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A Simple Nonparametric Approach to Derivative Security Valuation

MICHAEL STUTZER*

ABSTRACT

Canonical valuation uses historical time series to predict the probability distribution of the discounted value of primary assets' discounted prices plus accumulated dividends at any future date. Then the axiomatically-rationalized *maximum entropy principle* is used to estimate risk-neutral (equivalent martingale) probabilities that correctly price the primary assets, as well as any predesignated subset of derivative securities whose payoffs occur at this date. Valuation of other derivative securities proceeds by calculation of its discounted, risk-neutral expected value. Both simulation and empirical evidence suggest that canonical valuation has merit.

SINCE THE ADVENT OF the justly celebrated Black-Scholes option pricing formula, intense interest has focused on testing its implications. There is consensus that, in contradiction to the Black-Scholes model predictions, implied volatilities of European options are not independent of their exercise prices.[1] While one might argue that the statistically significant deviations found in older studies may not be economically significant, Rubinstein (1994, pp. 773–774) recently concluded that "there has been a very marked and rapid deterioration" since 1986 in the applicability of Black-Scholes to S&P 500 index options. In particular, he notes that:

> Out-of-the-money puts (and hence in-the-money calls perforce by put-call parity) became valued much more highly, eventually leading to the 1990–1992 (as well as current) situation where low striking price options had significantly higher implied volatilities than high striking price options.

Rubinstein conjectures that this inverse relationship of Black-Scholes implied volatility to exercise price is caused by investors' fear of a repeat of the Crash of 1987. The inverse relationship has also been confirmed and studied by Derman and Kani (1994). Both Rubinstein, and Derman and Kani focus on the possibility that local volatility of the index depends on the contemporaneous index value and time. More formally, under the usual frictionless, contin-

* Carlson School of Management, University of Minnesota. Many thanks to Michael Dothan for several helpful conversations and much general encouragement. Useful suggestions were made by the editor and the referee, and by Steve LeRoy, Yacine Aït-Sahalia, Mark Rubinstein, Anlong Li, and others attending my presentation at the Chicago Board of Trade Research Conference. Tom Lee of the Clifton Group provided data and practitioner sensibility.

[1] See, for example, MacBeth and Merville (1979) Rubinstein (1985), Chance (1986), and Swidler and Diltz (1992), all of which documented statistically significant, Black-Scholes pricing biases.

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uous time, complete market trading assumptions with a known constant instantaneous riskless interest rate, suppose (just for simplicity) the index stocks paid no dividends and that its value $P(t)$ were governed by the Ito process:

$$dP/P = \mu dt + \sigma(P, t)dW. \tag{1}$$

If an analyst knew the local volatility function $\sigma(P, t)$ in equation (1), the analyst could compute (or numerically approximate) the probability distribution of the future value of the index at any time T. For example, if $\sigma(P, t)$ were a constant σ, as assumed in the Black-Scholes model, it is well known that the future value distribution at T is lognormal. The risk-neutral (equivalent martingale) probability measure can then be computed, and used to predict the current price of any European option expiring at T (see, e.g., Dothan (1990)). As noted in Hull (1993, pp. 438–439), the aforementioned inverse relationship between Black-Scholes implied volatility and exercise price is consistent with the S&P 500 index distribution having a fatter left tail and a thinner right tail than a lognormal distribution has, that could be caused by local index volatility that is negatively correlated with the index value, e.g., $\partial\sigma/\partial P < 0$.[2]

But suppose that an analyst does *not* know $\sigma(P, t)$, but does observe the current value of the index and the current prices of a subset of European options expiring at T. Both Rubinstein, and Derman and Kani advocate different procedures for estimating a function $\sigma(P, t)$ *implied* by this data. They do so in the setting of a binomial tree. Each article uncovers a discrete time estimator of $\sigma(P, t)$ (i.e., a node-dependent pattern of local volatility), that will generate a probability distribution of time T future index values, and risk-neutral valuations consistent with the current value of the index and the predesignated subset of current option prices.

In the context of a single factor (the short interest rate) model of interest rate derivatives, Aït-Sahalia (1996) assumes a functional form for the drift and that the unknown volatility function $\sigma(r)$ is independent of time. He then uses interest rate data to construct a nonparametric estimator of the latter. This permits calculation of the equivalent martingale measure needed to price derivatives in that context.

These nonparametric approaches depend heavily on the assumption that there is just one state variable, and that it is governed by an Ito process whose parametric form is unknown. These are still strong assumptions. For example, Bates (1995a, p. 6) has noted that some of the corresponding S&P 500 futures

[2] Hull's intuition for this result is as follows. Consider an out-of-the-money European put. When index volatility is negatively correlated with the index, the index volatility will rise as the index value falls closer to the exercise price. Compared to the case of constant volatility (which implies a lognormal index value distribution at expiration), this increases the probability that the option will expire in-the-money (i.e., it thickens the left-hand tail of the index value distribution at expiration). Accordingly, the out-of-the-money put option will be priced higher than under Black-Scholes lognormality, so its Black-Scholes implied volatility will be higher than for an in-the-money put.

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option anomalies might be better explained by "jump-diffusion models with substantial negative-mean jumps (crashes), and stochastic volatility models with substantial negative correlations between volatility and market shocks." In fact, he concludes (p. 28) that "a volatility-jump model is clearly necessary." Furthermore, these approaches are relatively complicated procedures, because they attempt to use the underlying asset and/or option price data to uncover the *structure* of the unknown Ito process.

Different nonparametric approaches to this problem are enabled by using neural network (see Hutchinson, Lo, and Poggio (1994) or kernel regression (see Aït-Sahalia and Lo (1995)) techniques to fit functional "black boxes" that map market option data (e.g., time to expiration, strike price, underlying asset price, dividend yields, and interest rates) into option prices. Armed with a functional black box, one can then make use of the familiar finding that the second partial derivative of the call option pricing function, taken with respect to the exercise price, may be used to compute contingent claims prices needed to price other derivative securities (see, e.g., Breeden and Litzenberger (1978)). But this approach uses a massive amount of market option price data to predict the prices of other options and derivative securities, and hence also does not constitute a general predictive *theory* of pricing.

