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APPROXIMATE OPTION VALUATION FOR ARBITRARY STOCHASTIC PROCESSES*

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We show how a given probability distribution can be approximated by an arbitrary distribution in terms of a series expansion involving second and higher moments. This theoretical development is specialized to the problem of option valuation where the underlying security distribution, if not lognormal, can be approximated by a lognormally distributed random variable. The resulting option price is expressed as the sum of a Black–Scholes price plus adjustment terms which depend on the second and higher moments of the underlying security stochastic process. This approach permits the impact on the option price of skewness and kurtosis of the underlying stock's distribution to be evaluated.

1. Introduction

The Black–Scholes (1973) formula is rightly regarded by both practitioners and academics as the premier model of option valuation. In spite of its preeminence it has some well-known deficiencies. Empirically, model prices appear to differ from market prices in certain systematic ways [see, e.g., Black (1975)]. These biases are usually ascribed to the strong assumption that the underlying security follows a stationary geometric Brownian motion. The implication is that at the end of any finite interval the stock price is lognormally distributed and that over succeeding intervals the variance is constant.

The majority of the empirical evidence [e.g., Rosenberg (1972), Oldfield, Rogalski and Jarrow (1977)] suggests that this assumption does not hold. Consequently, many theoretical option valuation models have been derived utilizing different and (arguably) more realistic assumptions for the underlying security stochastic process [e.g., Cox (1975), Cox and Ross (1976), Merton (1976), Geske (1979), Rubinstein (1980)]. An alternative approach is to estimate the underlying security distribution and then to use numerical

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integration techniques to obtain the option price [e.g., the method of Gastineau and Madansky reported in Gastineau (1975)]. Many of these 'second generation' models (either by comparative statics or empirical testing) have been shown to (partially) explain the biases in the original Black–Scholes model.

These approaches have some obvious shortcomings. For instance, the evaluation of an option price for an arbitrary underlying distribution may be possible only by numerical integration techniques because of the analytical intractability of the distribution function. Empirically, it may be straightforward to estimate the moments and other properties of the underlying distribution, but using that information to compute option prices may be considerably more complex. Moreover, in both approaches it is far from clear how the specification of the underlying stock's distribution manifests itself in the resulting option price, short of actually computing the option value for all distributions of interest.

This consideration is important because it arises for a large class of valuation problems where the underlying distribution is itself a convolution of other distributions. For example, the valuation of an option on a portfolio or index where the component securities are distributed lognormally. Another example is the valuation of an option on a stock whose distribution is convoluted from the distributions of the real assets held by the corporation [Rubinstein (1980)]. In these problems, partial information concerning the underlying distribution may be known (for instance, its moments may be tabulated) but the distribution function itself may be so complex as to prevent direct integration. The central question addressed here is how may this partial information be used in obtaining option prices.

In this paper we attempt to explicitly examine the impact of the underlying distribution, summarized by its moments, on the option price. The approach is to approximate the underlying distribution with an alternate (more tractable) distribution. In section 2 we derive the series expansion of a given distribution in terms of an unspecified approximating distribution. This approach, similar to the familiar Taylor series expansion for an analytic function, is called a generalized Edgeworth series expansion. It has the desirable property that the coefficients in the expansion are simple functions of the moments of the given and approximating distributions.

In section 3 we apply the Edgeworth series expansion to the problem of option valuation. It is straightforward to obtain the expected value of an option where early exercise is not optimal. In situations where riskless arbitrage is possible, Cox–Ross (1976) reasoning permits the (approximate) value of the option to be inferred from its expected value at maturity. The approximation in this approach results from the size of the error term in the expansion.

The final theoretical step is taken in section 4. Here we specify the approximating distribution to be the lognormal. Applying the results of section 3, the (approximate) option price is seen to be the Black–Scholes price plus three adjustments which depend, respectively, on the differences between the variance, skewness, and kurtosis of the underlying and the lognormal distribution.

The presence of the error term in the Edgeworth expansion indicates that our results are only approximate. Intuition suggests that for practical purposes the first four moments of the underlying distribution should capture the majority of its influence as it affects option pricing. In section 5 we provide some results from simulations to provide guidance when using this technique. For Merton's (1976) jump-diffusion option model and for Cox's (1975) constant elasticity of diffusion model, we calculate the magnitude of the error term assuming, of course, that each is the true representation of reality. The final section, section 6, concludes the paper.

2. Approximating distribution

This section modestly generalizes a technique developed by Schleher (1977) for approximating a given probability distribution, $F(s)$, called the true distribution, with an alternative distribution, $A(s)$, called the approximating distribution. Our generalization allows for the possibility of non-existing moments. In the statistics literature, this technique is called a generalized Edgeworth series expansion. The original Edgeworth series expansion [see Cramer (1946) and Kendall and Stuart (1977)] concentrated on using the standard normal as the approximating distribution. The following approach considers any arbitrary distribution $A(s)$. For simplicity of presentation, the derivation will be performed for the restricted class of distributions where $\mathrm{d}A(s)/\mathrm{d}s = a(s)$ and $\mathrm{d}F(s)/\mathrm{d}s = f(s)$ exist, i.e., distributions with continuous density functions. The derivation is generalizable; however, the presentation involves measure theory and is outside the scope of this paper.

The following notation will be employed:

$$\alpha_j(F) = \int_{-\infty}^{\infty} s^j f(s)\,\mathrm{d}s,$$

$$\mu_j(F) = \int_{-\infty}^{\infty} (s - \alpha_1(F))^j f(s)\,\mathrm{d}s, \tag{1}$$

$$\phi(F,t) = \int_{-\infty}^{\infty} \mathrm{e}^{its} f(s)\,\mathrm{d}s,$$

where $i^2 = -1$, $\alpha_j(F)$ is the jth moment of distribution F, $\mu_j(F)$ is the jth central moment of distribution F, and $\phi(F,t)$ is the characteristic function of F. We assume that $\alpha_j(F)$ exists for $j \leqq n$.

