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STOCHASTIC VOLATILITY FOR LÉVY PROCESSES

Peter Carr

Courant Institute, New York University

Hélyette Geman

University Paris Dauphine, and ESSEC Business School

Dilip B. Madan

Robert H. Smith School of Business, University of Maryland

Marc Yor

Laboratoire de probabilités et Modeles aléatoires, Université Pierre et Marie Curie, Paris

Three processes reflecting persistence of volatility are initially formulated by evaluating three Lévy processes at a time change given by the integral of a mean-reverting square root process. The model for the mean-reverting time change is then generalized to include non-Gaussian models that are solutions to Ornstein-Uhlenbeck equations driven by one-sided discontinuous Lévy processes permitting correlation with the stock. Positive stock price processes are obtained by exponentiating and mean correcting these processes, or alternatively by stochastically exponentiating these processes. The characteristic functions for the log price can be used to yield option prices via the fast Fourier transform. In general mean-corrected exponentiation performs better than employing the stochastic exponential. It is observed that the mean-corrected exponential model is not a martingale in the filtration in which it is originally defined. This leads us to formulate and investigate the important property of martingale marginals where we seek martingales in altered filtrations consistent with the one-dimensional marginal distributions of the level of the process at each future date.

Key Words: variance gamma, static arbitrage, stochastic exponential, leverage, OU equation, Lévy marginal, martingale marginal.

1. INTRODUCTION

It has been clear that the standard option pricing model of Black and Scholes (1973) and Merton (1973) (hereafter BMS) has been inconsistent with options data for at least a decade. The model in fact implies that the informational content of the option surface

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Address correspondence to Dilip Madan, Robert H. Smith School of Business, University of Maryland, College Park, MD 20742; e-mail: dbm@rhsmith.umd.edu.

is one dimensional. Were the model to be empirically relevant, then one could infer the prices of options at all strikes and maturities from the price of any single option. This property calls into question the very existence of these markets and the associated activity of price discovery undertaken therein.

At the other extreme we have, for example, local volatility models that enhance the dimension of the informational content to a doubly indexed continuum specifying the volatility $\sigma(S, t)$that would prevail at any future time t were the spot asset to trade then at the price S. Furthermore, the implicit maintained hypothesis of a one-dimensional Markov process remains questionable. This hypothesis is called into question by trading in the options markets that presumably occurs to extract from these markets relevant information that is not available on observing the spot price of the underlying asset.

This paper is concerned primarily with synthesizing the informational content of option prices across strike and maturity at a point of time. The objective is accomplished by developing a parsimonious model that fits the surface at each point of time. The parameters of this synthesizing model are then viewed as efficient summaries of the surface and it is envisaged that the changes in these synthesizing parameters from day to day define a dynamic model that is ultimately Markovisan in the dimension of the spot price augmented by the synthesizing parameters.

Improvements in pricing performance over the BMS model have been sought by a majority of researchers by modifying the continuous-time stochastic process followed by the underlying asset. In particular, asset returns have been modeled as diffusions with stochastic volatility (e.g., Heston 1993; Hull and White 1987), as jump-diffusions (e.g., Kou and Wang 2001; Merton 1976), or as both (e.g., Bates 1996, 2000; Duffie, Pan, and Singleton 2000). Empirical work on these models has generally supported the need for both stochastic volatility to calibrate the longer maturities and jumps to reflect shorter maturity option prices.

On the theoretical side, arguments have been proposed by Geman, Madan, and Yor (2001) which suggest that price processes for financial assets must have a jump component but they need not have a diffusion component. Their argument rests on recognizing that all price processes of interest may be regarded as Brownian motion subordinated to a random clock. This clock may be regarded as a cumulative measure of economic activity, as conjectured by Clark (1973), and as estimated by Ané and Geman (2000). As time must be increasing, the random clock can be modeled as a pure jump increasing process, or alternatively as a time integral of a positive diffusion process, and thus devoid of a martingale component. If jumps are suppressed, then the clock is locally deterministic. Ruling out a locally deterministic clock a priori, it is concluded that the required jumps in the clock induce jumps in the price process. There is no similar argument requiring that prices have a diffusion component. Furthermore, the use of jump processes with an infinite arrival rate can adequately encompass the contribution of any diffusion component, rendering its explicit employment vacuous.

Three examples of infinite activity pure jump Lévy processes are employed in this paper. First, we have the normal inverse Gaussian (NIG) model of Barndorff-Nielsen (1998), and its generalization to the generalized hyperbolic class by Eberlein, Keller, and Prause (1998). Second, we have the symmetric variance gamma (VG) model studied by Madan and Seneta (1990) and its asymmetric extension studied by Madan, Carr, and Chang (1998), and Madan and Milne (1991). Finally, we have the model developed by Carr, Geman, Madan, and Yor (2002; hereafter, the CGMY model); which further generalizes the VG model.

The empirical success of pure jump Lévy processes is not maintained when one considers the variation of option prices across maturity. It has been observed in Konikov and Madan (2002) that these homogeneous Lévy processes impose strict conditions on the term structure of the risk-neutral variance, skewness, and kurtosis. Specifically, the variance rate is constant over the term, skewness is inversely proportional to the square root of the term, and kurtosis is inversely proportional to the term. In contrast, the data suggest that these risk-neutral moments are often rising with term. Collectively, these considerations suggest that it may be desirable to incorporate a richer behavior across maturity than is implied by homogeneous Lévy processes.

In a parallel development in the literature, it has been observed by several authors (e.g., Barndorff-Nielsen and Shephard 2001; Bates 1996, 2000; Duan 1995; Engle 1982; Heston 1993) that volatilities estimated from the time series are both stochastic and usually clustered. The phenomenon of clustering is commonly referred to as *volatility persistence*. For these reasons, the objective of this paper is to extend the otherwise fairly successful Lévy process models cited above by incorporating stochastic and mean-reverting volatilities.

We take the three homogeneous Lévy processes cited above, the NIG, VG, and CGMY models, and generate the desired volatility properties by subordinating them, in the first instance to the time integral of a Cox, Ingersoll, and Ross (1985; CIR) process. The randomness of the CIR process induces stochastic volatility (SV), and the mean reversion in this process induces volatility clustering. We term the resulting processes NIGSV, VGSV, and CGMYSV in recognition of their synthesis with stochastic volatility.

Extensions to other mean-reverting processes expressed as solutions to Ornstein-Uhlenbeck (OU) equations driven by one-sided pure jump Lévy processes, termed *background driving Lévy processes* (BDLP) by Barndorff-Nielsen and Shephard (2001), are subsequently presented that accommodate leverage effects as well. This work is closely related to work by Barndorff-Nielsen, Nicolato, and Shepard (2001) and Nicolato and Venardos (2001) in which such methods were developed to model the accumulated variance of a diffusion; we employ similar processes to time change a discontinuous Lévy process. The specific BDLPs considered are those associated with a stationary solution to the OU equation that is gamma or inverse Gaussian (IG) and this leads to the BDLPs we refer to as SG and SIG. In addition we also develop the inverse Gaussian (IG) BDLP. The resulting alternative BDLP models are 18 in number, combining three Lévy processes with three pure jump Lévy BDLPs and then employing two architectures of exponentiation and stochastic exponentiation. Finally, we also add correlation or leverage to VGSA (the exponential form for combining VG with CIR) and develop the VGCSA model. In all, the paper develops 25 models with explicit forms for the log characteristic function in each case.

In constructing risk-neutral price processes from the stochastic volatility Lévy processes (e.g., NIGSV, VGSV, and CGMYSV), two approaches are followed. The first approach reflects the implications of prohibiting static arbitrage opportunities by constructing the risk-neutral distribution at each maturity as the exponential of NIGSV, VGSV, and CGMYSV processes normalized to reflect initial forward prices. This class of models is termed NIGSA, VGSA, and CGMYSA. The second approach follows the more stringent implications of excluding dynamic arbitrage by compensating the pure jump processes NIGSV, VGSV, and CGMYSV to form martingales. These martingales are then stochastically exponentiated to yield martingale candidates (in the enlarged filtration of the stock price and the integrated CIR time change) for forward prices. This class of models is termed NIGSAM, VGSAM, and CGMYSAM.

We note that the martingale properties of the SAM models are all with respect to the enlarged filtration, which includes knowledge of the subordinator given by the time-integrated CIR process. In this regard Geman, Madan, and Yor (2002) observed that for pure jump processes the time change is a random variable given a full realization of the price path; they want on to develop its conditional law in a number of cases. To the extent that the subordinator cannot be ascertained from a time series of prices, and to the extent that parameters must be reestimated to ensure pricing quality, serious issues arise as to the practical relevance of the associated martingale condition.

The models NIGSA, VGSA, and CGMYSA take a more conservative approach than the martingale models NIGSAM, VGSAM, and CGMYSAM. We find that these more conservative models consistently provide substantially superior empirical performance over their martingale counterparts. We are then led to making a deeper study of the properties of these more conservative models . In this regard, we introduce two important new concepts, which we term the *martingale marginal property* and the *Lévy marginal property*. We define a process as having the martingale marginal property if it has the same one-dimensional marginal distributions as some martingale process. We further show that the martingale marginal property is intimately connected with the absence of elementary arbitrages that amount to static arbitrages enhanced by a few stock trades triggered by observations on the price level at particular times.

We further define a process as having the Lévy marginal property if it has the martingale marginal property and if the martingale is derived from normalizing the exponential of a time-inhomogeneous Lévy process. We show first that if the CIR process is started at zero, then our conservative processes have this Lévy marginal property. When the starting value is not zero, we conjecture that these processes have the martingale marginal properties.

We report the results of estimating the six CIR-based models using S&P 500 option closing prices for the second Wednesday of each month of the year 2000. In the interest of brevity, for other BDLPs we provide results for just the VG model. The models are observed to be capable of adequately fitting a wide range of strikes and maturities consistently across the year.

The outline of the paper is as follows. In Section 2 we briefly summarize the three homogeneous Lévy processes, NIG, VG, and CGMY. Section 3 introduces the time change using an integrated CIR process and presents the characteristic functions for the processes NIGSV, VGSV, and CGMYSV. In Section 4 we introduce the two stock price model architectures and the martingale marginal property along with its connections to arbitrage. The characteristic functions for the log of the stock price for the six models NIGSA, VGSA, CGMYSA, NIGSAM, VGSAM, and CGMYSAM are then developed. Section 5 studies the martingale properties of the models NIGSA, VGSA, and CGMYSA. Section 6 introduces other BDLPs and presents the required construction of characteristic functions in these cases, including leverage effects. Section 7 adds correlation to VGSA. Section 8 describes the data and briefly reviews the estimation methodology. The results for our class of models are presented in Section 9. Section 10 summarizes the paper and provides suggestions for further research.

2. THE LÉVY PROCESSES

Three homogeneous Lévy processes employed in this paper to develop their stochastic volatility versions are briefly summarized in this section. These are NIG, VG, and CGMY. All three processes are pure jump with infinite activity. In the interest of notational parsimony, the reader is forewarned that "notational overloading" is employed in the discussion of the three models. Confusion is easily avoided by simply being aware of the context in which the notation is used.

2.1. The Normal Inverse Gaussian Model

The NIG process has a characteristic function defined by three parameters (see Barndorff-Nielsen 1998):

$$\phi_{\text{NIG}}(u;\alpha,\beta,t\delta) = \exp\left(-t\delta\left(\sqrt{\alpha^2-(\beta+iu)^2}-\sqrt{\alpha^2-\beta^2}\right)\right). \tag{2.1}$$

From the linearity of the log characteristic function in the time variable, we observe that this is an infinitely divisible process with stationary independent increments. For comparison with the variance gamma later we write the NIG process explicitly as Brownian motion with drift time changed by an independent inverse Gaussian process. Let T_t^ν be the first time that a Brownian motion with drift ν reaches the positive level t. It is well known that the Laplace transform of this random time is

$$E\left[\exp\left(-\lambda T_t^\nu\right)\right] = \exp\left(-t\left(\sqrt{2\lambda+\nu^2}-\nu\right)\right). \tag{2.2}$$

Now consider evaluating Brownian motion with drift θ and volatility σ at the inverse Gaussian process to define the new process

$$X_{\text{NIG}}(t;\sigma,\nu,\theta) = \theta T_t^\nu + \sigma W\left(T_t^\nu\right). \tag{2.3}$$

Suppressing the dependence of the process on its parameters, the characteristic function is

$$E\left[e^{iuX_{\text{NIG}}(t)}\right] = \exp\left(-t\sigma\left(\sqrt{\frac{\nu^2}{\sigma^2}+\frac{\theta^2}{\sigma^4}-\left(\frac{\theta}{\sigma^2}+iu\right)^2}-\frac{\nu}{\sigma^2}\right)\right).$$

Hence we may define

$$\beta = \frac{\theta}{\sigma^2}; \qquad \alpha^2 = \frac{\nu^2}{\sigma^2}+\frac{\theta^2}{\sigma^4}; \qquad \delta = \sigma$$

and observe that the NIG process is

$$X_{\text{NIG}}(t;\alpha,\beta,\delta) = \beta\delta^2 T_t^{\delta\sqrt{\alpha^2-\beta^2}} + \delta W\left(T_t^{\delta\sqrt{\alpha^2-\beta^2}}\right). \tag{2.4}$$

To obtain the NIG Lévy density, note that, conditioning on a jump of g in the time change, the move is Gaussian with mean $\beta\delta^2 g$ and variance $\delta^2 g$.The arrival rate for the jumps is given by the Lévy density for inverse Gaussian time:

$$k(g) = \frac{\exp\left(-\frac{\delta^2(\alpha^2-\beta^2)}{2}g\right)}{g^{3/2}}.$$

It follows that the Lévy density for NIG is

$$k_{\text{NIG}}(x) = \sqrt{\frac{2}{\pi}}\delta\alpha^2\frac{e^{\beta x}K_1(|x|)}{|x|},$$

where $K_a(x)$ is the Bessel function.