In contrast, the simple nonparametric method developed in Section II, that I call *canonical valuation*,[3] is a predictive theory of pricing which, like parametric approaches, does not strictly require the use of derivative security prices to predict other derivative security prices. By using past data to estimate the payoff distribution at expiration, canonical valuation permits more accurate assessment of the likely pricing impact caused by investors' data-based beliefs about the future value distribution. In addition, the approach permits easy handling of stochastic dividend yields and interest rates, as well as the pricing of derivative securities with more than one underlying asset (i.e., more than one state variable). Finally, the computer programming burden is not onerous.

To illustrate the utility of canonical valuation in the simplest setting, a simulation test adopted by Hutchinson *et al.* (1994) and by Aït-Sahalia and Lo (1995) is conducted.[4] The test is for whether a valuation method will make Black-Scholes predictions in a simulated market governed by the Black-Scholes parametric assumptions, i.e., frictionless, continuous time trading of a stock whose price process is a geometric Brownian motion. It will be seen that the canonical valuation approach also produces prices close to Black-Scholes prices, without using *any* of the simulated market Black-Scholes prices in the valuation process. However, to demonstrate the ability of canonical valuation to incorporate option price data when desired, we also document that some

[3] The term "canonical" is motivated by the critical role played by a particular probability distribution that was named the canonical distribution by the physicist J.W. Gibbs. It plays a central role in statistical mechanics, and as mentioned later, in the field of Bayesian information theoretic statistics.

[4] The author is also indebted to Anlong Li for making this suggestion.

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improvement is possible when a limited amount of current market option prices are incorporated into the valuation process.

To further illustrate the validity of canonical valuation, Section III will show that canonical valuation of S&P 500 index options does predict that deep in-the-money, short to medium term calls will have prices substantially higher than Black-Scholes predicts. This finding occurs when the historical data required for canonical valuation include at least the Crash of 1987. This verifies the conjecture of Rubinstein, attributing the relationship to market participants' beliefs that another market crash is possible. Moreover, Bates (1991) has produced a "skewness premium" statistic that is positive in many estimated option pricing models, but that is negative for market option prices. It will be shown that canonical price predictions also have negative skewness premia, consistent with market index option prices.

Section I briefly reviews risk-neutral valuation and the associated equivalent martingale probability measure. Section II defines and motivates an approach to estimating and using this measure, called canonical valuation. Then the aforementioned simulation test is conducted, establishing some evidence in favor of canonical valuation. Section III canonically values S&P 500 index options to explain some deviations of market prices from Black-Scholes prices. Section IV summarizes the results and lists some topics left for further research.

I. Risk-Neutral Valuation

We start from the following well-known proposition upon which derivative security valuation is based (Huang and Litzenberger (1988, p. 231)):

> In summary, we have proved that if a price system admits no arbitrage opportunities, then, after a normalization, there must exist a probability π^* that assigns a strictly positive probability to every state of nature such that price processes plus the accumulated dividends are martingales under π^*. A probability having the above property is termed an *an equivalent martingale measure*.

This quotation refers to a normalization. The normalization used here is a commonly used one in discrete time models. At each future time period T, each price process is discounted by the product of one-period gross riskless interest rates $r(t)$ up to that time.[5] Denoting the current price of primary asset i by P_i and its future (possibly stochastic) dividend (or other intermediate payout)

[5] For example, this normalization is described by Rubinstein (1994, p. 794), and is also used in the popular one-factor term structure models (see e.g., Hull (1993, Chap. 15)).

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stream by $D_i(t)$, the equivalent martingale probabilities π^* at time T must satisfy:

$$\begin{aligned} P_i &= \mathrm{E}_{\pi^*}\left[\frac{P_i(T)+D_i(T)+\sum_{t=1}^{T-1} D_i(t)\prod_{s=t}^{T-1} r(s)}{\prod_{t=1}^{T} r(t)}\right] \\ &= \mathrm{E}_{\pi}\left[\frac{P_i(T)+D_i(T)+\sum_{t=1}^{T-1} D_i(t)\prod_{s=t}^{T-1} r(s)}{\prod_{t=1}^{T} r(t)}\,\frac{\mathrm{d}\pi^*}{\mathrm{d}\pi}\right], \qquad i = 1, \ldots, N \end{aligned} \tag{2}$$

where π denotes the actual probability measure, $d\pi^*/d\pi$ denotes the Radon-Nykodym density of the martingale measure with respect to it, and the subscript on the mathematical expectation operator E denotes the probability measure used to compute the expectation.

To implement the risk-neutral valuation of a derivative security with N underlying assets, one must be able to estimate the equivalent martingale measure satisfying the constraints (2), and use it to evaluate the expected discounted value of the derivative security. Different valuation models do so in different ways. For example, the simplest and most widely known discrete time model is the venerable binomial stock option model (see, e.g., Huang and Litzenberger (1988, pp. 248–254)) for a nondividend paying stock and a constant interest rate. It uses $N = 1$ primary asset (i.e., the underlying stock), and constructs a binomial tree producing a distribution of T possible values of $P_1(T)$, with T corresponding positive probabilities comprising the martingale measure π^*. As is well-known, it is possible to produce numerous specifications of the binomial tree, each of which produces risk-neutral option values that converge to the Black-Scholes price as the time period shrinks to zero. It is also possible to approximate a wide variety of other continuous time parametric processes by different specifications of a binomial tree (see Nelson and Ramaswamy (1990)).

II. Canonical Valuation

But what is to be done when one does not want to assume a particular continuous time process, or equivalently to estimate a particular binomial model for the underlying asset?