Given $\alpha_n(F)$ exists, the first $n-1$ cumulants (or semi-invariants) $\kappa_j(F)$ from $j=1,\ldots,n-1$ also exist [see Kendall and Stuart (1977)]. These are defined by

$$\log \phi(F,t) = \sum_{j=1}^{n-1} \kappa_j(F)((it)^j/j!) + \mathrm{o}(t^{n-1}), \tag{2}$$

where

$$\mathrm{o}(t^{n-1}) \quad \text{satisfies} \quad \lim_{t \to 0} \mathrm{o}(t^{n-1})/t^{n-1} = 0.$$

The relationship between the moments and the cumulants can be found by expanding $\phi(F,t)$ in terms of moments, substituting into eq. (2), and then equating the coefficients in the resulting polynomials [see Kendall and Stuart (1977, p. 73)]. For reference, the first four cumulants are

$$\begin{aligned} &\kappa_1(F) = \alpha_1(F), \qquad \kappa_2(F) = \mu_2(F), \\ &\kappa_3(F) = \mu_3(F), \qquad \kappa_4(F) = \mu_4(F) - 3\mu_2(F)^2. \end{aligned} \tag{3}$$

The first cumulant is the mean, the second the variance, the third a measure of skewness, and the fourth a measure of kurtosis.

Analogous notation will be employed for the moments, cumulants, and characteristic function of A, i.e., $\alpha_j(A)$, $\mu_j(A)$, $\kappa_j(A)$, and $\phi(A,t)$. It is assumed that both $\alpha_j(A)$ and $\mathrm{d}^j A(s)/\mathrm{d}s^j$ exist for $j \leqq m$ (where m can differ from n).

The following series expansion for $f(s)$ in terms of $a(s)$ is proven in the appendix, given $n, m \geqq 5$,

$$\begin{aligned} f(s) = a(s) &+ \frac{(\kappa_2(F) - \kappa_2(A))}{2!} \frac{\mathrm{d}^2 a(s)}{\mathrm{d}s^2} - \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{\mathrm{d}^3 a(s)}{\mathrm{d}s^3} \\ &+ \frac{((\kappa_4(F) - \kappa_4(A)) + 3(\kappa_2(F) - \kappa_2(A))^2)}{4!} \frac{\mathrm{d}a^4(s)}{\mathrm{d}s^4} + \varepsilon(s), \end{aligned} \tag{4}$$

where

$$\kappa_1(A) \equiv \kappa_1(F).$$

By construction, the first moment of the approximating distribution is set equal to the first moment of the true distribution. The difference between $f(s)$ and $a(s)$ is then expressible as a series expansion involving the higher-order cumulants of both distributions, and the derivatives of $a(s)$. The first term adjusts $a(s)$ to reflect any differences in variance between $f(s)$ and $a(s)$. The weighting factor is the second derivative of $a(s)$. The second term adjusts $a(s)$ to account for the difference in skewness between $f(s)$ and $a(s)$. The weighting factor is the third derivative. Similarly the fourth term compensates for the difference in kurtosis and variance between the two distributions with a weighting factor of the fourth derivative.[1] Depending on the existence of the higher moments, this expansion could be continued.

The residual error, $\varepsilon(s)$, contains any remaining difference between the left- and right-hand sides of (4) after the series expansion. Given arbitrary true and approximating distributions (where some moments may not exist), no general analytic bounds for this error $\varepsilon(s,N)$ as a function of N (the number of terms included) are available. For the case where all moments exist, it can be shown (see appendix) that $\varepsilon(s,N)\to 0$ uniformly in s as $N\to\infty$. In either case, for finite N, the relative size of this error needs to be examined using numerical analysis. We will return to this issue in section 5.

3. Approximate option valuation formula

This section employs the generalized Edgeworth series expansion (4) discussed in section 2 to obtain an approximate option valuation formula. The logic of the approach is simple. Using $f(s)$ as the true distribution of the stock price at maturity, the expected value at maturity of a payout protected option on that stock can be obtained.[2] The generalized Edgeworth series expansion then gives us an approximate expected value for the option at maturity in terms of the approximating distribution $a(s)$. To obtain the value of the option prior to maturity, we need to restrict the model such that a risk-neutrality valuation argument is valid.

Under continuous time models with no restrictions on preferences except non-satiation, given frictionless markets and a constant term structure, necessary and sufficient conditions for this argument being valid are that the change in both the stock's value and option's value over Δt are perfectly correlated as $\Delta t\to 0$ [see Garman (1976)]. This condition is satisfied by Markov diffusion processes where the variance component depends on at most the stock price and time. This includes the constant elasticity diffusion processes as a special case [see Cox (1975)]. Simple jump processes may also satisfy this condition [see Cox and Ross (1976)]. In addition, the argument has also been employed in Merton (1976) given a combined jump-diffusion

[1]For any distribution, $\mu_4(F)=\kappa_4(F)+3\kappa_2(F)^2$. Consequently, the third adjustment term reflects the differing 'unadjusted' kurtosis between the two distributions.

[2]See Cox and Rubinstein (1978) for the definition of a payout protected option. For organized exchanges, this would correspond to an American call option whose underlying stock has no dividend payments over the life of the option. The above approach could easily be generalized to include constant dividend yields for European options or American options where the conditions are such that it is never optimal to exercise early.

process, however, also necessary is the additional restriction that the jump risk is diversifiable.[3]

Assuming that the option model involving $f(s)$ belongs to this class of distributions, the valuation formula can be obtained by discounting the expected value of the option at maturity by the risk-free rate, while simultaneously setting the expected return on the underlying stock to be the risk-free rate.

Consider a payout protected call option at time t with striking price K, maturity date t, and where its underlying stock's value at time 0 (today) is denoted S_0. The distribution of the stock price at the maturity of the option, S_t, given the current price, S_0, will ge denoted by

$$\Pr[S_t \leqq s/S_0] \equiv F(s),$$

and represents the true underlying distribution for the stock price over $[0,t]$. Let the risk free rate, r, be constant over $[0,t]$. Consequently, under the risk neutrality argument, the true value for the call option, $C(F)$, is (using the boundary condition at maturity)

$$C(F) = e^{-rt} \int_{-\infty}^{\infty} \max[0, S_t - K]\, dF(S_t), \tag{5}$$

where

$$\alpha_1(F) \equiv S_0\, e^{rt}.$$

From the generalized Edgeworth series expansion (4), we can rewrite the valuation formula as[4]

$$\begin{aligned} C(F) = C(A) &+ e^{-rt} \frac{(\kappa_2(F) - \kappa_2(A))}{2} \int_{-\infty}^{\infty} \max[0, S_t - K] \frac{d^2 a(S_t)}{dS_t^2}\, dS_t \\ &- e^{-rt} \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \int_{-\infty}^{\infty} \max[0, S_t - K] \frac{d^3 a(S_t)}{dS_t^3}\, dS_t \\ &+ e^{-rt} \frac{((\kappa_4(F) - \kappa_4(A)) + 3(\kappa_2(F) - \kappa_2(A))^2)}{4!} \\ &\times \int_{-\infty}^{\infty} \max[0, S_t - K] \frac{d^4 a(S_t)}{dS_t^4}\, dS_t + \varepsilon(K), \end{aligned} \tag{6}$$

[3]Alternatively, in discrete time (frictionless) models given restrictions on both preferences and distributions, necessary and sufficient conditions for the validity of the risk-neutrality argument have been examined by Brennan (1979). They fall into two classes, either constant proportional risk aversion is needed when asset and market returns are jointly lognormal or constant absolute risk aversion is needed when asset and market returns are jointly normal. For this class of cases our approach is not as useful since the stock's distribution is already predetermined.

