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ABSTRACT

This article develops a new method for inferring risk-neutral probabilities (or state-contingent prices) from the simultaneously observed prices of European options. These probabilities are then used to infer a unique fully specified recombining binomial tree that is consistent with these probabilities (and, hence, consistent with all the observed option prices). A simple backwards recursive procedure solves for the entire tree. From the standpoint of the standard binomial option pricing model, which implies a limiting risk-neutral lognormal distribution for the underlying asset, the approach here provides the natural (and probably the simplest) way to generalize to arbitrary ending risk-neutral probability distributions.

ONE OF THE CENTRAL IDEAS of economic thought is that, in properly functioning markets, prices contain valuable information that can be used to make a wide variety of economic decisions. At the simplest level, a farmer learns of increased demand (or reduced supply) for his crops by observing increases in prices, which in turn may motivate him to plant more acreage. In financial economics, for example, it has been argued that future spot interest rates, predictions of inflation, or even anticipation of turns in the business cycle, can be inferred from current bond prices. The efficacy of such inferences depends on four conditions:

—A satisfactory model that relates prices to the desired inferred information,
—A model which can be implemented by timely and low-cost methods,
—Correct measurement of the exogenous inputs required by the model, and
—The efficiency of markets.

Indeed, given the right model, a fast and low-cost method of implementation, correctly specified inputs, and market efficiency, usually it will not be possible to obtain a superior estimate of the variable in question by any other method.

In this spirit, financial economists have tried to infer the volatility of underlying assets from the prices of their associated options. In the classic

* University of California at Berkeley. Presidential address to the American Finance Association, January 1994, Boston, Massachusetts. I would like to give special thanks to Jack Hirshleifer, who while he has not commented specifically on this article, nonetheless as my mentor in my formative years, propelled me in its direction. William Keirstead, while he has been a Ph.D. student at Berkeley, has helped implement the nonlinear programming algorithms described in this article. I am also grateful for recent conversations with Hua He, John Hull, Hayne Leland, and Alan White, and earlier conversations with Ray Hawkins and David Shimko.

example, the Black-Scholes formula for calls requires measurement of the underlying asset price and its payout rate, the riskless interest rate, and an associated option price, its striking price, and time-to-expiration.[1] The formula can be implemented in a fraction of a second on widely available low-cost computers and calculators. In many situations of practical relevance, the inputs can be easily measured and the related securities are traded in highly efficient markets. This model is widely viewed as one of the most successful in the social sciences and has perhaps (including its binomial extension) the most widely used formula, with embedded probabilities, in human history.

Despite this success, it is the thesis of this research that not only has the Black-Scholes formula become increasingly unreliable over time in the very markets where one would expect it to be most accurate, but, moreover, attempts by financial economists to extract probabilistic information from option prices have been puny in comparison to what is clearly possible.

I. Recent Evidence Concerning S&P 500 Index Options

The market for S&P 500 index options on the Chicago Board Options Exchange provides an arena where the common conditions required for the Black-Scholes formula would seem to be be best approximated in practice. The market is the second most active options market in the United States and has the largest open interest, the underlying is a cash asset rather than a future, the options are European rather than American, the options do not have the "wildcard" feature, which seriously complicates the valuation of the more active S&P 100 index options, the options can be easily hedged using S&P 500 index futures, the index payout can be reliably estimated or inferred from index futures, unlike bond prices the underlying index can a priori be assumed to follow a risk-neutral lognormal process, unlike currency exchange rates the index does not have an obvious non-competitive trader in its market (i.e., the government), and finally the underlying asset is an index which is therefore less likely to experience jumps than probably any of its component equities and most other underlying assets such as commodities, currencies and bonds.

In early research on 30 of its component equities using all reported trades and quotes on their options covering a two-year period during 1976 to 1978, I found that the Black-Scholes formula seemed to provide reasonably accurate values.[2] A minimal prediction of the Black-Scholes formula is that all options on the same underlying asset with the same time-to-expiration but with different striking prices should have the same implied volatility. While not strictly true, the formula passed this test with remarkable fidelity. While I showed that biases from the Black-Scholes predictions were *statistically* significant and there were long periods of time during which another option

[1] See Black and Scholes (1973).
[2] See Rubinstein (1985).

model would have worked better, there was no evidence that the biases were *economically* significant. Moreover, while the alternative model might have worked better for awhile, it would have performed worse at other times.

I used a minimax statistic to measure the economic significance of the bias. The idea behind this statistic is to place a lower bound on the performance of the formula without having to estimate volatility, either implied or statistical. Here is how it works. Select any two options on the same underlying asset with the same time-to-expiration, but with different striking prices. For a given volatility, for each option calculate the absolute difference between its market price and it corresponding Black-Scholes value based on the assumed volatility ("dollar error"). Record the maximum difference. Now repeat this procedure but each time alter the assumed volatility, and span the domain of volatilities from zero to infinity. We will end up with a function mapping the assumed volatility into the maximum dollar error. The minimax statistic is the minimum of these errors. We can say then that comparing just these two options, for one of them the Black-Scholes formula must have at least this dollar error, irrespective of the volatility.

Because the Black-Scholes formula is monotonicly increasing in volatility, the volatility at which such a minimum is reached always lies between the implied volatilities of each of the two options and, moreover, will be the volatility that equalizes the dollar errors for each of the two options. As a result, the minimax statistic can be computed quite easily. I will call this the *minimax dollar error*. To correct for the possibility that, other things equal, we might expect a larger dollar error the higher the underlying asset value, the minimax dollar error is scaled to an underlying asset price of 100 by multiplying it by 100 divided by the concurrent underlying asset price.[3] A negative sign is appended to the errors if, in the option pair, the higher striking price option has a lower implied volatility than the lower striking price option.

We could also measure percentage errors at the volatility that equalizes the absolute values of the ratio of the dollar error divided by the corresponding option market price. This, I will call the *minimax percentage error*.

During 1976 to 1978, looking at a variety of pairs of options, minimax percentage errors were on the order of 2 percent—a figure I would regard as sufficiently low to make the Black-Scholes formula a good working guide in the equity options market (although not of sufficient accuracy to satisfy professionals who make markets in options). More recently, I had occasion to measure minimax errors again during 1986 for S&P 500 index options, and again minimax percentage (as well as dollar) errors were quite low. However, since 1986 there has been a very marked and rapid deterioration for these options. Tables I and II list the minimax percentage and dollar errors by striking price ranges for S&P 500 index calls with time-to-expiration of 125 to 215 days.

[3] This scaling reflects that the Black-Scholes formula is homogeneous of degree one in the underlying asset price and the striking price, so that doubling each of these variables, other things equal, doubles the values of puts and calls.

Table I

Signed Minimax Percentage Errors

Data are from the S&P 500 index 125 to 215-day maturity calls, 4/2/86 to 8/31/92. The striking price range indicates the striking prices of the two options used to construct the minimax statistic. For example, −9 percent to +9 percent indicates that the first call was sampled from in-the-money (ITM) options with striking prices between 12 and 6 percent less than the concurrent index and the second call was sampled from out-of-the-money (OTM) options with striking prices between 6 and 12 percent more than the concurrent index. In both cases, options were chosen as close as possible to the midpoint of their respective intervals. The numbers for each year represent the median signed minimax error from sampling once every trading day over the year.

	Striking Price Range			
	ITM	ATM	OTM	ITM/OTM
Year	− 9% to −3%	− 3% to +3%	+ 3% to +9%	− 9% to +9%
1986	−0.3	−0.5	−0.3	−0.7
1987	−0.7	−1.0	−0.8	−1.6
1988	−2.5	−3.5	−4.1	−7.0
1989	−2.5	−4.8	−6.4	−7.7
1990	−3.4	−5.9	−8.7	−11.2
1991	−4.0	−7.0	−10.3	−13.1
1992	−4.9	−8.8	−14.2	−15.3

Using just these statistics, the Black-Scholes model worked quite well during 1986. Even in the worst case, comparing calls that were 9 percent in-the-money with calls that were 9 percent out-of-the-money, the minimax percentage error was less than 1 percent, and the scaled minimax dollar error was about 4 cents per $100 of the index. At an average index level during that year of about 225, that can be translated into an unscaled error of about 10 cents. That means that, if the Black-Scholes formula were correct, the market mispriced one of those options by at least 10 cents. If we assume that the "true" implied volatility lay between the implied volatilities of these options, then we can also say that if one of the options was mispriced by more than 10 cents, the other must have been mispriced by less than 10 cents.

However, during 1987 this situation began to deteriorate with percentage errors approximately doubling. 1988 represents a kind of discontinuity in the rate of deterioration, and each subsequent year shows increased percentage errors over the previous year. One is tempted to hypothesize that the stock market crash of October 1987 changed the way market participants viewed index options. Out-of-the-money puts (and hence in-the-money calls perforce by put-call parity) became valued much more highly, eventually leading to the 1990 to 1992 (as well as current) situation where low striking price options had significantly higher implied volatilities than high striking price options. In this domain, during 1986 the span of implied volatilities over the −9 percent to +9 percent striking price range was about $1\frac{1}{2}$ percent (roughly $18\frac{1}{2}$ to 17 percent). In contrast during 1992, this range was about $6\frac{1}{2}$ percent (roughly 19 to $12\frac{1}{2}$ percent). Anyone who had purchased out-of-

Table II

Signed Scaled Minimax Dollar Errors

Data are from the S&P 500 index 125 to 215-day maturity calls, 4/2/86 to 8/31/92. The striking price range indicates the striking prices of the two options used to construct the minimax statistic. For example −9 percent to +9 percent indicates that the first call was sampled from in-the-money (ITM) options with striking prices between 12 and 6 percent less than the concurrent index and the second call was sampled from out-of-the-money (OTM) options with striking prices between 6 and 12 percent more than the concurrent index. In both cases, options were chosen as close as possible to the midpoint of their respective intervals. The numbers for each year represent the median signed minimax error from sampling once every trading day over the year.

	Striking Price Range				
	ITM	ATM	OTM	ITM/OTM	S&P 500
Year	− 9% to −3%	− 3% to +3%	+ 3% to +9%	− 9% to +9%	low high
1986	−0.025	−0.025	−0.007	−0.044	203.49–254.73
1987	−0.070	−0.056	−0.031	−0.118	223.92–336.77
1988	−0.251	−0.212	−0.144	−0.551	242.63–283.66
1989	−0.248	−0.266	−0.191	−0.599	275.31–359.80
1990	−0.364	−0.382	−0.297	−0.908	295.46–368.95
1991	−0.371	−0.382	−0.250	−0.887	311.49–417.09
1992	−0.422	−0.389	−0.221	−0.858	394.50–441.28

the-money puts before the crash and held them during the week of the crash would have made huge profits: not only did put prices rise because the index fell by about 20 percent, but put prices rose because implied volatilities typically tripled or quadrupled. The market's pricing of index options since the crash seems to indicate an increasing "crash-o-phobia," a phenomenon that we will subsequently document in other ways.

The tendency for the graph of implied volatility as a function of striking price for otherwise identical options to depart from a horizontal line has become popularly known among market professionals as the "smile." Typical pre- and postcrash smiles are shown in Figures 1 and 2. The increased concern about smiles across options markets generally, and the conferences and even academic papers concerning them, is rough anecdotal evidence that similar problems with the Black-Scholes formula reported here for S&P 500 index options in recent years, may pervade options on many other underlying assets.

Of course, the estimation of minimax statistics across otherwise identical options with *different striking prices* is just one way to test whether the market is pricing options according to the Black-Scholes formula. Apart from general arbitrage tests such as put-call parity, I consider it the most basic test, since among alternatives it is the easiest to verify. However, it clearly does not test all implications of the Black-Scholes formula. One might also compare in a similar way otherwise identical options with *different time-to-expirations*. This can provide useful information, but may not be helpful in testing a slight generalization of the Black-Scholes formula that allows

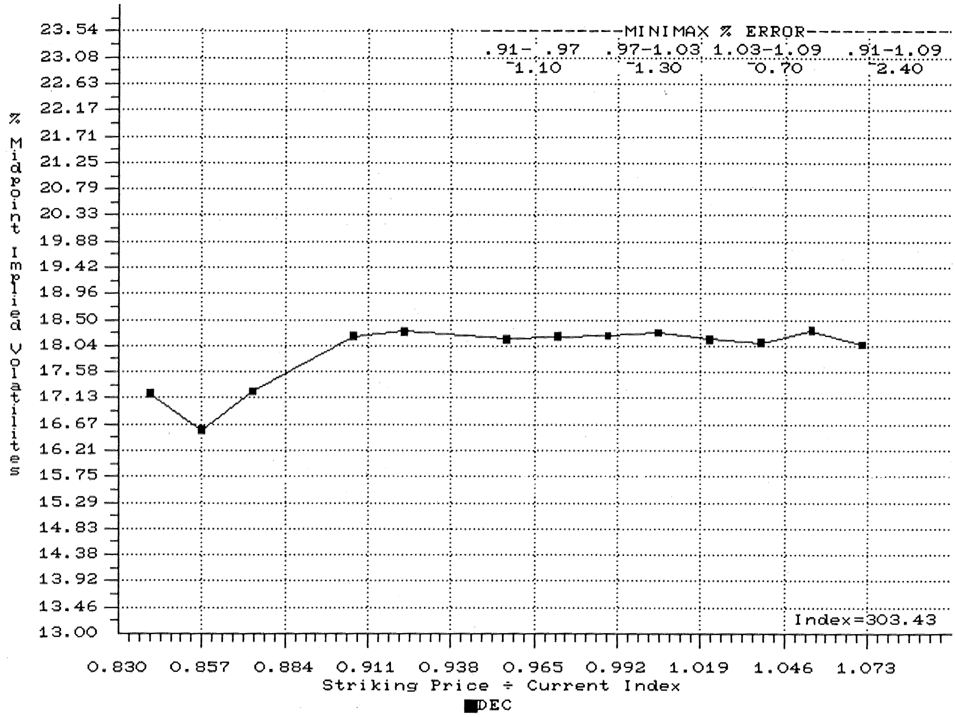


Figure 1. Typical precrash smile. Implied combined volatilities of S&P 500 index options (July 1, 1987; 9:00 A.M.).

time-dependent implied volatility. These two cross-section tests can be usefully supplemented by a third time-series test, which compares the implied volatilities measured today with implied volatilities of the same options measured tomorrow.[4] If the constant-volatility Black-Scholes formula is true, these implied volatilities should be the same. Even the more general formula, allowing for time-dependent volatility, can be tested by looking for variables other than time that are correlated with changing implied volatility. Along these lines, a very interesting working paper by David Shimko reports very high negative correlations during the period 1987 to 1989 between changes in implied volatilities on S&P 100 index options and the concurrent return of the index—a correlation that should be zero according to the Black-Scholes formula.[5]

[4] For the constant volatility Black-Scholes model, these time-series tests are the same as tests based on hedging using option deltas, since to know an option's delta is the same as knowing the difference between today's and tomorrow's option prices, conditional on knowing the change in its underlying asset price. Thus, if the Black-Scholes formula produces the correct deltas, time-series changes in implied volatilities must not be significant.

[5] See David Shimko (1991).