This section describes a simple nonparametric method of incorporating the no-arbitrage principle embodied in the constraint (2). This *canonical valuation* method proceeds in three parts. First, time series of past underlying asset returns, dividend yields, and riskless interest rates are used to compute the *empirical probability distribution* $\hat{\pi}$ of the bracketed term in equation (2). Second, we will describe and justify use of the *maximum entropy principle* of information theory and its numerous successful applications,[6] to transform the estimated probability distribution $\hat{\pi}$ into an estimate $\hat{\pi}^*$ of the unknown

[6] Kapur (1989) lists hundreds of applications across the physical and social sciences.

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martingale measure π^* satisfying equation (2). Third, the derivative security value is estimated by using $\hat{\pi}^*$ to calculate its expected discounted payoff.

A. The Simplest Example

To illustrate the canonical method, consider the simplest possible case of one underlying asset, that does not pay dividends, used to price derivative securities expiring T periods from now. To simplify this section's notation, the notation for the particular asset and time to expiration are suppressed.

The first step of the method starts with the underlying asset's current price P, and a historical time series of its previous prices $P(t)$, $t = -1, -2, \ldots, -H$. The (rolling) historical time series of T-period gross returns is then constructed by computing $R(-h) = P(-h)/P(-h - T)$, $h = 1, 2, \ldots, H - T$. We use the historical returns to construct the following $H - T$ possible values for the asset's price T-periods from now:

$$P^h = PR(-h), \qquad h = 1, 2, \ldots, H - T. \tag{3}$$

That is, we use the previous realized returns to construct possible prices at T for the underlying asset, each with estimated actual probability $\hat{\pi}(h) = 1/H - T$. In the special case of one period, the probability distribution of returns would be the familiar ungrouped histogram, or *empirical distribution*, of the underlying asset's single-period returns. Successive multiplication of T of those returns together is an alternative way of producing a possible T-period return in equation (3).

Under the general assumption that the returns are generated by an unknown ergodic Markov chain, the empirical distribution $\hat{\pi}$ is a consistent estimator of the invariant, unknown actual distribution π, converging exponentially to the invariant distribution π. Furthermore, it is an optimal nonparametric estimator, in the sense that its rate of convergence is the fastest possible among all such consistent estimators.[7]

Aït-Sahalia (1996) has utilized a kernel-smoothed empirical distribution in pricing interest rate derivatives. In that article, smoothing is required because some of its analysis requires additional assumptions of differentiability. But smoothness assumptions are not required to implement the canonical valuation method, i.e., only the entirely data-based histogram is used, without ad-hoc smoothing. This simplification eliminates the burdens of computing various kernels and their bandwidths, and deciding which choice of the two is "optimal."

Section II.D will describe the straightforward generalization proposed for the general case with stochastic dividends and interest rates. But before doing

[7] For a proof of this proposition, see Bahadur *et al.* (1979, sec. 3). Of course, *if* one knew the functional form of π, parametric estimation methods might be more efficient. For example, a functional form of π is delivered by *assuming* a particular parametric stochastic price process for the underlying asset(s). In order to avoid making this assumption, nonparametric methods are used.

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so, the second and third steps of canonical valuation are detailed for this simple example.

The second step of canonical valuation transforms the empirical probabilities $\hat{\pi}(h) \equiv 1/H - T$ of the T-period ahead prices in equation (3) into estimated risk-neutral (martingale) probabilities. To do so, we first rewrite the constraint (2) for this simple example, using the estimated probability distribution $\hat{\pi}$:

$$1 = \sum_{h=1}^{H-T} \frac{R(-h)}{r^T} \frac{\pi^*(h)}{\hat{\pi}(h)} \hat{\pi}(h) \tag{4}$$

where r denotes the (temporarily assumed) constant one-period gross riskless interest rate. We must select strictly positive probabilities π^* satisfying equation (4) as an approximation to martingale probabilities, which will do a good job in valuing the derivative securities of interest. But in general, there will be many probability measures satisfying equation (4). Physical scientists and statisticians have found that a straightforward, axiomatically rationalized, and economically interpretable way of solving this *linear inverse problem* is to solve the following convex minimization problem:

$$\hat{\pi}^* = \arg \min_{\pi^*(h)>0,\ \Sigma_h \pi^*(h)=1} I(\pi^*, \hat{\pi}) = \sum_{h=1}^{H-T} \pi^*(h) \log \frac{\pi^*(h)}{\hat{\pi}(h)} \quad \text{s.t.} \quad (4). \tag{5}$$

The objective function I in equation (5) is the Kullback-Leibler Information Criterion (KLIC) distance of the positive probabilities π^* to the empirical probabilities $\hat{\pi}$.

This procedure has an insightful interpretation in Bayesian, information-theoretic econometrics. Before gathering market data, one might in ignorance be reluctant to assume a specific process assumption, and just assume that the empirical probabilities $\hat{\pi}$ are also the unknown martingale probabilities. Having gathered past returns required to impose the self-consistent, no-arbitrage pricing constraint (4), one would naturally update this prior assessment. It is not unreasonable to demand that the update of the prior be that probability distribution which incorporates no information other than the constraint (4). Of course, additional constraints could also be incorporated if desired, as shown later, but it is interesting to first investigate the consequences of just the single self-consistency constraint.

Formalizing this estimation concept requires a well-accepted, quantitative definition of the *amount* of information gained during a change of probability measure required by the presence of constraints that the measure must satisfy, so that constrained minimization of it will have the above interpretation. The axiomatic rationalization for using $I(\pi^*, \hat{\pi})$ for this purpose was provided by Hobson (1971, sec. 2.3), who generalized a famous rationalization for maximizing the *Shannon entropy* (our case because $\hat{\pi} \equiv 1/H - T$) devised by Khinchin (1957). The first axiom is that any information gain function $I(\pi^*, \hat{\pi})$

