Local Edgeworth expansions

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Abstract

We derive local (over a small time interval Δ) Edgeworth expansions of the bivariate conditional characteristic function of the level/volatility Δ-increments of a Brownian semi-martingale. We do so without and with compound Poisson discontinuities (idiosyncratic and joint) in levels and volatility. In a first-order expansion (in $\sqrt{\Delta}$), the proposed local approximation to the bivariate Gaussian characteristic function of the level/volatility changes adds skewness through the time-varying correlations between the level/volatility changes and the volatility/volatility-of-volatility changes. In a second-order expansion (in Δ), the local approximation adds kurtosis through, among other quantities, the volatility-of-volatility and the volatility of the volatility-of-volatility. All discontinuities only affect the second order. We show how Fourier-inversion of the proposed expansion (and recovery of the local conditional density) is intimately related to the differentiability properties of the assumed process. We formalize these properties through the concept of W-differentiability, with W defining the leading Brownian motion.

Keywords: Semi-martingales, Poisson jumps, characteristic function, Edgeworth expansion.

JEL classification: C32, C46, G12, G13.

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1 Introduction

The work of Edgeworth (1883), Tchebycheff (1890), Cramér (1937) and Cornish and Fisher (1938) is broadly viewed as providing initial logical and mathematical foundations for expansions of probability distributions later defined as "Edgeworth expansions."

Since those early contributions, these expansions have been formalized by many authors, including Chibisov (1973), Bhattacharya and Rao (1976), Sargan (1976), Phillips (1977), Bhattacharya and Ghosh (1978) and Barndorff-Nielsen and Cox (1979).

Following, as it is natural, initial applications to independent and identically distributed observations, Edgeworth expansions have been employed for weakly dependent data (e.g., Götze and Hipp, 1983), ARMA processes (e.g., Taniguchi, 1984), Harris recurrent Markov chains (e.g., Malinovskii, 1987) and martingales (e.g., Mykland, 1992).

We work with continuous-time semi-martingales driven by Brownian motion. For this family of processes, as well as for processes in the same family to which compound Poisson processes (idiosyncratic, joint or both) are appended, we derive Edgeworth expansions of the bivariate conditional characteristic function of their level/volatility Δ-increments, as $\Delta \to 0$. The expansions are second-order (in $\sqrt{\Delta}$) local adjustments to the Gaussian characteristic function. Focusing on the univariate characteristic function of the level increments, the first order adds skewness through the time-varying correlation between level changes and volatility changes. The second order adds kurtosis (as well as second and sixth moment adjustments) through, among other quantities, the time-varying volatility-of-volatility. Turning to the univariate characteristic function of the volatility changes, the logic is similar and corresponding quantities appear naturally. Volatility skewness is driven by the the correlation between volatility and the volatility-of-volatility, whereas volatility kurtosis depends on the volatility of the volatility-of-volatility, among other quantities. When appended to diffusive dynamics, irrespective of whether they are idiosyncratic or joint, compound Poisson processes only affect the second order.

Our method of proof hinges on the notions of W-transform and W-differentiability, with W representing the leading Brownian motion driving the level dynamics (a W-Itô process).[1] A W-transform is, in essence, a Fourier-type transform of the process. W-differentiability refers, instead, to the depth of a cascade in which the drift and the diffusion of the original W-Itô process are W-Itô processes themselves, their own drift and diffusion are W-Itô processes, and so on. We show that the tail behavior of the W-transform is intimately related to the

[1] When focusing on volatility (or both levels and volatility) one should define analogous notions also based on the driving volatility Brownian motion(s). For conciseness, we will not be specific and use the terminology without loss of generality.

depth of this cascade, i.e., to the extent of W-differentiability. The tail behavior of the W-transform, in turn, affects the integrability of the characteristic function expansion. We discuss conditions on W-differentiability under which the proposed characteristic function expansions can be integrated to compute densities, cumulative distributions and the like.

The availability of high-frequency data justifies our asymptotic environment, in which $\Delta \to 0$, instead of a more traditional asymptotic environment, in which $1/n \to 0$, where n is the number of observations used to compute an estimator for which asymptotic refinements are seeked. We are, however, not concerned with deriving superior finite-sample distributional approximations for *estimators* based on high-frequency data (as in, e.g., the work of Gonçalves and Meddahi, 2009, Zhang, Mykland, and Aït-Sahalia, 2011, and Hounyo and Veliyev, 2016). We are, instead, concerned with the local distributional properties of the process itself. In this sense, our work relates to the recent work of Todorov (2021). However, our emphasis is on joint properties. The univariate characteristic function of the level changes in Todorov (2021) is - barring differences in the proposed formalizations - consistent with ours when we specialize our findings to the univariate (level only) case. Second, our method of proof (based on W-differentiability, W-transforms and their properties) is of separate interest. Among other benefits, it is naturally conducive to multivariate expansions, such as the one we propose. Third, we place emphasis on the Fourier-integrability of the proposed characteristic expansion(s) - which our method of proof facilitates - for the purpose of obtaining well-posed densities and related quantities.

Since the first draft of this article, our findings have been used to derive asymptotic results for high-frequency realized measures (Bandi, Pirino, and Renò, 2017) as well as to price and hedge structured financial products with short-term expirations (Bandi, Fusari, and Renò, 2023a, and Bandi, Fusari, and Renò, 2023b). The latter objective, in particular, requires computation of the local density of the price process underlying the financial instrument. Ignoring necessary measure changes due to no-arbitrage considerations, characterizing conditional local densities and distributions is - as pointed out above - a natural application of the methods we put forward, one to which we will return in Section 3. Existing work has, however, been univariate and focused on the level dynamics. Our central applied interest in this article is, instead, the *nonparametric* identification of rich (e.g., *non-affine*) price models, something that - as shown in Section 5 - the proposed bivariate expansions afford. To this extent, we introduce (local) moment-based estimation using closed-form characteristic function expansions. Our discussion centers around the point-wise identification of “processes” (namely, those driving the dynamics of the assumed W-Itô semi-martingales) - rather than “parameters” - and the role played by the proposed closed-form expansions in facilitating it.

Our approach - the notion of W-differentiability, in particular, and its implications for *iterated* Brownian integrals - relates to work on *multiple* Brownian integrals (Wiener, 1938, and Itô, 1951), Wiener chaos and Malliavin calculus (see, e.g., Nualart, 2006). It also relates to work on Wagner-Platen expansions for stochastic differential equations (Wagner and Platen, 1978). Kloeden and Platen (1992) and Platen and Bruti-Liberati (2010) provide a rich account and applications to processes without and with compound Poisson discontinuities, respectively. These expansions are typically applied to discretization of processes. We, too, work with a discretized process but are concerned with its local distributional properties rather than with its numerical solutions, as in Kloeden and Platen (1992), Platen and Bruti-Liberati (2010) and the references therein.

The article proceeds as follows. Section 2 offers preliminary results. We introduce the notions of W-differentiability and W-transform. Emphasis is placed on the impact of the former on the representation and order of the latter. In Section 3 we formalize the model in its diffusive form and present the corresponding expansion(s) in Theorem 3. This section provides intuition for the role played by W-transforms in our method of proof. Applications of Theorem 3 to expansions of the conditional density and the conditional distribution are, also, discussed. Section 4 is about the mapping between our framework and parametric stochastic volatility models. We show how our assumptions can be modified to allow for state variables (like volatility) and provide a simple adjustment of our bivariate expansion when volatility is the state variable. Section 5 introduces local moment-based estimation of the driving processes of W-Itô semi-martingales using the proposed characteristic function expansions. In Section 6 we add compound Poisson discontinuities to the process. We allow for idiosyncratic discontinuities in levels and volatility as well as for joint discontinuities. Theorem 3 contains the corresponding expansion(s). Section 7 concludes. All proofs are in the Appendix.

2 Preliminaries

This section introduces the process of interest in general terms. We will add details in Section 3. We focus on differentiability. The impact of differentiability on the rate of decay of the tails of the process' Fourier transform will be central to our treatment.

We work with the usual filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ (Protter, 2005).

Definition 1. *An adapted, real stochastic W-Itô process χ_t is defined as being W-differentiable (i.e., differentiable with respect to the standard Brownian motion W) if it admits the representation*

$$d\chi_t = a_t^{\chi} dt + b_t^{\chi} dW_t, \tag{1}$$

where the scalar processes a_t^{χ} and b_t^{χ} are adapted and càdlàg.

If $a_t^\chi = a^\chi$ and $b_t^\chi = b^\chi$, then the process is W-differentiable of degree 1. More generally, the degree of W-differentiability of χ_t depends on the W-differentiability of the processes a_t^χ and b_t^χ. By induction, we may therefore generalize the above definition and "W-differentiate" the process χ_t k times.

Definition 2. *An adapted, real stochastic W-Itô process χ_t is k times W-differentiable if it admits the representation*

$$
\begin{aligned}
d\chi_t &= a_t^\chi dt + b_t^\chi dW_t, \\
d\mathcal{S}_t^j &= \mathcal{A}_t^{\chi(j)} dt + \mathcal{B}_t^{\chi(j)} dW_t,
\end{aligned}
$$

where the 2^j-dimensional vector process $\mathcal{S}_t^j$ is defined as

$$
\mathcal{S}_t^j = (\mathcal{A}_t^{\chi(j-1)}, \mathcal{B}_t^{\chi(j-1)}), \quad \forall j \geq 1
$$

with $(\mathcal{A}_t^{\chi(0)}, \mathcal{B}_t^{\chi(0)}) = (a_t^\chi, b_t^\chi)$ and

$$
\mathcal{S}_t^k = (\mathcal{A}^{\chi(k-1)}, \mathcal{B}^{\chi(k-1)}),
$$

for $j = 1, \ldots, k$.

We denote by $\mathcal{D}_W^{(k)}$ the family of k times W-differentiable processes. In essence, if $\chi_t \in \mathcal{D}_W^{(k)}$, then χ_t admits a representation in terms of a W-Itô diffusion in which the two coefficients (drift and diffusion) are also W-Itô diffusions, the four coefficients of the drift and diffusion are themselves W-Itô diffusions and so on ($k-3$ times additional times). In other words, if the coefficients (starting with $\mathcal{S}_t^1$) are $k-1$ times W-differentiable, the process is k times differentiable.

Example 1. *Consider a real function $f(\cdot)$ which is differentiable $2k$ times in its domain. Then, the process $f(W_t) \in \mathcal{D}_W^{(k+1)}$. Indeed, if $f(\cdot)$ is twice differentiable, by Itô's lemma:*

$$
df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt,
$$

which implies $f(W_t) \in \mathcal{D}_W^{(2)}$. By iterating on $f'(W_t)$ and $f''(W_t)$, we verify the statement.

Example 2. *Consider two real functions $f(\cdot)$ and $g(\cdot)$ which are differentiable $2k$ times in their domain and are such that the stochastic differential equation*

$$
d\chi_t = f(\chi_t)dt + g(\chi_t)dW_t
$$

admits a unique solution for χ_t. Then, $\chi_t \in \mathcal{D}_W^{(k+2)}$. Indeed, by Itô's lemma:

$$
\begin{aligned}
df(\chi_t) &= f'(\chi_t)d\chi_t + \frac{1}{2}f''(\chi_t)g^2(\chi_t)dt = \left(f'(\chi_t)f(\chi_t) + \frac{1}{2}f''(\chi_t)g^2(\chi_t)\right)dt + f'(\chi_t)g(\chi_t)dW_t, \\
dg(\chi_t) &= g'(\chi_t)d\chi_t + \frac{1}{2}g''(\chi_t)g^2(\chi_t)dt = \left(g'(\chi_t)f(\chi_t) + \frac{1}{2}g''(\chi_t)g^2(\chi_t)\right)dt + g'(\chi_t)g(\chi_t)dW_t,
\end{aligned}
$$

implying that $\chi_t \in \mathcal{D}_W^{(3)}$ when $f(.)$ and $g(.)$ are twice differentiable. By iterating, again, we verify the statement.

We are interested in the following function

$$
f_{\chi_t}(u) = \mathbb{E}_0\left[e^{iuW_t}\chi_t\right], \tag{2}
$$

where $\mathbb{E}_0$ denotes expectation conditional on $\mathcal{F}_0$. The function $f_{\chi_t}(u)$ is a sort of (Brownian) Fourier transform (with signed reversed in the complex exponential) of the process χ_t. It will arise repeatedly in our proofs. We will sometimes call it a "W-transform".

The following lemma is a first step towards a bound in t and u for $f_{\chi_t}(u)$. The bound will lead to a general method of proof for local approximations to the characteristic function of W-Itô diffusions in the next section.

Lemma 1. *Consider a W-differentiable process χ_t defined as in Eq. (1). Its W-transform satisfies*

$$
\mathbb{E}_0\left[e^{iuW_t}\chi_t\right] = e^{-\frac{u^2}{2}t}\left(\chi_0 + \int_0^t e^{\frac{u^2}{2}s}\mathbb{E}_0\left[e^{iuW_s}\left(d\chi_s + iud[\chi, W]_s\right)\right]\right),
$$

where $[\chi, W]_t$ is the quadratic covariation process between χ_t and W_t.

Proof. *See Appendix.*

Corollary 1 (to Lemma 1). *Given Lemma 1, if $\chi_t \in \mathcal{D}_W^{(1)}$, then*

$$
\mathbb{E}_0\left[e^{iuW_t}\chi_t\right] = e^{-\frac{u^2}{2}t}\left(\chi_0 + \int_0^t \mathbb{E}_0\left[e^{iuW_s}\left(a^\chi + iub^\chi\right)\right]e^{\frac{u^2}{2}s}ds\right),
$$

since $a_t^\chi = a^\chi$ and $b_t^\chi = b^\chi$. If $\chi_t \in \mathcal{D}_W^{(2)}$, the expression can be iterated once:

$$
\begin{aligned}
&\mathbb{E}_0\left[e^{iuW_t}\chi_t\right] \\
= \;& e^{-\frac{u^2}{2}t}\left(\chi_0 + \int_0^t \mathbb{E}_0\left[e^{iuW_s}\left(a_s^\chi + iub_s^\chi\right)\right]e^{\frac{u^2}{2}s}ds\right)
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-\frac{u^2}{2}t}\left(\chi_0+\int_0^t \underbrace{\mathbb{E}_0\left[e^{iuW_s}a_s^\chi\right]}_{f_{a_s^\chi}(u)} e^{\frac{u^2}{2}s}ds+iu\int_0^t \underbrace{\mathbb{E}_0\left[e^{iuW_s}b_s^\chi\right]}_{f_{b_s^\chi}(u)} e^{\frac{u^2}{2}s}ds\right)\\
&= e^{-\frac{u^2}{2}t}\left(\chi_0+(a_0^\chi+iub_0^\chi)t+\int_0^t\int_0^{s_1}\mathbb{E}_0\left[e^{iuW_{s_2}}(a^{a^\chi}+iub^{a^\chi})\right]e^{\frac{u^2}{2}s_2}ds_2+\right.\\
&\quad \left.+\,iu\int_0^t\int_0^{s_1}\mathbb{E}_0\left[e^{iuW_{s_2}}(a^{b^\chi}+iub^{b^\chi})\right]e^{\frac{u^2}{2}s_2}ds_2\right),
\end{aligned}
$$

since $a_{s_2}^{a^\chi}=a^{a^\chi}$, $b_{s_2}^{a^\chi}=b^{a^\chi}$, $a_{s_2}^{b^\chi}=a^{b^\chi}$ and $b_{s_2}^{b^\chi}=b^{b^\chi}$. If $\chi\in\mathcal{D}_W^{(k)}$, with $k>2$ the expression can, of course, be iterated further.

Corollary 2 (to Lemma 1). *Write $d\widetilde{\chi}_t=d\chi_t+c_t^{\widetilde{\chi}}d\widetilde{W}_t$, where $\widetilde{W}_t$ is a Brownian motion independent of W_t and $c_t^{\widetilde{\chi}}$ is an adapted càdlàg process. The W-transform of $\widetilde{\chi}_t$ is*

$$
\mathbb{E}_0\left[e^{iuW_t}\widetilde{\chi}_t\right]=e^{-\frac{u^2}{2}t}\left(\widetilde{\chi}_0+\int_0^t e^{\frac{u^2}{2}s}\mathbb{E}_0\left[e^{iuW_s}\left(d\chi_s+iud[\chi,W]_s\right)\right]\right).
$$

Proof. The result is immediate given that $\mathbb{E}_0\left[e^{iuW_s}d\widetilde{\chi}_s\right]=\mathbb{E}_0\left[e^{iuW_s}d\chi_s\right]$ by the independence between W and $\widetilde{W}$, the law of iterated expectations, and the conditional zero mean property of the Brownian increments. Also, $d[\widetilde{\chi},W]_s=d[\chi,W]_s$, by the independence between W and $\widetilde{W}$, again.

Remark 1. ***(Extended W-differentiability)*** *We could define W-differentiability by adding local martingale increments driven by Brownian motions - independent of all other Brownian motions, including W - to the χ_t process and to the processes in $\mathcal{S}_t^j$. Doing so would allow us to capture meaningful (i.e., different from one) local correlations between χ_t and its coefficients, something which is important, e.g., in continuous-time asset pricing. We write that $\chi_t\in\mathcal{D}_W^{(k)}$ if*

$$
\begin{aligned}
d\chi_t &= a_t^\chi dt+b_t^\chi dW_t+c_t^\chi d\widetilde{W}_t,\\
d\mathcal{S}_t^j &= \mathcal{A}_t^{\chi(j)}dt+\mathcal{B}_t^{\chi(j)}dW_t+\mathcal{C}_t^{\chi(j)}d\widetilde{W}_t^{(j)},
\end{aligned}
\tag{3}
$$

where

$$
\mathcal{S}_t^j=(\mathcal{A}_t^{\chi(j-1)},\mathcal{B}_t^{\chi(j-1)}),\quad \forall j\geq 1
$$

with $(\mathcal{A}_t^{\chi(0)},\mathcal{B}_t^{\chi(0)})=(a_t^\chi,b_t^\chi)$ and

$$
\mathcal{S}_t^k=(\mathcal{A}^{\chi(k-1)},\mathcal{B}^{\chi(k-1)}),
$$

for $j=1,\ldots,k$. The processes $\mathcal{C}_t^{\chi(j)}$ with $\mathcal{C}_t^{\chi(0)}=c_t^\chi$ are càdlàg, adapted. The only common Brownian motion is W, which, again, justifies the terminology W-differentiability.

Remark 2. ***(W-transform under extended W-differentiability)*** *Corollary 2 to Lemma 1 implies that we can add local martingale increments driven by an independent Brownian motions to the χ_t process without affecting the W-transform. Because we will iterate the result in Lemma 1 (so as to exploit higher-order W-differentiability) as in Corollary 1 to Lemma 1, Corollary 2 also implies that we can add similar local martingale increments to the coefficients themselves. In other words, the iteration in Corollary 1 to Lemma 1 applies under the extended notion of W-differentiability in Remark 1, Eq. (3). The tail properties (for large u) of the W-transform will depend crucially on coefficients associated with innovations to W, a result which provides another justification (deriving now from the smoothness of Fourier-like transforms) for defining differentiability in terms of W, i.e., W-differentiability.*

Before bounding $f_{\chi_t}(u)$ in both t and u, we introduce a subset of $\mathcal{D}_W^{(k)}$. We note that the number - and order in t - of the terms χ_0, $(a_0^\chi + iub_0^\chi)t$, ... , depends on the degree of differentiability of the process. The subset of $\mathcal{D}_W^{(k)}$ in the next definition contains processes which reduce the impact of these terms.

Definition 3. *Consider an adapted, real stochastic process $\chi_t \in \mathcal{D}_W^{(k)}$ as in Definition 2. We say that this process is "standard" W-differentiable (k times) if $\chi_0 = 0$ and $\mathcal{A}_0^{\chi(j)} = \mathcal{B}_0^{\chi(j)} = 0$ for all $j = 1, \ldots, k$.*

The class of k times standard W-differentiable processes is denoted by $\mathcal{SD}_W^{(k)}$. When $\chi_t \in \mathcal{D}_W^{(1)}$ and $\chi_0 = 0$, then $\chi_t \in \mathcal{SD}_W^{(1)}$. When $\chi_t \in \mathcal{D}_W^{(2)}$ and $\chi_0 = 0$, $a_0^\chi = 0$ and $b_0^\chi = 0$, then $\chi_t \in \mathcal{SD}_W^{(2)}$.

Example 3. *Consider the process $\chi_t \in \mathcal{D}_W^{(1)}$. By definition, $\chi_t - \chi_0 \in \mathcal{SD}_W^{(1)}$.*

We note that this trivial example is central to our treatment. We will compute W-transforms for differences, like $\chi_t - \chi_0$. Hence, assuming W-differentiability will immediately give us standard W-differentiability.

After introducing the concept of standard W-differentiability, we present Lemma 2 (below) for a generic process with some level of W-differentiability (k) and some level of standard W-differentiability (m, with m smaller than k).

Example 4. *Consider a geometric Brownian motion:*

$$d\chi_t = \mu\chi_t dt + \sigma\chi_t dW_t,$$

with $\chi_0 = 0$, where μ and σ are real constants. We have $\chi_t \in \mathcal{D}_W^{(\infty)}$ and $\chi_t \in \mathcal{SD}_W^{(\infty)}$.

Example 5. *Consider (a version of) the Heston (1993) stochastic volatility model:*

$$d\chi_t = \mu\chi_t dt + \sqrt{V_t}\chi_t dW_t,$$

$$dV_t = (\alpha - \beta V_t)dt + \eta\sqrt{V_t}\left(\rho_t dW_t + \sqrt{1-\rho_t^2}dW_t^{(1)}\right), \tag{4}$$

where μ, α, β, η are real constants and ρ_t (time-varying leverage) is an adapted, real process. We immediately have that $\chi_t \in \mathcal{D}_W^{(\infty)}$ and, without other assumptions, that $\chi_t - \chi_0 \in \mathcal{SD}_W^{(1)}$.

We now turn to our main result regarding a bound on $f_{\chi_t}(u)$ in t and u. The class $\mathcal{SD}_W^{(m)}$ will be used in the statement of Lemma 2. Its role will be discussed in Remark 4.

Lemma 2. *Consider a process $\chi_t \in \mathcal{D}_W^{(k)}$ with $k \geq 1$ and such that $\chi_t \in \mathcal{SD}_W^{(m)}$ with $m < k$. Assume the coefficients are bounded. The function $f_{\chi_s}(u)$, defined in Eq. (2), satisfies*

$$|f_{\chi_t}(u)| \leq C\widetilde{g}_{k,m}(u,t),$$

where C is a suitable constant which does not depend on k, m, u, t, and

$$\widetilde{g}_{k,m}(u,t) = e^{-\frac{u^2}{2}t}\sum_{j=m+1}^{k-1}(1+|u|)^j\frac{t^j}{j!} + g_k(u,t),$$

with

$$g_k(u,t) = 2^k\frac{(1+|u|)^k}{u^{2k}}\left(1 - \frac{\Gamma\left(k, \frac{u^2}{2}t\right)}{(k-1)!}\right).$$

Proof. See Appendix.

Remark 3. *Some examples of the function $g_k(u,t)$, for alternative values of k, are:*

$$g_1(u,t) = 2(1+|u|)\frac{\left(1-e^{-\frac{u^2}{2}t}\right)}{u^2},$$

$$g_2(u,t) = 4(1+|u|)^2\frac{\left(1-e^{-\frac{u^2}{2}t}(1+\frac{u^2}{2}t)\right)}{u^4},$$

$$g_3(u,t) = 8(1+|u|)^3\frac{\left(1-e^{-\frac{u^2}{2}t}\left(1+\frac{u^2}{2}t+\frac{1}{2}\left(\frac{u^2}{2}t\right)^2\right)\right)}{u^6}.$$

Remark 4. *The bound on the function $f_{\chi_t}(u)$ has two objects which mix terms in t and terms in u. Using the expressions in Remark 3 and a Taylor expansion around 0 for small t, it is easy to see that the bound in t of the second term is $O(t^k)$. For small t, the bound in t of the first term is $O(t^{m+1})$. Thus, because $m < k$, the higher the order of standard W-differentiability of χ_t, the tighter the bound in t. Importantly, $g_k(u,t) \sim |u|^{-k}$, when $|u| \to \infty$, for a fixed*

$t > 0$. *The function* $\widetilde{g}_{k,m}(u,t)$ *is, therefore, integrable on* $\mathbb{R}$, *with respect to* u *and for every fixed* $t > 0$, *when* $k \geq 2$. *We conclude that, for the purpose of bounding in* u, *it is the order of* W*-differentiability of* χ_t *that matters.*

Remark 5. *The role of the* W*-differentiability of* χ_t *in controlling the tail behavior of* $f_{\chi_t}(u)$ *is analogous to the role of the differentiability of a deterministic function in controlling the tail behavior of its Fourier transform. Consider a function* $g(x)$. *By repeated applications of integration by parts, we have*

$$\begin{aligned} f_g(u) &= \int g(x)e^{-iux}dx \\ &= -\frac{1}{-iu}\int g'(x)e^{-iux}dx \\ &= \frac{1}{(-iu)^2}\int g''(x)e^{-iux}dx \\ &= \ldots \end{aligned}$$

which implies that $f_g(u) = O(1/u^k)$ *with* k *denoting the order of differentiability of* $g(x)$.