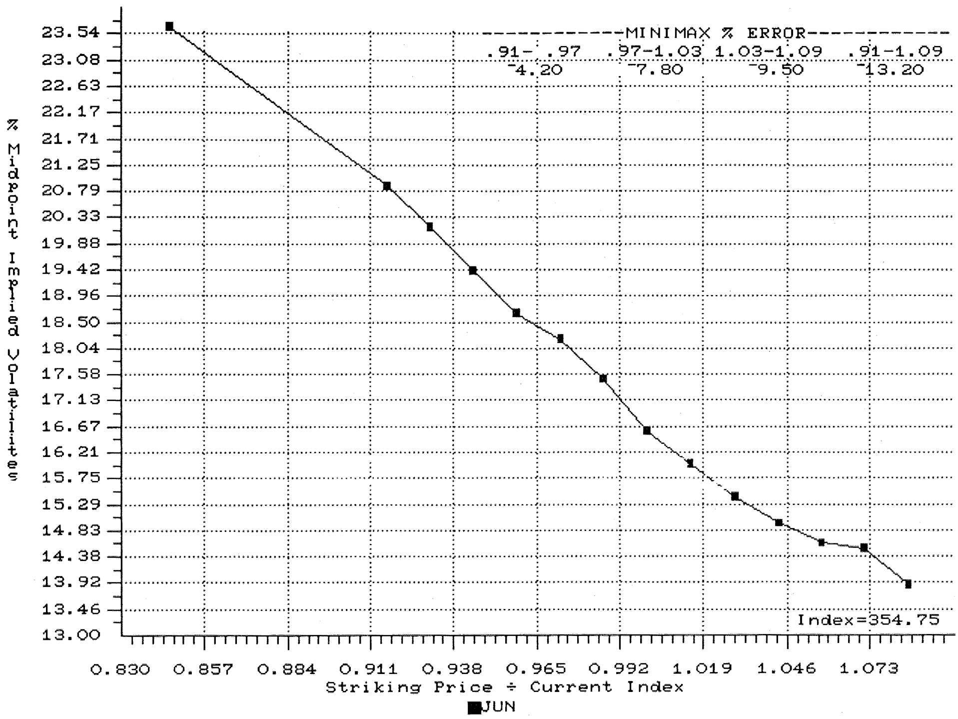


Figure 2. Typical postcrash smile. Implied combined volatilities of S&P 500 index options (January 2, 1990; 10:00 A.M.).

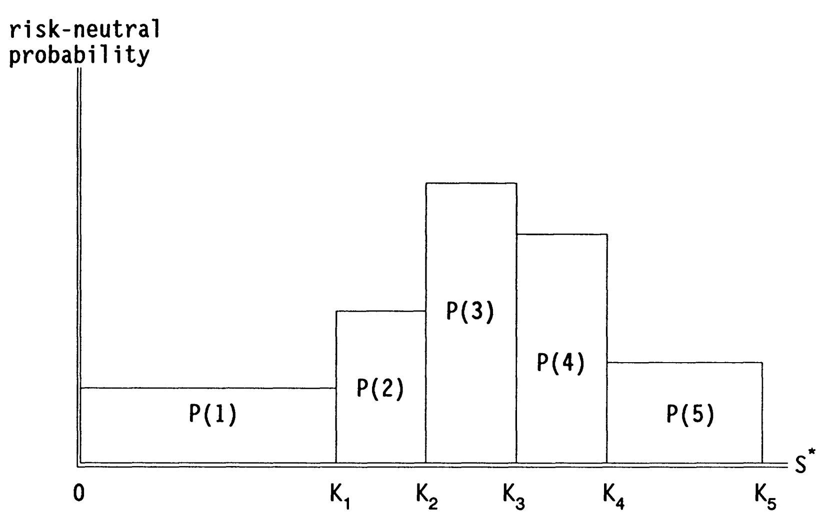


Figure 3. Risk-neutral probability distribution.

Although this article will discuss the use of these three types of tests, it will not investigate what I would call statistical tests. These tests usually take the form of comparing implied volatility with historically sampled volatility. Since the "true" stochastic process of historical volatility is not known, these tests are not as convincing to me as the three outlined above.

This discussion overlooks one possibility: the Black-Scholes formula is true but the market for options is inefficient. This would imply that investors using the Black-Scholes formula and simply following a strategy of selling low striking price index options and buying high striking price index options during the 1988 to 1992 period should have made considerable profits. While I have not tested for this possibility, given my priors concerning market efficiency and in the face of the large profits that would have been possible under this hypothesis, I will suppose that it would be soundly rejected and not pursue the the matter further, or leave it to skeptics whose priors would justify a different research strategy.

The constant volatility Black-Scholes model, as distinguished from its formula, which can be justified on other grounds,[6] will fail under any of the following four violations of its assumptions:

1. The local volatility of the underlying asset, the riskless interest rate, or the asset payout rate is a function of the concurrent underlying asset price or time.
2. The local volatility of the underlying asset, the riskless interest rate, or the asset payout rate is a function of the prior path of the underlying asset price.
3. The local volatility of the underlying asset, the riskless interest rate, or the asset payout rate is a function of a state-variable which is not the concurrent underlying asset price or the prior path of the underlying asset price; or the underlying asset price, interest rate or payout rate can experience jumps in level between successive opportunities to trade.
4. The market has imperfections such as significant transactions costs, restrictions on short selling, taxes, noncompetitive pricing, etc.

The violations of Black-Scholes assumptions at each of these levels becomes increasingly serious and difficult to remedy as we move down the list. Although, violations of types 1 and 2 still leave the arbitrage reasoning—the essence of the Black-Scholes argument—intact, type 2 violations lead perhaps to insurmountable computational problems. Violations of type 3 are far more serious, since they destroy the arbitrage foundations of the Black-Scholes model and have left researchers so far with two unpalatable alternatives: either an equilibrium model in which investor preferences explicitly enter, or other securities in addition to the underlying and riskless assets must be included in the arbitrage strategy. Violations of type 4 are the worst, because their effects are notoriously difficult to model and they typically lead only to bands within which the option price should lie.

[6] See Rubinstein (1976).

In this context, here is one way to think of the contribution of this article. It will provide a computationally effective way to *value* options, even in the presence of violations 1, 2, and 3, and a computationally effective way to *hedge* options even under violation 1. Other work has dealt with violation 1, most notably the constant elasticity of variance diffusion model developed by John Cox and Stephen Ross.[7] But this work begins with a parameterization of the function relating local volatility to the underlying asset price. The model here derives this function (which may depend on time as well) endogenously, and it can be thought of as an attempt to exhaust the potential for violation 1 to explain observed option prices.

II. Implied Ending Risk-Neutral Probabilities

The approach we will use to value and hedge options involves three steps. First, we must somehow estimate the ending risk-neutral probabilities of the underlying asset return. The approach emphasized here will be to infer these from the riskless interest rate and concurrent market prices of the underlying asset and its associated otherwise identical European options with different striking prices. Second, we infer a unique, fully specified stochastic process of the underlying asset price from these risk-neutral probabilities. Third, armed with this process, we can calculate the value and hedging parameters of any derivative instrument maturing with or before the European options.

Longstaff's Method (Amended)

Ever since the work of Stephen Ross,[8] it has been well known that in principle it should be possible to infer state-contingent prices, or their close relatives, risk-neutral probabilities,[9] from option prices. The first version of a recent working paper by Francis Longstaff describes a way of doing this.[10] Here I will describe a somewhat modified version of his method. Let:

S ≡ current price of underlying asset

C_1, C_2, C_3, C_4 ≡ concurrent prices of associated call options with striking prices $K_1 < K_2 < K_3 < K_4$, all with the same time-to-expiration

S^* ≡ price of underlying asset on the expiration date

r^n ≡ one plus the riskless rate of interest through the expiration date

δ^n ≡ one plus the payout rate on the underlying asset through the expiration date

[7] See Cox and Ross (1976).

[8] See Ross (1976).

[9] Jacques Dreze (1970) was probably the first to realize the significance of this correspondence between state-contingent prices and probabilities.

[10] See Longstaff (1990). His method is quite similar to the first attempt of which I am aware described by Banz and Miller (1978).

Assume that, conditional on S^* being between adjoining striking prices (including 0), all levels of S^* have equal risk-neutral probabilities. Also assume that there exists a number $K_5 > K_4$ such that the probability that $S^* > K_5$ is zero, and that, conditional on S^* being between K_4 and K_5, all levels of S^* have the same risk-neutral probability. Figure 3 depicts this situation.

Taking the market prices S, C_1, C_2, C_3, C_4 and the return r^n as exogenous, Appendix I derives the following solution for P_1, P_2, P_3, P_4, P_5, and K_5:

$$P_1 = 2\left[1 - r^n(S\delta^{-n} - C_1)K_1^{-1}\right]$$
$$P_2 = 2\left[1 - P_1 - r^n(C_1 - C_2)(K_2 - K_1)^{-1}\right]$$
$$P_3 = 2\left[1 - P_1 - P_2 - r^n(C_2 - C_3)(K_3 - K_2)^{-1}\right]$$
$$P_4 = 2\left[1 - P_1 - P_2 - P_3 - r^n(C_3 - C_4)(K_4 - K_3)^{-1}\right]$$
$$P_5 = 1 - P_1 - P_2 - P_3 - P_4$$
$$K_5 = K_4 + (2r^nC_4 \div P_5)$$

Thus, the implied risk-neutral probabilities can be derived in triangular fashion by solving the first equation for P_1, using this value for P_1, and solving the second equation for P_2, using these values for P_1 and P_2 and solving the third equation for P_3, etc. This reveals an interesting structure of the solution. The relation between the underlying asset price, which can be thought of as a payout-protected zero-strike option, and the lowest striking price option essentially determines the probability in the lower tail. Then each successive option contributes information about the probability from there to the next striking price. The upper tail probability then follows from the constraint that the total probability must be 1.

As a test of this method, I calculated risk-neutral implied probabilities from eleven options assumed to be priced according to the Black-Scholes formula and therefore under a lognormal risk-neutral density function. Table III compares the Black-Scholes probability in each striking price interval with the discrete probabilities derived using the amended Longstaff method.

Not only are the discrete probabilities quite different than lognormal, jumping from very low to very high over adjoining intervals, even worse, many are negative. Observe that nothing in the amended Longstaff method precludes negative probabilities. Clearly, assuming a uniform distribution between strikes and a finite upper bound to the underlying asset price does not produce satisfactory results.

This method also highlights the critical problem: in current option markets, we only observe striking prices at discrete intervals. Moreover, we have considerable identification problems in the tails because the distance between zero (the "striking price" of the underlying asset) and the lowest option striking price is usually quite large, and we have no striking prices above some maximum level. To solve this problem, we need to find some way to

Table III

Implied Risk-Neutral Probabilities Using an Amended Longstaff Method

Time-to-expiration was assumed to be one year, and the volatility used in the Black-Scholes formula was assumed to be 20 percent. Other variables were $S = 100$, $r^n = 1.1$, and $\delta^n = 1.05$.

		Implied Risk-Neutral Probability	
Strike	Black-Scholes Call Price	Black-Scholes	Discrete (P_j)
0	100.00	0.0581	0.0092
75	27.37	0.0479	0.1427
80	23.19	0.0663	−0.0283
85	19.27	0.0826	0.1776
90	15.69	0.0938	−0.0008
95	12.51	0.0986	0.1944
100	9.78	0.0971	0.0042
105	7.49	0.0902	0.1809
110	5.62	0.0798	−0.0085
115	4.15	0.0676	0.1562
120	3.01	0.0552	−0.0338
125	2.15	0.1628	0.2062
148 or ∞	0.00		

interpolate between striking prices and extrapolate to provide satisfactory tail probabilities.

Shimko's Method

Douglas Breeden and Robert Litzenberger[11] showed that if a continuum of European options with the same time-to-expiration existed on a single underlying asset spanning striking prices from zero to infinity, the entire risk-neutral probability distribution for that expiration date can be inferred by calculating the second derivative of each option price with respect to its striking price.

David Shimko provides a way to implement this idea.[12] He first plots the smile and fits a smooth curve to it between the lowest and highest option striking prices. This provides him with interpolated Black-Scholes implied volatilities. Using the Black-Scholes formula, he inverts the implied volatilities, solving for the option price as a continuous function of the striking price. Then, taking the second derivative of this function, he determines the implied risk-neutral probability distribution between the lowest and the highest strike options. Although he uses the Black-Scholes formula, Shimko's method does not require it to be true. He has merely used the formula as a translation device that allows him to interpolate implied volatilities rather than the

[11] See Breeden and Litzenberger (1978).

[12] See Shimko (1993).

observed option prices themselves. He supplies tail probabilities by grafting lognormal distributions onto each of the tails.[13]

Using S&P 100 index options, Shimko then creates graphs of the risk-neutral distribution for various option maturities on selected dates from 1987 to 1989. His method passes the test I applied to the Longstaff method, since if the smile is exactly horizontal, he will imply a lognormal risk-neutral probability distribution with the correct volatility.

An Optimization Method

Here I propose yet another method. First, we establish a prior guess of the risk-neutral probabilities. In general, it could be anything; but for working purposes, I will suppose that our prior is the result of constructing an n-step standard binomial tree using the average of the Black-Scholes implied volatilities of the two nearest-the-money call options.[14] Denote the nodal underlying asset prices at the end of the tree from lowest to highest by S_j for $j = 0, \ldots, n$. Denote the ending nodal derived risk-neutral probabilities by P_j' where it will be the case that $\Sigma_j P_j' = 1$. For example, if p' is the risk-neutral probability of an up move over each binomial period, then $P_j' = [n!/j!(n-j)!]p'^j(1-p')^{n-j}$. For sufficiently large n, this probability distribution will be approximately lognormal. Let r and δ represent, respectively, the riskless interest return and underlying asset payout return over each binomial period.[15] Let S^b (S^a) be the current bid (ask) price of the underlying asset and C_i^b (C_i^a) the bid (ask) price simultaneously observed on European call $i = 1, \ldots, m$ maturing at the end of the tree, assumed not to be protected against payouts. Choose $n \gg m$.

The implied posterior risk-neutral probabilities, P_j, are then the solution to the following quadratic program:

$$\min_{P_j} \Sigma_j (P_j - P_j')^2 \quad \text{subject to:}$$

$$\Sigma_j P_j = 1 \quad \text{and} \quad P_j \geq 0 \quad \text{for } j = 0, \ldots, n$$

$$S^b \leq S \leq S^a \quad \text{where } S = (\delta^n \Sigma_j P_j S_j)/r^n$$

$$C_i^b \leq C_i \leq C_i^a \quad \text{where } C_i = (\Sigma_j P_j \max[0, S_j - K_i])/r^n \quad \text{for } i = 1, \ldots, m$$

[13] Shimko does not use the information contained in the underlying asset price to help identify the lower tail. He seems to do this because he is worried about errors that would be created by nonsimultaneity in the reporting of the index due to the familiar problem of lagged trading of components of the S&P 100 index. Except for rare moments (such as occurred on October 19 to 20, 1987), my own cursory research indicates that the error created by this nonsimultaneity should be very small. For an entire month in 1986, I constructed an index of the average time to the last trade for the S&P 500, with market value proportions as weights. After the first half hour of the trading day, the index was typically only about 5 minutes old. I therefore believe that his method can be improved by treating the underlying asset as just another option, but payout protected with a zero striking price.

