Journal of the American Statistical Association

ISSN: 0162-1459 (Print) 1537-274X (Online) Journal homepage: www.tandfonline.com/journals/uasa20

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The leverage effect has become an extensively studied phenomenon that describes the (usually) negative relation between stock returns and their volatility. Although this characteristic of stock returns is well acknowledged, most studies of the phenomenon are based on cross-sectional calibration with parametric models. On the statistical side, most previous works are conducted over daily or longer return horizons, and few of them have carefully studied its estimation, especially with high-frequency data. However, estimation of the leverage effect is important because sensible inference is possible only when the leverage effect is estimated reliably. In this article, we provide nonparametric estimation for a class of stochastic measures of leverage effect. To construct estimators with good statistical properties, we introduce a new stochastic leverage effect parameter. The estimators and their statistical properties are provided in cases both with and without microstructure noise, under the stochastic volatility model. In asymptotics, the consistency and limiting distribution of the estimators are derived and corroborated by simulation results. For consistency, a previously unknown bias correction factor is added to the estimators. Applications of the estimators are also explored. This estimator provides the opportunity to study high-frequency regression, which leads to the prediction of volatility using not only previous volatility but also the leverage effect. The estimator also reveals a theoretical connection between skewness and the leverage effect, which further leads to the prediction of skewness. Furthermore, adopting the ideas similar to the estimation of the leverage effect, it is easy to extend the methods to study other important aspects of stock returns, such as volatility of volatility.

KEY WORDS: Consistency; Discrete observation; Efficiency; Itô process; Microstructure noise; Realized volatility; Skewness; Stable convergence.

1. INTRODUCTION

The leverage effect has become an extensively studied empirical phenomenon in the form of the (usually negative) correlation between (current) returns and (current and future) volatility (Engle and Ng 1993; Zakoian 1994; Wu and Xiao 2002, etc.). It is one of the many stylized facts of the security return distribution, along with the well-known fat tails, skewness, excess kurtosis, and heteroscedasticity. The discovery of leverage effect closely relates to the study of stochastic volatility. Although for very low frequency data, such as monthly or yearly asset returns, the assumption of homogeneity seems not to be entirely unreasonable (Mandelbrot 1963; Fama 1965; Officer 1973), the increasing frequency of observed data in studies suggests heterogeneity in volatility, in other words, time-varying volatility (Engle 1982; Bollerslev 1986; Andersen and Bollerslev 1998; Engle 2000; Andersen et al. 2001). This finding has had profound implications in both the theory and practice of financial economics and econometrics. It has inspired new model building, such as the emergence of ARCH models and the later stochastic volatility models. Modeling volatility as a separate process allows the study of its relation with the associated return process, which leads to the discovery of asymmetric volatility. Time varying volatility is also of substantial importance in modeling for options pricing, as in Hull and White (1987), Stein and Stein (1991), Heston (1993), and Ball and Roma (1994).

Black (1976) and Christie (1982) were among the first to document the volatility asymmetry, and gave an explanation based on the "leverage effect" hypothesis: A drop in the value of the stock (negative return) increases the financial leverage (debt-to-equity ratio), which makes the stock riskier and increases its volatility. Since then, leverage effect has been taken to be synonymous with asymmetric volatility. Financial leverage itself, however, seems not enough to explain either the large magnitude of the effect of declines in current price on future volatility (Figlewski and Wang 2001), or the phenomenon that the asymmetry of market index returns is generally larger than that of individual stocks (Kim and Kon 1994; Tauchen and Zhang 1996; Andersen et al. 2001). In another point of view, the asymmetric nature of volatility to return shocks simply reflects time-varying risk premium (Pindyck 1984; French et al. 1987; Campbell and Hentschel 1991). This explanation is often referred to as the "volatility feedback effect": If volatility is priced, an anticipated increase in volatility raises the required return on equity, leading to an immediate stock price decline.

Many later works either compare the two effects or seek to argue that they can both be at work (Nelson 1991; Engle and Ng 1993; Glosten et al. 1993; Bekaert and Wu 1997; Wu 2001; Hasanhodzic and Lo 2011). While there is little agreement concerning the fundamental causes behind the leverage effect, that is not the focus of this article.

As most early studies are conducted over daily or longer time horizons, it is worthwhile to examine this phenomenon with high frequency data, which provides the opportunity to explore more closely the relation between stock price and its own volatility. Some recent work has demonstrated that the volatility asymmetry still appears over fairly small time intervals. But some new aspects are added as both very good and bad news increase volatility, with the latter having a more severe effect

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Journal of the American Statistical Association
March 2014, Vol. 109, No. 505, Theory and Methods
DOI: 10.1080/01621459.2013.864189

(Barndorff-Nielsen et al. 2008b; Chen and Ghysels 2011). Also, the leverage effect decays exponentially as the time lag between return and volatility increases. In the literature, the peak effect is obtained at the instantaneous correlation between return and volatility (Bouchaud et al. 2001; Bollerslev et al. 2006).[1] This corresponds to our definition of the leverage effect as being instantaneous. For further discussion of this effect; see the end of Section 2.2.

Although many papers deal with the source or new properties of the leverage effect, few have tried to rigorously estimate it, which is critical for supporting any conclusive claims. Simple correlation estimators may be applied to the estimation of the leverage effect with caution. Those estimators lose consistency with high-frequency data (Aït-Sahalia et al. 2013). It is the purpose of this article to construct nonparametric estimators of leverage effects in the stochastic volatility model. To study the estimation, we define a new leverage effect parameter as the covariance (a covariation, to be precise) between the stock return and a function of its volatility. To construct the estimator for the new parameter, we first study the classical equi-distant sampling case without microstructure noise as the foundation for the study (later in the article) of more complicated cases, such as the case with microstructure noise. As is emphasized in several studies (Mykland and Zhang 2006; Barndorff-Nielsen et al. 2008a; Renault and Werker 2011), it is more natural to work with irregularly spaced data in practice. Based on the results in equidistant cases, the extension of estimators to irregularly spaced data can be constructed in a similar way, with some adaptations.

Even with equidistant observations without microstructure noise, we discover a previously unknown bias correction factor. This bias correction factor is critical to obtaining consistency. The factor may have a substantial impact on the estimated value since it functions as a magnifier, especially when estimates are close to zero. The bias correction factor may play an even more important role in the estimation, when the situation becomes more complicated as market microstructure noise is present. Indeed, in the case with microstructure noise, the bias correction factor is found to be bigger than that in the case with uncontaminated continuous price paths.

Statistical properties such as consistency and asymptotic distribution are carefully studied in different settings. The theoretical findings of these statistical properties are corroborated by the simulation results. These asymptotic properties have applications to hypothesis testing (e.g., for model checking) and constructing confidence intervals.

There are many ways to apply the estimators of the leverage effect depending on the practical purpose. One way to explore the potential application of the estimators is embedded in the definition of the stochastic parameter of the leverage effect. According to the definition, one specific choice of the function imposed on volatility gives rise to a unique relation between the leverage effect and skewness, which will help to estimate skewness consistently. This relation may introduce further applications in hedging strategy or new product design; see Neuberger (2011). Another carefully chosen function of volatility can simplify the estimation of high-frequency regression coefficients. This leads to an interesting discovery in volatility prediction. The empirical study with Microsoft stock data (2008–2011) shows strong predictive power of a term containing the leverage effect on the next period volatility. The power is comparable to that of current period volatility which is believed to be the most significant term in volatility prediction.

The main results of this article will be given in Sections 2 and 4. The data-generating mechanism and model setting can be found in Section 2.1. The (stochastic) parameter of the leverage effect is defined in Section 2.2. Based on this, for the case without microstructure noise, the estimator and limit theorems can be found in Sections 2.3 and 2.4. Simulation results are provided in Section 3. Results that corroborate the theorems can be found in Section 3.1. Section 4 studies the case where market microstructure noise is present in the data. The estimator and limit theorems for this case are provided in Sections 4.1 and 4.2. Simulation results for this case are provided in Section 5. The extension to irregularly spaced data can be found in Section 6. The relation between leverage effect and skewness is shown in Section 7. An application of the leverage effect in high-frequency regression is implemented in Section 8. The details of empirical studies are in Section 9. The conclusion is provided in Section 10. Proofs are in the Appendix.

[1]The relative change between time lag and lagged leverage effect should be maintained, even if the consistency may be an concern, since the consistency can be achieved by a bias correction multiplier from our study shown later.

2. MAIN RESULTS

2.1 Data-Generating Mechanism

In general, we shall work with a broad class of continuous semimartingales, namely *Itô processes*. In econometrics and financial mathematics studies, this is the most popular model for log price processes due to nonarbitrage considerations (Delbaen and Schachermayer 1994, 1995, 1998).

Definition 1. A process X_t is called an Itô process provided it satisfies

$$dX_t = \mu_t\, dt + \sigma_t\, dW_t,\ X_0 = x_0, \tag{1}$$

where μ_t and σ_t are adapted *càdlàg* locally bounded random processes, and W_t is a Wiener process. The underlying filtration will be called $(\mathcal{F}_t)$. The probability measure will be called P.

The integrated variance process is given as

$$\langle X, X\rangle_t = \int_0^t \sigma_u^2\, du. \tag{2}$$

The process (2) is also known as the quadratic variation of X. We shall sometimes also use the term "integrated volatility."

We further assume that σ_t is also an Itô process (see the next section for discussion of this)

$$d\sigma_t = a_t\, dt + f_t\, dW_t + g_t\, dB_t, \tag{3}$$

where B_t is another Wiener process independent of W_t, and a_t, f_t, and g_t are all assumed to be Itô processes.

Clearly, in this stochastic volatility (SV) model, X_t corresponds to the log price process and σ_t is its own volatility process. Both processes have a common driving Wiener process W_t, which accommodates the leverage effect.

To summarize the technical requirements, we specify exact conditions as follows:

Assumption 1. The system satisfies (1) and (3), where X and σ are continuous processes (the continuous modification). We assume that all the coefficients f_t, g_t, a_t, μ_t are locally bounded in absolute value. We also assume that σ_t is locally bounded away from zero.[2]

2.2 The Parameter: A Definition of Leverage Effect

As we have seen in the Introduction, the literature offers various perspectives on how to specify a parameter for this effect; see also the discussion below in this section. We concentrate on the estimation of the following stochastic parameter:

Definition 2. The stochastic parameter of the contemporaneous leverage effect is defined as the quadratic covariation between X_t and $F(\sigma_t^2)$

$$\langle X, F(\sigma^2)\rangle_T = \int_0^T 2F'\big(\sigma_t^2\big)\sigma_t^2 f_t\, dt, \tag{4}$$

where we suppose that

Assumption 2. $x \mapsto F(x)$ is twice continuously differentiable, monotone on $(0, \infty)$.

The incorporation of the function F allows more flexibility and a wider range of applications when different forms of volatility are of interest, such as log-volatility processes which tend to be more stationary over time as implied by many empirical studies. The actual choice of F will depend on the practical purpose and empirical evidence. The inclusion of this function can also help reveal some interesting connections between leverage effects and other statistics. Further interpretation of this specification will be seen in Sections 7 and 8.

We stop here for a moment to reflect on the definitions. First of all, we work with continuous processes. The interface with jump processes remains to be explored. The latter permits additional concepts of asymmetry, in particular the semivariance of Barndorff-Nielsen, Kinnebrock, and Shephard (2008b). The connection between semivariance and the leverage effect (and skewness, see Section 7) in this article is an important question which we leave for future investigation. This is necessarily a complex matter, as it involves a different model of the price process (continuous paths vs. jumps).

Once one works with continuous paths, the assumption that the leverage effect is instantaneous is both natural in a semimartingale model for σ_t, and is empirically supported by the finding in Bollerslev, Litvinova, and Tauchen (2006), where it was shown that the connection between return and volatility is most significant when the time lag is 0. This does not contradict the fact that the effect can appear at a greater time lag, as documented by Chen and Ghysels (2011).

As far as the Itô process (continuous semimartingale) assumption is concerned, this assumption appears frequently both on the options pricing side (Hull and White 1987; Stein and Stein 1991; Heston 1993; Ball and Roma 1994), and on the econometric side (Barndorff-Nielsen and Shephard 2002; Barndorff-Nielsen et al. 2006; Jacod 2008; Barndorff-Nielsen and Veraart 2009, Aït-Shalia and Jacod 2009; Mykland and Zhang 2011b). A parallel development can be carried out under assumptions of fractional Brownian motion (Comte and Renault 1998; Gloter and Hoffmann 2004; Brockwell and Marquardt 2005; Nualart 2006; Comte et al. 2010).

The above is, of course, a set of theoretical considerations. We finally appeal to the results in Section 9 to show that our current definition of volatility asymmetry does find something empirically relevant; we substantially improve the prediction of next-period volatility using the current-period leverage effect.

2.3 Estimation in the Absence of Microstructure Noise

As the first step, we shall work with the equally spaced case for the process (X_t); specifically it is observed every $\Delta t_{n,i+1} = \Delta t = \frac{T}{n}$ units of time, at times $0 = t_{n,0} < t_{n,1} < t_{n,2} < \cdots < t_{n,n} = T$. Furthermore, we divide observed values into K_n blocks, with block size $M_n = [c\sqrt{n}]$ (except possibly for the first and last block, which does not matter for the asymptotics), for some constant c. The boundary points are on the grid $\mathcal{H} = \{0 < \tau_{n,1} < \tau_{n,2} < \cdots < \tau_{n,K_n-1} \le T\}$, where $K_n = [\frac{n}{M_n}]$.

Define[3]

$$\widehat{\langle X, F(\sigma^2)\rangle}_T = 2\sum_{i=0}^{K_n-2} \big(X_{\tau_{n,i+1}} - X_{\tau_{n,i}}\big)\big(F\big(\hat\sigma^2_{\tau_{n,i+1}}\big) - F\big(\hat\sigma^2_{\tau_{n,i}}\big)\big), \tag{5}$$

and

$$\hat\sigma^2_{\tau_{n,i}} = \frac{1}{M_n \times \Delta t}\sum_{t_{n,j}\in(\tau_{n,i},\tau_{n,i+1}]} \big(X_{t_{n,j+1}} - X_{t_{n,j}}\big)^2.$$

The factor 2 in the first part of Equation (5) might look unnatural to be included. However, it is crucial for the consistency of the estimator (see Remark 3 for a discussion of this previously unknown factor).

Theorem 1. Under Assumptions 1–2, as $n \to \infty$ and T fixed,

$$\begin{aligned} &n^{1/4}(\widehat{\langle X, F(\sigma^2)\rangle}_T - \langle X, F(\sigma^2)\rangle_T) \\ &\quad\xrightarrow{\mathcal{L}} Z\left(\frac{16}{c}\int_0^T \big(F'\big(\sigma_t^2\big)\big)^2\sigma_t^6\, dt \right.\\ &\qquad\left. + cT\int_0^T \big(F'\big(\sigma_t^2\big)\big)^2\sigma_t^4\left(\frac{44}{3}f_t^2 + \frac{32}{3}g_t^2\right) dt\right)^{1/2}, \end{aligned} \tag{6}$$

stably in law,[4] where Z is a standard normal random variable and independent of $\mathcal{F}_T$.

[2]To get from local boundedness to results that cover the whole time interval, use arguments as in chap. 2.4.5 (p. 160-161) of Mykland and Zhang (2012). $|\sigma_t|$ is locally bounded from above by continuity. The assumptions guarantee that the equivalent martingale measure for X exists locally. This is used in the proofs; see the beginning of Section A.1.

[3]One can also consider a kernel estimator of the spot volatility in (5), by applying the methods in Kristensen (2010), with some adaptation. A detailed study is beyond the scope of this article.

[4]Suppose that all relevant processes $(X_t, \sigma_t,$ etc.) are adapted to the filtration $(\mathcal{F}_t)$. Let Z_n be a sequence of $\mathcal{F}_t$-measurable random variables. We say that Z_n converges stably in law to Z as $n \to \infty$ if Z is measurable with respect to an extension of $\mathcal{F}_T$ so that for all $A \in \mathcal{F}_T$ and for all bounded continuous g, $EI_A g(Z_n) \to EI_A g(Z)$ as $n \to \infty$. The same definition applies to triangular arrays.