For later use, we record here the unit time log characteristic function expressed in terms of the parameters of the time-changed Brownian motion

$$\psi_{\mathrm{NIG}}(u;\sigma,\nu,\theta) = \sigma\left(\frac{\nu}{\theta} - \sqrt{\frac{\nu^2}{\theta^2} - 2\frac{\theta i u}{\sigma^2} + u^2}\right).$$

2.2. The Variance Gamma Model

The variance gamma process is defined by evaluating Brownian motion with drift θ and volatility σ at an independent gamma time. Specifically, we have

$$X_{\mathrm{VG}}(t;\sigma,\nu,\theta) = \theta G_t^\nu + \sigma W\big(G_t^\nu\big),$$

where G_t^ν is a gamma process with mean rate t and variance rate νt, independent of W. The Laplace transform of the gamma process is

(2.5) $$E\big[\exp\big(-\lambda G_t^\nu\big)\big] = (1+\lambda\nu)^{-t/\nu}.$$

The characteristic function of the VG process is easily evaluated as

$$E\big[e^{iuX_{\mathrm{VG}}(t)}\big] = (1 - iu\theta\nu + \sigma^2\nu u^2/2)^{-t/\nu}.$$

The Lévy density for the variance gamma process may be derived directly from the Lévy Khintchine theorem. Alternatively, one may exploit the representation of the variance gamma process as the difference of two independent gamma processes. It is shown in Carr et al. (2002) that

$$k_{\mathrm{VG}}(x) = \begin{cases} \frac{C\exp(Gx)}{|x|} & x<0 \\ \frac{C\exp(-Mx)}{x} & x>0, \end{cases}$$

where

(2.6) $$C = \frac{1}{\nu}$$

(2.7) $$G = \left(\sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} - \frac{\theta\nu}{2}\right)^{-1}$$

(2.8) $$M = \left(\sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} + \frac{\theta\nu}{2}\right)^{-1}.$$

For later use, we will need the following unit time log characteristic function in the Lévy measure parametrization:

(2.9) $$\psi_{\mathrm{VG}}(u;C,G,M) = C\log\left(\frac{GM}{GM + (M-G)iu + u^2}\right).$$

2.3. The CGMY model

The specific form for the CGMY Lévy density is

$$k_{\mathrm{CGMY}}(x) = \begin{cases} \frac{C\exp(Gx)}{(-x)^{1+Y}} & x < 0 \\ \frac{C\exp(-Mx)}{x^{1+Y}} & x > 0. \end{cases}$$

This process has also come to be known in the literature on turbulence as the *truncated Lévy flight model* (Mantegna and Stanley 1994). Boyarchenko and Levendorskii (2000) and Hagan and Woodward (2002) also consider option pricing with such a process. Furthermore, these processes play an important role in the construction of certain Poisson-Dirichlet laws as studied in Pitman and Yor (1997).

The characteristic function is

$$(2.10)\qquad \begin{aligned} &E[\exp(iuX_{\mathrm{CGMY}}(t))] \\ &\quad = \exp(tC\Gamma(-Y)[(M-iu)^Y + (G+iu)^Y - M^Y - G^Y]). \end{aligned}$$

The case $G = M$ was also studied by Koponen (1995).

In what follows, the C and Y parameters will be allowed to take different values for positive and negative outcomes in x. Letting C_p, Y_p denote the parameters for $x > 0$ and C_n, Y_n denote the parameters for $x < 0$, the generalized characteristic function is

$$\begin{aligned} &E[\exp(iuX_{\mathrm{CGMY}}(t)] \\ &\quad = \exp\left(tC_p\Gamma(-Y_p)\left((M-iu)^{Y_p} - M^{Y_p}\right) + C_n\Gamma(-Y_n)\left((G+iu)^{Y_n} - G^{Y_n}\right)\right). \end{aligned}$$

For later use, we record here the unit time log characteristic function

$$\begin{aligned} &\psi_{\mathrm{CGMY}}(u; C_p, G, M, Y_p, Y_n, \zeta) \\ &\quad = C_p\left(\Gamma(-Y_p)\left((M-iu)^{Y_p} - M^{Y_p}\right) + \zeta\Gamma(-Y_n)\left((G+iu)^{Y_n} - G^{Y_n}\right)\right) \end{aligned}$$

with ζ defined as the ratio of C_n to C_p.

3. CLUSTERING TIME OR ACTIVITY PERSISTENCE

The basic intuition underlying our approach to stochastic volatility arises from the Brownian scaling property. This property relates changes in scale to changes in time and thus random changes in volatility can alternatively be captured by random changes in time. The instantaneous rate of time change must be positive if the new clock is to be increasing. Furthermore, this rate of time change must be mean reverting if the random time changes are to persist. The classic example of a mean-reverting positive process is the so-called square root process of Cox, Ingersoll, and Ross. We first focus on this candidate and in section 6 we address the use of other Lévy processes driving the stochastic volatility time change. In section 7 we also incorporate the effects of leverage along the lines followed by Barndorff-Nielsen et al. (2001), but applied here to a discontinuous Lévy process requiring exact integrals of certain functionals of the Lévy cumulant that we develop.

Hence, we consider first the process $y(t)$ defined as the solution to the stochastic differential equation

$$(3.1)\qquad dy = \kappa(\eta - y)dt + \lambda\sqrt{y}\,dW,$$

where $W(t)$ is a standard Brownian motion independent of any processes encountered thus far. The parameter η has the usual interpretation as the long run rate of time change, κ is the rate of mean reversion, and λ governs the volatility of the time change.

The process $y(t)$ is the instantaneous rate of time change and so the new clock is given by its integral

$$Y(t) = \int_0^t y(u)\,du. \tag{3.2}$$

The characteristic function for $Y(t)$ is well known from the work of CIR and from the literature on Brownian motion because it is closely associated with Lévy's stochastic area formula (e.g., see Lamberton and Lapeyre 1996; Pitman and Yor 1982; Yor 1992). Other important references include Taylor (1975) and Williams (1976). We recall that the characteristic function for $Y(t)$ is explicitly given by

$$\begin{aligned}
E[\exp(iuY(t))] &= \phi(u, t, y(0); \kappa, \eta, \lambda)\\
&= A(t,u)\exp(B(t,u)y(0)),\\
A(t,u) &= \frac{\exp\left(\frac{\kappa^2\eta t}{\lambda^2}\right)}{\left(\cosh\left(\frac{\gamma t}{2}\right) + \frac{\kappa}{\gamma}\sinh\left(\frac{\gamma t}{2}\right)\right)^{2\kappa\eta/\lambda^2}},\\
B(t,u) &= \frac{2iu}{\kappa + \gamma\coth\left(\frac{\gamma t}{2}\right)},\\
\gamma &= \sqrt{\kappa^2 - 2\lambda^2 iu}.
\end{aligned}$$

3.1. The Generic Stochastic Volatility Lévy Process

Let $X(t)$ be a Lévy process, so that it has stationary independent increments. Its characteristic function is thus of the form

$$E[\exp(iuX(t))] = \exp(t\psi_X(u)). \tag{3.3}$$

For simplicity, we assume a Lévy density exists and denote it by $k(x)$. When $X(t)$ is a pure jump, zero drift process of finite variation, the log characteristic function at unit time $\psi_X(u)$ is related to $k(x)$ by

$$\psi_X(u) = \int_{-\infty}^{\infty}(e^{iux} - 1)k(x)\,dx. \tag{3.4}$$

Explicit forms for $\psi_X(u)$ in the case of the NIG, the VG, and the CGMY models were exhibited in Section 2.

The class of stochastic volatility Lévy processes (SVLP) is defined by

$$Z(t) = X(Y(t)), \tag{3.5}$$

where Y is independent of X. Thus, Z is obtained by Bochner's procedure of subordinating X to Y.

The process $Z(t)$ is a semimartingale with a martingale component that is a compensated jump process. Specifically, we may write that

$$Z(t) = Z(0) + \int_0^t \int_{-\infty}^{\infty} x\nu(dx, ds) + \int_0^t \int_{-\infty}^{\infty} x(\mu(dx, ds) - \nu(dx, ds))$$

$$\nu(dx, ds) = y(s)k(x)\, ds\, dx,$$

where $\mu(dx, ds)$ is the integer-valued random measure associated with the jumps of Z and $\nu(dx, ds)$ is the compensator. The factorization of the compensator occurs as a result of the time change on noting that the speed of the economy is $y(s)\, ds$.

The compensator for the jumps of the stochastic volatility Lévy process (SVLP) models is the measure $\nu(dx, ds) = y(s)k(x)\, dx\, ds$. This representation suggests that any scaling constant in the Lévy density that is σ for the NIG model and C for the VG or CGMY models may be absorbed into a scaling of the process $y(t)$. The parameterizations we develop below incorporate this scaling absorption of the Lévy density into the process for $y(t)$ where we identify the scaling constant with the initial value for the y process.

We note that the quadratic variation expected at time s in the interval (s, t) is given by

$$\int_{-\infty}^{\infty} x^2 k(x)\, dx E\left[\int_s^t y(u)\, du\right]$$

$$= \frac{\int_{-\infty}^{\infty} x^2 k(x)\, dx}{i}\left[\frac{\partial}{\partial u} A(t-s, u)|_{u=0} + \frac{\partial}{\partial u} B(t-s, u)|_{u=0} \times y(s)\right]$$

and, unlike Lévy processes, the conditional expected quadratic variations are stochastic and adapted to the process for the time change, $y(t)$. The autocorrelation in the process for y is then expected to lead to autocorrelation in squared returns.

One could also consider a factor structure of a vector of CIR processes and also allow for CIR processes that drive the long-term volatility η, but as our interest is in synthesizing the information content of the option surface at a point of time in as few parameters as possible, we did not make this further development on observing that we had an adequate synthesis on using a single time change process.

Yet another possibility is to combine differently the time changes in the first subordination resulting in the Lévy process, for the case of NIG, and for VG where this time change is known, with the second time change for stochastic volatility. We have composed the time changes, but alternatively they could have been added to form a single time change displaying a continuous mean-reverting component and a discontinuous inverse Gaussian or gamma component. The characteristic functions for the time change would in this case be products of the characteristic function of the Heston stochastic volatility model and the NIG or VG characteristic function. This approach has recently been observed in Barndorff-Nielsen et al. (2001) and has been pursued by Sin (2002) in the context of term structure models. We did not take this route as we were interested in providing architectures for stochastic volatility for Lévy processes generally, without first writing them in the form of a subordinated Brownian motion.

The characteristic functions for these stochastic volatility enhanced Lévy processes are obtained simply as follows

$$\begin{aligned} E[\exp(iuZ(t))] &= E[\exp(Y(t)\psi_X(u))] \\ &= \phi(-i\psi_X(u), t, y(0); \kappa, \eta, \lambda). \end{aligned} \tag{3.6}$$

The specific parametrizations for the three Lévy processes are developed next.