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should be a continuous function of its arguments, so that the information changes only a small amount when the probabilities change by a small amount. The second axiom requires that a mere relabelling of the states (i.e., which of the *possible* returns is dubbed the first possible return, the second possible return, etc.) should not change the value of I. Third, it is required that $I(\hat{\pi}, \hat{\pi}) = 0$. That is, no information is gained unless there *is* a change of measure. Fourth, suppose that $\hat{\pi}$ were uniformly distributed on a subset of m outcomes (zero elsewhere), while π^* is also uniformly distributed, but on only n of those outcomes, $n \leq m$. Then, I should be increasing in m, because more information is gained when π^* rules out more of the outcomes possible under $\hat{\pi}$. Furthermore, it is required that I should be decreasing in n, as less information is gained when π^* is more diffuse. Finally, Hobson postulates a complicated "composition rule," that is also not unreasonable, but too complicated to compactly summarize here. Then, Hobson (1971, Appendix A) shows that the only functions satisfying these axioms are proportional to the KLIC $I(\pi^*, \hat{\pi}) = \Sigma_h \pi^*(h) \log(\pi^*(h)/\hat{\pi}(h))$ used in (5).[8]

As mentioned above, substituting $\hat{\pi}(h) \equiv 1/H - T$ into equation (5) and simplifying shows that the constrained minimization of I is equivalent to constrained maximization of the Shannon entropy $-\Sigma_h \pi^*(h) \log \pi^*(h)$. Using the Lagrange multiplier method, the solution to equation (5) is the following *Gibbs canonical distribution*:

$$\hat{\pi}^*(h) = \frac{\exp\left[\gamma^* \dfrac{R(-h)}{r^T}\right]}{\sum_h \exp\left[\gamma^* \dfrac{R(-h)}{r^T}\right]}, \qquad h = 1, \ldots, H - T. \tag{6}$$

But equation (6) requires us to find a Lagrange multiplier γ^*. It is known (e.g., Ben-Tal (1985)) that γ^* may be found by solving the following unconstrained convex problem:

$$\gamma^* = \arg\min_{\gamma} \sum_h \exp\left[\gamma\left(\frac{R(-h)}{r^T} - 1\right)\right] \tag{7}$$

Substituting equation (7) into equation (6) produces estimated martingale probabilities that by construction satisfy equation (4). Under the above interpretation, the probabilities (6) have the interpretation of being those satisfying equation (4) that embody no additional, i.e., unmodelled, information other than the particular no-arbitrage constraint (4). Note that this simple unconstrained optimization procedure produces strictly positive probabilities, without having to solve the nonnegatively constrained, quadratic programming

[8] It would take us too far afield to provide more details about this rationalization or the many uses of the KLIC in analogous applications in and outside economics. Readers interested in another use of this idea in asset pricing should see Stutzer (1995), and should see the survey article by Maasoumi (1993) or the book by Sengupta (1993) for other econometric applications.

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(least-squares) problem described in Rubinstein (1994). And unlike that approach and the related ones developed in Jackwerth and Rubinstein (1996) (see also Mayhew (1995)), canonical valuation makes use of the data-based empirical distribution as a prior, rather than some parametric class of priors.

The third and final step in the canonical valuation of a derivative security is to use the estimated martingale probabilities $\hat{\pi}^*$ to compute its expected discounted payoff. For example, the value of a call option with exercise price X expiring at T would be predicted to be

$$C = \sum_h \frac{\max[PR(-h)-X,\ 0]}{r^T}\,\hat{\pi}^*(h). \tag{8}$$

It is easy to extend the analysis to the case of $N - 1$ additional underlying assets. $N - 1$ additional constraints of the form (4) would be added to the constrained minimization problem (5), whose solution would be the multivariate canonical distribution

$$\hat{\pi}^*(h) = \frac{\exp[\sum_{i=1}^{N} \gamma_i^*(R_i(-h)/r^T)]}{\sum_h \exp[\sum_{i=1}^{N} \gamma_i^*(R_i(-h)/r^T)]}, \qquad h = 1, \ldots, H - T \tag{9}$$

where the N-component multiplier vector γ^* satisfies

$$\gamma^* = \arg\min_{\gamma} \sum_h \exp\left[\sum_{i=1}^{N} \gamma_i^*(R_i(-h)/r^T - 1)\right]. \tag{10}$$

If desired, one can also easily include the key feature of the Rubinstein (1994), Jackwerth and Rubinstein (1996), and Derman and Kani (1994) approaches, i.e., one can ensure that a subset of derivative securities is also correctly priced on a particular date. In our example, suppose again that there is only one underlying asset, and that we wish to ensure correct pricing of a particular call option expiring on date T, with exercise price X, whose market option price is C. Then, we need to find two multipliers, γ_1^* and γ_2^*, satisfying

$$\gamma^* = \arg\min_{\gamma} \sum_h \exp\left[\gamma_1\left(\frac{R(-h)}{r^T} - 1\right) + \gamma_2\left(\frac{\max[PR(-h)-X,\ 0]}{r^T} - C\right)\right] \tag{11}$$

which are substituted into the following bivariate canonical distribution for the estimated martingale probabilities:

$$\hat{\pi}^*(h) = \frac{\exp[\gamma_1(R(-h)/r^T) + \gamma_2(\max[PR(-h)-X,\ 0]/r^T)]}{\sum_h \exp[\gamma_1(R(-h)/r^T) + \gamma_2(\max[PR(-h)-X,\ 0]/r^T)]} \qquad h = 1, \ldots, H - T. \tag{12}$$

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In the following simulation test, market option price constraints like this were not needed to get good predicted prices in most cases. But in the exceptional cases where they were needed, an additional constraint like this did significantly improve pricing performance.

B. An Illustrative Test

Hutchinson *et al.* (1994) simulated 2 years of daily stock returns (253 trading days per year) from a geometric Brownian process with a 10 percent drift and 20 percent annual volatility, and produced Black-Scholes call option prices consistent with this data. They then used this data to train a neural network, producing a generalized quadratic form $C/X = f(P/X, T)$ for predicting the ratio of the call option price to the strike price. The estimated function does a good job of predicting other Black-Scholes prices not used in the training set. A similar exercise using kernel regression was reported in Aït-Sahalia and Lo (1995).