Another natural estimator analogous to $\widehat{\langle X, F(\sigma^2)\rangle}_T$ is

$$\widetilde{\langle X, F(\sigma^2)\rangle}_T = 2\sum_{i=0}^{K_n-2}\left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}}\right)\left(F\left(\widetilde{\sigma}^2_{\tau_{n,i+1}}\right) - F\left(\widetilde{\sigma}^2_{\tau_{n,i}}\right)\right),$$

$$\widetilde{\sigma}^2_{\tau_{n,i}} = \frac{1}{(M_n-1)\times\Delta t}\sum_{t_{n,j}\in(\tau_{n,i},\tau_{n,i+1}]} \times\left(\Delta X_{t_{n,j+1}} - \overline{\Delta X_{\tau_{n,i+1}}}\right)^2, \tag{7}$$

and

$$\overline{\Delta X_{\tau_{n,i+1}}} = \frac{1}{M_n}\sum_{t_{n,j}\in(\tau_{n,i},\tau_{n,i+1}]}\Delta X_{t_{n,j+1}} = \frac{1}{M_n}\left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}}\right).$$

Noticing the relation between the two estimators when $F(x) = x$: $\widetilde{\langle X, \sigma^2\rangle}_T = \frac{M_n}{M_n-1}\widehat{\langle X, \sigma^2\rangle} - \sum_i \frac{2}{M_n(M_n-1)\Delta t}\Delta X_{\tau_{n,i+1}}(\Delta X_{\tau_{n,i+2}})^2 + \sum_i \frac{2}{M_n(M_n-1)\Delta t}(\Delta X_{\tau_{n,i+1}})^3$, the following theorem can be easily derived:

Theorem 2. Under Assumptions 1–2 as in Theorem 1, as $n\to\infty$ and T fixed,

$$n^{1/4}(\widetilde{\langle X, F(\sigma^2)\rangle}_T - \langle X, F(\sigma^2)\rangle_T) \xrightarrow{\mathcal{L}} Z\left(\frac{16}{c}\int_0^T \left(F'\left(\sigma_t^2\right)\right)^2\sigma_t^6\,dt + cT\int_0^T\left(F'\left(\sigma_t^2\right)\right)^2\sigma_t^4\left(\frac{44}{3}f_t^2 + \frac{32}{3}g_t^2\right)dt\right)^{1/2}, \tag{8}$$

stably in law,[5] where Z is a standard normal random variable and independent of $\mathcal{F}_T$.

Remark 1. From the limit theorems, it is not hard to see that by properly choosing c, one can minimize the limit variance. The optimal value is

$$c^2 = \frac{16\int_0^T\left(F'\left(\sigma_t^2\right)\right)^2\sigma_t^6\,dt}{T\int_0^T\left(F'\left(\sigma_t^2\right)\right)^2\sigma_t^4\left(\frac{44}{3}f_t^2 + \frac{32}{3}g_t^2\right)dt}. \tag{9}$$

See Section 2.4 for an estimator[6] of c.

Remark 2. The two estimators have the same asymptotic properties. Even though the centered version gives slightly more symmetric results, it does not behave very differently from the noncentered version. In practice, the noncentered version can be applied with less programming effort. Therefore, our later simulation mainly adopts the noncentered version of estimators.

Remark 3. The origin of the factor 2 in the estimator can be found in the proof of Theorem 1. For intuition, however, we give here a verbal explanation of the source of this adjustment constant. Let us consider the case where $F(x) = x$. Since $\hat{\sigma}_t^2$ is a consistent estimator of σ_t^2, then the first multiplication $(X_{\tau_{i+1}} - X_{\tau_i})(\hat{\sigma}^2_{\tau_{i+1}} - \sigma^2_{\tau_i})$ already gives a consistent (though infeasible) estimator of the leverage effect in the interval $(\tau_i, \tau_{i+1}]$. Then one may expect the remainder term $(X_{\tau_{i+1}} - X_{\tau_i})(\hat{\sigma}^2_{\tau_i} - \sigma^2_{\tau_i})$ to have mean zero. However, since $\hat{\sigma}^2_{\tau_i}$ employs data in the time interval $(\tau_i, \tau_{i+1}]$, as does $(X_{\tau_{i+1}} - X_{\tau_i})$, the product does not converge to zero but to one half of the leverage effect. To see why it is one half, note that each increment $\frac{\Delta X^2_{t_j}}{\Delta t}$ term is roughly an (inconsistent) estimator of $\sigma^2_{t_j}$. Thus the cross product gives an average of leverage effects over $(\tau_i, t_1], (\tau_i, t_2], \ldots, (\tau_i, \tau_{i+1}]$. If $\langle X, \sigma^2\rangle'_t$ is considered to be constant over $(\tau_i, \tau_{i+1}]$, that average of those leverage effects will give a value of about half of the leverage effect over the entire interval. Hence, we have reduced the estimation of leverage effect by half. An adjustment factor of 2 therefore needs to be added to achieve consistency.

2.4 Estimation of Asymptotic Variance

Let

$$G_n^1 = 2n^{\frac{1}{2}}\sum_{i=0}^{K_n-2}(X_{\tau_{n,i+1}} - X_{\tau_{n,i}})^2\left(F\left(\widehat{\sigma}^2_{\tau_{n,i+1}}\right) - F\left(\widehat{\sigma}^2_{\tau_{n,i}}\right)\right)^2,$$

and (10)

$$G_n^2 = 2n^{\frac{1}{2}}M_n\Delta t\sum_{i=0}^{K_n-2}\widehat{\sigma}^2_{\tau_{n,i}}\left(F\left(\widehat{\sigma}^2_{\tau_{n,i+1}}\right) - F\left(\widehat{\sigma}^2_{\tau_{n,i}}\right)\right)^2.$$

By the same methods as in the proof of Theorem 1, we have the following convergences in probability:

$$G_n^1 \xrightarrow{p} \frac{8}{c}\int_0^T\left(F'\left(\sigma_t^2\right)\right)^2\sigma_t^6\,dt + cT\int_0^T\left(F'\left(\sigma_t^2\right)\right)^2\sigma_t^4\left(\frac{28}{3}f_t^2 + \frac{16}{3}g_t^2\right)dt, \tag{11}$$

$$G_n^2 \xrightarrow{p} \frac{8}{c}\int_0^T\left(F'\left(\sigma_t^2\right)\right)^2\sigma_t^6\,dt + cT\int_0^T\left(F'\left(\sigma_t^2\right)\right)^2\sigma_t^4\frac{16}{3}\left(f_t^2 + g_t^2\right)dt, \tag{12}$$

and

$$G_n^1 + G_n^2 \xrightarrow{p} \frac{16}{c}\int_0^T\left(F'\left(\sigma_t^2\right)\right)^2\sigma_t^6\,dt + cT\int_0^T\left(F'\left(\sigma_t^2\right)\right)^2\sigma_t^4\left(\frac{44}{3}f_t^2 + \frac{32}{3}g_t^2\right)dt. \tag{13}$$

Equation (13) gives the estimation of the asymptotic variance. With this estimation, a feasible version of the central limit distribution can be derived.

Theorem 3. Under Assumptions 1–2, as $n\to\infty$ and T fixed,

$$\frac{n^{1/4}(\widehat{\langle X, F(\sigma^2)\rangle}_T - \langle X, F(\sigma^2)\rangle_T)}{\sqrt{G_n^1 + G_n^2}} \xrightarrow{\mathcal{L}} Z_1,$$

and (14)

$$\frac{n^{1/4}(\widetilde{\langle X, F(\sigma^2)\rangle}_T - \langle X, F(\sigma^2)\rangle_T)}{\sqrt{G_n^1 + G_n^2}} \xrightarrow{\mathcal{L}} Z_1$$

stably in law,[7] where Z_1 is a standard normal random variable and independent of $\mathcal{F}_T$.

Notice that the limiting distribution Z_1 is the same in both limits. In other words, the difference between the two statistics

[5] See Footnote 4.

[6] Here and in the continuation of Remark 1, we assume that the denominator in (9) is nonzero.

[7] See Footnote 4.

Table 1. The summary statistics do exhibit the target normality

	MSE	Mean	Median	Q_1	Q_3
$n = 390$, $T = 1/250$ infeasible	1.097112	−0.02108	−0.005707	−0.676	0.6496
$n = 390$, $T = 1/250$ feasible	1.051725	0.006831	−0.004835	−0.754	0.7399
$n = 23400$, $T = 1/250$ infeasible	1.009285	−0.006430	0.003727	−0.6982	0.6796
$n = 23400$, $T = 1/250$ feasible	1.002125	0.002964	0.004267	−0.6964	0.6917
$n \to \infty$, fixed T (asymptotic value)	1	0	0	−0.674	0.674

NOTE: This corroborates the theorems and shows that the asymptotics can predict small sample behavior. For sample size 390, both the mean and median are very close to 0. The MSE is close to 1 and the quartiles are close to the theoretical values from $N(0, 1)$. As sample size increases, the MSE decreases further closer to 1.

converges to zero in probability. With this feasible CLT, one can conduct hypothesis testing and construct confidence interval for the leverage effect parameter.

Remark 1 (continued). The result (13) opens paths to estimating the tuning parameter c in (9). We here outline two approaches.

Method 1: The conceptually simplest possibility is to pick $c = \arg\min\{G_n^1 + G_n^2\}$ over a suitable grid of c's. If the grid is nested and becomes dense as $n \to \infty$, this automatically provides a consistent estimator of c.

Method 2: Since Method 1 is computationally heavy, we here also propose an alternative two-step method. Fix an initial value c_0, and compute $(G_n^1 + G_n^2)(c_0)$. On the other hand, we can reduce estimation of $\gamma^2 = \int_0^T (F'(\sigma_t^2))^2 \sigma_t^6\, dt$ to the local estimation of volatility by the methods in Section 4.1 in Mykland and Zhang (2009). Call this latter estimate $\hat{\gamma}^2$. We thus obtain that $c_0^{-1}(G_n^1 + G_n^2)(c_0) - 16\hat{\gamma}^2/c_0^2$ consistently estimates $T \int_0^T (F'(\sigma_t^2))^2 \sigma_t^4 (\frac{44}{3} f_t^2 + \frac{32}{3} g_t^2)\, dt$. A consistent estimate of c^2 is thus given from (9) as

$$\hat{c}^2 = 16\hat{\gamma}^2 \left(c_0^{-1}(G_n^1 + G_n^2)(c_0) - 16\hat{\gamma}^2/c_0^2\right)^{-1}.$$

The two methods can be used together, with the second providing a starting point for seaching for a minimum in Method 1.

3. SIMULATION RESULTS

All simulation results are based on 10,000 sample paths while varying the sample size n, function F, and optimal choice of c (path dependent). In the simulation, the properties of the estimator are studied with the Heston model (Heston 1993). To examine the theoretical limit distribution, the distribution of the statistics in Theorem 1

$$\frac{n^{1/4}(\widehat{\langle X, F(\sigma^2)\rangle}_T - \langle X, F(\sigma^2)\rangle_T)}{\left(\frac{16}{cT}\int_0^T F'\left(\sigma_t^2\right)^2 \sigma_t^6\, dt + c\int_0^T \left(F'\left(\sigma_t^2\right)\right)^2 \sigma_t^4 \left(\frac{44}{3} f_t^2 + \frac{32}{3} g_t^2\right) dt\right)^{1/2}},$$

and the statistics in Theorem 3,

$$\frac{n^{1/4}(\widehat{\langle X, F(\sigma^2)\rangle}_T - \langle X, F(\sigma^2)\rangle_T)}{\sqrt{G_n^1 + G_n^2}}$$

and

$$\frac{n^{1/4}(\widetilde{\langle X, F(\sigma^2)\rangle}_T - \langle X, F(\sigma^2)\rangle_T)}{\sqrt{G_n^1 + G_n^2}},$$

are studied. So if the asymptotics correctly predict small sample behavior, the distributions should be close to the standard normal distribution.

The Heston model used in the simulation is of the form:

$$\begin{aligned} dX_t &= \sigma_t\, dW_t, \\ d\sigma_t^2 &= \kappa\left(\theta - \sigma_t^2\right) dt + \gamma\sigma_t(\rho\, dW_t + \sqrt{1-\rho^2}\, dB_t), \\ &\qquad \text{where } W_t \perp\!\!\!\perp B_t. \end{aligned} \quad (15)$$

3.1 Normality Demonstration

In the simulation, the true log price is simulated from the Heston model with broadly realistic parameter values: $\kappa = 5$, $\theta = 0.04$, $\gamma = 0.5$, $\rho = -\sqrt{0.5}$ over 1 trading day. Two different sampling frequencies are studied to examine the small sample behavior. The first is when the data are observed at 1-min frequency, which corresponds to sample size 390. The second is when the data are observed at every second, which corresponds to sample size 23,400. The results are given in Table 1.

4. ESTIMATION WITH MICROSTRUCTURE NOISE

It is well known that markets are not so ideal that log price processes can be simply represented by pure semimartingales. This has long been thought about as "microstructure" see, e.g., Roll (1984), O'Hara (1995), Harris (1990), and Hasbrouck (1996). In the context of high-frequency data, such microstructure was originally observed through the so-called signature plot (introduced by Andersen et al. (2000); see also the discussion in Mykland and Zhang (2005)). This led researchers to investigate a model where the efficient price is latent, and one actually observes

$$Y_t = X_t + \epsilon_t. \quad (16)$$

Several approaches[8] seek to deal with microstructure noise while estimating integrated volatility, and they shed light on how to proceed in the estimation of leverage effects in the similar situation. Among these approaches, we have focused on preaveraging. The preaveraging method (Jacod et al. 2009; Podolskij and Vetter 2009a; Mykland and Zhang 2011a) provides a plausible way to solve the problem with microstructure. Therefore, all of the following discussion will be in the framework of preaveraging and the blocking method will be adjusted as follows:

[8]Such as Zhang, Mykland, and Aït-Sahalia (2005), Zhang (2006), Barndorff-Nielsen et al. (2008a), Reiss (2010), and Xiu (2010), as well as the preaveraging papers cited in the text.

The contaminated log price process (Y_t) is observed every $\Delta t_{n,i} = \frac{T}{n}$ units of time, at times $0 = t_{n,0} < t_{n,1} < t_{n,2} < \cdots < t_{n,n} = T$.

Assumption 3.

$$Y_t = X_t + \epsilon_t, \text{ where } \epsilon_t\text{'s are iid. } N(0, a^2) \text{ and } \epsilon_t \perp\!\!\!\perp \text{ the } W \text{ and } B \text{ processes, for all } t. \tag{17}$$

We also assume that ϵ_t's have finite fourth moment, and are independent of both return and volatility processes.

We can also relax these assumptions; see the development in Mykland and Zhang (2011a).

Here two nested levels of blocks will be required. The first level of blocks defines the range of preaveraging and the second one implements a blocking idea similar to that in the case without noise in Section 2.3.

Blocks are defined on a much less dense grid of $\tau_{n,i}$, also spanning $[0, T]$, so that

$$\text{block \# } i = \{t_{n,j} : \tau_{n,i} \leq t_{n,j} < \tau_{n,i+1}\} \tag{18}$$

(the last block, however, includes T). We define the block size by

$$M_{n,i} = \#\{j : \tau_{n,i} \leq t_{n,j} < \tau_{n,i+1}\}. \tag{19}$$

In principle, the block size $M_{n,i}$ can vary across the trading period $[0, T]$, but for this development we take $M_{n,i} = M_n$; it depends on the sample size n, but not on the block index i.

We then use as an estimated value of the efficient price in the time period $[\tau_{n,i}, \tau_{n,i+1})$:

$$\hat{X}_{\tau_{n,i}} = \frac{1}{M_n} \sum_{t_{n,j} \in [\tau_{n,i}, \tau_{n,i+1})} Y_{t_{n,j}}.$$

Treating the estimated efficient price as a new data frame, we proceed as in Section 2.3 but with X_t replaced by $\hat{X}_t$, n by $n' = n/M_n$ (up to rounding), and $t_{n,i}$ by $\tau_{n,i}$. Furthermore, we divide $\hat{X}_t$ values into K_n blocks, with block size $L = L_n = [c\sqrt{n'}]$ (except possibly for the first and last block, which does not matter for the asymptotics), for some constant c. The boundary points are on the grid $\mathcal{G} = \{0 < \lambda_{n,1} < \lambda_{n,2} < \cdots < \lambda_{n,K_n-1} \leq T\} \subset \mathcal{H}$.

4.1 The Case With Microstructure Noise

In the case with microstructure noise, the data blocking mechanism will be similar to that just stated, but less complicated where $M = M_n = [c_1\sqrt{n}]$, $\tau_{n,i} = iM_n\frac{T}{n}$, and $L = L_n = [\frac{cn^{1/4}}{\sqrt{c_1}}]$. The interval between successive observations is now $\Delta t = \Delta t_n = t_{n,j+1} - t_{n,j} = T/n$.

Define

$$\widehat{\langle X, F(\sigma^2)\rangle}_T = 3 \sum_{i=0}^{K_n-2} \left(\hat{X}_{\lambda_{n,i+1}} - \hat{X}_{\lambda_{n,i}}\right)\left(F\left(\hat{\sigma}^2_{\lambda_{n,i+1}}\right) - F\left(\hat{\sigma}^2_{\lambda_{n,i}}\right)\right),$$

$$\hat{X}_{\tau_{n,i}} = \frac{1}{M} \sum_{t_{n,j} \in [\tau_{n,i}, \tau_{n,i+1})} Y_{t_{n,j}}, \tag{20}$$

and

$$\hat{\sigma}^2_{\lambda_{n,i}} = \frac{1}{L \times M \times \Delta t} \sum_{\tau_{n,j+1} \in (\lambda_{n,i}, \lambda_{n,i+1}]} \left(\hat{X}_{\tau_{n,j+1}} - \hat{X}_{\tau_{n,j}}\right)^2.$$

Note that the factor 2 in the previous proposed estimator in Equation (5) is now changed to 3 instead. This change is due to the preaveraging method we adopted first to asymptotically eliminate the impact of noise on the estimation. The change is consistent with the adjustment to the realized volatility estimated by preaveraging; see Jacod et al. (2009).

