=\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)\zeta.
\end{aligned}$$

Therefore, we obtain

$$E\left[\pi_1|\int_0^T V(\sigma_{0t})dW_{1,t}=x\right]=\zeta\left(\frac{1}{\Sigma^3}x^3-\frac{1}{\Sigma^2}x^2-\frac{1}{\Sigma^2}3x+\frac{1}{\Sigma}\right). \qquad \blacksquare \tag{70}$$

A.1. Second-order approximation. This subsection lists the results necessary for computing push-down of the Malliavin weights of ϵ^2-order terms of (68). The details of the computation will be given upon request.

A.1.1. $H_1(X_T^{(0)},\int_0^T v_t^{(2)}dW_{1,t})$.

$$\begin{aligned}
&E\left[H_1\left(X_T^{(0)},\int_0^T v_t^{(2)}dW_{1,t}\right)|\int_0^T V(\sigma_{0s})dW_{1,s}=x\right]\\
&=\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}\right)(2b_{11}+b_{12})+\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)b_{13},
\end{aligned} \tag{71}$$

where

$$\begin{aligned}
b_{11}&=\rho^2\int_0^T V'(\sigma_{0t})V(\sigma_{0t})\int_0^t A_1'(\sigma_{0s})\eta_s V(\sigma_{0s})\int_0^s \eta_u^{-1}A_1(\sigma_{0u})V(\sigma_{0u})dudsdt,\\
b_{12}&=\int_0^T\left(\int_0^t V(\sigma_{0s})\eta_s^{-1}A_1(\sigma_{0s})ds\right)^2 V''(\sigma_{0t})\eta_t^2 V(\sigma_{0t})dt,\\
b_{13}&=\int_0^T V''(\sigma_{0t})V(\sigma_{0t})\left(\int_0^t \eta_t\eta_t^{-1}A_1(\sigma_{0s})\right)^2 dsdt.
\end{aligned}$$

A.1.2. $H_1(X_T^{(0)}, \int_0^T (v_t^{(1)})^2 dt)$.

$$E\left[H_1\left(X_T^{(0)}, \int_0^T (v_t^{(1)})^2 dt\right) \mid \int_0^T V(\sigma_{0s}) dW_{1,s} = x\right] = \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right) b_{21} + \frac{x}{\Sigma} b_{22}, \tag{72}$$

where

$$b_{21} = \rho^2 \int_0^T (V'(\sigma_{0t})\eta_t)^2 \left(\int_0^t \eta_s^{-1} A_1(\sigma_{0s}) V(\sigma_{0s}) ds\right)^2 dt,$$
$$b_{22} = \int_0^T (V'(\sigma_{0t})\eta_t)^2 \int_0^t (\eta_s^{-1} A_1(\sigma_{0s}))^2 ds dt.$$

A.1.3. $H_1(X_T^{(0)}, \int_0^T V(\sigma_{0t}) v_t^{(2)} dt)$.

$$E\left[H_1\left(X_T^{(0)}, \int_0^T V(\sigma_{0t}) v_t^{(2)} dt\right) \mid \int_0^T V(\sigma_{0s}) dW_{1,s} = x\right] = 2\left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right) b_{31} + \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right) b_{32} + \frac{x}{\Sigma} b_{33}, \tag{73}$$

where

$$b_{31} = \rho^2 \int_0^T V'(\sigma_{0t}) V(\sigma_{0t}) \int_0^t A_1'(\sigma_{0s}) \eta_s V(\sigma_{0s}) \int_0^s \eta_u^{-1} A_1(\sigma_{0u}) V(\sigma_{0u}) du ds dt,$$
$$b_{32} = \rho^2 \int_0^T (V''(\sigma_{0t}) \eta_t^2 V(\sigma_{0t})) \left(\int_0^t \eta_s^{-1} A_1(\sigma_{0s}) V(\sigma_{0s}) ds\right)^2 dt,$$
$$b_{33} = \int_0^T (V''(\sigma_{0t}) \eta_t^2 V(\sigma_{0t})) \int_0^t (\eta_s^{-1} A_1(\sigma_{0s}))^2 ds dt.$$

A.1.4. $H_2(X_T^{(0)}, (\int_0^T v_t^{(1)} dW_{1,t})^2)$.