[14] This uses the well-known method described in Cox, Ross, and Rubinstein (1979).

[15] If it is known how the payout return depends on the ending nodal asset prices, then stochastic payouts can be easily handled by replacing the equation for S with:

$$S = (\Sigma_j P_j S_j \delta_j^n)/r^n.$$

The P_j are therefore the risk-neutral probabilities, which are, in the least squares sense, closest to lognormal that cause the present values of the underlying asset and all the options calculated with these probabilities to fall between their respective bid and ask prices.

This technique also passes the earlier test. That is, if all the options are priced with bid and ask prices surrounding their standard binomial values, then $P_j = P'_j$ for all j. In addition, the technique passes a second test. If a solution exists, then the denser the set of options, other things equal, the less sensitive P_j will be to the prior guess. In the limit, as the number of options becomes increasingly dense on the real line, P_j will become independent of the prior.

Using NAG nonlinear programming software, for a 200-step tree for S&P 500 index options observed three times a day from 1986 to 1992, in a few seconds for each time, the algorithm always converged to a solution whenever there were no general arbitrage opportunities among the underlying asset, riskless asset, and the options—that is, whenever there existed values of S and $\{C_i\}$ such that if transactions to buy and sell could have been effected at those same prices (without paying the bid-ask spread), no general arbitrage opportunities existed.[16]

Although I adopted a specific minimization function and a specific prior guess, the optimization method is quite flexible. As long as a solution exists and the number of probabilities n is greater than the number of options m, the solution will depend on the prior guess and minimization function chosen. The least squares form of the minimization function is just one of a number of candidates. For example, one might instead minimize the "goodness of fit" function $\Sigma_j(P_j - P'_j)^2/P'_j$ or, as suggested to me by Ray Hawkins, the "maximum entropy" function $-\Sigma_j P_j \log(P_j/P'_j)$ or, as Ron Dembo has suggested, the absolute difference function, $\Sigma_j|P_j - P'_j|$.[17]

For most *precrash* days, there is very little difference if any between the risk-neutral lognormal prior guess and the implied posterior probabilities. Figure 4 provides an illustration of this procedure for a typical *postcrash* day January 2, 1990 at 10:00 A.M. in Chicago, using S&P 500 index options maturing in June 1990. Using a 200-step tree, the risk-neutral prior guess is

[16] The menu of general arbitrage opportunities is described in Chapter 4 of Cox and Rubinstein (1985). By far the most frequent arbitrage opportunities present in the data are butterfly spreads.

[17] Kendall and Stuart (1979) in Chapter 30 discuss the related problem of measurement of the "closeness" of an observed frequency distribution to an hypothesized probability distribution, including the "goodness of fit" and "maximum entropy" criteria. In addition, they discuss strategies for spacing observations. For example, an alternative to spacing the S_j at the end of a standard binomial tree, is to space the ending asset prices, separated by areas of equal probability. The absolute difference criteria has the advantage that the optimization problem can be reduced to a linear rather than a quadratic, or more generally, nonlinear program.

In independent research, Stutzer (1993) uses the maximum entropy criterion to solve for state-contingent prices in a manner similar to mine. He provides a three-state example involving a single option, using exact equality constraints for the current underlying asset and option prices.

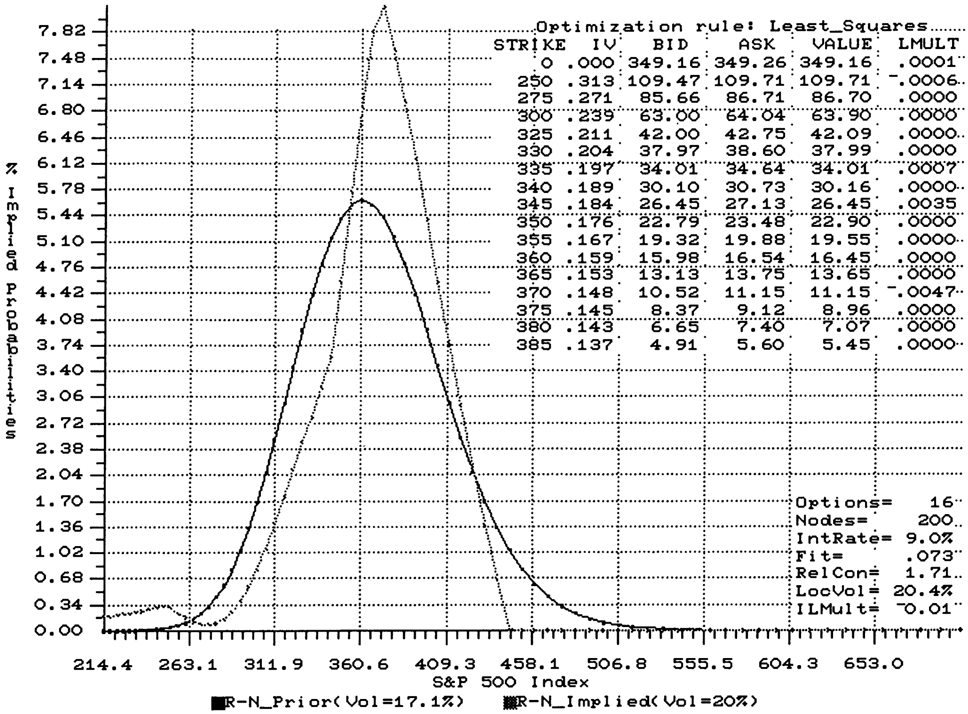


Figure 4. Risk-neutral 164-day probabilities of S&P 500 index options (January 2, 1990; 11:00 A.M.).

assumed to be approximately lognormal with an implied volatility of 17 percent, equal to the average of the two nearest-the-money options (strikes 350 and 355). The risk-neutral implied posterior distribution is slightly bimodal and more highly skewed and kurtotic. The bimodality coming from the lower tail ("crash-o-phobia") is quite common during the postcrash period. The table on the upper right shows the Black-Scholes implied volatilities (column 2) and the concurrent bid $(S^b, \{C_i^b\})$ (column 3) and ask $(S^a, \{C_i^a\})$ (column 4) prices for each option.[18] The index itself was assumed to have a two-tick bid-ask spread around the 10:00 A.M. reported level of the index (354.75) after deducting anticipated dividends prior to the options' expiration date.

Using the best fitting risk-neutral probabilities, column 5 reports values $(S, \{C_i\})$. Note that in each case, the values lie between the corresponding quotes. When a quotation constraint is binding so that the value is equal to either the bid or the ask, the NAG software also reports the associated

[18] The CBOE options market does not perform on command. In particular, the 13 puts and 15 calls, which were used to construct the bid and ask prices, did not trade simultaneously at 10:00 A.M. To surmount this problem, elaborate procedures were followed to adjust nonsimultaneously observed option prices to their probable 10:00 A.M. levels, had they all been quoted at that time.

Lagrangian multiplier (or shadow price, to use the language of economics), shown in column 6. Positive multipliers indicate that the bid is binding, and negative multipliers indicate that the ask is binding; and the absolute size of the multiplier indicates how important its associated option was, given the quotes of the other options, as a cause of squared differences between prior and posterior probabilities. High absolute multipliers indicate options priced under the more extreme departures from risk-neutral lognormality. This suggests the first of three potential empirical tests: examine the profitability of buying options with high negative multipliers and selling options with high positive multipliers. Even though the model is designed to fit the prices of all the options of a given expiration, information from it can be used to isolate the most extreme deviations from our prior guess.

From this point, by whatever method—Shimko's, optimization, or some other way—we assume that we have satisfactorily estimated risk-neutral probabilities P_j associated with asset returns $R_j = S_j/S$.

III. Implied Stochastic Process

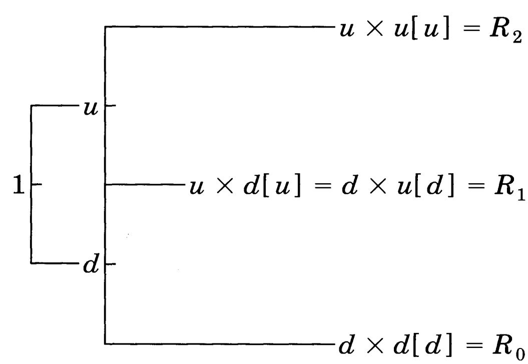
We now take as exogenous the discretized risk-neutral probability distribution of the underlying asset returns at some specified time in the future. For example, suppose there are three possible ending discrete returns R_0, R_1, and R_2 where $0 < R_0 < R_1 < R_2$. Each of these returns has known associated risk-neutral probabilities P_0, P_1, and P_2, where all the probabilities are positive numbers which sum to one.[19]

ASSUMPTION 1: *The underlying asset return follows a binomial process.*

ASSUMPTION 2: *The binomial tree is recombining.*

ASSUMPTION 3: *The ending nodal values are ordered from lowest to highest.*

For our example, with only three possible outcomes, we must then have an $n = 2$ step binomial tree:



(A)

[19] If some probabilities are zero, replace them with very small numbers.

Here the notation indicates that the move at any step can depend on the node. For example, $u[d]$ is the up move immediately following a down move. Because the tree is assumed to be recombining, $u \times d[u] = d \times u[d]$.

Discussion of Assumption 1

The assumption of a binomial process, as a practical matter,[20] places the model, in the continuous-time limit, in the context of a single state-variable diffusion process where both the drift and volatility can be quite general functions of the state variable, the prior path of the state variable, and time.

Discussion of Assumption 2

A recombining tree implies *path-independence of returns* in the sense that all paths containing the same number of up moves and the same number of down moves lead to the same nodal return, irrespective of the ordering of the moves along the paths. This restricts the limiting diffusion to one which does not depend on the prior path of the state variable. It would seem, in addition, to rule out diffusions that while path-independent, nonetheless must apparently be mimicked discretely by a nonrecombining tree. A well-known example of this is the constant elasticity of variance diffusion process.[21] However, it has been shown under fairly general conditions that any path-independent tree that is nonrecombining can, by adjusting the move sizes, be transformed into a tree that is recombining without changing the limiting form of the resulting diffusion.[22] To this extent, Assumption 2 is therefore not, as a practical matter, a restriction on the tree but is, rather, a matter of computational convenience.

Discussion of Assumption 3

The ordering assumption is equivalent to requiring that after an up (a down) move, from that point on, the most extreme remaining ending return in the down (up) direction is dropped from consideration. While this assumption will not change the current value of European derivatives maturing at the end of the tree, it will affect the interior structure of the tree and hence other tree properties such as local volatility and option deltas, as well as the value of American options.

[20] The other limiting possibility is the binomial jump process discussed in Cox, Ross, and Rubinstein (1979). This limiting possibility, however, has such bizarre implications that subsequent academic work has shown little interest in it. Of course, in the more general setting of this article, where the binomial move size is itself stochastic, it would be possible for the limiting form to take on a combination of a diffusion and a two-state jump process, where at each instant of time, it would be known in advance just which of these two processes would be mimicked next.

[21] See Cox and Rubinstein (1985).

[22] See Nelson and Ramaswamy (1990).

Also associate with each move, risk-neutral probabilities, where $p[\bullet](1 - p[\bullet])$ is the probability of an up (a down) move after the previous sequence of realized moves indicated in the brackets:

$$
1 \begin{cases} p \\ 1 - p \end{cases}
\qquad
\begin{cases}
p \times p[u] = P_2 \\
p \times (1 - p[u]) + (1 - p) \times p[d] = P_1 \\
(1 - p) \times (1 - p[d]) = P_0
\end{cases}
\tag{B}
$$

Because the "move probabilities" $p[\bullet]$ are risk-neutral:

$$r[\bullet] = ((1 - p[\bullet]) \times d[\bullet]) + (p[\bullet] \times u[\bullet])$$

where $r[\bullet]$ is one plus the riskless rate of interest over the associated binomial step, which at this point may be dependent on the previous combination of realized moves.

Therefore, each probability must be related to its associated up and down moves as follows:

$$p[\bullet] = (r[\bullet] - d[\bullet]) \div (u[\bullet] - d[\bullet])$$

For our example:

$$
\begin{aligned}
p &= (r - d) \div (u - d) \\
p[d] &= (r[d] - d[d]) \div (u[d] - d[d]) \\
p[u] &= (r[u] - d[u]) \div (u[u] - d[u])
\end{aligned}
\tag{C}
$$

Payouts

If the underlying asset has payouts that do not accrue to the holder of a derivative, then we may want to measure the returns R_0, R_1, and R_2 exclusive of payouts and reinterpret the variable $r[\bullet]$ as the ratio of one plus the riskless interest rate divided by one plus the payout rate over the corresponding binomial step.

Our goals is to infer uniquely the entire tree from the ending nodal returns (R_0, R_1, and R_2) and ending nodal risk-neutral probabilities (P_0, P_1,and P_2). Assessing our progress, we must determine 12 unknowns:

$$d, u, r, p, d[d], u[d], r[d], p[d], d[u], u[u], r[u] \text{ and } p[u]$$

from 10 equations, four from equations A, three from equations B, and three from equations C. Since the number of unknowns exceeds the number of equations, we will need to impose other conditions. One possible condition is:

ASSUMPTION 4: *The interest rate is constant (per unit of time).*

In our example, this means that $r = r[d] = r[u]$; so henceforth we shall usually refer to one plus the interest rate simply as r.

Generalization: This assumption is not required if we know (exogeneous to the model) the different node-dependent interest rates. These might be inferred from the current prices of default-free bonds of different maturities, from the interest rates implied in futures contracts with different delivery dates, or from a maturity series of otherwise identical European puts and calls. Should this information be supplied exogenously, in our example, we can let $r \neq r[d] \neq r[u]$.

It is noteworthy that so far we have said nothing about the elapsed time for each move. For example, while the total time of all moves must be equal to the prespecified time from the beginning to the end of the tree, we can divide that time up across moves any way we like. For example, we might suppose that the second move is twice as long as the first move. This flexibility may prove quite useful in some situations. Successively lengthening the time for each move may lead to faster convergence to the continuous-time process, because a wider span of ending returns can be reached with fewer steps.[23] This flexibility also provides a means of handling differences between trading and calendar time—for example, differences between intraweek and weekend volatility.

In this context, I like to think of the interest return r as the "clock" running the tree. Tying it to a specific interval or intervals of time (not simply saying, as we have so far, that it is the interest return over a binomial step of indeterminant length) determines the speed of the tree. There is a kind of "equivalence principle" at work here: an individual who can only view the tree cannot distinguish changes in interest returns $r[\bullet]$ from changes in the elapsed time of different moves. For concreteness, we will suppose that $r[\bullet]$ is always measured over the same interval of calendar time.

Assumption 4 reduces the problem to 10 equations in 10 unknowns. However, it is easy to show that equations B are not independent of each other. Thus, we will need to add further structure.

ASSUMPTION 5: *All paths that lead to the same ending node have the same risk-neutral probability.*

[23] The recent working paper by Amin and Bodurtha (1993) argues that the difficult numerical problem of the valuation of path-dependent American options can be efficiently handled by a nonrecombining binomial tree where the elapsed time per move is successively lengthened as the end of the tree is approached.