Theorem 4. Under Assumptions 1–3, as $n \to \infty$ and T fixed,

$$\begin{aligned} &n^{1/8}(\widehat{\langle X, F(\sigma^2)\rangle}_T - \langle X, F(\sigma^2)\rangle_T) \\ &\xrightarrow{\mathcal{L}} Z\Bigg(c\sqrt{c_1}T\int_0^T \left(F'\left(\sigma_t^2\right)\right)^2\sigma_t^4\left(\frac{44}{3}f_t^2 + \frac{32}{3}g_t^2\right)dt \\ &\quad + \frac{16\sqrt{c_1}}{c}\int_0^T \left(F'\left(\sigma_t^2\right)\right)^2\sigma_t^6\,dt \\ &\quad + \frac{96a^2}{cc_1^{3/2}T}\int_0^T \left(F'\left(\sigma_t^2\right)\right)^2\sigma_t^4\,dt \\ &\quad + \frac{216a^4}{cc_1^{7/2}T^2}\int_0^T \left(F'\left(\sigma_t^2\right)\right)^2\sigma_t^2\,dt\Bigg)^{1/2}, \end{aligned} \tag{21}$$

stably in law,[9] where Z is a standard normal random variable and independent of $\mathcal{F}_T$.

The optimal c and c_1 that minimize the asymptotic variance are derived as follows:

$$c = \sqrt{\frac{-C^2 + 12AD + C\sqrt{C^2 + 12AD}}{9BD}}, \tag{22}$$

and

$$c_1 = \sqrt{\frac{C + \sqrt{C^2 + 12AD}}{2A}}, \tag{23}$$

where $A = 16\int_0^T (F'(\sigma_t^2))^2\sigma_t^6\,dt$, $B = T\int_0^T (F'(\sigma_t^2))^2\sigma_t^4 (\frac{44}{3}f_t^2 + \frac{32}{3}g_t^2)\,dt$, $C = \frac{96a^2}{T}\int_0^T (F'(\sigma_t^2))^2\sigma_t^4\,dt$, and $D = \frac{216a^4}{T^2}\int_0^T (F'(\sigma_t^2))^2\sigma_t^2\,dt$.

In practice, c and c_1 can be estimated by minimizing $G_n^1 + G_n^2$ defined in the next section, over a suitable grid of c's and c_1's. If the grid is nested and becomes dense as $n \to \infty$, this automatically provides a consistent estimator of c and c_1.

4.2 Estimation of Asymptotic Variance

Let

$$G_n^1 = \frac{9}{2}n^{\frac{1}{4}} \sum_{i=0}^{K_n-2} \left(\hat{X}_{\lambda_{n,i+1}} - \hat{X}_{\lambda_{n,i}}\right)^2\left(F\left(\hat{\sigma}^2_{\lambda_{n,i+1}}\right) - F\left(\hat{\sigma}^2_{\lambda_{n,i}}\right)\right)^2,$$

and (24)

$$G_n^2 = \frac{9}{2}n^{\frac{1}{4}} L_n M_n \Delta t \sum_{i=0}^{K_n-2} \widehat{\sigma}^2_{\lambda_{n,i+1}}\left(F\left(\widehat{\sigma}^2_{\lambda_{n,i+1}}\right) - F\left(\widehat{\sigma}^2_{\lambda_{n,i+1}}\right)\right)^2.$$

[9]See Footnote 4.

Table 2. Because of the much slower convergence rate, the simulation results are not as good as in the case without microstructure noise

	MSE	Mean	Median	Q_1	Q_3
$n = 5$ days, $T = 1/50$ infeasible	1.315581	−0.05502	−0.01236	−0.71910	0.66330
$n = 5$ days, $T = 1/50$ feasible	1.142911	0.02566	−0.02703	−0.79680	0.80940
$n = 20$ days, $T = 2/25$ infeasible	1.193074	−0.03032	−0.003359	−0.6793	0.6578
$n = 20$ days, $T = 2/25$ feasible	1.125859	0.02167	−0.05247	−0.77740	0.76390
$n \to \infty$, fixed T (asymptotic value)	1	0	0	−0.674	0.674

NOTE: Even so, with reasonably large sample size, the mean and median are still close to 0. The MSE is not very far from 1, and the quartiles are reasonably close to the theoretical values from the standard normal distribution.

By the same methods as in the proof of Theorem 1, we have the following convergences in probability

$$\begin{aligned} G_n^1 \xrightarrow{p} & \frac{8\sqrt{c_1}}{c}\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^6\,dt \\ &+ c\sqrt{c_1}\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^4 \left(\frac{28}{3}f_t^2 + \frac{16}{3}g_t^2\right)dt \\ &+ \frac{48a^2}{cc_1^{3/2}T}\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^4\,dt \\ &+ \frac{108a^4}{cc_1^{7/2}T^2}\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^2\,dt, \end{aligned} \tag{24}$$

$$\begin{aligned} G_n^2 \xrightarrow{p} & \frac{8\sqrt{c_1}}{cT}\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^6\,dt \\ &+ c\sqrt{c_1}\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^4 \frac{16}{3}\left(f_t^2 + g_t^2\right)dt \\ &+ \frac{48a^2}{cc_1^{3/2}T^2}\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^4\,dt \\ &+ \frac{108a^4}{cc_1^{7/2}T^3}\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^2\,dt, \end{aligned} \tag{25}$$

and

$$\begin{aligned} G_n^1 + G_n^2 \xrightarrow{p} & \frac{16\sqrt{c_1}}{cT}\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^6\,dt \\ &+ c\sqrt{c_1}\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^4 \left(\frac{44}{3}f_t^2 + \frac{32}{3}g_t^2\right)dt \\ &+ \frac{96a^2}{cc_1^{3/2}T^2}\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^4\,dt \\ &+ \frac{216a^4}{cc_1^{7/2}T^3}\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^2\,dt. \end{aligned} \tag{26}$$

Equation (27) gives the estimation of the asymptotic variance. With this estimation, a feasible version of the central limit distribution can be derived.

Theorem 5. Under Assumptions 1–3, as $n \to \infty$ and T fixed,

$$\frac{n^{1/8}(\widehat{\langle X, F(\sigma^2)\rangle_T} - \langle X, F(\sigma^2)\rangle_T)}{\sqrt{G_n^1 + G_n^2}} \xrightarrow{\mathcal{L}} Z_1, \tag{28}$$

stably in law,[10] where Z_1 is a standard normal random variable and independent of $\mathcal{F}_T$.

[10]See Footnote 4.

5. SIMULATION FOR THE CASE WITH MICROSTRUCTURE NOISE

Similarly to the case without microstructure noise, the small sample behavior of the asymptotic normality can be demonstrated by simulating the statistics

$$\frac{n^{1/8}(\widehat{\langle X, F(\sigma^2)\rangle_T} - \langle X, F(\sigma^2)\rangle_T)}{\sqrt{\begin{array}{l}\frac{16\sqrt{c_1}}{c}\int_0^T (F'(\sigma_t^2))^2\sigma_t^6\,dt + c\sqrt{c_1}T\int_0^T (F'(\sigma_t^2))^2 \\ \quad \sigma_t^4(\frac{44}{3}f_t^2 + \frac{32}{3}g_t^2)\,dt + \frac{96a^2}{cc_1^{3/2}T}\int_0^T (F'(\sigma_t^2))^2\sigma_t^4\,dt \\ \quad + \frac{216a^4}{cc_1^{7/2}T^2}\int_0^T (F'(\sigma_t^2))^2\sigma_t^2\,dt\end{array}}}$$

and

$$\frac{n^{1/8}(\widehat{\langle X, F(\sigma^2)\rangle_T} - \langle X, F(\sigma^2)\rangle_T)}{\sqrt{G_n^1 + G_n^2}}.$$

The Heston model is once again adopted in the simulation. The parameterization is the same with $\kappa = 5, \theta = 0.04, \gamma = 0.5, \rho = -\sqrt{0.5}$. The true log-price process is latent. It is contaminated by market microstructure as in Equation (17). The standard deviation of noise is set to be $a = 0.0005$. This is also a realistic value in practice. Since the first step of preaveraging consumes part of the data and reduces the sample size for the second step of estimation, the choices of n are bigger than those in the case without noise. The frequency is chosen as 1 sec, which produces 23, 400 observations in each trading day. The results corroborate the theorem and are demonstrated in Table 2.

Even though only the simulations with $F(x) = x$ are presented, the results with other functions satisfying the condition in definition (2) such as $F(x) = \log x$ have been investigated. The results look very similar and the tables are omitted for the reasons of space.

6. IRREGULARLY SPACED DATA

So far our analysis in cases both with and without microstructure noise has been based on measuring prices in regularly spaced intervals. In some ways it is more natural to work with prices measured in tick time and so it would be desirable to extend the above theory to cover irregularly spaced data. This is emphasized by Zhang, Mykland, and Aït-Sahalia (2005), Barndorff-Nielsen et al. (2008a), and Renault and Werker (2011) in their studies. We here use the framework from Barndorff-Nielsen et al. (2008a).

Assumption 4. The observation times $(t_{n,i})$ satisfy the condition

$$t_{n,i} = G\left(i\frac{T}{n}\right) = \int_0^{i\frac{T}{n}} G'(s)\,ds, \quad i = 0, 1, \ldots, n,$$

where $G : [0, T] \to [0, T]$ is a strictly increasing, twice differentiable function with $G(0) = 0$, $G(T) = T$. $G'(s)$ is locally bounded away from 0, and G'' is bounded.

With the change of time under the Assumption 4, the stochastic volatility model can be written as:

$$dZ_t = dX \circ G(t) = \mu_{G(t)} G'(t)\,dt + \sigma_{G(t)}\sqrt{G'(t)}\,dW_t^e,$$

and (29)

$$ds_t = d\sigma \circ G(t) = a_{G(t)} G'(t)\,dt + f_{G(t)}\sqrt{G'(t)}\,dW_t^e + g_{G(t)}\sqrt{G'(t)}\,dB_t^e,$$

where W^e and B^e are independent Wiener processes.

Proposition 1. The leverage effect in Definition 2 satisfies $\langle X, F(\sigma^2)\rangle_T = \langle Z, F(s^2)\rangle_T$.

Proof:

$$\begin{aligned}\langle Z, F(s^2)\rangle_T &= \int_0^T F'(s^2)d\langle s^2, Z\rangle_t \\ &= \int_0^T F'\big(\sigma^2_{G(t)}\big)2\sigma^2_{G(t)} f_{G(t)} G'(t)\,dt \\ &= \int_0^T F'\big(\sigma^2_{G(t)}\big)2\sigma^2_{G(t)} f_{G(t)}\,dG(t) \\ &= \int_0^T F'\big(\sigma^2_v\big)2\sigma^2_v f_v\,dv \\ &= \langle X, F(\sigma^2)\rangle_T.\end{aligned}$$

With all index notation kept the same as in Section 2.3, the estimator of $\langle Z, F(s^2)\rangle_T$ can be constructed similar to Equation (5), with one adaptation:

$$\widehat{\langle Z, F(s^2)\rangle}_T = 2\sum_{i=0}^{K_n-2}(X \circ G((i+1)M_n\Delta t) - X \circ G(iM_n\Delta t)) \times \big(F\big(\hat{s}^2_{(i+1)M_n\Delta t}\big) - F\big(\hat{s}^2_{iM_n\Delta t}\big)\big)$$

and (30)

$$\hat{s}^2_{iM_n\Delta t} = \frac{1}{\Delta\tau_{n,i+1}} \sum_{\substack{t_{n,j+1} \in \\ (iM_n\Delta t,\ (i+1)M_n\Delta t]}} \times (X \circ G(t_{n,j+1}) - X \circ G(t_{n,j}))^2.$$

The CLT, estimation of asymptotic variance, and feasible CLT follow for the estimator $\widehat{\langle Z, F(s^2}\rangle_T$ as in the equidistant case without microstructure noise. The results for the case with microstructure noise can be derived analogously. Considering the contaminated process $Y \circ G(t) = X \circ G(t) + \epsilon \circ G(t) = Z_t + \epsilon \circ G(t)$, with all index notation kept the same as in section 4, the estimator of the leverage effect can be constructed as follows:

$$\widehat{\langle Z, F(s^2)\rangle}_T = 3\sum_{i=0}^{K_n-2}\big(\hat{Z}_{\lambda_{n,i+1}} - \hat{Z}_{\lambda_{n,i}}\big)\big(F\big(\hat{s}^2_{\lambda_{n,i+1}}\big) - F\big(\hat{s}_{\lambda_{n,i}}\big)\big),$$

$$\hat{Z}_{\tau_{n,j}} = \frac{1}{M}\sum_{\substack{t_{n,p+1} \in \\ (jM\Delta t, (j+1)M\Delta t]}} (Y \circ G(t_{n,p+1}) - Y \circ G(t_{n,p})),$$

and

$$\hat{s}^2_{\lambda_{n,i}} = \frac{1}{\Delta\lambda_{n,i+1}} \sum_{\substack{\tau_{n,j+1} \in \\ (iLM\Delta t,\ (i+1)LM\Delta t]}} (\hat{Z}_{\tau_{n,j+1}} - \hat{Z}_{\tau_{n,j}})^2. \quad (30)$$

We emphasize that this estimator is feasible (observable), since both contaminated price Y_t and observation times $t_{n,i}$ and $\lambda_{n,i}$ are directly observable from the market.

The CLT, estimator of asymptotic variance and feasible CLT can be derived in a similar manner as when observations are regularly spaced.

7. LEVERAGE EFFECT AND SKEWNESS

From sec. 2 of Mykland and Zhang (2009), leverage effect($F(x) = x$) and skewness have a close relationship. For equidistant data, the skewness of returns in high-frequency data satisfies (as $n \to \infty$)

$$\begin{aligned}&\frac{n}{T}\lim \sum_{t_{n,i+1}\le T} \Delta X^3_{t_{n,i+1}} \\ &\quad \xrightarrow{\mathcal{L}} \frac{3}{2}\langle\sigma^2, X\rangle_T + 3\int_0^T \sigma_t^3\big(dW_t + \sigma_t^{-1}\mu_t\,dt\big) \\ &\qquad + \left(6\int_0^T \sigma_t^6\,dt\right)^{1/2} Z,\end{aligned}$$

where Z is a standard normal random variable. This is a biased and inconsistent estimator, but it is interesting to find that leverage effect appears on the right-hand side. When the mean is removed from blocks of size M, this empirical skewness converges to the leverage effect plus a mixed normal error:

$$\begin{aligned}&\frac{n}{T}\lim \sum_{t_{n,i+1}\le T} (\Delta X_{t_{n,i+1}} - \text{local mean of } X)^3 \\ &\xrightarrow{\mathcal{L}} \frac{3}{2}\langle\sigma^2, X\rangle_T + \left(\frac{M-1}{M}\left(6 + \frac{18}{M} - \frac{15}{M^2}\right)\int_0^T \sigma_t^6\,dt\right)^{1/2} Z.\end{aligned}$$

M is chosen differently in Mykland and Zhang (2009) from that in this article. It is a constant instead (i.e., M does not grow with n). This relationship tells us that in the case where skewness is hard to estimate directly, the consistent estimation of leverage effect proposed by this article provides an alternative way to estimate skewness.

To further emphasize that we are indeed estimating a form of skewness by the leverage effect, we now consider the *predictable instantaneous skewness:*

$$\text{p-skew} := \frac{n}{T}\sum_{t_{n,i+1}\le T} E\big(\Delta X^3_{t_{n,i+1}}|\mathcal{F}_{t_i}\big).$$

We obtain

Proposition 2. Subject to regularity conditions, as $n \to \infty$,

$$\text{p-skew} \xrightarrow{p} \frac{3}{2}\langle \sigma^2, X\rangle_T.$$

It should be noted that since $E(\Delta X^3_{t_{n,i+1}} | \mathcal{F}_{t_i})$ is an unobservable quantity, this proposition does not yield a method of estimation. It does, however, clarify the relationship between skewness and leverage effect.

The existence of a connection between skewness and the leverage effect has previously been noted in Meddahi and Renault (2004); see the discussion following Proposition 3.4 (p. 370).

8. LEVERAGE EFFECTS AND REGRESSION COEFFICIENTS

The estimation of leverage effects also has an application to estimating the regression coefficient of the volatility on its own log return. On one hand, the existence of leverage effect implies the relation of volatility and the log return as stated below:

$$d\sigma_t^2 = 2f_t\, dX_t + 2\sigma_t g_t\, dB_t + \left(2\sigma_t a_t - 2f_t\mu_t + f_t^2 + g_t^2\right) dt, \tag{31}$$

and

$$\frac{d\langle X, \sigma_t^2\rangle_t}{d\langle X, X\rangle_t} = 2f_t. \tag{32}$$

On the other hand, the leverage effect specified as $\langle X, \log\sigma\rangle$ takes the following form:

$$\frac{2d\langle X, \log\sigma\rangle}{dt} = 2f_t. \tag{33}$$

Equations (33) and (34) suggest two ways of applying the estimation of leverage effects ($F(x) = x$ and $F(x) = \frac{1}{2}\log(x)$) to estimating the regression coefficient of the volatility process on its own log-return process. The second method only involves lower orders of the volatility process, and is thus comparatively robust. We will use this method in the next section.

