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β in the tails[☆]

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ABSTRACT

Do hedge funds hedge? In negative states of the world, often not as much as they should. For several styles, we report larger market betas when market returns are low (i.e., "beta in the tails"). We justify this finding through a combination of *negative-mean* jumps in the market returns and *large* market jump betas: when moving to the left tail of the market return distribution jump dynamics dominate continuous dynamics and the overall systematic risk of the fund is driven by the higher systematic risk associated with return discontinuities. Methodologically, the separation of continuous and discontinuous dynamics is conducted by exploiting the informational content of the high-order infinitesimal cross-moments of hedge-fund and market returns.

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1. Introduction

Hedge funds should hedge. They should offer additional diversification to investors holding "market-like" portfolios and should, therefore, have low correlation with the market portfolio. Said differently, their systematic risk, as measured by the classical market beta,[1] should be limited. This is somewhat true for most styles and is generally true for styles whose explicit aim is to reduce market exposure, as is the case, e.g., for Equity Market Neutral funds.

In spite of this observation, should one condition on large negative market shocks, the resulting hedge-fund betas would often appear larger (in absolute value) than they would be in the absence of conditioning, i.e., when considering the full sample. More precisely, when sorting market returns from the smallest to the largest and computing betas associated with returns up to a specific, maximum, market return level, the betas often decline in absolute value (irrespective of style) toward zero. This phenomenon is dubbed "beta in the tails", *positive* (resp. *negative*) "beta in the tails" being associated with *larger* (resp. *lower*) betas in the left tail of the market return distribution. The presence of positive "beta in the tails" is, of course, such that diversification fails precisely when it is needed, i.e., during more extreme market conditions.[2]

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1 Because the object of interest will always be a "market beta" in this paper, in what follows we will ignore the qualifier "market", unless it is required by the context, and simply write "beta".

2 We are thankful to Jimmy Liew for bringing positive "beta in the tails" to our attention as an intriguing empirical finding whose regularity we evaluate in this study before turning to modeling. Early evidence is contained in Liew (2003).

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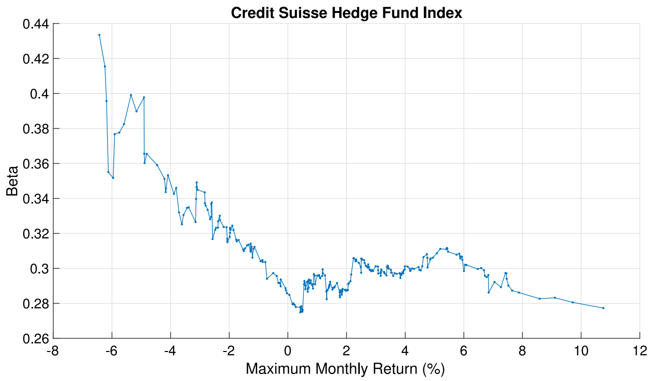

Fig. 1. Credit Suisse Hedge Fund Index betas computed on sorted market returns. Each point in the figure corresponds to an observed market return. The corresponding beta is obtained by using all market returns below that value along with the associated returns on the Credit Suisse Hedge Fund Index. The returns are monthly from January 1994 to August 2018, for a total of 295 observations.

Fig. 1 illustrates the described phenomenon for the Credit Suisse Hedge Fund Index.[3] The index has positive beta in the left tail (with betas in excess of 0.4) but the beta values decline (to numbers lower than 0.28) as larger market returns are included in the sample. The decline is reasonably steep but is not as steep as that of some of the styles comprising the index.

Fig. 2 provides the same plot for the 10 styles in the index and 3 Event Driven sub-styles. The returns on Convertible Arbitrage strategies, for instance, have betas in the left tail as high as 0.8. The betas, however, decline virtually monotonically to values lower than 0.2. Similarly, e.g., Emerging Markets funds may have overall betas around 0.5 but their betas in the left tail are as high as 1.3.

While generally positive, "beta in the tails" appears to be negative in two cases: Managed Futures funds and, to some extent, Equity Market Neutral funds. Managed Futures funds, or Commodity Trading Advisor (CTA) pools,[4] seem to be effective hedges in the left tail of the market return distribution. Their tendency not to be net long or short in any specific market leads to betas which converge to small (absolute) values as one sorts through larger market returns. The Equity Market Neutral betas hover between −0.25 and 0.3, which is coherent with the idea that Market Neutral Funds should be able to control market exposure. If anything, it is interesting to notice that these funds also offer some diversification in the presence of large negative market shocks. We will return to the robustness of this finding.

This preliminary evidence offers a clear message. The systematic risk of alternative (alt) investments in times of market drops may be considerably different from its unconditional level. A positive "beta in the tails", in particular, translates into a form of tail risk with adverse implications for investors holding diversified portfolios and looking to hedge market exposure with these alt investments.

While suggestive, this evidence is, however, not model-based. To this extent, in what follows we assume hedge-fund returns are the result of two components: a *diffusive* component capturing dynamics in the small price changes and a *discontinuous* (jump) component capturing the temporal evolution of the large price changes. Both components are further broken down into an element correlated with the market and one idiosyncratic in nature.

The assumed data generating process for hedge-fund returns leads to two notions of market beta, one associated with the diffusive return component and one associated with the discontinuous return component. Given the model, the *overall beta* is a weighted average of *diffusive beta* and *jump beta* with weights given by the contributions to the overall market variance of the variances of the individual (diffusive and jump) market components.

We show that the proposed model can generate the effects in Figs. 1 and 2. The intuition is simple. The market return unconditional distribution is a mixture of the distributions of continuous returns and jump returns. If the jump distribution is in the left tail of the overall market return distribution (something which is empirically compelling in light of the negative mean of the market return jumps), a positive (resp. negative) jump beta larger, in absolute value, than the diffusive beta will drive the overall beta in the market's left tail and generate positive (resp. negative) "beta in the tails".[5]

Importantly, a combination of jump betas larger, in absolute value, than diffusive betas and negative-mean market jumps can, *alone*, generate "beta in the tails" without the need for *any* time-variation in the model parameters, including

[3] The index comprises 10 styles: Convertible Arbitrage, Dedicated Short Bias, Emerging Markets, Equity Market Neutral, Event Driven, Fixed Income Arbitrage, Global Macro, Long/Short Equity, Managed Futures and Multistrategy. The same styles, along with a breakdown of Event Driven into Distressed Event Driven, Multistrategy Event Driven and Risk Arbitrage Event Driven, will be evaluated individually in what follows.

[4] An early account of the process of assimilation, due to the relaxation of regulatory constraints, of CTAs in the hedge-fund space is contained in Fung and Hsieh (1999).

[5] We will document that, for the majority of the funds in our sample, our positive estimates of the jump betas are larger, and often markedly so, than the corresponding diffusive beta estimates.

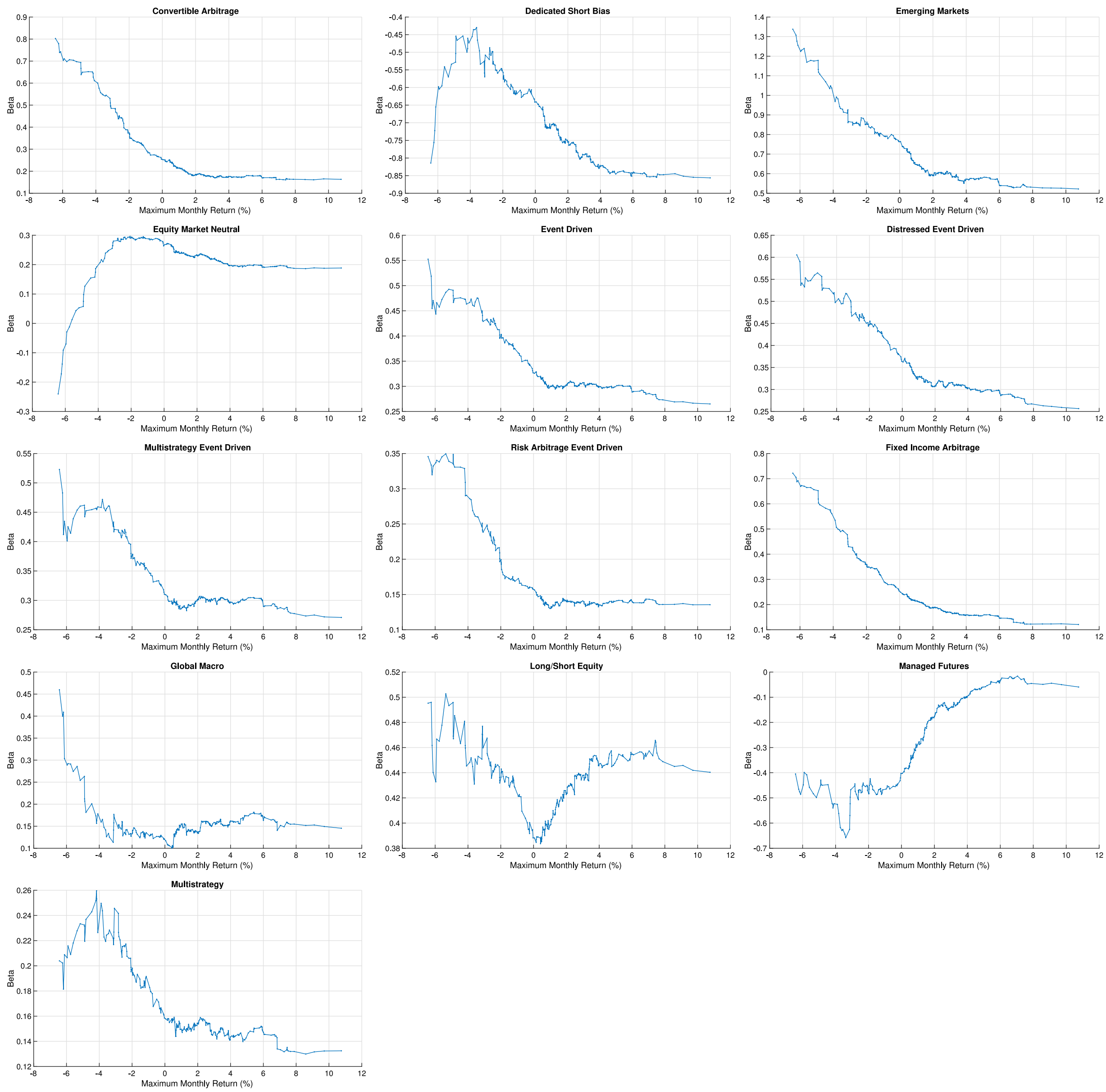

Fig. 2. Betas computed on sorted market returns for 10 hedge-fund styles (Convertible Arbitrage, Dedicated Short Bias, Emerging Markets, Equity Market Neutral, Event Driven, Fixed Income Arbitrage, Global Macro, Long/Short Equity, Managed Futures and Multistrategy) and 3 sub-styles (Distressed Event Driven, Multistrategy Event Driven and Risk Arbitrage Event Driven). Each point in the panels corresponds to an observed market return. The corresponding beta is obtained by using all market returns below that value along with the associated returns on each specific style/sub-style. The returns are monthly from January 1994 to August 2018, for a total of 295 observations.

the jump and diffusive betas themselves. This said, "beta in the tails" may not just be jump beta. It could also potentially be *dynamic beta*, an increasing (diffusive and, hence, overall) beta in adverse states of nature associated with market returns in the left tail of their distribution. In light of this observation, we allow for unrestricted dependence of all betas (and all functions of the assumed bivariate system for hedge-fund returns and the market) on a variable capturing the state of the economy. Conditioning on the state of economy adds flexibility to the model and makes it less prone to misspecification. We report findings associated with market (diffusive) volatility as the conditioning variable but, as we discuss below, the results are robust to other (financial and macroeconomic) variables.