Before introducing the small-time expansion of the process' characteristic function in the diffusive case, we provide details on the assumed diffusive process. We will then offer intuition for the role of the W-transform in leading to the proposed expansion.

3 W-Itô semi-martingale: the expansions

We work with an Itô diffusion process X_t in which the stochastic coefficient associated with the Brownian shock is expressed explicitly as another Itô diffusion process. This process is known, in finance, as a stochastic volatility model.

Assumption 1. *The adapted, real stochastic process* X_t *defined on the filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *satisfies*

$$\begin{aligned} X_t &= X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s, && (5) \\ \sigma_t &= \sigma_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s dW_s + \int_0^t \beta'_s dW'_s, && (6) \\ \mu_t &= \mu_0 + \int_0^t \gamma_s ds + \int_0^t \delta_s dW_s + \int_0^t \delta'_s dW^*_s, && (7) \\ \beta_t &= \beta_0 + \int_0^t \zeta_s ds + \int_0^t \eta_s dW_s + \int_0^t \eta'_s dW^{**}_s. && (8) \end{aligned}$$

$$\alpha_t = \alpha_0 + \int_0^t \xi_s ds + \int_0^t \theta_s dW_s + \int_0^t \theta'_s dW_s^{***}. \tag{9}$$

$$\eta_t = \eta_0 + \int_0^t \xi'_s ds + \int_0^t \psi_s dW_s + \int_0^t \psi'_s dW_s^{****}. \tag{10}$$

$$\beta'_t = \beta'_0 + \int_0^t \zeta'_s ds + \int_0^t \kappa_s dW_s + \int_0^t \kappa'_s dW_s^{*****}. \tag{11}$$

$$\kappa_t = \kappa_0 + \int_0^t \xi''_s ds + \int_0^t \iota_s dW_s + \int_0^t \iota'_s dW_s^{******}. \tag{12}$$

The processes W, W', W^*, W^{**}, W^{***}, W^{****}, W^{*****}, W^{******} are independent Brownian motions. All stochastic coefficients are bounded càdlàg. Also, $\sigma_t, \beta_t, \beta'_t, \delta_t, \delta', \eta, \eta', \theta, \theta', \psi, \psi', \kappa, \kappa', \iota, \iota' > 0$ almost surely, $\forall t \geq 0$.

If we were just focused on the level dynamics, in Assumption 1 we would specify Eq. (6), Eq. (7) and Eq. (8) in addition to the level process in Eq. (5). In other words, not only would we specify the dynamics of the volatility (σ_t) process - something which is typical in the stochastic volatility literature - we would also specify the dynamics of the drift coefficient (μ_t) and those of the W-volatility-of-volatility coefficients (β_t), i.e., the portion of the volatility-of-volatility associated with the driving Brownian motion W. We would do so because the corresponding coefficients will play a role (to the first and the second order) in the proposed expansions (c.f., e.g., Corollary 3).

Because we are interested in the joint distributional properties of levels and volatility, we specify analogous processes for volatility. Specifically, Eq. (9) is the volatility drift (α_t), Eq. (38) is the W'-volatility-of-volatility, i.e., the portion of the volatility-of-volatility associated with the remaining (other than W) driving Brownian motion W' of the volatility process, the first portion being already accounted for. Both the W-volatility of the W-volatility-of-volatility (in Eq. (10)) and the W-volatility of the W'-volatility-of-volatility (in Eq. (12)) also need to be specified in analogy with the specification of the W-volatility-of-volatility coefficients (β_t) in the level case.

With diffusive assumptions on the coefficients of the drift process in Eq. (7), the process would, at least, be $D_W^{(3)}$. The exact degree of W-differentiability will be specified - and justified - in the statement of the theorems.

Because of the presence of an array of independent Brownian motions (other than W), we apply the extended notion of W-differentiability in Remark 1. The role of the independent Brownian motions is to break the perfect dependence between layers of the process which would, otherwise, be generated by the driving (level) Brownian motion W.

Remark 6. *We emphasize that the specification of the process is nonparametric in the sense that no parametric assumptions are imposed on the process' coefficients. All coefficients are assumed to be processes for which we only specific their (extended) W-differentiability properties.*

Write, now, the demeaned, standardized, discretized level/volatility processes as

$$Z_\Delta = \frac{X_\Delta - X_0 - \mu_0\Delta}{\sigma_0\sqrt{\Delta}},$$

$$V_\Delta = \frac{\sigma_\Delta - \sigma_0 - \alpha_0\Delta}{\beta_0\sqrt{\Delta}}.$$

The following theorem provides the conditional (given time-0 information) characteristic function of the pair (Z_Δ, V_Δ) for small Δ values.

Theorem 1. *Let Assumption 1 hold. If $X_t \in \mathcal{D}_W^{(6)}$, the conditional characteristic function of Z_Δ and V_Δ with respect to time-0 information is expressed as follows:*

$$\begin{aligned}
\mathbb{E}_0\left[e^{iuZ_\Delta + ivV_\Delta}\right] =& e^{-\frac{(u+v)^2}{2}} e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2} \Bigg\{1 - i\frac{1}{2}(u+v)^2\left(u\frac{\beta_0}{\sigma_0} + v\frac{\eta_0}{\beta_0}\right)\sqrt{\Delta} \\
& - i(u+v)\frac{1}{2}v\frac{\beta_0'}{\beta_0}\left(u\frac{\beta_0'}{\sigma_0}\right)\sqrt{\Delta} - i(u+v)\frac{1}{2}v^2\frac{\beta_0'}{\beta_0}\frac{\kappa_0}{\beta_0}\sqrt{\Delta} \\
& - \frac{1}{2}(u+v)\left(u\frac{\delta_0}{\sigma_0} + v\frac{\theta_0}{\beta_0}\right)\Delta - \frac{1}{2}(u+v)\left(u\frac{\alpha_0}{\sigma_0} + v\frac{\zeta_0}{\beta_0}\right)\Delta \\
& + \frac{1}{6}(u+v)^3\left(u\frac{\eta_0}{\sigma_0} + v\frac{\psi_0}{\beta_0}\right)\Delta - \frac{1}{2}v^2\frac{\beta_0'}{\beta_0}\frac{\zeta_0'}{\beta_0}\Delta + \frac{(u+v)^2v^2}{6}\frac{\iota_0}{\beta_0}\frac{\beta_0'}{\beta_0}\Delta \\
& - \frac{1}{8}\left(2 - 4(u+v)^2 + (u+v)^4\right)\left(\left(u\frac{\beta_0}{\sigma_0}\right)^2 + \left(v\frac{\eta_0}{\beta_0}\right)^2\right)\Delta \\
& + \frac{1}{2}(u+v)^2\left(u\frac{\beta_0'}{\sigma_0}\right)^2\left(\frac{1}{3} - \frac{1}{4}\left(v\frac{\beta_0'}{\beta_0}\right)^2\right)\Delta - \frac{1}{2}\left(u\frac{\beta_0'}{\sigma_0}\right)^2\left(\frac{1}{2} - \frac{1}{3}\left(v\frac{\beta_0'}{\beta_0}\right)^2\right)\Delta \\
& + \frac{1}{2}v^2\left(\frac{\beta_0'}{\beta_0}\right)^2\left(v\frac{\kappa_0}{\beta_0}\right)^2\left(\frac{1}{3} - \frac{1}{4}(u+v)^2\right)\Delta - \frac{1}{2}\left(v\frac{\kappa_0}{\beta_0}\right)^2\left(\frac{1}{2} - \frac{1}{3}(u+v)^2\right)\Delta \\
& - \frac{1}{2}v^2\left(\frac{\eta_0'}{\beta_0}\right)^2\left(\frac{1}{2} - \frac{1}{3}(u+v)^2\right)\Delta - \frac{1}{2}v^2\left(\frac{\kappa_0'}{\beta_0}\right)^2\left(\frac{1}{2} - \frac{1}{3}\left(v\frac{\beta_0'}{\beta_0}\right)^2\right)\Delta \\
& + \frac{1}{6}(u+v)^2uv\frac{\kappa_0}{\sigma_0}\frac{\beta_0'}{\beta_0}\Delta \\
& - \frac{1}{4}u^2(u+v)v\frac{(\beta_0')^2}{\sigma_0^2}\left((u+v)^2 - 2\right)\Delta - \frac{1}{4}v^3(u+v)\frac{\eta_0\kappa_0\beta_0'}{\beta_0^3}\left((u+v)^2 - 2\right)\Delta \\
& - \frac{1}{4}uv\frac{\eta_0}{\sigma_0}(2 - 4(u+v)^2 + (u+v)^4)\Delta - \frac{1}{4}uv^2(u+v)\frac{\beta_0'}{\sigma_0}\frac{\kappa_0}{\beta_0}\left((u+v)^2 - 2\right)\Delta
\end{aligned} \tag{13}$$

$$
\begin{aligned}
&-\frac{1}{4}uv^2(u+v)\left(\frac{\beta_0'}{\beta_0}\right)^2\frac{\eta_0}{\sigma_0}\left((u+v)^2-2\right)\Delta \\
&-\frac{1}{2}uv\frac{\beta_0'\kappa_0}{\sigma_0\beta_0}\left(v\frac{\beta_0'}{\beta_0}\right)^2\left(\frac{1}{4}(u+v)^2-\frac{1}{3}\right)\Delta \\
&-\frac{1}{2}uv\frac{\beta_0'\kappa_0}{\sigma_0\beta_0}(u+v)^2\left(\frac{1}{4}\left(v\frac{\beta_0'}{\beta_0}\right)^2-\frac{1}{3}\right)\Delta\Bigg\}+O(\Delta^{3/2})\widetilde{\phi}(u,v)
\end{aligned}
$$

where $\widetilde{\phi}(u,v)$ is an integrable function over $\mathbb{R}^2$ of order $u^{-3}v^{-2}$, as $|u|\to\infty$ and $|v|\to\infty$.

The characteristic function of Z_Δ (respectively V_Δ) can, of course, be recovered by just setting $v=0$ (respectively $u=0$). For clarity, in what follows we will largely comment on the $v=0$ sub-case of the more general expansion. A similar logic will apply to the bivariate case. We begin with the formal result.

Corollary 3 (to Theorem 1). *Let Assumption 1 hold. If $X_t\in\mathcal{D}_W^{(5)}$, the conditional characteristic function of Z_Δ with respect to time-0 information is expressed as follows:*

$$
\begin{aligned}
\mathbb{E}_0\left[e^{iuZ_\Delta}\right]=&e^{-\frac{u^2}{2}}\Bigg(1-iu^3\frac{\beta_0}{2\sigma_0}\sqrt{\Delta}-\frac{1}{2}u^2\frac{\alpha_0+\delta_0}{\sigma_0}\Delta-\frac{1}{8}\frac{\beta_0^2}{\sigma_0^2}u^2\left(2-4u^2+u^4\right)\Delta \\
&+\frac{(\beta_0')^2}{\sigma_0^2}\left(\frac{1}{6}u^4-\frac{1}{4}u^2\right)\Delta+\frac{\eta_0}{6\sigma_0}u^4\Delta\Bigg)+O(\Delta^{3/2})\widetilde{\phi}(u),
\end{aligned}
\tag{14}
$$

where $\widetilde{\phi}(u)$ is an integrable function over $\mathbb{R}$ of order u^{-2}, as $|u|\to\infty$.

Eq. (14) is a second-order (in $\sqrt{\Delta}$) expansion of the characteristic function of the (demeaned, standardized) discretized process Z_Δ around the standard Gaussian characteristic function. The term in $\sqrt{\Delta}$ depends on the portion of the volatility-of-volatility associated with W (β_0) as well as on volatility (σ_0). The terms in Δ depend on the same quantities, as well as on the portion of the volatility-of-volatility associated with W' (β_0'), on the portion of the volatility of the drift associated with W (δ_0), on the drift of the volatility process (α_0) and on the portion of the volatility of the volatility-of-volatility associated with W (η_0). The residual term is of order $\Delta^{3/2}$. The expansion justifies the need to express certain dynamics explicitly in Eq. (5), Eq. (7), Eq. (6) and Eq. (8).

Importantly, the order in u (for $|u|\to\infty$) is u^{-2}, i.e., the minimum possible order for Fourier-integrability of the characteristic function and, therefore, for deriving the implied density, something that we do below. As discussed in Remark 4, this order is intimately related to the W-differentiability of the process, which we assume to be $D_W^{(5)}$.

Focusing, again for clarity, on the characteristic function of the levels only, Corollary 4 to Theorem 1 provides a more transparent representation which makes explicit the role played in the expansion but the overall volatility-of-volatility (i.e., the volatility-of-volatility induced by *both* Brownian motions), which we denote by $\widetilde{\beta}_0$, and the role played by time-varying leverage (i.e., the correlation between shocks in levels and shocks in volatility), which we denote by ρ_0. The resulting expansion is of particular interest in asset pricing (c.f., Bandi, Fusari, and Renò, 2023a, and Bandi, Fusari, and Renò, 2023b).

Corollary 4 (to Theorem 1). *Writing, similarly to what is done for the traditional Heston model (see, e.g., Eq. (4)),*

$$\beta_t = \widetilde{\beta}_t \rho_t,$$
$$\beta_t' = \widetilde{\beta}_t \sqrt{1-\rho_t^2},$$

the expansion in Corollary 3 can be re-expressed as follows:

$$\begin{aligned}\mathbb{E}_0\left[e^{iuZ_\Delta}\right] =& e^{-\frac{u^2}{2}}\left(1 - iu^3\frac{\widetilde{\beta}_0\rho_0}{2\sigma_0}\sqrt{\Delta} - \frac{1}{2}u^2\frac{\alpha_0+\delta_0}{\sigma_0}\Delta\right.\\ &\left.-\frac{1}{24}\frac{\widetilde{\beta}_0^2}{\sigma_0^2}u^2\left(6-4u^2+\rho_0^2u^2\left(3u^2-8\right)\right)\Delta + \frac{\eta_0}{6\sigma_0}u^4\Delta\right) + O(\Delta^{3/2})\widetilde{\phi}(u).\end{aligned}$$

This representation offers a clearer illustration of the role of specific processes in driving the dynamics of specific moments. Write the expansion as

$$\mathbb{E}_0\left[e^{iuZ_\Delta}\right] = e^{-\frac{u^2}{2}}(1 + au^2\Delta + ibu^3\sqrt{\Delta} + cu^4\Delta + du^6\Delta),$$

where a, b, c and d are the coefficients associated with, respectively, u^2, u^3, u^4 and u^6 in the expansion. We note that the centered second moment is

$$1 - 2a\Delta.$$

The centered third moment is

$$-6b\sqrt{\Delta}.$$

The centered fourth moment is, instead,

$$3(1 - 4a\Delta + 8c\Delta).$$

Thus, the lowest-order (i.e., $\sqrt{\Delta}$) term in the expansion is a third-moment (skewness) correction driven by the correlation between level and volatility shocks (ρ_0). The highest-order (i.e., Δ) term is a second, fourth and sixth moment correction. The second moment correction is specific to our problem and induced by the drift and volatility dynamics. The fourth and sixth moment corrections are more standard Edgeworth-like terms. The former is a kurtosis correction, the latter represents the (second-order) contribution of squared skewness to even (sixth) moment adjustments (c.f., e.g., Hall, 2013, page 45, for a similar adjustment in traditional expansions).

We now turn to an intuitive discussion of the role of the W-transform, introduced in Section 2, in deriving the result in Theorem 1. For clarity, again, we focus on the univariate case Z_Δ. We note that the characteristic function of Z_Δ may be easily represented in terms of the expectation of the product of two exponentials. The first is the exponential of the discretized driving Brownian motion W. The second is the exponential of the (standardized) discretized X process in which the coefficients are taken as distances from their values at zero:

$$\mathbb{E}_0\left[e^{iu\left(\frac{X_\Delta - X_0 - \mu_0\Delta}{\sigma_0\sqrt{\Delta}}\right)}\right] = \mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}e^{iuY_\Delta\sqrt{\Delta}}\right],$$

where

$$Y_\Delta = \frac{1}{\sigma_0\Delta}\left(\int_0^\Delta(\mu_s-\mu_0)ds + \int_0^\Delta(\sigma_s-\sigma_0)dW_s\right). \tag{15}$$

Now, by Taylor expansion around 0 for small Δ, we may write

$$e^{iuY_\Delta\sqrt{\Delta}} = \sum_{k=0}^{+\infty}\frac{(iu)^k}{k!}\Delta^{\frac{k}{2}}Y_\Delta^k = 1 + iuY_\Delta\sqrt{\Delta} + \frac{1}{2}(iu)^2Y_\Delta^2\Delta + \sum_{k=3}^{+\infty}\frac{(iu)^k}{k!}\Delta^{\frac{k}{2}}Y_\Delta^k.$$

It follows that

$$\begin{aligned}
\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}e^{iuY_\Delta\sqrt{\Delta}}\right] &= \mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\left(1 + iuY_\Delta\sqrt{\Delta} + \frac{1}{2}(iu)^2Y_\Delta^2\Delta + \sum_{k=3}^{+\infty}\frac{(iu)^k}{k!}\Delta^{\frac{k}{2}}Y_\Delta^k\right)\right] \\
&= e^{-u^2/2} + \underbrace{\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}iuY_\Delta\sqrt{\Delta}\right]}_{A_\Delta} + \underbrace{\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\frac{1}{2}(iu)^2Y_\Delta^2\Delta\right]}_{B_\Delta} + \underbrace{\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\sum_{k=3}^{+\infty}\frac{(iu)^k}{k!}\Delta^{\frac{k}{2}}Y_\Delta^k\right]}_{C_\Delta}. \qquad (16)
\end{aligned}$$

The terms A_Δ, B_Δ and C_Δ expand the characteristic function of the standard Gaussian random variable ($e^{-u^2/2}$) around 0. We observe that they are W-transforms of specific processes, a remark which justifies the role played by a generic W-transform in our proofs (see Appendix). Theorem 1 generalizes this logic to the bivariate case.

Next, we apply our methods to two natural objects having to do with the process Z_Δ: its conditional density and its conditional distribution. Needless to say, similar considerations apply to the process V_Δ as well as to the joint process Z_Δ, V_Δ.

3.1 The conditional density of Z_Δ

We derive a (small-Δ) expansion of the conditional density of the (standardized, demeaned) discretized process Z_Δ by Fourier inversion of the expansion of the process' conditional characteristic function. As discussed, the integrability in u of the characteristic function is a consequence of the assumed W-differentiability of the process.

The expansion (around a standard Gaussian density) depends on Hermite polynomials with weights given by key features of the volatility process. One such feature is leverage (ρ_0), a quantity whose presence adds skewness to the resulting density.

Corollary 5 (to Theorem 1). *The expansion of the conditional density function of Z_Δ is given by*

$$f_{Z_\Delta}(z) = \varphi(z)\left(1 + \frac{\widetilde{\beta}_0\rho_0}{2\sigma_0}\underbrace{z(z^2-3)}_{He_3(z)}\sqrt{\Delta} + \frac{1}{2}\frac{\alpha_0+\delta_0}{\sigma_0}\underbrace{(z^2-1)}_{He_2(z)}\Delta \right.$$
$$+\frac{1}{24}\frac{\widetilde{\beta}_0^2}{\sigma_0^2}\left(\underbrace{4z^4-18z^2+6}_{4He_4(z)+6He_2(z)}+\rho_0^2\left(\underbrace{3z^6-37z^4+87z^2-21}_{3He_6(z)+8He_4(z)}\right)\right)\Delta$$
$$\left.+\frac{\eta_0}{6\sigma_0}\left(\underbrace{z^4-6z^2+3}_{He_4(z)}\right)\Delta\right) + O(\Delta^{3/2})\widetilde{f}(z),$$

where $\varphi(z)$ is the standard Gaussian density, $\widetilde{f}(z)$ is the value of an integrable function at z, and the term $He_j(z)$ defines a generic Hermite polynomial of order j.

Analogously to traditional Edgeworth expansions in which the asymptotic order depends on the number of observations used to define the estimator n (with $n \to \infty$) rather than on the interval Δ (with $\Delta \to 0$), the first order (in $\sqrt{\Delta}$ here, rather than in $1/\sqrt{n}$) is an adjustment to skewness through an odd function ($He_3(z)$). The second order (in Δ here, rather than in $1/n$) is, instead, an adjustment to kurtosis through even functions ($He_4(z)$ and $He_2(z)$ here, $He_4(z)$ in the standard case) as well as second and sixth moments. As emphasized earlier, consistent with traditional expansions, the second order also corrects for the residual impact of squared "skewness" in the sixth moment.

3.2 The conditional distribution function of Z_Δ

Given the expansion of the conditional density, it is immediate to obtain an expansion of the corresponding distribution function. It is sufficient to use the well-known property

$$\frac{\partial H_k(z)\varphi(z)}{\partial z} = -H_{k+1}(z)\varphi(z)$$

to obtain the following.

Corollary 6 (**to Theorem 1**). *The expansion of the conditional cumulative distribution function of Z_Δ is given by*

$$F_{Z_\Delta}(z) = \Phi(z) - \varphi(z)\left(\frac{\widetilde{\beta}_0\rho_0}{2\sigma_0}He_2(z)\sqrt{\Delta}\right.$$
$$+\left(\frac{1}{2}\frac{\alpha_0+\delta_0}{\sigma_0}He_1(z) + \frac{1}{24}\frac{\widetilde{\beta}_0^2}{\sigma_0^2}\left(4He_3(z)+6He_1(z)+\rho_0^2\left(3He_5(z)+8He_3(z)\right)\right) + \frac{\eta_0}{6\sigma_0}He_3(z)\right)\Delta\Bigg)$$
$$+\,O(\Delta^{3/2})\widetilde{F}(z),$$

where $\Phi(z)$ is the cumulative distribution function of the standard Gaussian random variable, $\widetilde{F}(z)$ is the value of an integrable function at z, and the term $He_j(z)$ defines a generic Hermite polynomial of order j.

4 Parametric stochastic volatility models

Classical parametric stochastic volatility models have a specific structure in which the latent variable is stochastic volatility and the corresponding shocks carry over to other processes. We use the Heston's model (Heston, 1993) as an example, with the understanding that the same logic applies to alternative specifications.

The Heston's model reads as follows:

$$\begin{aligned} dX_t &= (\mu - \sigma_t^2/2)dt + \sqrt{\sigma_t^2}dW_t, && (17)\\ d\sigma_t^2 &= \kappa(\omega-\sigma_t^2)dt + \eta\sqrt{\sigma_t^2}\rho dW_t + \eta\sqrt{\sigma_t^2}\sqrt{1-\rho^2}dW_t', && (18)\end{aligned}$$

where $\mu,\kappa,\omega,\eta,\rho$ are parameters. Using Itô's lemma, we obtain

$$d\sigma_t = \frac{1}{\sigma_t}\left(\frac{1}{2}\kappa(\omega-\sigma_t^2) - \frac{1}{8}\eta^2\right)dt + \frac{1}{2}\eta\rho dW_t + \frac{1}{2}\eta\sqrt{1-\rho^2}dW_t', \quad (19)$$

and

$$d\mu_t = -\frac{1}{2}d\sigma_t^2. \tag{20}$$

Thus, $d\beta_t = 0$ and $d\beta_t' = 0$, while

$$d\alpha_t = (...)dt + \frac{\eta}{16\sigma_t^2}\left(\eta^2 - 4\kappa(\omega + \sigma_t^2)\right)(\rho dW_t + \sqrt{1-\rho^2}dW_t').$$

In essence, the two Brownian motions driving the volatility dynamics (i.e., W_t and W_t') enter different processes (e.g., μ_t and α_t). Assumption 1 may, however, be easily modified to allow for correlation between processes induced by both Brownian motions, rather than solely by W_t. To this extent, we write the following.

Assumption 2. *The adapted, real stochastic process X_t defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfies*

$$\begin{aligned}
X_t &= X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s, & (21)\\
\sigma_t &= \sigma_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s dW_s + \int_0^t \beta_s' dW_s', & (22)\\
\mu_t &= \mu_0 + \int_0^t \gamma_s ds + \int_0^t \delta_s dW_s + \int_0^t \delta_s^\sigma dW_s', & (23)\\
\beta_t &= \beta_0 + \int_0^t \zeta_s ds + \int_0^t \eta_s dW_s + \int_0^t \eta_s^\sigma dW_s' & (24)\\
\alpha_t &= \alpha_0 + \int_0^t \xi_s ds + \int_0^t \theta_s dW_s + \int_0^t \theta_s^\sigma dW_s' & (25)\\
\beta_t' &= \beta_0' + \int_0^t \zeta_s' ds + \int_0^t \kappa_s dW_s + \int_0^t \kappa_s^\sigma dW_s'. & (26)\\
\eta_t &= \eta_0 + \int_0^t \xi_s' ds + \int_0^t \psi_s dW_s + \int_0^t \psi_s^\sigma dW_s'. & (27)\\
\kappa_t &= \kappa_0 + \int_0^t \xi_s'' ds + \int_0^t \iota_s dW_s + \int_0^t \iota_s^\sigma dW_s'. & (28)
\end{aligned}$$

The processes W, W' are independent Brownian motions. All stochastic coefficients are bounded càdlàg. Also, all the coefficients are positive almost surely, $\forall t \geq 0$.