To replicate this test with far less data, only 1 year of daily stock data was used to produce a distribution of time T historical gross returns, and initially *no* Black-Scholes option prices were used in producing the canonical values, i.e., no additional constraints were imposed. Because no option prices were used in producing canonical values, it is unfair to compare them with Black-Scholes prices computed with an at-the-money option's *implied* volatility. By incorporating the at-the-money option price into the Black-Scholes calibration process, the latter practice is more akin to nonparametric approaches described earlier that use some option prices to predict other option prices. A completely analogous implementation of Black-Scholes is one based on an historical estimate of volatility, because that also relies on the historical returns of the underlying stock to predict option prices. We are thus motivated to compare the canonical and historical volatility-based Black Scholes values to the actual Black-Scholes price.

To make this comparison, one must simulate T-day holding period stock returns R from a geometric Brownian process with 10 percent drift and 20 percent volatility, for substitution in equations (7) and (6). For example, to price an option with 126 trading days (i.e., ½ year) to expiration, we must produce 127 historical ½ year gross returns $R(-1) = P(-1)/P(-127)$, $R(-2) = P(-2)/P(-128)$, out to $R(-127) = P(-127)/P(-253)$. To do so requires the fact that the log of the ½ year gross returns will be independent and normally distributed, with a mean of 0.04 and a standard deviation of $0.2\sqrt{1/2}$ (see Hull (1993, p. 212)). Exponentiating a random sample of these returns produces the 127 simulated historical ½ year returns for substitution into equation (7). The standard deviation of the log returns is used in the usual way to produce the historical volatility estimate required for the corresponding Black-Scholes valuation.[9] All valuations assumed a constant continuously compounded interest rate of 5 percent, used to compute r in equation (7). The Newton-

[9] See Hull (1993, p. 215) for details.

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Table I

Simulation of Canonical Values in a Black-Scholes Market

Canonical call option values are compared to historical volatility-based Black-Scholes values in a Black-Scholes market with $\mu = 0.1$, $\sigma = 0.2$, and continuously compounded interest rate of 5 percent. The current index value is denoted by P and the exercise price by X. Relative exercise prices (P/X) were chosen in accord with Rubinstein's (1985, p. 466) classifications for out-of-the-money options through in-the-money options. For each combination of P/X and time to expiration, the top number is the mean absolute percentage error (MAPE) of the canonical values compared to the ideal Black-Scholes price (i.e., with $\sigma = 0.2$). The middle number reports the MAPE of the historical volatility-based Black-Scholes values. The bottom number in each cell is the canonical MAPE when an additional constraint is imposed, forcing the out-of-the-money option (i.e., $P/X = 0.95$) to be priced correctly for each expiration time.

200 Reps	Expiration (Years)		
P/X	1/13	1/4	1/2
9/10	0.339	0.122	0.107
	0.214	0.112	0.089
	0.229	0.059	0.035
1	0.038	0.038	0.047
	0.035	0.034	0.041
	0.025	0.016	0.013
1 1/8	0.001	0.007	0.014
	0.001	0.005	0.011
	0.001	0.007	0.011

Raphson method is used to find the solution γ^* to equation (7), which is substituted into equation (6) to produce the estimated martingale probabilities. These were then substituted into equation (8) to produce a canonical call option value.

For each relative exercise price (P/X) and time to expiration, the whole process was repeated 200 times, producing 200 pairs of canonical and historical volatility-based Black-Scholes values. These were used to compute both the mean absolute percentage error (MAPE) of the 200 canonical values relative to the ideal Black-Scholes model price (i.e., using $\sigma = 0.2$), and the analogous MAPE for the 200 historical volatility-based Black-Scholes values.

Typical results of this experiment are reported in Table I. Comparing the middle and top numbers in each cell, we see that the MAPEs of the historical volatility-based Black-Scholes model values are generally only a bit lower than the MAPEs of the corresponding canonical values. This performance edge is attributable to historical volatility-based Black Scholes valuation's implicit incorporation of the additional information that the stock price process has constant volatility. The canonical values make no use of this structural knowledge, which in actual practice is never known with certainty. Yet Black-Scholes' performance edge, caused by this inherent bias in the experiment, is generally small.

The worst relative performance in Table I occurs when the option is deepest out-of-the-money ($P/X = 0.9$) and closest to expiration (4 weeks). In that case,

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the historical volatility-based values are subject to a 21.4 percent error, because the historical volatility estimator has a larger standard error when the data interval is smaller, and the Black-Scholes percentage pricing errors are highly sensitive with respect to volatility.[10] The corresponding canonical values are even worse. In any short return series, there are only a few returns large enough to make a deep out-of-the-money option pay at expiration. This small number varies across the simulated series, causing the percentage pricing errors to be large.

To remedy this problem, an additional constraint is imposed for each time-to-maturity, guaranteeing the correct pricing of an option with a relative exercise price of 0.95, i.e., midway between deep-out-of-the-money and at-the-money. To incorporate this additional constraint, appropriate current, exercise, and call prices are substituted into the second constraint embodied in problem (11), which is then solved and substituted into equation (12). Comparing the bottom number of each cell in Table I with the middle number, we see that this lowers the canonical MAPE considerably, in many cases outperforming historical volatility-based valuation.[11] *In summary, the accuracy of canonical valuation in an artificial Black-Scholes world is comparable to the accuracy of historical volatility-based Black-Scholes valuation in that world.*

C. Discussion

However, it is important to note that the canonical method might not perform as well in other hypothetical worlds, where the correct martingale probabilities are radically different from the actual probabilities. This problem would arise if the single constraint (4) did not restrict the feasible set of measures enough to enable the canonical measure (6) to be close enough to the correct martingale measure. As seen above, a remedy for this problem would be to incorporate additional constraints forcing a subset of options to be priced correctly.[12] Adding these constraints shrinks the feasible set tighter around the correct martingale measure, and is consistent with the conventional maximum entropy interpretation of prediction errors, i.e. that errors are caused by the presence of additional constraints that should have been modelled (see, e.g., Jaynes (1979)). The ease of incorporating such constraints, as well as any others that can be expressed as constraints on the distribution, facilitates finding an approximation to the actual martingale probabilities.