Do hedge funds offer diversification to investors holding well-diversified portfolios? Our findings suggest a positive (albeit qualified) answer when focusing on unconditional systematic risk. When conditioning on times of extreme market stress, however, their level of systematic risk (i.e., their "beta in the tails") may be such that the advertised diversification potential of these alt investments should rightfully be called into question.

Having made this point, we provide some empirical support to the idea that *tail systematic risk* may, in fact, be *priced*. Indeed, a CAPM model in which systematic risk is measured using jump betas is found to capture the cross-sectional variation in expected hedge-fund returns better than a CAPM model in which systematic risk is measured using diffusive betas. Some observations are in order. First, in light of our limited (cross-sectional and time-series) sample size, these results should only be viewed as suggestive, rather than as conclusive. Second, the sole goal of our pricing exercise is to offer some evidence regarding the reaction of market participants to tail market risk. Said differently, our goal is *not* to search for the best pricing model for hedge-fund returns, a task which would require broadening the set of factors beyond the market (see, e.g., Bali et al., 2011, 2012; Fung and Hsieh, 2004; Hasanhodzic and Lo, 2007). Third, even though the CAPM is likely to be an overly simplistic model for hedge-fund returns, there is ample evidence that it represents a key benchmark. The work of Agarwal et al. (2018), for example, is explicit about the role that CAPM alpha still plays in providing information about fund performance and, as a result, in driving capital flows into the hedge-fund industry. Similar evidence has been offered about mutual funds (Berk and Van Binsbergen, 2016; Barber et al., 2016).

The paper proceeds as follows. Section 2 discusses the related literature and further positions this paper. In Section 3, we present a continuous-time bivariate model for hedge-fund returns and market returns with systematic and idiosyncratic jumps. In the context of the model, we define diffusive beta and jump beta and discuss the representation of the overall beta in terms of their weighted average. Section 4 illustrates the proposed identification procedure. We rely on the bivariate process' *high-order infinitesimal cross-moments*, their closed-form expressions for every state of the conditioning variable and, in consequence, their suitability for moment-based estimation. The empirical findings are in Section 5. Some asset pricing implications are discussed in Section 6. Section 7 concludes.

2. The related literature and positioning

We address two related literatures. On the one hand, a vast recent body of work in finance has focused on the specificities of alt investments, hedge funds in particular, from a variety of perspectives. By focusing on three notions of systematic risk (as represented by diffusive, jump and overall beta), we study the ability of alternative hedge-fund styles to provide diversification in times of market stress. On the other hand, a successful literature in financial econometrics has provided, as we do, measurements of systematic (diffusive and jump) risk. A critical feature of our work, one which separates us from the bulk of the existing contributions, is the nonparametric nature of our beta measurements. We will argue that identification based on high-order infinitesimal cross-moments of the market and hedge-fund return processes is conducive to the evaluation of flexible forms of dependence between systematic risk and the state of the economy. Allowing for such flexibility improves model specification.

We cannot attempt to review the two literatures in their entirety here but only provide a few key references to position our contribution.

In a classical paper, Fung and Hsieh (1997) emphasize the ability of hedge-fund managers to generate, through dynamic trading strategies, returns which have low correlation with those of traditional asset classes. After adjusting for the non-synchronicity between hedge-fund returns and market returns, Asness et al. (2001) find that hedge funds have larger exposure to the market than previously believed. They argue that large monthly returns may, in fact, be due to such exposure rather than to skill. A skeptical view of the statistical significance of skill is also offered by Liew (2003), who provides an early account of increased systematic risk in times of "market dislocation". We too argue that market exposure may be larger than believed (and priced). Focusing on "market dislocation" as in Liew (2003), rather than on measurement issues as in Asness et al. (2001), we study the mechanics of enhanced market exposure associated with adverse market events and formalize the role of systematic jump risk.

The importance of time-variation in risk exposure (e.g., for performance evaluation) is central to the work of, e.g., Bollen and Whaley (2009) and Patton and Ramadorai (2013). Consistent with the implications of this line of inquiry, we allow for flexible nonlinear dependence on the state of the economy. As in Billio et al. (2012), who study the risk of alternative hedge-fund styles in times of crisis, our proposed separation of the overall market beta into diffusive and jump betas, combined with beta dependence on the state of the economy, assists our study of the risk exposure of different hedge-fund styles in times of market stress. For a recent review paper on the hedge-fund industry, and additional references, we refer the reader to Getmansky et al. (2015).

On the measurement side, Bollerslev et al. (2016) study the separation of the traditional (overall) beta into continuous and discontinuous beta using high-frequency data. As in their work, this separation is central to what we do. We, however, operate within an alternative identification procedure, one which permits conditioning on state variables as well as joint estimation of all of the system's driving functions. Li et al. (2017) introduce an efficient inferential theory to estimate discontinuous betas interpreted as slopes in suitably-defined regressions of filtered asset price jumps on filtered market price jumps. As we will discuss, borrowing their jargon, the model we propose will have an interpretation in terms of a *mixed* diffusive/jump regression with two slopes, one associated with diffusive dynamics and one associated with jump dynamics. The identification procedure will differ from that in Li et al. (2017). In particular, it will not require preliminary filtering of the large discontinuities, a feature which is helpful in the absence of high-frequency data. We, in fact, use monthly data in this study. As emphasized, our procedure will also allow for conditioning on the state of the economy, something which will aid model specification and facilitate economic interpretation.

3. The model

We consider a bivariate continuous-time model for excess continuously-compounded hedge-fund returns (dY_t) and excess continuously-compounded market returns (dX_t). Given Brownian shocks $\{W, W'\}$ and Poisson shocks $\{J, J'\}$ with (state-dependent) intensities $\lambda_X(z_t)$ and $\lambda_Y(z_t)$, the model is expressed as follows:

$$dY_t = \mu_Y(z_t)dt + \underbrace{\beta(z_t)\sigma_X(z_t)dW_t}_{dY_t^{\text{dif}}} + \underbrace{\tilde{\beta}(z_t)c_X(z_t)dJ_t}_{dY_t^{\text{jump}}} + \underbrace{s(z_t)dW'_t + c_Y(z_t)dJ'_t}_{dY_t^{\text{id}}}, \tag{1}$$

$$dX_t = \mu_X(z_t)dt + \underbrace{\sigma_X(z_t)dW_t}_{dX_t^{\text{dif}}} + \underbrace{c_X(z_t)dJ_t}_{dX_t^{\text{jump}}}, \tag{2}$$

where $c_X(z_t)$ and $c_Y(z_t)$ are random jump sizes with expected values $\mu_X^{\text{jump}}(z_t)$ and $\mu_Y^{\text{jump}}(z_t)$ and standard deviations $\sigma_X^{\text{jump}}(z_t)$ and $\sigma_Y^{\text{jump}}(z_t)$. All shocks are independent. The quantities dY_t^{jump} and dX_t^{jump} are systematic co-jump components leading to a notion of *jump beta* denoted by $\tilde{\beta}(z_t)$. Similarly, dY_t^{dif} and dX_t^{dif} are systematic diffusive components leading to a notion of *diffusive beta* denoted by $\beta(z_t)$. The term Y_t^{id} represents the (idiosyncratic) portion of the hedge-fund return dynamics which is independent of market dynamics and may be viewed as an orthogonal residual in a regression of dY_t on dX_t^{dif} and dX_t^{jump} (c.f. Eq. (3)). Both betas, and all other functions of the bivariate system, are assumed to depend on the state of the economy, as summarized by the variable z_t.[6]

The adopted specification is parsimonious and, yet, rich. First, the excess market return process is modeled as having *small* and *large* components. Second, we allow for *idiosyncratic* and *systematic*, *small* and *large*, changes in the hedge-fund prices. The systematic, small and large, price changes and, in particular, their associated betas will be the object of our analysis.

In order to facilitate interpretation, it is convenient to re-write the model as follows:

$$\begin{aligned} dY_t &= \mu_Y(z_t)dt + dY_t^{\text{dif}} + dY_t^{\text{jump}} + dY_t^{\text{id}}, \\ &= \mu_Y(z_t)dt + \beta(z_t)dX_t^{\text{dif}} + \tilde{\beta}(z_t)dX_t^{\text{jump}} + dY_t^{\text{id}}. \end{aligned} \tag{3}$$

The result is a *mixed* diffusive/jump regression in which the intercept (i.e., the drift of the excess hedge-fund return process) and the slopes (i.e., the betas) are functions of the state variable z_t. In the next section, we discuss a moment-based local procedure designed to identify the regression slopes $\beta(z_t)$ and $\tilde{\beta}(z_t)$ for all levels z_t.