Given Assumption 2, the characteristic function (up to the first order in $\sqrt{\Delta}$) is given by the following theorem:[2]

[2] An analogous, but much lenghtier, expression may be provided to incorporate the second (and higher) order terms. The extended expression is available upon request.

Theorem 2. *Let Assumption 2 hold. If $X_t \in \mathcal{D}_W^{(6)}$, the conditional characteristic function of Z_Δ and V_Δ with respect to time-0 information is expressed as follows:*

$$\begin{aligned}\mathbb{E}_0\left[e^{iuZ_\Delta+ivV_\Delta}\right] =& e^{-\frac{(u+v)^2}{2}} e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2} \left\{1 - i\frac{1}{2}(u+v)^2\left(u\frac{\beta_0}{\sigma_0}+v\frac{\eta_0}{\beta_0}\right)\sqrt{\Delta}\right.\\ & - i(u+v)\frac{1}{2}v\frac{\beta_0'}{\beta_0}\left(u\frac{\beta_0'}{\sigma_0}\right)\sqrt{\Delta} - i(u+v)\frac{1}{2}v^2\frac{\beta_0'}{\beta_0}\frac{\kappa_0}{\beta_0}\sqrt{\Delta}\\ & \left. -i\frac{1}{2}(u+v)v^2\frac{\beta_0'}{\beta_0}\frac{\eta_0^\sigma}{\beta_0}\sqrt{\Delta} - i\frac{1}{2}v^3\left(\frac{\beta_0'}{\beta_0}\right)^2\frac{\kappa_0^\sigma}{\beta_0}\sqrt{\Delta}\right\} + O(\Delta)\widetilde{\phi}(u,v), \qquad (29)\end{aligned}$$

where $\widetilde{\phi}(u,v)$ is an integrable function over $\mathbb{R}^2$ of order $u^{-3}v^{-2}$, as $|u| \to \infty$ and $|v| \to \infty$.

Relative to the expansion in Theorem 1 stopped at the first order, we have two extra terms (the last two terms). These are terms induced by the correlation between the volatility-specific Brownian shock W' entering the various processes and the same Brownian shock W' in the W'-transform associated with volatility.

We emphasize that the latter transform would not affect the univariate expansion of the characteristic function of the level process (since $v = 0$). Such a univariate expansion would, therefore, be the same under Assumption 1 and under Assumption 2. In fact, it would also be unchanged under a version of Assumption 2 which encompasses Assumption 1 by adding orthogonal Brownian motions to each process (from Eq. (23) to Eq. (28)).

Using Eq. (30) through Eq. (38), the Heston's time-0 values of the various processes map into the time-0 value of the processes in the expansion as follows:

$$\begin{aligned}\alpha_0 =& \frac{1}{\sigma_0}\left(\frac{1}{2}\kappa(\omega-\sigma_0^2) - \frac{1}{8}\eta^2\right)\\ \beta_0 =& \frac{1}{2}\eta\rho\\ \beta_0' =& \frac{1}{2}\eta\sqrt{1-\rho^2}\\ \delta_0 =& -\frac{1}{2}\eta\sigma_0\rho\\ \delta_0^\sigma =& -\frac{1}{2}\eta\sigma_0\sqrt{1-\rho^2}\\ \theta_0 =& \frac{\eta\rho}{16\sigma_0^2}\left(\eta^2 - 4\kappa(\omega+\sigma_0^2)\right)\\ \theta_0^\sigma =& \frac{\eta\sqrt{1-\rho^2}}{16\sigma_0^2}\left(\eta^2 - 4\kappa(\omega+\sigma_0^2)\right)\\ \eta_0 =& \eta_0^\sigma = \kappa_0 = \kappa_0^\sigma = \psi_0 = \psi_0^\sigma = \iota_0 = \iota_0^\sigma = 0.\end{aligned}$$

Because the sole state variable is volatility, the time-0 processes in the expansion depend on the time-0 value of the volatility process (σ_0) and the parameter vector ($\mu, \kappa, \omega, \eta, \rho$).

We now provide a non-affine example. Consider the log-volatility model (discussed, e.g., by Chernov, Gallant, Ghysels, and Tauchen, 2003):

$$
\begin{aligned}
dX_t &= (\mu - \sigma_t^2/2)dt + \sigma_t dW_t, && (30)\\
d\log\sigma_t &= \kappa(\omega - \log\sigma_t)dt + \eta\rho dW_t + \eta\sqrt{1-\rho^2}dW_t', && (31)
\end{aligned}
$$

Another application of Ito's lemma now yields:

$$
\begin{aligned}
\alpha_0 &= \frac{1}{2}\sigma_0\left(\eta^2 + 2\kappa\omega - 2k\log\sigma_0\right) && \\
\beta_0 &= \sigma_0\eta\rho & \beta_0' &= \sigma_0\eta\sqrt{1-\rho^2}\\
\delta_0 &= -\sigma_0^2\eta\rho & \delta_0^\sigma &= -\sigma_0^2\eta\sqrt{1-\rho^2}\\
\theta_0 &= \frac{1}{2}\sigma_0\eta\rho\left(\eta^2 + 2\kappa(\omega-1) - 2k\log\sigma_0\right) & \theta_0^\sigma &= \frac{1}{2}\sigma_0\eta\sqrt{1-\rho^2}\left(\eta^2 + 2\kappa(\omega-1) - 2k\log\sigma_0\right)\\
\eta_0 &= \sigma_0\eta^2\rho^2 & \eta_0^\sigma &= \sigma_0\eta^2\rho\sqrt{1-\rho^2}\\
\kappa_0 &= \sigma_0\eta^2\rho\sqrt{1-\rho^2} & \kappa_0^\sigma &= \sigma_0\eta^2(1-\rho^2)\\
\psi_0 &= \sigma_0\eta^3\rho^3 & \psi_0^\sigma &= \sigma_0\eta^3\rho^2\sqrt{1-\rho^2}\\
\iota_0 &= \sigma_0\eta^3\rho^2\sqrt{1-\rho^2} & \iota_0^\sigma &= \sigma_0\eta^3\rho(1-\rho^2).
\end{aligned}
$$

5 Characteristic function-based local inference

We are interested in estimating - locally - key features of a level process X_t (among which, σ_t, β_t and ρ_t) for each time t. To this extent, write

$$
\begin{aligned}
& \mathbb{E}_t\left[e^{iu\left(\frac{\widetilde{X}_{t+\Delta}-\widetilde{X}_t-\mu_t\Delta}{\sigma_t\sqrt{\Delta}}\right)+iv\left(\frac{\widehat{\sigma}_{t+\Delta}-\widehat{\sigma}_t-\alpha_t\Delta}{\beta_t\sqrt{\Delta}}\right)}\right] && (32)\\
=\;& \mathbb{E}_t\left[e^{iu\left(\frac{X_{t+\Delta}-X_t+\epsilon^X_{t+\Delta}-\epsilon^X_t-\mu_t\Delta}{\sigma_t\sqrt{\Delta}}\right)+iv\left(\frac{\sigma_{t+\Delta}-\sigma_t+\epsilon^\sigma_{t+\Delta}-\epsilon^\sigma_t-\alpha_t\Delta}{\beta_t\sqrt{\Delta}}\right)}\right] && (33)\\
=\;& \mathbb{E}_t\left[e^{iu\left(\frac{X_{t+\Delta}-X_t-\mu_t\Delta}{\sigma_t\sqrt{\Delta}}\right)+iv\left(\frac{\sigma_{t+\Delta}-\sigma_t-\alpha_t\Delta}{\beta_t\sqrt{\Delta}}\right)}e^{iu\left(\frac{\epsilon^X_{t+\Delta}-\epsilon^X_t}{\sigma_t\sqrt{\Delta}}\right)}e^{iv\left(\frac{\epsilon^\sigma_{t+\Delta}-\epsilon^\sigma_t}{\beta_t\sqrt{\Delta}}\right)}\right] && (34)
\end{aligned}
$$

$$= \quad \phi_t^c(u,v)\mathbb{E}_t\left[e^{iu\left(\frac{\epsilon_{t+\Delta}^X-\epsilon_t^X}{\sigma_t\sqrt{\Delta}}\right)}\right]\mathbb{E}_t\left[e^{iv\left(\frac{\epsilon_{t+\Delta}^\sigma-\epsilon_t^\sigma}{\beta_t\sqrt{\Delta}}\right)}\right], \tag{35}$$

where $\phi_t^c(u,v)$ denotes the characteristic function expansion reported in Theorem 1.

Eq. (32) is the bivariate characteristic function of (1) microstructure noise-contaminated price differences $(\widetilde{X}_{t+\Delta}-\widetilde{X}_t)$ and (2) differences of spot volatility estimates affected by estimation error $(\widehat{\sigma}_{t+\Delta}-\widehat{\sigma}_t)$. In Eq. (33) we separate microstructure noise and estimation error from $X_{t+\Delta}-X_t$ and $\sigma_{t+\Delta}-\sigma_t$. Eq. (35) hinges on independence.

Assuming, e.g., normality for uncorrelated $\{\epsilon_t^\sigma\}$ and $\{\epsilon_t^X\}$ errors - but with the understanding that alternative specifications/assumptions on the errors are easily implemented - leads to the following local moment conditions

$$\mathbb{E}_t\left[e^{iu\left(\frac{\widetilde{X}_{t+\Delta}-\widetilde{X}_t-\mu_t\Delta}{\sigma_t\sqrt{\Delta}}\right)+iv\left(\frac{\widehat{\sigma}_{t+\Delta}-\widehat{\sigma}_t-\alpha_t\Delta}{\beta_t\sqrt{\Delta}}\right)}-\phi_t^c(u,v)e^{-iu^2\left(\frac{\sigma_{\epsilon X}^2}{\sigma_t^2\Delta}\right)}e^{-iv^2\left(\frac{\sigma_{\epsilon\sigma}^2}{\beta_t^2\Delta}\right)}\right]=0,$$

with $u,v\in\mathbb{R}^2$. The corresponding local empirical moments are

$$\sum_{s=1}^T\frac{\mathbb{K}\left(\frac{s-t}{h}\right)}{\sum_{s=1}^T\mathbb{K}\left(\frac{s-t}{h}\right)}\left[e^{iu\left(\frac{\widetilde{X}_{s+\Delta}-\widetilde{X}_s-\mu_t\Delta}{\sigma_t\sqrt{\Delta}}\right)+iv\left(\frac{\widehat{\sigma}_{s+\Delta}-\widehat{\sigma}_s-\alpha_t\Delta}{\beta_t\sqrt{\Delta}}\right)}-\phi_t^c(u,v)e^{-iu^2\left(\frac{\sigma_{\epsilon X}^2}{\sigma_t^2\Delta}\right)}e^{-iv^2\left(\frac{\sigma_{\epsilon\sigma}^2}{\beta_t^2\Delta}\right)}\right],$$

where $\mathbb{K}(\cdot)$ is a kernel and h is a bandwidth.

Implementation of GMM on a set of suitably-chosen local moments - for each t - would deliver time-t estimates of the processes driving X_t as well as (possibly time-varying) estimates of the variances of the noise in prices and the measurement error in spot variances.

The use of the characteristic function in a method-of-moments procedure traces back at least to the work of Feuerverger and McDunnough (1981). More recent articles on the use of the characteristic function (and characteristic function-based method-of-moments procedures) for the estimation of continuous-time models are Singleton (2001), Jiang and Knight (2002), Chacko and Viceira (2003) and Carrasco, Chernov, Florens, and Ghysels (2007). Issues of efficiency are discussed in all papers. Carrasco, Chernov, Florens, and Ghysels (2007) provide a particularly appealing solution which exploits the continuity of the moment conditions delivered by the characteristic exponent(s) (u and v, in our notation above).

Differently from this line of work, our interest is not in estimating *parametric* continuous-time models for which the characteristic function is known in closed form, e.g., the class of affine models of which we provide an example in Section 4. Rather, consistent with recent developments in high-frequency econometrics (e.g. Bandi, Fusari, and Renò, 2023a, and the references therein), our objective is to treat the driving processes of the price process X_t as such (i.e., as processes), identify them point-wise (on a pre-specified time grid) and study their dynamics/interactions without imposing any specific structure, other then the W-differentiability properties spelled out earlier. The procedure is, therefore, nonparametric. It is made nonparametric by a conditional characteristic function expansion which is valid for all Brownian semi-martingales conforming with Assumption 1 or Assumption 2 and depends - in closed-form - to any specified order - on time-t values of the underlying processes (our object of econometric interest).

6 Adding discontinuities (in levels and volatility)

We now append discontinuities to the original Itô process in Section 3. In particular, we allow for co-jumps in levels and volatility (captured by the Poisson process $J^{X,\sigma}$) as well as for idiosyncratic discontinuities in both levels and volatility (captured by the Poisson processes J^X and J^σ). The corresponding intensities are $\lambda^{X,\sigma}$, λ^X and λ^σ. The jumps sizes are, instead, $c^{X,\sigma}$ (price co-jumps), $c^{\sigma,X}$ (volatility co-jumps), c^X (idiosyncratic price jumps) and c^σ (idiosyncratic volatility jumps).

Analogously to the diffusive characteristics in Assumption 1, we emphasize that all intensities, as well as the moments of the jump sizes, are processes. Their continuity properties are presented in Assumption 3, Eq. (38) and Eq. (39).

Assumption 3. *The adapted, real stochastic process X_t defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfies*

$$
\begin{aligned}
X_t &= X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t c_s^{X,\sigma} dJ_s^{X,\sigma} + \int_0^t c_s^X dJ_s^X, && (36)\\
\sigma_t &= \sigma_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s dW_s + \int_0^t \beta'_s dW'_s, + \int_0^t \widetilde{\beta}_t dW_s^\sigma \\
&\quad + \int_0^t c_s^{\sigma,X} dJ_s^{X,\sigma} + \int_0^t c_s^\sigma dJ_s^\sigma, && (37)\\
\mu_t &= \mu_0 + \int_0^t \gamma_s ds + \int_0^t \delta_s dW_s + \int_0^t \delta'_s dW_s^*, \\
\beta_t &= \beta_0 + \int_0^t \zeta_s ds + \int_0^t \eta_s dW_s + \int_0^t \eta'_s dW_s^{**}. \\
\alpha_t &= \alpha_0 + \int_0^t \xi_s ds + \int_0^t \theta_s dW_s + \int_0^t \theta'_s dW_s^{***}.
\end{aligned}
$$

$$\eta_t = \eta_0 + \int_0^t \xi'_s ds + \int_0^t \psi_s dW_s + \int_0^t \psi'_s dW_s^{****}.$$
$$\beta'_t = \beta'_0 + \int_0^t \zeta'_s ds + \int_0^t \kappa_s dW_s + \int_0^t \kappa'_s dW_s^{*****}.$$
$$\kappa_t = \kappa_0 + \int_0^t \xi''_s ds + \int_0^t \iota_s dW_s + \int_0^t \iota'_s dW_s^{******}.$$

The Brownian shocks and the driving characteristics satisfy the same properties as in Assumption 1. The processes J^X, $J^{X,\sigma}$ and J^σ are mutually independent Poisson processes (independent of W) with stochastic intensities λ_t^X, $\lambda_t^{X,\sigma}$ and λ_t^σ, respectively. The quantities c_t^X, $c_t^{X,\sigma}$, $c_t^{\sigma,X}$ and c_t^σ are stochastic, time-varying, jump sizes. For $0 \le s < t \le \Delta$ and for Δ small enough, there exists constants C, $\Gamma_\lambda > 1$ and $\Gamma_c > 2$ such that

$$\mathbb{E}_0\left[\left|\lambda_t^X - \lambda_s^X\right|^2 + \left|\lambda_t^{X,\sigma} - \lambda_s^{X,\sigma}\right|^2 + \left|\lambda_t^\sigma - \lambda_s^\sigma\right|^2\right] \le C|t-s|^{\Gamma_\lambda}, \tag{38}$$

$$\mathbb{E}_0\left[\left|c_t^X - c_s^X\right|^2 + \left|c_t^{X,\sigma} - c_s^{X,\sigma}\right|^2 + \left|c_t^{\sigma,X} - c_s^{\sigma,X}\right|^2 + \left|c_t^\sigma - c_s^\sigma\right|^2\right] \le C|t-s|^{\Gamma_c}. \tag{39}$$

The corresponding expansion is contained in Theorem 3.

Theorem 3. *Let Assumption 3 hold. If $\mu_t, \sigma_t \in \mathcal{D}_W^{(5)}$, the conditional characteristic function of Z_Δ and V_Δ with respect to time-0 information is expressed as follows:*

$$\begin{aligned}
\mathbb{E}_0\left[e^{iuZ_\Delta + ivV_\Delta}\right] &= \phi_0^c(u,v) + \Delta e^{-\frac{(u+v)^2}{2}} e^{-\frac{v^2}{2}\left(\frac{\beta'_0}{\beta_0}\right)^2}\left(\int_{\mathbb{R}}\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}x} - 1\right)\lambda_0^X f_0^X(dx)\right. \\
&\quad + \int_{\mathbb{R}^2}\int_0^1\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}x} e^{-\frac{u^2}{2}\left(\frac{y^2}{\sigma_0^2} + 2\frac{y}{\sigma_0}\right)z - \frac{uvy}{\sigma_0}z} - 1\right)\lambda_0^{X,\sigma} f_0^{X,\sigma}(dx,dy)dz \\
&\quad + \int_{\mathbb{R}}\int_0^1\left(e^{-\frac{u^2}{2}\left(\frac{y^2}{\sigma_0^2} + 2\frac{y}{\sigma_0}\right)z - \frac{uvy}{\sigma_0}z} - 1\right)\lambda_0^\sigma f_0^\sigma(dy)dz \\
&\quad \left. + \int_{\mathbb{R}}\left(e^{\frac{iv\beta'_0}{\beta_0^2\sqrt{\Delta}}x} - 1\right)\left(\lambda_0^\sigma f_0^\sigma(dx) + \lambda_0^{X,\sigma} f_0^{X,\sigma}(dx)\right)\right) + O(\Delta^{3/2})\widetilde{\phi}(u,v),
\end{aligned} \tag{40}$$

where $\phi_0^c(u,v)$ is the bivariate characteristic function expansion reported in Theorem 1, $f_0^X(\cdot,\cdot)$, $f_0^{X,\sigma}(\cdot,\cdot)$ and $f_0^\sigma(\cdot)$ are the density of c_0^X, the joint density of $c_0^{X,\sigma}$ and $c_0^{\sigma,X}$ and the density of c_0^σ, respectively. $\widetilde{\phi}(u,v)$ is an integrable function in u, v over $\mathbb{R}^2$ of order $u^{-3}v^{-2}$ as $|u| \to \infty$ and $|v| \to \infty$.

As earlier, for clarity, we are explicit about the univariate level case, i.e., the case $v = 0$.

Corollary 7 (to Theorem 3). *Let Assumption 3 hold. If $\mu_t, \sigma_t \in \mathcal{D}_W^{(4)}$, the conditional* *characteristic function of Z_Δ with respect to time-0 information is expressed as follows*

$$
\begin{aligned}
&\mathbb{E}_0\left[e^{iuZ_\Delta}\right] \\
= \quad & e^{-\frac{u^2}{2}}\left(1 - iu^3\frac{\widetilde{\beta}_0\rho_0}{2\sigma_0}\sqrt{\Delta} - \frac{1}{2}u^2\frac{\alpha_0+\delta_0}{\sigma_0}\Delta - \frac{1}{24}\frac{\widetilde{\beta}_0^2}{\sigma_0^2}u^2\left(6-4u^2+\rho_0^2u^2\left(3u^2-8\right)\right)\Delta + \frac{\eta_0}{6\sigma_0}u^4\Delta\right. \\
& +\Delta\int_{\mathbb{R}}\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}x}-1\right)\lambda_0^X f_0^X(dx) \\
& +\Delta\int_{\mathbb{R}^2}\int_0^1\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}x}e^{-\frac{u^2}{2}\left(\frac{y^2}{\sigma_0^2}+2\frac{y}{\sigma_0}\right)v}-1\right)\lambda_0^{X,\sigma}f_0^{X,\sigma}(dx,dy)dv \\
& \left.+\Delta\int_{\mathbb{R}}\int_0^1\left(e^{-\frac{u^2}{2}\left(\frac{z^2}{\sigma_0^2}+2\frac{z}{\sigma_0}\right)v}-1\right)\lambda_0^\sigma f_0^\sigma(dz)dv\right) + O(\Delta^{3/2})\widetilde{\phi}(u),
\end{aligned}
\tag{41}
$$

where $f_0^X(\cdot,\cdot)$, $f_0^{X,\sigma}(\cdot,\cdot)$ and $f_0^\sigma(\cdot)$ are the density of c_0^X, the joint density of $c_0^{X,\sigma}$ and $c_0^{\sigma,X}$ and the density of c_0^σ, respectively. $\widetilde{\phi}(u)$ is an integrable function in u over $\mathbb{R}$ of order u^{-2} as $|u|\to\infty$.

All discontinuities enter the expansion to the second order, i.e., the order Δ. They add kurtosis, as well as skewness, to the driving Gaussian characteristic function. The term in Eq. (41), in particular, provides an additional second-order skewness component, operating through the correlation between the jump sizes in levels and volatility when non-zero, which has been shown to be empirically-warranted in the case of market returns (Bandi and Renò, 2016).

We note that implementation of the expansion in Theorem 3 requires choices of the 0-densities of all jump sizes. The jump intensities, on the other hand, can be treated as parameters, like all other diffusive characteristics. In this sense, the expansion in Theorem 1 may be viewed as being fully nonparametric whereas the one in Theorem 3 - by necessity - hinges on a specification of the 0-densities. Issues of implementation are discussed in Bandi, Fusari, and Renò (2023a) and Bandi, Fusari, and Renò (2023b).

7 Conclusions

This article contributes to the rich literature on Edgeworth expansions, referenced in Section 1, by providing expansions of the conditional characteristic function of standardized level/volatility Δ-increments of continuous and discontinuous martingales for a vanishing Δ.

To date, the expansions have been used (in their univariate level-only versions) to derive asymptotic approximations (Bandi, Pirino, and Renò, 2017) as well as to price and hedge financial products, like options with short-term expirations, instruments whose traded volume has been exponentially increasing over the last few years (Bandi, Fusari, and Renò, 2023a, and Bandi, Fusari, and Renò, 2023b).

As previewed in Section 5, the full bivariate expansion(s) open up new possibilities, e.g., in the nonparametric identification of flexible (e.g., *non-affine*) price models. Section 4 provides the logic. A complete analysis is, however, currently under way. We will report on it at a later stage.

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A Appendix: Proofs

Proof of Lemma 1. Using Itô's Lemma,

$$e^{iuW_t}\chi_t = \chi_0 + \int_0^t e^{iuW_s} d\chi_s + \int_0^t \chi_s d\left(e^{iuW_s}\right) + \int_0^t d\left[\chi, e^{iuW}\right]_s .$$

Using Itô's Lemma again,

$$d\left(e^{iuW_s}\right) = iue^{iuW_s} dW_s - \frac{1}{2}u^2 e^{iuW_s} ds,$$

which implies

$$f_{\chi_t}(u) = \chi_0 + \mathbb{E}_0\left[\int_0^t e^{iuW_s} d\chi_s\right] - \frac{1}{2}u^2 \int_0^t f_{\chi_s}(u) ds + iu \int_0^t \mathbb{E}_0\left[e^{iuW_s} d[\chi, W]_s\right].$$

Solving the above ODE for $f_{\chi_t}(u)$ - as a function of t, for a fixed u - with initial condition $f_{\chi_0}(u) = \chi_0$, we obtain Eq. (1). □

We now present a result which will be used in the proof of Lemma 2.

Proposition 1. *Define the multiple integral:*

$$I_k(u,t) = \int_0^t \int_0^{s_1} \dots \int_0^{s_{k-1}} e^{\frac{u^2}{2}s_k} ds_1 \dots ds_k.$$

We have

$$I_k(u,t) = \frac{2^k}{u^{2k}}\left(e^{\frac{u^2}{2}t} - \sum_{j=0}^{k-1} \frac{1}{j!}\left(\frac{u^2}{2}t\right)^j\right) = \frac{2^k}{u^{2k}} e^{\frac{u^2}{2}t}\left(1 - \frac{\Gamma\left(k, \frac{u^2}{2}t\right)}{(k-1)!}\right),$$

where $\Gamma(k,x) = \int_x^\infty y^{k-1}e^{-y}dy$ is the upper incomplete gamma function.