This implies we can write down the following equations B′:

$$p \times p[u] = P_2 \equiv P_{uu}$$

$$(1-p) \times p[d] = P_1 \div 2 \equiv P_{du} \qquad p \times (1 - p[u]) = P_1 \div 2 \equiv P_{ud}$$

$$(1-p) \times (1 - p[d]) = P_0 \equiv P_{dd}$$

The variables P_{dd}, $P_{du} = P_{ud}$, and P_{uu}, then, are probabilities associated with single paths through the tree ("path probabilities").

Considering equations B′ by themselves, since there are 4 equations in 3 unknowns, p, $p[d]$, and $p[u]$, they would seem to be overdetermined. However, using the special structure of their right-hand sides, namely that $P_{dd} + P_{du} + P_{ud} + P_{uu} = 1$, it is easy to show that any three of the equations can be used to derive the fourth.

Generalization: This assumption is not required if we know (exogenous to the model) the different path-dependent probabilities $(1-p) \times p[d] = P_{du}$ and $p \times (1 - p[u]) = P_{ud}$. These might be inferred from the current prices of options maturing before the ending date, from the prices of American options, or from options with path-dependent payoffs. In any case, we continue to require $P_{du} + P_{ud} = P_1$.

This ends the specification of the model. Given these equations, the known ending nodal returns $R_0, \ldots, R_2$ and nodal probabilities $P_0, \ldots, P_2$, we show below that it is possible to solve for a unique binomial implied tree: d, u, $d[d]$, $u[d]$, $d[u]$, $u[u]$, and r. Of course, from equations C, we can then immediately determine p, $p[d]$, and $p[u]$, should we wish to do so. Moreover, we also show that the solution is consistent with the nonexistence of riskless arbitrage as we work backwards in the tree.

Before we do this, note that the standard binomial option pricing model is a special case, since all five assumptions hold for that model as well but with the added requirement that u and d are constant throughout the tree. An implication of constant move size is that $P_0 = (1-p)^2$, $P_1 = 2p(1-p)$, $P_2 = p^2$. In contrast, the model here allows these ending probabilities to take on arbitrary values and, therefore, represents a significant generalization.

IV. The Solution

The implied binomial tree can now be solved conveniently by working backwards recursively from the end of the tree.

Here is the general method; it's as simple as One-Two-Three. The unsubscripted P variables below represent *path* probabilities, and the R variables represent nodal values. Say you are working backwards from the end of a tree, and you have worked out (P^+, R^+) and (P^-, R^-) and want to figure out the prior node (P, R):

$$(P, R)\begin{cases}(P^+, R^+)\\[2em] (P^-, R^-)\end{cases}$$

$$\text{One:}\quad P = P^- + P^+$$
$$\text{Two:}\quad p = P^+/P$$
$$\text{Three:}\quad R = [(1-p)R^- + pR^+]/r$$

That's it! and you are now ready for the next backwards recursive step.

To start everything rolling, go the end of the tree and attach to each node its nodal value R_j and *nodal* probability P_j. Now take each ending nodal probability and divide it by the number of paths to that node to get the path probability, which is in general:

$$P_j \div [n!/j!(n-j)!]$$

Also, define the interest return r as the nth root of the sum of P_jR_j, so that:

$$r^n = \Sigma_j P_j R_j \quad \left(\text{with payouts: } (r/\delta)^n = \Sigma_j P_j R_j\right)$$

Each of these three steps makes simple economic sense. The first simply says that an interior path probability equals the sum of the subsequent path probabilities that can eminate from it. The second step is the simple probabilistic rule that allocates total probability across up and down moves since:

$$p = P^+/P \quad \text{and} \quad (1-p) = P^-/P$$

The third step uses the risk-neutral move probability so calculated to determine the interior nodal value R by setting it equal to the discounted value of its risk-neutral expected value at the end of the move.

Interior arbitrage

Starting with positive ending path probabilities, step one insures that all interior path probabilities are also positive, since the sum of two positive numbers is obviously positive. Step two then implies that all move probabilities $p[\bullet]$ are positive, since they are the ratio of two positive numbers, and moreover they are less than one since the $P^+ < P$. This fully justifies our habit of referring to the $p[\bullet]$ as move "probabilities." A necessary and sufficient condition for there to be no riskless arbitrage opportunities between the riskless asset and the underlying asset at any point in the tree is that $r[\bullet]$ (or $r[\bullet]/\delta[\bullet]$ with payouts) must always lie between the corresponding values of $d[\bullet]$ and $u[\bullet]$ at each node. Indeed, from equations C, the fact that the corresponding $p[\bullet]$ qualify as probabilities guarantees that this will be so.

Significance of Assumption 3

With this solution, it is quite easy to see that Assumption 3, which governs the ordering of the returns at the end of the tree, affects the structure of the tree. For example, consider a 3-step tree where $P_0 = P_1 = P_2 = 1/3$, and we permute the ordering of R_0, R_1, R_2 at the end of the tree to R_1, R_0, R_2. Where before $d = [(2/3)R_0 + (1/3)R_1]/r$ and $u = [(1/3)R_1 + (2/3)R_2]/r$, with the permuted ordering $d' = [(2/3)R_1 + (1/3)R_0]/r$ and $u' = [(1/3)R_0 + (2/3)R_2]/r$. Clearly:

$$d < d' < u' < u$$

so that the local volatility over the first move will be smaller under the permuted ordering.

Some n-Step Tree Properties

See Appendix II for a 3-step numerical example. In the third step of the tree, the eight possible move sizes are $d[dd]$, $u[dd]$, $d[du]$, $u[du]$, $d[ud]$, $u[ud]$, $d[uu]$, and $u[uu]$. For example, $d[du]$ means the down move following a sequence of first a down move followed by an up move. By assumption, since the tree is recombining, it must be path independent in the sense that at any node in the tree, for the next move, while its size may depend on the number of up or down moves that gave rise to that node, its size must be independent of the ordering of these prior moves. This is an implication of the return path independence we discussed before. In this case, it means operationally that $d[du] = d[ud]$ and $u[du] = u[ud]$.

In addition, it is quite easy to verify the following result.[24]

Given only Assumption 1: If all ending paths containing the same numbers of up and down moves have the same ending risk-neutral probability, then all interior paths containing the same numbers of up and down moves also have the same interior risk-neutral probability.

Since, as a special case, the supposition to this result clearly holds under Assumptions 2 and 5, its conclusion will as well.

For example, in a 3-step tree, the equations replacing equations B' would be:

$$p \times p[u] \times p[uu] = P_3 \equiv P_{uuu} \tag{1}$$

$$p \times p[u] \times (1 - p[uu]) = P_2 \div 3 \equiv P_{uud} \tag{2}$$

$$p \times (1 - p[u]) \times p[ud] = P_2 \div 3 \equiv P_{udu} \tag{3}$$

$$(1 - p) \times p[d] \times p[du] = P_2 \div 3 \equiv P_{duu} \tag{4}$$

$$p \times (1 - p[u]) \times (1 - p[ud]) = P_1 \div 3 \equiv P_{udd} \tag{5}$$

[24] One may be tempted to believe the complementary result: given only Assumption 1, if all ending paths containing the same numbers of up and down moves lead to the same ending node, this will be true of interior paths as well (in other words, the tree will be recombining). However, such is *not* the case.

$$(1-p) \times p[d] \times (1-p[du]) = P_1 \div 3 \equiv P_{dud} \tag{6}$$

$$(1-p) \times (1-p[d]) \times p[dd] = P_1 \div 3 \equiv P_{ddu} \tag{7}$$

$$(1-p) \times (1-p[d]) \times (1-p[dd]) = P_0 \equiv P_{ddd} \tag{8}$$

In this case there is only one interior node that can be reached by more than one path: the middle node at the end of step 2 which is reached by two paths. We need to show that these path probabilities are equal. That is:

$$p \times (1-p[u]) = (1-p) \times p[d]$$

To see this, equations (3) and (5) imply $p[ud] = P_2/(P_1 + P_2)$. Equations (4) and (6) imply $p[du] = P_2/(P_1 + P_2)$. Therefore, $p[ud] = p[du]$. Substitute this into equations (3) and (4) [or equations (5) and (6)], and we have our desired result.

This 3-step example also highlights the special role played by the interest rate. In addition to providing a clock, it is also the primary route through which the model communicates with and is bound to the external world. I conjecture that in any reasonable economy, it is always possible to design a security whose stochastic process obeys Assumptions 1 and 2. But, given only these two assumptions, we are no longer free to impose whatever structure we wish on the risk-neutral probabilities since:

Given only Assumptions 1 and 2: The risk-neutral move probabilities will be path-independent if and only if the riskless interest rate is path-independent.

This follows immediately from equations of type C, which provide expressions for $p[du]$ and $p[ud]$:

$$p[du] = (r[du] - d[du]) \div (u[du] - d[du])$$
$$p[ud] = (r[ud] - d[ud]) \div (u[ud] - d[ud]).$$

Since Assumptions 1 and 2 imply that $d[du] = d[ud]$ and $u[du] = u[ud]$, we must then have $p[du] = p[ud]$ if and only if $r[du] = r[ud]$, the conditions which embody the notion of path independence. Since we have just shown that Assumptions 1, 2, and 5 imply $p[du] = p[ud]$, these must also imply $r[du] = r[ud]$.

As a result of these properties, the qualitative features of the tree that we have noted for the ending nodes are recursively reproduced as we work backwards in the tree.

This reduction of the solution to a simple recursive procedure is quite fortunate. For example, in a 200-step lattice, we need to determine a total of 60,301 unknowns: 40,200 potentially different move sizes, 20,100 potentially different move probabilities, and 1 interest rate from 60,301 independent equations, many of which are nonlinear in the unknowns. Despite this, the solution procedure is only slightly more time consuming than for a standard binomial tree with given constant move sizes and move probabilities.[25]

V. Extensions

As mentioned earlier, Assumptions 4 and 5 can be completely dropped if we have some way of knowing how one plus the interest rate $r[\bullet]$ varies with the underlying asset return and time and how the individual ending risk-neutral probabilities P_j for $j = 0, \ldots, n$ are divvied up among different paths leading to the same ending node.

Node-dependent interest rates

In some situations, we may be willing to infer the time dependence of $r[\bullet]$ from the forward rates implied in the current prices of riskless bonds of different maturities. For example if B_1 and B_2 were the current prices of zero coupon bonds yielding \$1 at the end of steps 1 and 2, respectively, then we could preset $r = 1/B_1$ and $r[d] = r[u] = B_1/B_2$. It is much more difficult to see where we could obtain reliable information about the dependence of $r[\bullet]$ on the combination of prior moves—that is, to justify by inference from the prices of securities that $r[d] \neq r[u]$.

However, suppose we interpret $r[\bullet]$ as one plus the rate of interest divided by one plus the underlying asset payout rate. We may then come to appreciate this added flexibility. This gives us a simple way of incorporating into the tree, at minimal computational cost, a payout rate that can be a very general function of the concurrent underlying asset return and time, provided this function can be exogenously specified.

Introducing dividends into the standard binomial model by adjusting the risk-neutral move probabilities can lead to incorrect results when dividends are highly state or date dependent. For example, for some common equities with quarterly dividends, one might try to include the effect of dividends by calculating the move probability as $p[\bullet] = ((r/\delta[\bullet]) - d) \div (u - d)$. However, for accurate trees with a large number of moves, with $\delta[\bullet]$ sufficiently large and discrete, it is quite easy for $r/\delta[\bullet] < d$ over some moves, producing negative and nonsensical "probabilities." Fortunately, as Bruno Dupire has mentioned to me, this is not a problem with the implied tree constructed here, because the move sizes $d[\bullet]$ and $u[\bullet]$ will automatically be adjusted in

[25] In his notes on which he has based several talks during the last two years, Hayne Leland at Berkeley has solved a similar problem. He shows that under certain assumptions, given ending and possibly path-dependent desired personal wealth levels, ending *subjective* (not risk-neutral) probabilities of an investor relative to the market consensus investor can be implied. It is then possible to work backwards from the end of a recombining binomial tree and derive the tree structure of the investor's subjective beliefs. He uses his analysis to answer the puzzling question: who should buy exotic options such as lookbacks and Asians? Although he assumes a standard constant move size binomial tree, his work gave me the faith to pursue the research reported here—that I should not be daunted by the number of variables to be determined in the tree.

the recursive procedure to insure that $d[\bullet] < (r/\delta[\bullet]) < u[\bullet]$ at every move in the tree.

So, if we know how $r[\bullet]$ and $\delta[\bullet]$ depend on the underlying asset and time, we can use these rates in constructing our tree. However, we are not completely free to choose them, since, to avoid arbitrage opportunities, the one-plus interest rates (possibly divided by one-plus payout rates) must be individually positive and jointly satisfy:

$$\begin{aligned} 1 = P_{dd}(R_0/(r \times r[d])) + P_{du}(R_1/(r \times r[d])) \\ + P_{ud}(R_1/(r \times r[u])) + P_{uu}(R_2/(r \times r[u])) \end{aligned}$$

This is an obvious generalization of the above solution for r under constant interest rates, which, for purposes of comparison, can be restated as:

$$1 = P_{dd}(R_0/r^2) + P_{du}(R_1/r^2) + P_{ud}(R_1/r^2) + P_{uu}(R_2/r^2)$$

In words, more generally, each ending return must be discounted by its associated path of interest rates before talking risk-neutral expectations.

While this generalization extends easily to an n-step tree, to maintain the benefits of a recombining tree, we cannot go so far as to allow the interest rate structure to be path dependent. In the absence of Assumption 4 or 5, we no longer have a way of guaranteeing that the interest rate will be path independent. So for example, in a 3-step tree we must then add the requirement that $r[du] = r[ud]$.

Different Risk-Neutral Path Probabilities at the Same Ending Node

A natural way to infer these probabilities is from standard options with maturities *prior* to the ending date. In our 2-step example, suppose through options that mature at the end of step 1, we are able to infer the risk-neutral nodal probabilities (which in this special case are also path probabilities) P_d and P_u at that time, where $P_d + P_u = 1$. It turns out that this gives us just enough information to infer the individual path probabilities P_{du} and P_{ud}, which, since we have dropped Assumption 5, are no longer assumed to be equal.

To see this, in a 2-step tree, over the first step, we must have $p = P_u$. We can now reconsider equations B, which before left the move probabilities indeterminant. With this added restriction, they can be solved easily for:

$$p[d] = 1 - P_0 \div P_d \quad \text{and} \quad p[u] = P_2 \div P_u$$

Note that in this case the two path probabilities leading to the middle ending node at step 2 are not generally equal; that is:

$$P_{du} = (1-p)p[d] = P_d - P_0 \quad \text{and} \quad P_{ud} = p(1-p[u]) = P_u - P_2$$

Of course, jointly they continue to satisfy the requirement that:

$$P_{du} + P_{ud} = (P_d - P_0) + (P_u - P_2) = 1 - P_0 - P_2 = P_1$$

Extending this example to n-steps, remember that even though it is no longer true that all paths that lead to the same interior or ending node have the same probability, the move probabilities must remain independent of the prior path. For example, in a 3-step tree, it remains the case that $p[ud] = p[du]$. This follows from the assumption that the binomial tree is recombining. As previously noted, a recombining tree implies that $d[du] = d[ud]$, $u[du] = u[ud]$. In addition, to maintain the benefits of a recombining tree, in the absence of Assumptions 4 and 5, we must in their place assume $r[du] = r[ud]$. We then have by our earlier result: $p[ud] = p[ud]$.