Before doing so, we observe that the overall (conditional) market beta can be expressed as follows:

$$\begin{aligned} \frac{\mathrm{C}_t(dY_t, dX_t)}{\mathrm{V}_t(dX_t)} &= \frac{\mathrm{V}_t(dX_t^{\text{dif}})}{\mathrm{V}_t(dX_t)}\beta(z_t) + \frac{\mathrm{V}_t(dX_t^{\text{jump}})}{\mathrm{V}_t(dX_t)}\tilde{\beta}(z_t) \\ &= \frac{\mathrm{V}_t(dX_t^{\text{dif}})}{\mathrm{V}_t(dX_t^{\text{dif}} + dX_t^{\text{jump}})}\beta(z_t) + \frac{\mathrm{V}_t(dX_t^{\text{jump}})}{\mathrm{V}_t(dX_t^{\text{dif}} + dX_t^{\text{jump}})}\tilde{\beta}(z_t) \\ &= \frac{\sigma_X^2(z_t)}{\sigma_X^2(z_t) + ((\sigma_X^{\text{jump}})^2(z_t) + (\mu_X^{\text{jump}})^2(z_t))\lambda_X(z_t)}\beta(z_t) \\ &\quad + \frac{((\sigma_X^{\text{jump}})^2(z_t) + (\mu_X^{\text{jump}})^2(z_t))\lambda_X(z_t)}{\sigma_X^2(z_t) + ((\sigma_X^{\text{jump}})^2(z_t) + (\mu_X^{\text{jump}})^2(z_t))\lambda_X(z_t)}\tilde{\beta}(z_t) \\ &:= (1 - w(z_t))\beta(z_t) + w(z_t)\tilde{\beta}(z_t) \\ &= \beta^{\text{overall}}(z_t). \end{aligned} \tag{4}$$

The overall beta is a weighted average of the diffusive and jump betas with weights given by the relative (with respect to the overall market variance) variance of the corresponding diffusive and discontinuous market return components. The presence of the mean of the jump sizes, i.e., $\mu_X^{\text{jump}}(z_t)$, in $\mathrm{V}_t(dX_t^{\text{jump}})$ should not be surprising. In the absence of jump size randomness (i.e., when $\sigma_X^{\text{jump}}(z_t) = 0$), "deterministic" jumps continue to induce variability in the market return process through their likelihood of occurring.

3.1. β in the tails: the mechanism

Eq. (4) is suggestive of the fact that, in general, the relative contribution of jump beta to the overall beta is the result of complex dynamics driven by the dependence between key quantities of the system in Eqs. (1) and (2) and the state variable z_t. In order to relate the model directly to Figs. 1 and 2, one may, therefore, wish to use market returns as the conditioning variable z_t. Because of their "flow" nature, the use of contemporaneous market returns (dX_t) is, however, not

[6] In what follows, the conditioning variable will be market diffusive volatility but we will discuss robustness to alternative choices.

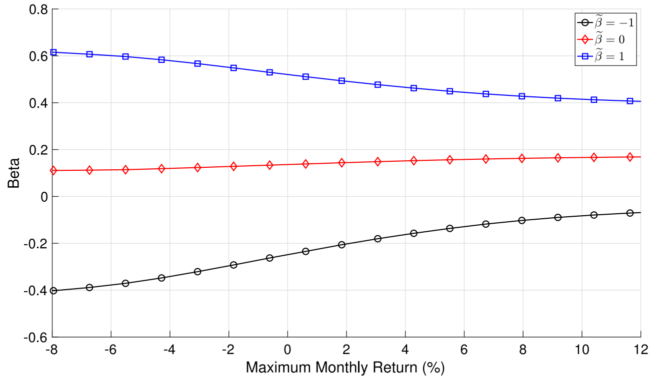

Fig. 3. Overall betas of simulated (from the model in Eqs. (1) and (2)) returns on the market and the Credit Suisse Hedge Fund Index. The parameter estimates used in the simulations are averages over the empirical distribution of the conditioning variable, i.e., $z_t = \sigma_t$, where σ_t is market diffusive volatility. The estimated overall betas are plotted with respect to sorted market returns (as in Figs. 1 and 2) for three different values of the jump beta, i.e., $\tilde{\beta}$.

possible in continuous time. At the same time, using "stock" variables for conditioning would make interpretation difficult and dependent on the relation between the adopted variable and market returns (i.e., the conditioning variable in Figs. 1 and 2). In light of these observations, while we do condition on the state of the economy and allow for unrestricted dependence on z_t in order to improve model specification, we emphasize a key reason for "beta in the tails" which does not hinge on rich (or *any*, in fact) time-variation driven by the state variable z_t.

The logic is the following. A first-order implication of the model is that *negative-mean* market jumps (i.e., jumps in the left tail of the market return distribution) and jump betas *larger* (in absolute value) than the diffusive betas may generate large (again, in absolute value) overall betas in the left tail of the market return distribution *even* if all of the system's functions are constant. The intuition is simple. When conditioning on returns in the left tail of the market return distribution (as done in Figs. 1 and 2 for low market returns), the dominant beta is the one associated with jumps (being the market jumps concentrated, because of their negative mean, in the left tail of the market return distribution). As we progress toward the center of the market return distribution and, subsequently, toward its right tail, diffusive beta plays a larger and larger role, thereby attenuating the overall beta.

This logic is consistent with Fig. 3. The figure is obtained by simulating market return data and data on the Credit Suisse Hedge Fund Index returns. As in the motivational Fig. 1, we plot overall betas as a function of sorted market returns using the model in Eqs. (1)–(2) and *average* parameter estimates (from the model) for three different values of the jump beta $\tilde{\beta}$ (-1, 0 and 1). The averages of the parameter estimates (obtained as in Section 4) are taken over the empirical distribution of $z_t = \sigma_t$, where σ_t is market diffusive volatility. When $\tilde{\beta}$ is large in absolute value, we reproduce "beta in the tails" with all of the system's functions set as being constant. As pointed out, allowing for dependence on a variable correlated with the state of the economy solely adds flexibility to the model (by changing the conditional and unconditional distribution of the diffusive and jump return components) and improves model specification. While such dependence could contribute to "beta in the tails", it is neither needed nor a likely first-order driver.

Simulations reveal another implication of the model. In any finite sample, interesting beta dynamics (possibly leading to non-monotonic patterns) in the extreme left tail of the market return distribution could also be induced by an interaction between extreme *diffusive* tail observations and *jump* tail observations. The former may even lead to overall betas which are temporarily *larger* (resp. *lower*) than the jump betas by levering the jump betas *up* (resp. *down*) in just the same way in which outliers would affect regression slopes. As one moves toward the center of the return distribution, these effects would quickly disappear. In agreement with this observation, both Figs. 1 and 2 reveal some non-monotonicity at the extreme left limit of the market return range. We will show that, even with a jump beta larger than the diffusive beta, the overall beta will sometime be estimated as being larger than the jump beta.

4. Identification through nonparametric infinitesimal method of moments

Define the generic *infinitesimal cross-moment* of order p_1, p_2, with $p_1 \geqslant p_2 \geqslant 0$, between the hedge-fund excess return process and the market excess return process:

$$\vartheta_{p_1,p_2}(z) = \lim_{\Delta\to 0} \frac{1}{\Delta} \mathrm{E}\left[(Y_{t+\Delta} - Y_t)^{p_1} (X_{t+\Delta} - X_t)^{p_2} \, | z_t = z \right]. \tag{5}$$

The infinitesimal cross-moments have closed-form representations in terms of the functions defining the bivariate system. Alternative choices of p_1 and p_2 will, therefore, highlight different aspects of the system's bivariate distribution and provide useful restrictions (conditional on the state z) for identification.

We assume all discontinuities are normally distributed. In order to avoid clutter, in what follows we suppress the dependence on z, when ambiguity does not arise. All moments (and the corresponding functions) should, however, be interpreted as being state-dependent.

Consider, first, the hedge-fund return infinitesimal moments. Write

$$\vartheta_{1,0} = \mu_Y + \vartheta_{1,0}^{\text{jump}}, \tag{6}$$

$$\vartheta_{2,0} = (\beta^2\sigma_X^2 + s^2) + \vartheta_{2,0}^{\text{jump}}, \tag{7}$$

$$\vartheta_{p_1,0} = \vartheta_{p_1,0}^{\text{jump}} \qquad p_1 \geqslant 3 \tag{8}$$

with

$$\vartheta_{p_1,0}^{\text{jump}} = \lambda_X \sum_{j=0}^{p_1} \binom{p_1}{j} G_{0,j} \left(\tilde{\beta}\sigma_X^{\text{jump}}\right)^j \left(\tilde{\beta}\mu_X^{\text{jump}}\right)^{p_1-j} + \lambda_Y \sum_{j=0}^{p_1} \binom{p_1}{j} G_{0,j} \left(\sigma_Y^{\text{jump}}\right)^j \left(\mu_Y^{\text{jump}}\right)^{p_1-j},$$

where $G_{0,0} = 1$ and, for $g \geqslant 1$, $G_{0,2g} = (2g-1)!!$ and $G_{0,2g-1} = 0$.