9. EMPIRICAL STUDY

In the empirical study, we employ Microsoft stock trades data from the New York Stock Exchange (NYSE TAQ). The years under study are 2008 through 2011. Even though the stock is traded between 9:30 am and 4:00 pm, the window 9:45 am–3:45 pm is chosen in the empirical analysis. The reason for choosing this window is that a vast body of empirical studies documents increased return volatility and trading volume at the open and close of the stock market (Wood et al. 1985; Chan et al. 2000). A 15-min cushion at the open and close may strike a good balance between avoiding abnormal trading activities in the market and preserving enough data points to perform the estimation procedures in a consistent way. On average, there are currently several hundred thousand trades of Microsoft during each trading day. There are frequently multiple trades in each second.

In Section 8, we explored intraday high-frequency regression. It is clear that two forms of the leverage effect reveal the relation between volatility and return in the regression model. One way to extrapolate this intraday behavior to between-day volatility prediction is to include the previous day's return but scaled by a time-varying leverage effect. Technically, all regressors are now in the drift term.

Since we are not trying to discover the best model calibration for volatility prediction, but rather to investigate the predictive power of return scaled by leverage effect, the prediction model is simply a linear regression (or AR(2)). Though this may not be a very sophisticated model, the results can still improve understanding the role of leverage effects in volatility prediction:

$$\int_{t_i}^{t_{i+1}} \sigma_t^2\, dt = \alpha_0 + \alpha_1 \int_{t_{i-1}}^{t_i} \sigma_t^2\, dt + \alpha_2 \int_{t_{i-2}}^{t_{i-1}} \sigma_t^2\, dt + \alpha_3 \Delta X^2_{t_i-} + \alpha_4 \int_{t_{i-1}}^{t_i} 2f_t\, dt \times \Delta X_{t_i} + \epsilon_i.$$

- The integrated volatility $\int_{t_i}^{t_{i+1}} \sigma_t^2\, dt$ can be estimated by various methods. In this empirical study, the preaveraging method (Jacod et al. 2009) is adopted.
- ΔX_{t_i-} denotes the overnight log return.
- $\int_{t_{i-1}}^{t_i} 2f_t\, dt$ can be estimated by the proposed leverage effect estimator in this article by setting $F(x) = \frac{1}{2}\log(x)$.
- The inclusion of lagged volatilities and overnight returns is due to the empirical finding of volatility clustering.

In this study, since we do not consider the case with jumps involved, we first remove the days with jump activities by the jump test from Lee and Mykland (2012).[11] Alternatively, one can apply the jump tests as in Aït-Shalia and Jacod (2009), Barndorff-Nielsen and Shephard (2004), Barndorff-Nielsen and Shephard (2006), Mancini (2001), Podolskij and Ziggel (2010), and other works by the same authors.

We also follow the convention by preaveraging the data over every 5 min in the first step. After preaveraging, we take $c = 1$. We explore the volatility prediction over a two-day period, a longer period than one day, because of the comparatively slow convergence rate of the estimators as discussed in Section 5. To check the robustness, we also repeat the regression replacing return scaled the leverage effect by return itself. The prediction results are shown in Table 3, and the time series plot of the estimated leverage effect is given in Figure 1.

"SS explained" is the main indicator to show whether return scaled by leverage effect has a big contribution to the prediction of leverage effect. For each year, the first column of p-valued based on t-test is given only as reference. Of course, since the covariates are not independent from the response variable, these statistics cannot really tell whether the corresponding covariate should be included in the model. The third column gives the collinearity diagnostic by variance inflation factors (vif) (see, e.g., Weisberg 2004).

Since almost all vif values are close to 1, one can consider the covariates not to be collinear with each other. The sum of squares explained by the return scaled by the leverage effect (RLE) are substantial even when this term is included into the model

[11]The total numbers of days removed are 21 for 2008, 27 for 2009, 60 for 2010, and 43 for 2011.

Table 3. Two-day ahead volatility prediction results with Microsoft 2007–2010 data

	2008			2009		
	P(> |t|)	SS explained	vif	P(> |t|)	SS explained	vif
α_0	0.003909			0.035226		
RV_{t-1}	$1.75 \cdot 10^{-7}$	$7.05 \cdot 10^{-5}*$	2.050335	0.000752	$1.7633 \cdot 10^{-5} * **$	5.603527
RV_{t-2}	0.369974	$1.05 \cdot 10^{-5}*$	1.461129	0.658205	$1.9960 \cdot 10^{-6}$	1.349047
R^2_{t-}	0.0493	$9.198 \cdot 10^{-6}$	1.142936	0.570849	7.09×10^{-7}	1.062036
RLE_{t-1}	0.000454	$2.0697 \cdot 10^{-5}*$	1.551306	0.027969	$4.827 \cdot 10^{-6}*$	4.855324
or R_{t-1}	0.821	$9.0 \cdot 10^{-8}*$	1.003209	0.280060	$1.1866 \cdot 10^{-6}$	1.198528
	2010			2011		
	P(> |t|)	SS explained	vif	P(> |t|)	SS explained	vif
α_0	0.000706			0.000747		
RV_{t-1}	0.833737	$1.18 \cdot 10^{-7}$	1.453209	$5.61 \cdot 10^{-5}$	$2.2341 \cdot 10^{-6} * **$	1.622136
RV_{t-2}	0.180635	$2.066 \cdot 10^{-7}$	1.008225	0.215383	$8.6345 \times 10^{-7}*$	1.178455
R^2_{t-}	$4.32 \cdot 10^{-10}$	$1.4117 \cdot 10^{-5} * **$	1.011785	0.270385	$1.7367 \cdot 10^{-7}$	1.034753
RLE_{t-1}	0.5254	$1.172 \cdot 10^{-7}$	1.455641	0.000486	$2.1891 \cdot 10^{-6} * **$	1.463627
or R_{t-1}	0.870061	$7.8 \cdot 10^{-9}$	1.068818	0.102010	$5.0273 \cdot 10^{-7}$	1.049329

RV_t denotes the estimated integrated volatility at day t; R_{t-} denotes overnight return for day t; *RLE* denotes the log return scaled by leverage effect at day t (estimated leverage effect $\times$ log return). R_{t-1} denotes the previous period return itself without scaling. “SS explained” denotes the sum of squares *gained* by adding each covariate in the order presented in the table.

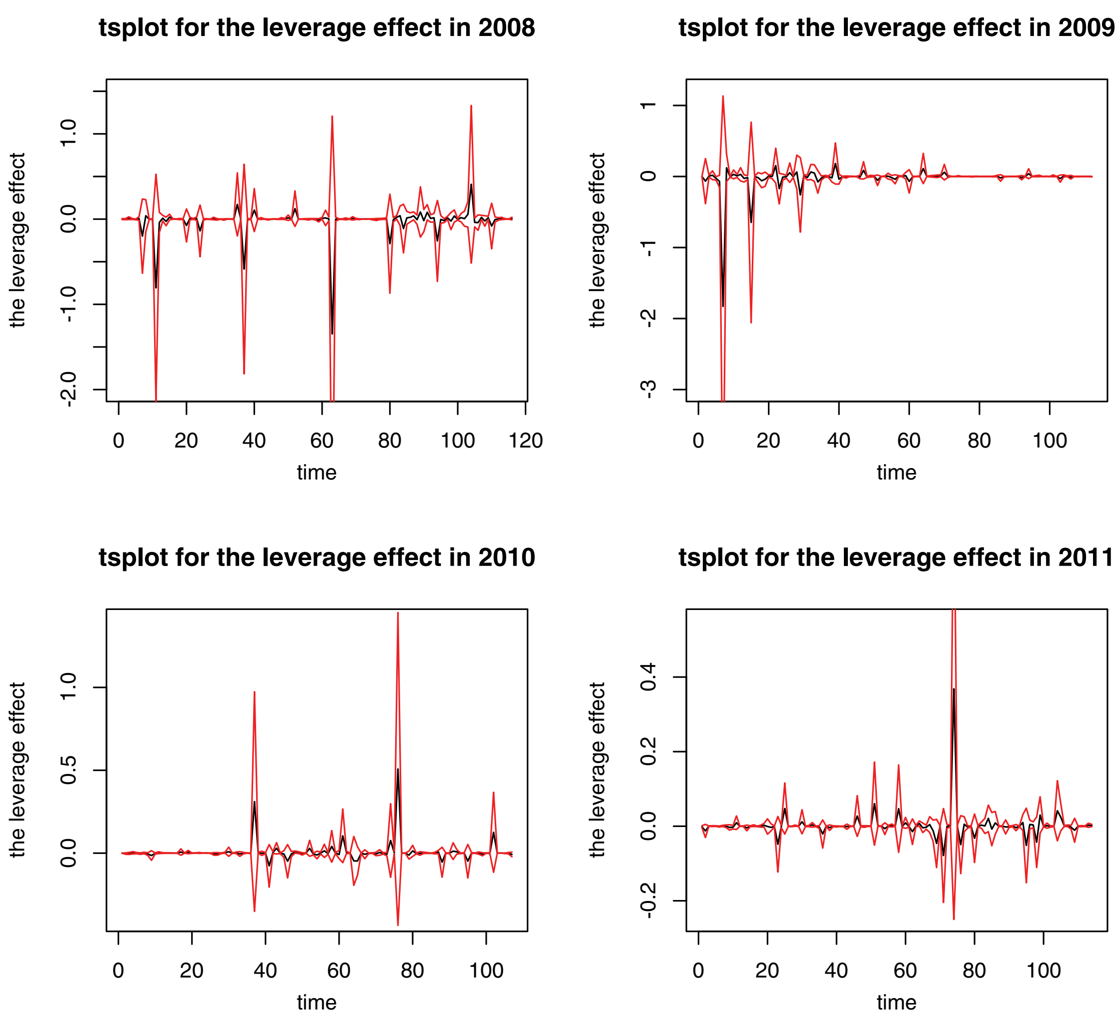

Figure 1. TS-plot of the estimated leverage effects: The black curves present the time series plots of the estimated leverage effects. The red curves give the 95% confidence intervals of the estimated values. The values on the vertical axes are different from one year to another. That is due to the different magnitudes of the estimated leverage effects. Apparently year 2008 and 2009 display the biggest negative leverage effects. This observation coincides with the empirical fact during the financial crisis.

last. Most of these sums of squares are comparable to the sum of squares explained by the previous day's volatility, which is believed to be the most significant factor for volatility prediction (Engle 1982; Bollerslev 1986). In all cases, RLE has stronger predictive power than the two-period ahead integrated volatility does. This strongly suggests the inclusion of return scaled by leverage effect into any model trying to predict next-period's volatility. The predictive power of a time-varying leverage effect estimator[12] is consistent with the earlier work by Engle and Ng (1993) and Chen and Ghysels (2011), but here appears in a new form. In addition, the previous period return does not contribute to the sum of squares as much as the one scaled by the leverage effect. In some case, the previous period return is not significant while the return scaled by the leverage effect explains significant amount of sum of squares.

10. CONCLUSION

This article provides nonparametric estimators of the leverage effect, and analyzes them both theoretically and in simulation. The definition of the stochastic parameter of the leverage effect involves a twice differentiable monotone function. Even though the reliance of the estimation on higher moments of volatility is of concern in practice, the carefully chosen function F can help to reduce the order of moments required and provide robust results. Other benefits of this function can be easily seen in the discussion of the connection between the leverage effect and skewness. While the sum of intraday cubic returns is not a consistent estimator of skewness, the p-skewness (Section 7) can instead be estimated consistently by the leverage effect estimators proposed by this article. The related properties of p-skewness can also be studied by the properties of the leverage effect. Clever choices of the function F can also reduce the work of estimation, such as the estimation of high-frequency regression coefficients. Instead of estimating both leverage effects and realized volatility, a different form of the leverage effect can serve as the estimated coefficient (Section 8). If the properties of the estimated coefficient are of interest, it is more attractive to apply the method in Equation (34), whose statistical properties have already been studied in this article, than to apply the first ratio statistic in Equation (33) whose statistical properties require further efforts to investigate.

The bias correction factors in the estimators contribute to an important finding in this article and are previously unknown. They not only provide the consistency of the estimation, but also imply that simple covariance estimators tend to underestimate the leverage effect, especially when the values of the leverage effect are close to zero. The amplifying factors play a vital role of bias correction in the estimation.

The empirical studies demonstrate the importance of the leverage effect in volatility prediction. Even though the simple regression (or AR(2)) model is adopted in the study, the explanatory power of RLE is surprisingly high. The power is almost of the same magnitude as the predictive power of the previous period volatility which is widely considered to be the main source of variation in volatility prediction. This high explanatory power suggests that time-varying leverage effects should be included additionally to explain the variation and clustering in volatility prediction models.

Even though we have provided a way to deal with irregularly spaced data, it is important to study the estimation of leverage effects and the asymptotic properties of estimators when time is endogenous (as in Li et al. 2013). Different methods of dealing with microstructure noise should also be studied and compared with the ones in this article. Our findings create the important necessary foundation for further analysis both theoretically and empirically, as well as an investigation of how to carry out risk management in the presence of leverage effects.

Finally, as discussed in Section 2.2, many open questions remain in terms of model specification (continuous vs. jumps, semimartingale vs. long range dependence), with reference to the papers cited in that section. In particular, the connection to semivariance remains to be explored.

APPENDIX A: PROOF OF THEOREM 1

A.1 Preliminaries

In the following, by $\mathcal{F}_j$ we mean $\mathcal{F}_{t_{n,j}}$, and p is a positive integer. Without loss of generality, we will set $\mu_t = 0$, see Mykland and Zhang (2009), sec. 2.2, as well as our current Assumption 1 and associated Footnote 1. Recall that

$$\widehat{\langle X, F(\sigma^2)\rangle}_T = 2\sum_i \left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}}\right)\left(F\left(\hat{\sigma}^2_{\tau_{n,i+1}}\right) - F\left(\hat{\sigma}^2_{\tau_{n,i}}\right)\right) \quad (A.1)$$

and

$$\hat{\sigma}^2_{\tau_{n,i}} = \frac{1}{M_n \times \Delta t} \sum_{t_{n,j+1}\in(\tau_{n,i},\tau_{n,i+1}]} \Delta X^2_{t_{n,j+1}}. \quad (A.2)$$

Lemma 1. Under Assumption 1,

$$\begin{aligned} E\left(\left(X_{t_{n,k}} - X_{t_{n,j}}\right)^2 \sigma^p_{t_{n,m}} \middle| \mathcal{F}_j\right) &= \sigma^{p+2}_{t_{n,j}}(t_{n,k} - t_{n,j}) + p^2 f^2_{t_{n,j}} \sigma^p_{t_{n,j}} (t_{n,k} - t_{n,j})^2 \\ &\quad + p\sigma^p_{t_{n,j}} \langle X, f\rangle_{t_{n,j}} (t_{n,k} - t_{n,j})^2 \\ &\quad + \frac{1}{2} p(p-1)\left(f^2_{t_{n,j}} + g^2_{t_{n,j}}\right)\sigma^p_{t_{n,j}}(t_{n,k} - t_{n,j})(t_{n,m} - t_{n,j}) \\ &\quad + \left(p + \frac{1}{2}\right)\left(f^2_{t_{n,j}} + g^2_{t_{n,j}}\right)\sigma^p_{t_{n,j}}(t_{n,k} - t_{n,j})^2 \\ &\quad + a_{t_{n,j}}\sigma^{p+1}_{t_{n,j}}(t_{n,k} - t_{n,j})^2 \\ &\quad + p a_{t_{n,j}}\sigma^{p+1}_{t_{n,j}}(t_{n,k} - t_{n,j})(t_{n,m} - t_{n,j}) + O_p(\Delta t^{5/2}), \end{aligned} \quad (A.3)$$

$$E\left(\left(X_{t_{n,k}} - X_{t_{n,j}}\right)\sigma^p_{t_{n,m}} \middle| \mathcal{F}_j\right) = p\sigma^p_{t_{n,j}} f_{t_{n,j}}(t_{n,k} - t_{n,j}) + O_p(\Delta t^{3/2}), \quad (A.4)$$

[12] The main motivation for us to include "leverage effect scaled returns" is the earlier empirical findings on asymmetric impact of positive or negative returns on the volatility process. To capture this asymmetric impact in the prediction model, we include the extra term. We realize that Jacod (1994, 1996), Barndorff-Nielsen, and Shephard (2005), and others, showed that the correlation (leverage effect) has no impact on the asymptotic distribution. These observations seem to suggest that asymmetries do not matter for forecasting, but that is not so. The concept of the news impact curve (Engle and Ng 1993) was originally formulated within the context of daily ARCH-type models. In the models of Engle and Ng (1993), the returns are included in the volatility prediction models by differently scaling the positive or negative returns. Here we applied the intraday data over small time interval to estimate integrated volatility, but predict the next volatility over a much longer time interval (two-day ahead). Therefore, the prediction falls into a comparative low frequency setting. The leverage effect turns out to have a significant impact on the volatility prediction which is supported by our empirical finding. This inclusion of leverage effect scaled returns in the volatility prediction model is further supported by the findings in Chen and Ghysels (2011), where they are dealing with two different frequencies and reaching a similar conclusion.

and

$$
\begin{aligned}
&E\big(2\big(X_{t_{n,j}}-X_{t_{n,i}}\big)\sigma^2_{t_{n,j+1}}\big(X_{t_{n,k}}-X_{t_{n,j}}\big)\sigma^2_{t_{n,k+1}}\big|\mathcal{F}_i\big)\\
&\quad= 16\sigma^4_{t_{n,i}}f^2_{t_{n,i}}(t_{n,k}-t_{n,j+1})(t_{n,j}-t_{n,i})\\
&\qquad+4\sigma^4_{t_{n,i}}\langle X,f\rangle_{t_{n,i}}(t_{n,k}-t_{n,j+1})(t_{n,j}-t_{n,i})+O_p(\Delta t^{5/2}), \quad \text{(A.5)}
\end{aligned}
$$

where $t_{n,i}<t_{n,j}<t_{n,k}\le t_{n,m}$, $t_{n,m}-t_{n,j}=O_p(\Delta t^{\frac{1}{2}})$ and $\Delta t=\frac{T}{n}$.