$$\begin{aligned} &E\left[H_2\left(X_T^{(0)}, \left(\int_0^T v_t^{(1)} dW_{1,t}\right)^2\right) \mid \int_0^T V(\sigma_{0s}) dW_{1,s} = x\right] \\ &= \left(\frac{x^6}{\Sigma^6} - \frac{15x^4}{\Sigma^5} + \frac{45x^2}{\Sigma^4} - \frac{15}{\Sigma^3}\right) b_{41} \\ &+ \left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma}\right) (2b_{42} + 2b_{43} + b_{44}) \\ &+ \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma^2}\right) b_{45}, \end{aligned} \tag{74}$$

where

$$
\begin{aligned}
b_{41} &= \left(\rho \int_0^T V'(\sigma_{0t})\eta_t V(\sigma_{0t}) \int_0^t \eta_s^{-1} A_1(\sigma_{0s}) V(\sigma_{0s}) ds dt\right)^2, \\
b_{42} &= \int_0^T V'(\sigma_{0t})\eta_t V(\sigma_{0t}) \int_0^t V'(\sigma_{0s})\eta_s V(\sigma_{0s}) \int_0^s (\eta_u^{-1} A_1(\sigma_{0u}))^2 du ds dt, \\
b_{43} &= \rho^2 \int_0^T V'(\sigma_{0t})\eta_t V(\sigma_{0t}) \int_0^t V'(\sigma_{0s}) A_1(\sigma_{0s}) \int_0^s V(\sigma_{0u})\eta_u^{-1} A_1(\sigma_{0u}) du ds dt, \\
b_{44} &= \rho^2 \int_0^T (V'(\sigma_{0t})\eta_t)^2 \left(\int_0^t \eta_s^{-1} A_1(\sigma_{0s}) V(\sigma_{0s}) ds\right)^2 dt, \\
b_{45} &= \int_0^T (V'(\sigma_{0t})\eta_t)^2 \int_0^t (\eta_s^{-1} A_1(\sigma_{0s}))^2 ds dt.
\end{aligned}
$$

A.1.5. $H_2(X_T^{(0)}, \int_0^T v_t^{(1)} dW_t \int_0^T V(\sigma_{0t}) v_t^{(1)} dt)$.

$$
\begin{aligned}
&E\left[H_2\left(X_T^{(0)}, \int_0^T v_t^{(1)} dW_{1,t} \int_0^T V(\sigma_{0t}) v_t^{(1)} dt\right) \mid \int_0^T V(\sigma_{0t}) dW_{1,t} = x\right] \\
&= \left(\frac{x^5}{\Sigma^5} - \frac{10x^3}{\Sigma^4} + \frac{15x}{\Sigma^3}\right) b_{51} + \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right)(b_{52} + b_{53}),
\end{aligned} \tag{75}
$$

where

$$
\begin{aligned}
b_{51} &= \rho^2 \left(\int_0^T q_{3t} q_{1t} \int_0^t q_{2s} q_{1s} ds dt\right)\left(\int_0^T q_{4u}\left(\int_u^T q_{5s} ds\right) q_{1u} du\right), \\
b_{52} &= \left(\int_0^T q_{3t} q_{1t} \int_0^t q_{4s} q_{2s} \left(\int_s^T q_{5u} du\right) ds dt\right), \\
b_{53} &= \left(\rho^2 \int_0^T q_{4t} q_{3t} \left(\int_t^T q_{5u} du\right) \int_0^t q_{2s} q_{1s} ds dt\right), \\
q_{1t} &= V(\sigma_{0t}), \\
q_{2t} &= q_{4t} = \eta_t^{-1} A_1(\sigma_{0t}), \\
q_{3t} &= V'(\sigma_{0t})\eta_t, \\
q_{5t} &= V'(\sigma_{0t}) V(\sigma_{0t})\eta_t.
\end{aligned}
$$

A.1.6. $H_2(X_T^{(0)}, (\int_0^T V(\sigma_{0t}) v_t^{(1)} dt)^2)$.