To imply all the path probabilities in an n-step tree requires that we know exogenously all the nodal interior and ending probabilities in the tree. In turn, these nodal probabilities can be inferred from standard European options, provided that their maturities span all nodal dates in the tree.[26] That is, we would need currently available options maturing at the end of step 1, step 2, step 3, etc. Given the nature of traded options, this is clearly an unrealistic expectation for trees sufficiently fine to be of practical value. That is why, for application purposes, we will continue to maintain Assumption 5.

However, in future research, it should prove useful to investigate methods of interpolating across the available option maturities to fill in the missing expiration dates, or to find some way to use the current prices of American options for the same purpose.

Even without this, what shorter maturity options that do exist can be put to good purpose. Here is a second potential empirical test: compare their market prices to the value for them that we would infer from the early portion of our binomial tree implied from the prices of longer maturity options. Given Assumptions 1 to 4, this gives us a way of separately testing Assumption 5 concerning path-independent risk-neutral probabilities. If the tree successfully predicts the contemporaneous prices of the shorter maturity options, Assumption 5 need not concern us.

VI. Volatility and Mean Structure

Move (Local) Volatility

Knowing the full binomial tree, at any node we can measure the move volatility $\sigma[\bullet]$ as follows:

$$\mu[\bullet] \equiv ((1 - p[\bullet]) \times \log d[\bullet]) + (p[\bullet] \times \log u[\bullet])$$
$$\sigma^2[\bullet] \equiv \left((1 - p[\bullet]) \times [\log d[\bullet] - \mu[\bullet]]^2\right) + \left(p[\bullet] \times [\log u[\bullet] - \mu[\bullet]]^2\right)$$

Holding fixed the time remaining to the ending node, if limits are taken properly, as the number of moves increases and the move size goes to zero, the move volatility will approach the instantaneous diffusion volatility, com-

[26] Note that for there to be no arbitrage opportunities, the interior nodal probabilities derived from option prices would need to be consistent across dates. In our 2-step example, we would require that $P_d > P_0$ and $P_u > P_2$.

monly written as $\sigma(S, t)$, in the continuous-time literature. In the Black-Scholes model the diffusion volatility is assumed to be constant, or at most a function of time—either known in advance, or implied in some way from option prices. In other models (such as the constant elasticity of the variance diffusion model), the diffusion volatility is assumed to be a known function of S, where perhaps a free parameter might be inferred from option prices. By contrast, not only does the model here permit dependence of σ on t as well as S, but the model does not even begin with a specific parameterization of $\sigma(S, t)$; instead, using the rules to construct our binomial tree—and despite the fact that $\sigma(S, t)$ is likely to be a very complex function unique for each different assumed ending return distribution—$\sigma[\bullet]$, as an approximation to $\sigma(S, t)$, can nonetheless be easily determined by using the above backwards recursive solution procedure.

Table IV examines a "typical" day from the postcrash period. Bid (ask) quotes from 13 June puts and 15 June calls on January 2, 1990 at 10:00 A.M. on the S&P 500 index were averaged for each striking price and input into the quadratic program to estimate risk-neutral probabilities for the June expiration date. These probabilities were then taken as inputs to create an implied binomial tree. Finally, at each node in the tree the local volatility was calculated using the above equation. Although the annualized local volatility on January 2 was virtually the same as the annualized global volatility over the life of the options, the predicted pattern of local volatility over time shows considerable variation. Using the model as a lens to peer into the future, we would expect the local volatility to rise dramatically should the index decline rapidly. For example, if the index fell from 355 to 302 (an 18 percent decline) over the next twelve calendar days, the annualized local volatility should quadruple from about 20 to 77 percent. But, if the same drop in the index occurred over a longer time interval, say three months, then the local volatility would only double. On the other hand, increases in the index should be accompanied by significant decreases in local volatility, although again this effect tends to be attenuated the longer the increase takes to occur. If, instead, the index remains relatively flat over the next three months, then local volatility should decline to about 15 percent. On the downside, it is tempting to conclude that the market has built into option prices a repeat of the experience during the crash: a sudden decline in prices led to a tripling or even quadrupling of Black-Scholes implied volatility, which gradually subsided as the market stabilized in the months following the crash.

These results are consistent with Shimko's time-series empirical observation that Black-Scholes implied volatility varies strongly and inversely with the contemporaneous index return. The constant elasticity of variance diffusion formula would also predict that local volatility should be inversely correlated with a contemporaneous index return. However, since that model is stationary, it will not predict the time-dependent nature of this relation.

Table IV suggests a third and final empirical test: if we really take the model seriously, we should be able to march into the future along the realized binomial path and find that options continue to imply what remains of the original tree. Of course, the Black-Scholes model demonstrably fails this test, and realistically we cannot expect a very close fit here, since the real world is certainly much more complex than any model could be. So the interesting question to answer is one of comparative models: can predictions of local volatility be improved by using this approach compared to existing alternatives?

Global Volatility

Remember that we are looking at *local* volatilities, not Black-Scholes-implied volatilities that summarize the level of uncertainty over the *entire life* of an option. The very high (low) local volatilities shown in the table could imply much lower (higher) Black-Scholes-implied volatilities over longer intervals. This is evident from the table, which indicates that the effects of extreme volatility levels, other things equal, are attenuated as one moves into the future. This is an issue that can be resolved within the structure of the model. As we work backwards in the tree, at each interior node we can calculate the ending nodal probabilities conditional upon arriving at that interior node (see the formula is Section VII). Using these conditional probabilities, at each interior node we then calculate the risk-neutral *global* volatility from that node looking forward to the end of the tree. Table V displays the annualized global volatility structure. Indeed, the sensitivity of global volatility to the level of the index is about half the sensitivity of the local volatility. So, for example, if the index falls from 355 to 302 over the next twelve days, while the local volatility rises from about 20 to 77 percent, the global volatility rises from 20 to 38 percent. Similarly, on the upside, if the index moves from 355 to 400 in twenty-seven days, the local volatility falls from 20 to 4 percent, but the global volatility only falls from 20 to 9 percent.

It is also possible to calculate a tree of annualized *at-the-money* Black-Scholes-implied volatilties. To do this, at each interior node calculate the Black-Scholes value of an option (see Section VII) with a striking price set equal to the index level at that node and a time-to-expiration equal to the remaining time to the end of the tree. Then invert the Black-Scholes formula to obtain the implied volatility at each node. Such a tree shows that the current (time 0) implied volatility is 17 percent. If, after twelve days, the index falls from 355 to 302, the implied volatility rises to 39 percent. On the other hand, if, after twenty-seven days, the index rises from 355 to 400, the implied volatility falls to 9 percent. So the behavior of the at-the-money Black-Scholes-implied volatility is quite similar to the global volatility.

Consensus Mean

In the diffusion continuous-time limit, the move volatility calculated from risk-neutral probabilities and the move volatility calculated from the "true" market-wide (consensus) subjective probabilities converge to the same number as the move size approaches zero. However, this is not true for the mean.

Table IV

Annualized Percentage Implied Local Volatility Structure ($\sigma[\bullet]$)

Based on S&P 500 index June call and put options maturing in 164 days.
January 2, 1990: 10:00 A.M.

S&P 500	Days into the Future																			
Index	0	3	7	12	17	22	27	32	37	42	47	52	57	61	66	71	76	81	86	91
407									3.3	3.5	3.7	3.9	4.1	4.2	4.4	4.7	5.0	5.2	5.6	5.9
405								3.7	3.8	4.0	4.2	4.4	4.6	4.7	5.0	5.3	5.5	5.8	6.2	6.5
402								4.1	4.3	4.5	4.7	4.9	5.1	5.3	5.5	5.8	6.1	6.4	6.7	7.1
400							4.4	4.6	4.7	4.9	5.1	5.3	5.6	5.8	6.0	6.3	6.6	6.9	7.3	7.6
397							4.8	5.0	5.2	5.4	5.6	5.8	6.1	6.2	6.5	6.8	7.1	7.4	7.8	8.2
394						5.1	5.2	5.4	5.6	5.8	6.0	6.3	6.5	6.7	7.0	7.3	7.6	7.9	8.3	8.6
392						5.5	5.7	5.8	6.0	6.2	6.5	6.7	7.0	7.1	7.4	7.7	8.0	8.4	8.7	9.1
389					5.7	5.9	6.1	6.3	6.5	6.7	6.9	7.1	7.4	7.6	7.8	8.2	8.5	8.8	9.1	9.5
386					6.1	6.3	6.5	6.7	6.9	7.1	7.3	7.5	7.8	8.0	8.2	8.6	8.9	9.2	9.5	9.8
384					6.5	6.7	6.9	7.1	7.3	7.5	7.7	7.9	8.2	8.4	8.6	9.0	9.2	9.5	9.8	10.2
381				6.8	7.0	7.2	7.3	7.5	7.7	7.9	8.1	8.3	8.6	8.8	9.0	9.3	9.6	9.9	10.2	10.5
378				7.3	7.4	7.6	7.8	8.0	8.1	8.3	8.5	8.8	9.0	9.2	9.4	9.7	10.0	10.3	10.5	10.8
376				7.8	8.0	8.1	8.3	8.4	8.6	8.8	9.0	9.2	9.4	9.6	9.8	10.1	10.4	10.6	10.9	11.2
373			8.4	8.5	8.6	8.7	8.8	8.9	9.1	9.3	9.5	9.6	9.9	10.0	10.2	10.5	10.8	11.0	11.3	11.5
371			9.2	9.2	9.3	9.3	9.4	9.5	9.6	9.8	9.9	10.1	10.3	10.5	10.7	10.9	11.2	11.4	11.6	11.9
368			10.2	10.1	10.1	10.1	10.1	10.2	10.3	10.4	10.5	10.6	10.8	11.0	11.1	11.4	11.6	11.8	12.0	12.3
365		11.6	11.5	11.3	11.1	11.0	11.0	11.0	11.0	11.0	11.1	11.2	11.4	11.5	11.6	11.8	12.0	12.2	12.5	12.7
363		13.3	13.1	12.7	12.4	12.2	12.0	11.9	11.8	11.8	11.8	11.9	12.0	12.0	12.2	12.3	12.5	12.7	12.9	13.1
360		15.3	14.8	14.4	14.0	13.6	13.3	13.1	12.9	12.7	12.6	12.6	12.7	12.7	12.8	12.9	13.0	13.2	13.4	13.6
357		17.5	17.0	16.4	15.8	15.2	14.8	14.4	14.1	13.9	13.7	13.5	13.4	13.4	13.4	13.5	13.6	13.7	13.9	14.1
355*	20.4	20.0	19.3	18.5	17.8	17.2	16.6	16.0	15.5	15.1	14.7	14.5	14.3	14.2	14.2	14.1	14.2	14.3	14.4	14.5

Table IV—*Continued*

S&P 500 Index	Days into the Future																			
	0	3	7	12	17	22	27	32	37	42	47	52	57	61	66	71	76	81	86	91
342		33.7	32.6	31.4	30.0	28.7	27.5	26.3	25.1	23.8	22.7	21.8	20.7	20.1	19.3	18.6	18.0	17.6	17.3	17.1
338		38.6	37.5	36.1	34.7	33.3	31.9	30.3	29.0	27.7	26.4	25.1	23.7	23.0	21.9	20.7	19.9	19.2	18.6	18.2
334			42.4	41.0	39.5	38.0	36.4	34.9	33.3	31.8	30.1	28.7	27.1	26.1	24.7	23.4	22.2	21.2	20.2	19.5
330			47.3	45.9	44.4	42.7	41.1	39.4	37.7	36.0	34.3	32.7	30.8	29.7	28.2	26.3	24.9	23.6	22.3	21.2
326			52.3	50.8	49.1	47.4	45.8	43.9	42.1	40.4	38.6	36.7	34.7	33.5	31.6	29.7	28.0	26.3	24.6	23.1
322			57.2	55.6	53.8	52.2	50.4	48.4	46.6	44.7	42.8	40.8	38.6	37.3	35.4	33.2	31.3	29.4	27.6	25.8
318			62.1	60.1	58.5	56.9	54.8	52.9	51.1	49.0	46.9	45.0	42.6	41.3	39.3	36.9	34.8	32.8	30.6	28.5
314			66.3	64.6	63.2	61.1	59.1	57.4	55.3	53.1	51.1	49.1	46.5	45.2	43.0	40.6	38.4	36.2	34.0	31.8
310			70.4	69.1	67.5	65.2	63.5	61.7	59.2	57.1	55.2	52.8	50.5	48.9	46.6	44.2	41.9	39.7	37.5	35.1
306			74.6	73.6	71.2	69.3	67.9	65.3	63.1	61.2	58.8	56.4	53.9	52.3	50.2	47.5	45.4	43.2	40.9	38.4
302			78.7	77.2	74.9	73.5	71.4	68.9	67.0	64.9	62.1	60.0	57.2	55.7	53.8	50.9	48.8	46.3	43.9	41.7
297			82.9	80.4	78.6	77.6	74.5	72.4	70.9	67.8	65.4	63.5	60.4	59.2	56.6	54.2	51.5	49.1	47.0	45.0
293				83.5	82.3	80.2	77.6	76.0	73.4	70.7	68.7	66.3	63.7	61.8	59.0	56.4	54.0	52.0	50.1	47.4
289				86.7	86.1	82.7	80.8	79.3	75.9	73.6	71.9	68.4	65.8	63.8	61.5	58.6	56.5	54.7	51.9	49.6
285				89.8	87.9	85.2	83.9	81.0	78.3	76.5	73.4	70.5	67.5	65.9	64.0	60.7	58.9	55.9	53.6	51.9
281				93.0	89.7	87.7	86.5	82.8	80.7	78.6	75.0	72.7	69.3	67.9	65.4	62.8	59.6	57.1	55.3	53.3
277				94.4	91.4	90.3	87.5	84.6	83.1	79.5	76.6	74.8	71.0	69.6	66.0	63.2	60.3	58.4	56.3	53.6
273				95.2	93.2	92.5	88.4	86.4	84.2	80.4	78.1	75.3	72.4	69.6	66.6	63.4	61.1	59.1	56.0	53.8

* Exact S&P 500 index on January 2, 1990, reported at 10:00 A.M. in Chicago is 354.75.

Table V

Annualized Percentage Implied Global Volatility Structure

Based on S&P 500 index June call and put options maturing in 164 days.
January 2, 1990: 10:00 A.M.