Proof of Proposition 1.

$$\begin{aligned}
I_k(u,t) &= \int_0^t \int_0^{s_1} \dots \int_0^{s_{k-1}} e^{\frac{u^2}{2}s_k} ds_1 \dots ds_k \\
&= \frac{2}{u^2}\int_0^t \int_0^{s_1} \dots \int_0^{s_{k-2}} \left(e^{\frac{u^2}{2}s_{k-1}} - 1\right) ds_1 \dots ds_{k-1} \\
&= \frac{2^2}{u^4}\int_0^t \int_0^{s_1} \dots \int_0^{s_{k-3}} \left(e^{\frac{u^2}{2}s_{k-2}} - 1 - \frac{u^2}{2}s_{k-2}\right) ds_1 \dots ds_{k-2} \\
&= \frac{2^3}{u^6}\int_0^t \dots \int_0^{s_{k-4}} \left(e^{\frac{u^2}{2}s_{k-3}} - 1 - \frac{u^2}{2}s_{k-3} - \frac{1}{2}\left(\frac{u^2}{2}s_{k-3}\right)^2\right) ds_1 \dots ds_{k-3} \\
&\dots \\
&= \frac{2^k}{u^{2k}}\left(e^{\frac{u^2}{2}t} - \sum_{j=0}^{k-1} \frac{1}{j!}\left(\frac{u^2}{2}t\right)^j\right) = \frac{2^k}{u^{2k}} e^{\frac{u^2}{2}t}\left(1 - \frac{\Gamma\left(k, \frac{u^2}{2}t\right)}{(k-1)!}\right).
\end{aligned}$$

$\square$

Proof of Lemma 2. For convenience, we use the following notation for the process χ_t in Assumption 1:

$$d\chi_t = a(\chi)_t dt + b(\chi)_t dW_t.$$

Assume $\chi_t \in \mathcal{D}_W^k$. Iterating Lemma 1 k times, we have:

$$\begin{aligned}
f_{\chi_t}(u) &= e^{-\frac{u^2}{2}t}\left(\chi_0 + \int_0^t e^{\frac{u^2}{2}s_1}\mathbb{E}_0\left[e^{iuW_{s_1}}\left(a(\chi)_{s_1} + iub(\chi)_{s_1}\right)\right]ds_1\right) \\
&= e^{-\frac{u^2}{2}t}\left(\chi_0 + \left(\underbrace{a(\chi)_0 + iub(\chi)_0}_{\Xi_0^{(1)\chi}(u)}\right)t\right. \\
&\quad \left. + \int_0^t\int_0^{s_1} e^{\frac{u^2}{2}s_2}\mathbb{E}_0\left[e^{iuW_{s_2}}\left(\underbrace{a(a(\chi)+iub(\chi))_{s_2} + iub(a(\chi)+iub(\chi))_{s_2}}_{\Xi_{s_2}^{(2)\chi}(u)}\right)\right]ds_1 ds_2\right) \\
&= e^{-\frac{u^2}{2}t}\left(\sum_{j=0}^{k-1}\Xi_0^{(j)\chi}(u)\frac{t^j}{j!} + \int_0^t\int_0^{s_1}\cdots\int_0^{s_{k-1}} e^{\frac{u^2}{2}s_k}\mathbb{E}_0\left[e^{iuW_{s_k}}\Xi_{s_k}^{(k)\chi}(u)\right]ds_1\ldots ds_k\right) \\
&\le e^{-\frac{u^2}{2}t}\left(\sum_{j=m+1}^{k-1}\left|\Xi_0^{(j)\chi}(u)\right|\frac{t^j}{j!} + \int_0^t\int_0^{s_1}\cdots\int_0^{s_{k-1}} e^{\frac{u^2}{2}s_k}\mathbb{E}_0\left[\underbrace{\left|e^{iuW_{s_k}}\right|}_{=1}\left|\Xi_{s_k}^{(k)\chi}(u)\right|\right]ds_1\ldots ds_k\right),
\end{aligned}$$

where we defined $\Xi_t^{(0)\chi}(u) = \chi_t$ and

$$\Xi_t^{(k)\chi}(u) = a(\Xi_t^{(k-1)\chi}(u)) + iub(\Xi_t^{(k-1)\chi}(u)) \tag{42}$$

iteratively and, in the last equality, we used the fact that $\chi_t \in \mathcal{SD}_W^{(m)}$ with $m < k$. Next, we notice that

$$\left|\Xi_t^{(k)\chi}(u)\right| \le C(1+|u|)^k, \tag{43}$$

uniformly in t, since $\Xi_t^{(k)\chi}(u)$ is a polynomial in u of degree k with bounded coefficients. Thus, we have

$$|f_{\chi_t}(u)| \le Ce^{-\frac{u^2}{2}t}\left(\sum_{j=m+1}^{k-1}(1+|u|)^j\frac{t^j}{j!} + \underbrace{(1+|u|)^k\underbrace{\int_0^t\int_0^{s_1}\cdots\int_0^{s_{k-1}} e^{\frac{u^2}{2}s_k}ds_1\ldots ds_k}_{I_k(u,t)}}_{g_k(u,t)}\right).$$

This result, together with the representation of $I_k(u,t)$ in Lemma 1, completes the proof. $\square$

Lemma 3. *Under the Assumptions of Theorem 3, there exists a constant C such that, for Δ sufficiently small,*

$$\left|\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\Delta}}\left(\int_0^\Delta(\sigma_s-\sigma_0)dW_s\right)^k\right]\right|\leq Ck^{10}\Delta^k g_5(u,1), \tag{44}$$

where $k\geq 3$ and C does not depend on k, u or Δ.

Proof of Lemma 3. Define

$$Y_\Delta=M_\Delta^k=\left(\int_0^\Delta(\sigma_s-\sigma_0)dW_s\right)^k.$$

We have

$$\begin{aligned}
f_{Y_\Delta}(u)&=e^{-\frac{u^2}{2}}\int_0^\Delta e^{\frac{u^2}{2}\frac{s}{\Delta}}\mathbb{E}_0\left[e^{iu\frac{W_s}{\sqrt{\Delta}}}\left(\frac{k(k-1)}{2}M_s^{k-2}(\sigma_s-\sigma_0)^2+\frac{iu}{\sqrt{\Delta}}\left(kM_s^{k-1}(\sigma_s-\sigma_0)\right)\right)\right]ds\\
&=e^{-\frac{u^2}{2}}\int_0^\Delta e^{\frac{u^2}{2}\frac{s}{\Delta}}\mathbb{E}_0\left[e^{iu\frac{W_s}{\sqrt{\Delta}}}\Xi_s^{(1)Y}\left(\frac{u}{\sqrt{\Delta}}\right)\right]ds\\
&\leq e^{-\frac{u^2}{2}}\int_0^\Delta e^{\frac{u^2}{2}\frac{s}{\Delta}}\mathbb{E}_0\left[\left|e^{iu\frac{W_s}{\sqrt{\Delta}}}\right|\left|\Xi_s^{(1)Y}\left(\frac{u}{\sqrt{\Delta}}\right)\right|\right]ds.\\
&\leq\sup_{0\leq s\leq\Delta}\mathbb{E}_0\left[\left|\Xi_s^{(1)Y}\left(\frac{u}{\sqrt{\Delta}}\right)\right|\right]\frac{2\Delta}{u^2}\left(1-e^{-\frac{u^2}{2}}\right)\\
&\leq\left(\sup_{0\leq s\leq\Delta}\mathbb{E}_0\left[\left|a\left(\Xi_s^{(0)Y}\left(\frac{u}{\sqrt{\Delta}}\right)\right)\right|\right]+\left|\frac{u}{\sqrt{\Delta}}\right|\sup_{0\leq s\leq\Delta}\mathbb{E}_0\left[\left|b\left(\Xi_s^{(0)Y}\left(\frac{u}{\sqrt{\Delta}}\right)\right)\right|\right]\right)\frac{2\Delta}{u^2}\left(1-e^{-\frac{u^2}{2}}\right),
\end{aligned}$$

where the first equality is due to Lemma 1 and the last inequality uses the notation in Eq. (42). Now, write

$$\begin{aligned}
&\sup_{0\leq s\leq\Delta}\mathbb{E}_0\left[\left|\left(\int_0^s(\sigma_u-\sigma_0)dW_u\right)^p(\sigma_s-\sigma_0)^v\right|\right]\\
&\leq\left(\sup_{0\leq s\leq\Delta}\mathbb{E}_0\left[\left(\int_0^s(\sigma_u-\sigma_0)dW_u\right)^{2p}\right]\right)^{1/2}\left(\sup_{0\leq s\leq\Delta}\mathbb{E}_0\left[((\sigma_s-\sigma_0)^{2v})\right]\right)^{1/2}\\
&\sim\Delta^{p+v/2},
\end{aligned}\tag{45}$$

where the inequality is Hölder's and the asymptotic order derives from Eq. (46) and Eq. (47) below. First, note that

$$\begin{aligned}
\mathbb{E}_0\left[\left|\int_0^s(\sigma_u-\sigma_0)dW_u\right|^{2p}\right]&\leq\left(\sqrt{8p}\right)\mathbb{E}_0\left[\left(\int_0^s(\sigma_u-\sigma_0)^2du\right)^p\right]\\
&\leq\left(\sqrt{8p}\right)\Delta^{p-1}\int_0^s\mathbb{E}_0\left[(\sigma_u-\sigma_0)^{2p}\right]du\\
&\sim(8p)\,\Delta^{p-1}\int_0^s\left(\int_0^u\widetilde{\beta}_a da\right)^p du\sim\frac{8p}{p+1}\Delta^{2p}\sim\Delta^{2p},
\end{aligned}\tag{46}$$

where the first inequality and the first asymptotic order are Burkholder-Davis-Gundy's and the second inequality is Jensen's. Similarly, given Burkholder-Davis-Gundy's inequality, we have

$$\sup_{0\le s\le\Delta}\mathbb{E}_0\left[\left((\sigma_s-\sigma_0)^{2v}\right)\right]\le\sup_{0\le s\le\Delta}\left(\int_0^s\widetilde{\beta}_u du\right)^v\sim\Delta^v. \tag{47}$$

Using Eq. (45), we conclude that

$$\sup_{0\le s\le\Delta}\frac{k(k-1)}{2}\mathbb{E}_0\left[\left|M_s^{k-2}(\sigma_s-\sigma_0)^2\right|\right]=\sup_{0\le s\le\Delta}\mathbb{E}_0\left[\left|a\left(\Xi_s^{(0)Y}\left(\frac{u}{\sqrt{\Delta}}\right)\right)\right|\right]=O(k^2\Delta^{k-1}) \tag{48}$$

and

$$\sup_{0\le s\le\Delta}k\mathbb{E}_0\left[\left|M_s^{k-1}(\sigma_s-\sigma_0)\right|\right]=\sup_{0\le s\le\Delta}\mathbb{E}_0\left[\left|b\left(\Xi_s^{(0)Y}\left(\frac{u}{\sqrt{\Delta}}\right)\right)\right|\right]=O(k\Delta^{k-1/2}). \tag{49}$$

Hence, using Eq. (48) and Eq. (49), for $\Delta<1$ and small enough,

$$\begin{aligned}
f_{Y_\Delta}(u)\le&\left(\sup_{0\le s\le\Delta}\mathbb{E}_0\left[\left|a\left(\Xi_s^{(0)Y}\left(\frac{u}{\sqrt{\Delta}}\right)\right)\right|\right]+\left|\frac{u}{\sqrt{\Delta}}\right|\sup_{0\le s\le\Delta}\mathbb{E}_0\left[\left|b\left(\Xi_s^{(0)Y}\left(\frac{u}{\sqrt{\Delta}}\right)\right)\right|\right]\right)\frac{2\Delta}{u^2}\left(1-e^{-\frac{u^2}{2}}\right),\\
\sim&\left(k^2\Delta^{k-1}+\left|\frac{u}{\sqrt{\Delta}}\right|k\Delta^{k-1/2}\right)\frac{2\Delta}{u^2}\left(1-e^{-\frac{u^2}{2}}\right)\\
\sim&\, k^2\Delta^k(1+|u|)\frac{2}{u^2}\left(1-e^{-\frac{u^2}{2}}\right)\\
\sim&\, k^2\Delta^k g_1(u,1).
\end{aligned}$$

It is, now, easy to show that $Y_\Delta\in\mathcal{D}_W^{(5)}$. Thus, iterating the argument above four more times, we have

$$\begin{aligned}
f_{Y_\Delta}(u)&=e^{-\frac{u^2}{2}}\int_0^\Delta\int_0^{s_1}\cdots\int_0^{s_4}e^{\frac{u^2}{2}\frac{s_k}{\Delta}}\mathbb{E}_0\left[e^{iu\frac{W_{s_5}}{\sqrt{\Delta}}}\Xi_{s_5}^{(5)Y}\left(\frac{u}{\sqrt{\Delta}}\right)\right]ds_1\dots ds_5\\
&\le Ck^{10}\Delta^k g_5(u,1).
\end{aligned}$$

□

Lemma 4. *Consider a W-differentiable process χ_t defined as in Eq. (1). Consider, also, an independent Brownian motion W'_t. It holds that*

$$\mathbb{E}_0\left[e^{iuW_t}e^{ivW'_t}\chi_t\right]=e^{-\frac{u^2+v^2}{2}t}\left(\chi_0+\int_0^t e^{\frac{u^2+v^2}{2}s}\mathbb{E}_0\left[e^{iuW_s}e^{ivW'_s}\left(d\chi_s+iud[\chi,W]_s\right)\right]\right),$$

where $[\chi,W]_t$ is the quadratic covariation process between χ_t and W_t.

Proof of Theorem 1 and Corollary 3. In the proof, C denotes a generic constant which may change from line to line. We will denote the "first order" of the expansion as being the order in $\sqrt{\Delta}$ and the "second order" as being the order in Δ. The proof proceeds in two steps. In the first step, we prove the expansion for $v=0$, thereby illustrating how to use the W-transforms to derive all leading orders. We then generalize to the case $v\neq0$ in the second step.

Step 1. Set $v = 0$.

We begin with Eq. (16) in the main text, which we repeat here for convenience:

$$
\begin{aligned}
&\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}e^{iuY_\Delta\sqrt{\Delta}}\right]\\
&=\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\left(1+iuY_\Delta\sqrt{\Delta}+\frac{1}{2}(iu)^2Y_\Delta^2\Delta+\sum_{k=3}^{+\infty}\frac{(iu)^k}{k!}\Delta^{\frac{k}{2}}Y_\Delta^k\right)\right]\\
&=e^{-u^2/2}+\underbrace{\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}iuY_\Delta\sqrt{\Delta}\right]}_{A_\Delta}+\underbrace{\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\frac{1}{2}(iu)^2Y_\Delta^2\Delta\right]}_{B_\Delta}+\underbrace{\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\sum_{k=3}^{+\infty}\frac{(iu)^k}{k!}\Delta^{\frac{k}{2}}Y_\Delta^k\right]}_{C_\Delta}.
\end{aligned}
$$

We now study each term, one by one. First, consider the term A_Δ and write

$$
\begin{aligned}
A_\Delta &= \frac{iu}{\sigma_0\sqrt{\Delta}}\mathbb{E}_0\left[\left(\int_0^\Delta(\mu_s-\mu_0)ds\right)e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right]\\
&\quad+\frac{iu}{\sigma_0\sqrt{\Delta}}\mathbb{E}_0\left[\left(\int_0^\Delta\left(\int_0^s\alpha_{s_1}ds_1+\int_0^s\beta_{s_1}dW_{s_1}+\int_0^s\beta'_{s_1}dW'_{s_1}\right)dW_s\right)e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right]\\
&= A_{\Delta,0}+\frac{iu}{\sigma_0\sqrt{\Delta}}\mathbb{E}_0\left[\left(\int_0^\Delta(\beta_0W_s+\alpha_0 s+\right.\right.\\
&\quad\left.\left.+\int_0^s(\alpha_{s_1}-\alpha_0)ds_1+\int_0^s(\beta_{s_1}-\beta_0)dW_{s_1}+\int_0^s\beta'_{s_1}dW'_{s_1}\right)dW_s\right)e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right]\\
&= A_{\Delta,0}+A_{\Delta,1}+A_{\Delta,2}+A_{\Delta,3}+A_{\Delta,4}+A_{\Delta,5},
\end{aligned}
$$

where the second equality is the result of Eq. (37). We note that, using the definition of W-transform in Eq. (2), we may write

$$
f_{\int_0^\Delta(\mu_s-\mu_0)ds}\left(\frac{u}{\sqrt{\Delta}}\right)=\mathbb{E}_0\left[\left(\int_0^\Delta(\mu_s-\mu_0)ds\right)e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right],
$$

and apply Lemma 1. Similar functions f will be defined throughout, before applying Lemma 1. We will, therefore, not repeat their definition, which will be obvious, and only invoke the lemma. By Lemma 1, we have

$$
\begin{aligned}
A_{\Delta,0} &= \frac{iu}{\sigma_0\sqrt{\Delta}}e^{-\frac{u^2\Delta}{2\Delta}}\int_0^\Delta\mathbb{E}_0\left[e^{iu\frac{W_s}{\sqrt{\Delta}}}(\mu_s-\mu_0)\right]e^{\frac{u^2s}{2\Delta}}ds\\
&= \frac{iu}{\sigma_0\sqrt{\Delta}}e^{-\frac{u^2}{2}}\int_0^\Delta\mathbb{E}_0\left[e^{iu\frac{W_s}{\sqrt{\Delta}}}\left(\delta_0W_s+\int_0^s\gamma_{s_1}ds_1+\int_0^s(\delta_{s_1}-\delta_0)dW_{s_1}+\int_0^s\delta'_{s_1}dW^*_{s_1}\right)\right]e^{\frac{u^2s}{2\Delta}}ds\\
&= \overline{A}^0_{\Delta,0}+\overline{A}^1_{\Delta,0}+\overline{A}^2_{\Delta,0}+\overline{A}^3_{\Delta,0},
\end{aligned}
$$

where the second equality derives from Eq. (38). The dominating term (i.e., the term converging to zero at the slowest rate in Δ) among these four is the first:

$$
\overline{A}^0_{\Delta,0}=\frac{u\delta_0}{\sigma_0\sqrt{\Delta}}e^{-\frac{u^2}{2}}\int_0^\Delta\mathbb{E}_0\left[e^{iu\frac{W_s}{\sqrt{\Delta}}}iW_s\right]e^{\frac{u^2s}{2\Delta}}ds
$$

$$
\begin{aligned}
&= \frac{u\delta_0}{\sigma_0\sqrt{\Delta}} e^{-\frac{u^2}{2}} \int_0^\Delta \left[-\frac{su}{\sqrt{\Delta}} e^{\frac{-su^2}{2\Delta}} \right] e^{\frac{u^2 s}{2\Delta}} ds \\
&= -\frac{u^2\delta_0}{2\sigma_0} e^{-\frac{u^2}{2}} \Delta,
\end{aligned}
$$

where the first equality is definitional and the second equality derives from the properties of the characteristic function of the Gaussian random variable. We know, in fact, that if $X \sim N(\mu, \sigma^2)$, the first derivative (with respect to u) of the characteristic function of X is equal to

$$
\frac{\partial \phi_X(u)}{\partial u} = \mathbb{E}_0 \left[iX e^{iuX} \right] = (i\mu - \sigma^2 u) e^{iu\mu - \frac{\sigma^2 u^2}{2}}. \tag{50}
$$

The term $\overline{A}^0_{\Delta,0}$ will, therefore, matter to the second order. We now show that all other terms converge to zero faster than $\overline{A}^0_{\Delta,0}$. We begin with $\overline{A}^2_{\Delta,0}$. Lemma 1, applied to this term, yields

$$
\begin{aligned}
\overline{A}^2_{\Delta,0} &= \frac{iu}{\sigma_0\sqrt{\Delta}} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \mathbb{E}_0 \left[e^{iu\frac{W_{s_1}}{\sqrt{\Delta}}} \left(i\frac{u}{\sqrt{\Delta}} (\delta_{s_1} - \delta_0) ds_1 \right) \right] e^{\frac{u^2 s_1}{2\Delta}} ds_1 ds \\
&= -\frac{u^2}{\sigma_0 \Delta} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \mathbb{E}_0 \left[e^{iu\frac{W_{s_1}}{\sqrt{\Delta}}} (\delta_{s_1} - \delta_0) \right] e^{\frac{u^2 s_1}{2\Delta}} ds_1 ds.
\end{aligned}
$$

It is now sufficient to assume that $\delta_t - \delta_0 \in \mathcal{D}^{(1)}_W$ (which is guaranteed by the assumption $X_t \in \mathcal{D}^{(5)}_W$) to show that this term is negligible. Using, again, Lemma 1 and the representation in Proposition 1, we have

$$
\left| \overline{A}^2_{\Delta,0} \right| \leq C \frac{u^2}{\sigma_0 \Delta} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s g_1 \left(\frac{u}{\sqrt{\Delta}}, s_1 \right) e^{\frac{u^2 s_1}{2\Delta}} ds_1 ds. \tag{51}
$$

Now, write

$$
\begin{aligned}
\int_0^\Delta \int_0^s g_1 \left(\frac{u}{\sqrt{\Delta}}, s_1 \right) e^{\frac{u^2}{2}\frac{s_1}{\Delta}} ds_1 ds &= \int_0^\Delta \int_0^s \frac{2\Delta}{u^2} \left(1 + \frac{|u|}{\sqrt{\Delta}} \right) \left(e^{\frac{u^2 s_1}{2\Delta}} - 1 \right) ds_1 ds \\
&= \left(1 + \frac{|u|}{\sqrt{\Delta}} \right) \int_0^\Delta \frac{4\Delta^2}{u^4} \left(e^{\frac{u^2 s}{2\Delta}} - 1 - \frac{u^2 s}{2\Delta} \right) ds \\
&= \left(1 + \frac{|u|}{\sqrt{\Delta}} \right) \frac{8\Delta^3}{u^6} \left(e^{\frac{u^2 \Delta}{2\Delta}} - 1 - \frac{u^2 \Delta}{2\Delta} - \frac{u^4 \Delta^2}{2\Delta^2} \right) \\
&= \left(1 + \frac{|u|}{\sqrt{\Delta}} \right)^{-2} e^{\frac{u^2 \Delta}{2\Delta}} g_3 \left(\frac{u}{\sqrt{\Delta}}, \Delta \right),
\end{aligned}
$$

which yields

$$
\left| \overline{A}^2_{\Delta,0} \right| \leq C \frac{u^2}{\sigma_0 \Delta} \left(1 + \frac{|u|}{\sqrt{\Delta}} \right)^{-2} g_3 \left(\frac{u}{\sqrt{\Delta}}, \Delta \right).
$$

Notice that, for integers $m \geq 0$ and $k \geq m$, and for $\Delta < 1$,

$$
\left(1 + \frac{|u|}{\sqrt{\Delta}} \right)^{-m} g_k \left(\frac{u}{\sqrt{\Delta}}, \Delta \right) = 2^k \Delta^k \frac{\left(1 + \frac{|u|}{\sqrt{\Delta}} \right)^{k-m}}{u^{2k}} \left(1 - \frac{\Gamma\left(k, \frac{u^2}{2}\right)}{(k-1)!} \right)
$$

$$
\begin{aligned}
&\leq \; 2^k \Delta^k \frac{\left(\frac{1+|u|}{\sqrt{\Delta}}\right)^{k-m}}{u^{2k}} \left(1 - \frac{\Gamma\left(k, \frac{u^2}{2}\right)}{(k-1)!}\right) \\
&\leq \; \Delta^{\frac{m+k}{2}} (1+|u|)^{-m} g_k(u,1), \qquad (52)
\end{aligned}
$$

where, in the second inequality, we used the fact that, for $\Delta < 1$, $1 + \frac{u}{\sqrt{\Delta}} \leq \frac{1+u}{\sqrt{\Delta}}$. We conclude that

$$
\left|\overline{A}^2_{\Delta,0}\right| \leq C \frac{u^2}{\sigma_0} (1+|u|)^{-2} g_3(u,1)\, \Delta^{3/2}.
$$

Thus, $\overline{A}^2_{\Delta,0}$ is of order $\Delta^{3/2}$ and we can control the bound in u.

We now turn to $\overline{A}^1_{\Delta,0}$. By using Lemma 1, before adding and subtracting γ_0, we obtain

$$
\begin{aligned}
\overline{A}^1_{\Delta,0} =& \frac{iu}{\sigma_0\sqrt{\Delta}} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \mathbb{E}_0 \left[e^{iu\frac{W_{s_1}}{\sqrt{\Delta}}} \gamma_{s_1} \right] e^{\frac{u^2}{2}\frac{s_1}{\Delta}} ds_1 ds \\
=& \underbrace{\frac{iu}{\sigma_0\sqrt{\Delta}} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \mathbb{E}_0 \left[e^{iu\frac{W_{s_1}}{\sqrt{\Delta}}} (\gamma_{s_1} - \gamma_0) \right] e^{\frac{u^2}{2}\frac{s_1}{\Delta}} ds_1 ds}_{\overline{A}^{1,a}_{\Delta,0}} + \gamma_0 \frac{iu}{\sigma_0\sqrt{\Delta}} e^{-\frac{u^2}{2}} \frac{1}{2}\Delta^2.
\end{aligned}
$$

The second term is of order $\Delta^{3/2}$ and, because $\gamma_t - \gamma_0 \in \mathcal{D}^{(1)}_W$ (which is, again, guaranteed by the assumption $X_t \in \mathcal{D}^{(5)}_W$), we have

$$
\begin{aligned}
\left|\overline{A}^{1,a}_{\Delta,0}\right| \leq& \; C \frac{|u|}{\sigma_0\sqrt{\Delta}} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s g_1\left(\frac{u}{\sqrt{\Delta}}, s_1\right) e^{\frac{u^2}{2}\frac{s_1}{\Delta}} ds_1 ds \\
=& \; C \frac{|u|}{\sigma_0\sqrt{\Delta}} e^{-\frac{u^2}{2}} \left(1 + \frac{|u|}{\sqrt{\Delta}}\right)^{-2} e^{\frac{u^2}{2}} g_3\left(\frac{u}{\sqrt{\Delta}}, \Delta\right) \\
\leq& \; C \frac{|u|}{\sigma_0} (1+|u|)^{-2} g_3(u,1)\, \Delta^2,
\end{aligned}
$$

where, in the last inequality, we used the reasoning leading to Eq. (52). Thus, the full term $\overline{A}^1_{\Delta,0}$ is of order $\Delta^{3/2}$ and we can control the bound in u.