$$
\begin{aligned}
&E\left[H_2\left(X_T^{(0)}, \left(\int_0^T V(\sigma_{0t}) v_t^{(1)} dt\right)^2\right) \mid \int_0^T V(\sigma_{0s}) dW_{1,s} = x\right] \\
&= 2\left[\left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma}\right) b_{61} + \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) b_{62}\right],
\end{aligned} \tag{76}
$$

where

$$
\begin{aligned}
b_{61} &= \rho^2 \int_0^T q_{3t} \left(\int_0^t q_{2u} q_{1u} du \right) \left(\int_0^t \left(\int_u^t q_{3s} ds \right) q_{2u} q_{1u} du \right) dt, \\
b_{62} &= \int_0^T q_{3t} \int_0^t q_{2u}^2 \left(\int_u^t q_{3s} ds \right) dudt, \\
q_{1t} &= V(\sigma_{0t}), \\
q_{2t} &= \eta_t^{-1} A_1(\sigma_{0t}), \\
q_{3t} &= V'(\sigma_{0t}) V(\sigma_{0t}) \eta_t.
\end{aligned}
$$

Appendix B. Proof of Proposition 5.2. Let C_1^M denote the coefficient of ϵ-order in the asymptotic expansion of C^{SVJ} around $\epsilon = 0$:

$$
\begin{aligned}
C_1^M &= \frac{\partial}{\partial \epsilon} E\left[(S_T^{(\epsilon)} - K)^+ \right] |_{\epsilon=0} \\
&= E\left[\partial_x \left\{ \left(S_T^{(0)} - K \right)^+ \right\} \frac{\partial}{\partial \epsilon} S_T^{(\epsilon)} |_{\epsilon=0} \right] \\
&= \sum_{n=0}^{\infty} E\left[\partial_x \left\{ \left(S_T^{(0)} - K \right)^+ \right\} \frac{\partial}{\partial \epsilon} S_T^{(\epsilon)} |_{\epsilon=0} \mathbf{1}_{\{N(T)=n\}} \right],
\end{aligned} \tag{77}
$$

where

$$
\partial_x \left\{ \left(S_T^{(0)} - K \right)^+ \right\} = \frac{\partial}{\partial x} (x - K)^+ |_{x = S_T^{(0)}}.
$$

Next, by the integration-by-parts formula including the jump amplitudes (e.g., Bavouzet and Messaoud [2]), the right-hand side of the last equality in the above equation is expressed as

$$
C_1^M = \sum_{n=0}^{\infty} E\left[\left(S_T^{(0)} - K \right)^+ H_{1,n} \mathbf{1}_{\{N(T)=n\}} \right], \tag{78}
$$

where $H_{1,n}$ is the Malliavin weight on $\{N(T) = n\}$, which will be given below.

Let D_0 be the Malliavin derivative with respect to Brownian motion for W_1 and D_i be the Malliavin derivative with respect to the jump amplitudes Δ_i, $i = 1, \dots, n$. Then, on $\{N(T) = n\}$,

$$
D_{0,t} S_T^{(0)} = S_T^{(0)} D_{0t} X_T^{(0)} = S_T^{(0)} V(\sigma_t^{(0)}), \quad t \in [0, T], \tag{79}
$$

$$
D_i S_T^{(0)} = S_T^{(0)} D_i X_T^{(0)} = S_T^{(0)} b, \quad i = 1, \dots, n, \tag{80}
$$

$$
X_T^{(0)} = \int_0^T V(\sigma_t^{(0)}) dW_{1,t} - \frac{1}{2} \int_0^T V(\sigma_t^{(0)})^2 dt + \sum_{j=1}^n Y_j,
$$

where we use $D_i X_T^{(0)} = b$ (see, e.g., Bavouzet and Messaoud [2, p. 280]) as well as $D_{0,t} X_T^{(0)} = V(\sigma_t^{(0)})$.

Then, the Malliavin covariance of $S_T^{(0)}$, $\sigma_{S_T^{(0)}}$ is given by

$$
\begin{aligned}
\sigma_{S_T^{(0)}} &= \int_0^T \left(D_{0,t}S_T^{(0)}\right)^2 dt + \sum_{i=1}^n \left(D_i S_T^{(0)}\right)^2 \\
&= (S_T^{(0)})^2 \int_0^T V(\sigma_t^{(0)})^2 dt + (S_T^{(0)})^2 b^2 n \\
&= (S_T^{(0)})^2 \Sigma_n,
\end{aligned}
\tag{81}
$$

where

$$
\Sigma_n := \int_0^T V(\sigma_t^{(0)})^2 dt + (S_T^{(0)})^2 b^2 n
$$

and

$$
\sum_{i=1}^n \left(D_i S_T^{(0)}\right)^2 = (S_T^{(0)})^2 (b^2 n)
$$

is the contribution by the jump part (see, e.g., Bavouzet and Messaoud [2, p. 282]).