[4]By construction, the approximating distribution of S_t has the identical mean as the true distribution of S_t. In particular circumstances it is possible to set additional parameters of the two distributions equal as well; for an example, see section 4.

where

$$\alpha_1(A) \equiv e^{+rt} S_0 \quad \text{and} \quad C(A) \equiv e^{-rt} \int_{-\infty}^{\infty} \max[0, S_t - K] a(S_t)\, dS_t.$$

Expression (6) approximates the option's value with a formula, $C(A)$, based on the distribution $a(s)$ and corresponding adjustment terms. The contribution of expression (6) is an explicit representation of the adjustment terms. The first adjustment term corrects for differing variance between the approximate and true distributions. The second and third adjustment terms correct for skewness and kurtosis differences respectively. Any residual error is contained in $\varepsilon(K)$.

For any application of (6), the relative size of the error $\varepsilon(K)$ needs to be examined. An example of this analysis is performed in section 5 below.

Expression (6) is valid for any distribution $a(s)$ satisfying the assumptions behind the generalized Edgeworth series expansion. This includes the constant elasticity diffusion process distributions as a special case. To be of practical use, however, the choosen distribution should give a closed form solution for $C(A)$ and the adjustment terms. In this light, (6) can also be viewed as a technique for evaluating complicated integrals. Offering an alternative procedure to the existing techniques proposed by Parkinson (1977) and Brennan and Schwartz (1977) when the associated partial differential equation for $C(F)$ cannot be solved directly.

4. Approximating option values with the Black–Scholes formula

Given its widespread use in academics and professional trading, an obvious candidate for the approximating distribution, $a(s)$, is the lognormal distribution. In this case $C(A)$ will correspond to Black–Scholes (1973) formula. Expression (6) will then give an explicit expression for the adjustment terms between the true option value, $C(F)$, and Black–Scholes formula, $C(A)$.

The approximating lognormal distribution for the stock price, S_t, is a function of two parameters: the first and second cumulants of the random variable $\log(S_t)$. In the Edgeworth series expansion, the first cumulant of the lognormal distribution, $\alpha_1(A)$, is set equal to the first cumulant of the true distribution, $\alpha_1(F) \equiv S_0 e^{rt}$. This choice is predetermined by the risk-neutrality argument. However, this still leaves the second parameter for the lognormal distribution, the second cumulant for $\log(S_t)$, unrestricted.

In setting the value for this second parameter, the goal is to obtain that lognormal distribution 'closest' to the true distribution. The approach employed is somewhat arbitrary.[5] However, three particular methods stand

[5]In mathematical notation [using (8)], choose $(\sigma^2 t)$ to minimize $\|a(s) - f(s)\|$, where $\|\cdot\|$ is some norm on the space of continuous functions. Given different norms, in general different choices for $(\sigma^2 t)$ will be optimal.

Without restricting preferences, arbitrage arguments alone would suggest the supremum norm out. The first is to equate the second cumulant of the approximating lognormal to the true distribution's second cumulant, i.e., $\kappa_2(A) \equiv \kappa_2(F)$. This completely specifies the second parameter for the lognormal distribution. The second approach is to directly equate the second cumulants of $\log(S_t)$ for the approximating lognormal and the true distribution. Both of these approaches consider cumulants over the entire interval $[0, t]$. A third approach is to equate the instantaneous variances over $[0, \Delta t]$ as $\Delta t \to 0$. In this case both the instantaneous return variance and the instantaneous logarithmic variance are equal.

The three approach will give different approximating distributions, $a(s)$. This paper demonstrates the second approach, equating the cumulants of $(\log S_t)$. This is done, in part, so that the resulting approximating formula is useful in explaining the empirical evidence contained in Black and Scholes (1975), Macbeth and Merville (1980), and most other empirical tests. These studies use sample statistics to obtain estimates for the second cumulant of $\log(S_t)$. These estimates will reflect the true underlying distribution's second cumulant of $\log(S_t)$. In this case, the distribution is given by[6]

$$a(S_t) = (S_t \sigma \sqrt{t 2\pi})^{-1} \exp[-\{\log S_t - (\log \alpha_1(A) - \sigma^2 t/2)\}^2 / 2\sigma^2 t], \tag{7}$$

where

$$\alpha_1(A) \equiv S_0 \, e^{rt}$$

$$\sigma^2 t \equiv \int_{-\infty}^{\infty} (\log S_t)^2 \, dF(S_t) - \left[\int_{-\infty}^{\infty} \log S_t \, dF(S_t) \right]^2.$$

Defining $q^2 \equiv e^{\sigma^2 t} - 1$, the cumulants are [see Mitchell (1968)]

$$\begin{aligned} \kappa_1(A) &= \alpha_1(A), \\ \kappa_2(A) &= \mu_2(A) = \kappa_1(A)^2 q^2, \\ \kappa_3(A) &= \kappa_1(A)^3 q^3 (3q + q^3), \\ \kappa_4(A) &= \kappa_1(A)^4 q^4 (16q^2 + 15q^4 + 6q^6 + q^8). \end{aligned} \tag{8}$$

is appropriate [see Cox and Rubinstein (1978, p. 397)]. Together with expression (4), the optimal parameter, $\sigma^2 t$, should minimize $\sup_s |a(s) - f(s)|$. Without exact knowledge of $f(s)$, this problem cannot be solved directly.