As for the term driven by the orthogonal Brownian motion W^*, Lemma 1 yields

$$
\overline{A}^3_{\Delta,0} = -\frac{u^2}{\sigma_0\Delta} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \mathbb{E}_0 \left[e^{iu\frac{W_{s_1}}{\sqrt{\Delta}}} \delta'_{s_1} d[W, W^*]_{s_1} \right] e^{\frac{u^2}{2}\frac{s_1}{\Delta}} ds = 0.
$$

We now turn to $A_{\Delta,1}$. We make use again of Lemma 1 to obtain

$$
\begin{aligned}
A_{\Delta,1} =& \frac{iu\beta_0}{\sigma_0\sqrt{\Delta}} \mathbb{E}_0 \left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}} \left(\int_0^\Delta W_s dW_s \right) \right] \\
=& \frac{iu\beta_0}{\sigma_0\sqrt{\Delta}} e^{-\frac{u^2\Delta}{2\Delta}} \int_0^\Delta e^{\frac{u^2 s}{2\Delta}} \mathbb{E}_0 \left[\frac{iu}{\sqrt{\Delta}} e^{\frac{iuW_s}{\sqrt{\Delta}}} W_s ds \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{iu^2\beta_0}{\sigma_0\Delta} e^{-\frac{u^2}{2}} \int_0^\Delta e^{\frac{u^2 s}{2\Delta}} \mathbb{E}_0\left[iW_s e^{\frac{iuW_s}{\sqrt{\Delta}}}\right] ds \\
&= \frac{iu^2\beta_0}{\sigma_0\Delta} e^{-\frac{u^2}{2}} \int_0^\Delta e^{\frac{u^2 s}{2\Delta}} \left[-\frac{su}{\sqrt{\Delta}} e^{\frac{-su^2}{2\Delta}}\right] ds \\
&= \frac{\beta_0}{2\sigma_0}\sqrt{\Delta}\left(-iu^3 e^{-\frac{u^2}{2}}\right),
\end{aligned}
$$

where the penultimate equality hinges, again, on Eq. (50). This term will matter to the first order. Turning to $A_{\Delta,2}$, using Lemma 1, write

$$
\begin{aligned}
A_{\Delta,2} &= \frac{iu\alpha_0}{\sigma_0\sqrt{\Delta}} \mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}} \int_0^\Delta s dW_s\right] \\
&= \frac{iu\alpha_0}{\sigma_0\sqrt{\Delta}} \frac{iu}{\sqrt{\Delta}} e^{-\frac{u^2\Delta}{2\Delta}} \int_0^\Delta \mathbb{E}_0\left[s e^{iu\frac{W_s}{\sqrt{\Delta}}}\right] e^{\frac{u^2}{2}\frac{s}{\Delta}} ds \\
&= \frac{iu\alpha_0}{\sigma_0\sqrt{\Delta}} \frac{iu}{\sqrt{\Delta}} e^{-\frac{u^2}{2}} \int_0^\Delta s e^{-\frac{u^2}{2}\frac{s}{\Delta}} e^{\frac{u^2}{2}\frac{s}{\Delta}} ds \\
&= -\frac{\alpha_0}{2\sigma_0}\Delta u^2 e^{-\frac{u^2}{2}}.
\end{aligned}
$$

This term will enter the second order of the expansion. Next, using Lemma 1 twice,

$$
\begin{aligned}
A_{\Delta,3} =& \frac{iu}{\sigma_0\sqrt{\Delta}} \mathbb{E}_0\left[\left(\int_0^\Delta \left(\int_0^s (\alpha_{s_1}-\alpha_0) ds_1\right) dW_s\right) e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right] \\
=& \frac{iu}{\sigma_0\sqrt{\Delta}} \frac{iu}{\sqrt{\Delta}} e^{-\frac{u^2}{2}} \int_0^\Delta \mathbb{E}_0\left[\left(\int_0^s (\alpha_{s_1}-\alpha_0) ds_1\right) e^{iu\frac{W_s}{\sqrt{\Delta}}}\right] e^{\frac{u^2}{2}\frac{s}{\Delta}} ds \\
=& -\frac{u^2}{\sigma_0\Delta} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \mathbb{E}_0\left[(\alpha_{s_1}-\alpha_0) e^{iu\frac{W_{s_1}}{\sqrt{\Delta}}}\right] e^{\frac{u^2}{2}\frac{s_1}{\Delta}} ds_1 ds.
\end{aligned}
$$

Because $\alpha_s - \alpha_0 \in \mathcal{D}_W^{(1)}$ (since $X_t \in \mathcal{D}_W^{(5)}$), this term can be handled exactly as the term $\overline{A}^2_{\Delta,0}$ above. Like in the case of $\overline{A}^2_{\Delta,0}$, its order is $\Delta^{3/2}$. Similarly, using Lemma 1 twice again, we obtain

$$
\begin{aligned}
&A_{\Delta,4} = \frac{-iu^3}{\sigma_0\Delta^{3/2}} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \mathbb{E}_0\left[(\beta_{s_1}-\beta_0) e^{iu\frac{W_{s_1}}{\sqrt{\Delta}}}\right] e^{\frac{u^2}{2}\frac{s_1}{\Delta}} ds_1 ds \\
&= \frac{-iu^3}{\sigma_0\Delta^{3/2}} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \mathbb{E}_0\left[\left(\eta_0 W_{s_1} + \int_0^{s_1} (\eta_{s_2}-\eta_0) dW_{s_2} + \int_0^{s_1} \zeta_{s_2} ds_2 + \int_0^{s_1} \eta'_{s_2} dW^{**}_{s_2}\right) e^{iu\frac{W_{s_1}}{\sqrt{\Delta}}}\right] e^{\frac{u^2}{2}\frac{s_1}{\Delta}} ds_1 ds \\
&= \overline{A}^0_{\Delta,4} + \overline{A}^1_{\Delta,4} + \overline{A}^2_{\Delta,4} + \overline{A}^3_{\Delta,4},
\end{aligned}
$$

where the second equality is the result of Eq. (38). For the first term, we have

$$
\begin{aligned}
\overline{A}^0_{\Delta,4} =& \frac{-iu^3\eta_0}{\sigma_0\Delta^{3/2}} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \mathbb{E}_0\left[W_{s_1} e^{iu\frac{W_{s_1}}{\sqrt{\Delta}}}\right] e^{\frac{u^2 s_1}{2\Delta}} ds_1 ds \\
=& \frac{-u^3\eta_0}{\sigma_0\Delta^{3/2}} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \left[-\frac{s_1 u}{\sqrt{\Delta}} e^{\frac{-s_1 u^2}{2\Delta}}\right] e^{\frac{u^2 s_1}{2\Delta}} ds_1 ds \\
=& \frac{u^4\eta_0}{6\sigma_0} e^{-\frac{u^2}{2}} \Delta,
\end{aligned}
$$

where Eq. (50) is used in the second equality. This term will enter the expansion to the second order. The remaining terms in $A_{\Delta,4}$ are, instead, negligible. Using Lemma 1 once more,

$$\overline{A}^1_{\Delta,4} = \frac{u^4}{\sigma_0 \Delta^2} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \int_0^{s_1} \mathbb{E}_0 \left[(\eta_{s_2} - \eta_0) e^{iu \frac{W_{s_2}}{\sqrt{\Delta}}} \right] e^{\frac{u^2 s_2}{2\Delta}} ds_2 ds_1 ds.$$

Now, following the same steps leading to Eq. (51), the condition $\eta_t - \eta_0 \in \mathcal{D}^{(1)}_W$ (which is implied by $X_t \in \mathcal{D}^{(5)}_W$) yields

$$\begin{aligned}
\left| \overline{A}^1_{\Delta,4} \right| &\leq C \frac{u^4}{\sigma_0 \Delta^2} \left(1 + \frac{|u|}{\sqrt{\Delta}}\right)^{-3} g_4\left(\frac{u}{\sqrt{\Delta}}, \Delta\right) \\
&\leq C \frac{u^4}{\sigma_0 \Delta^2} \Delta^{\frac{4+3}{2}} (1 + |u|)^{-3} g_4(u, 1) \\
&\leq C \frac{u^4}{\sigma_0} \Delta^{\frac{3}{2}} (1 + |u|)^{-3} g_4(u, 1),
\end{aligned}$$

where the second inequality, in particular, derives from Eq. (52). Now, write

$$\begin{aligned}
\overline{A}^2_{\Delta,4} &= \frac{-iu^3}{\sigma_0 \Delta^{3/2}} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \mathbb{E}_0 \left[\left(\int_0^{s_1} \zeta_{s_2} ds_2 \right) e^{iu \frac{W_{s_1}}{\sqrt{\Delta}}} \right] e^{\frac{u^2}{2} \frac{s_1}{\Delta}} ds_1 ds \\
&= \frac{-iu^3}{\sigma_0 \Delta^{3/2}} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \int_0^{s_1} \mathbb{E}_0 \left[e^{iu \frac{W_{s_2}}{\sqrt{\Delta}}} \zeta_{s_2} \right] e^{\frac{u^2}{2} \frac{s_2}{\Delta}} ds_2 ds_1 ds \\
&= \frac{-iu^3}{\sigma_0 \Delta^{3/2}} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \int_0^{s_1} \mathbb{E}_0 \left[e^{iu \frac{W_{s_2}}{\sqrt{\Delta}}} (\zeta_{s_2} - \zeta_0) \right] e^{\frac{u^2}{2} \frac{s_2}{\Delta}} ds_2 ds_1 ds + \\
&\quad + \frac{-iu^3}{\sigma_0 \Delta^{3/2}} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \int_0^{s_1} \mathbb{E}_0 \left[e^{iu \frac{W_{s_2}}{\sqrt{\Delta}}} \zeta_0 \right] e^{\frac{u^2}{2} \frac{s_2}{\Delta}} ds_2 ds_1 ds \\
&= \frac{-iu^3}{\sigma_0 \Delta^{3/2}} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \int_0^{s_1} \mathbb{E}_0 \left[e^{iu \frac{W_{s_2}}{\sqrt{\Delta}}} (\zeta_{s_2} - \zeta_0) \right] e^{\frac{u^2}{2} \frac{s_2}{\Delta}} ds_2 ds_1 ds + \\
&\quad + \frac{-iu^3 \zeta_0}{6\sigma_0} e^{-\frac{u^2}{2}} \Delta^{3/2} \\
&\leq \frac{|u^3|}{\sigma_0 \Delta^{3/2}} \left(1 + \frac{|u|}{\sqrt{\Delta}}\right)^{-3} g_4\left(\frac{u}{\sqrt{\Delta}}, \Delta\right) + \frac{|u^3| \zeta_0}{6\sigma_0} e^{-\frac{u^2}{2}} \Delta^{3/2} \\
&\leq \frac{|u^3|}{\sigma_0 \Delta^{3/2}} \Delta^{\frac{4+3}{2}} (1 + |u|)^{-3} g_4(u, 1) + \frac{|u^3| \zeta_0}{6\sigma_0} e^{-\frac{u^2}{2}} \Delta^{3/2} \\
&\leq \frac{|u^3|}{\sigma_0} \Delta^2 (1 + |u|)^{-3} g_4(u, 1) + \frac{|u^3| \zeta_0}{6\sigma_0} e^{-\frac{u^2}{2}} \Delta^{3/2},
\end{aligned}$$

where the first equality is due to Lemma 1, the second equality derives from adding and subtracting ζ_0, the first inequality is the result of the same argument leading to Eq. (51) and the second inequality derives from Eq. (52). Also, by Lemma 1 again

$$\begin{aligned}
\overline{A}^3_{\Delta,4} &= \frac{-iu^3}{\sigma_0 \Delta^{3/2}} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \mathbb{E}_0 \left[\left(\int_0^{s_1} \eta'_{s_2} dW^{**}_{s_2} \right) e^{iu \frac{W_{s_1}}{\sqrt{\Delta}}} \right] e^{\frac{u^2}{2} \frac{s_1}{\Delta}} ds_1 ds \\
&= \frac{u^4}{\sigma_0 \Delta^2} e^{-\frac{u^2}{2}} \int_0^\Delta \int_0^s \int_0^{s_1} \mathbb{E}_0 \left[e^{iu \frac{W_{s_2}}{\sqrt{\Delta}}} \eta'_{s_2} d[W, W^{**}]_{s_2} \right] e^{\frac{u^2}{2} \frac{s_2}{\Delta}} ds_2 ds_1 ds
\end{aligned}$$

$$= 0,$$

because of the orthogonality between W and W^{**}. Finally, the term $A_{\Delta,5}$ can be treated in a similar fashion. By applying Lemma 1 twice, we have

$$\begin{aligned}
A_{\Delta,5} &= \frac{iu}{\sigma_0\sqrt{\Delta}}\mathbb{E}_0\left[\left(\int_0^\Delta\left(\int_0^s \beta'_{s_1}dW'_{s_1}\right)dW_s\right)e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right]\\
&= \frac{-u^2}{\sigma_0\Delta}e^{-\frac{u^2}{2}}\int_0^\Delta \mathbb{E}_0\left[e^{iu\frac{W_s}{\sqrt{\Delta}}}\int_0^s \beta'_{s_1}dW'_{s_1}\right]e^{\frac{u^2 s}{2\Delta}}ds\\
&= \frac{-iu^3}{\sigma_0\Delta^{3/2}}e^{-\frac{u^2}{2}}\int_0^\Delta\int_0^s \mathbb{E}_0\left[e^{iu\frac{W_{s_1}}{\sqrt{\Delta}}}\beta'_{s_1}d[W,W']_{s_1}\right]e^{\frac{u^2 s_1}{2\Delta}}ds_1 ds\\
&= 0,
\end{aligned}$$

because of the orthogonality between W and W'.

We now turn to the term B_Δ. Write

$$\begin{aligned}
B_\Delta =& -\frac{1}{2}\frac{u^2}{\sigma_0^2\Delta}\mathbb{E}_0\left[\left(\int_0^\Delta(\mu_s-\mu_0)ds+\int_0^\Delta(\sigma_s-\sigma_0)dW_s\right)^2 e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right]\\
=& -\frac{1}{2}\frac{u^2}{\sigma_0^2\Delta}\mathbb{E}_0\left[\left(\alpha_0^2\left(\int_0^\Delta sdW_s\right)^2+\beta_0^2\left(\int_0^\Delta W_sdW_s\right)^2+(\beta'_0)^2\left(\int_0^\Delta W'_sdW_s\right)^2\right.\right.\\
&+\left(\int_0^\Delta \psi_s^{d\sigma}dW_s\right)^2+2\alpha_0\beta_0\left(\int_0^\Delta sdW_s\right)\left(\int_0^\Delta W_sdW_s\right)\\
&+2\alpha_0\beta'_0\left(\int_0^\Delta sdW_s\right)\left(\int_0^\Delta W'_sdW_s\right)\\
&+2\beta_0\beta'_0\left(\int_0^\Delta W'_sdW_s\right)\left(\int_0^\Delta W_sdW_s\right)\\
&+2\left(\int_0^\Delta \psi_s^{d\sigma}dW_s\right)\left(\int_0^\Delta\left(\alpha_0 s+\beta_0W_s+\beta'_0W'_s\right)dW_s\right)+\left(\int_0^\Delta(\mu_s-\mu_0)ds\right)^2\\
&\left.\left.+2\left(\int_0^\Delta(\mu_s-\mu_0)ds\right)\left(\int_0^\Delta(\sigma_s-\sigma_0)dW_s\right)\right)e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right]\\
=&B_{\Delta,1}+B_{\Delta,2}+B_{\Delta,3}+B_{\Delta,4}+B_{\Delta,5}+B_{\Delta,6}+B_{\Delta,7}+B_{\Delta,8}+B_{\Delta,9}+B_{\Delta,10}.
\end{aligned}$$

where

$$\psi_s^{d\sigma}=\int_0^s(\alpha_{s_1}-\alpha_0)ds_1+\int_0^s(\beta_{s_1}-\beta_0)dW_{s_1}+\int_0^s(\beta'_{s_1}-\beta'_0)dW'_{s_1}.$$

Given Itô's lemma, $d\left(\int_0^s udW_u\right)^2=2\left(\int_0^s udW_u\right)sdW_s+s^2ds$. Thus, by Lemma 1, which we use twice, we obtain

$$B_{\Delta,1}=-\frac{1}{2}\frac{u^2}{\sigma_0^2\Delta}\mathbb{E}_0\left[\alpha_0^2\left(\int_0^\Delta sdW_s\right)^2 e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right]$$

$$
\begin{aligned}
=&-\frac{1}{2}\frac{u^2}{\sigma_0^2\Delta}\alpha_0^2 e^{-\frac{u^2\Delta}{2\Delta}}\int_0^\Delta \mathbb{E}_0\left[s^2 e^{iu\frac{W_s}{\sqrt{\Delta}}}\right]e^{\frac{u^2 s}{2\Delta}}ds\\
&-\frac{2}{2}\frac{u^2}{\sigma_0^2\Delta}\alpha_0^2\frac{iu}{\sqrt{\Delta}}e^{-\frac{u^2\Delta}{2\Delta}}\int_0^\Delta \mathbb{E}_0\left[se^{iu\frac{W_s}{\sqrt{\Delta}}}\left(\int_0^s udW_u\right)\right]e^{\frac{u^2 s}{2\Delta}}ds\\
=&-\frac{1}{6}\frac{u^2}{\sigma_0^2\Delta}\alpha_0^2 e^{-\frac{u^2}{2}}\left(s^3|_0^\Delta\right)\\
&-\frac{iu^3}{\sigma_0^2\Delta^{3/2}}\alpha_0^2 e^{-\frac{u^2}{2}}\int_0^\Delta s\frac{iu}{\sqrt{\Delta}}\int_0^s \mathbb{E}_0\left[ue^{iu\frac{W_u}{\sqrt{\Delta}}}\right]e^{\frac{u^2 s_2}{2\Delta}}duds\\
=&-\frac{1}{6}\frac{u^2}{\sigma_0^2}\alpha_0^2 e^{-\frac{u^2}{2}}\Delta^2+\frac{u^4}{2\sigma_0^2\Delta^2}\alpha_0^2 e^{-\frac{u^2}{2}}\int_0^\Delta s^3 ds\\
=&-\frac{1}{6}\frac{u^2}{\sigma_0^2}\alpha_0^2 e^{-\frac{u^2}{2}}\Delta^2+\frac{u^4}{8\sigma_0^2}\alpha_0^2 e^{-\frac{u^2}{2}}\Delta^2\\
=&\frac{\alpha_0^2}{\sigma_0^2}e^{-\frac{u^2}{2}}\left(\frac{u^4}{8}-\frac{u^2}{6}\right)\Delta^2.
\end{aligned}
$$

We now turn to $B_{\Delta,2}$. Using the fact that $\int_0^\Delta W_s dW_s=\frac{1}{2}W_\Delta^2-\frac{1}{2}\Delta$, we have

$$
\begin{aligned}
B_{\Delta,2}=&-\frac{1}{2}\frac{u^2}{\sigma_0^2\Delta}\mathbb{E}_0\left[\beta_0^2\left(\int_0^\Delta W_s dW_s\right)^2 e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right]\\
=&-\frac{1}{2}\frac{u^2}{\sigma_0^2\Delta}\mathbb{E}_0\left[\beta_0^2\left(\frac{1}{2}W_\Delta^2-\frac{1}{2}\Delta\right)^2 e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right]\\
=&-\frac{1}{2}\frac{\beta_0^2}{\sigma_0^2}\frac{u^2}{\Delta}\mathbb{E}_0\left[\left(\frac{1}{4}W_\Delta^4+\frac{1}{4}\Delta^2-\frac{1}{2}W_\Delta^2\Delta\right)e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right]\\
=&-\frac{1}{2}\frac{\beta_0^2}{\sigma_0^2}\frac{u^2}{\Delta}\Delta^2\mathbb{E}_0\left[\left(\frac{1}{4}\left(\frac{i}{\sqrt{\Delta}}\right)^4 W_\Delta^4+\frac{1}{4}+\frac{1}{2}\left(\frac{i}{\sqrt{\Delta}}\right)^2 W_\Delta^2\right)e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right]\\
=&-\frac{1}{2}\frac{\beta_0^2}{\sigma_0^2}u^2\Delta e^{-\frac{u^2}{2}}\left[\frac{1}{4}\left(3-6u^2+u^4\right)+\frac{1}{4}+\frac{1}{2}\left(u^2-1\right)\right]\\
=&-\frac{1}{2}\frac{\beta_0^2}{\sigma_0^2}u^2\Delta e^{-\frac{u^2}{2}}\left[\frac{1}{4}u^4+\frac{1}{2}-u^2\right]\\
=&-\frac{1}{8}\frac{\beta_0^2}{\sigma_0^2}e^{-\frac{u^2}{2}}u^2(2-4u^2+u^4)\Delta,
\end{aligned}
$$

where the third equality from the bottom derives (like in the case of Eq. (50)) from the properties of the derivatives (second and fourth, here) of the Gaussian characteristic function. This term will matter to the second order.

We now turn to $B_{\Delta,3}$. Recall that, by Itô's lemma, we may write

$$
\left(\int_0^\Delta W_s' dW_s\right)^2=2\int_0^\Delta\left(\int_0^s W_{s_1}' dW_{s_1}\right)W_s' dW_s+\int_0^\Delta\left(W_s'\right)^2 ds, \tag{53}
$$

and

$$
W_s'\left(\int_0^s W_{s_1}' dW_{s_1}\right)=\int_0^s\left(W_{s_1}'\right)^2 dW_{s_1}+\int_0^s\left(\int_0^{s_1} W_{s_2}' dW_{s_2}\right)dW_{s_1}', \tag{54}
$$

where the second equation uses the orthogonality between W and W'. We, therefore, have

$$\begin{aligned}
B_{\Delta,3} =& -\frac{1}{2}\frac{u^2}{\sigma_0^2\Delta}(\beta_0')^2\mathbb{E}_0\left[\left(\int_0^\Delta W_s' dW_s\right)^2 e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right] \\
=& -\frac{1}{2}\frac{u^2}{\sigma_0^2\Delta}(\beta_0')^2 e^{-\frac{u^2}{2}}\int_0^\Delta \mathbb{E}_0\left[\left(W_s'\right)^2 e^{iu\frac{W_s}{\sqrt{\Delta}}}\right] e^{\frac{u^2 s}{2\Delta}} ds \\
&-\frac{2u^2}{2\sigma_0^2\Delta}(\beta_0')^2\left(\frac{iu}{\sqrt{\Delta}}\right) e^{-\frac{u^2}{2}}\int_0^\Delta \mathbb{E}_0\left[e^{iu\frac{W_s}{\sqrt{\Delta}}} W_s'\left(\int_0^s W_{s_1}' dW_{s_1}\right)\right] e^{\frac{u^2 s}{2\Delta}} ds \\
=& -\frac{1}{4}\frac{u^2}{\sigma_0^2}(\beta_0')^2 e^{-\frac{u^2}{2}}\Delta \\
&-\frac{2u^2}{2\sigma_0^2\Delta}(\beta_0')^2\left(\frac{iu}{\sqrt{\Delta}}\right)^2 e^{-\frac{u^2}{2}}\int_0^\Delta\int_0^s e^{\frac{u^2 s_1}{2\Delta}}\mathbb{E}_0\left[\left(W_{s_1}'\right)^2 e^{iu\frac{W_{s_1}}{\sqrt{\Delta}}}\right] ds_1 ds \\
=& -\frac{1}{4}\frac{u^2}{\sigma_0^2}(\beta_0')^2 e^{-\frac{u^2}{2}}\Delta + \frac{1}{6}\frac{u^4}{\sigma_0^2}(\beta_0')^2 e^{-\frac{u^2}{2}}\Delta,
\end{aligned}$$

where the first equality derives from Lemma 1 applied to Eq. (53) and the second equality derives from Lemma 1 applied to Eq. (54). This term, again, contributes to the second order.

A straightforward comparison of term $B_{\Delta,4}$ with the previous three suggests that $B_{\Delta,4}$ is $o(\Delta)$ Tedious applications of Lemma 1 confirm this logic.