Using the Skorohod integral for the Brownian motion and the jump amplitude, on $\{N(T) = n\}$ we can compute the Malliavin weight $H_{1,n}$ as follows:

$$
\begin{aligned}
H_{1,n} &= \sum_{i=0}^n D_i^* \left(\frac{\frac{\partial}{\partial \epsilon} S_T^{(\epsilon)}|_{\epsilon=0} D_i S_T^{(0)}}{\sigma_{S_T^{(0)}}} \right) = \sum_{i=0}^n D_i^* \left(\frac{S_T^{(0)} X_T^{(1)} S_T^{(0)} D_i X_T^{(0)}}{(S_T^{(0)})^2 \Sigma_n} \right) \\
&= \frac{1}{\Sigma_n} \sum_{i=0}^n D_i^* \left(X_T^{(1)} D_i X_T^{(0)} \right) \\
&= \frac{1}{\Sigma_n} \left(X_T^{(1)} D_0^* (D_0 X_T^{(0)}) - \int_0^T (D_{0,t} X_T^{(1)})(D_{0,t} X_T^{(0)}) dt \right) \\
&+ \frac{1}{\Sigma_n} \sum_{i=1}^n \left((X_T^{(1)} D_i^* (D_i X_T^{(0)}) - (D_i X_T^{(1)})(D_i X_T^{(0)}) \right),
\end{aligned}
\tag{82}
$$

where each D_i^* $(i = 0, 1, \ldots, n)$ is the adjoint operator of D_i $(i = 0, 1, \ldots, n)$ and

$$
\begin{aligned}
X_T^{(1)} &= \frac{\partial}{\partial \epsilon} X_T^{(\epsilon)}|_{\epsilon=0} \\
= \int_0^T V'(\sigma_{0t}) \int_0^t \eta_t \eta_s^{-1} A_1(\sigma_{0s}) dB_s dW_{1,t} &- \int_0^T V(\sigma_{0t}) V'(\sigma_{0t}) \int_0^t \eta_t \eta_s^{-1} A_1(\sigma_{0s}) dB_s dt,
\end{aligned}
$$

with

$$
B_t = \rho W_{1,t} + \sqrt{1-\rho^2} W_{2,t}.
$$

Note that $D_0^*(D_0 X_T^{(0)})$ is expressed by the usual Itô integral,

$$
D_0^*(D_0 X_T^{(0)}) = \int_0^T V(\sigma_t^{(0)}) dW_{1,t}.
$$

Recall that Δ_i is Gaussian random variable $N(0,1)$. Hence, on p. 280 in Bavouzet and Messaoud [2], $D_i^*(D_i X_T^{(0)})$ for $i=1,\ldots,n$ is given by

$$D_i^*(D_i X_T^{(0)}) = D_i^*(b) = -b\frac{\partial}{\partial \Delta_i}\log p(\Delta_i) = b\Delta_i, \tag{83}$$

where

$$p(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

Because $X_T^{(1)}$ is a stochastic integral driven by the Brownian motion which is independent of Δ_i, we have for $t \in [0,T]$,

$$\begin{aligned} D_{0,t}X_T^{(1)} = V'(\sigma_t^{(0)})\int_0^t \eta_t\eta_s^{-1}A_1(\sigma_s^{(0)})dB_s + \rho\eta_t^{-1}A_1(\sigma_t^{(0)})\int_t^T V'(\sigma_u^{(0)})\eta_u dW_{1,u} \\ - \rho\eta_t^{-1}A_1(\sigma_t^{(0)})\int_t^T V'(\sigma_u^{(0)})V(\sigma_u^{(0)})\eta_u du, \end{aligned}$$

and for $i=1,\ldots,n$,

$$D_i X_T^{(1)} = 0.$$

Hence, (82) is expressed as

$$H_{1,n} = \frac{1}{\Sigma_n}\left(X_T^{(1)}\left(\int_0^T V(\sigma_t^{(0)})dW_{1,t} + \sum_{i=1}^n b\Delta_i\right) - \int_0^T D_{0,t}X_T^{(1)}V(\sigma_t^{(0)})dt\right).$$