Proof of Lemma 1:

1. For (A.3), note first that

$$
\begin{aligned}
&\sigma^p_{t_{n,m}}-\sigma^p_{t_{n,j}}\\
&\quad= p\int_{t_{n,j}}^{t_{n,m}}\sigma_t^{p-1}\,d\sigma_t+\frac{p(p-1)}{2}\int_{t_{n,j}}^{t_{n,m}}\sigma_t^{p-2}\big(f_t^2+g_t^2\big)\,dt,
\end{aligned}
$$

and

$$
\begin{aligned}
&\big(X_{t_{n,k}}-X_{t_{n,j}}\big)^2\\
&\quad=\int_{t_{n,j}}^{t_{n,k}}2\big(X_t-X_{t_{n,j}}\big)\,dX_t+\int_{t_{n,j}}^{t_{n,k}}\sigma_t^2\,dt\\
&\quad=\int_{t_{n,j}}^{t_{n,k}}2\big(X_t-X_{t_{n,j}}\big)\,dX_t+\int_{t_{n,j}}^{t_{n,k}}(t_{n,k}-t)d\big(\sigma_t^2-\sigma^2_{t_{n,j}}\big)\\
&\qquad+\sigma^2_{t_{n,j}}(t_{n,k}-t_{n,j})\\
&\quad=\int_{t_{n,j}}^{t_{n,k}}2\big(X_t-X_{t_{n,j}}\big)\,dX_t+\int_{t_{n,j}}^{t_{n,k}}(t_{n,k}-t)2\sigma_t d\sigma_t\\
&\qquad+\int_{t_{n,j}}^{t_{n,k}}(t_{n,k}-t)\big(f_t^2+g_t^2\big)\,dt\\
&\qquad+\sigma^2_{t_{n,j}}(t_{n,k}-t_{n,j}).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&E\big(\big(X_{t_{n,k}}-X_{t_{n,j}}\big)^2\sigma^p_{t_{n,m}}\big|\mathcal{F}_j\big)\\
&\quad=E\Bigg(\Bigg[\sigma^p_{t_{n,j}}+\frac{p(p-1)}{2}\int_{t_{n,j}}^{t_{n,m}}\sigma_t^{p-2}\big(f_t^2+g_t^2\big)\,dt\Bigg]\\
&\qquad\times\Bigg[\int_{t_{n,j}}^{t_{n,k}}2\big(X_t-X_{t_{n,j}}\big)dX_t+\sigma^2_{t_{n,j}}(t_{n,k}-t_{n,j})\\
&\qquad+\int_{t_{n,j}}^{t_{n,k}}(t_{n,k}-t)2\sigma_t a_t\,dt+\int_{t_{n,j}}^{t_{n,k}}(t_{n,k}-t)\big(f_t^2+g_t^2\big)\,dt\Bigg]\\
&\qquad+p\int_{t_{n,j}}^{t_{n,k}}2\big(X_t-X_{t_{n,j}}\big)\sigma_t^{p-1}d\langle X,\sigma\rangle_t\\
&\qquad+p\int_{t_{n,j}}^{t_{n,k}}2(t_{n,k}-t)\sigma_t^p d\langle\sigma,\sigma\rangle_t+\sigma^2_{t_{n,j}}(t_{n,k}-t_{n,j})\\
&\qquad\times p\int_{t_{n,j}}^{t_{n,m}}\sigma_t^{p-1}a_t\,dt|\mathcal{F}_j\Bigg)+O_p(\Delta t^{5/2})\\
&\quad=E\Bigg(\sigma^{p+2}_{t_{n,j}}(t_{n,k}-t_{n,j})+\sigma^{p+1}_{t_{n,j}}a_{t_{n,j}}(t_{n,k}-t_{n,j})^2\\
&\qquad+\frac{1}{2}\big(f^2_{t_{n,j}}+g^2_{t_{n,j}}\big)\sigma^p_{t_{n,j}}(t_{n,k}-t_{n,j})^2\\
&\qquad+\frac{p(p-1)}{2}\big(f^2_{t_{n,j}}+g^2_{t_{n,j}}\big)\sigma^p_{t_{n,j}}(t_{n,k}-t_{n,j})(t_{n,m}-t_{n,j})\\
&\qquad+pf_{t_{n,j}}\int_{t_{n,j}}^{t_{n,k}}2\big(X_t-X_{t_{n,j}}\big)\sigma_t^p\,dt\\
&\qquad+p\sigma^p_{t_{n,j}}\int_{t_{n,j}}^{t_{n,m}}2\big(X_t-X_{t_{n,j}}\big)\big(f_t-f_{t_{n,j}}\big)\,dt\\
&\qquad+p\int_{t_{n,j}}^{t_{n,k}}2(t_{n,k}-t)\sigma_t^p\big(f_t^2+g_t^2\big)\,dt\\
&\qquad+p\sigma^{p+1}_{t_{n,j}}a_{t_{n,j}}(t_{n,k}-t_{n,j})(t_{n,m}-t_{n,j})|\mathcal{F}_j\Bigg)+O_p(\Delta t^{5/2})\\
&\quad=E\Bigg(\sigma^{p+2}_{t_{n,j}}(t_{n,k}-t_{n,j})+\sigma^{p+1}_{t_{n,j}}a_{t_{n,j}}(t_{n,k}-t_{n,j})^2\\
&\qquad+\Big(\frac{1}{2}+p\Big)\big(f^2_{t_{n,j}}+g^2_{t_{n,j}}\big)\sigma^p_{t_{n,j}}(t_{n,k}-t_{n,j})^2\\
&\qquad+\frac{p(p-1)}{2}\big(f^2_{t_{n,j}}+g^2_{t_{n,j}}\big)\sigma^p_{t_{n,j}}(t_{n,k}-t_{n,j})(t_{n,m}-t_{n,j})\\
&\qquad+p^2f_{t_{n,j}}\int_{t_{n,j}}^{t_{n,k}}(t_{n,k}-t)2\sigma_t^p f_t\,dt\\
&\qquad+p\sigma^p_{t_{n,j}}\langle X,f\rangle_{t_{n,j}}(t_{n,k}-t_{n,j})^2+p\sigma^{p+1}_{t_{n,j}}a_{t_{n,j}}(t_{n,k}-t_{n,j})\\
&\qquad\times(t_{n,m}-t_{n,j})|\mathcal{F}_j\Bigg)+O_p(\Delta t^{5/2})\\
&\quad=\sigma^{p+2}_{t_{n,j}}(t_{n,k}-t_{n,j})+a_{t_{n,j}}\sigma^{p+1}_{t_{n,j}}(t_{n,k}-t_{n,j})^2\\
&\qquad+\Big(p+\frac{1}{2}\Big)\big(f^2_{t_{n,j}}+g^2_{t_{n,j}}\big)\sigma^p_{t_{n,j}}(t_{n,k}-t_{n,j})^2\\
&\qquad+\frac{p(p-1)}{2}\big(f^2_{t_{n,j}}+g^2_{t_{n,j}}\big)\sigma^p_{t_{n,j}}(t_{n,k}-t_{n,j})(t_{n,m}-t_{n,j})\\
&\qquad+p^2f^2_{t_{n,j}}\sigma^p_{t_{n,j}}(t_{n,k}-t_{n,j})^2\\
&\qquad+p\sigma^p_{t_{n,j}}\langle X,f\rangle_{t_{n,j}}(t_{n,k}-t_{n,j})^2\\
&\qquad+pa_{t_{n,j}}\sigma^{p+1}_{t_{n,j}}(t_{n,k}-t_{n,j})(t_{n,m}-t_{n,j})+O_p(\Delta t^{5/2}).
\end{aligned}
$$

2. For (A.4):

$$
\begin{aligned}
&E\big((X_{t_{n,k}}-X_{t_{n,j}})\sigma^p_{t_{n,m}}\big|\mathcal{F}_j\big)\\
&\quad=E\Bigg(p\int_{t_{n,j}}^{t_{n,k}}\sigma_t^{p-1}d\langle X,\sigma\rangle_t|\mathcal{F}_j\Bigg)+O_p(\Delta t^{3/2})\\
&\quad=E\Bigg(p\int_{t_{n,j}}^{t_{n,k}}\sigma_t^p f_t\,dt|\mathcal{F}_j\Bigg)+O_p(\Delta t^{3/2})\\
&\quad=p\sigma^p_{t_{n,j}}f_{t_{n,j}}(t_{n,k}-t_{n,j})+O_p(\Delta t^{3/2}).
\end{aligned}
$$

3. (A.5) is the direct consequence of (A.3) and (A.4). This completes the proof.

Later in the derivation of limit theorem, it is easy to see that the proof and calculations strongly depend on Lemma 1, which will be applied recursively.

A.2 Main Martingale Representation and Argument to Prove Theorem

The proof of Theorem 1 is provided in this section, with supporting details in the following sections. Construct an approximate martingale (MG) on the grid of the $\tau_{n,i}$'s as follows:

$$
\begin{aligned}
&\frac{1}{\sqrt{M_n\Delta t}}\widehat{\langle X,F(\sigma^2)\rangle}_t\\
&\quad=\frac{2}{\sqrt{M_n\Delta t}}\Bigg\{\sum_{i=0,\tau_{n,i+1}\le t}^{K_n-2}\big(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\big)\big(F\big(\hat\sigma^2_{\tau_{n,i+1}}\big)-F\big(\hat\sigma^2_{\tau_{n,i}}\big)\big)\Bigg\}\\
&\quad=\frac{2}{\sqrt{M_n\Delta t}}\Bigg\{\sum_{i=0,\tau_{n,i+1}\le t}^{K_n-2}\big(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\big)\big(F\big(\hat\sigma^2_{\tau_{n,i+1}}\big)-F\big(\sigma^2_{\tau_{n,i+1}}\big)\\
&\qquad+F\big(\sigma^2_{\tau_{n,i+1}}\big)-F\big(\sigma^2_{\tau_{n,i}}\big)+F\big(\sigma^2_{\tau_{n,i}}\big)-F\big(\hat\sigma^2_{\tau_{n,i}}\big)\big)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{\sqrt{M_n\Delta t}}\left\{\sum_{i=0,\tau_{n,i+1}\le t}^{K_n-2}\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)\left(F\left(\sigma^2_{\tau_{n,i+1}}\right)-F\left(\sigma^2_{\tau_{n,i}}\right)\right)\right.\\
&\quad+\sum_{i=0,\tau_{n,i+1}\le t}^{K_n-2}\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)\left(F\left(\hat\sigma^2_{\tau_{n,i+1}}\right)-F\left(\sigma^2_{\tau_{n,i+1}}\right)\right)\\
&\quad\left.-\sum_{i=0,\tau_{n,i+1}\le t}^{K_n-2}\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)\left(F\left(\hat\sigma^2_{\tau_{n,i}}\right)-F\left(\sigma^2_{\tau_{n,i}}\right)\right)\right\}\\
&= \frac{2}{\sqrt{M_n\Delta t}}\left\{\sum_{i=0,\tau_{n,i+1}\le t}^{K_n-2}\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)\left(F\left(\sigma^2_{\tau_{n,i+1}}\right)-F\left(\sigma^2_{\tau_{n,i}}\right)\right)\right.\\
&\quad+\sum_{i=1,\tau_{n,i+1}\le t}^{K_n-1}\left(X_{\tau_{n,i}}-X_{\tau_{n,i-1}}\right)\left(F\left(\hat\sigma^2_{\tau_{n,i}}\right)-F\left(\sigma^2_{\tau_{n,i}}\right)\right)\\
&\quad\left.-\sum_{i=0,\tau_{n,i+1}\le t}^{K_n-2}\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)\left(F\left(\hat\sigma^2_{\tau_{n,i}}\right)-F\left(\sigma^2_{\tau_{n,i}}\right)\right)\right\}\\
&= \frac{2}{\sqrt{M_n\Delta t}}\left\{\underbrace{\sum_{i=0,\tau_{n,i+1}\le t}^{K_n-2}\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)\left(\sigma^2_{\tau_{n,i+1}}-\sigma^2_{\tau_{n,i}}\right)F'\left(\sigma^2_{\tau_{n,i}}\right)}_{(1)}\right.\\
&\quad+\underbrace{\sum_{i=1,\tau_{n,i+1}\le t}^{K_n-1}\left(X_{\tau_{n,i}}-X_{\tau_{n,i-1}}\right)\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)F'\left(\sigma^2_{\tau_{n,i}}\right)}_{(2)}\\
&\quad\left.-\underbrace{\sum_{i=0,\tau_{n,i+1}\le t}^{K_n-2}\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)F'\left(\sigma^2_{\tau_{n,i}}\right)}_{(3)}\right\}+o_p(1).
\end{aligned}
$$

It follows that

$$
\frac{1}{\sqrt{M_n\Delta t}}(\widehat{\langle X, F(\sigma^2)\rangle}_t-\langle X, F(\sigma^2)\rangle_t)=\sum_{i=0,\tau_{n,i+1}\le t}\Delta V^n_{\tau_{n,i+1}}+o_p(1),
$$

where, except for a few terms at the edge,

$$
\begin{aligned}
\Delta V^n_{\tau_{n,i+1}} &= \frac{2}{\sqrt{M_n\Delta t}}\left\{\underbrace{\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)\left(\sigma^2_{\tau_{n,i+1}}-\sigma^2_{\tau_{n,i}}\right)F'\left(\sigma^2_{\tau_{n,i}}\right)}_{(1)}\right.\\
&\quad+\underbrace{\left(X_{\tau_{n,i}}-X_{\tau_{n,i-1}}\right)\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)F'\left(\sigma^2_{\tau_{n,i}}\right)}_{(2)}\\
&\quad-\underbrace{\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)F'\left(\sigma^2_{\tau_{n,i}}\right)}_{(3)}\\
&\quad-\underbrace{\int_{\tau_i}^{\tau_{i+1}}F'\left(\sigma^2_t\right)\sigma^2_t f_t\,dt}_{(4)}.
\end{aligned}
$$

And so the martingale increment becomes

$$
\Delta M^n_{\tau_{i+1}} = \Delta V^n_{\tau_{n,i+1}} - E\left(\Delta V^n_{\tau_{n,i+1}}|\mathcal{F}_i\right). \tag{A.6}
$$

The martingale up to time t is

$$
M^n_t = \sum_{\tau_{n,i+1}\le t}\left\{\Delta V^n_{\tau_{n,i+1}} - E\left(\Delta V^n_{\tau_{n,i+1}}|\mathcal{F}_i\right)\right\}. \tag{A.7}
$$

Although M^n above is observed in discrete time, we can interpolate the martingale into a continuous martingale up to any time t; see, in particular, Heath (1977) as well as Mykland (1995) and the references therein. This interpolation is closely related to Skorokhod embedding in Brownian motion; see, for example, Appendix I of Hall and Heyde (1980). Then we only need to prove the CLT for the interpolated continuous martingale. Therefore, we can apply Theorem 2.28 (p. 152) in Mykland and Zhang (2012) to prove the CLT. The conditions of cited theorem will follow from the development in the rest of Appendix 10, in particular (A.16) and (A.19), and the approximation in Lemma 2. Alternatively, one can develop a functional argument for the CLT as in Jacad (2009), Jacod and Shiryaev (2003), Jacod and Protter (2011), and Podolskij and Vetter (2009b).