Regarding $B_{\Delta,5}$, because

$$\begin{aligned}
&\left(\int_0^\Delta s dW_s\right)\left(\int_0^\Delta W_s dW_s\right) \\
=& \left(\int_0^\Delta\left(\int_0^s s_1 dW_{s_1}\right) W_s dW_s\right) + \left(\int_0^\Delta\left(\int_0^s W_{s_1} dW_{s_1}\right) s dW_s\right) + \left(\int_0^\Delta s W_s ds\right),
\end{aligned} \tag{55}$$

$$\begin{aligned}
&\left(W_s\int_0^s s_1 dW_{s_1}\right) \\
=& \left(\int_0^s\left(\int_0^{s_1} s_2 dW_{s_2}\right) dW_{s_1}\right) + \left(\int_0^s W_{s_1} s_1 dW_{s_1}\right) + \left(\int_0^s s_1 ds_1\right)
\end{aligned} \tag{56}$$

and

$$\left(s\int_0^s W_{s_1} dW_{s_1}\right) = \left(\int_0^s\left(\int_0^{s_1} W_{s_2} dW_{s_2}\right) ds_1\right) + \left(\int_0^s W_{s_1} s_1 dW_{s_1}\right) \tag{57}$$

we readily have

$$\begin{aligned}
B_{\Delta,5} =& -\frac{2}{2}\frac{u^2}{\sigma_0^2\Delta}\mathbb{E}_0\left[\alpha_0\beta_0\left(\int_0^\Delta s dW_s\right)\left(\int_0^\Delta W_s dW_s\right) e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right] \\
=& -\frac{u^2}{\sigma_0^2\Delta}\mathbb{E}_0\left[\alpha_0\beta_0\left(\int_0^\Delta\left(\int_0^s s_1 dW_{s_1}\right) W_s dW_s\right) e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{u^2}{\sigma_0^2\Delta}\mathbb{E}_0\left[\alpha_0\beta_0\left(\int_0^\Delta\left(\int_0^s W_{s_1}dW_{s_1}\right)sdW_s\right)e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right]\\
&- \frac{u^2}{\sigma_0^2\Delta}\mathbb{E}_0\left[\alpha_0\beta_0\left(\int_0^\Delta sW_sds\right)e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\right]\\
=&- \frac{iu^3}{\sigma_0^2\Delta^{3/2}}\alpha_0\beta_0e^{-\frac{u^2}{2}}\int_0^\Delta\mathbb{E}_0\left[\left(W_s\int_0^s s_1dW_{s_1}\right)e^{iu\frac{W_s}{\sqrt{\Delta}}}\right]e^{\frac{u^2s}{2\Delta}}ds\\
&- \frac{iu^3}{\sigma_0^2\Delta^{3/2}}\alpha_0\beta_0e^{-\frac{u^2}{2}}\int_0^\Delta\mathbb{E}_0\left[\left(s\int_0^s W_{s_1}dW_{s_1}\right)e^{iu\frac{W_s}{\sqrt{\Delta}}}\right]e^{\frac{u^2s}{2\Delta}}ds\\
&- \frac{u^2}{\sigma_0^2\Delta}\alpha_0\beta_0e^{-\frac{u^2}{2}}\int_0^\Delta\mathbb{E}_0\left[sW_se^{iu\frac{W_s}{\sqrt{\Delta}}}\right]e^{\frac{u^2s}{2\Delta}}ds\\
=&- \frac{iu^3}{\sigma_0^2\Delta^{3/2}}\alpha_0\beta_0e^{-\frac{u^2}{2}}\int_0^\Delta\mathbb{E}_0\left[\left(\int_0^s\left(\int_0^{s_1} s_2dW_{s_2}\right)dW_{s_1}\right)e^{iu\frac{W_s}{\sqrt{\Delta}}}\right]e^{\frac{u^2s}{2\Delta}}ds\\
&- \frac{2iu^3}{\sigma_0^2\Delta^{3/2}}\alpha_0\beta_0e^{-\frac{u^2}{2}}\int_0^\Delta\mathbb{E}_0\left[\left(\int_0^s W_{s_1}s_1dW_{s_1}\right)e^{iu\frac{W_s}{\sqrt{\Delta}}}\right]e^{\frac{u^2s}{2\Delta}}ds\\
&- \frac{iu^3}{\sigma_0^2\Delta^{3/2}}\alpha_0\beta_0e^{-\frac{u^2}{2}}\int_0^\Delta\mathbb{E}_0\left[\left(\int_0^s s_1ds_1\right)e^{iu\frac{W_s}{\sqrt{\Delta}}}\right]e^{\frac{u^2s}{2\Delta}}ds\\
&- \frac{iu^3}{\sigma_0^2\Delta^{3/2}}\alpha_0\beta_0e^{-\frac{u^2}{2}}\int_0^\Delta\mathbb{E}_0\left[\left(\int_0^s\left(\int_0^{s_1} W_{s_2}dW_{s_2}\right)ds_1\right)e^{iu\frac{W_s}{\sqrt{\Delta}}}\right]e^{\frac{u^2s}{2\Delta}}ds\\
&- \frac{u^2}{\sigma_0^2\Delta}\alpha_0\beta_0e^{-\frac{u^2}{2}}\int_0^\Delta\mathbb{E}_0\left[sW_se^{iu\frac{W_s}{\sqrt{\Delta}}}\right]e^{\frac{u^2s}{2\Delta}}ds,
\end{aligned}
$$

where the first equality derives from Itô's lemma (as in Eq. (55)), the second equality is an application of Lemma 1 and the third equality derives, again, from Itô's lemma (as in Eqs. (56) and (57)). Thus,

$$
\begin{aligned}
&- \frac{iu^3}{\sigma_0^2\Delta^{3/2}}\alpha_0\beta_0e^{-\frac{u^2}{2}}\int_0^\Delta\mathbb{E}_0\left[\left(\int_0^s\left(\int_0^{s_1} s_2dW_{s_2}\right)dW_{s_1}\right)e^{iu\frac{W_s}{\sqrt{\Delta}}}\right]e^{\frac{u^2s}{2\Delta}}ds\\
&- \frac{2iu^3}{\sigma_0^2\Delta^{3/2}}\alpha_0\beta_0e^{-\frac{u^2}{2}}\int_0^\Delta\mathbb{E}_0\left[\left(\int_0^s W_{s_1}s_1dW_{s_1}\right)e^{iu\frac{W_s}{\sqrt{\Delta}}}\right]e^{\frac{u^2s}{2\Delta}}ds\\
&- \frac{iu^3}{\sigma_0^2\Delta^{3/2}}\alpha_0\beta_0e^{-\frac{u^2}{2}}\int_0^\Delta\mathbb{E}_0\left[\left(\int_0^s s_1ds_1\right)e^{iu\frac{W_s}{\sqrt{\Delta}}}\right]e^{\frac{u^2s}{2\Delta}}ds\\
&- \frac{iu^3}{\sigma_0^2\Delta^{3/2}}\alpha_0\beta_0e^{-\frac{u^2}{2}}\int_0^\Delta\mathbb{E}_0\left[\left(\int_0^s\left(\int_0^{s_1} W_{s_2}dW_{s_2}\right)ds_1\right)e^{iu\frac{W_s}{\sqrt{\Delta}}}\right]e^{\frac{u^2s}{2\Delta}}ds\\
&- \frac{u^2}{\sigma_0^2\Delta}\alpha_0\beta_0e^{-\frac{u^2}{2}}\int_0^\Delta\mathbb{E}_0\left[sW_se^{iu\frac{W_s}{\sqrt{\Delta}}}\right]e^{\frac{u^2s}{2\Delta}}ds\\
=&\frac{iu^5}{\sigma_0^2\Delta^{5/2}}\alpha_0\beta_0e^{-\frac{u^2}{2}}\int_0^\Delta\int_0^s\int_0^{s_1}\mathbb{E}_0\left[s_2e^{iu\frac{W_{s_2}}{\sqrt{\Delta}}}\right]e^{\frac{u^2s_2}{2\Delta}}ds_2ds_1ds\\
&- \frac{2iu^4}{\sigma_0^2\Delta^2}\alpha_0\beta_0e^{-\frac{u^2}{2}}\int_0^\Delta\int_0^s\mathbb{E}_0\left[iW_{s_1}s_1e^{iu\frac{W_{s_1}}{\sqrt{\Delta}}}\right]e^{\frac{u^2s_1}{2\Delta}}ds_1ds\\
&- \frac{iu^3}{\sigma_0^2\Delta^{3/2}}\alpha_0\beta_0e^{-\frac{u^2}{2}}\int_0^\Delta\int_0^s\mathbb{E}_0\left[s_1e^{iu\frac{W_{s_1}}{\sqrt{\Delta}}}\right]e^{\frac{u^2s_1}{2\Delta}}ds_1ds\\
&- \frac{iu^4}{\sigma_0^2\Delta^2}\alpha_0\beta_0e^{-\frac{u^2}{2}}\int_0^\Delta\int_0^s\int_0^{s_1}\mathbb{E}_0\left[iW_{s_2}e^{iu\frac{W_{s_2}}{\sqrt{\Delta}}}\right]e^{\frac{u^2s_2}{2\Delta}}dsds_1ds_2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{iu^2}{\sigma_0^2\Delta}\alpha_0\beta_0 e^{-\frac{u^2}{2}}\int_0^\Delta \mathbb{E}_0\left[siW_s e^{iu\frac{W_s}{\sqrt{\Delta}}}\right]e^{\frac{u^2 s}{2\Delta}}ds\\
=&\Delta^{3/2}\frac{1}{24}\frac{iu^5}{\sigma_0^2}\alpha_0\beta_0 e^{-\frac{u^2}{2}}-\frac{2iu^4}{\sigma_0^2\Delta^2}\alpha_0\beta_0 e^{-\frac{u^2}{2}}\int_0^\Delta\int_0^s s_1\left[-\frac{s_1 u}{\sqrt{\Delta}}e^{\frac{-s_1u^2}{2\Delta}}\right]e^{\frac{u^2 s_1}{2\Delta}}ds_1 ds\\
&-\Delta^{3/2}\frac{1}{6}\frac{iu^3}{\sigma_0^2}\alpha_0\beta_0 e^{-\frac{u^2}{2}}-\frac{iu^4}{\sigma_0^2\Delta^2}\alpha_0\beta_0 e^{-\frac{u^2}{2}}\int_0^\Delta\int_0^s\int_0^{s_1}\left[-\frac{s_2 u}{\sqrt{\Delta}}e^{\frac{-s_2u^2}{2\Delta}}\right]e^{\frac{u^2 s_2}{2\Delta}}ds ds_1 ds_2\\
&+\frac{iu^2}{\sigma_0^2\Delta}\alpha_0\beta_0 e^{-\frac{u^2}{2}}\int_0^\Delta s\mathbb{E}_0\left[-\frac{su}{\sqrt{\Delta}}e^{\frac{-su^2}{2\Delta}}\right]e^{\frac{u^2 s}{2\Delta}}ds\\
=&\Delta^{3/2}\frac{1}{24}\frac{iu^5}{\sigma_0^2}\alpha_0\beta_0 e^{-\frac{u^2}{2}}+\Delta^{3/2}\frac{4}{24}\frac{iu^5}{\sigma_0^2}\alpha_0\beta_0 e^{-\frac{u^2}{2}}-\Delta^{3/2}\frac{1}{6}\frac{iu^3}{\sigma_0^2}\alpha_0\beta_0 e^{-\frac{u^2}{2}}\\
&+\Delta^{3/2}\frac{1}{24}\frac{iu^5}{\sigma_0^2}\alpha_0\beta_0 e^{-\frac{u^2}{2}}-\Delta^{3/2}\frac{1}{3}\frac{iu^3}{\sigma_0^2}\alpha_0\beta_0 e^{-\frac{u^2}{2}}\\
=&-\Delta^{3/2}\frac{1}{2}\frac{iu^3}{\sigma_0^2\Delta}\alpha_0\beta_0 e^{-\frac{u^2}{2}}+\Delta^{3/2}\frac{1}{6}\frac{iu^5}{\sigma_0^2}\alpha_0\beta_0 e^{-\frac{u^2}{2}}.
\end{aligned}
\tag{58}
$$

We conclude that $B_{\Delta,5}$ is negligible. Using the same method of proof as for $B_{\Delta,5}$, we can also verify that $B_{\Delta,6}=0$ and $B_{\Delta,7}=0$. Repeated applications of Lemma 1 and Itô's lemma (such as those leading to Eq. (58)) also imply that $B_{\Delta,8}$, $B_{\Delta,9}$ and $B_{\Delta,10}$ are negligible.

Finally, let us turn to the infinite sum in C_Δ. Write

$$
\begin{aligned}
C_\Delta(u)=&\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\sum_{k=3}^{+\infty}\frac{(iu)^k}{k!}\Delta^{\frac{k}{2}}Y_\Delta^k\right].\\
\sim&\sum_{k=3}^{+\infty}\frac{(iu)^k}{k!}\frac{\Delta^{-\frac{k}{2}}}{\sigma_0^k}\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\left(\int_0^\Delta(\sigma_s-\sigma_0)dW_s\right)^k\right]\\
\leq&\sum_{k=3}^{+\infty}\frac{(iu)^k}{k!}\frac{\Delta^{-\frac{k}{2}}}{\sigma_0^k}\left|\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\left(\int_0^\Delta(\sigma_s-\sigma_0)dW_s\right)^k\right]\right|.
\end{aligned}
$$

Applying Lemma 3, for small Δ, we have

$$
\begin{aligned}
C_\Delta(u)\leq&C\sum_{k=3}^{+\infty}\frac{|u|^k}{k!}\frac{\Delta^{-\frac{k}{2}}}{\sigma_0^k}k^{10}\Delta^k g_5(u,1)\\
\leq&\ Cg_5(u,1)\sum_{k=3}^{+\infty}\frac{k^{10}}{k!}\left(\frac{|u|\,\Delta^{1/2}}{\sigma_0}\right)^k\\
\sim&\ Cg_5(u,1)\frac{|u|^3\,\Delta^{3/2}}{\sigma_0^{3/2}},
\end{aligned}
$$

and the component of order $\Delta^{3/2}$ is integrable with respect to u.

Step 2. We now deal with the full characteristic function. Write

$$\mathbb{E}_0\left[e^{iuZ_\Delta+ivV_\Delta}\right]=\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}e^{iuY_\Delta\sqrt{\Delta}}e^{ivY_\Delta''\sqrt{\Delta}}\right],$$

where

$$Y_\Delta=\frac{1}{\sigma_0\Delta}\left(\int_0^\Delta(\mu_s-\mu_0)ds+\int_0^\Delta(\sigma_s-\sigma_0)dW_s\right),$$

and

$$Y_\Delta''=\frac{1}{\beta_0\Delta}\left(\int_0^\Delta(\alpha_s-\alpha_0)ds+\int_0^\Delta\left(\beta_s-\beta_0\right)dW_s+\int_0^\Delta\left(\beta_s'-\beta_0'\right)dW_s'\right).$$

Using Taylor expansion, we have

$$\begin{aligned}
\mathbb{E}_0\left[e^{iuZ_\Delta+ivV_\Delta}\right]=&\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\left(1+iuY_\Delta\sqrt{\Delta}+\frac{1}{2}(iu)^2Y_\Delta^2\Delta+\sum_{k=3}^{+\infty}\frac{(iu)^k}{k!}\Delta^{\frac{k}{2}}Y_\Delta^k\right)\cdot\right.\\
&\left.\left(1+ivY_\Delta''\sqrt{\Delta}+\frac{1}{2}(iv)^2(Y_\Delta'')^2\Delta+\sum_{k=3}^{+\infty}\frac{(iv)^k}{k!}\Delta^{\frac{k}{2}}(Y_\Delta'')^k\right)\right]\\
=&e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}+\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\left(iuY_\Delta\sqrt{\Delta}+\frac{1}{2}(iu)^2Y_\Delta^2\Delta\right)\right]\\
&+\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\left(ivY_\Delta''\sqrt{\Delta}+\frac{1}{2}(iv)^2(Y_\Delta'')^2\Delta\right)\right]\\
&+\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}iuY_\Delta ivY_\Delta''\Delta\right]+C_\Delta'(u,v)\\
=&e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}+A_\Delta''+B_\Delta''+C_\Delta''(u,v),
\end{aligned}$$

where

$$\begin{aligned}
A_\Delta''=&e^{-\frac{1}{2}\left(v\frac{\beta_0'}{\beta_0}\right)^2}\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}\frac{i}{\sqrt{\Delta}}\left(u\frac{\delta_0}{\sigma_0}+v\frac{\theta_0}{\beta_0}\right)\int_0^\Delta W_sds\right]\\
&+e^{-\frac{1}{2}\left(v\frac{\beta_0'}{\beta_0}\right)^2}\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}\frac{i}{\sqrt{\Delta}}\left(u\frac{\alpha_0}{\sigma_0}+v\frac{\zeta_0}{\beta_0}\right)\int_0^\Delta sdW_s\right]\\
&+e^{-\frac{1}{2}\left(v\frac{\beta_0'}{\beta_0}\right)^2}\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}\frac{i}{\sqrt{\Delta}}\left(u\frac{\beta_0}{\sigma_0}+v\frac{\eta_0}{\beta_0}\right)\int_0^\Delta W_sdW_s\right]\\
&+\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\frac{i}{\sqrt{\Delta}}\left(u\frac{\beta_0'}{\sigma_0}\right)\int_0^\Delta W_s'dW_s\right]\\
&+e^{-\frac{1}{2}\left(v\frac{\beta_0'}{\beta_0}\right)^2}\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}\frac{i}{\sqrt{\Delta}}\left(u\frac{\eta_0}{\sigma_0}+v\frac{\psi_0}{\beta_0}\right)\int_0^\Delta\int_0^s W_{s_1}dW_{s_1}dW_s\right]\\
&+e^{-\frac{1}{2}\left(v\frac{\beta_0'}{\beta_0}\right)^2}\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}\frac{i}{\sqrt{\Delta}}\left(v\frac{\eta_0'}{\beta_0}\right)\int_0^\Delta W_s^{**}dW_s\right]
\end{aligned}$$

$$
\begin{aligned}
&+ e^{-\frac{1}{2}(u+v)^2}\mathbb{E}_0\left[e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\frac{i}{\sqrt{\Delta}}\left(v\frac{\zeta_0'}{\beta_0}\right)\int_0^\Delta s dW_s'\right] \\
&+ \mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\frac{i}{\sqrt{\Delta}}\left(v\frac{\kappa_0}{\beta_0}\right)\int_0^\Delta W_s dW_s'\right] \\
&+ e^{-\frac{1}{2}(u+v)^2}\mathbb{E}_0\left[e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\frac{i}{\sqrt{\Delta}}\left(v\frac{\kappa_0'}{\beta_0}\right)\int_0^\Delta W_s^{*****} dW_s'\right] \\
&+ \mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\frac{i}{\sqrt{\Delta}}\left(v\frac{\iota_0}{\beta_0}\right)\int_0^\Delta\int_0^s W_{s_1} dW_{s_1} dW_s'\right] \\
&+ \mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\frac{i}{\sqrt{\Delta}}\left(u\frac{\kappa_0}{\sigma_0}\right)\int_0^\Delta\int_0^s W_{s_1} dW_{s_1}' dW_s\right] = \sum_{k=1}^{11}\widetilde{A}_{k,\Delta},
\end{aligned}
$$

$$
\begin{aligned}
B_\Delta'' =& -\frac{1}{2}e^{-\frac{1}{2}\left(v\frac{\beta_0'}{\beta_0}\right)^2}\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}\frac{1}{\Delta}\left(\left(u\frac{\beta_0}{\sigma_0}\right)^2+\left(v\frac{\eta_0}{\beta_0}\right)^2\right)\left(\int_0^\Delta W_s dW_s\right)^2\right] \\
&-\frac{1}{2}\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\frac{1}{\Delta}\left(u\frac{\beta_0'}{\sigma_0}\right)^2\left(\int_0^\Delta W_s' dW_s\right)^2\right] \\
&-\frac{1}{2}e^{-\frac{(u+v)^2}{2}}\mathbb{E}_0\left[e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\frac{1}{\Delta}\left(v\frac{\eta_0'}{\beta_0}\right)^2\left(\int_0^\Delta W_s^{**} dW_s\right)^2\right] \\
&-\frac{1}{2}\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\frac{1}{\Delta}\left(v\frac{\kappa_0}{\beta_0}\right)^2\left(\int_0^\Delta W_s dW_s'\right)^2\right] \\
&-\frac{1}{2}e^{-\frac{(u+v)^2}{2}}\mathbb{E}_0\left[e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\frac{1}{\Delta}\left(v\frac{\kappa_0'}{\beta_0}\right)^2\left(\int_0^\Delta W_s^{*****} dW_s'\right)^2\right] \\
&-\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\frac{1}{\Delta}u^2\frac{\beta_0\beta_0'}{\sigma_0^2}\int_0^\Delta W_s dW_s\int_0^\Delta W_s' dW_s\right] \\
&-\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\frac{v^2}{\beta_0^2}\frac{1}{\Delta}\left(\int_0^\Delta\eta_0 W_s dW_s\int_0^\Delta\eta_0' W_s^{**} dW_s+\int_0^\Delta\eta_0 W_s dW_s\int_0^\Delta\kappa_0 W_s dW_s'+\right.\right. \\
&\int_0^\Delta\eta_0 W_s dW_s\int_0^\Delta\kappa_0' W_s^{*****} dW_s'+\int_0^\Delta\eta_0' W_s^{**} dW_s\int_0^\Delta\kappa_0 W_s dW_s'+ \\
&\left.\left.\int_0^\Delta\eta_0' W_s^{**} dW_s\int_0^\Delta\kappa_0' W_s^{*****} dW_s'+\int_0^\Delta\kappa_0 W_s dW_s'\int_0^\Delta\kappa_0' W_s^{*****} dW_s'\right)\right] \\
&-\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\frac{uv}{\sigma_0\beta_0}\frac{1}{\Delta}\int_0^\Delta(\beta_0 W_s+\beta_0' W_s')dW_s\cdot\right. \\
&\left.\cdot\left(\int_0^\Delta(\eta_0 W_s+\eta_0' W_s^{**})dW_s+\int_0^\Delta(\kappa_0 W_s+\kappa_0' W_s^{*****})dW_s'\right)\right] = \sum_{k=1}^{8}\widetilde{B}_{k,\Delta},
\end{aligned}
$$

and $C_\Delta''(u,v)$ is negligible and bounded in u and v as stated in the theorem.