S&P 500 Index	Days into the Future																			
	0	3	7	12	17	22	27	32	37	42	47	52	57	61	66	71	76	81	85	91
407									7.6	7.8	8.0	8.2	8.4	8.6	8.8	9.0	9.3	9.5	9.8	10.1
405								8.0	8.1	8.3	8.5	8.7	8.9	9.1	9.3	9.6	9.8	10.1	10.3	10.6
402								8.5	8.6	8.8	9.0	9.2	9.4	9.6	9.8	10.1	10.3	10.5	10.8	11.1
400							8.7	8.9	9.1	9.3	9.5	9.6	9.9	10.0	10.2	10.5	10.7	11.0	11.2	11.5
397							9.2	9.3	9.5	9.7	9.9	10.1	10.3	10.4	10.6	10.9	11.1	11.4	11.6	11.9
394						9.4	9.6	9.7	9.9	10.1	10.3	10.4	10.7	10.8	11.0	11.3	11.5	11.7	12.0	12.2
392						9.8	9.9	10.1	10.3	10.4	10.6	10.8	11.0	11.1	11.3	11.6	11.8	12.0	12.3	12.5
389					10.0	10.1	10.3	10.5	10.6	10.8	11.0	11.1	11.3	11.5	11.7	11.9	12.1	12.3	12.6	12.8
386					10.4	10.5	10.6	10.8	10.9	11.1	11.3	11.4	11.7	11.8	12.0	12.2	12.4	12.6	12.8	13.0
384					10.7	10.8	11.0	11.1	11.3	11.4	11.6	11.8	11.9	12.1	12.2	12.5	12.6	12.8	13.1	13.3
381				10.9	11.1	11.2	11.3	11.5	11.6	11.7	11.9	12.1	12.2	12.3	12.5	12.7	12.9	13.1	13.3	13.5
378				11.3	11.4	11.5	11.7	11.8	11.9	12.1	12.2	12.4	12.5	12.6	12.8	13.0	13.2	13.3	13.5	13.7
376				11.8	11.8	11.9	12.0	12.1	12.3	12.4	12.5	12.7	12.8	12.9	13.1	13.2	13.4	13.6	13.7	13.9
373			12.2	12.3	12.3	12.4	12.4	12.5	12.6	12.7	12.8	13.0	13.1	13.2	13.3	13.5	13.6	13.8	13.9	14.1
371			12.9	12.8	12.8	12.9	12.9	12.9	13.0	13.1	13.2	13.3	13.4	13.5	13.6	13.8	13.9	14.0	14.2	14.3
368			13.6	13.5	13.5	13.4	13.4	13.4	13.4	13.5	13.6	13.6	13.7	13.8	13.9	14.1	14.2	14.3	14.4	14.6
365		14.7	14.5	14.3	14.2	14.1	14.0	14.0	13.9	14.0	14.0	14.0	14.1	14.1	14.2	14.3	14.5	14.6	14.7	14.8
363		15.8	15.6	15.3	15.1	14.9	14.7	14.6	14.5	14.5	14.4	14.4	14.5	14.5	14.6	14.7	14.7	14.8	14.9	15.1
360		17.0	16.7	16.4	16.1	15.8	15.6	15.4	15.2	15.1	15.0	14.9	14.9	14.9	14.9	15.0	15.0	15.1	15.2	15.3
357		18.3	18.0	17.6	17.2	16.8	16.5	16.2	16.0	15.8	15.6	15.5	15.4	15.3	15.3	15.3	15.4	15.4	15.5	15.6
355*	20.0	19.7	19.3	18.8	18.3	17.9	17.5	17.1	16.8	16.5	16.2	16.1	15.9	15.8	15.8	15.7	15.7	15.7	15.8	15.9

Table V—*Continued*

S&P 500 Index	Days into the Future																			
	0	3	7	12	17	22	27	32	37	42	47	52	57	61	66	71	76	81	85	91
342		25.7	25.3	24.7	24.1	23.5	22.9	22.3	21.7	21.1	20.5	19.9	19.3	19.0	18.6	18.1	17.8	17.6	17.4	17.2
338		27.7	27.1	26.6	25.9	25.3	24.7	24.1	23.4	22.7	22.1	21.5	20.8	20.4	19.8	19.2	18.8	18.4	18.0	17.8
334			29.0	28.3	27.6	27.1	26.4	25.7	25.1	24.4	23.7	23.0	22.3	21.8	21.2	20.5	19.9	19.3	18.9	18.5
330			30.4	29.9	29.4	28.7	28.0	27.4	26.7	26.0	25.3	24.6	23.8	23.3	22.6	21.8	21.1	20.5	19.8	19.3
326			31.9	31.5	30.8	30.1	29.6	29.0	28.2	27.5	26.8	26.1	25.3	24.8	24.1	23.2	22.4	21.7	20.9	20.2
322			33.3	32.8	32.1	31.6	31.1	30.3	29.6	29.0	28.3	27.5	26.7	26.2	25.4	24.5	23.7	23.0	22.2	21.4
318			34.7	34.0	33.4	33.0	32.2	31.5	31.0	30.3	29.5	28.8	27.9	27.5	26.8	25.8	25.1	24.3	23.4	22.5
314			35.7	35.1	34.7	34.0	33.3	32.8	32.2	31.4	30.7	30.1	29.2	28.8	28.0	27.1	26.4	25.5	24.7	23.8
310			36.6	36.2	35.8	35.0	34.5	34.0	33.1	32.5	31.9	31.2	30.4	29.9	29.1	28.3	27.4	26.7	25.9	25.0
306			37.5	37.3	36.6	36.0	35.6	34.8	34.1	33.6	32.9	32.1	31.3	30.8	30.1	29.2	28.5	27.8	27.0	26.2
302			38.4	38.0	37.3	36.9	36.4	35.6	35.1	34.5	33.6	33.0	32.1	31.7	31.1	30.2	29.6	28.8	27.9	27.2
297			39.3	38.6	38.1	37.9	37.0	36.4	36.0	35.1	34.4	33.9	32.9	32.6	31.8	31.1	30.3	29.5	28.9	28.2
293				39.1	38.8	38.3	37.6	37.2	36.5	35.7	35.1	34.5	33.8	33.2	32.4	31.6	30.9	30.3	29.8	28.9
289				39.7	39.6	38.7	38.2	37.8	36.9	36.3	35.9	34.9	34.2	33.6	32.9	32.1	31.5	31.0	30.1	29.5
285				40.2	39.8	39.1	38.8	38.0	37.3	36.9	36.1	35.3	34.5	34.0	33.5	32.6	32.1	31.2	30.5	30.0
281				40.8	40.0	39.5	39.2	38.3	37.7	37.2	36.3	35.6	34.8	34.4	33.7	33.0	32.1	31.4	30.9	30.3
277				40.9	40.1	39.8	39.2	38.5	38.1	37.2	36.5	36.0	35.0	34.7	33.7	33.0	32.2	31.6	31.1	30.3
273				40.8	40.3	40.2	39.2	38.7	38.2	37.3	36.7	36.0	35.2	34.5	33.7	32.9	32.2	31.7	30.8	30.2

* Exact S&P 500 index on January 2, 1990, reported at 10:00 A.M. in Chicago is 354.75.

The risk-neutral mean for a single move is obviously r. If $q[\bullet]$ $(1 - q[\bullet])$ is the consensus subjective probability of an up (down) move after the previous sequence of realized moves indicated in the brackets, then the corresponding consensus move mean is:

$$m[\bullet] \equiv [((1 - q[\bullet]) \times d[\bullet]) + (q[\bullet] \times u[\bullet])] \times \delta[\bullet]$$

where $\delta[\bullet]$ is one plus the payout rate over the next binomial step after the previous sequence of realized moves indicated in the brackets, and we must be careful to interpret $d[\bullet]$ and $u[\bullet]$ as only the capital gain portion of the underlying asset return.

In general, not only is $m[\bullet]$ not equal to r in discrete time, but neither will it converge to r in the continuous-time limit. In fact, if the underlying asset is the market portfolio, market-wide risk aversion implies that $m[\bullet] > r$ throughout the entire tree. This is consistent with the common observation that the subjective probability distribution of ending returns cannot be inferred only from knowledge of its risk-neutral distribution.

Ending Node-Dependent Mean

However, in a fully specified utility theory framework, at least for the marketwide portfolio, knowing the implied risk-neutral binomial tree goes a long way toward a full specification of the consensus stochastic move process.[27] To take a very special but classic example, consider a complete markets economy with a representative investor who maximizes the expected utility of ending wealth with constant relative risk aversion subject to the usual budget constraint that she invest all her wealth. Since, in this case, her proportional investment choice is invariant to her level of wealth, we can regard her as having a utility function $U(\delta^n R_j)$, measured in terms of return, and an initial wealth of 1. In brief, she chooses R_j by solving the following Lagrangian problem:[28]

$$\max \Sigma_j Q_j U(\delta^n R_j) - \lambda\left[\Sigma_j (P_j/r^n)\delta^n R_j - 1\right]$$

where $Q_j > 0$ is the subjective probability she attaches to state j (so that $\Sigma_j Q_j = 1$). The P_j/r^n are often called "state-contingent prices."

[27] A recent complementary article by He and Leland (1993), is also in the context of a single state-variable path-independent diffusion process for the market portfolio return and the riskless interest rate, together with an exogenously specified utility of terminal wealth. Their key result is a differential equation that the local drift and volatility must follow to be consistent with equilibrium. Knowing one, say the drift, it is then possible to solve the differential equation for the other, in this case the volatility. In this context, the work here can be viewed as a binomial implementation of this differential equation by using risk-neutral probabilities inferred from market prices as a wedge between the drift and volatility, permitting the solution of either one without first knowing the other.

[28] Actually, if r, δ, and the R_j are taken as exogenous, the problem amounts to determining the equilibrium risk-neutral ending nodal probabilities P_j.

Solving the first-order conditions that arise after differentiating with respect to R_j:

$$Q_j = \lambda\left[(P_j/r^n) \div U'(\delta^n R_j)\right]$$

where

$$\lambda = 1 \div \Sigma_j\left[(P_j/r^n) \div \left(U'(\delta^n R_j)\right)\right]$$

Now, returning to our original binomial tree where we assumed that we somehow knew both the R_j and P_j, we now have a formula for converting the ending nodal risk-neutral probabilities P_j into the ending consensus subjective nodal probabilities Q_j.

This allows us to say a little more about Assumption 5. Under our utility theory framework, since the utility function of return does not depend on the path that leads to the return R_j, and as long as interest rates are not node dependent (although they may be time dependent), then a necessary and sufficient condition for Assumption 5 (which restricts *risk-neutral probabilities*) is that all paths that lead to the same ending node have the same *consensus subjective probability*.

Figure 5, matched to Figure 4, displays the difference between risk-neutral probabilities and consensus subjective probabilities, assuming logarithmic

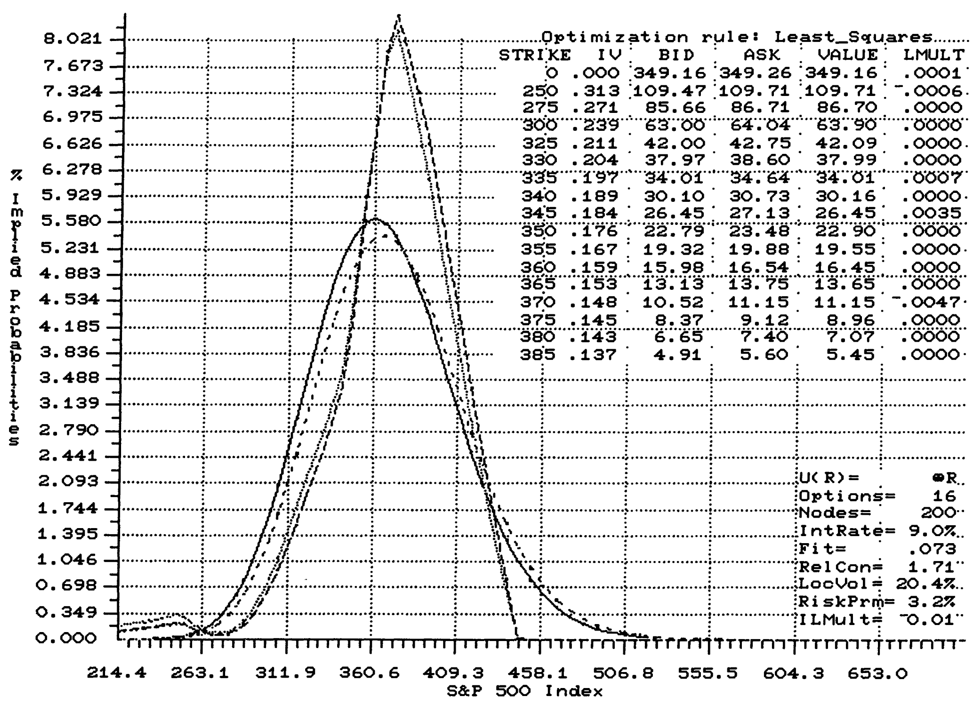


Figure 5. Consensus 164-day probabilities of S&P 500 index options (January 2, 1990; 11:00 A.M.).

utility, $U(\delta^n R_j) = \log(\delta^n R_j)$. Although the posterior distributions are shifted to the right since the market risk premium would then be about 3.3 percent, the qualitative shapes of the distributions are quite similar. Indeed, even if the utility function were $-(\delta^n R_j)^{-.65}$, which produces a market risk premium of 5 percent, the two distributions remain less so, but are still, to the naked eye, quite similar in shape. This tempts me to suggest that, despite warnings to the contrary, we can justifiably suppose a rough similarity between the risk-neutral probabilities implied in option prices and subjective beliefs.

Move-Dependent Mean

It is well known, for the special case of logarithmic utility, that over time our representative investor will follow a myopic decision rule even in the face of a changing set of investment opportunities. Therefore, as we work forward in the tree, at each interior node she chooses investments $d[\bullet]$ and $u[\bullet]$ to solve the following problem:

$$\max((1 - q[\bullet]) \times U(\delta[\bullet]d[\bullet])) + (q[\bullet] \times U(\delta[\bullet]u[\bullet]))$$
$$\text{subject to } (((1 - p[\bullet])/r) \times \delta[\bullet]d[\bullet]) + ((p[\bullet]/r) \times \delta[\bullet]u[\bullet]) = 1$$

where, for our special case, we have $U(\delta[\bullet]d[\bullet]) = \log(\delta[\bullet]d[\bullet])$ and $U(\delta[\bullet]u[\bullet]) = \log(\delta[\bullet]u[\bullet])$. Since utility functions are unique up to increasing linear transformations, with logarithmic utility we can safety regard $U(\delta[\bullet]d[\bullet]) \simeq \log d[\bullet]$ and $U(\delta[\bullet]u[\bullet]) \simeq \log u[\bullet]$.[29] Since $\lambda = 1$ in this case, at each interior node:

$$q[\bullet] = p[\bullet] \times (u[\bullet]/r) \times \delta[\bullet]$$

We now have an easy way at each interior node of the tree to convert our derived risk-neutral move probabilities $p[\bullet]$ into the consensus subjective move probabilities $q[\bullet]$. Moreover, this gives us all the information we need to calculate the consensus mean structure $m[\bullet]$ over all steps in the tree.