[6]To get the approximating distribution under the first approach, the definition for σ^2 would change to the solution of

$$\kappa_2(F) = \kappa_2(A) = \alpha_1^2(A)[e^{\sigma^2 t} - 1].$$

To get the approximating distribution under the third approach, i.e. equating instantaneous

The next step is to substitute the lognormal distribution, (7), into (6). This involves terms such as

$$\int_K^{\infty} (S_t - K)(\mathrm{d}^j a(S_t)/\mathrm{d}S_t^j)\,\mathrm{d}S_t \quad \text{for} \quad j \geqq 2,$$

which can be integrated by parts to give

$$\int_K^{\infty} (S_t - K)\frac{\mathrm{d}^j a(S_t)}{\mathrm{d}S_t^j}\,\mathrm{d}S_t = \lim_{x\to\infty} x\,\frac{\mathrm{d}a^{j-1}(x)}{\mathrm{d}S_t^{j-1}} - \lim_{x\to\infty}\frac{\mathrm{d}a^{j-1}(x)}{\mathrm{d}S_{t-1}^{j-1}}$$

$$+\frac{\mathrm{d}a^{j-2}(K)}{\mathrm{d}S_t^{j-2}} - K\lim_{x\to\infty}\frac{\mathrm{d}a^{j-1}(x)}{\mathrm{d}S_t^{j-1}} \quad \text{for} \quad j \geqq 2. \tag{9}$$

For the lognormal distribution, it is known [see Kendall and Stuart (1977, p. 180)] that

$$\lim_{x\to\infty} x^u a(x) = 0 \quad \text{for} \quad u > 0. \tag{10}$$

This limit implies all but the third term in expression (9) goes to zero, i.e.,

$$\int_K^{\infty} (S_t - K)\frac{\mathrm{d}^j a(S_t)}{\mathrm{d}S_t^j}\,\mathrm{d}S_t = \frac{\mathrm{d}a^{j-2}(K)}{\mathrm{d}S_t^{j-2}} \quad \text{for} \quad j \geqq 2. \tag{11}$$

Collecting terms, the expression for the approximate option price is

$$C(F) = C(A) + \mathrm{e}^{-rt}\frac{(\kappa_2(F)-\kappa_2(A))}{2!}a(K) - \mathrm{e}^{-rt}\frac{(\kappa_3(F)-\kappa_3(A))}{3!}\frac{\mathrm{d}a(K)}{\mathrm{d}S_t}$$

$$+\mathrm{e}^{-rt}\frac{((\kappa_4(F)-\kappa_4(A))+3(\kappa_2(F)-\kappa_2(A))^2)}{4!}\frac{\mathrm{d}^2 a(K)}{\mathrm{d}S_t^2} + \varepsilon(K), \tag{12}$$

where

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variances, the definition for $\sigma^2 t$ would change to

$$\sigma^2 \equiv \lim_{t\to 0}\left(\int_{-\infty}^{\infty} (\log S_t)^2\,\mathrm{d}F(S_t) - \left[\int_{-\infty}^{\infty} \log S_t\,\mathrm{d}F(S_t)\right]^2\right)\Big/ t.$$

For example, if the true distribution follows the stochastic process

$$\mathrm{d}S_0 = \psi S_0\,\mathrm{d}t + \delta S_0^{\rho}\,\mathrm{d}z, \quad \rho < 1, \quad \text{then} \quad \sigma^2 \equiv [\delta S_0^{\rho-1}]^2.$$

$$C(A) = S_0 N(d) - K\,\mathrm{e}^{-rt} N(d - \sigma\sqrt{t}),$$

$$d = \frac{\log(S_0/K\,\mathrm{e}^{-rt}) + \sigma^2 t/2}{\sigma\sqrt{t}},$$

$N(\cdot)$ = cumulative standard normal.

Expression (12) gives three adjustment terms to the Black–Scholes valuation formula which will bring its value closer to the true option value [up to error $\varepsilon(K)$].

The first adjustment term corrects for differing variance. If the true distribution has a larger variance than the approximating lognormal, then this term is positive. The size of the adjustment term depends on the magnitude of the approximating density function at the exercise price, $a(K)$. This value depends on whether the stock price is in or out of the money, and how deep it is in or out of the money. An option is in the money if $S_0 > K\,\mathrm{e}^{-rt}$, at the money if $S_0 = K\,\mathrm{e}^{-rt}$, or out of the money if $S_0 < K\,\mathrm{e}^{-rt}$. Since the mean of the distribution is $S_0\,\mathrm{e}^{rt}$, one can classify in/at/out of money as to whether

$$\begin{aligned} &K > \alpha_1(A) && \text{[out of the money]}, \\ &K = \alpha_1(A) && \text{[at the money]}, \\ &K < \alpha_1(A) && \text{[in the money]}. \end{aligned} \tag{13}$$

Using fig. 1 for reference, the at the money options will have a larger first adjustment term [due to the factor $a(K)$] than deep in or deep out of the money options.

The second term corrects for differing skewness. Its sign depends on whether the true distribution is more skewed than the approximating lognormal and the sign of the lognormal's first derivative evaluated at K. Using fig. 1, this derivative changes signs. It is positive if $K <$ mode of lognormal, where

$$\text{Mode} = \alpha_1(A)\,\mathrm{e}^{-3\sigma^2 t/2}. \tag{14}$$

The maximum absolute adjustment due to this term occurs at the first and second inflection points, i.e.,

$$\text{Inflection points} = (\text{mode})\,\mathrm{e}^{(-\sigma^2 t/2)[1 + [1 \pm 4/\sigma^2 t]^{\frac{1}{2}}]}. \tag{15}$$

These will correspond to deep in (1st inflection) and deep out (2nd inflection)