The analogy with the maximum entropy (MaxEnt) formulation of statistical mechanics (see Jaynes (1957)) is straightforward. In statistical physics, the

[10] That is, the partial derivative "vega" divided by the option price grows large as P/X and time to expiration shrink.

[11] A comparison of the constrained canonical valuation with *implied* volatility-based Black-Scholes valuation is not necessary, because in this constant volatility market, any option's implied volatility is the actual constant volatility, so the pricing of other options will be perfect. In real markets, implied volatility-based Black-Scholes may also perform poorly for many reasons, e.g., actual volatility is not a constant, as described earlier.

[12] The author is indebted to Yacine Aït-Sahalia for making these points.

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law of conservation of energy constrains the relevant probability measure needed to predict the thermodynamics of gases, while in financial theory the law of no-arbitrage constrains the martingale measure to satisfy the constraint(s) (2) for the underlying asset(s). As in statistical mechanics, it is a testable working hypothesis that the number of *additional* constraints needed to make good canonical valuations will be few or zero. It is this *entropic hypothesis*—i.e., use of the empirical distribution as a "prior" distribution leads to a KLIC-minimizing "posterior" risk-neutral distribution requiring the imposition of fewer derivative security pricing constraints—that sharply differentiates canonical valuation from other methods.[13] Should the hypothesis prove useful in practical pricing applications, there will be little value in identifying hypothetical economic worlds where the hypothesis would not work well. In addition to the positive simulation findings above, encouraging empirical evidence from actual stock index options is presented in Section III, again without incorporation of market option prices in the valuation process. An earlier entropic option pricing paper (Stutzer (1994)) provides additional corroborating evidence. Of course, it is possible that other applications will require more incorporation of derivative security pricing constraints in order to predict the prices of others.

Before presenting the empirical evidence, we turn to a more detailed description of the use of discrete time data on prices, dividends, and interest rates needed to price S&P 500 index options.

D. The General Case

It is easy to generalize the canonical valuation procedure to incorporate the use of historical data on dividend yields d and interest rates r. As described earlier, one uses H historical prices from the underlying asset to produce the empirical distribution of T-period ahead returns, which when multiplied by the current price produces the possible time T prices of the underlying asset. The corresponding historical dividend yields and short interest rates are used to produce possible accumulated dividend totals and discount factors needed in equation (2).

For the sole purpose of illustration, suppose there are only $H = 4$ periods of historical data. Suppose one desires to price derivatives expiring in $T = 2$ periods. Start by producing $H - T = 2$ possible values for $P(T)$ in equation (2), i.e., $P \times P(-1)/P(-3)$, and $P \times P(-2)/P(-4)$, as done earlier. The two possible accumulated dividend totals corresponding to these are $P \times (d(-2)r(-2) +$

[13] For example, Jackwerth and Rubinstein (1996) and Aït-Sahalia and Lo (1995) make use of large option price databases to infer risk-neutral distributions consistent with them. These contributions are properly thought of as methods for interpolating an option pricing function or its implied risk-neutral distribution from a large set of option data consistent with it. During April 1996, it was brought to my attention that Buchen and Kelly (1996, p. 155) simulated some risk-neutral distributions, and then demonstrated that a Gibbs distribution "can make an excellent reconstruction of any highly skewed distribution" once the analyst imposes a "relatively small" subset of constraints forcing correct pricing of options over "a reasonable spread of strike prices covering the range of variation of the asset distribution."

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Table II

Canonical S&P 500 Index Call Option Prices: 1/88 to 7/94

Canonical index call option values are compared to Black-Scholes prices computed using the historical volatility (12.6 percent) and average dividend yield (3.1 percent) over the period 01/01/88 to 07/01/94, and a fixed continuously compounded interest rate of 5 percent. The current index value is $P = 450$ (a typical July 1994 index value), and the exercise price is denoted by X. Values of X in the table were again chosen in accord with Rubinstein's (1985, p. 466) classifications for out-of-the-money options through in-the-money options. In each row, the **bold faced** number is the canonical price, followed by the Black-Scholes price. The number below is the Black-Scholes volatility implied by the *canonical* value. While the tabled prices were rounded off for readability, the higher accuracy computed prices were used to compute the implied volatilities.

$P = 450$	Expiration (Years)			
X	1/13	1/4	1/2	1
500	**0.02**; 0	**0.29**; 0.68	**1.24**; 2.94	**2.59**; 8.32
	0.137	0.107	0.098	0.080
450	**5.7**; 6.6	**10.3**; 12.2	**14.3**; 17.8	**21.9**; 26.0
	0.109	0.104	0.097	0.102
400	**50.5**; 50.5	**52.3**; 51.7	**53.1**; 54.2	**56.0**; 9.3
	0.145	0.156	0.087	0.067

$d(-1))$ and $P \times (d(-3)r(-3) + d(-2))$, respectively. Each is added to its corresponding possible terminal price, producing the two possible values for the numerator in equation (2). The corresponding two values of the denominator are $r \times r(-2)$ and $r \times r(-3)$, respectively. Finally, one divides the $H - T = 2$ numerators by their corresponding denominators, substitutes these ratios for the terms $R(-h)/r^T$ in equations (4), (7), and (6), and solves equation (7) and substitutes into equation (6) to produce the estimated martingale probabilities $\hat{\pi}^*$.

Note that even when $H - T$ is large, both stochastic dividends and stochastic interest rates are treated with little increase in computational complexity.

III. Canonical Valuation of S&P 500 Index Options

To evaluate the impact of the Crash of 1987 on canonical option valuation, I produce separate canonical values using weekly historical index returns and dividend yields back from 07/01/94 to the following three dates: 01/01/88 ($H = 339$ weeks), 01/02/87 ($H = 391$ weeks), and 09/06/85 ($H = 460$ weeks).[14] To remove the influence of stochastic interest rates on the following comparison with Black-Scholes valuation, a continuously compounded interest rate of 5 percent was assumed in producing the results reported in Tables II through IV.