We now notice that $\widetilde{A}_{1,\Delta}$ can be treated like $\overline{A}_{\Delta,0}^0$, so that

$$
\widetilde{A}_{1,\Delta} = -\frac{1}{2}(u+v)e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\left(u\frac{\delta_0}{\sigma_0}+v\frac{\theta_0}{\beta_0}\right)\Delta.
$$

Also, $\widetilde{A}_{2,\Delta}$ can be treated like $A_{\Delta,2}$:

$$\widetilde{A}_{2,\Delta} = -\frac{1}{2}(u+v)e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\left(u\frac{\alpha_0}{\sigma_0}+v\frac{\zeta_0}{\beta_0}\right)\Delta.$$

Similarly, $\widetilde{A}_{3,\Delta}+\widetilde{B}_{\Delta,1}$ can be treated like $A_{\Delta,1}+B_{\Delta,2}$:

$$\begin{aligned}\widetilde{A}_{3,\Delta}+\widetilde{B}_{\Delta,1} =& -i\frac{1}{2}(u+v)^2e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\left(u\frac{\beta_0}{\sigma_0}+v\frac{\eta_0}{\beta_0}\right)\sqrt{\Delta}\\ &-\frac{1}{8}\left(2-4(u+v)^2+(u+v)^4\right)e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\left(\left(u\frac{\beta_0}{\sigma_0}\right)^2+\left(v\frac{\eta_0}{\beta_0}\right)^2\right)\Delta.\end{aligned}$$

Turning to $\widetilde{A}_{4,\Delta}$, by Lemma 4 we have

$$\begin{aligned}\widetilde{A}_{4,\Delta} =& \frac{i}{\sqrt{\Delta}}\left(u\frac{\beta_0'}{\sigma_0}\right)\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\int_0^\Delta W_s'dW_s\right]\\ =& \frac{i}{\sqrt{\Delta}}\left(u\frac{\beta_0'}{\sigma_0}\right)e^{-\frac{1}{2}\left((u+v)^2+v^2\left(\frac{\beta_0'}{\beta_0}\right)^2\right)}\cdot\\ &\cdot\int_0^\Delta e^{\left((u+v)^2+v^2\left(\frac{\beta_0'}{\beta_0}\right)^2\right)\frac{s}{2\Delta}}\mathbb{E}_0\left[e^{i(u+v)\frac{W_s}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_s'}{\sqrt{\Delta}}}\left(W_s'dW_s+\frac{i(u+v)}{\sqrt{\Delta}}W_s'ds\right)\right]\\ =& \frac{i(u+v)}{\sqrt{\Delta}}\frac{i}{\sqrt{\Delta}}\left(u\frac{\beta_0'}{\sigma_0}\right)e^{-\frac{1}{2}\left((u+v)^2+v^2\left(\frac{\beta_0'}{\beta_0}\right)^2\right)}\int_0^\Delta e^{\left((u+v)^2+v^2\left(\frac{\beta_0'}{\beta_0}\right)^2\right)\frac{s}{2\Delta}}e^{-(u+v)^2\frac{s}{2\Delta}}\frac{isv\frac{\beta_0'}{\beta_0}}{\sqrt{\Delta}}e^{-\left(v\frac{\beta_0'}{\beta_0}\right)^2\frac{s}{2\Delta}}ds\\ =& -i(u+v)\frac{1}{2}v\frac{\beta_0'}{\beta_0}\left(u\frac{\beta_0'}{\sigma_0}\right)e^{-\frac{1}{2}\left((u+v)^2+v^2\left(\frac{\beta_0'}{\beta_0}\right)^2\right)}\sqrt{\Delta}.\end{aligned}$$

The term $\widetilde{A}_{5,\Delta}$ can be treated like $\overline{A}^0_{\Delta,4}$:

$$\widetilde{A}_{5,\Delta} = \frac{1}{6}(u+v)^3e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\left(u\frac{\eta_0}{\sigma_0}+v\frac{\psi_0}{\beta_0}\right)\Delta.$$

The term $\widetilde{A}_{6,\Delta}$ is zero, while $\widetilde{A}_{7,\Delta}$ can be treated like $\widetilde{A}_{2,\Delta}$:

$$\widetilde{A}_{7,\Delta} = -\frac{1}{2}v^2\frac{\beta_0'}{\beta_0}\frac{\zeta_0'}{\beta_0}e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\Delta.$$

The term $\widetilde{A}_{8,\Delta}$ can be treated like $\widetilde{A}_{4,\Delta}$:

$$\widetilde{A}_{8,\Delta} = - i(u+v)\frac{1}{2}v\frac{\beta_0'}{\beta_0}\left(v\frac{\kappa_0}{\beta_0}\right)e^{-\frac{1}{2}\left((u+v)^2+v^2\left(\frac{\beta_0'}{\beta_0}\right)^2\right)}\sqrt{\Delta}.$$

The term $\widetilde{A}_{9,\Delta}$ is zero like $\widetilde{A}_{6,\Delta}$. Finally,

$$\begin{aligned}
\widetilde{A}_{10,\Delta} =&\frac{i}{\sqrt{\Delta}}\left(v\frac{\iota_0}{\beta_0}\right)\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\left(\int_0^\Delta\int_0^s W_{s_1}dW_{s_1}dW_s'\right)\right]\\
=&\frac{i}{\sqrt{\Delta}}\left(v\frac{\iota_0}{\beta_0}\right)e^{-\frac{(u+v)^2}{2}-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\cdot\\
&\cdot\int_0^\Delta e^{\left((u+v)^2+v^2\left(\frac{\beta_0'}{\beta_0}\right)^2\right)\frac{s}{2\Delta}}\mathbb{E}_0\left[e^{i(u+v)\frac{W_s}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_s'}{\sqrt{\Delta}}}iv\frac{\beta_0'}{\beta_0}\frac{1}{\sqrt{\Delta}}\left(\int_0^s W_{s_1}dW_{s_1}\right)\right]ds\\
=&-\frac{v^2}{\Delta}\frac{\iota_0}{\beta_0}\frac{\beta_0'}{\beta_0}e^{-\frac{(u+v)^2}{2}-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\int_0^\Delta e^{\left((u+v)^2+v^2\left(\frac{\beta_0'}{\beta_0}\right)^2\right)\frac{s}{2\Delta}}\mathbb{E}_0\left[e^{i(u+v)\frac{W_s}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_s'}{\sqrt{\Delta}}}\left(\frac{W_s^2}{2}-\frac{s}{2}\right)\right]ds\\
=&-\frac{v^2}{\Delta}\frac{\iota_0}{\beta_0}\frac{\beta_0'}{\beta_0}e^{-\frac{(u+v)^2}{2}-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\int_0^\Delta -\frac{s^2}{2\Delta}(u+v)^2ds\\
=&\frac{(u+v)^2v^2}{6}\frac{\iota_0}{\beta_0}\frac{\beta_0'}{\beta_0}e^{-\frac{(u+v)^2}{2}-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\Delta,
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\widetilde{A}_{11,\Delta} =&\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\frac{i}{\sqrt{\Delta}}\left(u\frac{\kappa_0}{\sigma_0}\right)\int_0^\Delta\int_0^s W_{s_1}dW_{s_1}'dW_s\right]\\
=&\frac{(u+v)^2uv}{6}\frac{\kappa_0}{\sigma_0}\frac{\beta_0'}{\beta_0}e^{-\frac{(u+v)^2}{2}-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\Delta.
\end{aligned}$$

Now, we handle the $\widetilde{B}_{k,\Delta}$ terms. The quantity $\widetilde{B}_{\Delta,1}$ has already been studied in Step 1. Regarding the term $\widetilde{B}_{\Delta,2}$, we have

$$\begin{aligned}
\widetilde{B}_{\Delta,2} =& -\frac{1}{2\Delta}\left(u\frac{\beta_0'}{\sigma_0}\right)^2\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\left(\int_0^\Delta W_s'dW_s\right)^2\right]\\
=& -\frac{1}{2\Delta}\left(u\frac{\beta_0'}{\sigma_0}\right)^2\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\left(2\int_0^\Delta\int_0^s W_{s_1}'dW_{s_1}W_s'dW_s+\int_0^\Delta(W_s')^2ds\right)\right]\\
=& -\frac{1}{2\Delta}\left(u\frac{\beta_0'}{\sigma_0}\right)^2e^{-\frac{1}{2}\left((u+v)^2+v^2\left(\frac{\beta_0'}{\beta_0}\right)^2\right)}\cdot\\
&\cdot\int_0^\Delta e^{\left((u+v)^2+v^2\left(\frac{\beta_0'}{\beta_0}\right)^2\right)\frac{s}{2\Delta}}\mathbb{E}_0\left[e^{i(u+v)\frac{W_s}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_s'}{\sqrt{\Delta}}}\left(2\frac{i(u+v)}{\sqrt{\Delta}}W_s'\int_0^s W_{s_1}'dW_{s_1}+(W_s')^2\right)\right]ds\\
=&\frac{1}{2}(u+v)^2\left(u\frac{\beta_0'}{\sigma_0}\right)^2e^{-\frac{1}{2}\left((u+v)^2+v^2\left(\frac{\beta_0'}{\beta_0}\right)^2\right)}\left(\frac{1}{3}-\frac{1}{4}\left(v\frac{\beta_0'}{\beta_0}\right)^2\right)\Delta
\end{aligned}$$

$$
-\frac{1}{2}\left(u\frac{\beta_0'}{\sigma_0}\right)^2 e^{-\frac{1}{2}\left((u+v)^2+v^2\left(\frac{\beta_0'}{\beta_0}\right)^2\right)}\left(\frac{1}{2}-\frac{1}{3}\left(v\frac{\beta_0'}{\beta_0}\right)^2\right)\Delta.
$$

The term $\widetilde{B}_{3,\Delta}$ can be treated like the term $B_{\Delta,3}$:

$$
\begin{aligned}
\widetilde{B}_{3,\Delta} =& -\frac{1}{2}\frac{1}{\Delta}\left(v\frac{\eta_0'}{\beta_0}\right)^2 e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}\left(\int_0^\Delta W_s^{**}dW_s\right)^2\right] \\
=& -\frac{1}{2}\frac{1}{\Delta}\left(v\frac{\eta_0'}{\beta_0}\right)^2 e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\left(\frac{1}{2}-\frac{1}{3}(u+v)^2\right)\Delta.
\end{aligned}
$$

The term $\widetilde{B}_{4,\Delta}$ can be treated like the term $\widetilde{B}_{2,\Delta}$:

$$
\begin{aligned}
\widetilde{B}_{\Delta,2} =& \frac{1}{2}v^2\left(\frac{\beta_0'}{\beta_0}\right)^2\left(v\frac{\kappa_0}{\beta_0}\right)^2 e^{-\frac{1}{2}\left((u+v)^2+v^2\left(\frac{\beta_0'}{\beta_0}\right)^2\right)}\left(\frac{1}{3}-\frac{1}{4}(u+v)^2\right)\Delta \\
& -\frac{1}{2}\left(v\frac{\kappa_0}{\beta_0}\right)^2 e^{-\frac{1}{2}\left((u+v)^2+v^2\left(\frac{\beta_0'}{\beta_0}\right)^2\right)}\left(\frac{1}{2}-\frac{1}{3}(u+v)^2\right)\Delta.
\end{aligned}
$$

The term $\widetilde{B}_{5,\Delta}$ can be treated like the term $\widetilde{B}_{3,\Delta}$:

$$
\widetilde{B}_{\Delta,5} = -\frac{1}{2}\left(v\frac{\kappa_0'}{\beta_0}\right)^2 e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\left(\frac{1}{2}-\frac{1}{3}\left(v\frac{\beta_0'}{\beta_0}\right)^2\right)\Delta.
$$

Regarding the term $\widetilde{B}_{6,\Delta}$, using Itô's lemma and Lemma 4, we have

$$
\begin{aligned}
\widetilde{B}_{\Delta,6} =& -\frac{1}{\Delta}u^2\frac{\beta_0\beta_0'}{\sigma_0^2}\;\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\int_0^\Delta W_s dW_s\int_0^\Delta W_s' dW_s\right] \\
=& -\frac{1}{\Delta}u^2\frac{\beta_0\beta_0'}{\sigma_0^2}\mathbb{E}_0\Bigg[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\Bigg(\int_0^\Delta\int_0^s W_{s_1}dW_{s_1}W_s' dW_s+\int_0^\Delta\int_0^s W_{s_1}' dW_{s_1}W_s dW_s \\
& +\int_0^\Delta W_s W_s' ds\Bigg)\Bigg] \\
=& \widetilde{B}_{\Delta,6}^a+\widetilde{B}_{\Delta,6}^b+\widetilde{B}_{\Delta,6}^c.
\end{aligned}
$$

For the term $\widetilde{B}_{\Delta,6}^a$ we use

$$
\begin{aligned}
W_s'\int_0^s W_{s_1}dW_{s_1} =& \int_0^s\int_0^{s_1} W_{s_2}dW_{s_2}dW_{s_1}'+\int_0^s W_{s_1}' W_{s_1}dW_{s_1} \\
=& \int_0^s\int_0^{s_1} W_{s_2}dW_{s_2}dW_{s_1}'+\int_0^s\int_0^{s_1} W_{s_2}dW_{s_2}' dW_{s_1}+\int_0^s\int_0^{s_1} W_{s_2}' dW_{s_2}dW_{s_1},
\end{aligned}
$$

so that

$$\widetilde{B}^a_{\Delta,6} = -\frac{1}{24}u^2\frac{\beta_0\beta_0'}{\sigma_0^2}e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}3(u+v)^3v\frac{\beta_0'}{\beta_0}\Delta.$$

For the term $\widetilde{B}^b_{\Delta,6}$ we use

$$\begin{aligned}W_s\int_0^s W'_{s_1}dW_{s_1} &= \int_0^s\int_0^{s_1} W'_{s_2}dW_{s_2}dW_{s_1} + \int_0^s W_{s_1}W'_{s_1}dW_{s_1} + \int_0^s W'_{s_1}ds_1\\ &= \int_0^s\int_0^{s_1} W'_{s_2}dW_{s_2}dW_{s_1} + \int_0^s\int_0^{s_1} W_{s_2}dW'_{s_2}dW_{s_1} + \int_0^s\int_0^{s_1} W'_{s_2}dW_{s_2}dW_{s_1} + \int_0^s W'_{s_1}ds_1,\end{aligned}$$

so that

$$\widetilde{B}^b_{\Delta,6} = -\frac{1}{24}u^2\frac{\beta_0\beta_0'}{\sigma_0^2}e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\left(3(u+v)^3v\frac{\beta_0'}{\beta_0} - 4(u+v)v\frac{\beta_0'}{\beta_0}\right)\Delta.$$

For the term $\widetilde{B}^c_{\Delta,6}$, we use

$$W_sW'_s = \int_0^s W'_{s_1}dW_{s_1} + \int_0^s W_{s_1}dW'_{s_1},$$

so that

$$\widetilde{B}^c_{\Delta,6} = \frac{1}{6}u^2\frac{\beta_0\beta_0'}{\sigma_0^2}e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}2(u+v)v\frac{\beta_0'}{\beta_0}\Delta.$$

Summarizing, we obtain

$$\widetilde{B}_{\Delta,6} = -\frac{1}{4}u^2(u+v)v\frac{(\beta_0')^2}{\sigma_0^2}e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\left((u+v)^2-2\right)\Delta.$$

Using the same argument for $\widetilde{B}_{\Delta,6}$, only the term in $\eta_0\kappa_0$ of $\widetilde{B}_{\Delta,7}$ is non zero. We, thus, have

$$\begin{aligned}\widetilde{B}_{\Delta,7} &= -\frac{1}{\Delta}v^2\frac{\eta_0\kappa_0}{\beta_0^2}\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W'_\Delta}{\sqrt{\Delta}}}\int_0^\Delta W_sdW_s\int_0^\Delta W_sdW'_s\right]\\ &= -\frac{1}{\Delta}v^2\frac{\eta_0\kappa_0}{\beta_0^2}\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W'_\Delta}{\sqrt{\Delta}}}\left(\int_0^\Delta\int_0^s W_{s_1}dW_{s_1}W_sdW'_s + \int_0^\Delta\int_0^s W_{s_1}dW'_{s_1}W_sdW_s\right)\right]\\ &= \widetilde{B}^a_{\Delta,7} + \widetilde{B}^b_{\Delta,7}.\end{aligned}$$

For $\widetilde{B}^a_{\Delta,7}$ we use

$$\begin{aligned}W_s\int_0^s W_{s_1}dW_{s_1} &= \int_0^s\int_0^{s_1} W_{s_2}dW_{s_2}dW_{s_1} + \int_0^s W_{s_1}^2dW_{s_1} + \int_0^s W_{s_1}ds_1\\ &= \int_0^s\int_0^{s_1} W_{s_2}dW_{s_2}dW_{s_1} + 2\int_0^s\int_0^{s_1} W_{s_2}dW_{s_2}dW_{s_1} + \int_0^s s_1dW_{s_1} + \int_0^s W_sds,\end{aligned}$$

which implies

$$\widetilde{B}^a_{\Delta,7} = -v^2\frac{\eta_0\kappa_0}{\beta_0^2}v\frac{\beta_0'}{\beta_0}\left(\frac{(u+v)^3}{8} - \frac{(u+v)}{3}\right)e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\Delta.$$

For $\widetilde{B}^b_{\Delta,7}$, we use

$$\begin{aligned}W_s\int_0^s W_{s_1}dW'_{s_1} &= \int_0^s\int_0^{s_1} W_{s_2}dW'_{s_2}dW_{s_1} + \int_0^s W_{s_1}^2 dW'_{s_1}\\ &= \int_0^s\int_0^{s_1} W_{s_2}dW'_{s_2}dW_{s_1} + 2\int_0^s\int_0^{s_1} W_{s_2}dW_{s_2}dW'_{s_1} + \int_0^s s_1 dW'_{s_1},\end{aligned}$$

which implies

$$\widetilde{B}^b_{\Delta,7} = -v^2\frac{\eta_0\kappa_0}{\beta_0^2}(u+v)\left(\frac{(u+v)^2 v\frac{\beta_0'}{\beta_0}}{8} - \frac{v\frac{\beta_0'}{\beta_0}}{6}\right)e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\Delta.$$

Summarizing, again, we have

$$\widetilde{B}_{\Delta,7} = -\frac{1}{4}v^3(u+v)\frac{\eta_0\kappa_0\beta_0'}{\beta_0^3}\left((u+v)^2 - 2\right)e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\Delta.$$

The final term is $\widetilde{B}_{8,\Delta}$, which comprises 8 terms. The term in $\beta_0\eta_0$ is

$$\begin{aligned}\widetilde{B}^a_{8,\Delta} &= -\frac{uv}{\sigma_0\beta_0}\frac{1}{\Delta}\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W'_\Delta}{\sqrt{\Delta}}}\int_0^\Delta \beta_0 W_s dW_s\int_0^\Delta \eta_0 W_s dW_s\right]\\ &= -uv\frac{\eta_0}{\sigma_0}\frac{1}{\Delta}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}\left(\frac{W_\Delta^2}{2} - \frac{\Delta}{2}\right)^2\right]\\ &= -\frac{1}{4}uv\frac{\eta_0}{\sigma_0}e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}(2 - 4(u+v)^2 + (u+v)^4)\Delta.\end{aligned}$$

The terms in $\beta_0\eta_0'$ and $\beta_0\kappa_0'$ are zero. The term in $\beta_0\kappa_0$ can be treated like the term $\widetilde{B}_{7,\Delta}$, so that

$$\widetilde{B}^b_{8,\Delta} = -\frac{1}{4}uv^2(u+v)\frac{\beta_0'}{\sigma_0}\frac{\kappa_0}{\beta_0}\left((u+v)^2 - 2\right)e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\Delta.$$

The terms in $\beta_0'\eta_0'$ and $\beta_0'\kappa_0'$ are zero. The term in $\beta_0'\eta_0$ can be treated like the term $\widetilde{B}_{6,\Delta}$:

$$\widetilde{B}^c_{8,\Delta} = -\frac{1}{4}uv^2(u+v)\left(\frac{\beta_0'}{\beta_0}\right)^2\frac{\eta_0}{\sigma_0}e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\left((u+v)^2 - 2\right)\Delta.$$

Finally, regarding the term in $\beta_0'\kappa_0$, we have

$$\widetilde{B}^d_{\Delta,8} = -\frac{1}{\Delta}uv\frac{\beta_0'\kappa_0}{\sigma_0\beta_0}\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W'_\Delta}{\sqrt{\Delta}}}\int_0^\Delta W'_s dW_s\int_0^\Delta W_s dW'_s\right]$$

$$
\begin{aligned}
&= -\frac{1}{\Delta}uv\frac{\beta_0'\kappa_0}{\sigma_0\beta_0}\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\left(\int_0^\Delta\int_0^s W_{s_1}'dW_{s_1}W_s dW_s' + \int_0^\Delta\int_0^s W_{s_1}dW_{s_1}'W_s'dW_s\right)\right]\\
&:= \widetilde{B}_{\Delta,8}^{d,a} + \widetilde{B}_{\Delta,8}^{d,b}.
\end{aligned}
$$

For the term $\widetilde{B}_{\Delta,8}^{d,a}$, we use

$$
\begin{aligned}
W_s\int_0^s W_{s_1}'dW_{s_1} &= \int_0^s\int_0^{s_1} W_{s_2}'dW_{s_2}dW_{s_1} + \int_0^s W_{s_1}'W_{s_1}dW_{s_1} + \int_0^s W_{s_1}'ds_1\\
&= \int_0^s\int_0^{s_1} W_{s_2}'dW_{s_2}dW_{s_1} + \int_0^s\int_0^{s_1} W_{s_2}'dW_{s_2}dW_{s_1} + \int_0^s\int_0^{s_1} W_{s_2}dW_{s_2}'dW_{s_1} + \int_0^s W_{s_1}'ds_1,
\end{aligned}
$$

so that

$$
\widetilde{B}_{\Delta,8}^{d,a} = -uv\frac{\beta_0'\kappa_0}{\sigma_0\beta_0}v\frac{\beta_0'}{\beta_0}\left(\frac{1}{8}(u+v)^2 v\frac{\beta_0'}{\beta_0} - \frac{1}{6}v\frac{\beta_0'}{\beta_0}\right)e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\Delta.
$$

For the term $\widetilde{B}_{\Delta,8}^{d,b}$, we use

$$
W_s'\int_0^s W_{s_1}dW_{s_1}' = \int_0^s\int_0^{s_1} W_{s_2}dW_{s_2}'dW_{s_1}' + \int_0^s\int_0^{s_1} W_{s_2}dW_{s_2}'dW_{s_1}' + \int_0^s\int_0^{s_1} W_{s_2}'dW_{s_2}dW_{s_1}' + \int_0^s W_{s_1}ds_1,
$$

so that

$$
\widetilde{B}_{\Delta,8}^{d,b} = -uv\frac{\beta_0'\kappa_0}{\sigma_0\beta_0}(u+v)\left(\frac{1}{8}(u+v)\left(v\frac{\beta_0'}{\beta_0}\right)^2 - \frac{1}{6}(u+v)\right)e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\Delta.
$$

Summarizing, we obtain

$$
\begin{aligned}
\widetilde{B}_{\Delta,8} = &-e^{-\frac{(u+v)^2}{2}}e^{-\frac{v^2}{2}\left(\frac{\beta_0'}{\beta_0}\right)^2}\left(\frac{1}{4}uv\frac{\eta_0}{\sigma_0}(2-4(u+v)^2+(u+v)^4) + \frac{1}{4}uv^2(u+v)\frac{\beta_0'}{\sigma_0}\frac{\kappa_0}{\beta_0}\left((u+v)^2-2\right)\right.\\
&+\frac{1}{4}uv^2(u+v)\left(\frac{\beta_0'}{\beta_0}\right)^2\frac{\eta_0}{\sigma_0}\left((u+v)^2-2\right) + uv\frac{\beta_0'\kappa_0}{\sigma_0\beta_0}v\frac{\beta_0'}{\beta_0}\left(\frac{1}{8}(u+v)^2 v\frac{\beta_0'}{\beta_0} - \frac{1}{6}v\frac{\beta_0'}{\beta_0}\right)\\
&\left.+uv\frac{\beta_0'\kappa_0}{\sigma_0\beta_0}(u+v)\left(\frac{1}{8}(u+v)\left(v\frac{\beta_0'}{\beta_0}\right)^2 - \frac{1}{6}(u+v)\right)\right)\Delta.
\end{aligned}
$$

This completes the proof.

□

Proof of Corollary 1. The density is obtained from the characteristic function using the classical Fourier inversion formula:

$$
f_{Z_\Delta}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuz}\mathbb{E}_0\left[e^{iuZ_\Delta}\right]du,
$$

provided integrability, i.e., $\int_{-\infty}^{\infty}\left|\mathbb{E}_0\left[e^{iuZ_\Delta}\right]\right|du < \infty$ is satisfied. Focusing, e.g., on the last term in the proof of Theorem 3, namely $C_\Delta(u)$, we have

$$\int_{-\infty}^{\infty} C_\Delta(u)du \sim C\Delta^{3/2} \int_{-\infty}^{\infty} g_5(u,1)\,|u|^3\,du \sim O(\Delta^{3/2}).$$

The same argument is used for all other terms by appealing to the integrability - over u - of (functions of) the function $g_k(u,1)$ for suitable k values.

The final expression, written in terms of Hermite polynomials, is obtained by using a classical property of these polynomials:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuz}\,(-iu)^k\,e^{-\frac{u^2}{2}}du = H_k(z)\varphi(z) \qquad \forall k.$$

□

Proof of Theorem 3. In the proof, for conciseness, we will ignore idiosyncratic discontinuities in the price process (given by J^X). It will however be apparent that their addition (in the statement of the theorem) is straightforward given the way in which $J^{X,\sigma}$ and J^σ are handled below. Consistent with the proof of Theorem 1, we start with $v = 0$.

Step 1. Set $v = 0$.