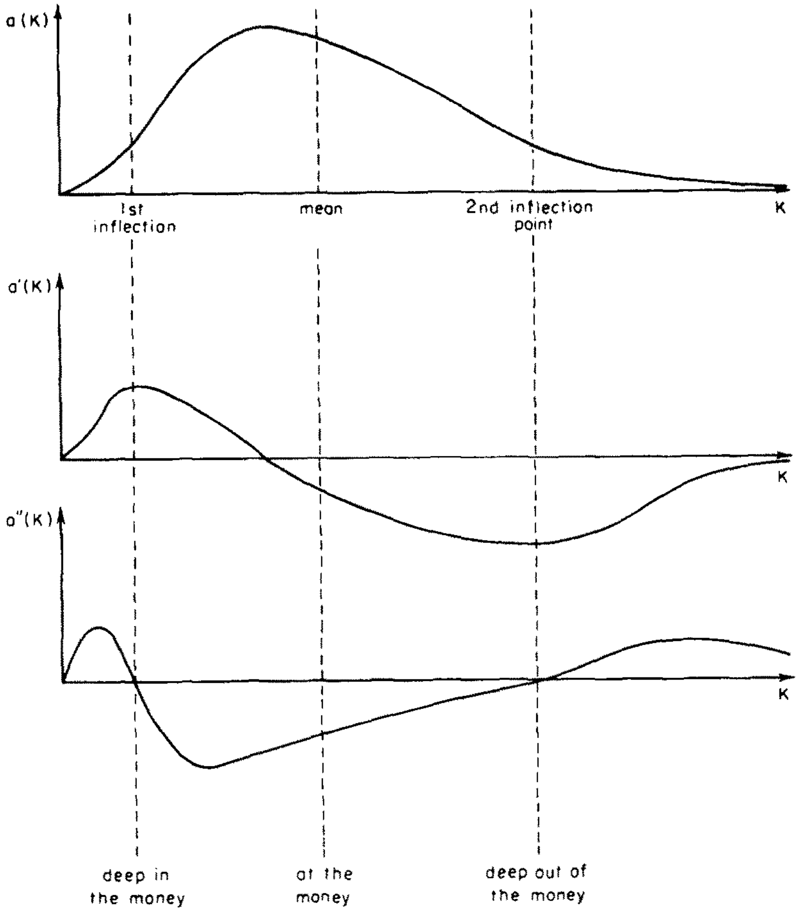

Fig. 1. Graphs of the lognormal distribution $(K, a(K))$, its first $(K, a'(K))$ and its second $(K, a''(K))$ derivatives. The lognormal distribution with parameters (b, σ^2) is defined by $a(K) = (K\sigma\sqrt{t2\pi})^{-1}\exp[-(\log K - b + \sigma^2 t/2)^2/2\sigma^2 t]$.

of the money options. Conversely, the adjustment due to skewness is zero for near in the money options (at the mode), and approximately zero for at and near out of the money options.

Finally, the third adjustment term reflects both the differing kurtosis and variance with a weighting factor equal to the third derivative of the approximating lognormal evaluated at K. The sign of this derivative changes at the first and second inflection points. It is positive for deep in and out of money options, and negative otherwise.

The closeness of the approximation due to the adjustment terms depends on the relative size of the error $\varepsilon(K)$. If $\varepsilon(K)$ is small, the approximation is good. By adding additional terms, the approximation should improve. A numerical analysis of the error term is contained in the next section.

Assuming the error term, $\varepsilon(K)$, is small, the sum of the adjustment terms will have differing influence for deep in/out money options versus at the money options. For deep in/out of the money options, due to both $a(K)$ and $a''(K)$ being approximately zero, both the variance and kurtosis adjustment terms will be small (near zero). The skewness term will be the dominating adjustment to the Black–Scholes call price. Conversely, for (near or) at the money options, both the variance and kurtosis terms will dominate. Here the skewness adjustment will be approximately zero (see fig. 1).

This comparative statics analysis can be used to understand the conflicting empirical evidence contained in Black (1975) and Macbeth and Merville (1980). Black found the Black–Scholes formula overprices deep in the money options and underprices deep out of the money options, while Macbeth and Merville found just the opposite. For deep in/out of the money options the skewness term will dominate. For reference, Bhattacharya (1980, table 3, p. 1102) presents sample statistics for selected stock distributions. Although the kurtosis of the true distribution exceeds the approximating lognormal's kurtosis, the skewness difference is positive in about half the sample, and negative in the other half. Using this evidence as representative, for different samples of options the skewness adjustment can be either positive or negative. This dominant effect can cause the Black–Scholes formula to be either over- or undervalued for deep in/out of the money options.

5. Numerical analysis of residual error

As discussed in the preceding section, no general analytic bounds can be derived for the residual error in the approximate option valuation technique. To investigate the magnitude of the residual error, this section contains a numerical analysis. The numerical analysis simulates errors in the approximate option valuation technique, expression (12), by prespecifying the true stock price distribution to be generated by the jump-diffusion process used by Merton (1976) and Cox's (1975) constant elasticity of variance diffusion process. These two processes are useful as a comparison for testing the approximate option formula since they have features which several studies have found to be more characteristic of actual stock price movements than the lognormal process of Black–Scholes [see, e.g., Cox and Rubinstein (1978)]. Moreover, both processes within these two classes contain the Black–Scholes model as a limiting case, and give rise to a closed form option valuation formula. Thus these distributions are ideal for simulating the likely magnitude of the errors that may be found by implementing the model on actual prices.

The first simulation assumes the true stock price distribution satisfies the following jump-diffusion process:

$$
dS_0/S_0=\psi\,dt+v\,dz+\left\langle\begin{array}{ll}\overset{\lambda\,dt}{\rule{3cm}{0.4pt}} & (Y-1)\\[1em] \underset{1-\lambda\,dt}{\rule{3cm}{0.4pt}} & 0\end{array}\right., \tag{16}
$$

where

$\psi, v, \lambda=$ constants, $\quad 0\leqq v, \lambda,$

$dz=$ Brownian motion,

$\log Y=$ random variable distributed normal $(-\gamma^2/2, \gamma^2)$ and independent of dz.

Assuming the jump component to be diversifiable, Merton (1975) obtains the following option valuation formula:

$$
C(F)=\sum_{n=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^n}{n!}W(S_0,t,K,v_n,r),
$$

where

$$
\begin{aligned}
&v_n^2=v^2+(n\gamma^2/t),\\
&W(S_0,t,K,v_n,r)=S_0N(h)-K\,e^{-rt}N(h-v_n\sqrt{t}),\\
&h=[\log(S_0/K\,e^{-rt})+v_n^2t/2]/v_n\sqrt{t},\\
&N(\cdot)=\text{cumulative standard normal.}
\end{aligned}
\tag{17}
$$

To completely specify the residual error in expression (12), the moments of the true stock price distribution based on (16) are needed. Consistent with the risk-neutrality argument [see expression (5)], it can be shown that under (16)

$$
\alpha_j(F)=S_0^j\exp([r-v^2/2]tj+v^2tj^2/2+\lambda\,t[e^{\gamma^2 j(j-1)/2}-1]). \tag{18}
$$

Finally, the second parameter of the approximating lognormal distribution (σ^2) needs to be specified. Two different methods will be examined: (i) setting σ^2 equal to the solution of $\mu_2(F)=\mu_2(A)$,[7] and (ii) setting $\sigma^2=v^2+\lambda(e^{\gamma^2}-1)$, i.e., equating instantaneous variances.

The numerical analysis is contained in table 1. The mean absolute dollar error in approximating Merton's jump-diffusion formula, (17), with the

[7]The details of this calculation are contained in footnote 6.