A.3 The Aggregate Conditional Variance

To calculate the quadratic variation, we will calculate the aggregate conditional variance of M^n_t

$$
\begin{aligned}
&\sum_{\tau_{i+1}\le t}\operatorname{var}\left(\Delta M^n_{\tau_{i+1}}|\mathcal{F}_i\right)\\
&\quad=\sum_{\tau_{i+1}\le t}E\left(\left(\Delta M^n_{\tau_{i+1}}\right)^2|\mathcal{F}_i\right)\\
&\quad=\sum_{\tau_{i+1}\le t}\left\{E\left(\left(\Delta V^n_{\tau_{n,i+1}}\right)^2|\mathcal{F}_i\right)-\left(E\left(\Delta V^n_{\tau_{n,i+1}}|\mathcal{F}_i\right)\right)^2\right\}.
\end{aligned} \tag{A.8}
$$

We will first prove that $\sum_{\tau_{i+1}\le t}(E(\Delta V^n_{\tau_{n,i+1}}|\mathcal{F}_i))^2$ is negligible, of order $O_p(M_n^2\Delta t^2)$. Except a few terms at the edge, the following could be established:

$$
\begin{aligned}
&E\left(\Delta V^n_{\tau_{n,i+1}}|\mathcal{F}_i\right)\\
&\quad=\frac{2}{\sqrt{M_n\Delta t}}E\left(\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)\left(\sigma^2_{\tau_{n,i+1}}-\sigma^2_{\tau_{n,i}}\right)F'\left(\sigma^2_{\tau_i}\right)\right.\\
&\qquad+\left(X_{\tau_{n,i}}-X_{\tau_{n,i-1}}\right)\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)F'\left(\sigma^2_{\tau_i}\right)\\
&\qquad\left.-\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)F'\left(\sigma^2_{\tau_i}\right)|\mathcal{F}_i\right)\\
&\quad=\frac{2F'\left(\sigma^2_{\tau_{n,i}}\right)}{\sqrt{M_n\Delta t}}E\left(\int_{\tau_i}^{\tau_{i+1}}\sigma^2_t f_t\,dt-\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)\right.\\
&\qquad\left.+\left(X_{\tau_{n,i}}-X_{\tau_{n,i-1}}\right)\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)|\mathcal{F}_i\right)+O_p(M_n\Delta t)\\
&\quad=\frac{2F'\left(\sigma^2_{\tau_{n,i}}\right)}{\sqrt{M_n\Delta t}}E\left(\int_{\tau_i}^{\tau_{i+1}}\left(\sigma^2_t f_t-\sigma^2_{\tau_{n,i}}f_{\tau_{n,i}}\right)dt\right.\\
&\qquad\left.-\sum_{j=1}^{M}\int_{\tau_{n,i}}^{t_{n,j}}2\left(\sigma^2_t f_t-\sigma^2_{\tau_{n,i}}f_{\tau_{n,i}}\right)dt|\mathcal{F}_i\right)\\
&\qquad+\frac{2F'\left(\sigma^2_{\tau_{n,i}}\right)}{\sqrt{M_n\Delta t}}\left(X_{\tau_{n,i}}-X_{\tau_{n,i-1}}\right)E\left(\int_{\tau_{n,i}}^{\tau_{n,i+1}}\sigma^2_t-\sigma^2_{\tau_{n,i}}\,dt|\mathcal{F}_i\right)\\
&\qquad+O_p(M_n\Delta t).
\end{aligned} \tag{A.9}
$$

We derive that

$$
\begin{aligned}
&E\left(\sum_{\tau_{n,i+1}\le t}\left(E\left(\Delta V^n_{\tau_{n,i+1}}|\mathcal{F}_i\right)\right)^2\right)\\
&\quad\le\frac{12F'\left(\sigma^2_{\tau_i}\right)^2}{M_n\Delta t}\sum_{\tau_{i+1}\le t}E\left\{\left(E\int_{\tau_i}^{\tau_{i+1}}\left(\sigma^2_t f_t-\sigma^2_{\tau_{n,i}}f_{\tau_{n,i}}\right)dt|\mathcal{F}_i\right)^2\right.\\
&\qquad+\left(\sum_{j=1}^{M}E\left(\int_{\tau_{n,i}}^{t_{n,j}}2\left(\sigma^2_t f_t-\sigma^2_{\tau_{n,i}}f_{\tau_{n,i}}\right)dt|\mathcal{F}_i\right)\right)^2\\
&\qquad\left.+\left(X_{\tau_{n,i}}-X_{\tau_{n,i-1}}\right)^2\left(E\left(\int_{\tau_{n,i}}^{\tau_{n,i+1}}\sigma^2_t-\sigma^2_{\tau_{n,i}}\,dt|\mathcal{F}_i\right)\right)^2\right\}\\
&\qquad+O_p\left((M_n\Delta t)^2\right)\\
&\quad\le K(M_n\Delta t)^2.
\end{aligned} \tag{A.10}
$$

With this result, we can calculate the conditional variance by only considering conditional second moments instead.

A.4 Aggregate Conditional Second Moment

$$\begin{aligned}
&\sum_{\tau_{n,i+1}\le t} E\left(\left(\Delta V^n_{\tau_{i+1}}\right)^2 \middle| \mathcal{F}_i\right)\\
&= \frac{4}{M_n\Delta t}\sum_{\tau_{i+1}\le t} E\Bigg(\Bigg[\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)\left(\sigma^2_{\tau_{n,i+1}}-\sigma^2_{\tau_{n,i}}\right)F'\left(\sigma^2_{\tau_i}\right)\\
&\quad + \left(X_{\tau_{n,i}}-X_{\tau_{n,i-1}}\right)\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)F'\left(\sigma^2_{\tau_i}\right)\\
&\quad - \left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)F'\left(\sigma^2_{\tau_i}\right)\\
&\quad - \int_{\tau_i}^{\tau_{i+1}} F'\left(\sigma^2_t\right)\sigma^2_t f_t\,dt\Bigg]^2 \Bigg|\mathcal{F}_i\Bigg)\\
&= \frac{4F'\left(\sigma^2_{\tau_i}\right)^2}{M_n\Delta t}\sum_{\tau_{i+1}\le t} E((X_{\tau_{n,i+1}}-X_{\tau_{n,i}})^2\left(\sigma^2_{\tau_{n,i+1}}-\sigma^2_{\tau_{n,i}}\right)^2\\
&\quad + \left(X_{\tau_{n,i}}-X_{\tau_{n,i-1}}\right)^2\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)^2\\
&\quad + \left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)^2\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)^2\\
&\quad - 2\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)^2\left(\sigma^2_{\tau_{n,i+1}}-\sigma^2_{\tau_{n,i}}\right)\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)\\
&\quad + \left(\int_{\tau_i}^{\tau_{i+1}}\sigma^2_t f_t\,dt\right)^2 - 2(X_{\tau_{n,i+1}}-X_{\tau_{n,i}})\left(\sigma^2_{\tau_{n,i+1}}-\sigma^2_{\tau_{n,i}}\right)\\
&\quad \times \int_{\tau_i}^{\tau_{i+1}}\sigma^2_t f_t\,dt|\mathcal{F}_i\Bigg) + O_p(M_n\Delta t). \qquad \text{(A.11)}
\end{aligned}$$

Applying Lemma (1), we can prove the following results

$$\begin{aligned}
&\frac{1}{M_n\Delta t}\sum_{\tau_{n,i+1}\le t} E\left(\left(X_{\tau_{n,i}}-X_{\tau_{n,i-1}}\right)^2\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)^2\middle|\mathcal{F}_i\right)\\
&= \frac{2}{M_n^2\Delta t}\sum_{\tau_{i+1}\le t}\sigma^6_{\tau_{n,i}}M_n\Delta t + \frac{4}{3}\sum_{\tau_{i+1}\le t}\sigma^4_{\tau_{n,i}}\left(f^2_{\tau_{n,i}}+g^2_{\tau_{n,i}}\right)M_n\Delta t\\
&\quad + O_p(M_n\Delta t), \qquad \text{(A.12)}
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{M_n\Delta t}\sum_{\tau_{n,i+1}\le t} E\Bigg(\left(\int_{\tau_i}^{\tau_{i+1}}\sigma^2_t f_t\,dt\right)^2 - 2(X_{\tau_{n,i+1}}-X_{\tau_{n,i}})\\
&\quad \times \left(\sigma^2_{\tau_{n,i+1}}-\sigma^2_{\tau_{n,i}}\right)\int_{\tau_i}^{\tau_{i+1}}\sigma^2_t f_t\,dt|\mathcal{F}_i\Bigg)\\
&\quad = -\sum_{\tau_{i+1}\le t}\sigma^4_{\tau_{n,i}}f^2_{\tau_{n,i}}M_n\Delta t + O_p(M_n\Delta t), \qquad \text{(A.13)}
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{M_n\Delta t}\sum_{\tau_{n,i+1}\le t} E\left(\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)^2\left(\sigma^2_{\tau_{n,i+1}}-\sigma^2_{\tau_{n,i}}\right)^2\right.\\
&\quad \left. - 2\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)^2\left(\sigma^2_{\tau_{n,i+1}}-\sigma^2_{\tau_{n,i}}\right)\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)\middle|\mathcal{F}_i\right)\\
&\quad = O_p(M_n\Delta t), \qquad \text{(A.14)}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{M_n\Delta t}\sum_{\tau_{n,i+1}\le t} E\left(\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)^2\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)^2\middle|\mathcal{F}_i\right)\\
&\quad = \sum_{\tau_{i+1}\le t}\frac{2}{M_n^2\Delta t}\sigma^6_{\tau_{n,i}}M_n\Delta t + \frac{4}{3}\sigma^4_{\tau_{n,i}}\left(f^2_{\tau_{n,i}}+g^2_{\tau_{n,i}}\right)M_n\Delta t\\
&\quad\quad + 2\sigma^4_{\tau_{n,i}}f^2_{\tau_{n,i}}M_n\Delta t + O_p(M_n\Delta t), \qquad \text{(A.15)}
\end{aligned}$$

The above three equations lead to the following convergence:

$$\begin{aligned}
\sum_{\tau_{i+1}\le t}\operatorname{var}\left(\Delta M^n_{\tau_{i+1}}\middle|\mathcal{F}_i\right) &\xrightarrow{p} \frac{16}{c^2 t}\int_0^t\left(F'\left(\sigma^2_s\right)\right)^2\sigma^6_s\,ds\\
&\quad + \int_0^t\left(F'\left(\sigma^2_s\right)\right)^2\sigma^4_s\left(\frac{44}{3}f^2_s+\frac{32}{3}g^2_s\right)ds. \qquad \text{(A.16)}
\end{aligned}$$

To finalize the proof of Theorem 1, we need the quadratic variation of the *interpolated* martingale. The following lemma shows that the asymptotics of the quadratic variation is the same as that of the aggregate conditional variance as in Equation (A.16)

Lemma 2.

$$[M^n, M^n]_t - \sum_{\tau_{n,i}\le t}\operatorname{var}\left(\Delta M^n_{\tau_{n,i}}\middle|\mathcal{F}_i\right) = o_p(1) \qquad \text{(A.17)}$$

Proof of Lemma 2:

$$\begin{aligned}
&E\left([M^n, M^n]_t - \sum_{\tau_{n,i}\le t}\operatorname{var}\left(\Delta M^n_{\tau_{n,i}}\middle|\mathcal{F}_i\right)\right)^2\\
&= E\left(\sum_{\tau_{n,i}\le t}\left(\Delta M^n_{\tau_{n,i}}\right)^2 - \sum_{\tau_{n,i}\le t}\operatorname{var}\left(\Delta M^n_{\tau_{n,i}}\middle|\mathcal{F}_i\right)\right)^2\\
&= \sum_{\tau_{n,i}\le t} E\left(\left(\Delta M^n_{\tau_{n,i}}\right)^2 - E\left[\left(\Delta M^n_{\tau_{n,i}}\right)^2\middle|\mathcal{F}_i\right]\right)^2\\
&\quad \text{(by MG property, cross product term has expectation 0)}\\
&\le \sum_{\tau_{n,i}\le t} E\left(\Delta M^n_{\tau_{n,i}}\right)^4 + E\left(E\left[(\Delta M^n_{\tau_{n,i}})^2\middle|\mathcal{F}_i\right]\right)^2\\
&\le \sum_{\tau_{n,i}\le t} E\left(\Delta M^n_{\tau_{n,i}}\right)^4 \text{ (by Jensen's inequality for}\\
&\quad \text{conditional expectation)}\\
&\le E\left(\sup\left(\Delta M^n_{\tau_{n,i}}\right)^2\times\left[M^n, M^n\right]_t\right)\\
&\to 0, \qquad \text{(A.18)}
\end{aligned}$$

since σ_{τ_i}, f_{τ_i}, g_{τ_i}, $[M^n, M^n]_t$ are assumed to be bounded, $M_n\Delta t = O_p(n^{-1/2})$.

A.5 Elimination of the Bias Term

The bias in the limit of $\langle X, \widehat{F(\sigma^2)}\rangle_T$ also depends on the limit of $[M^n, W^{(i)}]_t$, where either $W^{(i)}_t = W_t$ or $W^{(i)}$ is orthogonal to W_t, for any $t \in (0, T]$. For the second case, it is obvious that $[M^n, W^{(i)}]_t = 0$. We only need to study the first case $W^{(i)}_t = W_t$. In this case, since the covariance between $O_p(M_n\Delta t)$ terms and W_t will be of even higher order (at least $O_p((M_n\Delta t)^{3/2})$), those are negligible in the limit. Thus we only need to consider the following aggregate conditional expectation:

$$\begin{aligned}
&\frac{1}{\sqrt{M_n\Delta t}}\sum_{\tau_{i+1}\le t}\operatorname{cov}\left(\Delta M^n_{\tau_{n,i+1}}, \Delta W_{\tau_{n,i+1}}\middle|\mathcal{F}_i\right)\\
&= \frac{1}{\sqrt{M_n\Delta t}}F'\left(\sigma^2_{\tau_{n,i}}\right)\\
&\quad \times \sum_{\tau_{i+1}\le t} E\Bigg(\sum_i \Delta W_{\tau_{n,i+1}}\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)\left(\sigma^2_{\tau_{n,i+1}}-\sigma^2_{\tau_{n,i}}\right)\\
&\quad + \Delta W_{\tau_{n,i+1}}\left(X_{\tau_{n,i}}-X_{\tau_{n,i-1}}\right)\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)\\
&\quad - \Delta W_{\tau_{n,i+1}}\left(X_{\tau_{n,i+1}}-X_{\tau_{n,i}}\right)\left(\hat\sigma^2_{\tau_{n,i}}-\sigma^2_{\tau_{n,i}}\right)\\
&\quad + O_p((M_n\Delta t)^{3/2})\\
&= O_p((M_n\Delta t)^{3/2}) \text{ (By Itô's formula and Lemma 1).} \qquad \text{(A.19)}
\end{aligned}$$

Thus, $[M^n, W]_t = O_p((M_n\Delta t)^{3/2})$ for any $t \in (0, T]$.

All the proofs above can be easily extended to the case for any $t \in (0, T]$. Thus, as outlined at the end of Section A.2, with (A.3), (A.16), (A.18), and (A.19), the proof of Theorem 1 is completed by applying the Central Limit Theorem for semimartingales (refer to the version in Mykland and Zhang 2012, thm 2.28).

APPENDIX B: PROOF OF THEOREM 2

With Taylor expansion, the first-order difference of the two estimators is: $\sum_{\tau_{i+1}\le t} \frac{2}{M_n(M_n-1)\Delta t} F'(\sigma_{\tau_{n,i}})(\Delta X_{\tau_{n,i+1}})^3$ and $\sum_{\tau_{i+1}\le t} \frac{2}{M_n(M_n-1)\Delta t} F'(\sigma_{\tau_{n,i}})\Delta X_{\tau_{n,i+1}}(\Delta X_{\tau_{n,i+2}})^2$. In order for any term to make a difference in the limit, the term must be of order lower than $O_p(M_n\Delta t)$. However, the difference terms are of order higher than $O_p(M_n\Delta t)$ by BDG inequality as shown in the proof of Proposition 2 at the end of the article:

$$\sum_{\tau_{i+1}\le t} \frac{2}{M_n(M_n-1)\Delta t} F'\big(\sigma_{\tau_{n,i}}\big) E\Big(\big(\Delta X_{\tau_{n,i+1}}\big)^3 \big| \mathcal{F}_i\Big) = O_p\big(n^{-\frac{1}{2}}\big),$$
$$\sum_{\tau_{i+1}\le t} \frac{2}{M_n(M_n-1)\Delta t} F'\big(\sigma_{\tau_{n,i}}\big)\Delta X_{\tau_{n,i+1}}\big(\Delta X_{\tau_{n,i+2}}\big)^2 = O_p\big(n^{-\frac{1}{2}}\big).$$

Theorem 2 is easily proved with this result and a slight variation of the proof of Theorem 1.