3.1.1. The Process NIGSV. The stochastic volatility version of the NIG process is

$$Z_{\mathrm{NIG}}(t) = X_{\mathrm{NIG}}(Y(t); \sigma, \nu, \theta).$$

We note that $y(0) = \sigma$ and so we can write

$$E\exp(iuZ_{\mathrm{NIG}}(t)) = \phi(-i\psi_{\mathrm{NIG}}(1, \nu, \theta), t, \sigma; \kappa, \eta, \lambda).$$

This is a six-parameter process with parameters

$$\sigma, \nu, \theta, \kappa, \eta, \lambda.$$

3.1.2. The Process VGSV. The stochastic volatility version of the VG process is

$$\begin{aligned} Z_{\mathrm{VG}}(t) &= X_{\mathrm{VG}}(Y(t); \sigma, \nu, \theta) \\ &= X_{\mathrm{VG}}(Y(t); C, G, M), \end{aligned}$$

where the second representation employs the parameters of the Lévy density as defined in equations (2.6), (2.7), and (2.8). As commented earlier here we identify the parameter C with $y(0)$. Hence, we may write

$$E[\exp(iuZ_{\mathrm{VG}}(t))] = \phi(-i\psi_{\mathrm{VG}}(u; 1, G, M), t, C; \kappa, \eta, \lambda). \tag{3.7}$$

This is a six-parameter process with parameters

$$C, G, M, \kappa, \eta, \lambda.$$

3.1.3. The Process CGMYSV. The stochastic volatility version of the CGMY process is

$$Z_{\mathrm{CGMY}}(t) = X_{\mathrm{CGMY}}(Y(t); C_p, G, M, Y_p, Y_n, \zeta),$$

where we have replaced C_n by its ratio to C_p. The identification in this case is between C_p and $y(0)$ and we continue to use the notation C. We thus obtain that

$$E[\exp(iuZ_{\mathrm{CGMY}}(t))] = \phi(-i\psi_{\mathrm{CGMY}}(1, G, M, Y_p, Y_n, \zeta), t, C; \kappa, \eta, \lambda).$$

This is a nine-parameter process with parameters

$$C, G, M, Y_p, Y_n, \zeta, \kappa, \eta, \lambda.$$

4. THE STOCK PRICE PROCESSES

This section considers two approaches for obtaining a positive stock price process. The first approach uses the ordinary exponential function normalized to the right forward price, the second uses the stochastic exponential to construct discounted martingales. The second approach is a little more involved and has some desirable and possibly undesirable features from an economic point of view. The most desirable feature is that one easily obtains the martingale laws required by the exclusion of dynamic arbitrage. The undesirable feature is that the validity of the martingale representation is tied to the relevant filtration for the dynamics of the option surface being the space of two-dimensional paths generated by movements in the spot price and the hidden mean-reverting time change. To the extent that the time change is not observable, and to the extent that option markets are actually characterized by an informational filtration that is either higher dimensional than the specified two dimensions or is in fact completely different from this filtration, the associated martingale property is called into question in any case.

The first approach produces models that are superior in their ability to capture the information content of the option surface and aid in the task of providing an efficient synthesis of this data. This led us to inquire into the martingale properties of these models and to formulate the important concept of martingale marginals as opposed to and distinct from the idea of an equivalent martingale measure. The next section presents this concept and relates it to the absence of essentially static arbitrage, marginally enhanced to permit certain dynamic stock trades that may be arranged in advance and activated on rules written at time 0 and triggered by movements in the spot price. We subsequently inquire into whether the models obtained on a normalized ordinary exponentiation have the martingale marginal property.

4.1. Martingale Marginals and Semistatic Arbitrage

The property of martingale marginals differs substantially from the idea of martingale measures. For the latter, one begins with a complete probability space on which we seek to find martingales via measure changes. In studying martingale marginals we need not have at the start a probability space or even a stochastic process, but rather just a family of densities of random variables $\mathbb{Q} = \{q(X, t), t > 0\}$ indexed by the real number t. We say that the family of densities $\mathbb{Q}$ has the martingale marginal property if there exists a probability space on which one may define a martingale $M(t)$ such that for each t the law of $M(t)$ is given by the density $q(M, t)$. Both the filtration and the probability measure are created as part of the construction procedure of the martingale. For a more detailed discussion of the martingale marginal property with explicit constructions of martingale marginals for a variety of processes in a variety of ways we refer the reader to Madan and Yor (2002).

We now relate the property of martingale marginals to the absence of arbitrage in certain market structures. The market structure we are concerned with is that of trading vanilla options on a single underlying asset for all strikes and maturities at a fixed date in calendar time. Let $C(t; K, T)$ be the price, at calendar time t, of a call option of strike K and maturity T on an underlying asset that trades at time t for the price of $S(t)$. Also trading are bonds of all maturities and unit face value with time t prices of one dollar. For the sake of simplicity we restrict attention to the case of zero interest rates and dividend yields.

In the absence of arbitrage between options, stocks, and bonds at each maturity one may deduce from standard arguments the existence of a risk-neutral density for the stock price at future time T, $q(S, T)$ such that

$$C(t; K, T) = \int_K^\infty (S - K) q(S, T)\, dS. \tag{4.1}$$

In fact we may identify the risk-neutral density from option prices at time t using the Breeden and Litzenberger (1978) formula when interest rates and dividend yield are zero as

$$q(S, T) = \frac{\partial^2}{\partial K^2}\bigg|_{K=S} C(t; K, T). \tag{4.2}$$

It follows in particular that all the densities in the family of risk-neutral densities have a constant expectation, for with zero interest rates and dividend yields,

$$S(t) = \int_0^\infty Sq(S, T)\,dS. \tag{4.3}$$

We now enhance the collection of assets one may trade to include dynamic trades in stock where the trade is triggered by the level of the price of the asset at the earlier date. Hence, given two future time points $T_1 < T_2$ we may invest in an asset that pays at time T_2 the quantity

$$\mathbf{1}_{S(T_1)>K}(S(T_2) - S(T_1)). \tag{4.4}$$

These are well-defined European-type payoffs that are the result of zero cost positions in forwards triggered by simple rules depending on the level of the stock price at the earlier date of the forward contract. In this sense we define semistatic trading by marginally enhancing static trading and introducing a simple set of assets replicating a limited form of dynamic trading. One may explicitly introduce these payoffs as part of the time t space of European traded assets or allow for the limited dynamic trade. The prices of these assets are zero at time t.

From the inequality

$$(S(T_2) - K)^+ - (S(T_1) - K)^+ - \mathbf{1}_{S(T_1)>K}(S(T_2) - S(T_1)) \geq 0$$

we deduce on computing valuations that

$$\int_0^\infty (S - K)^+ q(S, T_2)\,dS \geq \int_0^\infty (S - K)^+ q(S, T_1)\,dS \tag{4.5}$$

or that call prices rise with maturity for a fixed strike.

It follows from a result of Rothschild and Stiglitz (1970, 1971, 1972) and Kellerer (1972) that if the family of distributions $q(S, T)$ with constant expectation has the positive calendar spread property (4.5) then there exists a martingale $M(T)$ such that for all T the density of $M(T)$ is $q(M, T)$. We see then that the absence of static arbitrage in our marginally enhanced space of static trading assets implies that the family of risk-neutral densities has the martingale marginal property. The converse is easily seen as an application of Jensen's inequality applied to the convex payoff $(S(T) - K)^+$.

Since the absence of dynamic arbitrage always precludes the absence of marginally enhanced static arbitrages as described here, the martingale marginal property is more fundamental than the existence of an equivalent martingale measure. In fact the former can be verified from a knowledge of the risk-neutral densities at a point of time in contexts where the latter is not even defined due to a lack of knowledge of the appropriate filtration with respect to which the martingale is to be formulated. It is even possible that the martingale marginal property holds and there are no equivalent martingale measures. A simple example on a binomial tree illustrates this situation.

Consider the two-period process defined on the tree of Figure (4.1). This tree displays a process with no equivalent martingale measure as there is arbitrage with the stock only rising from the time 1 down state of \$90. However, as the final probabilities for the three states are $1/4$ each for the extreme states of \$120 and \$80 at time 2, and $1/2$ for the intermediate state of \$100, a standard binomial tree with path probabilities of $1/2$ and movement from \$90 to \$100 and \$80 provides a martingale marginal process.

This example also highlights the fact that dynamic arbitrages are intimately linked with the filtrations with respect to which the process is defined. In this regard we note

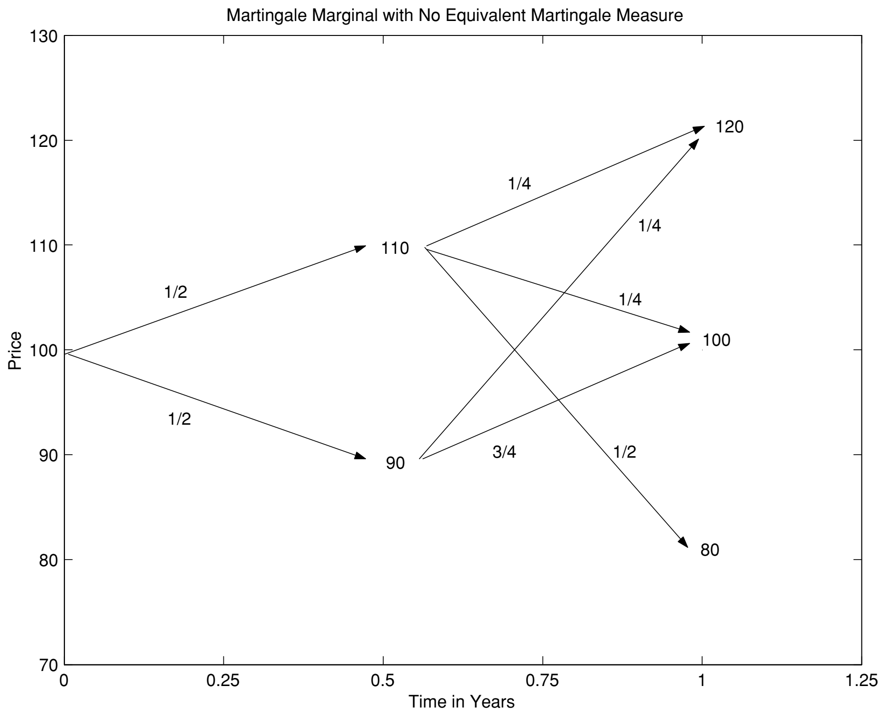

FIGURE 4.1. Two-period tree displaying a process with no equivalent martingale measure but one with martingale marginals.

that many models in the literature involve stochastic processes for hidden variables like stochastic volatility. To the extent these hidden variables are not observed, dynamic trading considerations based on their observation may well admit arbitrage that investors cannot access. If the dimension of the hidden Markov filtration is small, then one may be tempted to view the hidden filtration as observable via the prices of traded options. However, if one is correct about the size of the hidden filtration but wrong about the dynamics, either as a result of estimation error or more broadly model error (e.g., with the dynamics specified as a diffusion instead of a non-Gaussian model), then the value of the observed option prices as an indirect reading of the hidden Markov filtration is called into question. Indirect observation and direct observation must never really ever be viewed as at par or equivalent.

4.2. Ordinary Exponentials of SVLP Processes

Under this approach, the risk-neutral stock price process is given by mean correcting the exponential of a stochastic volatility Lévy process. Let $S(t)$ denote the stock price at time t and let r and q denote the constant continuously compounded interest rate and dividend yield respectively. Let $Z(t)$ be a generic SVLP as described in (3.5). We define the stock price at time t by the random variable

$$S(t) = S(0)\frac{\exp((r-q)t + Z(t))}{E[\exp(Z(t)]}. \tag{4.6}$$

Noting that

$$E[\exp(Z(t))] = \phi(-i\psi_X(-i), t, y(0); \kappa, \eta, \lambda),$$

we get that the characteristic function for the log of the stock price at time t is given by

$$E[\exp(iu\log(S(t)))] = \exp(iu(\log(S(0) + (r-q)t)) \times \frac{\phi(-i\psi_X(u), t, y(0); \kappa, \eta, \lambda)}{\phi(-i\psi_X(-i), t, y(0); \kappa, \eta, \lambda)^{iu}}. \tag{4.7}$$

The exponential stock price models obtained on using NIG, VG, and CGMY for the process $X(t)$ in defining $Z(t)$ are termed, respectively, NIGSA, VGSA, and CGMYSA. Details for these models are presented next.

4.2.1. NIGSA Characteristic Function for Log Stock Price. This characteristic function for the NIGSA process at time t is explicitly given as

$$\mathrm{NIGSACF}(u) = \exp(iu(\log(S(0) + (r-q)t)) \times \frac{\phi(-i\psi_{\mathrm{NIG}}(u; 1, \nu, \theta), t, \sigma; \kappa, \eta, \lambda)}{\phi(-i\psi_{\mathrm{NIG}}(-i; 1, \nu, \theta), t, \sigma; \kappa, \eta, \lambda)^{iu}}. \tag{4.8}$$

4.2.2. VGSA Characteristic Function for Log Stock Price. This characteristic function for the VGSA process at time t is explicitly given as

$$\mathrm{VGSACF}(u) = \exp(iu(\log(S(0) + (r-q)t)) \times \frac{\phi(-i\psi_{\mathrm{VG}}(u; 1, G, M), t, C; \kappa, \eta, \lambda)}{\phi(-i\psi_{\mathrm{VG}}(-i; 1, G, M), t, C; \kappa, \eta, \lambda)^{iu}}. \tag{4.9}$$

4.2.3. CGMYSA Characteristic Function for Log Stock Price. This characteristic function for the CGMYSA process at time t is explicitly given as

$$\mathrm{CGMYSACF}(u) = \exp(iu(\log(S(0) + (r-q)t)) \times \frac{\phi(-i\psi_{\mathrm{CGMY}}(u; 1, G, M, Y_p, Y_n, \zeta), t, C; \kappa, \eta, \lambda)}{\phi(-i\psi_{\mathrm{CGMY}}(-i; 1, G, M, Y_p, Y_n, \zeta), t, C; \kappa, \eta, \lambda)^{iu}}. \tag{4.10}$$

4.3. Stochastic Exponentials of SVLP Processes

Under this approach, martingale models for the discounted stock price are obtained by stochastically exponentiating martingales. Let $Z(t)$ be a generic SVLP. The process $Z(t)$ is a pure jump process with a predictable compensator given by

$$\rho(dx, dt) = y(t)k(x)\,dt\,dx.$$

It follows that

$$n(t) = Z(t) - \int_0^t \int_{-\infty}^{\infty} x\rho(dx, ds)$$

is a martingale. Let $\mu(dx, dt)$ be the integer-valued random measure associated with the jumps of the process $Z(t)$, so that

$$Z(t) = \int_0^t \int_{-\infty}^{\infty} x\mu(dx, ds).$$

Then $n(t)$ is the compensated jump martingale

$$n(t) = x * (\mu - \rho).$$

Now define the compensated jump martingale $m(t)$ by

$$m(t) = (e^x - 1) * (\mu - \rho)$$

and consider the stochastic exponential of $m(t)$ given by

$$M(t) = \exp\left(Z(t) - \int_0^t \int_{-\infty}^{\infty} (e^x - 1)k(x)y(s)\,dx\,ds\right).$$

Employing (3.4), we have with $Y(t) = \int_0^t y(s)\,ds$ that

$$M(t) = \exp(Z(t) - Y(t)\psi_X(-i)). \tag{4.11}$$

We may also write $M(t)$ as

$$M(t) = \exp(X(Y(t)) - Y(t)\psi_X(-i)),$$

which is the martingale

$$\exp(X(u) - u\psi_X(-i))$$

evaluated at an independent random time change $Y(t)$, and hence is also a martingale. Development of (4.11) shows that the relationship to the stochastic volatility process $Z(t)$ is precisely one of stochastically exponentiating $m(t)$. (For a related development see Carr and Wu 2002.)