Then,

$$\begin{aligned} C_1^M &= \sum_{n=0}^\infty E\left[\left(S_T^{(0)}-K\right)^+ H_{1,n}\mathbf{1}_{\{N(T)=n\}}\right] \\ &= \sum_{n=0}^\infty \frac{(\lambda T)^n e^{-\lambda T}}{n!}E\left[\left(S_T^{(0)}-K\right)^+ H_{1,n}\right] \\ &= \sum_{n=0}^\infty \frac{(\lambda T)^n e^{-\lambda T}}{n!}\int_{\mathbf{R}}\left(se^{x-\lambda mT-\frac{1}{2}\int_0^T V(\sigma_t^{(0)})^2dt+na} - K\right)^+ \vartheta_n(x)\frac{1}{\sqrt{2\pi\Sigma_n}}e^{-\frac{1}{2\Sigma_n}x^2}dx, \end{aligned}$$

where $\vartheta_n(x)$ is the push-down of the Malliavin weight $H_{1,n}$:

$$\vartheta_n(x) = E\left[H_{1,n}|\int_0^T V(\sigma_t^{(0)})dW_{1,t} + \sum_{i=1}^n b\Delta_i = x\right].$$

We easily evaluate $\vartheta_n(x)$ as

$$\vartheta_n(x) = \zeta\left(\frac{1}{\Sigma_n^3}x^3 - \frac{1}{\Sigma_n^2}x^2 - \frac{1}{\Sigma_n^2}3x + \frac{1}{\Sigma_n}\right). \tag{84}$$

Thus, we obtain (66):

$$C^{SVJ}(T,K) = C^M(T,K,\sigma^{BS}) + \epsilon C_1^M + O(\epsilon^2). \tag{85}$$

Next, let us regard C^M as a function of σ^{BS}:

$$\sigma^{BS} \mapsto C^M(T,K,\sigma^{BS}) = \sum_{n=0}^{\infty} \frac{(\lambda(m+1)T)^n e^{-\lambda(m+1)T}}{n!} C_n^{BS}\left(\left(\left(\sigma^{BS}\right)^2 + (b^2 n/T)\right)^{1/2}\right).$$

Also, the *vega* of C^M with respect to σ^{BS}, $vega^M$ is given by

$$\begin{aligned} vega^M &= \frac{\partial}{\partial \sigma^{BS}} C^M(T,K,\sigma^{BS}) \\ &= \sum_{n=0}^{\infty} \frac{(\lambda(m+1)T)^n e^{-\lambda(m+1)T}}{n!} \frac{\sigma^{BS}}{\sigma_n} s\sqrt{T} n(d_n). \end{aligned} \tag{86}$$

Suppose that $\sigma^{SVJ.IV}$ satisfies

$$C^{SVJ}(T,K) = C^M(T,K,\sigma^{SVJ.IV}(T,K))$$

and that $\sigma^{SVJ.IV}(T,K)$ is expanded around $\epsilon = 0$:

$$\sigma^{SVJ.IV}(T,K) = \sigma^{BS} + \epsilon \cdot \sigma^1 + O(\epsilon^2).$$

Then, we have

$$C^M(\sigma^{SVJ.IV}) = C^M(T,K,\sigma^{BS}) + \epsilon \cdot vega^M \sigma^1 + O(\epsilon^2). \tag{87}$$

Finally, comparing (66) with (87), we obtain

$$\sigma^1 = \frac{C_1^M}{vega^M}, \tag{88}$$

which leads to the result (67).

REFERENCES

[1] F. Antonelli and S. Scarlatti, *Pricing options under stochastic volatility: A power series approach*, Finance Stoch., 13 (2009), pp. 269–303.

[2] M.-P. Bavouzet and M. Messaoud, *Computation of Greeks using Malliavin's calculus in jump type market models*, Electron. J. Probab., 11 (2006), pp. 276–300.

[3] G. Ben Arous and P. Laurence, *Second Order Expansion for Implied Volatility in Two Factor Local Stochastic Volatility Models and Applications to the Dynamic λ-SABR Model*, preprint, 2009.

[4] J.-P. Fouque, G. Papanicolaou, and K. R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, Cambridge, UK, 2000.