All infinitesimal moments depend on the corresponding moments of the systematic and idiosyncratic jumps. The first and the second infinitesimal moments also depend on the drift and the diffusive variance, respectively. The same expressions apply to the market (without the idiosyncratic jump and diffusive components) and imply that, e.g., the first six infinitesimal moments of the market return process are:

$$\vartheta_{0,1} = \mu_X + \lambda_X \mu_X^{\text{jump}}, \tag{9}$$

$$\vartheta_{0,2} = \sigma_X^2 + \lambda_X \left(\left(\mu_X^{\text{jump}}\right)^2 + \left(\sigma_X^{\text{jump}}\right)^2 \right), \tag{10}$$

$$\vartheta_{0,3} = \lambda_X \left(\left(\mu_X^{\text{jump}}\right)^3 + 3\left(\mu_X^{\text{jump}}\right)\left(\sigma_X^{\text{jump}}\right)^2 \right), \tag{11}$$

$$\vartheta_{0,4} = \lambda_X \left(\left(\mu_X^{\text{jump}}\right)^4 + 6\left(\mu_X^{\text{jump}}\right)^2\left(\sigma_X^{\text{jump}}\right)^2 + 3\left(\sigma_X^{\text{jump}}\right)^4 \right), \tag{12}$$

$$\vartheta_{0,5} = \lambda_X \left(\left(\mu_X^{\text{jump}}\right)^5 + 10\left(\mu_X^{\text{jump}}\right)^3\left(\sigma_X^{\text{jump}}\right)^2 + 15\left(\mu_X^{\text{jump}}\right)\left(\sigma_X^{\text{jump}}\right)^4 \right), \tag{13}$$

$$\vartheta_{0,6} = \lambda_X \left(\left(\mu_X^{\text{jump}}\right)^6 + 15\left(\mu_X^{\text{jump}}\right)^4\left(\sigma_X^{\text{jump}}\right)^2 + 45\left(\mu_X^{\text{jump}}\right)^2\left(\sigma_X^{\text{jump}}\right)^4 + 15\left(\sigma_X^{\text{jump}}\right)^6 \right). \tag{14}$$

We now turn to the genuine infinitesimal cross-moments:

$$\vartheta_{1,1} = \beta\sigma_X^2 + \vartheta_{1,1}^{\text{jump}} \tag{15}$$

and

$$\vartheta_{1+p_1,1+p_2} = \vartheta_{1+p_1,1+p_2}^{\text{jump}} \qquad p_1 \geqslant 1, p_2 = 0 \text{ or } p_2 \geqslant 1, p_1 = 0 \text{ or } p_1, p_2 \geqslant 1, \tag{16}$$

with

$$\vartheta_{p_1,p_2}^{\text{jump}} = \lambda_X \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2} \binom{p_1}{j_1}\binom{p_2}{j_2} G_{j_1,j_2} \left(\tilde{\beta}\sigma_X^{\text{jump}}\right)^{j_1} \left(\sigma_X^{\text{jump}}\right)^{j_2} \left(\tilde{\beta}\mu_X^{\text{jump}}\right)^{p_1-j_1} \left(\mu_X^{\text{jump}}\right)^{p_2-j_2}, \tag{17}$$

where, for $g_1, g_2 \geqslant 1$, $G_{g_1,g_2} = (g_1+g_2-1)G_{g_1-1,g_2-1}$. The cross-moment expressions imply, for instance, that

$$\vartheta_{1,1} = \beta\sigma_X^2 + \lambda_X \left(\tilde{\beta}\left(\sigma_X^{\text{jump}}\right)^2 + \tilde{\beta}\left(\mu_X^{\text{jump}}\right)^2 \right),$$

and

$$\vartheta_{2,2} = \lambda_X \left(\tilde{\beta}^2\left(\mu_X^{\text{jump}}\right)^4 + 6\tilde{\beta}^2\left(\sigma_X^{\text{jump}}\right)^2\left(\mu_X^{\text{jump}}\right)^2 + 3\tilde{\beta}^2\left(\sigma_X^{\text{jump}}\right)^4 \right).$$

In essence, by setting $p_1 = 0$ and varying p_2, one may identify all functions of the market excess return process (c.f., Eqs. (9) through (14)). By setting $p_2 = 0$ and varying p_1, and by employing the genuine cross-moments with $p_1 \geqslant p_2 \geqslant 1$, one may identify the diffusive beta, the jump beta and the idiosyncratic (jump and diffusive) features of the hedge-fund excess return process.

The generic infinitesimal cross-moments in Eq. (5) lend themselves to moment-based estimation, a procedure we have called Nonparametric Infinitesimal Method of Moments, or NIMM, in other contexts (Bandi and Renò, 2016). Once a nonparametric estimator for Eq. (5) is defined, the closed-form expressions in Eqs. (6) through (8), the analogous expressions with $p_1 = 0$ and $p_2 \neq 0$, and Eqs. (15) through (17) can be brought to bear to identify all system's functions by optimizing a metric which minimizes the distance between the nonparametric estimates and their closed-form expressions. We note that the procedure is conducted for all states z. Hence, even though, for each z level, the method is akin to a parametric procedure, the final outcome is the provision of nonparametric functions of the assumed state variable. We now provide specifics.

4.1. NIMM

Let $\{Y_{i\Delta_{n,T}}, X_{i\Delta_{n,T}}\}$ be a discrete bivariate sample of hedge-fund and market excess returns. We assume availability of n equally-spaced observations (with spacing in time denoted by $\Delta_{n,T}$) over a time span T. Next, we define the following Nadaraya–Watson kernel estimator:

$$\widehat{\vartheta}_{p_1,p_2}(z) = \frac{\sum_{i=1}^{n-1} \mathbf{K}\left(\frac{z_{i\Delta_{n,T}} - z}{h_{n,T}(z)}\right) \left(Y_{(i+1)\Delta_{n,T}} - Y_{i\Delta_{n,T}}\right)^{p_1} \left(X_{(i+1)\Delta_{n,T}} - X_{i\Delta_{n,T}}\right)^{p_2}}{\Delta_{n,T} \sum_{i=1}^{n} \mathbf{K}\left(\frac{z_{i\Delta_{n,T}} - z}{h_{n,T}(z)}\right)}, \tag{18}$$

where $\mathbf{K}(.)$ is a kernel function whose role is to guarantee proper conditioning on z and $h_{n,T}(z)$ is a (state-specific) smoothing parameter. The quantity $\widehat{\vartheta}_{p_1,p_2}(z)$ is an infinitesimal cross-moment estimator. In what follows, we use $\widehat{\vartheta}_{p_1,p_2}(z)$ to identify $\vartheta_{p_1,p_2}(z)$, for all choices of p_1 and p_2.

Denote, now, by $f_1(z), \ldots, f_M(z)$ the M functions of the bivariate system in Eqs. (1) and (2) ($\mu_X(z)$, $\mu_Y(z)$ and so on). These functions ($\beta(z)$ and $\tilde{\beta}(z)$, in particular) are the objects of econometric interest. We employ S infinitesimal cross-moment estimates $\widehat{\vartheta}_{p_1,p_2}(z)$, with $S \geqslant M$ for identification, collected in a column vector $\widehat{\vartheta}_{p_m,p_n}(z)$. As discussed earlier, each infinitesimal cross-moment $\vartheta_{p_1,p_2}(z) = g_{p_1,p_2}(f_1(z), \ldots, f_M(z))$ is a mapping g_{p_1,p_2} from the functions $f_1(z), \ldots, f_M(z)$ to $\mathbb{R}$. For each value of z, the M-vector of estimates $(\widehat{f}_1(z), \ldots, \widehat{f}_M(z))$ is, therefore, naturally defined as

$$(\widehat{f}_1(z), \ldots, \widehat{f}_M(z)) = \underset{(f_1(z),\ldots,f_M(z))}{\arg\min} \, (\widehat{\vartheta}_{p_m,p_n}(z) - \vartheta_{p_m,p_n}(z))^\top U(z)(\widehat{\vartheta}_{p_m,p_n}(z) - \vartheta_{p_m,p_n}(z)),$$

where $U(z)$ is an $S \times S$ symmetrical and positive-definite weight matrix. For efficiency, we set $U(z) = V^{-1}(z)$, where $V_{(p_1,p_2),(p_3,p_4)}(z) = \frac{\vartheta_{p_1+p_3,p_2+p_4}(z)}{h_{n,T}(z)\widehat{L}_{n,T}(z)}$ defines the generic element of the matrix $V(z)$ and

$$\widehat{L}_{n,T}(z) = \frac{\Delta_{n,T}}{h_{n,T}(z)} \sum_{i=1}^{n} \mathbf{K}\left(\frac{z_{i\Delta_{n,T}} - z}{h_{n,T}(z)}\right) \tag{19}$$

is an estimate of the occupation density of the state variable z_t. The structure of the asymptotic variance–covariance matrix $V(z)$ of the infinitesimal cross-moments used in the choice of $U(z)$ is immediate given the method of proof leading to Theorem 2. The limiting variances in Eq. (20) are, in fact, the diagonal elements of $V(z)$.

An inferential theory for infinitesimal cross-moment estimation under a weak (recurrence) assumption for z_t is outlined next.

Theorem 1 (*Consistency*). *If $n, T \to \infty$ and $\Delta_{n,T} = T/n \to 0$ so that $h_{n,T}(z)\widehat{L}_{n,T}(z) \xrightarrow{p} \infty$ and $\frac{\Delta_{n,T}}{h^2_{n,T}(z)} \to 0$, then*

$$\widehat{\vartheta}_{p_1,0}(z) \xrightarrow{p} \begin{cases} \mu_Y(z) + \lambda_X(z)\tilde{\beta}(z)\mathrm{E}[c_X(z)] + \lambda_Y(z)\mathrm{E}[c_Y(z)] & p_1 = 1 \\ \beta^2(z)\sigma_X^2(z) + s^2(z) + \lambda_X(z)\tilde{\beta}^2(z)\mathrm{E}[c_X^2(z)] + \lambda_Y(z)\mathrm{E}[c_Y^2(z)] & p_1 = 2 \\ \lambda_X(z)\tilde{\beta}^{p_1}(z)\mathrm{E}[c_X^{p_1}(z)] + \lambda_Y(z)\mathrm{E}[c_Y^{p_1}(z)] & p_1 \geqslant 3 \end{cases},$$

$$\widehat{\vartheta}_{0,p_2}(z) \xrightarrow{p} \begin{cases} \mu_X(z) + \lambda_X(z)\mathrm{E}[c_X(z)] & p_2 = 1 \\ \sigma_X^2(z) + \lambda_X(z)\mathrm{E}[c_X^2(z)] & p_2 = 2 \\ \lambda_X(z)\mathrm{E}[c_X^{p_2}(z)] & p_2 \geqslant 3 \end{cases},$$

$$\widehat{\vartheta}_{1,1}(z) \xrightarrow{p} \beta(z)\sigma_X^2(z) + \lambda_X(z)\tilde{\beta}(z)\mathrm{E}[c_X^2(z)],$$

and, without loss of generality, for $p_1 \geqslant p_2 \geqslant 1$ (with $p_1 > p_2$ if $p_2 = 1$),

$$\widehat{\vartheta}_{p_1,p_2}(z) \xrightarrow{p} \lambda_X(z)\tilde{\beta}^{p_1}(z)\mathrm{E}[c_X^{p_1+p_2}(z)].$$

Proof. Adapting the theory in Bandi and Renò (2016, 2018) to our assumed model in Eqs. (1) and (2) yields the result.