This second approach to developing stock price processes adopts the formulation

$$S(t) = S(0)\exp((r - q)t)\exp(X(Y(t)) - Y(t)\psi_X(-i)). \tag{4.12}$$

In this case, the characteristic function for the log of the stock price is given by

$$\begin{aligned} E[\exp(iu\log(S(t)))] &= \exp(iu(\log(S(0) + (r - q)t)) \\ &\quad \times \phi(-i\psi_X(u) - u\psi_X(-i), t, y(0); \kappa, \eta, \lambda). \end{aligned} \tag{4.13}$$

The three special cases of interest are formulated next.

4.3.1. The NIGSAM Characteristic Function for Log Stock Price. For the NIG process at time t, the characteristic function is

$$\begin{aligned} \text{NIGSAMCF}(u) &= \exp(iu(\log(S(0) + (r - q)t)) \\ &\quad \times \phi(-i\psi_{\text{NIG}}(u, 1, \nu, \theta) - u\psi_{\text{NIG}}(-i, 1, \nu, \theta), t, \sigma; \kappa, \eta, \lambda). \end{aligned}$$

4.3.2. The VGSAM Characteristic Function for Log Stock Price. For the VG process at time t, the characteristic function is

$$\begin{aligned} \text{VGSAMCF}(u) &= \exp(iu(\log(S(0) + (r - q)t)) \\ &\quad \times \phi(-i\psi_{\text{VG}}(u, 1, G, M) - u\psi_{\text{VG}}(-i, 1, G, M), t, C; \kappa, \eta, \lambda). \end{aligned}$$

4.3.3. The CGMYSAM Characteristic Function for Log Stock Price. For the CGMY process at time t, the characteristic function is

$$\mathrm{CGMYSAMCF}(u) = \exp(iu(\log(S(0) + (r - q)t))\phi(-i\psi_{\mathrm{CGMY}}(u, 1, G, M, Y_p, Y_n, \zeta) - u\psi_{\mathrm{CGMY}}(-i, 1, G, M, Y_p, Y_n, \zeta), t, C; \kappa, \eta, \lambda).$$

5. EXPONENTIATION AND MARTINGALE MARGINALS

The NIGSA, VGSA, and CGMYSA models are formulated by writing the discounted stock price relative as a process of unit unconditional expectation obtained on exponentiating the NIGSV, VGSV, and CGMYSV processes and dividing the exponential by its mean. This formulation leads to prices free of static arbitrage since expectations are calculated with respect to a measure on the space of paths that respects spot forward arbitrage. If the log price processes had independent increments, then forward price processes would be (local) martingales because conditional expectations are now identified with unconditional expectations. However, the lack of independence in the increments of the SA processes implies that forward price processes need not be martingales, and hence these processes are subject to the possibility of dynamic arbitrage.

This section addresses the relatively deeper question of whether the SA construction has the property of martingale marginals. This question could be investigated from a computational perspective by constructing Lévy densities associated with characteristic functions for the processes, but a richer understanding of the possibilities is provided by the more structural representation of the processes pursued here.

We first ask whether there exist processes of independent increments, possibly inhomogeneous, with the same one-dimensional densities as the processes NIGSA, VGSA, or CGMYSA. Since European option prices only determine the one-dimensional density of the stock price at each maturity and we have seen that it is possible that two or more probability measures are consistent with the same option prices, it is also possible that one of these measures is a martingale measure and the other arises from the NIGSA, VGSA, or CGMYSA models. We show that there exists a very large class of martingale measures for which this is indeed true. More generally we investigate the nature of the departures when such a representation is not available.

To assist the discussion, we focus on the generic case where $(X(u), u \geq 0)$ is a homogeneous Lévy process and $Z(t) = X(Y(t))$ with $Y(t)$ defined in accordance with (3.2) and (3.1). Let $V(t)$ be a generic representation of our constant unconditional expectation process

$$V(t) = \frac{\exp(Z(t))}{E[\exp(Z(t)]}.$$

It is clear that if one constructs a process of independent and possibly inhomogeneous increments $(U(t), t \geq 0)$, that is independent of $(X(t), t \geq 0)$, but with the same one-dimensional distributions as those of $Y(t)$, then the one-dimensional distributions of $V(t)$ are those of

$$\widetilde{V}(t) = \frac{\exp(X(U(t))}{E[\exp(X(U(t))]},$$

where now $\widetilde{V}(t)$ is a process of independent multiplicative increments and a martingale. This leads us to focus our attention on representing the one-dimensional distributions of the process $Y(t)$.

The process $Y(t)$ has three parameters κ, η, and λ and is useful to employ scaling changes to relate the process to the case where $\lambda = 2$. In fact, if we let

$$h(t) = \frac{4}{\lambda^2} y(t)$$

then $H(t) = \int_0^t h(s)\, ds = \frac{4}{\lambda^2} Y(t)$ and an application of Ito's lemma shows that $h(t)$ satisfies the stochastic differential equation

$$dh = \left(\frac{4\kappa\eta}{\lambda^2} - \kappa h \right) dt + 2\sqrt{h} dW(t).$$

To simplify the notation and to better relate to the results in Pitman and Yor (1982), we introduce the stochastic differential equation

$$dh = (\delta + 2\beta h)\, dt + 2\sqrt{h} dW(t), \tag{5.1}$$

where our case of interest is $\delta = \frac{4\kappa\eta}{\lambda^2}$ and $\beta = -\frac{\kappa}{2}$. We denote by ${}^{\beta}Q_x^{\delta}$ the law of the process $h(t)$ satisfying (5.1) and starting at $h(0) = x$. It is well known that for fixed β, this two-parameter family enjoys the additivity property

$${}^{\beta}Q_x^{\delta} * {}^{\beta}Q_{x'}^{\delta'} = {}^{\beta}Q_{x+x'}^{\delta+\delta'}. \tag{5.2}$$

Furthermore, as shown by Shiga and Watanabe (1973), these diffusions are (up to a trivial homothetic change of variable) the only family of $\mathbb{R}_+$-valued diffusions to have this additivity property. Denoting the solution by $(h(u), u \geq 0)$, (5.2) implies that for every nonnegative measure $\mu(ds)$ on $\mathbb{R}_+$ and every $t \geq 0$, the random variable

$$I_{\mu,t}(h) = \int_0^t \mu(ds) h(s)$$

is infinitely divisible under the law ${}^{\beta}Q_x^{\delta}$, with parameters of infinite divisibility x and δ. Its Lévy Khintchine representation is studied in Pitman and Yor (1982). In fact, Pitman and Yor used classical Ray-Knight theorems on Brownian local times (among other arguments) to show the existence, for given β, of two σ-finite measures ${}^{\beta}M$, and ${}^{\beta}N$ on $C(\mathbb{R}_+, \mathbb{R}_+)$ such that

$${}^{\beta}Q_x^{\delta}(\exp(-\gamma I_{\mu,t})) = \exp\left(- \int (x\, {}^{\beta}M + \delta\, {}^{\beta}N)(dh)(1 - e^{-\gamma I_{\mu,t}}) \right)$$

The Lévy measure associated to $I_{\mu,t}$ under ${}^{\beta}Q_x^{\delta}$ is

$$x\, {}^{\beta}m_{\mu,t} + \delta\, {}^{\beta}n_{\mu,t},$$

where ${}^{\beta}m_{\mu,t}$, ${}^{\beta}n_{\mu,t}$ are the images of ${}^{\beta}M$, ${}^{\beta}N$ by the mapping $h \longrightarrow I_{\mu,t}(h)$. A number of computations of these Lévy measures are found in Pitman and Yor (1982).

We are interested here in yet another possible infinite divisibility property. Specifically, for a given "reasonable" μ, we wish to determine whether the marginals of the process $(I_{\mu,t}(h), t \geq 0)$ are those of a process with inhomogeneous independent increments. For simplicity, we take $\mu(ds) = ds$ as the Lebesgue measure and we say that the process $H(t) = \int_0^t h(u)\, du$ has the Lévy marginal (LM) property if there exists an inhomogeneous Lévy process $(\theta(t), t \geq 0)$ such that for any given t

$$H(t) \stackrel{(d)}{=} \theta(t).$$

Our main result is the following.

THEOREM 5.1. *Let $\beta \in \mathbb{R}$, and let p, q be two reals.*

(i) The process $(Y_{p,q}(t) = py(t) + q\int_0^t y(s)\,ds,\ t \geq 0)$, under ${}^{\beta}Q_0^{\delta}$ enjoys the LM property.
(ii) Let $x \neq 0$. The process $(Y_{0,1}(t) = \int_0^t y(s)\,ds,\ t \geq 0)$ considered under ${}^{\beta}Q_x^{\delta}$ does not enjoy the LM property.

We first deal with the case $\delta = 2$. For this case, the theorem is a consequence of the following theorem. (Proofs of both theorems are given in the Appendix.)

THEOREM 5.2. *Let $({}^{(-\mu)}y(a), a \geq 0)$ denote a process distributed as ${}^{(-\mu)}Q_0^2$. Then one has*
(a)

(5.3)

$$\left({}^{-(\mu)}y(b); \int_0^b da\, {}^{(-\mu)}y(a)\right) \stackrel{(d)}{=} \left(\ell^0_{T_b}\left(X^{(\mu)}\right); \int_0^{T_b(X^{(\mu)})} ds \mathbf{1}_{\left(X_s^{(\mu)}>\right)}\right), \quad \textit{for every } b \geq 0,$$

where $(X_t^{(\mu)}, t \geq 0)$ is the solution of

$$X_t = B_t + \mu \int_0^t ds \mathbf{1}_{(X_s>0)} \tag{5.4}$$

and $T_b(X^{\mu}) = \inf\{t \geq 0 : X_t^{\mu} = b\}$.
(b) There is the identity

$$\left(\int_0^{T_b(X^{(\mu)})} ds \mathbf{1}_{\left(X_s^{(\mu)}>0\right)}, b \geq 0\right) \stackrel{(d)}{=} \left(T_b\left(\left|Z^{(\mu)}\right|\right), b \geq 0\right),$$

where $(Z_t^{(\mu)}, t \geq 0)$ is the solution to

$$Z_t = \gamma_t + \mu \int_0^t sgn(Z_s)\,ds, \tag{5.5}$$

with $(\gamma_t, t \geq 0)$ a Brownian motion.
(c) The identity in law,

$$\left(\left|Z_t^{(\mu)}\right|, t \geq 0\right) \stackrel{(d)}{=} \left(S_t\left(\beta^{(-\mu)}\right) - \beta_t^{(-\mu)}, t \geq 0\right),$$

holds, where on the right-hand side $\beta_t^{(-\mu)} = \beta_t - \mu t$ for a Brownian motion β_t, and $S_t(\theta) = \sup_{s \leq t}\theta_s$

From these results, we may write the law

$${}^{\beta}Q_x^{\delta} = {}^{\beta}Q_0^{\delta} * {}^{\beta}Q_x^{0}$$

and, hence, we may write

$${}^{\beta}y_x^{\delta}(t) = {}^{\beta}y_0^{\delta}(t) + {}^{\beta}y_x^{0}(t),$$

where the processes ${}^{\beta}y_0^{\delta}$, ${}^{\beta}y_x^{0}$ are independent. On integrating, we obtain

$${}^{\beta}Y_0^{\delta}(t) = \int_0^t {}^{\beta}y_0^{\delta}(u)\,du$$

$$^{\beta}Y_x^0(t) = \int_0^t {}^{\beta}y_x^0(u)\,du.$$

The marginals of the process $X(Y(t))$ now agree with the marginals of $X({}^{\beta}Y_0^{\delta}(t)) + X({}^{\beta}Y_x^0(t))$ and hence we may write

$$\frac{\exp(X(Y(t)))}{E[\exp(X(Y(t)))]} \overset{(d)}{=} \frac{\exp\left(X\left({}^{\beta}Y_0^{\delta}(t)\right)\right)}{E\left[\exp\left(X\left({}^{\beta}Y_0^{\delta}(t)\right)\right)\right]}\,\frac{\exp\left(X\left({}^{\beta}Y_x^0(t)\right)\right)}{E\left[\exp\left(X\left({}^{\beta}Y_x^0(t)\right)\right)\right]}$$
$$\overset{def}{=} M(t)U(t).$$

The process $M(t)$ has the multiplicative LM property and there exists an inhomogeneous Levy process of independent multiplicative increments with unit unconditional expectations and the same marginal distributions. In particular, this inhomogeneous Lévy process is also a martingale. On the other hand, the process $U(t)$ does not have the LM property. Hence, its one-dimensional distributions may not be consistent with a martingale process by such an argument.