[5] E. Fournié, J.-M. Lasry, J. Lebuchoux, P.-L. Lions, and N. Touzi, *Applications of Malliavin calculus to Monte-Carlo methods in finance*, Finance Stoch., 3 (1999), pp. 391–412.

[6] E. Fournié, J. Lebuchoux, and N. Touzi, *Small noise expansion and importance sampling*, Asymptot. Anal., 14 (1997), pp. 361–376.

[7] J. Gatheral, *Further developments in volatility derivatives modeling*, presentation at Global Derivatives Trading & Risk Management, Paris, 2008.
[8] J. Gatheral, E. P. Hsu, P. Laurence, C. Ouyang, and T.-H. Wang, *Asymptotics of implied volatility in local volatility models*, Math. Finance, to appear.
[9] P. S. Hagan, D. Kumar, A. S. Lesniewski, and D. E. Woodward, *Managing smile risk*, Wilmott Magazine, September (2002), pp. 84–108.
[10] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd ed., North–Holland, Amsterdam, Kodansha, Tokyo, 1989.
[11] N. Kunitomo and A. Takahashi, *On validity of the asymptotic expansion approach in contingent claim analysis*, Ann. Appl. Probab., 13 (2003), pp. 914–952.
[12] P. H. Labordère, *Analysis, Geometry and Modeling in Finance: Advanced Methods in Options Pricing*, Chapman and Hall, CRC, Boca Raton, FL, 2008.
[13] A. L. Lewis, *Option Valuation under Stochastic Volatility with Mathematica Code*, Finance Press, Newport Beach, CA, 2000.
[14] P. Malliavin, *Stochastic Analysis*, Springer, Berlin, 1997.
[15] P. Malliavin and A. Thalmaier, *Stochastic Calculus of Variations in Mathematical Finance*, Springer, Berlin, 2006.
[16] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer, Berlin, 2006.
[17] Y. Sakamoto and N. Yoshida, *Asymptotic Expansions of Mixture Type Statistics Based on Generalized Wiener Functionals*, Cooperative Research Report 58, The Institute of Statistical Mathematics, Tokyo, Japan, 1994.
[18] K. Shiraya, A. Takahashi, and T. Yamada, *Pricing discrete barrier options under stochastic volatility*, Asia Pac. Financ. Market, to appear.
[19] M. Siopacha and J. Teichmann, *Weak and strong Taylor methods for numerical solutions of stochastic differential equations*, Quant. Finance, 11 (2011), pp. 517–528.
[20] A. Takahashi, *Essays on the Valuation Problems of Contingent Claims*, Ph.D. dissertation, Haas School of Business, University of California, Berkeley, CA, 1995.
[21] A. Takahashi, *An asymptotic expansion approach to pricing financial contingent claims*, Asia Pac. Financ. Market, 6 (1999), pp. 115–151.
[22] A. Takahashi, K. Takehara, and M. Toda, *Computation in an Asymptotic Expansion Method*, Preprint, CARF Working Paper Series CARF-F-149, 2009.
[23] A. Takahashi and N. Yoshida, *An asymptotic expansion scheme for optimal investment problems*, Stat. Inference Stoch. Process., 7 (2004), pp. 153–188.
[24] A. Takahashi and N. Yoshida, *Monte Carlo simulation with asymptotic method*, J. Japan Statist. Soc., 35 (2005), pp. 171–203.
[25] S. Watanabe, *Malliavin's calculus in terms of generalized Wiener functionals*, in Theory and Application of Random Fields, Lecture Notes in Control and Inform. Sci. 49, Springer, Berlin, 1983.
[26] S. Watanabe, *Lectures on Stochastic Differential Equations and Malliavin Calculus*, Springer, Berlin, 1984.
[27] S. Watanabe, *Analysis of Wiener functionals (Malliavin calculus) and its applications to heat kernels*, Ann. Probab., 15 (1987), pp. 1–39.
[28] N. Yoshida, *Asymptotic expansion for small diffusions via the theory of Malliavin-Watanabe*, Probab. Theory Related Fields, 92 (1992), pp. 275–311.
[29] N. Yoshida, *Asymptotic expansion for statistics related to small diffusions*, J. Japan Statist. Soc., 22 (1992), pp. 139–159.