Table VI, in parallel with Table IV, examines the annualized local consensus mean assuming logarithmic utility. At high mean levels, the local mean is quite nonstationary and very dependent on previous realized return. At low levels, the local mean becomes almost insensitive to the index and time as it approaches the 9.0 percent riskless interest rate that provides a lower bound. Since the local mean is inversely correlated with return, the return itself should be highly mean reverting. Of course, given Table IV, this behavior of the mean should come as no surprise, since the tradeoff between risk and return in equilibrium will typically imply that local consensus mean and volatility will be positively correlated.

[29] Notice that the budget constraint is simply a restatement of equations C defining "risk-neutral" probabilities.

VII. Options

American Options

As in a standard binomial tree, the current value of a standard *American* call option can be derived by working backwards recursively from the end of the tree using the following two rules:

$$\begin{aligned} \text{at the end:} \quad C[\bullet] &= \max[0, S[\bullet] - K] \\ \text{in the interior:} \quad C[\bullet] &= \max[S[\bullet] - K, (((1 - p[\bullet]) \times C_d[\bullet]) \\ &\quad + (p[\bullet] \times C_u[\bullet])) \div r] \end{aligned}$$

where K is the striking price of the option, $S[\bullet] \equiv SR[\bullet]$, $R[\bullet]$, and $C[\bullet]$ are the underlying path return and the call value after the previous sequence of realized moves indicated in the brackets, and $C_d[\bullet]$ ($C_u[\bullet]$) is the call value after a following down (up) move.

Below, we will also use the notation, $C_{(\bullet)}[\bullet]$ and $S_{(\bullet)}[\bullet]$, more generally to indicate the call value and underlying asset price after the previous sequence of realized moves indicated in the brackets and after the following sequence of moves ($\bullet$). So for example, $S_{du}[\bullet]$ means the underlying asset price observed after the sequence of moves indicated in the brackets and followed by down move followed by an up move. If the brackets are omitted, then the values are current, that is, measured at the beginning of the tree.

For standard call options, the current option "delta" ($\simeq \partial C/\partial S$) and "gamma" ($\simeq \partial^2 C/\partial S^2$) may be read directly from the first steps of the generalized binomial tree:

$$\begin{aligned} \Delta &\equiv (C_u - C_d) \div \delta(S_u - S_d) \\ \Delta[d] &\equiv (C_{du} - C_{dd}) \div \delta[d](S_{du} - S_{dd}) \\ \Delta[u] &\equiv (C_{uu} - C_{ud}) \div \delta[u](S_{uu} - S_{ud}) \\ \Gamma &\equiv (\Delta[u] - \Delta[d]) \div \delta(S_u - S_d) \end{aligned}$$

In a standard binomial tree, one naturally calculates the call option "theta" ($\simeq -\partial C/\partial t$) as:

$$\Theta \equiv (C_{ud} - C) \div 2h$$

where h is the elapsed time for a single binomial step. This works because in a *standard* binomial tree, $S_{ud} = S$, so in the above numerator we are comparing the values of two calls under identical situations except that their time-to-expirations are different. However, in our *generalized* binomial tree, it will typically be the case that $S_{ud} \neq S$, which can easily make the above measure of theta quite inaccurate.

In place of this technique, we can take advantage of the Black-Scholes differential equation, which holds even if σ is a function of S and t. This

Table VI

Annualized Percentage Implied Local Consensus Mean Structure ($m[\bullet]$)

Based on S&P 500 index June call and put options maturing in 164 days. Assumes logarithmic utility. Annualized as if the risk-neutral distribution were lognormal. Risk premia can be calculated by subtracting an annualized riskless rate of 9.0% from these numbers.

January 2, 1990: 10:00 A.M.

S&P 500	Days into the Future																			
Index	0	3	7	12	17	22	27	32	37	42	47	52	57	61	66	71	76	81	86	91
407									9.1	9.1	9.1	9.1	9.2	9.2	9.2	9.2	9.2	9.3	9.3	9.4
405								9.1	9.1	9.2	9.2	9.2	9.2	9.2	9.3	9.3	9.3	9.4	9.4	9.4
402								9.2	9.2	9.2	9.2	9.2	9.3	9.3	9.3	9.3	9.4	9.4	9.5	9.5
400							9.2	9.2	9.2	9.2	9.3	9.3	9.3	9.3	9.4	9.4	9.5	9.5	9.6	9.6
397							9.2	9.3	9.3	9.3	9.3	9.3	9.4	9.4	9.4	9.5	9.5	9.6	9.6	9.7
394						9.3	9.3	9.3	9.3	9.3	9.4	9.4	9.4	9.5	9.5	9.6	9.6	9.7	9.7	9.8
392						9.3	9.3	9.4	9.4	9.4	9.4	9.5	9.5	9.5	9.6	9.6	9.7	9.7	9.8	9.9
389					9.3	9.4	9.4	9.4	9.4	9.5	9.5	9.5	9.6	9.6	9.7	9.7	9.8	9.8	9.9	10.0
386					9.4	9.4	9.4	9.5	9.5	9.5	9.6	9.6	9.6	9.7	9.7	9.8	9.8	9.9	10.0	10.0
384					9.4	9.5	9.5	9.5	9.6	9.6	9.6	9.7	9.7	9.7	9.8	9.9	9.9	10.0	10.0	10.1
381				9.5	9.5	9.5	9.6	9.6	9.6	9.7	9.7	9.7	9.8	9.8	9.9	9.9	10.0	10.1	10.1	10.2
378				9.6	9.6	9.6	9.6	9.7	9.7	9.7	9.8	9.8	9.9	9.9	10.0	10.0	10.1	10.1	10.2	10.3
376				9.7	9.7	9.7	9.7	9.8	9.8	9.8	9.9	9.9	10.0	10.0	10.0	10.1	10.2	10.2	10.3	10.3
373			9.8	9.8	9.8	9.8	9.8	9.9	9.9	9.9	10.0	10.0	10.0	10.1	10.1	10.2	10.2	10.3	10.4	10.4
371			9.9	9.9	9.9	9.9	10.0	10.0	10.0	10.0	10.1	10.1	10.2	10.2	10.2	10.3	10.3	10.4	10.5	10.5
368			10.1	10.1	10.1	10.1	10.1	10.1	10.1	10.2	10.2	10.2	10.3	10.3	10.3	10.4	10.5	10.5	10.6	10.6
365		10.5	10.4	10.4	10.3	10.3	10.3	10.3	10.3	10.3	10.3	10.4	10.4	10.4	10.5	10.5	10.6	10.6	10.7	10.7
363		11.0	10.9	10.8	10.7	10.6	10.6	10.5	10.5	10.5	10.5	10.5	10.6	10.6	10.6	10.7	10.7	10.8	10.8	10.9
360		11.6	11.4	11.3	11.2	11.0	10.9	10.9	10.8	10.8	10.7	10.7	10.7	10.8	10.8	10.8	10.8	10.9	10.9	11.0
357		12.4	12.3	12.0	11.7	11.5	11.4	11.3	11.2	11.1	11.0	11.0	11.0	10.9	11.0	11.0	11.0	11.1	11.1	11.2
355*	13.6	13.5	13.2	12.8	12.6	12.3	12.1	11.8	11.6	11.5	11.4	11.3	11.2	11.2	11.2	11.2	11.2	11.2	11.3	11.3

Table VI—*Continued*

S&P 500 Index	Days into the Future																		
	0	3	7	12	17	22	27	32	37	42	47	52	57	61	66	71	76	81	86	91
342		22.7	21.8	20.7	19.4	18.5	17.9	17.0	16.2	15.4	14.8	14.3	13.8	13.5	13.1	12.8	12.6	12.4	12.3	12.2
338		26.7	26.1	24.2	23.5	22.3	20.9	19.5	18.8	17.9	17.0	16.2	15.4	15.0	14.4	13.8	13.4	13.1	12.8	12.6
334			30.5	29.9	28.3	26.2	24.8	23.7	22.2	20.7	19.4	18.5	17.3	16.7	15.9	15.2	14.5	14.0	13.6	13.2
330			37.9	35.9	33.1	31.6	30.0	27.9	25.6	24.5	23.1	21.6	20.1	19.3	18.2	16.9	16.1	15.3	14.6	14.0
326			45.4	41.9	39.9	38.0	35.2	32.4	31.0	29.0	26.8	24.7	23.2	22.1	20.4	19.2	17.9	16.8	15.8	14.9
322			52.9	49.5	47.8	44.5	40.7	39.1	36.5	33.6	31.2	29.4	26.6	25.6	24.0	21.7	20.4	19.0	17.7	16.5
318			60.4	59.3	55.8	50.9	48.9	45.9	42.1	39.2	36.9	34.2	31.3	29.8	27.5	25.3	23.3	21.5	19.7	18.2
314			71.9	69.1	63.8	60.3	57.1	52.7	48.7	45.9	42.6	39.0	36.1	34.0	31.3	28.8	26.3	24.4	22.6	20.8
310			83.6	78.8	73.1	70.0	65.3	59.9	56.7	52.7	48.3	44.8	40.8	38.8	36.2	32.7	30.4	28.0	25.7	23.4
306			95.3	88.6	84.4	79.9	73.5	69.0	64.6	59.5	54.9	51.2	46.5	44.3	41.0	37.3	34.4	31.6	28.9	26.6
302			107.0	99.9	95.7	89.6	83.0	78.1	72.6	66.6	62.0	57.5	52.3	49.7	45.8	41.8	38.5	35.3	32.6	30.1
297			118.7	112.1	106.9	99.4	92.9	87.2	80.5	74.3	69.1	63.9	58.2	55.2	50.5	46.4	42.5	39.3	36.4	33.5
293				124.4	118.2	109.4	102.9	96.3	88.2	81.9	76.2	69.7	64.0	60.0	55.2	50.3	46.5	43.2	40.1	36.7
289				136.6	129.5	119.3	112.8	104.9	95.9	89.5	83.1	75.2	68.5	64.6	59.9	54.2	50.5	47.0	42.9	39.8
285				148.8	138.0	129.3	122.8	111.7	103.5	97.1	87.8	80.6	72.8	69.1	64.6	58.1	54.4	49.2	45.6	42.9
281				161.0	146.4	139.2	131.2	118.5	111.2	102.8	92.5	86.0	77.0	73.6	67.5	62.0	55.8	51.4	48.3	44.9
277				167.6	154.8	149.2	135.7	125.2	118.9	106.0	97.2	91.4	81.2	77.4	68.8	62.8	57.2	53.5	49.8	45.3
273				172.4	163.3	157.9	140.1	132.0	122.3	109.2	101.9	92.8	84.6	77.5	70.2	63.1	58.5	54.7	49.3	45.8

* Exact S&P 500 index on January 2, 1990, reported at 10:00 A.M. in Chicago is 354.75.

equation gives us a way in discrete time of expressing Θ approximately as a function of C, Δ, and Γ:

$$\Theta \simeq \left[(\log r)C - (\log(r/\delta))S\Delta - {}^1\!/_2\,\sigma^2 S^2 \Gamma\right]/h$$

European Options

Just as in the case of a standard binomial tree, *European* options can be valued directly, without working backwards. For example, the current value of a European call may be stated in closed form as:

$$C = (\Sigma_j P_j \max[0, SR_j - K]) \div r^n$$

Despite the added complexity of the generalized binomial tree, European deltas, gammas, and thetas can also be evaluated using relatively simple closed-form expressions. We begin by deriving closed-form expressions for the ending nodal probabilities assessed at any interior node. From these, we can derive closed-form expressions for the underlying asset and European option values at any interior node. These, in turn, can be used to express in closed form the hedging parameters for European options maturing with or before the ending of the tree.

Let $P_j[\bullet]$ be the ending nodal probability assessed from the node after the previous sequence of realized moves indicated in the brackets. So, for example, $P_j[u]$ is the nodal probability attached to ending node j measured after it is already known that a one-up move has occurred. Using Assumption 5 and a little algebra, it can be shown that:

$$p = \Sigma_j (j/n) P_j \quad \text{and} \quad 1 - p = \Sigma_j [(n-j)/n] P_j$$

$$P_j[u] = \left[(j/n)P_j\right] \div p \quad \text{and} \quad P_j[d] = \left[((n-j)/n)P_j\right] \div (1-p)$$

$$p[u] = \Sigma_j ((j-1)/(n-1)) P_j[u] \quad \text{and} \quad p[d] = \Sigma_j (j/(n-1)) P_j[d]$$

$$P_j[uu] = \left[(j/n) \times ((j-1)/(n-1))P_j\right] \div (p \times p[u])$$

$$P_j[ud] = P_j[du] = \left[((n-j)/n) \times ((n-j-1)/(n-1))P_j\right] \div ((1-p) \times p[d])$$

$$P_j[dd] = \left[((n-j)/n) \times ((n-j-1)/(n-1))P_j\right] \div ((1-p) \times (1-p[d])$$

Now we can use these interior assessments of the nodal probabilities to determine the following interior nodal values for the call and the underlying asset (assuming the payout rate $\delta[\bullet]$ is constant):

$$S_d = (S\Sigma_j P_j[d] R_j) \div (r/\delta)^{n-1}$$

$$S_u = (S\Sigma_j P_j[u] R_j) \div (r/\delta)^{n-1}$$

$$C_d = (\Sigma_j P_j[d] \max[0, SR_j - K]) \div r^{n-1}$$

$$C_u = (\Sigma_j P_j[u]\max[0, SR_j - K]) \div r^{n-1}$$

$$S_{dd} = (S\Sigma_j P_j[dd]R_j) \div (r/\delta)^{n-2}$$

$$S_{du} = S_{ud} = (S\Sigma_j P_j[du]R_j) \div (r/\delta)^{n-2}$$

$$S_{uu} = (S\Sigma_j P_j[uu]R_j) \div (r/\delta)^{n-2}$$

$$C_{dd} = (\Sigma_j P_j[dd]\max[0, SR_j - K]) \div r^{n-2}$$

$$C_{du} = C_{ud} = (\Sigma_j P_j[du]\max[0, SR_j - K]) \div r^{n-2}$$

$$C_{uu} = (\Sigma_j P_j[uu]\max[0, SR_j - K]) \div r^{n-2}$$

Finally, substitute these expressions into the above equations for Δ, Γ, and Θ.[30]

We state one last result, which proved useful in constructing Table V. Here is the general formula (continuing to use Assumption 5) for interior assessments of ending nodal probabilities: let

$$X_j(k,l) \equiv [(j(j-1)\cdots(j-k+1)) \times ((n-j)(n-j-1)\cdots(n-j-l+1)]$$
$$\div [n(n-1)\cdots(n-k-l+1)]$$

then

$$P_j[u^k d^l] = \left[X_j(k,l)P_j\right] \div \Sigma_j X_j(k,l)P_j$$

Exotic Options

Several varieties of exotic options can be easily valued using the generalized binomial tree.