Some properties of the process $U(t)$ are worthy of note. First, we observe that ${}^{\beta}y_x^0(t)$ starts at x, but is eventually absorbed at 0. The distribution of the first hitting time of 0 by the process $y_x^0(t)$ is (see Yor 1992; Getoor 1979)

$$T^0\left(y_x^0\right) \overset{(d)}{=} \frac{x}{2e},$$

where e is a standard exponential random variable. More generally, for general β we have that

$$P[T^0 \leq s] = \exp\left(-\frac{\kappa x/2}{\exp(\kappa s) - 1}\right).$$

It follows that the numerator in the expression for $U(t)$ is eventually constant. The process is a smooth differentiable process that is unconditionally absent but serves conditionally as a random drift component perturbing the martingale $M(t)$. We may interpret this conditional drift as a conditional abnormal return that is unconditionally absent and eventually zero.

Leaving aside these considerations, we now introduce the property of martingale marginals (MM) for processes. We say that a process $H(t)$ of constant expectation has the property of MM just if there exists a martingale $N(t)$ with the same marginal distributions as those for $H(t)$ for each t. The process $U(t)$ may possess the property of martingale marginals, and by such a decomposition we could write martingale laws for the class of SA processes defined here.

The LM and MM properties introduced here are related in that if $(\widetilde{L}_t)$ satisfies the LM property, then $(\widetilde{M}_t) = \frac{\exp(\widetilde{L}_t)}{E[\exp(\widetilde{L}_t)]}$ satisfies the MM property. A priori, the converse does not hold; that is, if $(\widetilde{M}_t)$ satisfies the MM property, there does not necessarily exist $(\widetilde{L}_t)$ satisfying the LM property.

6. OTHER BDLPs AND LEVERAGE

We consider here the strategy of using a one-sided jump process for the background driving Lévy process (in the terminology of Barndorff-Nielsen and Shephard 2001) that drives an OU process for the economic speed or the rate at which time changes. Additionally, by allowing the jumps in the BDLP to directly impact the stock price, we incorporate the effects of leverage as well. The ideas of this section are closely related to those of Nicolato and Venardos (2001), with the difference that we time change a pure jump Lévy process but they time change a diffusion. For us this requires that certain functionals of Lévy cumulants, identified below, be integrable in closed form. In section 7 we extend VGSA to incorporate leverage in that model as well.

The formal structure is as follows. Let $X(t)$ be a homogeneous Lévy process. Furthermore let $U(t)$ be a one-sided jump process with independent and homogeneous increments—that is, a one-sided Lévy process. Examples that we shall use are the process with exponential jump sizes and a constant Poisson arrival rate under which the stationary distribution for speed of the economy is a gamma distribution and we refer to this as SG for stationary gamma. Other examples are the IG inverse Gaussian process or the time taken by a Brownian with drift to reach level t. Finally we consider SIG or a BDLP such that the stationary distribution for the speed of the economy is the inverse Gaussian distribution at unit time.

The process for the speed of the economy at time t, $y(t)$, is given by a solution to the OU equation

$$dy = -\kappa y dt + dU(t). \tag{6.1}$$

Solutions to such equations have also recently been extensively studied by Sato (1991, 1999) and Jurek and Vervaat (1983). In particular, the stationary solutions $V(t)$ to this equation characterize a subclass of infinitely divisible laws called *self-decomposable laws*. The process $y(t)$ may then be seen as a scaled and time changed form of $V(t)$; specifically we have (Lamperti 1962) that

$$y(t) = t^{\kappa} V(\log(t)).$$

We take the total elapsed economic time to be

$$Y(t) = \int_0^t y(s)\, ds. \tag{6.2}$$

The uncertainty embedded in the stock price is now given by $Z(t)$, where

$$Z(t) = X(Y(t)) + \rho U(t) \tag{6.3}$$

and the coefficient ρ allows for correlation between the Lévy volatility and the stock return.

We present details for both our approaches of exponentiating and using the stochastic exponential. In the first architecture, we have as before that $S(t)$ is given by equation (4.6) and the forward prices for fixed future delivery dates are processes of constant expectation. In the second architecture we follow the principle of equation (4.12).

The stock price process for the second architecture is given by

$$\begin{aligned} S(t) &= S(0)e^{(r-q)t} M(t) \\ &= S(0)e^{(r-q)t} \exp\left(Z(t) - \int_0^t \int_{-\infty}^{\infty} (e^x - 1)\, \nu(dx, ds) \right), \end{aligned} \tag{6.4}$$

where ν is the predictable compensator for the jumps in Z.

For a given specification for X, say NIG, VG, and CGMY, and choice of Z, say SG, IG, and SIG, we obtain on exponentiation the models VGSG, NIGSG, CGMYSG, NIGIG, . . . , CGMYSIG. For all these models when we use stochastic exponentiation we obtain the martingale models that we denote with the extension M, to get the models VGSGM, NIGSGM, CGMYSGM, NIGIGM, ..., CGMYSIGM. For the three homogeneous Lévy processes, three BDLPs, and two forms of exponentiation we have 18 models.

6.1. Generic Construction of Characteristic Function for the Log Stock Price

Suppose that the homogeneous Lévy process has the unit time log characteristic function $\psi_X(u)$ as defined by equation (3.4).

Further suppose that the BDLP has the characteristic function $\phi_U(u)$,

$$\begin{aligned}\phi_U(u) &= E\left[e^{iuU(t)}\right]\\ &= \exp(t\psi_U(u))\\ &= \exp\left(t\int_{-\infty}^{\infty}(e^{iux}-1)k_U(x)\,dx\right),\end{aligned}$$

where $\psi_U(u)$ is the log characteristic function and $k_U(x)$ is the Lévy density which is identically zero for $x < 0$.

The process $U(t)$ may be written in terms of its integer-valued random measure $\mu_U(dz, ds)$ as

$$\begin{aligned}U(t) &= (x * \mu_U)_t\\ &= \int_0^t\int_0^{\infty} x\mu_U(dx, ds).\end{aligned}$$

6.1.1. Characteristic Function for $Z(t)$. We begin by considering the construction of the characteristic function for $Z(t)$.

By definition we have that

$$\begin{aligned}E[e^{iuZ(t)}] &= E\left[e^{iuX(Y(t))+iu\rho U(t)}\right]\\ &= E\left[\exp\left(Y(t)\psi_X(u) + iu\rho U(t)\right)\right].\end{aligned}$$

We now develop an expression for the joint characteristic function of $Y(t)$ and $U(t)$,

$$\Phi_t(a, b) = E[\exp(iaY(t) + ibU(t))], \tag{6.5}$$

and note that the desired characteristic function is

$$\phi_Z(u) = \Phi_t(-i\psi_X(u), \rho u). \tag{6.6}$$

To construct the joint characteristic function we note that

$$y(s) = y(0)e^{-\kappa s} + \int_0^s\int_0^{\infty} e^{-\kappa(s-u)}x\mu_U(dx, du).$$

It follows that

$$\begin{aligned}Y(t) &= y(0)\frac{1-e^{-\kappa t}}{\kappa} + \int_0^t\int_0^s\int_0^{\infty} e^{-\kappa(s-u)}x\mu_U(dx, du)\,ds\\ &= y(0)\frac{1-e^{-\kappa t}}{\kappa} + \int_0^t\int_0^{\infty}\frac{1-e^{-\kappa(t-u)}}{\kappa}x\mu_U(dx, du).\end{aligned}$$

Hence

$$iaY(t) + ibZ(t) = iay(0)\frac{1-e^{-\kappa t}}{\kappa} + \int_0^t \int_0^\infty \left(ia\frac{1-e^{-\kappa(t-u)}}{\kappa} + ib\right) x\mu_U(dx, ds)$$

and

$$\Phi_t(a, b) = \exp\left(iay(0)\frac{1-e^{-\kappa t}}{\kappa}\right) E\left[\exp\left(\int_0^t \int_0^\infty \left(ia\frac{1-e^{-\kappa(t-u)}}{\kappa} + ib\right) x\mu_U(dx, ds)\right)\right].$$

Define the compensated jump martingale $n(s)$ by

$$n(s) = \left(\exp\left(\left(ia\frac{1-e^{-\kappa(t-u)}}{\kappa} + ib\right) x\right) - 1\right) * (\mu_U(dx, ds) - k_U(x)\, dxds)$$

and note that its stochastic exponential is the martingale

$$\begin{aligned} N(s) = \exp&\left(\int_0^s \int_0^\infty \left(ia\frac{1-e^{-\kappa(t-u)}}{\kappa} + ib\right) x\mu_U(dx, ds)\right) \\ &\times \exp\left(-\int_0^s \int_0^\infty \left(\exp\left(\left(ia\frac{1-e^{-\kappa(t-u)}}{\kappa} + ib\right) x\right) - 1\right) k_U(x)\, dxdu\right). \end{aligned}$$

In particular, noting that the expectation of $N(t) = 1$, we get that

$$\begin{aligned} E&\left[\exp\left(\int_0^t \int_0^\infty \left(ia\frac{1-e^{-\kappa(t-u)}}{\kappa} + ib\right) x\mu_U(dx, ds)\right)\right] \\ &= \exp\left(\int_0^t \int_0^\infty \left(\exp\left(\left(ia\frac{1-e^{-\kappa(t-u)}}{\kappa} + ib\right) x\right) - 1\right) k_U(x)\, dx\, du\right) \\ &= \exp\left(\int_0^t \psi_U\left(a\frac{1-e^{-\kappa(t-u)}}{\kappa} + b\right) du\right). \end{aligned}$$

Making the change of variable

$$v = b + a\frac{1-e^{-\kappa(t-u)}}{\kappa}$$

in the exponent we may write the result as

$$\begin{aligned} &\Phi_t(a, b) = \exp\left(iay(0)\frac{1-e^{-\kappa t}}{\kappa}\right) \exp\left(\int_L^U \frac{\psi_U(v)}{a + \kappa b - \kappa v}\, dv\right), \\ &L = b, \\ &U = b + a\frac{1-e^{-\kappa t}}{\kappa}. \end{aligned} \tag{6.7}$$

We have closed-form solutions for the characteristic function of $Z(t)$ provided we integrate the log characteristic function of $U(t)$ divided by a linear function in closed form. This is accomplished below for the BDLPs SG, IG, and SIG that are briefly described in the next subsection.

6.1.2. Characteristic Function for the Log Stock Price: Exponential Model. Here the stock is given by equation (4.6) and

$$\begin{aligned} E[\exp(iu\log(S(t))] &= \exp(iu(\log(S(0)+(r-q)t)) \\ &\times \Phi_t(-i\psi_X(u), \rho u) \\ &\times \exp(-iu\log(\Phi_t(-i\psi_X(-i), -i\rho)). \end{aligned}$$

6.1.3. Characteristic Function for the Log Stock Price: Stochastic Exponential Model. Here the stock price is given by equation (6.4) and we need to determine the compensator for the jump process $Z(t)$.

The process $X(Y(t))$ has compensator

$$y(s)k_X(x)\,dxds,$$

the process $\rho U(t)$ has compensator

$$k_U\left(\frac{x}{\rho}\right)\frac{1}{\rho}\,dxds.$$

It follows that the compensator for $Z(t)$ is

$$\nu(dx, ds) = \left[y(s)k_X(x) + k_U\left(\frac{x}{\rho}\right)\frac{1}{\rho}\right]dxds.$$

Computing the integral in the exponent of equation (6.4) we see that

$$M(t) = \exp(Z(t) - Y(t)\psi_X(-i) - t\psi_U(-i\rho)).$$

It follows on evaluation that

$$\begin{aligned} E[\exp(iu\log(S(t))] &= \exp(iu(\log(S(0)) + (r-q)t - \psi_U(-i\rho)t)) \\ &\times \Phi_t(-i\psi_X(u) - u\psi_X(-i), \rho u). \end{aligned}$$

This completes the generic construction of the characteristic function for the log of the stock price. We now develop the joint characteristic function for Y, U in the special cases of the stationary gamma, inverse Gaussian, and stationary inverse Gaussian.