Theorem 2 (*Weak Convergence*). *Let $n, T \to \infty$ and $\Delta_{n,T} = T/n \to 0$ so that $h_{n,T}(z)\widehat{L}_{n,T}(z) \xrightarrow{p} \infty$ and $\frac{\Delta_{n,T}\sqrt{\widehat{L}_{n,T}(z)}}{h^{3/2}_{n,T}(z)} \xrightarrow{p} 0$. If*

$$h^5_{n,T}(z)\widehat{L}_{n,T}(z) = O_p(1),$$

then

$$\sqrt{h_{n,T}(z)\widehat{L}_{n,T}(z)}\left\{\widehat{\vartheta}_{p_1,p_2}(z) - \vartheta_{p_1,p_2}(z) - \Gamma_{\vartheta_{p_1,p_2}}(z)\right\} \overset{d}{\Rightarrow} \mathcal{N}(0, \mathbf{K}_2\vartheta_{2p_1,2p_2}(z)), \tag{20}$$

with

$$\Gamma_{\vartheta_{p_1,p_2}}(z) = h^2_{n,T}(z)\mathbf{K}_1\left(\frac{\partial\vartheta_{p_1,p_2}(z)}{\partial z}\frac{\frac{\partial v(z)}{\partial z}}{v(z)} + \frac{1}{2}\frac{\partial^2\vartheta_{p_1,p_2}(z)}{\partial^2 z}\right),$$

where $v(\mathrm{d}z) = v(z)\mathrm{d}z$ is the invariant measure of the process z_t, $\mathbf{K}_1 = \int s^2\mathbf{K}(s)ds$ and $\mathbf{K}_2 = \int \mathbf{K}^2(s)ds$.

Proof. Adapting the theory in Bandi and Renò (2016, 2018) to our assumed model in Eqs. (1) and (2) yields the result.[7]

4.2. The data and details on implementation

We work with indices compiled by Credit Suisse Hedge Index LLC. The indices are portfolios of hedge funds sorted by trading style: Convertible Arbitrage, Dedicated Short Bias, Emerging Markets, Equity Market Neutral, Event Driven, Fixed Income Arbitrage, Global Macro, Long/Short Equity, Managed Futures and Multistrategy. We also consider 3 sub-styles (Distressed Event Driven, Multistrategy Event Driven and Risk Arbitrage Event Driven) along with the Credit Suisse Hedge Fund Index,[8] which is a portfolio of the above 10 styles, for a total of 14 series. The returns are monthly from January 1994 to August 2018, for a total of 295 observations.

The corresponding percentages of assets under management (AUM) vary considerably during the sample period.[9] Global Macro has more than 60% AUM at the beginning of the sample followed by Long/Short Equity (about 16%), Event Driven and Emerging Markets (between roughly 6% and 9%). By the end of the sample, we witness a redistribution in AUM away from Global Macro (about 15%) into Event Driven (about 18%), Multistrategy (about 17%), which is effectively inexistent at the beginning of the sample, and Managed Futures (around 9%). The AUM of Long/Short Equity increases drastically just after 2000 (to a percentage larger than 50%) before returning to a relative weight at the end of the sample similar to that at the beginning of the sample. Fixed Income Arbitrage and Emerging Markets are fairly stable and, on average, around 5% throughout the sample. Both Equity Market Neutral and Convertible Arbitrage increase to values between 5% and 8% around 2003 before decreasing steadily. Dedicated Short Bias has an almost negligible weight throughout the sample and virtually no AUM by the end of the sample. We will return to the AUM weights, when needed, in what follows.

The market return data is daily and aggregated to monthly figures. We work with the value-weighted returns of all US CRSP firms listed on the NYSE, AMEX and NASDAQ (share code 10 or 11). Monthly *excess* market returns and *excess* hedge-fund returns are obtained by subtracting the one-month Treasury bill rate from Ibbotson Associates. Summary statistics are reported in Table 1.

As emphasized, conditioning on the state of the economy is economically meaningful. “Beta in the tails”, in fact, may not just be jump beta. In principle, it could also be *dynamic beta*, i.e., an increasing (diffusive and, hence, overall) beta in adverse states of nature often characterized by market returns in the left tail of their distribution. We capture the state of the economy by conditioning on the most natural variable given Eq. (2), i.e., market diffusive volatility. We denote it by σ_t and write $\sigma_X(z_t) = z_t$ with $z_t = \sigma_t$.[10] The result of this specification is a stochastic volatility model for market returns with possible risk-return trade-offs (captured by $\mu_X(\sigma)$) and market jump sizes $c_X(\sigma)$ whose distribution is allowed, without being forced, to depend on the level of diffusive volatility.[11]

The monthly volatility estimates and the excess (market and fund) monthly returns are fed into Eq. (18) to implement NIMM, one fund at a time. Identification of the four functions driving the market dynamics hinges on the six moments

7 For both Theorems 1 and 2, we refer the reader to Bandi and Renò (2018) for regularity conditions imposed on the state variable z_t and the system’s functions.

8 The Credit Suisse Hedge Fund Index employs approximately 9000 funds (divided into 10 styles) and includes funds with a minimum track record of 12 months, audited financial statements and a minimum of US\$50 million under management. It is rebalanced monthly and is designed to represent performance *net* of performance fees.

9 We report *approximate* percentages deduced from the Index’s constituents.

10 In order to estimate σ_t, we use monthly bipower variation obtained from daily market return data. Bandi and Renò (2018) discuss how to handle asymptotically measurement error in the state variable.

11 Patton and Ramadorai (2013) study linear and piece-wise linear conditional risk exposures in which the conditioning variables are the change in the 3-month Treasury bill rate, the S&P 500 returns, volatility (as proxied by de-trended VIX) and the TED spread. We have estimated the model in Eqs. (1)–(2) using the same state variables (with the exception of the S&P 500 returns, which cannot be employed in continuous time) as well as global industrial production and Kilian’s index of real economic activity (Kilian, 2009). Results, available upon request, are in line with those reported in the paper.

Table 1
Summary statistics for monthly hedge-fund excess returns and market excess returns from January 1994 to August 2018, for a total of 295 observations. Legend: (0) Market, (1) Credit Suisse Hedge Fund Index, (2) Convertible Arbitrage, (3) Dedicated Short Bias, (4) Emerging Markets, (5) Equity Market Neutral, (6) Event Driven, (7) Distressed Event Driven, (8) Multistrategy Event Driven, (9) Risk Arbitrage Event Driven, (10) Fixed Income Arbitrage, (11) Global Macro, (12) Long/Short Equity, (13) Managed Futures, (14) Multistrategy.

Panel A: Summary statistics for monthly hedge-fund and market excess returns

	(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
Mean (%)	0.58	0.61	0.52	−0.50	0.55	0.35	0.64	0.72	0.60	0.46	0.42	0.77	0.69	0.36	0.61
Std (%)	4.31	1.94	1.83	4.64	3.86	3.25	1.74	1.76	1.88	1.12	1.51	2.49	2.56	3.29	1.41
Skewness	−0.94	−0.25	−3.05	0.53	−1.29	−14.07	−2.24	−2.31	−1.78	−0.97	−5.20	−0.09	−0.21	−0.06	−1.81
Kurtosis	4.95	6.44	24.49	3.85	12.02	225.91	14.74	16.97	11.19	7.97	46.43	8.18	7.19	2.93	10.57
Beta	1.00	0.28	0.16	−0.86	0.52	0.19	0.27	0.26	0.27	0.14	0.12	0.15	0.44	−0.06	0.13

Panel B: Correlation matrix

	(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
(1)	0.61													
(2)	0.38	0.56												
(3)	−0.83	−0.50	−0.29											
(4)	0.58	0.71	0.46	−0.55										
(5)	0.25	0.28	0.20	−0.15	0.15									
(6)	0.66	0.76	0.65	−0.60	0.69	0.28								
(7)	0.63	0.70	0.60	−0.57	0.64	0.32	0.93							
(8)	0.62	0.76	0.63	−0.56	0.68	0.23	0.97	0.82						
(9)	0.52	0.51	0.48	−0.46	0.50	0.17	0.67	0.59	0.66					
(10)	0.35	0.54	0.77	−0.21	0.41	0.31	0.51	0.50	0.49	0.30				
(11)	0.25	0.81	0.34	−0.11	0.45	0.06	0.40	0.35	0.43	0.23	0.40			
(12)	0.74	0.84	0.46	−0.70	0.67	0.20	0.74	0.68	0.73	0.59	0.38	0.45		
(13)	−0.08	0.21	−0.07	0.08	−0.01	−0.00	−0.01	−0.04	−0.00	−0.03	−0.05	0.32	0.09	
(14)	0.41	0.53	0.69	−0.26	0.31	0.35	0.57	0.50	0.57	0.36	0.61	0.29	0.49	0.10

listed in Eqs. (9) through (14). Identification of all other functions, including $\beta(\sigma)$ and $\tilde{\beta}(\sigma)$, is based on the infinitesimal cross-moments of order (1,0), (2,0), (3,0), (4,0), (1,1), (2,1), (3,1), (1,2), (1,3) and (2,2).

In order to implement the Nadaraya–Watson cross-moment estimator in Eq. (18), we use an Epanechnikov kernel.[12] Given stationarity of the volatility path, the occupation density $\widehat{L}_{n,T}(\sigma)$ is such that $\widehat{L}_{n,T}(\sigma) = O_p(T)$. Hence, the rate-optimal choice of the bandwidth is $T^{-1/5}$ (c.f., Theorem 2). We undersmooth slightly to dispense with the bias term and write $h_{n,T}(\sigma) = h_T(\sigma) = \phi(\sigma)\frac{1}{\ln(T)}T^{-1/5}$. The constant $\phi(\sigma)$ is chosen, for every σ level, as proportional to the occupation density of the volatility process, larger bandwidths being associated to volatility levels that are visited less often. As a result, the bandwidth ranges from 0.68 (expressed in units of the logarithmic bipower variation), in the center of the volatility's occupation density, to 2.62, in the tails.[13]

5. Empirical findings

We begin with the dynamics of the excess market return process. We then turn to the excess hedge-fund returns.