Table 1

Mean absolute dollar error in approximating Merton's jump-diffusion (JD) option values with the Black–Scholes (BS) option value plus adjustments.[a]

Partitioning of sample by parameters	Sample size[b]	Method 1 results[c]				Method 2 results[c]			
		(JD, BS)[d]	*(JD, BS1)*[e]	*(JD, BS2)*[f]	*(JD, BS3)*[g]	*(JD, BS)*	*(JD, BS1)*	*(JD, BS2)*	*(JD, BS3)*
Time to maturity in months									
1	135	0.017	0.017	0.024	0.030	0.131	0.041	0.050	0.104
4	135	0.018	0.018	0.024	0.038	0.395	0.099	0.122	0.188
7	135	0.018	0.018	0.025	0.046	0.568	0.163	0.218	0.349
Exercise price									
$35	135	0.009	0.009	0.012	0.056	0.267	0.131	0.062	0.288
$40	135	0.029	0.029	0.034	0.031	0.447	0.141	0.182	0.218
$45	135	0.015	0.015	0.027	0.028	0.379	0.032	0.146	0.136
Total volatility squared of jump and diffusion components									
0.2	135	0.010	0.010	0.012	0.012	0.204	0.046	0.053	0.057
0.3	135	0.017	0.017	0.022	0.033	0.364	0.096	0.115	0.176
0.4	135	0.026	0.026	0.039	0.070	0.525	0.162	0.223	0.408

Mean jumps per unit time									
1	135	0.053	0.053	0.076	0.134	0.588	0.186	0.259	0.524
3	135	0.022	0.022	0.029	0.037	0.614	0.163	0.202	0.292
5	135	0.014	0.014	0.018	0.021	0.620	0.158	0.189	0.253
Ratio of jump variance to diffusion variance									
0.1	81	0.003	0.003	0.004	0.005	0.145	0.031	0.034	0.030
0.2	81	0.009	0.009	0.013	0.017	0.269	0.063	0.074	0.086
0.3	81	0.017	0.017	0.023	0.035	0.377	0.098	0.122	0.175
0.4	81	0.025	0.025	0.035	0.056	0.473	0.136	0.178	0.303
0.5	81	0.034	0.034	0.047	0.078	0.558	0.178	0.243	0.474
Across all parameters									
	405	0.018	0.018	0.024	0.038	0.364	0.101	0.130	0.214

[a]Mean absolute error of $(x_i, y_i)_{i=1}^{n}$ equals $\sum_{i=1}^{n} |x_i - y_i|/n$. The range of parameters for the JD option values are: stock price $(S_0) = \$40$; risk-free rate $(r-1) = 0.05$; time to maturity in months $(t \times 12) = 1, 4, 7$; total volatility squared of jump and diffusion components $(v^2 + \lambda\gamma^2) = 0.2, 0.3, 0.4$; mean jumps per unit time $(\lambda) = 1, 3, 5$; ratio of jump component variance to diffusion component variance $(\lambda\gamma^2/v^2) = 0.1, 0.2, 0.3, 0.4, 0.5$.

[b]Sample size refers to the number of different parameter values used in the mean absolute error. For example, holding $t = 1$ fixed, the sample size 135 $(3 \times 3 \times 3 \times 5)$ represents varying K, $(v^2 + \lambda\gamma^2)$, λ, $\lambda\gamma^2/v^2$.

[c]Method 1 equates Black–Scholes variance $(\sigma^2 t)$ to the variance of the jump-diffusion process over the maturity of the option. Method 2 equates BS instantaneous variance (σ^2) to the instantaneous variance of the jump-diffusion process.

[d]JD = Merton's jump diffusion option value with parameters $(S_0, r, t, K, \lambda, v^2, \gamma^2)$. BS = Black–Scholes option value with parameters (S_0, r, t, K, σ^2).

[e]$BS1 = BS + e^{-rt}(\kappa_2(F) - \kappa_2(A))a(K)/2$.

[f]$BS2 = BS1 - e^{-rt}[(\kappa_3(F) - \kappa_3(A))\mathrm{d}a(K)/\mathrm{d}S_t]3!$.

[g]$BS3 = BS2 + e^{-rt}[(\kappa_4(F) - \kappa_4(A) + 3(\kappa_2(F) - \kappa_2(A))^2)\mathrm{d}^2 a(K)/\mathrm{d}S_t^2]/4!$.

Black–Scholes formula plus adjustments, (12), for various ranges of the parameters is contained therein. The parameter ranges examined are: $S_0 = 40$; $(r-1) = 0.05$; $K = 35, 40, 45$; $v^2 + \lambda\gamma^2 = 0.2, 0.3, 0.4$; $\lambda = 1, 3, 5$; and $\lambda\gamma^2/v^2 = 0.1$, 0.2, 0.3, 0.4, 0.5. The overall mean absolute dollar error is given in the last row. In method 1, where the variance of the approximating lognormal is set equal to the variance of the jump-diffusion process over the maturity of the option, the Black–Scholes model comes extremely close (mean absolute deviation of less than 2 cents) to the true price. Hence it is not surprising that the adjustments have little effect, although overall the mean absolute deviation increases to almost four cents. In contrast in method 2, when the instantaneous variances are equated, the Black–Scholes model does less well (mean absolute deviation of over 36 cents). In this case the adjustments work better; the variance adjustment reduces the error to almost 10 cents but the skewness and kurtosis adjustments are perverse. Even so, the approximation technique reduces the error approximately 40%.

The quality of the approximation, however, is not uniform across all ranges of the parameters. The approximating technique does best for long maturity options, out of the money options, and options where the jump component's variance is only a small fraction of the diffusion component's variance.

It appears that the approximation technique is beneficial in this situation. In method 1, it is comforting that the expansion is sufficiently robust that noise is not added in the lower terms. In the second method, the adjustment terms prove useful for nearly every option. Of the three adjustments, the variance term is the most important. Surprisingly, for the jump-diffusion process the skewness and kurtosis terms do not add any information (in fact, they add noise) to the impact of variance.