6.1.4. The SG, IG, and SIG BDLPs. We consider three models for the process $U(t)$. These are termed stationary gamma, inverse Gaussian, and stationary inverse Gaussian. The stationary gamma name refers to the fact that in this case the stationary solution to the OU equation is a gamma density. A similar motivation underlies the nomenclature SIG. For further details on these the reader is referred to Nicolato and Venardos (2001).

6.2. The SG Case

Consider by way of an example the process with Poisson arrival rate λ of positive jumps exponentially distributed with mean ζ. In this case the Lévy density for jumps in the process $Z(t)$ is

$$k_U(x) = \frac{\lambda}{\zeta}e^{-x/\zeta}.$$

The log characteristic function of the BDLP is

$$\psi_U(u) = \frac{iu\lambda}{1/\zeta - iu}.$$

We wish to integrate

$$\int \frac{iu\lambda}{(1/\zeta - iu)(a + \kappa b - \kappa u)}\, du.$$

We may write this as

$$\lambda \int \frac{u}{(1/(i\zeta) - u)(a + \kappa b - \kappa u)}\, du,$$

which is of the form

$$\lambda \int \frac{x}{(\alpha - x)(\gamma + \delta x)}\, dx,$$
$$\alpha = -i/\zeta$$
$$\gamma = a + \kappa b$$
$$\delta = -\kappa.$$

The indefinite integral is

$$-\frac{\lambda\alpha \log(x - \alpha)}{\gamma + \alpha\delta} - \frac{\lambda\gamma \log(\gamma + \delta x)}{\delta(\gamma + \alpha\delta)}$$
$$= \log\left[\left(\frac{1}{x - \alpha}\right)^{\frac{\lambda\alpha}{\gamma+\alpha\delta}} \left(\frac{1}{\gamma + \delta x}\right)^{\frac{\lambda\gamma}{\delta(\gamma+\alpha\delta)}}\right]$$
$$= \log\left[\left(x + \frac{i}{\zeta}\right)^{\frac{\lambda}{(\kappa - i\zeta(a+\kappa b))}} (a + \kappa b - \kappa x)^{\frac{\lambda(a+\kappa b)\zeta}{\kappa((a+\kappa b)\zeta + i\kappa)}}\right].$$

We evaluate this expression at $x = b + \frac{a(1 - e^{-\kappa t})}{\kappa}$ and subtract the evaluation at $x = b$.

6.3. The IG Case

The Laplace transform for inverse Gaussian time with drift ν for the Brownian motion is

$$E\left[\exp\left(-\lambda T_1^{\nu}\right)\right] = \exp\left(\nu - \sqrt{\nu^2 + 2\lambda}\right)$$

and the log characteristic function is

$$\psi_U(u) = \nu - \sqrt{\nu^2 - 2iu}.$$

In this case we wish to integrate

$$\int \frac{\nu - \sqrt{\nu^2 - 2iu}}{a + \kappa b - \kappa u}\, du.$$

This expression is of the form

$$\int \frac{\alpha - \sqrt{\alpha^2 + \beta x}}{\gamma + \delta x}\, dx,$$

$$\alpha = \nu$$
$$\beta = -2i$$
$$\gamma = a + \kappa b$$
$$\delta = -\kappa.$$

The indefinite integral is

$$-2\frac{\sqrt{\alpha^2+\beta x}}{\delta} - \frac{2(\beta\gamma-\alpha^2\delta)}{\delta^{3/2}\sqrt{\alpha^2\delta-\beta\gamma}}\operatorname{arctanh}\left[\frac{\sqrt{\delta}\sqrt{\alpha^2+\beta x}}{\sqrt{\alpha^2\delta-\beta\gamma}}\right] + \frac{\alpha\log(\gamma+\delta x)}{\delta}.$$

In terms of the original parameters we have

$$\frac{2\sqrt{\nu^2-2ix}}{\kappa} + \frac{2}{\kappa^{3/2}}\frac{\nu^2\kappa-2i(a+\kappa b)}{\sqrt{\nu^2\kappa-2i(a+\kappa b)}}\operatorname{arctanh}\left[\frac{\sqrt{\kappa}\sqrt{\nu^2-2ix}}{\sqrt{\nu^2\kappa-2i(a+\kappa b)}}\right]$$
$$-\frac{\nu\log(a+\kappa b-\kappa x)}{\kappa}.$$

6.4. The SIG Case

For this case Barndorff-Nielsen and Shephard (2001) show that the Lévy density is

$$k_U(x) = \frac{1}{2\sqrt{2\pi}}x^{-3/2}(1+\nu^2x)\exp\left(-\frac{\nu^2x}{2}\right).$$

The log characteristic function is

$$\psi_U(u) = \frac{iu}{\sqrt{\nu^2-2iu}}$$

and we wish to integrate

$$\int \frac{iu}{\sqrt{\nu^2-2iu}\,(a+\kappa b-\kappa u)}\,du.$$

This is of the form

$$i\int \frac{x}{\sqrt{\alpha^2+\beta x}(\gamma+\delta x)}\,dx,$$
$$\alpha = \nu$$
$$\beta = -2i$$
$$\gamma = a + \kappa b$$
$$\delta = -\kappa,$$

and the indefinite integral is

$$i\left[\frac{2\sqrt{\alpha^2+\beta x}}{\beta\delta} + \frac{2\gamma}{\delta^{3/2}\sqrt{\alpha^2\delta-\beta\gamma}}\operatorname{arctanh}\left[\frac{\sqrt{\delta}\sqrt{\alpha^2+\beta x}}{\sqrt{\alpha^2\delta-\beta\gamma}}\right]\right]$$
$$= \frac{\sqrt{\nu^2-2ix}}{\kappa} - \frac{2i(a+\kappa b)}{\kappa^{3/2}\sqrt{\nu^2\kappa-2i(a+\kappa b)}}\operatorname{arctanh}\left[\frac{\sqrt{\kappa}\sqrt{\nu^2-2ix}}{\sqrt{\nu^2\kappa-2i(a+\kappa b)}}\right].$$

6.5. Summary of SG, IG, and SIG Results

We have the following result

$$\int_L^U \frac{\psi_Z(v)}{a+\kappa b-\kappa v}\,dv = \Psi(U,a,b)-\Psi(L,a,b)$$

with analytic expressions for $\Psi(x,a,b)$ in the SG, IG, and SIG cases that we summarize as follows:

$$\Psi_{\mathrm{SG}}(x,a,b;\kappa,\lambda,\zeta) = \log\left[\left(x+\frac{i}{\zeta}\right)^{\frac{\lambda}{\kappa-i\zeta(a+\kappa b)}}(a+\kappa b-\kappa x)^{\frac{\lambda(a+\kappa b)\zeta}{\kappa((a+\kappa b)\zeta+i\kappa)}}\right],$$

$$\begin{aligned}\Psi_{\mathrm{IG}}(x,a,b;\kappa,\nu) &= \frac{2\sqrt{\nu^2-2ix}}{\kappa}+\frac{2\sqrt{\nu^2\kappa-2i(a+\kappa b)}}{\kappa^{3/2}}\\ &\quad\times\operatorname{arctanh}\left[\frac{\sqrt{\kappa}\sqrt{\nu^2-2ix}}{\sqrt{\nu^2\kappa-2i(a+\kappa b)}}\right]\\ &\quad-\frac{\nu\log(a+\kappa b-\kappa x)}{\kappa},\end{aligned}$$

$$\begin{aligned}\Psi_{\mathrm{SIG}}(x,a,b;\kappa,\nu) &= \frac{\sqrt{\nu^2-2ix}}{\kappa}-\frac{2i(a+\kappa b)}{\kappa^{3/2}\sqrt{\nu^2\kappa-2i(a+\kappa b)}}\\ &\quad\times\operatorname{arctanh}\left[\frac{\sqrt{\kappa}\sqrt{\nu^2-2ix}}{\sqrt{\nu^2\kappa-2i(a+\kappa b)}}\right].\end{aligned}$$

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7. LEVERAGE IN VGSA

We consider the introduction of correlation in VGSA along the following lines. Let $X(t)$ be the VG process and define the time change $Y(t)$ in accordance with equations (3.1) and (3.2). We take as the model for the uncertainty in the stock the process

$$Z(t) = X(Y(t)) + \rho y(t).$$

In this way the disturbances affecting the level of volatility also impact the stock directly. Any impact of the drifts will be absorbed by the constant forward or martingale construction and should be a serious effect. What makes this line of attack feasible is that the characteristic function for $Z(t)$, the fundamental entity for most of our work on identifying stochastic processes, has the structure

$$E\left[e^{iuZ(t)}\right] = E\left[e^{Y(t)\psi_X(u)+iu\rho y(t)}\right]$$

and we only need to evaluate the joint characteristic function for $Y(t)$, $y(t)$ at the point $(-i\psi_X(u), u\rho)$. Hence, defining

$$\Phi_t(a,b,x) = E[\exp(iaY(t)+iby(t)) \mid y(0)=x],$$

we have that

$$\phi_Z(u) = \Phi_t(-i\psi_X(u), \rho u).$$

We recall the solution for $\Phi_t(a, b, x)$ from Lamberton and Lapeyre (1996) and Pitman and Yor (1982) as

$$\begin{aligned}
\Phi_t(a, b, x) &= A(t, a, b)\exp(B(t, a, b)x),\\
A(t, a, b) &= \frac{\exp\left(\frac{\kappa^2\eta t}{\lambda^2}\right)}{\left[\cosh\left(\frac{\gamma t}{2}\right) + \frac{(\kappa - ib\lambda^2)}{\gamma}\sinh\left(\frac{\gamma t}{2}\right)\right]^{2\kappa\eta/\lambda^2}},\\
B(t, a, b) &= \frac{ib\left[\gamma\cosh\left(\frac{\gamma t}{2}\right) - \kappa\sinh\left(\frac{\gamma t}{2}\right)\right] + 2ia\sinh\left(\frac{\gamma t}{2}\right)}{\gamma\cosh\left(\frac{\gamma t}{2}\right) + (\kappa - ib\lambda^2)\sinh\left(\frac{\gamma t}{2}\right)},\\
\gamma &= \sqrt{\kappa^2 - 2\lambda^2 ia}.
\end{aligned}$$

We get the characteristic function for the model VGCSA where the letter C denotes correlated stochastic arrival by exponentiation as

$$\begin{aligned}
E[\exp(iu\log(S(t))] = {} & \exp(iu(\log(S(0)) + (r - q)t)\\
& \times \Phi_t(-i\psi_X(u), \rho u)\\
& \times \exp(-iu\log(\Phi_t(-i\psi_X(-i), -i\rho)).
\end{aligned}$$

8. THE DATA AND ESTIMATION PROCEDURE

We obtained data on out-of-the-money S&P 500 closing option prices for maturities between a month and a year for the second Wednesday of each month for the year 2000. This provides us with a monthly time series of option prices on a single but important underlying asset. The dates employed were Jan. 12, Feb. 9, Mar. 8, Apr. 12, May 10, Jun. 14, Jul. 12, Aug. 9, Sept. 13, Oct. 11, Nov. 8, and Dec. 13. Similar data were obtained for some 20 other underliers. By ticker symbol, they are BA, BKX, CSCO, DRG, GE, HWP, IBM, INTC, JNJ, KO, MCD, MSFT, ORCL, PFE, RUT, SUNW, WMT, XAU, XOI, and XOM.

For each model and each underlier, we followed a uniform procedure for constructing the option price. In particular, we used the fast Fourier transform (FFT) to invert the generalized Fourier transform of the call price, as developed in Carr and Madan (1998). This generalized Fourier transform is analytic whenever the characteristic function for the log of the stock price is analytic. More precisely, let $C(k, t)$ be the price of a call option with strike $\exp(k)$ and maturity t. Let a be a positive constant such that the ath moment of the stock price exists. Carr and Madan (1998) showed that

$$\begin{aligned}
\gamma(u, t) &= \int_{-\infty}^{\infty} e^{iuk}e^{\alpha k}C(k, t)\,dk\\
&= \frac{e^{-rt}\zeta(u - i(\alpha + 1), t)}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u},
\end{aligned}$$

where $\zeta(u, t)$ denotes the characteristic function for the log of the stock price. The call prices follow on performing the FFT integration

$$C(k, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuk-\alpha k}\gamma(u, t)\,du.$$

Put option prices are obtained using put-call parity.

One advantage of this procedure is that all models may be handled with a single code because a model change only involves changing the specific characteristic function that is called. Furthermore, as the FFT works equally well on matrix structures, all strikes and maturities may be simultaneously computed in a very efficient manner. This is a very desirable property when we consider that parameters have to be estimated within an optimization algorithm.

The model parameters in each case are estimated by minimizing the root mean square error between market close prices and model option prices. The root mean square error is taken here over all strikes and maturities. We also compute the average absolute error as a percentage of the mean price. For comparative purposes, we report this statistic as an overall measure of the quality of the fit.