5.1. Excess market returns

The estimated drift ($\mu_X(\sigma)$), the mean of the market jump sizes ($\mu_X^{\text{jump}}(\sigma)$), the standard deviation of the market jump sizes ($\sigma_X^{\text{jump}}(\sigma)$) and the intensity of the market jumps ($\lambda_X(\sigma)$) are reported in Fig. 4, as a function of the volatility level. The dots on the estimated functions correspond to twenty quantiles of the market volatility distribution. They indicate that the bulk of the volatility distribution is concentrated around an annualized value of about 8%. This value should not be viewed as being excessively low. We are, in fact, conditioning on a portion of the overall variation (diffusive volatility) during a tranquil time period. To add clarity, we plot the occupation density of the market diffusive volatility process in Fig. 5.

The drift $\mu_X(\sigma)$ yields a monthly average excess return due to diffusive dynamics between 1% and 2% around the center of the volatility distribution. A sizable negative mean of the jump sizes $\mu_X^{\text{jump}}(\sigma)$ (with values in the neighborhood of −2% – and lower – around the center of the volatility distribution) and a non-negligible likelihood of jump arrivals $\lambda_X(\sigma)$ give, however, rise to a large, negative, monthly average excess return due to the jump dynamics. Combining the two sources

12 Theorems 1 and 2 require the use of a kernel function expressed as a bounded, continuous, and symmetric density defined on a compact set and with bounded derivatives. In practice, the use of a Gaussian kernel would not modify our findings.

13 Automated bandwidth choice continues to be an unexplored area in continuous-time econometrics. In simulations using the model's estimated parameters, our choice of the level-specific constant $\phi(\sigma)$ is found to be satisfactory.

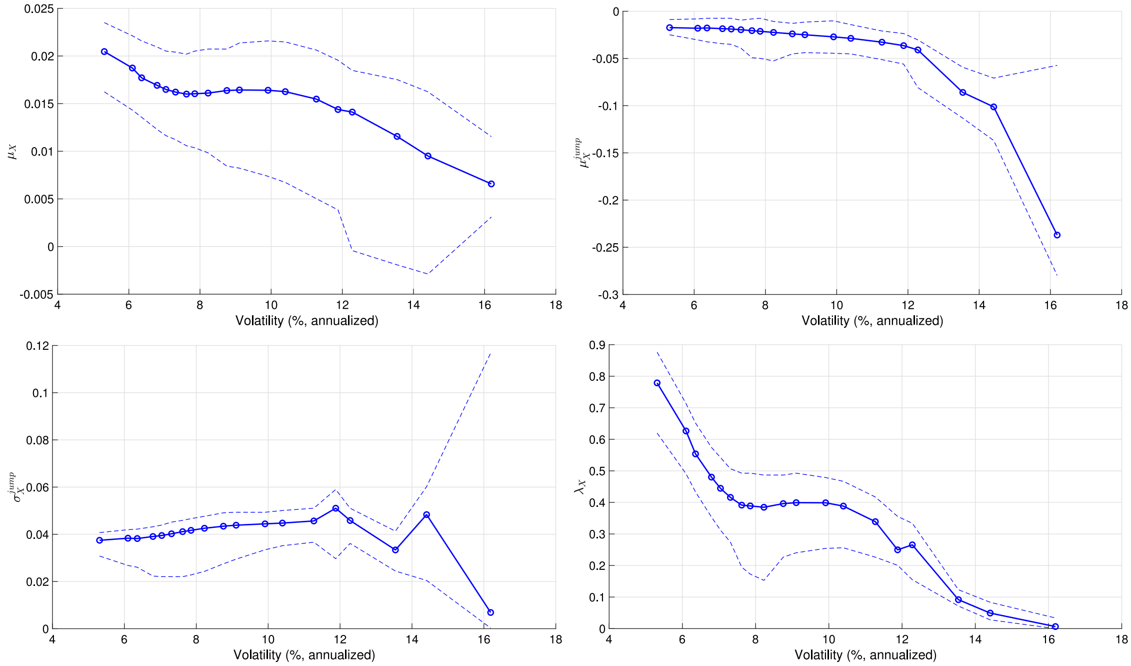

Fig. 4. We report estimates of the drift ($\mu_X(\sigma)$), the mean of the jump sizes ($\mu_X^{\text{jump}}(\sigma)$), the standard deviation of the jump sizes ($\sigma_X^{\text{jump}}(\sigma)$) and the intensity of the jumps ($\lambda_X(\sigma)$) for the market excess return process. The circles on the estimated lines correspond to 20 quantiles of monthly bipower variation, our estimate of market variance.

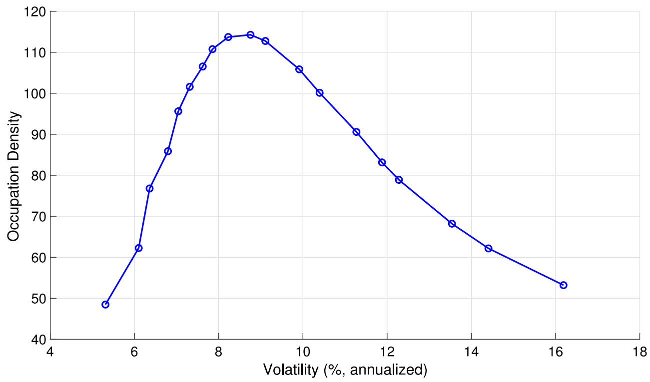

Fig. 5. The occupation density of the market volatility process.

of mean variation into $\mu_X(\sigma)+\mu_X^{\text{jump}}(\sigma)\lambda_X(\sigma)$ yields the average monthly excess return that we observe in the data (c.f., Table 1). For instance, for a volatility level of 8%, the diffusive drift is 1.6%, the mean of the jump sizes is −2.6% and the intensity of the jump arrivals is 0.39 leading to an overall average monthly value for excess market returns of 0.58% (and, as a consequence, a yearly mean value of roughly 7%).

Both the mean and the standard deviation of the market jump sizes are stable across highly-visited volatility states. In these states, the likelihood of a jump arrival is about 0.4 per month, thereby implying about 5 jumps per year. When they occur, the market jumps are generally large, with mean monthly values around −2%/−2.5% and standard deviations around 4%.

When market volatility is high, our estimates attribute a small portion of the return variation to the discontinuous dynamics ($\lambda_X(\sigma)$ is small). Should a jump occur, however, its magnitude would be even larger (due to large, negative, mean jump sizes $\mu_X^{\text{jump}}(\sigma)$ in the right tail of the volatility distribution) and with reduced dispersion around large values (because of a smaller standard deviation of the jump sizes $\sigma_X^{\text{jump}}(\sigma)$). Needless to say, estimation uncertainty on both $\mu_X^{\text{jump}}(\sigma)$ and $\sigma_X^{\text{jump}}(\sigma)$ is, however, considerably higher in these seldom-visited states with large market volatility.

The overall hedge-fund beta is a weighted average of diffusive and jump beta. As made explicit in Eq. (4), the weights depend on the market dynamics. We report their estimates in Fig. 6. The diffusive (resp. jump) weights increase (resp.

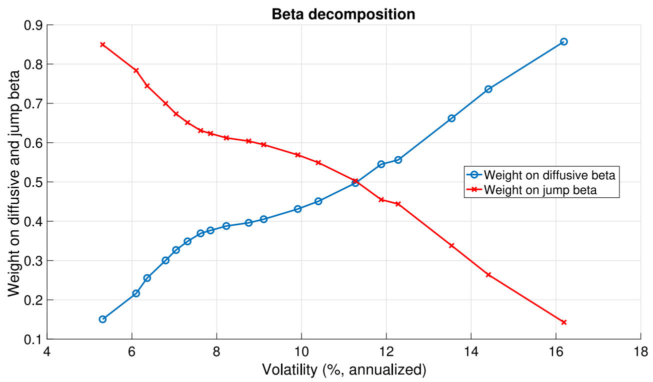

Fig. 6. Calculating the overall beta: the weights on diffusive and jump beta as a function of the volatility state.

decrease) with the volatility level due, also, to decreases in the likelihood of jumps and the standard deviation of the jump sizes (see Fig. 4). Around the center of the volatility occupation density, the weight attributed to jump beta (resp. diffusive beta) is about 60% (resp. 40%).

5.2. Excess hedge-fund returns

In Fig. 7 we plot estimated diffusive betas ($\beta(\sigma)$) and estimated jump betas ($\tilde{\beta}(\sigma)$) for each style (and all volatility levels). Visually, it is rather clear that the diffusive betas tend to be more concentrated around zero than the jump betas. This is true for most volatility levels, albeit less so at the lowest levels. While the jump betas are rather flat across volatility states, the diffusive betas appear to be attenuated particularly around high volatility states. All of the funds, with the exception of Dedicated Short Bias, have diffusive betas around the upper bound of the volatility distribution between -0.1 and 0.4. While the negative (diffusive and jump) betas of Dedicated Short Bias are unsurprising, we should view Dedicated Short Bias as being hardly representative of the industry. As indicated earlier, its AUM constitutes, throughout the sample, a negligible proportion of the AUM of the industry.

The attenuation of the diffusive betas for high volatility levels is, of course, not mechanical. While it is true that the diffusive betas have market variance in the denominator (since $\mathrm{V}_t(dX_t^{\mathrm{dif}}) = \sigma_t^2$), they have the covariance between diffusive excess market returns and diffusive excess hedge-fund returns in the numerator, i.e., $\mathrm{C}_t(dY_t^{\mathrm{dif}}, dX_t^{\mathrm{dif}})$. Given the assumed factor structure, however, $dY_t^{\mathrm{dif}} = \beta(\sigma_t)dX_t^{\mathrm{dif}}$ and $\mathrm{C}_t(dY_t^{\mathrm{dif}}, dX_t^{\mathrm{dif}}) = \beta(\sigma_t)\sigma_t^2$. Hence, the numerator is linearly related to market variance and the decrease to zero (in absolute value) of the diffusive betas cannot be viewed as a direct consequence of increasing levels of market variance.

The attenuation of the diffusive betas is economically interesting. If one of the goals of alt investments is to have low systematic risk, as far as the small or diffusive price changes are concerned, this goal appears to be accomplished more during times of market stress (as indicated by the associated higher volatility levels) than during normal times. When the focus is on diffusive dynamics, hedge-fund investing seems to offer, as advertised, some diversification to those holding well-diversified "market-like" portfolios, particularly in volatile times.