APPENDIX C: PROOF OF THEOREM 3

As we saw in the proof of Theorem 1 that

$$\sum_i \frac{1}{M_n\Delta t} E\Big(\big(X_{\tau_{n,i+1}} - X_{\tau_{n,i}}\big)^2\big(\hat\sigma^2_{\tau_{n,i+1}} - \hat\sigma^2_{\tau_{n,i}}\big)^2\big|\mathcal{F}_i\Big)$$
$$\xrightarrow{p} \frac{4}{c^2 T}\int_0^T \big(F'(\sigma_t^2)\big)^2\sigma_t^6\,dt + \int_0^T \big(F'(\sigma_t^2)\big)^2\sigma_t^4\left(\frac{14}{3}f_t^2 + \frac{8}{3}g_t^2\right)dt.$$

To prove the consistency of Equation (11), we can again apply the martingale convergence argument. According to Lemma 2, it suffices to check wether the conditional variance of the martingale converges to 0. By applying Lemma 1, we can obtain:

$$\begin{aligned}
&\sum_i E\left(\left(\frac{1}{M_n\Delta t}\big(X_{\tau_{n,i+1}} - X_{\tau_{n,i}}\big)^2\big(\hat\sigma^2_{\tau_{n,i+1}} - \hat\sigma^2_{\tau_{n,i}}\big)^2\right.\right.\\
&\quad\left.\left. - \frac{1}{M_n\Delta t}E\Big(\big(X_{\tau_{n,i+1}} - X_{\tau_{n,i}}\big)^2\big(\hat\sigma^2_{\tau_{n,i+1}} - \hat\sigma^2_{\tau_{n,i}}\big)^2\big|\mathcal{F}_i\Big)\right)^2\right|\mathcal{F}_i\Bigg)\\
&\quad = \sum_i E\left(\frac{1}{(M_n\Delta t)^2}\big(X_{\tau_{n,i+1}} - X_{\tau_{n,i}}\big)^4\big(\hat\sigma^2_{\tau_{n,i+1}} - \hat\sigma^2_{\tau_{n,i}}\big)^4\right|\mathcal{F}_i\Bigg)\\
&\quad\quad - E\left(\frac{1}{M_n\Delta t}\big(X_{\tau_{n,i+1}} - X_{\tau_{n,i}}\big)^2\big(\hat\sigma^2_{\tau_{n,i+1}} - \hat\sigma^2_{\tau_{n,i}}\big)^2\right|\mathcal{F}_i\Bigg)^2\\
&\quad = O_p\big(M_n^2\Delta t^2\big).
\end{aligned}\tag{C.20}$$

This proves $G_n^1 \xrightarrow{p} \frac{8}{c}\int_0^T (F'(\sigma_t^2))^2\sigma_t^6\,dt + cT\int_0^T (F'(\sigma_t^2))^2\sigma_t^4 (\frac{28}{3}f_t^2 + \frac{16}{3}g_t^2)\,dt$.

By similar argument, one can prove $G_n^2 \xrightarrow{p} \frac{8}{c}\int_0^T (F'(\sigma_t^2))^2\sigma_t^6\,dt + cT\int_0^T (F'(\sigma_t^2))^2\sigma_t^4\frac{16}{3}(f_t^2 + g_t^2)\,dt$.

These two convergences, together with Theorem 1, prove the stable convergence in Theorem 3 .

APPENDIX D: PROOF OF THEOREM 4

The proof of Theorem 4 will be done similarly to that of Theorem 1 in a manner to compare the difference between $(X_{t_{n,j+1}} - X_{t_{n,j}})$ and $(\bar X_{t_{n,j+1}} - \bar X_{t_{n,j}})$ in each step. As seen in the case without microstructure noise, it is enough to prove the case $F(x) = x$. For the convenience of later calculations, we always assume $\tau_{n,j+1} \in (\lambda_{n,i}, \lambda_{n,i+1}]$, $\tau'_{n,j+1} \in (\lambda_{n,i+1}, \lambda_{n,i+2}]$ and $\tau_{n,j+1}$ and $\tau'_{n,j+1}$ are corresponding $(j+1)$th observation time in the consecutive two big λ-blocks.

D.1 Aggregate Conditional Expectation of the Estimator

According to the contiguity to Gaussian noise (Mykland and Zhang 2011a), the process can be simplified as follows:

$$\begin{aligned}
\hat X_{\tau_{n,i}} &= \frac{1}{M}\sum_{t_{n,j}\in[\tau_i,\tau_{i+1})} Y_{t_{n,j}} = \frac{1}{M}\sum_{t_{n,j}\in[\tau_i,\tau_{i+1})} X_{t_{n,j}} + M^{-1/2}Z_{\tau_{n,i}}\\
&= \bar X_{\tau_{n,i}} + M^{-1/2}Z_{\tau_{n,i}},\\
\hat\sigma^2_{\lambda_{n,i+1}} &= \frac{1}{LM\Delta t}\sum_{\tau_{n,j+1}\in(\lambda_{n,i},\lambda_{n,i+1}]} \big(\hat X_{\tau_{n,j+1}} - \hat X_{\tau_{n,j}}\big)^2\\
&= \frac{1}{LM\Delta t}\sum_{\tau_{n,j+1}\in(\lambda_{n,i},\lambda_{n,i+1}]} \big(\Delta\bar X_{\tau_{n,j+1}} + \Delta Z_{\tau_{n,j+1}}M^{-1/2}\big)^2,
\end{aligned}\tag{D.21}$$

and

$$\begin{aligned}
\widehat{\langle X,\sigma^2\rangle}_T &= 3\sum_{i=0}^{K_n-2}\big(\hat X_{\lambda_{n,i+1}} - \hat X_{\lambda_{n,i}}\big)\big(\hat\sigma^2_{\lambda_{n,i+1}} - \hat\sigma^2_{\lambda_{n,i}}\big)\\
&= \frac{3}{LM\Delta t}\sum_{i=0}^{K_n-2}\big(\Delta\bar X_{\lambda_{n,i+1}} + \Delta Z_{\lambda_{n,i+1}}M^{-1/2}\big)\\
&\quad\times\left(\sum_{\tau'_{n,j+1}}(\Delta\bar X_{\tau'_{n,j+1}} + \Delta Z_{\tau'_{n,j+1}}M^{-1/2})^2\right.\\
&\quad\left. - \sum_{\tau_{n,j+1}}\big(\Delta\bar X_{\tau_{n,j+1}} + \Delta Z_{\tau_{n,j+1}}M^{-1/2}\big)^2\right),
\end{aligned}$$

where the $Z_{\tau_{n,i}}$'s are iid $N(0, a^2)$. This iid normality leads to the following aggregate conditional expectation of the estimator:

$$\begin{aligned}
&E\left(\frac{3}{LM\Delta t}\sum_{i=0}^{K_n-2}\Delta\bar X_{\lambda_{n,i+1}}\left(\sum_{\tau'_{n,j+1}\in(\lambda_{n,i+1},\lambda_{n,i+2}]}\Delta\bar X^2_{\tau'_{n,j+1}}\right.\right.\\
&\quad\left.\left. - \sum_{\tau_{n,j+1}\in(\lambda_{n,i},\lambda_{n,i+1}]}\Delta\bar X^2_{\tau_{n,j+1}}\right)\right|\mathcal{F}_i\Bigg) + O_p(LM\Delta t)\\
&= \frac{3}{LM\Delta t}\sum_{i=0}^{K_n-2}E\left(\frac{1}{M^3}\sum_{\substack{\tau'_{n,j+1}\in(\lambda_{n,i+1},\lambda_{n,i+2}]\\ t_{n,l}\in[\lambda_{n,i},\tau_{n,1})\\ t'_{n,l}\in[\lambda_{n,i+1},\tau'_{n,1})\\ t_{n,k}\in[\tau'_{n,j},\tau'_{n,j+1})\\ t'_{n,k}\in[\tau'_{n,j+1},\tau'_{n,j+2})}}\right.\\
&\quad\times\big(X_{t'_{n,l}} - X_{t_{n,l}}\big)\big(X_{t'_{n,k}} - X_{t_{n,k}}\big)^2\\
&\quad - \frac{1}{M^3}\sum_{\substack{t_{n,l}\in[\lambda_{n,i},\tau_{n,1})\\ t'_{n,l}\in[\lambda_{n,i+1},\tau'_{n,1})}}\big(X_{t'_{n,l}} - X_{t_{n,l}}\big)
\end{aligned}$$

$$\times \left(\sum_{\substack{\tau_{n,j+1} \in (\lambda_{n,i}, \lambda_{n,i+1}] \\ t_{n,k} \in [\tau_{n,j}, \tau_{n,j+1}) \\ t'_{n,k} \in [\tau_{n,j+1}, \tau_{n,j+2})}} \left(X_{t'_{n,k}} - X_{t_{n,k}}\right) \right)^2 \Bigg| \mathcal{F}_i \Bigg) + O_p(LM\Delta t)$$

$$= \sum_{i=0}^{K_n-2} 2\sigma^2_{\lambda_{n,i}} f_{\lambda_{n,i}} LM\Delta t + O_p(LM\Delta t). \qquad \text{(D.22)}$$

The last step applies Lemma 1 (A.3) and (A.4), and we conclude that:

$$\widehat{\langle X, \sigma^2 \rangle}_T \xrightarrow{p} \int_0^T 2\sigma_t^2 f_t dt = \langle X, \sigma^2 \rangle_T. \qquad \text{(D.23)}$$

There is another way to look at $\Delta \bar{X}_{\tau_{n,j+1}}$, which leads to a continuous approximation and provides a simpler way to prove the results.

$$\begin{aligned}
&\bar{X}_{\tau_{n,j+1}} - \bar{X}_{\tau_{n,j}} \\
&= X_{\tau_{n,j+1}} - X_{\tau_{n,j}} + \sum_{\substack{k=1 \\ t'_{n,k} \in [\tau_{n,j+1}, \tau_{n,j+2})}}^{M} \left(\frac{M-k}{M}\right) \left(X_{t'_{n,k+1}} - X_{t'_{n,k}}\right) \\
&\quad - \sum_{\substack{k=1 \\ t_{n,k} \in [\tau_{n,j}, \tau_{n,j+1})}}^{M} \left(\frac{M-k}{M}\right) \left(X_{t_{n,k+1}} - X_{t_{n,k}}\right) \\
&\simeq \Delta X_{\tau_{n,j+1}} + \frac{1}{M\Delta t} \int_{\tau_{n,j+1}}^{\tau_{n,j+2}} (\tau_{n,j+2} - t)\, dX_t \\
&\quad - \frac{1}{M\Delta t} \int_{\tau_{n,j}}^{\tau_{n,j+1}} (\tau_{n,j+1} - t)\, dX_t, \qquad \text{(D.24)}
\end{aligned}$$

and

$$\begin{aligned}
&\bar{X}_{\lambda_{n,i+1}} - \bar{X}_{\lambda_{n,i}} \\
&= X_{\lambda_{n,i+1}} - X_{\lambda_{n,i}} + \sum_{\substack{k=1 \\ t'_{n,k} \in [\lambda_{n,i+1}, \tau'_{n,1})}}^{M} \left(\frac{M-k}{M}\right) \left(X_{t'_{n,k+1}} - X_{t'_{n,k}}\right) \\
&\quad - \sum_{\substack{k=1 \\ t_{n,k} \in [\lambda_{n,i}, \tau_{n,1})}}^{M} \left(\frac{M-k}{M}\right) \left(X_{t_{n,k+1}} - X_{t_{n,k}}\right) \\
&\simeq X_{\lambda_{n,i+1}} - X_{\lambda_{n,i}} + \frac{1}{M\Delta t} \int_{\lambda_{n,i+1}}^{\tau'_{n,1}} (\tau'_{n,1} - t)\, dX_t \\
&\quad - \frac{1}{M\Delta t} \int_{\lambda_{n,i}}^{\tau_{n,1}} (\tau_{n,1} - t)\, dX_t. \qquad \text{(D.25)}
\end{aligned}$$

One can verify that, to the relevant order, the aggregate conditional expectation by this continuous approximation gives the same result but less complicated calculations.

D.2 Conditional Variance of the Approximate Martingale

Similarly as in the proof of Theorem 1,the martingale is constructed as follows:

Up to time t,

$$M_t^n = \sum_{\lambda_{n,i+1} \le t} \left\{ \Delta \widehat{V}^n_{\lambda_{n,i+1}} - E\left(\Delta \widehat{V}^n_{\lambda_{n,i+1}} \middle| \mathcal{F}_i\right) \right\}, \qquad \text{(D.26)}$$

where, except a few terms at the edge,

$$\begin{aligned}
\Delta \widehat{V}^n_{\lambda_{n,i+1}} = \frac{3}{\sqrt{LM\Delta t}} \Bigg\{ &\left(\widehat{X}_{\lambda_{n,i+1}} - \widehat{X}_{\lambda_{n,i}}\right)\left(\sigma^2_{\lambda_{n,i+1}} - \sigma^2_{\lambda_{n,i}}\right) \\
&+ \left(\widehat{X}_{\lambda_{n,i}} - \widehat{X}_{\lambda_{n,i-1}}\right) \left(\frac{1}{LM\Delta t} \sum_{\tau_i \in (\lambda_{n,i}, \lambda_{n,i+1}]} \Delta \bar{X}^2_{\tau_i} - \sigma^2_{\lambda_{n,i}} \right) \\
&- \left(\widehat{X}_{\lambda_{n,i+1}} - \widehat{X}_{\lambda_{n,i}}\right) \left(\frac{1}{LM\Delta t} \sum_{\tau_i \in (\lambda_{n,i}, \lambda_{n,i+1}]} \Delta \bar{X}^2_{\tau_i} - \sigma^2_{\lambda_{n,i}} \right) \\
&- \int_{\lambda_i}^{\lambda_{i+1}} \frac{2}{3} \sigma_t^2 f_t\, dt \\
&+ \left(\widehat{X}_{\lambda_{n,i+1}} - \widehat{X}_{\lambda_{n,i}}\right) \frac{1}{LM\Delta t} \Bigg(\sum_{\tau_i \in (\lambda_{n,i+1}, \lambda_{n,i+2}]} \Delta Z^2_{\tau_i} M^{-1} \\
&- \sum_{\tau_i \in (\lambda_{n,i}, \lambda_{n,i+1}]} \Delta Z^2_{\tau_i} M^{-1} \Bigg) \\
&+ (\widehat{X}_{\lambda_{n,i}} - \widehat{X}_{\lambda_{n,i-1}}) \left(\frac{1}{LM\Delta t} \sum_{\tau_i \in (\lambda_{n,i}, \lambda_{n,i+1}]} \Delta \bar{X}_{\tau_i} \Delta Z_{\tau_i} M^{-1/2} \right) \\
&- (\widehat{X}_{\lambda_{n,i+1}} - \widehat{X}_{\lambda_{n,i}}) \\
&\times \left(\frac{1}{LM\Delta t} \sum_{\tau_i \in (\lambda_{n,i}, \lambda_{n,i+1}]} \Delta \bar{X}_{\tau_i} \Delta Z_{\tau_i} M^{-1/2} \right) \Bigg\}.
\end{aligned}$$

Again, we will interpolate this martingale into a continuous martingale up to any time $t \in (0, T]$ as in the proof of Theorem 1.

From this point on, the continuous approximation of $\bar{X}$ with integral will be adopted for simplicity and transparency. However, all the calculation can be checked by applying Lemma 1 to the direct proof without continuous approximation.