9. RESULTS OF ESTIMATIONS FOR THE YEAR 2000

The results are presented in four tables and two graphs. With a view to a concise presentation we report results just for the S&P 500 Index options. Table 9.1 presents the results of regressing the absolute errors on moneyness and maturity for the three models VGSA, NIGSA, and CGMYSA. Then, in the interest of brevity, but with a view to displaying typical parameter values appropriate for each model, we present quarterly results for each of the SA and SAM models on the SPX in Tables 9.2 and 9.3 respectively. Finally, in Table 9.4 we present estimates for just one day, December 13, 2000, for only the VG Lévy process.

Each of the six CIR-based models was estimated on S&P 500 Index options using one day from each of the 12 months for the year 2000. A comparison of the average percentage pricing errors shows that the exponential method dominates the stochastic exponential method in all cases. The average improvements of the exponential over the stochastic exponential in the three cases of NIG, VG, and CGMY were 3.62%, 2.87%, and 2.70% respectively. We present in Figure 9.1 a graph of the percentage absolute pricing errors for each of the six models over the 12 months.

The domination of the mean-corrected exponential over the martingale stochastic exponential is markedly evident. Thus, for the rest of the reporting, we restrict attention to the ordinary exponential models.

TABLE 9.1
Results on Regressing the Absolute Percentage Errors for the Three Exponential Models on Moneyness, Its Square, and Its Maturity*

Model	Constant	Moneyness	Moneyness^2	Maturity	R^2
VGSA	7.7628	−15.6155	7.8784	−.1220	49.07
	(43.62)	(−42.01)	(40.40)	(−4.93)	
NIGSA	8.6899	−17.3529	8.7147	−.1813	51.62
	(45.99)	(−43.98)	(42.10)	(−6.89)	
CGMYSA	8.043	−16.2076	8.2097	−.1307	47.12
	(42.20)	(−40.72)	(39.31)	(−4.93)	

* t statistics are given in parentheses.

TABLE 9.2
Parameter Estimates for the SA Models on the SPX at the Quarterly Points

Quarter	Parameter estimates — VGSA model C	G	M	κ	η	λ	ape
Mar.	26.23	26.34	42.47	4.08	15.99	16.52	.0545
Jun.	48.98	40.62	62.14	7.24	32.15	24.81	.0394
Sep.	12.19	21.26	42.95	0.25	0	3.76	.0536
Dec.	15.75	19.14	38.69	2.18	5.71	5.67	.0242

	NIGSA model σ	ν	θ	κ	η	λ	ape
Mar.	1.14	24.24	−8.18	4.23	0.6925	3.51	.0551
Jun.	2.50	63.52	−11.28	8.71	1.63	6.53	.0412
Sep.	.6038	20.62	−10.65	.9225	.1734	.92	.0303
Dec.	.7878	18.93	−10.63	1.91	.2246	1.124	.0238

	CGMYSA model C	G	M	Y_P	Y_n	ζ	κ	η	λ	ape
Mar.	3.99	13.75	52.97	.5323	.5764	.5280	2.67	2.93	5.72	.0523
Jun.	5.48	21.87	66.41	.6423	.8376	.1827	7.64	3.57	8.47	.0392
Sep.	1.69	12.24	48.72	.6359	.7152	.4707	.4675	.6772	1.60	.0287
Dec.	37.66	19.78	192.20	.2893	.3291	.2022	1.63	22.89	10.83	.0206

The best-performing model by far was the CGMYSA model, as it consistently had the lowest pricing errors. The parameters of this model were also more stable across time. The mean pricing errors for the models NIGSA, VGSA, and CGMYSA were 4.1346%, 4.4251%, and 3.3738% respectively. Among these six models, the tentatively best model was the CGMYSA. The best overall fit was for December 13, 2000, for CGMYSA, and we present a graph (see Fig. 9.2) of the actual and predicted prices for that day.

9.1. Absolute Percentage Errors for the Year

For the three exponential models estimated for each of the 12 days in the year 2000, we stacked all of the absolute percentage pricing errors across strikes and maturities. The pricing errors are themselves orthogonal to strike and maturity, but the absolute pricing errors tend to be larger for shorter maturities and options that are further out-of-the-money. This is confirmed by regression results of the absolute errors on moneyness and maturity, where we employ moneyness and its square to capture the fact that we have out-of-the-money options on both sides of the forward. Table 9.1 presents the results.

We note that the coefficients for moneyness are significant in both their linear and quadratic terms. The shape is consistent with absolute errors rising as an option gets further out-of-the-money. The coefficient for maturity is also negative and significant, which is indicative of higher absolute errors for shorter maturity options. The R^2 coefficients are

Table 9.3
Parameter Estimates for the SAM Models on the SPX at the Quarterly Points

Quarter	Parameter estimates									
	VGSAM model									
Quarter	C	G	M	κ	η	λ	ape			
Mar.	3.71	7.33	32.40	12.66	3.38	7.35	.0839			
Jun.	5.30	11.02	30.11	0	4.15	1.14	.0678			
Sep.	5.17	12.35	33.61	.0762	16.94	.6939	.0794			
Dec.	16.43	11.72	42.67	58.51	6.17	24.00	.0529			
	NIGSAM model									
	σ	ν	θ	κ	η	λ	ape			
Mar.	.3214	9.28	−9.50	0	2.61	.1044	.0816			
Jun.	.4009	12.72	−9.98	0	11.52	.0006	.0641			
Sep.	.3566	14.56	−11.58	.0043	11.74	.0005	.0744			
Dec.	.4399	31.34	−51.35	0	19.43	.3861	.0602			
	CGMYSAM model									
	C	G	M	Y_P	Y_n	ζ	κ	η	λ	ape
Mar.	.1635	.6965	21.97	−3.65	1.45	.2883	8.51	.1497	.00022	.0785
Jun.	.3587	.4231	24.64	−4.51	1.67	.0526	6.65	.3469	.0006	.0612
Sep.	.4041	1.64	16.91	−2.90	1.54	.0676	4.85	.4474	2.78e-5	.0685
Dec.	2.044	3.68	52.86	−2.12	1.22	.0855	15.91	1.37	1.70	.0489

approximately 50%, with values of 51.62, 49.07, and 47.12 for the models NIGSA,VGSA, and CGMYSA respectively.

9.2. Parameter Estimates for the Various Models

The strong negative skew is captured by all three SA models. This is reflected by consistently strong negative estimates of θ for NIGSA, with an average value of −9.84. For the VGSA and CGMYSA models, this is reflected by a consistently lower value for G than for M. For the VGSA model, the average markup of M over G is 20.83; for the CGMYSA model it is 68.03.

The rates of mean reversion in volatility or activity are comparable for NIGSA, VGSA, and CGMYSA, averaging to 6.79, 4.27, and 3.34 respectively. These are associated with half lives of around 7.5 weeks.

All three models indicate a comparable long-term level, relative to the initial value of the time change process. For the NIGSA, VGSA, and CGMYSA models, these ratios average .5076, .4681, and .5118 respectively. The models are quite consistent in this regard.

The estimates for the volatility of the time change are consistent between NIGSA and CGMYSA, with mean values of 5.34 and 6.97 respectively. The values are somewhat higher for VGSA at 16.45.

We report the other BDLPs and leverage in Table 9.4 using the parameter key provided below the table for the values estimated for December 13, 2000, for the VG Lévy process.

TABLE 9.4
Parameter Estimates for the SA Models on the SPX at the Quarterly Points for Other BDLPs on Dec. 13, 2000

Model	P1	P2	P3	P4	P5	P6	P7	ape
VGSG	15.94	17.71	51.33	2.19	.6662	19.63	.0024	.0146
VGIG	13.21	2.49	68.66	.0282	−13.57	−7.16		.0406
VGSIG	16.46	3.17	96.54	.0112	−14.73	−8.54		.0441
VGSGM	96.56	51.06	225.4	3.394	1.795	205.23	−.0006	.0182
VGIGM	573.76	687.16	110.09	.0599	.0070	.0043		.0341
VGSIGM	31.601	26.707	468.73	.4046	−52.21	−.8696		.0337
VGCSA	14.27	16.91	40.93	2.12	3.486	5.475	.003	.0172

Parameter Key:

Model	P1	P2	P3	P4	P5	P6	P7
VGSG/VGSGM	C	G	M	κ	λ	ζ	ρ
VGIG/VGIGM	C	G	M	κ	ν	ρ	
VGSIG/VGSIGM	C	G	M	κ	ν	ρ	
VGCSA	C	G	M	κ	η	λ	ρ

We note that when we couple with the IG or SIG driver for volatility, then leverage is estimated negatively. However, for the other drivers of SG and CIR we get a slightly small and possibly positive value for the correlation term.

10. CONCLUSION

Twenty-five stochastic volatility models were formulated by time changing three homogeneous Lévy processes, using four stochastic processes to drive the volatility. The Lévy processes employed were the normal inverse Gaussian model of Barndorff-Nielsen (1998), the variance gamma of Madan et al. (1998), and the CGMY model of Carr et al. (2002). The time change used to induce stochastic volatility was the integral of the Cox, Ingersoll, and Ross (1985) process, and three one-sided pure jump processes: the inverse Gaussian IG, and two processes consistent with gamma and inverse Gaussian distributions for the stationary solution to the OU equation. These volatility drivers were termed SG and SIG. This resulted in 12 stochastic volatility transformations of Lévy processes with explicit solutions for the log characteristic in each case.

Stock price models were built by exponentiating these processes and correcting the mean in accordance with spot forward arbitrage considerations, leading to 12 models on the exponential architecture. A second class of discounted stock price models was obtained using stochastic exponentials, resulting in martingale models that were martingales in the joint filtration of the stock price and the stochastic time change.

The paper also introduces two new concepts connected with studying the martingale structure of the exponential models. We define the concept of martingale marginals as distinct from martingale measures that seek martingales in the original filtration. Instead, we emphasize the lack of knowledge with respect to the filtration, noting that in the first

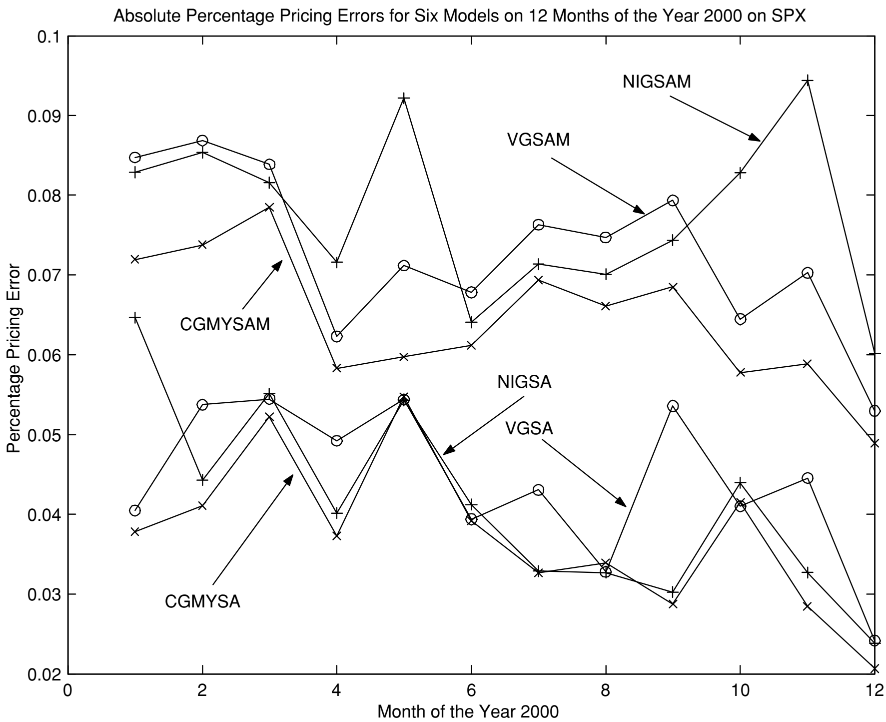

FIGURE 9.1. Graphs of absolute percentage errors for the six models across the 12 months of 2000 on the S&P 500 Index.

instance the object of option pricing models may be seen as synthesizing the information content of the option surface into a smaller dimension defined by the parameters of the model. The movement in these parameters through time, along with the spot price, can form the filtration for a Markov dynamics of the surface. With this in mind our interest turns to questions of no-static arbitrage marginally enhanced to include calendar spread arbitrages. We show that the absence of such elementary arbitrages is in fact consistent with the martingale marginal property that seeks martingales in altered filtrations consistent with the marginal risk-neutral distributions revealed at each maturity by the options market.

A procedure for constructing martingale marginals is the exploitation of the Lévy marginal property that seeks the construction of an inhomogeneous Lévy process as the solution to the martingale marginal problem. We show that the exponential constructions associated with the CIR process for the volatility satisfy the Lévy marginal property when the underlying CIR is started at zero. More generally, however, one may have the martingale marginal property for the SA processes without having the Lévy marginal property for the SV processes.