Table II contrasts the canonical call option values using the 339 weeks of

[14] The latter period was the longest one for which both historical returns and dividend yields could be easily obtained from the records of The Clifton Group. Also, options with fractional number of weeks to expiration could be valued by either interpolating the value from the two nearest weekly values or by using daily returns and dividend yields instead of weekly ones.

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Table III

Canonical S&P 500 Index Call Option Prices: 1/87 to 7/94

Canonical S&P 500 index call option values are compared to Black-Scholes prices computed using the historical volatility (14.7 percent) and the average dividend yield (3.1 percent) over the period 01/02/87 to 07/01/94, and a fixed continuously compounded interest rate of 5 percent. The current index value is $P = 450$ (a typical July 1994 index value), and the exercise price is denoted by X. Values of X in the table were again chosen in accord with Rubinstein's (1985, p. 466) classifications for out-of-the-money options through in-the-money options. In each row, the **bold faced** number is the canonical price, followed by the Black-Scholes price. The number below is the Black-Scholes volatility implied by the *canonical* value. While the tabled prices were rounded off for readability, the higher accuracy computed prices were used to compute the implied volatilities.

$P = 450$	Expiration (Years)			
X	1/13	1/4	1/2	1
500	**0.023**; 0.033 0.142	**0.73**; 1.34 0.127	**2.94**; 4.63 0.126	**5.4**; 11.5 0.104
450	**7.0**; 7.6 0.134	**13.6**; 14.1 0.140	**19.7**; 20.4 0.141	**26.7**; 29.6 0.129
400	**51.3**; 50.5 0.274	**55.4**; 52.1 0.237	**57.3**; 55.2 0.179	**58.6**; 61.4 0.116

historical data starting in 1988 (i.e., without the Crash of 1987) to the Black-Scholes (Merton) call option values computed using the historical average volatility (12.6 percent) and dividend yield (3.1 percent) over that period. To facilitate further comparison with Black-Scholes values, the Black-Scholes volatilities implied by the *canonical* values are also tabled. That is, *Table II also shows the volatility required to make the Black-Scholes price equal to the canonical option value*. An examination of Table II shows that there is no clear relationship between these implied volatilities and the exercise price, nor between deep in-the-money canonical call option values and their corresponding Black-Scholes prices. The empirical distribution of T-period returns may be used to *model* market participants' incorporation of historical data in formulating their beliefs about the distribution of future payoffs. With this interpretation, the canonical values in Table II do not exhibit the market prices' relationship to Black-Scholes prices *because* market participants forming expectations based *solely* on post-Crash returns would not anticipate the possibility of another Crash.

Table III shows that a markedly different pattern emerges when returns from 1987 are added to the data set. Here, an inverse relationship between implied volatilities and exercise price is evident for options expiring in three to six months, consistent with post-Crash observations. Furthermore, examination of Table III shows that the inverse pattern is less marked as the time to expiration increases from six months to one year, a phenomena also observed by practitioners studying market option prices.[15] Finally, deep in-the-money

[15] Personal communication with The Clifton Group.

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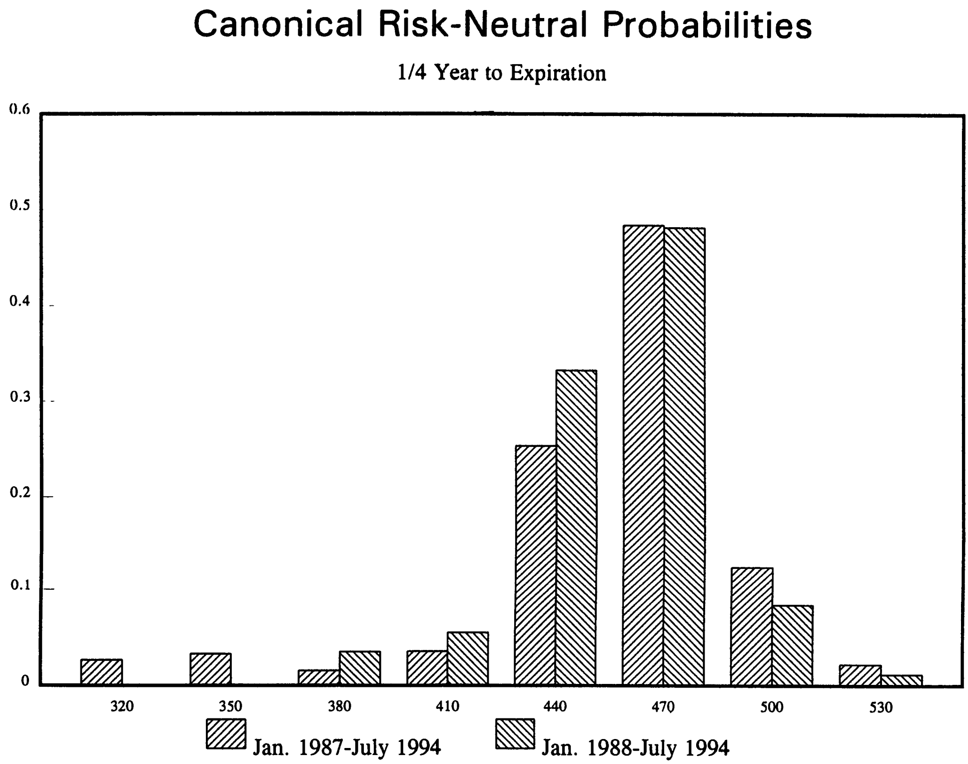

Figure 1. Comparison of Canonical Risk-Neutral Probabilities. The figure contrasts the canonical probabilities associated with the three-month-ahead distribution of the index plus accumulated dividends, from the data series starting in 1988 used to produce Table II, and from the data series starting in 1987 used to produce Table III, when the current index value is 450.

canonical call option values are indeed greater than their corresponding Black-Scholes prices for options with less than six months to maturity.