As in Theorem 7.4.1 in Daley and Vere-Jones (2003), write

$$J_t^{X,\sigma} = N^{X,\sigma}_{\int_0^t \lambda_s^{X,\sigma} ds}, \quad J_t^\sigma = N^\sigma_{\int_0^t \lambda_s^\sigma ds}, \tag{59}$$

where $N_t^{X,\sigma}$ and N_t^σ are Poisson processes with unit intensity. Consistent with the above embedding, define the same quantities with constant intensities (set at time 0):

$$\widetilde{J}_t^{X,\sigma} = N^{X,\sigma}_{\lambda_0^{X,\sigma} t}, \quad \widetilde{J}_t^\sigma = N^\sigma_{\lambda_0^\sigma t}. \tag{60}$$

Under Assumption 3, Z_Δ is now given by

$$\begin{aligned}
Z_\Delta &= \frac{W_\Delta}{\sqrt{\Delta}} + \frac{c_0^{X,\sigma} J_\Delta^{X,\sigma}}{\sigma_0\sqrt{\Delta}} + \frac{\int_0^\Delta (c_s^{X,\sigma} - c_0^{X,\sigma}) dJ_s^{X,\sigma}}{\sigma_0\sqrt{\Delta}} \\
&\quad + \frac{1}{\sigma_0\sqrt{\Delta}}\left(\int_0^\Delta (\mu_s - \mu_0)ds + \int_0^\Delta (\sigma_s - \sigma_0)dW_s\right) \\
&= \frac{W_\Delta}{\sqrt{\Delta}} + \frac{c_0^{X,\sigma} \widetilde{J}_\Delta^{X,\sigma}}{\sigma_0\sqrt{\Delta}} + \frac{\int_0^\Delta (c_s^{X,\sigma} - c_0^{X,\sigma}) dJ_s^{X,\sigma}}{\sigma_0\sqrt{\Delta}} + \frac{c_0^{X,\sigma}(J_\Delta^{X,\sigma} - \widetilde{J}_\Delta^{X,\sigma})}{\sigma_0\sqrt{\Delta}} \\
&\quad + \frac{\int_0^\Delta (\mu_s - \mu_0)ds}{\sigma_0\sqrt{\Delta}} + \frac{1}{\sigma_0\sqrt{\Delta}}\int_0^\Delta \left(\int_0^s \alpha_u du + \int_0^s \beta_u dW_u' + \int_0^s c_u^{\sigma,X} dJ_u^{X,\sigma} + \int_0^s c_u^\sigma dJ_u^\sigma\right) dW_s \\
&= \frac{W_\Delta}{\sqrt{\Delta}} + \frac{c_0^{X,\sigma} \widetilde{J}_\Delta^{X,\sigma}}{\sigma_0\sqrt{\Delta}} + \frac{\int_0^\Delta (c_s^{X,\sigma} - c_0^{X,\sigma}) dJ_s^{X,\sigma}}{\sigma_0\sqrt{\Delta}} + \frac{c_0^{X,\sigma}(J_\Delta^{X,\sigma} - \widetilde{J}_\Delta^{X,\sigma})}{\sigma_0\sqrt{\Delta}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\int_0^\Delta(\mu_s-\mu_0)ds}{\sigma_0\sqrt{\Delta}}+\frac{1}{\sigma_0\sqrt{\Delta}}\int_0^\Delta\left(\int_0^s\alpha_u du+\int_0^s\beta_u dW'_u\right)dW_s\\
&+\frac{1}{\sigma_0\sqrt{\Delta}}\int_0^\Delta\left(c_0^{\sigma,X}\widetilde{J}_s^{X,\sigma}+c_0^\sigma\widetilde{J}_s^\sigma\right)dW_s\\
&\frac{1}{\sigma_0\sqrt{\Delta}}\int_0^\Delta\left(c_0^{\sigma,X}(J_s^{X,\sigma}-\widetilde{J}_s^{X,\sigma})+c_0^\sigma(J_s^\sigma-\widetilde{J}_s^\sigma)\right)dW_s\\
&+\frac{1}{\sigma_0\sqrt{\Delta}}\int_0^\Delta\left(\int_0^s(c_u^{\sigma,X}-c_0^{\sigma,X})dJ_u^{X,\sigma}+\int_0^s(c_u^\sigma-c_0^\sigma)dJ_u^\sigma\right)dW_s.
\end{aligned}
$$

Integrating by parts, we have

$$
\int_0^\Delta\left(c_0^{\sigma,X}\widetilde{J}_s^{X,\sigma}+c_0^\sigma\widetilde{J}_s^\sigma\right)dW_s=c_0^{\sigma,X}\int_0^\Delta(W_\Delta-W_s)d\widetilde{J}_s^{X,\sigma}+c_0^\sigma\int_0^\Delta(W_\Delta-W_s)d\widetilde{J}_s^\sigma, \tag{61}
$$

so that

$$
Z_\Delta:=\frac{W_\Delta}{\sqrt{\Delta}}+U_\Delta+Y_\Delta\sqrt{\Delta},
$$

where

$$
\begin{aligned}
Y_\Delta=&\frac{c_0^{X,\sigma}(J_\Delta^{X,\sigma}-\widetilde{J}_\Delta^{X,\sigma})+\int_0^\Delta(c_s^{X,\sigma}-c_0^{X,\sigma})dJ_s^{X,\sigma}}{\sigma_0\Delta}\\
&+\frac{\int_0^\Delta(\mu_s-\mu_0)ds}{\sigma_0\Delta}\\
&+\frac{1}{\sigma_0\Delta}\int_0^\Delta\left(\int_0^s\alpha_u du+\int_0^s\beta_u dW'_u\right.\\
&\left.+\int_0^s(c_u^{\sigma,X}-c_0^{\sigma,X})dJ_u^{X,\sigma}+\int_0^s(c_u^\sigma-c_0^\sigma)dJ_u^\sigma+\left(c_0^{\sigma,X}(J_s^{X,\sigma}-\widetilde{J}_s^{X,\sigma})+c_0^\sigma(J_s^\sigma-\widetilde{J}_s^\sigma)\right)\right)dW_s,
\end{aligned}
$$

and

$$
U_\Delta=\frac{c_0^{X,\sigma}\widetilde{J}_\Delta^{X,\sigma}}{\sigma_0\sqrt{\Delta}}+\frac{c_0^{\sigma,X}\int_0^\Delta(W_\Delta-W_s)d\widetilde{J}_s^{X,\sigma}+c_0^\sigma\int_0^\Delta(W_\Delta-W_s)d\widetilde{J}_s^\sigma}{\sigma_0\sqrt{\Delta}}.
$$

By Taylor expansion, as in the proof of Theorem 1, we obtain

$$
e^{iuY_\Delta\sqrt{\Delta}}=\sum_{k=0}^{+\infty}\frac{(iu)^k}{k!}\Delta^{\frac{k}{2}}Y_\Delta^k=1+iuY_\Delta\sqrt{\Delta}+\frac{1}{2}(iu)^2Y_\Delta^2\Delta+\sum_{k=3}^{+\infty}\frac{(iu)^k}{k!}\Delta^{\frac{k}{2}}Y_\Delta^k.
$$

It follows that,

$$
\begin{aligned}
\phi_{Z_\Delta}(u)=&\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}e^{iuU_\Delta}\left(1+iuY_\Delta\sqrt{\Delta}+\frac{1}{2}(iu)^2Y_\Delta^2\Delta+\sum_{k=3}^{+\infty}\frac{(iu)^k}{k!}\Delta^{\frac{k}{2}}Y_\Delta^k\right)\right]\\
:=&\underbrace{\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}e^{iuU_\Delta}\right]}_{A_{\Delta,0}^J}+\underbrace{\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}e^{iuU_\Delta}iuY_\Delta\sqrt{\Delta}\right]}_{A_{\Delta,1}^J}+\underbrace{\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}e^{iuU_\Delta}\frac{1}{2}(iu)^2Y_\Delta^2\Delta\right]}_{A_{\Delta,2}^J}+A_{\Delta,3}^J.
\end{aligned}
$$

We begin with the term $A_{\Delta}^{J,0}$. Using the fact that $\widetilde{J}_s^{X,\sigma}$ and $\widetilde{J}_s^{\sigma}$ are Poisson process with constant intensity independent of W, conditioning on the path of W we have,

$$\begin{aligned}\mathbb{E}\left[e^{iuU_\Delta}\middle|\mathcal{F}^W\right] &= \mathbb{E}\left[e^{iu\left(\frac{c_0^{X,\sigma}\widetilde{J}_\Delta^{X,\sigma}+c_0^{\sigma,X}\int_0^\Delta(W_\Delta-W_s)d\widetilde{J}_s^{X,\sigma}}{\sigma_0\sqrt{\Delta}}\right)}\middle|\mathcal{F}^W\right]\mathbb{E}\left[e^{iu\left(\frac{c_0^{\sigma}\int_0^\Delta(W_\Delta-W_s)d\widetilde{J}_s^{\sigma}}{\sigma_0\sqrt{\Delta}}\right)}\middle|\mathcal{F}^W\right]\\
&= e^{\int_0^\Delta\int_{\mathbb{R}^2}\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}(x+(W_\Delta-W_s)y)}-1\right)\lambda_0^{X,\sigma}f_0^{X,\sigma}(dx,dy)ds}\, e^{\int_0^\Delta\int_{\mathbb{R}}\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}((W_\Delta-W_s)z)}-1\right)\lambda_0^{\sigma}f_0^{\sigma}(dz)ds}\\
&= e^{\int_0^\Delta\left(\int_{\mathbb{R}^2}\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}(x+(W_\Delta-W_s)y)}-1\right)\lambda_0^{X,\sigma}f_0^{X,\sigma}(dx,dy)+\int_{\mathbb{R}}\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}((W_\Delta-W_s)z)}-1\right)\lambda_0^{\sigma}f_0^{\sigma}(dz)\right)ds}.\end{aligned}$$

By the law of iterated expectations, after Taylor expanding the above term around 0 to the first order and neglecting higher (in Δ) order terms, we have

$$\begin{aligned}A_{\Delta,0}^{J} &= \mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\mathbb{E}\left[e^{iuU_\Delta}\middle|\mathcal{F}^W\right]\right]\\
&= e^{-u^2/2}+\underbrace{\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\int_0^\Delta\int_{\mathbb{R}^2}\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}(x+(W_\Delta-W_s)y)}-1\right)\lambda_0^{X,\sigma}f_0^{X,\sigma}(dx,dy)ds\right]}_{A_{\Delta,0,1}^{J}}\\
&+ \underbrace{\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\int_0^\Delta\int_{\mathbb{R}}\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}((W_\Delta-W_s)z)}-1\right)\lambda_0^{\sigma}f_0^{\sigma}(dz)ds\right]}_{A_{\Delta,0,2}^{J}}. \quad (62)\end{aligned}$$

Regarding the term $A_{\Delta,0,1}^{J}$, noticing that $W_\Delta - W_s$ is independent of W_s, we obtain

$$\begin{aligned}&\int_0^\Delta\int_{\mathbb{R}^2}\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}(x+(W_\Delta-W_s)y)}-1\right)\right]\lambda_0^{X,\sigma}f_0^{X,\sigma}(dx,dy)ds\\
&=\int_0^\Delta\int_{\mathbb{R}^2}\mathbb{E}_0\left[e^{iu\frac{W_s}{\sqrt{\Delta}}}\right]\mathbb{E}_0\left[e^{iu\frac{W_\Delta-W_s}{\sqrt{\Delta}}}\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}(x+(W_\Delta-W_s)y)}-1\right)\right]\lambda_0^{X,\sigma}f_0^{X,\sigma}(dx,dy)ds\\
&=\int_0^\Delta\int_{\mathbb{R}^2}e^{-\frac{u^2}{2}\frac{s}{\Delta}}\mathbb{E}_0\left[e^{iu\frac{W_\Delta-W_s}{\sqrt{\Delta}}}\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}(x+(W_\Delta-W_s)y)}-1\right)\right]\lambda_0^{X,\sigma}f_0^{X,\sigma}(dx,dy)ds\\
&=\int_0^\Delta\int_{\mathbb{R}^2}e^{-\frac{u^2}{2}\frac{s}{\Delta}}\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}x}e^{-\frac{u^2}{2}\left(1+\frac{y}{\sigma_0}\right)^2\frac{\Delta-s}{\Delta}}-e^{-\frac{u^2}{2}\frac{\Delta-s}{\Delta}}\right)\lambda_0^{X,\sigma}f_0^{X,\sigma}(dx,dy)ds\\
&=\int_0^\Delta\int_{\mathbb{R}^2}e^{-\frac{u^2}{2}\frac{s}{\Delta}}\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}x}e^{-\frac{u^2}{2}\left(1+\frac{y^2}{\sigma_0^2}+2\frac{y}{\sigma_0}\right)\frac{\Delta-s}{\Delta}}-e^{-\frac{u^2}{2}\frac{\Delta-s}{\Delta}}\right)\lambda_0^{X,\sigma}f_0^{X,\sigma}(dx,dy)ds\\
&=\int_0^\Delta\int_{\mathbb{R}^2}e^{-\frac{u^2}{2}\frac{s}{\Delta}}\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}x}e^{-\frac{u^2}{2}\left(\frac{y^2}{\sigma_0^2}+2\frac{y}{\sigma_0}\right)\frac{\Delta-s}{\Delta}}e^{-\frac{u^2}{2}\frac{\Delta-s}{\Delta}}-e^{-\frac{u^2}{2}\frac{\Delta-s}{\Delta}}\right)\lambda_0^{X,\sigma}f_0^{X,\sigma}(dx,dy)ds\\
&=\int_0^\Delta\int_{\mathbb{R}^2}\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}x}e^{-\frac{u^2}{2}\left(\frac{y^2}{\sigma_0^2}+2\frac{y}{\sigma_0}\right)\frac{\Delta-s}{\Delta}}e^{-\frac{u^2}{2}\frac{\Delta}{\Delta}}-e^{-\frac{u^2}{2}\frac{\Delta}{\Delta}}\right)\lambda_0^{X,\sigma}f_0^{X,\sigma}(dx,dy)ds\\
&=\Delta e^{-\frac{u^2}{2}}\int_{\mathbb{R}^2}\int_0^1\left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}x}e^{-\frac{u^2}{2}\left(\frac{y^2}{\sigma_0^2}+2\frac{y}{\sigma_0}\right)v}-1\right)\lambda_0^{X,\sigma}f_0^{X,\sigma}(dx,dy)dv. \quad (63)\end{aligned}$$

Regarding the term $A^J_{\Delta,0,2}$, using again the fact that $W_\Delta - W_s$ is independent of W_s, we have

$$
\begin{aligned}
&\int_0^\Delta \int_{\mathbb{R}} \mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}} \left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}((W_\Delta - W_s)z)} - 1 \right) \right] \lambda_0^\sigma f_0^\sigma(dz)ds \\
&= \int_0^\Delta \int_{\mathbb{R}} \mathbb{E}_0\left[e^{iu\frac{W_s}{\sqrt{\Delta}}} \right] \mathbb{E}_0\left[e^{iu\frac{W_\Delta - W_s}{\sqrt{\Delta}}} \left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}((W_\Delta - W_s)z)} - 1 \right) \right] \lambda_0^\sigma f_0^\sigma(dz)ds \\
&= \int_0^\Delta \int_{\mathbb{R}} e^{-\frac{u^2}{2}\frac{s}{\Delta}} \mathbb{E}_0\left[e^{iu\frac{W_\Delta - W_s}{\sqrt{\Delta}}} \left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}((W_\Delta - W_s)z)} - 1 \right) \right] \lambda_0^\sigma f_0^\sigma(dz)ds \\
&= \int_0^\Delta \int_{\mathbb{R}} e^{-\frac{u^2}{2}\frac{s}{\Delta}} \left(e^{-\frac{u^2}{2}\left(1+\frac{z}{\sigma_0}\right)^2 \frac{\Delta - s}{\Delta}} - e^{-\frac{u^2}{2}\frac{\Delta - s}{\Delta}} \right) \lambda_0^\sigma f_0^\sigma(dz)ds \\
&= \Delta e^{-\frac{u^2}{2}} \int_{\mathbb{R}} \int_0^1 \left(e^{-\frac{u^2}{2}\left(\frac{z^2}{\sigma_0^2} + 2\frac{z}{\sigma_0}\right)v} - 1 \right) \lambda_0^\sigma f_0^\sigma(dz)dv. \qquad (64)
\end{aligned}
$$

Eq. (63) and (64) lead to the expression for the jump component in the statement of the theorem. Because this component is of order Δ, excluding higher-order terms in the expansion leading to Eq. (62) is justified.

For the term $A^J_{\Delta,1}$, we use, once more, a Taylor expansion to write $e^{iuU_\Delta} = 1 + iuU_\Delta + O_p(\Delta^2)$, which gives

$$
A^J_{\Delta,1} = A_\Delta + \widetilde{A}^J_\Delta + iuA_\Delta U_\Delta + iu\widetilde{A}^J_{\Delta,2} U_\Delta + o_p(\Delta),
$$

where A_Δ is defined (and treated) as in the proof of Theorem 3 and

$$
\begin{aligned}
\widetilde{A}^J_\Delta =& \frac{iu}{\sigma_0\sqrt{\Delta}} \mathbb{E}_0\Bigg[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}} \Bigg(c_0^{X,\sigma}(J_\Delta^{X,\sigma} - \widetilde{J}_\Delta^{X,\sigma}) + \int_0^\Delta (c_s^{X,\sigma} - c_0^{X,\sigma}) dJ_s^{X,\sigma} \\
&+ \int_0^\Delta \left(\int_0^s (c_u^{\sigma,X} - c_0^{\sigma,X}) dJ_u^{X,\sigma} + \int_0^s (c_u^\sigma - c_0^\sigma) dJ_u^\sigma + \left(c_0^{\sigma,X}(J_s^{X,\sigma} - \widetilde{J}_s^{X,\sigma}) + c_0^\sigma (J_s^\sigma - \widetilde{J}_s^\sigma) \right) \right) dW_s \Bigg) \Bigg] \\
=& \widetilde{A}^J_{\Delta,1} + \widetilde{A}^J_{\Delta,2} + \widetilde{A}^J_{\Delta,3} + \widetilde{A}^J_{\Delta,4} + \widetilde{A}^J_{\Delta,5} + \widetilde{A}^J_{\Delta,6}.
\end{aligned}
$$

By independence of W and all Poisson processes, we have

$$
\begin{aligned}
\widetilde{A}^J_{\Delta,1} &= \frac{iuc_0^{X,\sigma}}{\sigma_0\sqrt{\Delta}} \mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}} \left(J_\Delta^{X,\sigma} - \widetilde{J}_\Delta^{X,\sigma} \right) \right] \\
&= \frac{iuc_0^{X,\sigma}}{\sigma_0\sqrt{\Delta}} \mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}} \right] \mathbb{E}_0\left[\left(J_\Delta^{X,\sigma} - \widetilde{J}_\Delta^{X,\sigma} \right) \right] \\
&\leq \frac{iuc_0^{X,\sigma}}{\sigma_0\sqrt{\Delta}} e^{-\frac{u^2}{2}} \mathbb{E}_0\left[\left| J_\Delta^{X,\sigma} - \widetilde{J}_\Delta^{X,\sigma} \right| \right].
\end{aligned}
$$

Given the representation in Eq. (59) and Eq. (60), write

$$
\left| J_\Delta^{X,\sigma} - \widetilde{J}_\Delta^{X,\sigma} \right| = N^{X,\sigma}_{\left| \int_0^\Delta (\lambda_s^{X,\sigma} - \lambda_0^{X,\sigma}) ds \right|},
$$

which yields

$$\begin{aligned}
& \mathbb{E}_0\left[\left|J_\Delta^{X,\sigma}-\widetilde{J}_\Delta^{X,\sigma}\right|\right] \\
= \; & \mathbb{E}_0\left[\left|\int_0^\Delta(\lambda_s^{X,\sigma}-\lambda_0^{X,\sigma})ds\right|\right] \\
\leq \; & \int_0^\Delta \mathbb{E}_0\left[\left|\lambda_s^{X,\sigma}-\lambda_0^{X,\sigma}\right|\right] ds \\
\leq \; & \int_0^\Delta \sqrt{\mathbb{E}_0\left[\left|\lambda_s^{X,\sigma}-\lambda_0^{X,\sigma}\right|^2\right]} ds \\
\leq \; & C\Delta^{1+\Gamma_\lambda/2}.
\end{aligned}$$

It follows that $\widetilde{A}_{\Delta,1}^J = O(\Delta^{1/2+\Gamma_\lambda/2})$ and, therefore, $o(\Delta)$ given Eq. (38). Now, by Cauchy-Schwartz inequality, write

$$\begin{aligned}
\widetilde{A}_{\Delta,2}^J &= \frac{iuc_0^{X,\sigma}}{\sigma_0\sqrt{\Delta}}\mathbb{E}_0\left[e^{iu\frac{W_\Delta}{\sqrt{\Delta}}}\int_0^\Delta(c_s^{X,\sigma}-c_0^{X,\sigma})dJ_s^{X,\sigma}\right] \\
&= \frac{iuc_0^{X,\sigma}}{\sigma_0\sqrt{\Delta}}\sqrt{\mathbb{E}_0\left[e^{iu\frac{2W_\Delta}{\sqrt{\Delta}}}\right]}\sqrt{\left[\int_0^\Delta \mathbb{E}_0(c_s^{X,\sigma}-c_0^{X,\sigma})^2\lambda_s^{X,\sigma}ds\right]} \\
&\leq C\frac{iuc_0^{X,\sigma}}{\sigma_0\sqrt{\Delta}}e^{-u^2}\Delta^{1/2+\Gamma_c/2}.
\end{aligned}$$

We conclude that $\widetilde{A}_{\Delta,2}^J = O(\Delta^{\Gamma_c/2})$ and, therefore, $o(\Delta)$, given Eq. (39). Similarly, we have

$$\widetilde{A}_{\Delta,3}^J = O(\Delta^{1/2+\Gamma_c/2}),$$

$$\widetilde{A}_{\Delta,4}^J = O(\Delta^{1/2+\Gamma_c/2}),$$

$$\widetilde{A}_{\Delta,5}^J = O(\Delta^{1+\Gamma_\lambda/2}),$$

$$\widetilde{A}_{\Delta,6}^J = O(\Delta^{1+\Gamma_\lambda/2}).$$

The remaining terms are, readily, o(Δ).

Step 2. Now, assume $v \neq 0$. The above reasoning implies that

$$\begin{aligned}
\mathbb{E}_0\left[e^{iuZ_\Delta+ivV_\Delta}\right] &= \phi^c(u,v)+\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}e^{iu\widetilde{U}_\Delta^1}e^{iv\widetilde{U}_\Delta^2}\right] \\
&=\mathbb{E}_0\left[e^{i(u+v)\frac{W_\Delta}{\sqrt{\Delta}}}e^{iv\frac{\beta_0'}{\beta_0}\frac{W_\Delta'}{\sqrt{\Delta}}}\left(iu\widetilde{U}_\Delta^1+iv\widetilde{U}_\Delta^2\right)\right]+o(\Delta) \\
&=\sum_{k=0}^5\widetilde{A}_\Delta^{J,k},
\end{aligned}$$

where

$$\widetilde{U}^1_\Delta = \frac{c_0^X \widetilde{J}^X_\Delta}{\sigma_0\sqrt{\Delta}} + \frac{c_0^{X,\sigma} \widetilde{J}^{X,\sigma}_\Delta}{\sigma_0\sqrt{\Delta}} + \frac{c_0^{\sigma,X} \int_0^\Delta (W_\Delta - W_s) d\widetilde{J}^{X,\sigma}_s + c_0^\sigma \int_0^\Delta (W_\Delta - W_s) d\widetilde{J}^\sigma_s}{\sigma_0\sqrt{\Delta}},$$

$$\widetilde{U}^2_\Delta = \frac{c_0^\sigma \widetilde{J}^\sigma_\Delta}{\beta_0\sqrt{\Delta}} + \frac{c_0^{X,\sigma} \widetilde{J}^{X,\sigma}_\Delta}{\beta_0\sqrt{\Delta}}.$$

Using the fact that Eq. (63) can be generalized using $u' \neq u$ as

$$\begin{aligned}
&\int_0^\Delta \int_{\mathbb{R}^2} \mathbb{E}_0 \left[e^{iu' \frac{W_\Delta}{\sqrt{\Delta}}} \left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}(x+(W_\Delta - W_s)y)} - 1 \right) \right] \lambda_0^{X,\sigma} f_0^{X,\sigma}(dx,dy)ds \\
&= \int_0^\Delta \int_{\mathbb{R}^2} \mathbb{E}_0 \left[e^{iu' \frac{W_s}{\sqrt{\Delta}}} \right] \mathbb{E}_0 \left[e^{iu' \frac{W_\Delta - W_s}{\sqrt{\Delta}}} \left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}(x+(W_\Delta - W_s)y)} - 1 \right) \right] \lambda_0^{X,\sigma} f_0^{X,\sigma}(dx,dy)ds \\
&= \int_0^\Delta \int_{\mathbb{R}^2} e^{-\frac{u'^2}{2}\frac{s}{\Delta}} \mathbb{E}_0 \left[e^{iu' \frac{W_\Delta - W_s}{\sqrt{\Delta}}} \left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}(x+(W_\Delta - W_s)y)} - 1 \right) \right] \lambda_0^{X,\sigma} f_0^{X,\sigma}(dx,dy)ds \\
&= \int_0^\Delta \int_{\mathbb{R}^2} e^{-\frac{u'^2}{2}\frac{s}{\Delta}} \left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}x} e^{-\frac{(uy+u'\sigma_0)^2}{2\sigma_0^2}\frac{\Delta - s}{\Delta}} - e^{-\frac{u'^2}{2}\frac{\Delta - s}{\Delta}} \right) \lambda_0^{X,\sigma} f_0^{X,\sigma}(dx,dy)ds \\
&= e^{-\frac{u'^2}{2}} \int_0^\Delta \int_{\mathbb{R}^2} \left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}x} e^{-\frac{u^2y^2 + 2u'uy\sigma_0}{2\sigma_0^2}\frac{\Delta - s}{\Delta}} - 1 \right) \lambda_0^{X,\sigma} f_0^{X,\sigma}(dx,dy)ds \\
&= \Delta e^{-\frac{u'^2}{2}} \int_{\mathbb{R}^2} \int_0^1 \left(e^{\frac{iu}{\sigma_0\sqrt{\Delta}}x} e^{-\frac{u^2y^2 + 2u'uy\sigma_0}{2\sigma_0^2}z} - 1 \right) \lambda_0^{X,\sigma} f_0^{X,\sigma}(dx,dy)dz
\end{aligned}$$

and choosing $u' = u + v$, the extra terms in the characteristic function are readily obtained as in the statement of the theorem. $\square$