The second simulation assumes the stock price distribution is generated by the constant elasticity of variance (CEV) diffusion process,

$$dS_0 = \psi S_0 \, dt + \delta S_0^{\rho} dz,$$

where

$$\psi, \delta, \rho = \text{constants}, \quad p < 1, \quad dz = \text{Brownian motion}. \tag{19}$$

The following option valuation formula based on (19) was first obtained by Cox (1975):

$$C(F) = S_0 \sum_{n=0}^{\infty} g(\lambda S_0^{-\phi}, n+1) G(\lambda (K\,e^{-rt})^{-\phi}, n+1-1/\phi)$$

$$- K\,e^{-rt} \sum_{n=0}^{\infty} g(\lambda S_0^{-\phi}, n+1-1/\phi) G(\lambda (K\,e^{-rt})^{-\phi}, n+1),$$

where

$$
\begin{aligned}
&\phi \equiv 2\rho - 2, \\
&\lambda \equiv 2r/\delta^2\phi(e^{\phi r t} - 1), \\
&\Gamma(n) \equiv \int_0^\infty e^{-v} v^{n-1}\, dv, \\
&g(z,n) \equiv e^{-z} z^{n-1}/\Gamma(n), \\
&G(w,n) \equiv \int_w^\infty g(z,n)\, dz.
\end{aligned}
\tag{20}
$$

The moments of the true stock price distribution based on (19), consistent with the risk-neutrality argument are

$$
\alpha_j(F) = [e^{rt}\lambda^{1/\phi}]^j \sum_{n=0}^{\infty} g(\lambda S_0^{-\phi}, n+1-1/\phi)\Gamma(n+1-j/\phi)/\Gamma(n+1) \qquad \text{for} \quad j \geqq 1. \tag{21}
$$

Again, the second parameter of the approximating distribution (σ^2) is specified in two ways: (i) setting σ^2 equal to the solution of $\mu_2(F) = \mu_2(A)$, and (ii) setting $\sigma^2 = [\delta S_0^{\rho-1}]^2$, i.e., equating instantaneous variances.

The error analysis is contained in table 2. The mean absolute dollar error in approximating Cox's constant elasticity of variance option model with the Black–Scholes formula plus adjustments, across all parameters, is contained in the last row of the table. The mean absolute error in cents when equating the total variances is 4.2 versus 4.1 cents when equating instantaneous stock price variances. As an aside, the Black–Scholes formula itself (in isolation from the adjustment terms) does best when equating instantaneous stock price variances: 5.0 cents versus 6.3 cents. We now restrict our discussion to the approximating formula based on equating stock variances over the option's maturity. The approximate option valuation technique (including all three adjustment terms) reduces the average absolute difference in cents between (CEV) valuation and (BS) valuation from 6.3 to 4.2 — a 33 percent reduction. The technique does best for short maturity options, low stock variance options, and in-the-money options. Across different (CEV) processes, the approximating technique does best for low ρ values in terms of percentage reduction in mean absolute deviation. This reflects the fact that a larger discrepancy exists between (CEV) valuation and (BS) valuation for low ρ values. However, the adjustment terms still improve the average absolute difference even for large ρ.

Table 2

Mean absolute dollar error in approximating constant elasticity of variance (CEV) option values with the Black–Scholes (BS) option value plus adjustments.[a]

Partitioning of sample by parameters	Sample size[b]	Method 1 results[c]				Method 2 results[c]			
		(CEV, BS)[d]	*(CEV, BS1)*[e]	*(CEV, BS2)*[f]	*(CEV, BS3)*[g]	*(CEV, BS)*	*(CEV, BS1)*	*(CEV, BS2)*	*(CEV, BS3)*
Time to maturity in months									
1	36	0.015	0.015	0.005	0.002	0.014	0.016	0.005	0.002
4	36	0.066	0.066	0.049	0.033	0.055	0.070	0.050	0.031
7	36	0.107	0.107	0.142	0.092	0.080	0.117	0.149	0.089
Exercise price									
\$35	36	0.103	0.103	0.046	0.056	0.070	0.112	0.052	0.051
\$40	36	0.045	0.045	0.090	0.022	0.002	0.047	0.094	0.025
\$45	36	0.039	0.039	0.059	0.049	0.077	0.043	0.058	0.047
Volatility (σ)									
0.2	36	0.029	0.029	0.011	0.006	0.025	0.030	0.011	0.006
0.3	36	0.060	0.060	0.046	0.031	0.049	0.065	0.048	0.030
0.4	36	0.099	0.099	0.138	0.089	0.075	0.108	0.145	0.087

Elasticity (ρ)									
0	27	0.097	0.097	0.093	0.058	0.080	0.102	0.094	0.053
0.25	27	0.075	0.075	0.078	0.050	0.060	0.081	0.081	0.047
0.5	27	0.052	0.052	0.058	0.038	0.039	0.057	0.062	0.038
0.75	27	0.027	0.027	0.032	0.023	0.020	0.030	0.036	0.024
Across all parameters									
	108	0.063	0.063	0.065	0.042	0.050	0.067	0.068	0.041

[a]Mean absolute error of $(x_i, y_i)_{i=1}^n$ equals $\sum_{i=1}^n |x_i - y_i|/n$. The range of parameters for the CEV option values are: stock price $(S_0)=\$40$; risk-free rate $(r-1)=0.05$; exercise price $(K)=\$35, \$40, \$45$; time to maturity in months $(t \times 12)=1, 4, 7$; elasticity $(\rho)=0, 0.25, 0.5, 0.75$; volatility $(\sigma)=0.2, 0.3, 0.4$

[b]Sample size refers to the number of different parameter values used in the mean absolute error. For example, holding $t=1$ fixed the sample size 36 $(3 \times 3 \times 4)$ represents varying K, σ, ρ.

[c]Method 1 equates Black–Scholes variance $(\sigma^2 t)$ to the variance of the CEV diffusion process over the maturity of the option. (Given σ is exogenous, δ is chosen such that this statement holds). Method 2 equates BS instantaneous variance (σ^2) to the instantaneous variance of the CEV diffusion process. (Given σ is exogenous, δ is chosen such that this statement holds).

[d]CEV = constant elasticity of variance option value with parameters $(S_0, r, t, K, \delta, \rho)$. BS = Black–Scholes option value with parameters (S_0, r, t, K, σ^2).

[e]$BS1 = BS + e^{-rt}(\kappa_2(F) - \kappa_2(A))a(K)/2$.

[f]$BS2 = BS1 - e^{-rt}[(\kappa_3(F) - \kappa_3(A))\mathrm{d}a(K)/\mathrm{d}S_t]3!$.

[g]$BS3 = BS2 + e^{-rt}[(\kappa_4(F) - \kappa_4(A) + 3(\kappa_2(F) - \kappa_2(A))^2)\mathrm{d}^2a(K)/\mathrm{d}S_t^2]/4!$.