Lemma 3.

$$\begin{aligned}
&E\left(\left(\frac{1}{M\Delta t} \int_{\tau_{n,j}}^{\tau_{n,j+1}} (\tau_{n,j+1} - t)\, dX_t \right)^2 \sigma^p_{\tau_{n,m}} \middle| \mathcal{F}_j \right) \\
&= \frac{1}{3} \sigma^{p+2}_{\tau_{n,j}} (\tau_{n,j+1} - \tau_{n,j}) + \Bigg[\frac{p(p-1)}{6} (\tau_{n,m} - \tau_{n,j})(\tau_{n,j+1} - \tau_{n,j}) \\
&\quad + \frac{p^2}{4} \sigma^p_{\tau_{n,j}} f^2_{\tau_{n,j}} (\tau_{n,j+1} - \tau_{n,j})^2 \\
&\quad + \frac{2p+1}{12} (\tau_{n,j+1} - \tau_{n,j})^2 \Bigg] \sigma^p_{\tau_{n,j}} \left(f_{\tau_{n,j}} + g^2_{\tau_{n,j}}\right) \\
&\quad + \frac{p}{4} \sigma^p_{\tau_{n,j}} \langle X, f \rangle_{\tau_{n,j}} (\tau_{n,j+1} - \tau_{n,j})^2 + O_p\left(\Delta t^{\frac{3}{2}}\right), \qquad \text{(D.27)}
\end{aligned}$$

$$\begin{aligned}
&E\left(\left(X_{\tau_{n,j}} - X_{\tau_{n,k}}\right) \frac{1}{M\Delta t} \int_{\tau_{n,k}}^{\tau_{n,k+1}} (\tau_{n,k+1} - t)\, dX_t\, \sigma^p_{\tau_{n,m}} \middle| \mathcal{F}_k \right) \\
&= \frac{1}{2} \sigma^{p+2}_{\tau_{n,k}} (\tau_{n,k+1} - \tau_{n,k}) + \sigma^p_{\tau_{n,k}} \left(\frac{p^2}{2} f^2_{\tau_{n,k}} + \frac{p}{2} \langle X, f \rangle_{\tau_{n,k}} \right) \\
&\quad \times \left[(\tau_{n,k+1} - \tau_{n,k})^2 + (\tau_{n,k+1} - \tau_{n,k})(\tau_{n,j} - \tau_{n,k+1}) \right] \\
&\quad + \frac{p(p-1)}{4} \sigma^p_{\tau_{n,k}} \left(f^2_{\tau_{n,k}} + g^2_{\tau_{n,k}}\right)(\tau_{n,k+1} - \tau_{n,k})(\tau_{n,m} - \tau_{n,k}) \\
&\quad + \frac{2p+1}{6} \sigma^p_{\tau_{n,k}} \left(f^2_{\tau_{n,k}} + g^2_{\tau_{n,k}}\right)(\tau_{n,k+1} - \tau_{n,k})^2 + O_p\left(\Delta t^{\frac{3}{2}}\right), \qquad \text{(D.28)}
\end{aligned}$$

$$E\left(\left(X_{\tau_{n,j}}-X_{\tau_{n,k}}\right)\frac{1}{M\Delta t}\int_{\tau_{n,j}}^{\tau_{n,j+1}}(\tau_{n,j+1}-t)\,dX_t\,\sigma_{\tau_{n,m}}^p\middle|\mathcal{F}_k\right)$$
$$=\sigma_{\tau_{n,k}}^p\left(\frac{p^2}{2}f_{\tau_{n,k}}^2+\frac{p}{2}\langle X,f\rangle_{\tau_{n,k}}\right)(\tau_{n,j+1}-\tau_{n,j})(\tau_{n,j}-\tau_{n,k})$$
$$+O_p\left(\Delta t^{\frac{3}{2}}\right), \tag{D.29}$$

and

$$E\left(\frac{1}{M\Delta t}\int_{\tau_{n,k}}^{\tau_{n,k+1}}(\tau_{n,k+1}-t)\,dX_t\right.$$
$$\left.\times\frac{1}{M\Delta t}\int_{\tau_{n,j}}^{\tau_{n,j+1}}(\tau_{n,j+1}-t)\,dX_t\,\sigma_{\tau_{n,m}}^p\middle|\mathcal{F}_k\right)$$
$$=\sigma_{\tau_{n,k}}^p\left(\frac{p^2}{4}f_{\tau_{n,k}}^2+\frac{p}{4}\langle X,f\rangle_{\tau_{n,k}}\right)(\tau_{n,j+1}-\tau_{n,j})(\tau_{n,k+1}-\tau_{n,k})$$
$$+O_p\left(\Delta t^{\frac{3}{2}}\right), \tag{D.30}$$

where $\tau_{n,k}<\tau_{n,j}\le\tau_{n,m}$, $\mathcal{F}_k=\mathcal{F}_{n,k}$.

The proof of this Lemma is similar to that of Lemma 1.

It is easy to see that the aggregate conditional expectation term in M_t^n is of order $O_p(LM\Delta t)$ from (D.22). So the aggregate conditional variance will be the same as the second moment of the term before the aggregate conditional expectation.

$$\sum_{\lambda_{n,i+1}\le t}\operatorname{var}\left(\Delta M_{\lambda_{n,i+1}}^n\middle|\mathcal{F}_i\right)$$
$$=\frac{9}{LM\Delta t}\sum_{\lambda_{n,i+1}\le t}E\left(\left(\widehat{X}_{\lambda_{n,i+1}}-\widehat{X}_{\lambda_{n,i}}\right)^2\left(\sigma_{\lambda_{n,i+1}}^2-\sigma_{\lambda_{n,i}}^2\right)^2\right.$$
$$+\left(\hat{X}_{\lambda_{n,i}}-\hat{X}_{\lambda_{n,i-1}}\right)^2\left(\hat{\sigma}_{\lambda_{n,i}}^2-\sigma_{\lambda_{n,i}}^2\right)^2$$
$$+\left(\hat{X}_{\lambda_{n,i+1}}-\hat{X}_{\lambda_{n,i}}\right)^2\left(\hat{\sigma}_{\lambda_{n,i}}^2-\sigma_{\lambda_{n,i}}^2\right)^2$$
$$-2\left(\hat{X}_{\lambda_{n,i+1}}-\hat{X}_{\lambda_{n,i}}\right)^2\left(\hat{\sigma}_{\lambda_{n,i}}^2-\sigma_{\lambda_{n,i}}^2\right)\left(\sigma_{\lambda_{n,i}}^2-\sigma_{\lambda_{n,i}}^2\right)$$
$$+\left(\int_{\lambda_i}^{\lambda_{i+1}}\frac{2}{3}\sigma_t^2f_t\,dt\right)^2-\frac{4}{3}\left(\int_{\lambda_i}^{\lambda_{i+1}}\sigma_t^2f_t\,dt\right)$$
$$\times\left(\widehat{X}_{\lambda_{n,i+1}}-\widehat{X}_{\lambda_{n,i}}\right)\left(\sigma_{\lambda_{n,i+1}}^2-\sigma_{\lambda_{n,i}}^2\right)$$
$$+\left(\widehat{X}_{\lambda_{n,i+1}}-\widehat{X}_{\lambda_{n,i}}\right)^2\frac{1}{L^2M^2\Delta t^2}\left(\sum_{\tau_i\in(\lambda_{n,i+1},\lambda_{n,i+2}]}\Delta Z_{\tau_i}^2M^{-1}\right.$$
$$\left.-\sum_{\tau_i\in(\lambda_{n,i},\lambda_{n,i+1}]}\Delta Z_{\tau_i}^2M^{-1}\right)^2$$
$$+\left(\widehat{X}_{\lambda_{n,i}}-\widehat{X}_{\lambda_{n,i-1}}\right)^2\left(\frac{1}{LM\Delta t}\sum_{\tau_i\in(\lambda_{n,i},\lambda_{n,i+1}]}\Delta\bar{X}_{\tau_i}\Delta Z_{\tau_i}M^{-1/2}\right)^2$$
$$\left.+(\widehat{X}_{\lambda_{n,i+1}}-\widehat{X}_{\lambda_{n,i}})^2\left(\frac{1}{LM\Delta t}\sum_{\tau_i\in(\lambda_{n,i},\lambda_{n,i+1}]}\Delta\bar{X}_{\tau_i}\Delta Z_{\tau_i}M^{-1/2}\right)^2\middle|\mathcal{F}_i\right)$$
$$+O_p(LM\Delta t).$$

We will separately calculate the conditional variance involving the microstructure noise $\Delta ZM^{-1/2}$ and the variance involving only the semimartingale process, denoted accordingly as $\operatorname{var}(\Delta M_{\lambda_{n,i+1}}^{n,\text{noise}}|\mathcal{F}_i)$ and $\operatorname{var}(\Delta M_{\lambda_{n,i+1}}^{n,\text{process}}|\mathcal{F}_i)$. Applying Lemma 1 and Lemma 3, we can derive:

$$\sum\operatorname{var}\left(\Delta M_{\lambda_{n,i+1}}^{n,\text{ noise}}\middle|\mathcal{F}_i\right)$$
$$=\frac{9}{L^3M^3\Delta t^3}\sum_{\lambda_{n,i+1}\le t}E\left(\left(\widehat{X}_{\lambda_{n,i+1}}-\widehat{X}_{\lambda_{n,i}}\right)^2\right.$$
$$\times\left(\sum_{\tau_i\in(\lambda_{n,i+1},\lambda_{n,i+2}]}\Delta Z_{\tau_i}^2M^{-1}-\sum_{\tau_i\in(\lambda_{n,i},\lambda_{n,i+1}]}\Delta Z_{\tau_i}^2M^{-1}\right)^2$$
$$+\left(\widehat{X}_{\lambda_{n,i}}-\widehat{X}_{\lambda_{n,i-1}}\right)^2\left(\sum_{\tau_i\in(\lambda_{n,i},\lambda_{n,i+1}]}\Delta\bar{X}_{\tau_i}\Delta Z_{\tau_i}M^{-1/2}\right)^2$$
$$\left.+\left(\widehat{X}_{\lambda_{n,i+1}}-\widehat{X}_{\lambda_{n,i}}\right)^2\left(\sum_{\tau_i\in(\lambda_{n,i},\lambda_{n,i+1}]}\Delta\bar{X}_{\tau_i}\Delta Z_{\tau_i}M^{-1/2}\right)^2\middle|\mathcal{F}_i\right)$$
$$+O_p(LM\Delta t)$$
$$=\sum_{\lambda_{n,i}\le t}\frac{96a^2}{L^3M^4\Delta t^3}\sigma_{\lambda_{n,i}}^4(LM\Delta t)^2$$
$$+\frac{324a^4}{L^2M^5\Delta t^3}\sum_{\lambda_{n,i+1}\le t}E\left(\Delta\bar{X}_{\lambda_{n,i+1}}^2\middle|\mathcal{F}_i\right)+O_p(LM\Delta t)$$
$$\xrightarrow{p}\frac{96a^2}{c^2c_1^2T^2}\int_0^T\sigma_t^4\,dt+\frac{216a^4}{c^2c_1^4T^3}\int_0^T\sigma_t^2\,dt, \tag{D.31}$$

and

$$\sum\operatorname{var}\left(\Delta M_{\lambda_{n,i+1}}^{n,\text{ process}}\middle|\mathcal{F}_i\right)$$
$$=\frac{9}{LM\Delta t}\sum_{\lambda_{n,i+1}\le t}E\left(\left(\widehat{X}_{\lambda_{n,i+1}}-\widehat{X}_{\lambda_{n,i}}\right)^2\left(\sigma_{\lambda_{n,i+1}}^2-\sigma_{\lambda_{n,i}}^2\right)^2\right.$$
$$+\left(\hat{X}_{\lambda_{n,i}}-\hat{X}_{\lambda_{n,i-1}}\right)^2\left(\hat{\sigma}_{\lambda_{n,i}}^2-\sigma_{\lambda_{n,i}}^2\right)^2$$
$$+\left(\hat{X}_{\lambda_{n,i+1}}-\hat{X}_{\lambda_{n,i}}\right)^2\left(\hat{\sigma}_{\lambda_{n,i}}^2-\sigma_{\lambda_{n,i}}^2\right)^2$$
$$-2\left(\hat{X}_{\lambda_{n,i+1}}-\hat{X}_{\lambda_{n,i}}\right)^2\left(\hat{\sigma}_{\lambda_{n,i}}^2-\sigma_{\lambda_{n,i}}^2\right)\left(\sigma_{\lambda_{n,i}}^2-\sigma_{\lambda_{n,i}}^2\right)$$
$$+\left(\int_{\lambda_i}^{\lambda_{i+1}}\frac{2}{3}\sigma_t^2f_t\,dt\right)^2-\frac{4}{3}\left(\int_{\lambda_i}^{\lambda_{i+1}}\sigma_t^2f_t\,dt\right)\left(\widehat{X}_{\lambda_{n,i+1}}-\widehat{X}_{\lambda_{n,i}}\right)$$
$$\left.\times\left(\sigma_{\lambda_{n,i+1}}^2-\sigma_{\lambda_{n,i}}^2\right)|\mathcal{F}_i\right)+O_p(LM\Delta t)$$
$$=\sum_{\lambda_{n,i}\le t}\left(\frac{16}{L^2M\Delta t}\sigma_{\lambda_{n,i}}^6LM\Delta t+\frac{44}{3}\sigma_{\lambda_{n,i}}^4f_{\lambda_{n,i}}^2LM\Delta t\right.$$
$$\left.+\frac{32}{3}\sigma_{\lambda_{n,i}}^4g_{\lambda_{n,i}}^2LM\Delta t\right)+O_p(LM\Delta t). \tag{D.32}$$

These results prove that, for any $t\in(0,T]$:

$$[M,M]_t\xrightarrow{p}\frac{16}{c^2t}\int_0^t\sigma_s^6\,ds+\int_0^t\sigma_s^4\left(\frac{44}{3}f_s^2+\frac{32}{3}g_s^2\right)ds$$
$$+\frac{96a^2}{c^2c_1^2t^2}\int_0^t\sigma_s^4\,ds+\frac{216a^4}{c^2c_1^4t^3}\int_0^t\sigma_s^2\,ds. \tag{D.33}$$

By similar methods, we can prove that the bias term also converges to zero (here we omit the lengthy proof):

$$\frac{1}{\sqrt{LM\Delta t}}\sum_i\operatorname{Cov}\left(\Delta M_{\lambda_{n,i+1}}^n,\Delta W_{\lambda_{n,i+1}}\middle|\mathcal{F}_i\right)=O_p((LM\Delta t)^{3/2}). \tag{D.34}$$

Thus, $[M^n, W]_t = O_p((LM\Delta t)^{3/2})$ for any $t \in (0, T]$. This completes the proof of Theorem 4.

APPENDIX E: PROOF OF THEOREM 5

Theorem 5 can be easily proved by the consistency of asymptotic variance (27) and stable convergence in Theorem 4. We only need to establish the convergence in probability of Equation (27). This convergence can be proved by adopting the similar martingale technique as in the proof of Theorme 3, and by applying Lemma 1, Lemma 2, and Lemma 3. We can prove: $E(\frac{81}{4L^2M^2\Delta t^2}\sum_i(\hat{X}_{\lambda_{n,i+1}} - \hat{X}_{\lambda_{n,i}})^4(F(\hat{\sigma}^2_{\lambda_{n,i+1}}) - F(\hat{\sigma}^2_{\lambda_{n,i}}))^4|\mathcal{F}_i) - (E(\frac{9}{2LM\Delta t}\sum_i(\hat{X}_{\lambda_{n,i+1}} - \hat{X}_{\lambda_{n,i}})^2(F(\hat{\sigma}^2_{\lambda_{n,i+1}}) - F(\hat{\sigma}^2_{\lambda_{n,i}}))^2|\mathcal{F}_i))^2 = O_p(LM\Delta t)$. This equation gives the consistency of $G_{n,1}$ in Equation (27). By similar argument, the consistency of $G_{n,2}$ can be proved and consequently completing the proof of the convergence of Equation (27) and Theorem 5.

APPENDIX F: PROOF OF PROPOSITION 2

We will assume the equivalent measure where both X_t and σ_t are martingales. This can be done in analogy with the development in Mykland and Zhang (2009), sec. 2.2, by a stopping argument and so long as the instantaneous correlation between X_t and σ_t is not ± 1. (In the latter case, the proof is different but straightforward.)

By the third Bartlett identity (Mykland 1994)

$$\begin{aligned} E\left(\Delta X^3_{t_{n,i+1}}|\mathcal{F}_{t_i}\right) &= 3E\left(\Delta X_{t_{n,i+1}}\Delta\langle X, X\rangle|\mathcal{F}_{t_i}\right) \\ &= 3E\left(\Delta X_{t_{n,i+1}}\int_{t_{n,i}}^{t_{n,i+1}} \sigma_u^2\,du|\mathcal{F}_{t_i}\right) \\ &= 3E\left(\Delta X_{t_{n,i+1}}\int_{t_{n,i}}^{t_{n,i+1}} \left(\sigma_u^2 - \sigma^2_{t_{n,i}}\right)du|\mathcal{F}_{t_i}\right) \\ &\quad \text{(by MG property of } X_t). \end{aligned} \tag{F.35}$$

By the Itô formula, $d(t_{n,i+1} - t)(\sigma_u^2 - \sigma^2_{t_{n,i}}) = -(\sigma_u^2 - \sigma^2_{t_{n,i}})\,dt + (t_{n,i+1} - t)\,d\sigma_u^2$, we obtain:

$$\begin{aligned} (F.35) &= 3E\left(\Delta X_{t_{n,i+1}}\int_{t_{n,i}}^{t_{n,i+1}} (t_{n,i+1} - t)\,d\sigma_u^2|\mathcal{F}_{t_i}\right) \\ &= 3E\left(\int_{t_{n,i}}^{t_{n,i+1}} (t_{n,i+1} - t)\,d\langle X, \sigma_u^2\rangle|\mathcal{F}_{t_i}\right). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{n}{T}\sum_{t_{n,i+1}\le T} E\left(\Delta X^3_{t_{n,i+1}}|\mathcal{F}_{t_i}\right) &= \frac{n}{T}\frac{3}{2}\sum_{t_{n,i+1}\le T} \Delta t^2_{n,i+1}\langle X, \sigma^2\rangle'_{t_{n,i}} \\ &\quad + \text{higher order terms} \\ &\xrightarrow{p} \frac{3}{2}\int_0^T \langle X, \sigma^2\rangle'_u\,du \\ &= \frac{3}{2}\langle X, \sigma^2\rangle_T. \end{aligned}$$

We have here used that $\Delta t_i = \frac{T}{n}$ (equidistant spacing); for a more general case, the expression will involve the quadratic variation of time.

[Received September 2011. Revised August 2013.]

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