The models were estimated for the second Wednesday of each month of the year 2000 on data for S&P 500 Index options and 20 other underlying assets. For the S&P 500 options, the exponential models were significantly better than their stochastic exponential counterparts in all three cases for the Lévy processes endowed with CIR stochastic volatility. The results for the S&P 500 options consistently reflected market skews and stochastic volatility, with mean-reversion rates of approximately seven

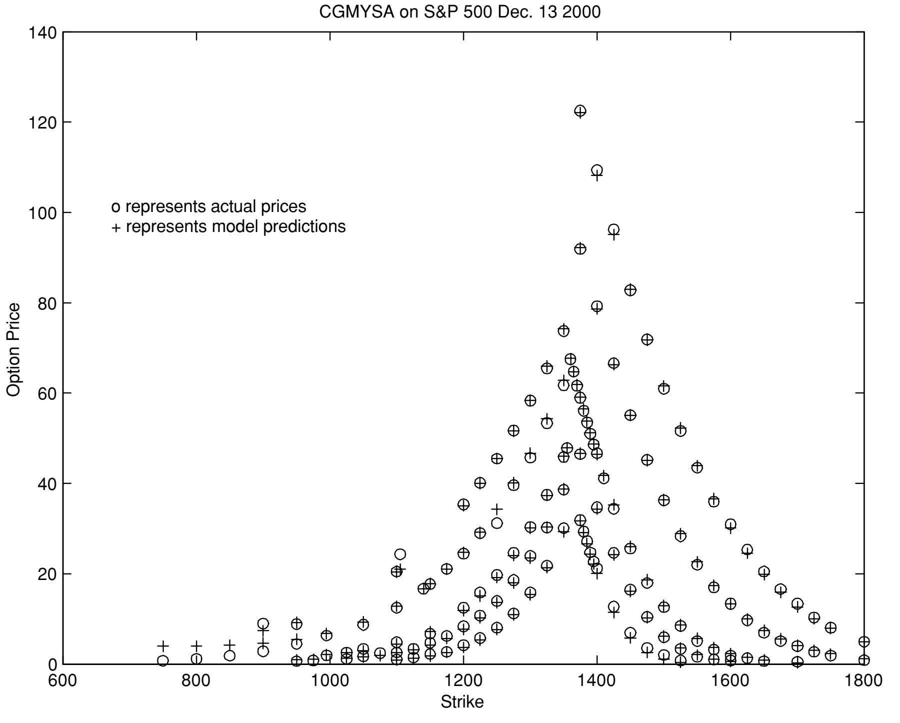

FIGURE 9.2. Actual and predicted prices of the CGMYSA model for December 13, 2000.

weeks. Similar patterns were observed for other underliers, but these are not reported here.

The best model by far was the CGMYSA model, with percentage errors across all strikes and maturities reaching as low as 2% for the S&P 500 Index options. For options on single names, the performance of the lower dimensional NIGSA and VGSA models was adequate. The class of models proposed here is for the first time providing us with a relatively parsimonious representation of the surface of option prices, with some stability over time in the parameter estimates. These structures lead to interesting applications on pricing exotic products and analyzing risk-management strategies in empirically realistic, yet tractable, contexts. We expect continuing research to shed further light on these interesting questions. Of particular interest is the study of the statistical dynamics in the same parametric class, with a view to effectively describing likelihoods at arbitrary horizons, using the analyticity of the characteristic functions defined here, and with a view to learning the nature of the measure change employed in markets.

APPENDIX

Proof of Theorem 5.2. Part (a) is a consequence of a slight generalization of Theorem 3.1 in Yor (1992). Let P denote the measure induced by the standard Wiener process and define $X_t^{(\mu)}$ by (5.4). By Girsanov's theorem, the law of this process has density with respect to P given by

$$P^{\mu,+}_{|\Im_t} = \exp\left\{\mu\int_0^t \mathbf{1}_{(X_s>0)}\,dX_s - \frac{\mu^2}{2}\int_0^t ds\mathbf{1}_{(X_s>0)}\right\}\cdot P_{|\Im_t}$$
$$= \exp\left\{\mu\left(X_t^+ - \frac{1}{2}\ell_t^0\right) - \frac{\mu^2}{2}\int_0^t ds\mathbf{1}_{(X_s>0)}\right\}\cdot P_{|\Im_t}.$$

We denote simply by $T_b : \inf\{t : X_t^{(\mu)} = b\}$ and consider a functional F of the local time at $b-a$ of $X^{(\mu)}$ up to time $T_{b,}$ $(\ell_{T_b}^{b-a}(X^{(\mu)}); 0 \le a \le b)$. We have that

$$E^{\mu,+}\left[F\left(\ell_{T_b}^{b-a}(X); 0 \le a \le b\right)\right]$$
$$= E\left[F\left(\ell_{T_b}^{b-a}(X); 0 \le a \le b\right)\exp\left(-\frac{\mu}{2}\left(\ell_{T_b}^0 - 2b\right) - \frac{\mu^2}{2}\int_0^b da\,\ell_{T_b}^a\right)\right]$$
$$= Q_0^2\left(F(Z_a, 0 \le a \le b)\exp\left(-\frac{\mu}{2}(Z_b - 2b)\right) - \frac{\mu^2}{2}\int_0^b da\,Z_a\right)$$
$$= {}^{(-\mu)}Q_0^2(F(Z_a, 0 \le a \le b)).$$

In particular, we have that

$$\mathcal{L}\left(Z_{b-a}; 0 \le a \le b; {}^{-\mu}Q_0^2\right) = \mathcal{L}\left(\ell_{T_b}^a(X; 0 \le a \le b; P^{\mu,+}\right),$$

where $\mathcal{L}(H; P)$ denotes the law of H under P. As a consequence, there is the identity in law between the pairs of two-dimensional variables:

$$\mathcal{L}\left\{Z_b, \int_0^b Z_a\,da; {}^{(-\mu)}Q_0^2\right\} = \mathcal{L}\left\{\ell_{T_b}^0(X), \int_0^b \ell_{T_b}^{b-a}(X)\,da; P^{\mu,+}\right\}$$
$$= \mathcal{L}\left\{\ell_{T_b}^0(X), \int_0^{T_b}\mathbf{1}_{(X_s>0)}\,ds; P^{\mu,+}\right\},$$

so that (5.3) holds.

For part (b), from equation (5.5) and Tanaka's formula we deduce

$$X_t^+ = \int_0^t \mathbf{1}_{(X_s>0)}(dB_s + \mu\,ds) + \frac{1}{2}\ell_t^0(X) \tag{A.1}$$
$$= \beta^{(\mu)}_{\int_0^t ds\mathbf{1}_{(X_s>0)}} + L_{\int_0^t ds\mathbf{1}_{(X_s>0)}},$$

where $L_{\int_0^t ds\mathbf{1}_{(X_s>0)}}$ is defined in accordance with Skorohod's lemma by $\sup_{s\le t}(-\beta^{(\mu)}_{\int_0^s du\mathbf{1}_{(X_u>0)}})$.

On the other hand, we have from (5.5) and Tanaka's formula:

$$|Z_t| = \int_0^t sgn(Z_s)\,d\gamma_s + \mu t + L_t(Z). \tag{A.2}$$

Comparing (A.1) and (A.2), we note that

$$X_t^+ = |Z_u|_{|u=\int_0^t ds\mathbf{1}_{(X_s>0)}} \tag{A.3}$$

for some process $(Z_u, u \ge 0)$.

Now the identity in law proposed in part (b) follows immediately from (A.3).
Part (c) is an immediate consequence of (5.5) and Skorohod's lemma. □

Proof of Theorem 5.1 continued. We come next to the general case for $\delta > 0$. Some important references for this development are Fitzsimmons (1987), Shiryaev and Cherny (1999), and Graversen and Shiryaev (2000). Here we note that with $x = \delta b/2$, and employing consequences of Ray-Knight theorems presented in Yor (1992), we have that

$$\begin{aligned}
&{}^{(-\mu)}Q_0^\delta\,(F(Z_a, 0 \le a \le b)\\
&= Q_0^\delta\left(F(Z_a, 0 \le a \le b)\exp\left\{-\frac{\mu}{2}(Z_b - \delta b) - \frac{\mu^2}{2}\int_0^b Z_a\,da\right\}\right)\\
&= E\left[F\left(\ell_{\tau_x}^{a-b}\left(|B| - \frac{2}{\delta}\ell\right); a \le b\right)\exp\left(-\frac{\mu}{2}\left(\ell_{\tau_x}^0\left(|B| - \frac{2}{\delta}\ell\right) - \delta b\right)\right.\right.\\
&\qquad\left.\left. - \frac{\mu^2}{2}\int_0^{\tau_x} ds\mathbf{1}_{(|B_s| - \frac{2}{\delta}\ell_s \le 0)}\right)\right],
\end{aligned}$$

where $\tau_x = \inf\{t \ge 0 : \ell_t^0 = x\}$.

We define

$$H_s = sgn(B_s)\mathbf{1}_{(|B_s| - \frac{2}{\delta}\ell_s \le 0)}$$

so that by Tanaka's formula we may write

$$\begin{aligned}
&{}^{(-\mu)}Q_0^\delta(F(Z_a, 0 \le a \le b)\\
&= E\left[F\left(\ell_{\tau_x}^{a-b}\left(|B| - \frac{2}{\delta}\ell\right); a \le b\right)\exp\left(-\mu\int_0^{\tau_x} H_s\,dB_s - \frac{\mu^2}{2}\int_0^{\tau_x} ds H_s^2\right)\right],
\end{aligned}$$

and it follows by Girsanov's theorem that this is

$$= E^{\mu,\delta}\left[F\left(\ell_{\tau_x}^{a-b}\left(|X| - \frac{2}{\delta}\ell\right); a \le b\right)\right],$$

where, under $P^{\mu,\delta}$, X solves

$$X_t = \beta_t - \mu\int_0^t ds\, sgn(X_s)\mathbf{1}_{(|X_s| - \frac{2}{\delta}\ell_s \le 0)}. \tag{A.4}$$

It follows in particular[1] that the law of $(\int_0^b da^{(-\mu)}y^\delta(a), b \ge 0)$ has the same one-dimensional marginals as the inhomogeneous Lévy process

$$\left(\int_0^{\tau_x} ds\mathbf{1}_{(|X_s| - \frac{2}{\delta}\ell_s \le 0)}, b \ge 0\right), \quad \text{under } P^{\mu,\delta}.$$

That this process is an inhomogeneous Lévy process in b follows from the fact that when we apply the Markov property in τ_x we obtain that $X_{\tau_x} = 0$ and $\ell_{\tau_x} = x$, hence the process $(X_{\tau_x+u}, u \ge 0)$ is independent from $(X_v, v \le \tau_x)$.

For part (ii) of Theorem 5.1 we note that by arguments similar to the ones used in the proof of part (i) we may show that for $x \ne 0$ we have that

$$\int_0^b da^{(-\mu)}y_x^\delta(a) \stackrel{(d)}{=} \int_0^{\tau_x^b(X)} ds\mathbf{1}_{(0 \le X_s \le b)},$$

[1] In accordance with Theorem 1, we ought to consider jointly ${}^{(-\mu)}y(b)$, but for the sake of simplicity we do not write this down.

where X solves (A.4) but now $\tau_x^b(X) = \inf\{t \geq 0 : \ell_t^b(X) > x\}$. That the LM property fails may be explained by the fact that $\tau_x^b(X)$ is no longer an increasing process in b. □

For the nonmonotonicity of $\tau_x^b(X)$ we note that for even Brownian motion, with $a < b$ we may write, on defining T_a, the first passage time of Brownian motion $(B(t), t \geq 0)$ to the level a, that

$$\tau_x^a(B) = T_a + \inf\left\{t > 0 \,\middle|\, L_{T_a+t}^a > x\right\}$$
$$\tau_x^b(B) = T_a + \inf\left\{t > 0 \,\middle|\, L_{T_a+t}^b > x\right\},$$

where L_t^a is the local time of $(B(t), t \geq 0)$ at the level a. On defining $\widehat{B}(t) = B(T_a + t) - a$ and noting the independence of $\widehat{B}(t)$ from the path of $(B(t), 0 \leq t \leq T_a)$, we observe that

$$\tau_x^a = T_a + \widehat{\tau}_x^{(0)}$$
$$\tau_x^b = T_a + \widehat{\tau}_x^{(b-a)}.$$

Hence, to see that $\tau_x^b(B)$ is not increasing in b one need only observe that $\widehat{\tau}_x^{(0)}$ is not necessarily less than $\widehat{\tau}_x^{(b-a)}$. In fact, the event

$$\left(\widehat{\tau}_x^{(0)} < \widehat{\tau}_x^{(b-a)}\right) \equiv \left(L_{\tau_x^{(0)}}^{(b-a)} < x\right)$$

and the probability that $L_{\tau_x^{(0)}}^{(b-a)} < x$ is (by the first Ray-Knight theorem) the probability that a squared Bessel process of dimension 0 started at x is at time $(b - a)$ below the level x. This probability is below unity and hence $\tau_x^b(B)$ is not increasing in b. Similar considerations apply to $\tau_x^b(X)$. □

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