The jump betas behave differently. Not only are they generally larger, they also tend to remain stable across volatility levels. In times of market stress, the level of systematic risk associated with large price moves is roughly as large as that during normal times. With the exception of Managed Futures and, once more, Dedicated Short Bias, all jump betas are positive whereas the diffusive betas appear to be more dispersed around zero. A natural implication of this observation is that the smaller systematic risk associated with diffusive dynamics can be more easily reduced (through funds of funds) than the larger systematic risk associated with discontinuous dynamics. A second implication is that the addition of hedge-fund investments to a well-diversified "market-like" portfolio may, in general, not yield – in times of large market's discontinuities – the diversification that alt investments are expected to offer.

For a clearer comparison between the diffusive betas and the jump betas, Fig. 8 reports both quantities, in a single panel, for the Credit Suisse Hedge Fund Index and, again, for all styles and the three Event Driven sub-styles. As emphasized, in the vast majority of cases, the jump betas are larger than their diffusive counterparts and more stable across volatility levels. In a few instances, the difference between the two beta notions is drastic, sometimes less so. For example, Emerging Markets funds have one-to-one discontinuities with the market (as represented by jump betas very close to 1). The corresponding diffusive betas around the bulk of the market volatility distribution, however, are smaller in absolute value and negative (around -0.25). Similar considerations can be made, e.g., for all of the Event Driven funds (i.e., the Event Driven index as well as the three sub-indices). In all four cases, the discontinuous betas are positive and sit above the corresponding diffusive betas. As said, there are exceptions. Around the center of the volatility distribution, the Equity Market Neutral funds have similarly small diffusive and jump betas. This finding is in line with market neutral strategies having limited exposure to both continuous and discontinuous market changes.

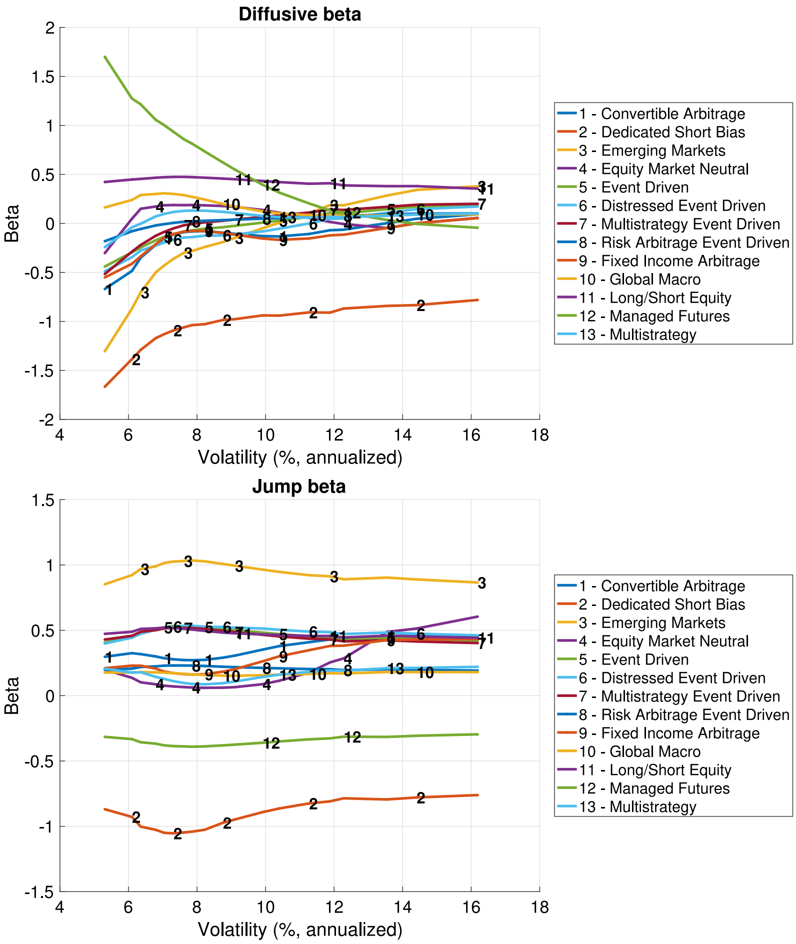

Fig. 7. Estimated diffusive and jump betas by style/sub-style.

5.3. β in the tails: the role of systematic jump risk

We have stressed, and shown in simulation (c.f., Section 3.1), that a combination of negative-mean market jumps and jump betas larger (in absolute value) than the diffusive betas can generate "beta in the tails" even in the absence of time-varying parameters for the system in Eqs. (1) and (2).

As reflected in Fig. 8, eight out of fourteen indices have jump betas which are positive (over the entire volatility range) and higher than the corresponding diffusive betas (sometimes by a large margin). The same indices have negative jump sizes, i.e., $\tilde{\beta}(\sigma)\mu_X^{\text{jump}}(\sigma)$, because $\tilde{\beta}(\sigma)$ is positive and the mean of the market jump sizes, i.e., $\mu_X^{\text{jump}}(\sigma)$, is, as shown in Fig. 4, negative. These indices are the aggregate Credit Suisse Hedge Fund Index, Convertible Arbitrage, Emerging Markets, Event Driven, Distressed Event Driven, Multistrategy Event Driven, Risk Arbitrage Event Driven and Fixed Income Arbitrage. Excluding the aggregate index, the style indices represent a percentage of AUM at the end of the sample around 35%.

All of them have overall betas in the left tail of the market return distribution which are higher than the corresponding betas in the right tail (c.f. Fig. 2). Importantly, the left-tail overall betas are, also, numerically in line with the jump betas estimated by NIMM. Consider the first three funds, but the same finding applies more generally. The Credit Suisse Hedge Fund Index, Convertible Arbitrage and Emerging Markets have positive "beta in the tails" hovering around 0.4, 0.7 and 1.2. These values agree with the magnitudes of the (stable, across volatility levels) jump betas estimated by NIMM (0.4, 0.4 and 1, respectively). If anything, they may be slightly higher than the NIMM estimates. As discussed earlier, our simulations in Section 3.1 reveal that this finding is not uncommon. It is simply the result of the interaction between extreme diffusive tail observations and jump tail observations. The former may operate as outliers and temporarily *lever up* the jump beta estimates, thereby yielding overall betas slightly higher than the jump betas (even when the latter are higher than the diffusive betas).

"Beta in the tails" can be negative. Managed Futures funds (whose percentage of AUM is about 10% at the end of the sample), e.g., have jump betas which are stable across volatility levels and smaller than the corresponding diffusive

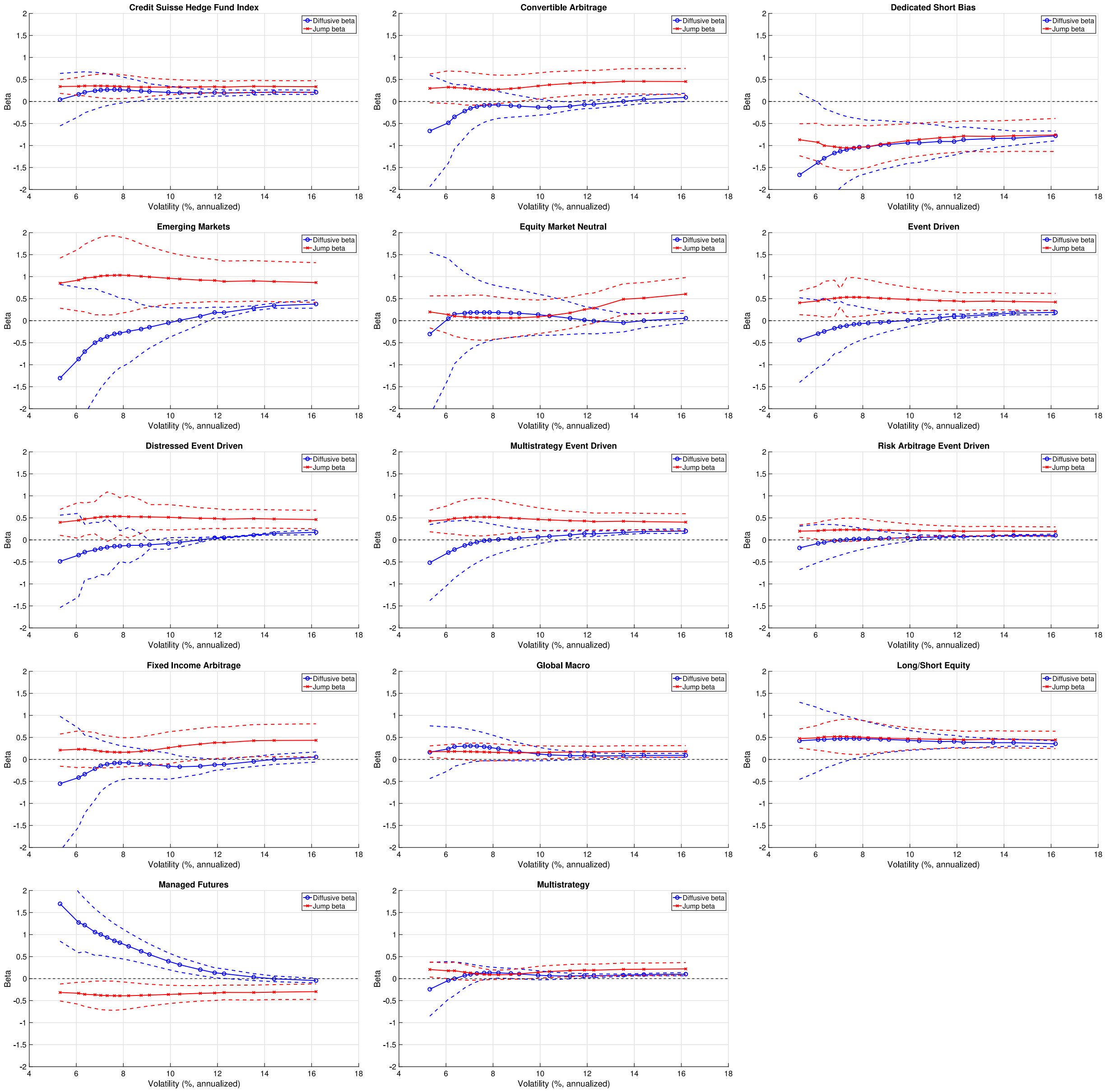

Fig. 8. Estimated diffusive and jump betas (with simulated standard errors) by style/sub-style.

betas (c.f., Fig. 8). The diffusive betas are positive and fluctuate between values as high as 1.5 and values as low as 0. The jump betas are, instead, concentrated between -0.3 and -0.4. In light of the suggested mechanism in Section 3.1, we would expect a negative overall beta in the left tail of the market return distribution and slow attenuation toward zero as we progress toward the right tail of the return distribution and more of the diffusive dynamics get incorporated. This is precisely what the corresponding panel in Fig. 2 reveals. Again, negative "beta in the tails" is symptomatic of the diversification potential offered, in times of market stress, by this specific style to investors holding "market-like" portfolios.