The aforementioned patterns arise because the incorporation of Crash returns from 1987 results in the assignment of more probability to sharply negative (net) returns, fattening the left-hand tail of the canonical distribution enough to produce the results. This is illustrated in Figure 1, which contrasts the post-Crash and pre-Crash data-based canonical probabilities associated with the three month-ahead distribution, again assuming a starting index value of 450. There, we see that the left-hand tail of the canonical distribution estimated with data including the Crash extends further than the tail of the distribution without Crash data.

A second diagnostic useful for illustrating this effect is Bates' (1991) Skewness Premium. Bates computed the percentage difference of the price of a call that is x percent > 0 out-of-the-money (relative to the current forward index value for delivery at the option's expiration) to the price of a put that is similarly x percent out-of-the-money. The ratio will of course depend on the relative thickness of the right and left hand tails of the martingale measure.

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Table IV

Canonical S&P 500 Index Call Option Prices: 9/85 to 7/94

Canonical S&P 500 index call option values are compared to Black-Scholes prices computed using the historical volatility (14.7 percent) and the average dividend yield (3.2 percent) over the longer period 09/06/85 to 07/01/94, and a fixed continuously compounded interest rate of 5 percent. The current index value is $P = 450$ (a typical July 1994 index value), and the exercise price is denoted by X. Values of X in the table were again chosen in accord with Rubinstein's (1985, p. 466) classifications for out-of-the-money options through in-the-money options. In each row, the **bold faced** number is the canonical price, followed by the Black-Scholes price. The number below is the Black-Scholes volatility implied by the *canonical* value. While the tabled prices were rounded off for readability, the higher accuracy computed prices were used to compute the implied volatilities.

$P = 450$	Expiration (Years)			
X	1/13	1/4	1/2	1
500	**0.017**; 0.033	**1.03**; 1.35	**4.05**; 4.63	**7.09**; 11.4
	0.137	0.138	0.140	0.117
450	**7.41**; 7.62	**14.8**; 14.1	**20.8**; 20.4	**27.5**; 29.4
	0.143	0.155	0.151	0.136
400	**51.3**; 50.4	**55.9**; 52.0	**57.7**; 55.1	**58.7**; 61.1
	0.280	0.249	0.188	0.123

He showed that many estimated option pricing models predict that this skewness premium will be positive, and not more than x percent. He then documented (for the most recent evidence, see Bates (1995a)) that the skewness premium of market S&P 500 futures option prices is significantly negative, and thereby concluded that most estimated option pricing models were inconsistent with market option prices.

We compute Bates' Skewness Premium for canonical valuation using the historical data starting in 1987. Canonical valuation for options in the three to six month range typically exhibits negative skewness premia for options in the three to six month range when $x > 0.02$. So canonical valuation passes this diagnostic test, which was failed by the Black-Scholes and several other option pricing models estimated by Bates. And it did so *without* incorporating market option prices in the valuation process.

Table IV shows that the results do not disappear when the additional returns back to September 1985 are included. Thus an inverse relationship between implied volatility and exercise price is likely to continue until the Crash of 1987 becomes a negligible part of past history.

Finally, the possible effects of stochastic interest rates are also considered. To estimate the marginal impact of a specific assumption about stochastic interest rates on the canonical values in Table IV, the assumption that the log of the weekly gross interest rate is a constant 0.05/52 was replaced by the assumption that it is independently, normally distributed with a mean equal to that constant, and a standard deviation 1/5 as large. The procedure described in Section II.D is used to produce new values of the numerators and denominators in equation (2), and the canonical values are recomputed. There

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is little percentage difference between them and the corresponding values in Table IV.

IV. Conclusions and Future Research

A simple nonparametric approach to derivative security valuation, called *canonical valuation*, has been proposed and implemented. The maximum entropy principle of Bayesian information theoretic statistics was used to transform a nonparametric estimator of the actual probability distribution into an estimate of the risk-neutral (equivalent martingale) probability distribution required for arbitrage-free valuation of derivative securities.

In a simulated market where the Black-Scholes model would (in theory) work well, it was shown that canonical valuation was about as accurate as historical volatility-based Black-Scholes valuation. To illustrate the use of canonical valuation in a real market, historical prices and dividend yields from the S&P 500 were used for valuing S&P 500 index options. It is well established that Black-Scholes has underpriced deep in-the-money calls since the Crash of 1987. Because canonical valuation depends on historical index returns, the impact of this Crash on canonical option values could be evaluated. It was shown that only when Crash-era data are utilized do the canonical option values exhibit this characteristic of index option prices. A related finding is that while market index option prices exhibit negative values for Bates' Skewness Premium, many estimated parametric option pricing models predict that the premium should be positive. Computations show that canonical option values do possess a negative skewness premium, also consistent with the observed market prices.

As in Rubinstein (1994), several important topics were left for future research, both theoretical and empirical. Like Monte Carlo approaches to option pricing, there is as yet no simple way to estimate the value of early exercise, a shortcoming in cases where that value is known to be significant. A second, difficult theoretical topic is to determine the optimal sampling interval and the length of the required historical time series. Computational demands increase with more frequent sampling and/or longer historical periods. In this introductory article, a sampling interval of one week and a series length dating back to before the Crash of 1987 seemed to yield sensible results. But it might be possible to use large deviations theory to quantify the nature of the results' dependence on both these variables. A third important topic is to examine the method's utility in more complex applications, such as options with more than one underlying asset, where its relative computational simplicity would be even more pronounced. In order to focus attention on the recent desire to find better ways to value stock index options, this is left for another paper.

Among the important empirical issues, there is need for a more detailed comparison of both canonical values and other nonparametric model values to actual transaction prices in important markets. This comparison must be done with great care, respecting the pitfalls noted by Rubinstein (1985) and Bates (1995b). Doing so will help determine the number and nature of additional

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self-consistent pricing constraints that are required for accurate canonical valuation in those markets. Finally, in some applications, it may be advantageous to restrict the historical returns to those periods where the underlying asset's price is close to its current price.

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