Across both types of stochastic processes (tables 1 and 2) the approximating technique does well. If the correct approximating distribution is selected (based on σ^2), then the Black–Scholes option value plus all three adjustment terms are on average within 4 cents of the true option value. The improvement due to the adjustment terms over the Black–Scholes formula depends on the magnitude of the error. If the error between the Black–Scholes model and the true option price is large (greater than 5 cents), then the approximating technique on average improves the estimate. If the error between the Black–Scholes model and the true option price is small (less than 5 cents), then the approximating technique on average adds noise. However, the additional noise is not large in an absolute sense.

6. Conclusion

This paper presents an approximate option valuation technique suitable for a wide class of stock price distributions. This approach was used to adjust the Black–Scholes option pricing formula to take into account discrepancies between the moments of the lognormal distribution on which the Black–Scholes model is based and the true stock price distribution. This allows a consistent interpretation of the seemingly contradictory results obtained by Black (1975) and Macbeth and Merville (1980).

The approximate option valuation technique contains a residual error. This residual error was examined numerically when the true stock price distribution was assumed to follow a jump-diffusion process and a constant elasticity of variance diffusion process. The adjustment terms on average adjust the Black–Scholes option value to within 4 cents of the true option price.

The ultimate test of this valuation approach must be based on market data. The numerical analysis performed here makes the strong assumption that the underlying security process belongs to the specialized class of jump-diffusion processes examined or the (CEV) diffusion processes. Since this class contains processes which are representative of stock price movements, the simulated errors and improvements over Black–Scholes would appear to be indicative of the performance of our approach, which is confirmed by preliminary results. This market test is the subject of another paper [Jarrow and Rudd (1982)].

Appendix 1: Proof of the generalized Edgeworth series expansion

From (2), letting $N=\inf(n,m)$,

$$\log \phi(F,t)=\sum_{j=1}^{N-1}(\kappa_j(F)-\kappa_j(A))\frac{(it)^j}{j!}+\sum_{j=1}^{N-1}\kappa_j(A)\frac{(it)^j}{j!}+\mathrm{o}(t^{N-1}). \tag{A.1}$$

But

$$\sum_{j=1}^{N-1} \kappa_j(A)\frac{(it)^j}{j!} = \log \phi(A,t) + \mathrm{o}(t^{N-1}),$$

substitution gives

$$\log \phi(F,t) = \sum_{j=1}^{N-1} (\kappa_j(F) - \kappa_j(A))\frac{(it)^j}{j!} + \log \phi(A,t) + \mathrm{o}(t^{N-1}). \tag{A.2}$$

Taking exponentials of this expression and using $\mathrm{e}^{\mathrm{o}(t^{N-1})} = 1 + \mathrm{o}(t^{N-1})$, transforms the equation into

$$\phi(F,t) = \exp\left\{\sum_{j=1}^{N-1} (\kappa_j(F) - \kappa_j(A))\frac{(it)^j}{j!}\right\}\phi(A,t) + \mathrm{o}(t^{N-1}). \tag{A.3}$$

Since $\exp\{\cdot\}$ is an analytic function, it can be expanded as an inifinite polynomial. Consequently, there exist E_j, $j=0,1,\ldots,N-1$, such that

$$\exp\left\{\sum_{j=1}^{N-1} (\kappa_j(F) - \kappa_j(A))\frac{(it)^j}{j!}\right\} = \sum_{j=0}^{N-1} E_j\frac{(it)^j}{j!} + \mathrm{o}(t^{N-1}). \tag{A.4}$$

For reference, the first four coefficients are

$$\begin{aligned}
E_0 &= 1,\\
E_1 &= (\kappa_1(F) - \kappa_1(A)),\\
E_2 &= (\kappa_2(F) - \kappa_2(A)) + E_1^2,\\
E_3 &= (\kappa_3(F) - \kappa_3(A)) + 3E_1(\kappa_2(F) - \kappa_2(A)) + E_1^3,\\
E_4 &= (\kappa_4(F) - \kappa_4(A)) + 4(\kappa_3(F) - \kappa_3(A))E_1 + 3(\kappa_2(F) - \kappa_2(A))^2\\
&\quad + 6E_1^2(\kappa_2(F) - \kappa_2(A)) + E_1^4.
\end{aligned} \tag{A.5}$$

Substituting (A.4) into (A.3) and using the fact that $\lim_{t\to 0} \phi(A,t) = 1$ gives

$$\phi(F,t) = \sum_{j=0}^{N-1} E_j\frac{(it)^j}{j!}\phi(A,t) + \mathrm{o}(t^{N-1}). \tag{A.6}$$

We now take the inverse Fourier transform of (A.6), using

$$f(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-its}\phi(F,t)\,dt,$$

$$a(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-its}\phi(A,t)\,dt, \tag{A.7}$$

$$(-1)^j\frac{d^j a(s)}{ds^j}=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-its}(it)^j\phi(A,t)\,dt,$$

gives

$$f(s)=a(s)+\sum_{j=1}^{N-1} E_j\frac{(-1)^j}{j!}\frac{d^j a(s)}{ds^j}+\varepsilon(s,N), \tag{A.8}$$

where

$$\varepsilon(s,N)=\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{its}o(t^{N-1})\,dt.$$

The error term exits since $\sum_{j=0}^{N-1} E_j((-1)^j/j!)(d^j a(s)/ds^j)$, $a(s)$ and $f(s)$ are finite.

No general bound on $\varepsilon(s,N)$ as a function of N is known for arbitrary $a(s)$ and $f(s)$. The known properties of $\varepsilon(s,N)$ have all been obtained for particular distributions by numerical analysis [see Schleher (1977) and Mitchell (1968)].

In the case where all moments of both $a(s)$ and $f(s)$ exist, it can be shown that

$$\lim_{N-1}\ \sup_{s\in[-\infty,\infty]}|\varepsilon(s,N)|=0.$$

This is seen by noting that

$$\varepsilon(s,N)=\sum_{j=N}^{\infty} E_j((-1)^j/j!)(d^j a(s)/ds^j),$$

where E_j, $d^j a(s)/ds^j$ are finite for all j.

For $\varepsilon(s,N)$ to exist for all s, it must be true that

$$\lim_{N\to\infty}|\varepsilon(s,N)|=0 \quad \text{for all } s.$$

To obtain (4), let $\kappa_1(A)=\kappa_1(F)$ and simplify (A.8) using (A.5). Q.E.D.

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