Unsurprisingly, given our discussion, Equity Market Neutral funds can have slightly negative "beta in the tails" just by virtue of having jump betas very close to zero and larger diffusive betas around the center of the volatility occupation density. Once more, the simulations in Section 3.1 reveal that, in any finite sample, the jump betas may be *levered down* and, in this case, translate into mildly negative overall beta estimates. Consistent with this finite sample effect, we report slightly negative overall beta estimates in the left tail of the market return distribution and overall beta estimates close to 0.2 (i.e., close to the estimated diffusive beta around the center of the volatility distribution) in the right tail of the return distribution (c.f., again, Fig. 2).

Finally, the behavior of the overall betas of Multistrategy, Long/Short Equity and Dedicated Short Bias are easily justifiable. They all have jump betas that closely resemble their diffusive betas with values in fairly small intervals between 0 and 0.2 (Multistrategy), between 0.4 and 0.5 (Long/Short Equity) and between −1.2 and −0.7 (Dedicated Short Bias). Their overall betas largely fluctuate in these same intervals.

6. Some asset pricing implications

It is now interesting to ask if *tail systematic risk*, such as the risk represented by a large jump beta, is likely to be compensated in excess hedge-fund returns. In order to do so, we regress cross-sectionally the average excess returns on the available 14 indices on the corresponding jump, diffusive and overall betas. We stress that, because the jump and the diffusive betas are functions of the volatility states, we compute *weighted* average betas by averaging across the volatility states using as weights the volatility's occupation density. We also emphasize that the overall beta is the *true* empirical beta of the 14 indices rather than the overall beta implied by the model (c.f. Eq. (4)). As we clarify below, this choice is conducive to more robust interpretations. If we had used a model-based overall beta, however, our findings would be unchanged. This is because of a strong correlation between overall empirical betas and overall model-based betas, an outcome which should be viewed as a useful diagnostic for the proposed model.

In Fig. 9 we provide results including and excluding Dedicated Short Bias. Dedicated Short Bias is designed to be an almost perfect hedge. The resulting diffusive and jump betas are, in fact, close to −1. When combined with negative average (excess) returns, these negative betas create an obvious lever in the regression (blue dashed line). One may argue that the strongly negative expected returns offered by Dedicated Short Bias are consistent with its "insurance" nature and, therefore, informative about risk pricing in the hedge-fund industry. We do not take a stand on the issue but recognize that, in light of the extremely limited AUM of Dedicated Short Bias, results which do not include Dedicated Short Bias are to be seen as more conservative.

When excluding Dedicated Short Bias, the diffusive betas do not align with average excess returns (Panel A of Fig. 9). The jump betas and the overall betas, however, continue to be positively correlated with average excess returns. In particular, the slope (i.e., the price of risk) associated with the jump beta is 0.2% (without Dedicated Short Bias) and 0.57% (with Dedicated Short Bias). The latter is very close to the average excess return on the market (0.58%), i.e., the theoretical price of risk in a CAPM world.

Because the overall beta is a weighted average of jump and diffusive betas, and diffusive risk may not be priced, the pricing performance of the overall beta ought to be driven by jump betas. Let us assume that the correct pricing model is a CAPM-style specification in which systematic risk is captured by the jump betas. Then, the relevant cross-sectional regression would be:

$$\mathrm{E}(Y_i) = \psi_0 + \psi_1 \tilde{\beta}_i + \varepsilon_i, \tag{21}$$

with i indexing individual hedge funds. Suppose that, instead, one runs a regression on the overall beta. If the correct model is Eq. (21), because of Eq. (4),[14] we would have

$$\mathrm{E}(Y_i) = \psi_0 + \underbrace{\frac{\psi_1}{w}}_{\widetilde{\psi}_1} \beta_i^{\text{overall}} + \underbrace{\varepsilon_i - \left(\frac{1-w}{w}\right)\psi_1 \beta_i}_{\widetilde{\varepsilon}_i} \tag{22}$$

Hence, the least-square estimate of the new slope ($\widetilde{\psi}_1$) would be inconsistent (because of the dependence between β_i^{overall} and the new error term $\widetilde{\varepsilon}_i$). Notwithstanding inconsistency, numerically we would also expect $\widetilde{\psi}_1 = \frac{\psi_1}{w} \geqslant \psi_1$ since $w \leqslant 1$. In agreement with this observation, for our data $\widetilde{\psi}_1 = 0.4\% > \psi_1 = 0.2\%$ (without Dedicated Short Bias) and $\widetilde{\psi}_1 = 0.87\% > \psi_1 = 0.57\%$ (with Dedicated Short Bias). We emphasize that this finding is not the result of the use of model-based overall betas. It is, instead, the result of the use of empirical overall betas. In this sense, we provide evidence which would, otherwise, appear to be more mechanical. For a pictorial representation, we refer again to Fig. 9, Panel B and C.

In sum, the tail systematic risk associated with large market moves may be priced. Because of our small (cross-sectional and time-series) sample size, these findings are, however, solely suggestive, rather than conclusive. The goal of this section was to evaluate whether investors may seek compensation for jump risk in spite of its limited saliency. It was *not* to search for the best pricing model, a goal which would require broadening the set of factors beyond the market[15] and increasing the number of test assets by working with individual funds. Having made this point, there is value to focusing on market risk, particularly in the case of these alt assets. In spite of its simplicity, there is ample evidence that the CAPM represents a key benchmark in the hedge-fund industry. Agarwal et al. (2018), for example, are explicit about the role that CAPM alphas play in providing information to market participants about hedge-fund performance and, as a result, in driving capital flows.

14 For the purpose of this discussion, the weights should be interpreted as constant and equal to average values across volatility levels.

15 See, e.g., the rich treatments in Bali et al. (2011, 2012), Fung and Hsieh (2004) and Hasanhodzic and Lo (2007) and the references therein.

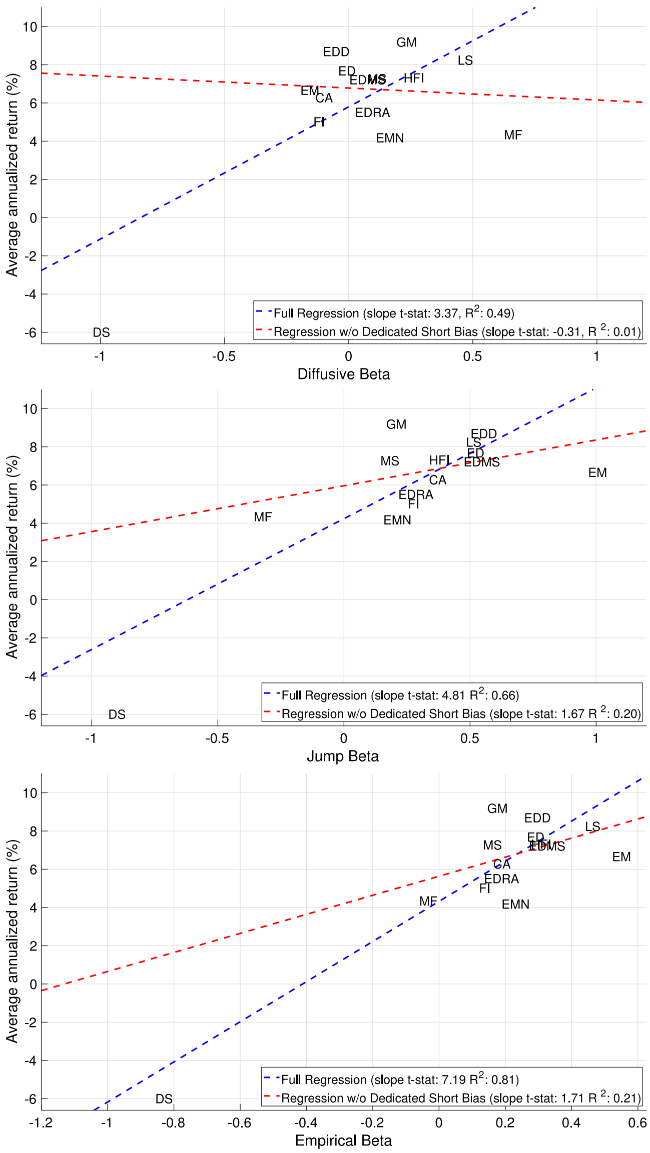

Fig. 9. The CAPM: using diffusive betas, jump betas and overall (empirical) betas as the source of systematic risk.

7. Conclusions

Hedge funds' unconditional betas often understate hedge funds' exposure to market risk during market drops. This exposure, which we represent through jump betas, is dubbed "beta in the tails". "Beta in the tails" leads to a portion of the funds' expected excess return which is likely not alpha (and manager-driven) but compensation for a hidden risk, one which may not be immediately salient to the econometrician but, in our data, appears to be salient to investors.

We discussed an infinitesimal moment-based procedure which is suited to yield empirical saliency for the above-mentioned hidden risk. The differential informational content of low-order and high-order infinitesimal cross-moments, with the former providing information about diffusive dynamics and the latter providing information about jump dynamics, was exploited to separate systematic risk in normal times from systematic risk in times of erratic market behavior. We did so as a function of a variable (volatility) capturing changes in the state of the economy, as is needed in order to evaluate the possibility of increasing diffusive betas in uncertain times. Increasing diffusive betas are a further (in addition to jump risk) source of "beta in the tails" which our data do not appear to support as a first-order driver.

Quoting, once more, Asness et al. (2001), we have addressed the following question: "Do hedge funds hedge?" Sometimes, but often *not* when it is necessary the most, namely in times of market stress.

In a recent study, Liew et al. (2019) report analogous phenomena in the cryptocurrency market. New work which employs NIMM in the study of cryptocurrencies' pricing dynamics is Chen et al. (2020). Exploring, through NIMM, our suggested mechanism in this novel alt asset class would be revealing.

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