Perhaps the simplest type of path-dependent option is one where the payoff depends not only on the final price of the underlying asset but also on whether or not the underlying asset has reached some other "barrier" price during the life of the option. For example, in a down-and-out call, a standard European or American call comes into existence when the down-and-out is issued, but the standard call is extinguished prior to expiration if the underlying asset price ever drops below the knock-out boundary, **H**. In that case the buyer of the option may be paid a fixed rebate, **B**, payable on the date the boundary is first reached. Otherwise, if the underlying asset price never drops below **H**, the down-and-out call will have the same payoff as a standard call. Valuing such an option is quite easy. Work backwards as usual on the generalized binomial tree; if the standard call is American, place at each node the present value of holding the call one more period or its current exercisable value, whichever is greater. However, whenever the current

[30] For a method to calculate other derivatives such as "rho" ($\simeq \partial C/\partial r$) see the technique suggested to me by Eric Reiner described in Rubinstein (1992). "Vega", as $\partial C/\partial\sigma$ is sometimes called, of course, no longer has a clear meaning in a model such as this that allows for stochastic local volatility.

underlying asset price is less than or equal to the barrier **H**, override this and place the rebate value **B** at the node.[31]

Somewhat more complex are "lookback" options, which are like standard options except that the minimum for calls (maximum for puts), **M**, of the underlying asset price over the life of the option is substituted for the striking price. European lookbacks can be valued using the implied binomial tree as a basis for Monte Carlo simulation. To construct a Monte Carlo path, at each time step in the tree along a single path, randomly select an up $u[\bullet]$ move with probability $p[\bullet]$ or down $d[\bullet]$ move with probability $1 - p[\bullet]$. This traces out one randomly selected path from the beginning to the end of the tree. For this path, record the ending underlying asset price as well as the minimum (or maximum) price that occurred along the sampled path. Use this to calculate the payoff to the option at the end of the path. Now, repeat this procedure thousands of times, compute a simple average of the resulting option payoffs, and discount this back to the present at the appropriate riskless rate of interest. This approach can be easily generalized to include lookbacks with extrema calculated over only some portion of the life of the option, for discrete extrema sampling intervals, and for lookback variations such as options paying off the maximum of zero or the difference between the maximum **M** and a fixed striking price K.

"Asian" options comprise yet another class of path-dependent options where the payoff depends not only possibly on the price at expiration of the underlying asset but also on the average price experienced by the underlying asset during at least some portion of the life of the option. Again, European Asians can be handled via Monte Carlo simulation in a similar way to the method described above for lookbacks.

VIII. Recent Related Research

Related work has recently been described in two articles appearing in *RISK*, one by Bruno Dupire and one by Emanuel Derman and Iraj Kani.[32] In both articles, an implied binomial or trinomial process is implied by the concurrent market prices of standard options of all striking prices *and* maturities available on a given underlying asset, using an approach similar to Shimko's but expanded to allow interpolation and extrapolation across maturities as well as striking prices. Dupire assumes a trinomial process with exogenously specified nodal values but fitted risk-neutral probabilities. Derman and Kani fit a binomial tree but where $n + 1$ nodal values are exogenously specified along the horizontal spine or trunk of the tree, rather than the vertical base at the end of the tree as in the technique described here. Because these approaches make use of options of different maturities, they can effectively dispense with Assumption 5, requiring equal path probabilities for all paths leading to the same ending node. This is consistent with

[31] For a development of all eight types of barrier options, see Rubinstein and Reiner (1991).

[32] See Dupire (1994) and Derman and Kani (1994).

our earlier assertion in Section V that Assumption 5 can be dropped if there is some way of knowing how the ending nodal probabilities are divvied up among their constituent paths—information that in principle can be inferred from options of shorter maturities.

An obvious advantage of these alternative approaches is that they can dispense with Assumption 5 yet still derive a unique tree. However, there are a number of reasons to be interested in trees implied only from options of a single maturity. First, since information concerning earlier maturity options has not been used to construct the tree, the tree shows how to use information contained in the prices of the options maturing at the end of the tree to infer consistent values for options maturing earlier in the tree. Second, once the ending nodal probabilities have been specified, the algorithm for constructing the tree given in Section IV is somewhat easier to understand and implement than the alternatives. Third, nothing in the procedure here requires that the ending risk-neutral distribution be completely consistent with available option prices. Some investors may have their own opinions concerning this distribution; if so, they can easily incorporate them directly as the ending nodal probabilities. Fourth, the optimization method used here for inferring ending nodal probabilities from option prices permits interpolation and extrapolation based on a subjective prior in combination with the prices of options. Fifth, unlike Shimko's technique, the optimization method can be easily modified to use the prices of nonstandard European options with payoffs that are not piecewise linear as the basis for inferring risk-neutral probabilities.[33]

IX. Future Research

It is never a good idea for an academic to carry a line of research so far that one has left nothing further to do. So in the spirit of generating citations for this paper, in addition to the aforementioned empirical tests, let me suggest the following agenda:

Implied risk-neutral probabilities:
- —How can American options be used to infer risk-neutral probabilities?
- —How can option maturities be interpolated, not just striking prices?
- —What is the best function to minimize, or the best prior to assume?
- —Is it better to interpolate implied volatilities or use optimization?

Implied stochastic process:
- —How can American options be used to drop the recombining assumption and deal with return path-dependent trees?
- —Can unique trinomial trees be designed to handle jumps or stochastic volatility?
- —Can similar trees be designed, consistent with current bond option and bond prices, to determine the stochastic process of bond prices?

[33] I thank Bill Keirstead for pointing out to me that the Breeden-Litzenberger second derivative approach can only be used with standard European call options.

Empirical issues:
- —How can $\sigma(S,t)$ and $m(S,t)$ be used to improve the way time-series data is interpreted?
- —How can $\sigma(S,t)$ and $m(S,t)$ be tested against historical time series?

Applications:
- —How can the best portfolio of standard options be found to hedge exotic options?

To my mind, since most listed options are American not European, the most pressing of these problems is the use of American options to infer risk-neutral probabilities. Here is one currently untested possibility. For each American option, fit a standard binomial tree to its current bid (ask) price. Using that tree, calculate the current bid (ask) price of an otherwise identical European option. Then follow either of the approaches outlined in this article to calculate the implied risk-neutral probabilities and from them, the implied binomial tree. Unfortunately, this approach is, at best, second best, since it does not do full justice to the information about the stochastic process contained in the price of American options. American options tell us something about the interior as well as ending nodal probabilities, or alternatively, about how the nodal probabilities are divvied up among their path probabilities. In fact, the tree resulting from this method should not exactly fit the current prices of all the American options, although the fit may be close enough for practical purposes.

Appendix 1: Amended Longstaff Method for Calculating Implied Risk-Neutral Probabilities

For C_4, as S^* goes from K_4 to K_5, the payoff to the call goes from 0 to $K_5 - K_4$, so the average payoff in the interval is $\frac{1}{2}(K_5 - K_4)$. Therefore:

$$r^n C_4 = \tfrac{1}{2}(K_5 - K_4)P_5$$

For C_3, consider three possible types of outcomes:

Outcome	Risk-neutral contribution to $r^n C_3$
$S^* < K_3$	0

As S^* goes from K_3 to K_4, the payoff to the call goes from 0 to $K_4 - K_3$, so the average payoff in the interval is $\frac{1}{2}(K_4 - K_3)$. Therefore:

$$K_3 \le S^* \le K_4 \qquad \tfrac{1}{2}(K_4 - K_3)P_4$$

The payoff in the region greater than K_4 is the same as an otherwise identical call C_4 with striking price K_4 plus an extra payoff $(K_4 - K_3)$ due to the lower striking price of call C_3. Therefore:

$$K_4 < S^* \qquad (K_4 - K_3)P_5 + r^n C_4$$

Adding up these payoff components:

$$\begin{aligned} r^n C_3 &= \tfrac{1}{2}(K_4 - K_3)P_4 + (K_4 - K_3)P_5 + r^n C_4 \\ &= (K_4 - K_3)\left[\tfrac{1}{2}P_4 + P_5\right] + r^n C_4 \end{aligned}$$

For C_2, again consider three possible types of outcomes:

Outcome	Risk-neutral contribution to $r^n C_2$
$S^* < K_2$	0
$K_2 \le S^* \le K_3$	$\tfrac{1}{2}(K_3 - K_2)P_3$
$K_3 < S^*$	$(K_3 - K_2)(P_4 + P_5) + r^n C_3$

Adding up these payoff components:

$$\begin{aligned} r^n C_2 &= \tfrac{1}{2}(K_3 - K_2)P_3 + (K_3 - K_2)(P_4 + P_5) + r^n C_3 \\ &= (K_3 - K_2)\left[\tfrac{1}{2}P_3 + P_4 + P_5\right] + r^n C_3 \end{aligned}$$

For C_1, again consider three possible types of outcomes:

Outcome	Risk-neutral contribution to $r^n C_1$
$S^* < K_1$	0
$K_1 \le S^* \le K_2$	$\tfrac{1}{2}(K_2 - K_1)P_2$
$K_2 < S^*$	$(K_2 - K_1)(P_3 + P_4 + P_5) + r^n C_2$

Adding up these payoff components:

$$\begin{aligned} r^n C_1 &= \tfrac{1}{2}(K_2 - K_1)P_2 + (K_2 - K_1)(P_3 + P_4 + P_5) + r^n C_2 \\ &= (K_2 - K_1)\left[\tfrac{1}{2}P_1 + P_2 + P_3 + P_4\right] + r^n C_2 \end{aligned}$$

Consider the underlying asset itself as a payout-protected call with striking price $K_0 = 0$. For S, again consider three possible types of outcomes:

Outcome	Risk-neutral contribution to $r^n S \delta^{-n}$
$S^* < 0$	0
$0 \le S^* \le K_1$	$\tfrac{1}{2}(K_1 - 0)P_1$
$K_1 < S^*$	$(K_1 - 0)(P_2 + P_3 + P_4 + P_5) + r^n C_1$

Adding up these payoff components:

$$\begin{aligned} r^n S\delta^{-n} &= \tfrac{1}{2}(K_1 - 0)P_1 + (K_1 - 0)(P_2 + P_3 + P_4 + P_5) + r^n C_1 \\ &= K_1\left[\tfrac{1}{2}P_1 + P_2 + P_3 + P_4 + P_5\right] + r^n C_1. \end{aligned}$$

To account for the fact that the associated options are not protected against payouts, we use the present value of the capital appreciation portion of the underlying asset $S\delta^{-n}$, where δ is one plus the underlying asset payout rate.

Restating earlier results:

$$r^n S\delta^{-n} = K_1\left[\tfrac{1}{2}P_1 + P_2 + P_3 + P_4 + P_5\right] + r^n C_1,$$

and because we have probabilities

$$1 = P_1 + P_2 + P_3 + P_4 + P_5,$$

we can put these together and solve for P_1:

$$\begin{aligned} r^n S\delta^{-n} &= K_1\left[\tfrac{1}{2}P_1 + (1 - P_1)\right] + r^n C_1 = K_1\left[1 - \tfrac{1}{2}P_1\right] + r^n C_1 \\ &\Rightarrow r^n(\delta^{-n}S - C_1)K_1^{-1} = 1 - \tfrac{1}{2}P_1 \\ &\Rightarrow P_1 = 2\left[1 - r^n(S\delta^{-n} - C_1)K_1^{-1}\right]. \end{aligned}$$

Similarly

$$\begin{aligned} P_2 &= 2\left[1 - P_1 - r^n(C_1 - C_2)(K_2 - K_1)^{-1}\right] \\ P_3 &= 2\left[1 - P_1 - P_2 - r^n(C_2 - C_3)(K_3 - K_2)^{-1}\right] \\ P_4 &= 2\left[1 - P_1 - P_2 - P_3 - r^n(C_3 - C_4)(K_4 - K_3)^{-1}\right] \\ P_5 &= 1 - P_1 - P_2 - P_3 - P_4. \end{aligned}$$

Thus, the implied risk-neutral probabilities can be derived by solving the equation for P_1, using this value for P_1 and solving the equation for P_2, using these values for P_1 and P_2 and solving the equation for P_3, using these values for P_1, P_2, and P_3 and solving the equation for P_4, and using these values for P_1, P_2, P_3, and P_4 and solving the final equation for P_5.

Restating an earlier result:

$$r^n C_4 = \tfrac{1}{2}(K_5 - K_4)P_5,$$

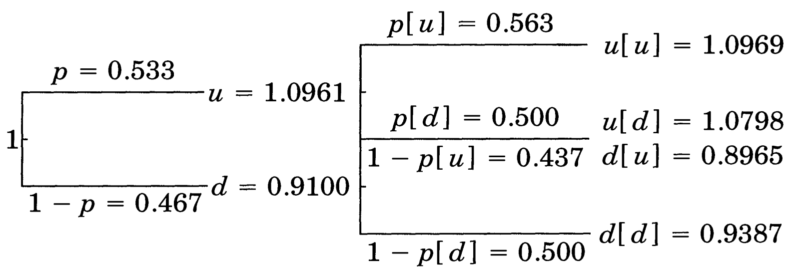
and having calculated P_5, we can solve this for K_5:

$$K_5 = K_4 + (2r^n C_4 \div P_5).$$

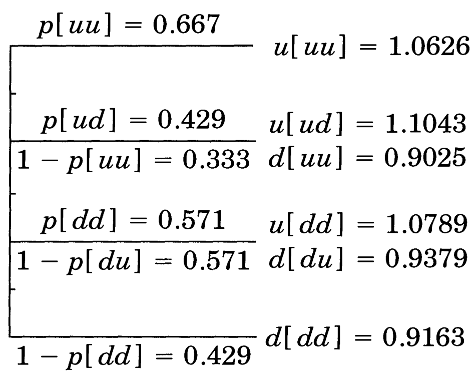
Appendix 2: Numerical Example of an Implied Binomial Tree

$R_0 = 0.7827$	$R_1 = 0.9216$	$R_2 = 1.0851$	$R_3 = 1.2776$
$P_0 = 0.1$	$P_1 = 0.4$	$P_2 = 0.3$	$P_3 = 0.2$
$n = 3$	$h = 0.167$	$r = 1.017$	$\delta = 1.008$

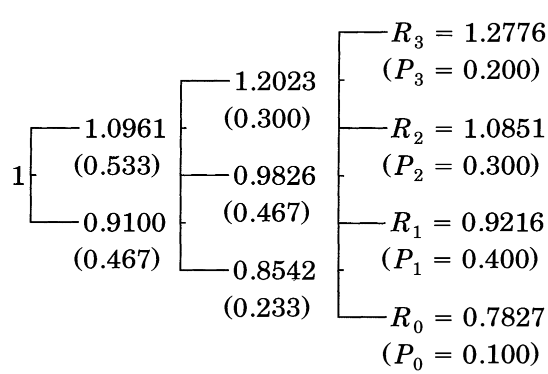
Move Return and Probability Tree



(continued)



Nodal Return and (Probability) Tree



Numerical Calculations

Interest / payout return:

$$(r/\delta) = [P_0R_0 + P_1R_1 + P_2R_2 + P_3R_3]^{1/n}$$
$$= [0.1(0.7827) + 0.4(0.9216) + 0.3(1.0851) + 0.2(1.2776)]^{1/3} = 1.0089$$

Path probabilities:

$$P_{uuu} = P_3/1 = 0.200$$
$$P_{uud} = P_{udu} = P_{duu} = P_2/3 = 0.300/3 = 0.100$$
$$P_{udd} = P_{dud} = P_{ddu} = P_1/3 = 0.400/3 = 0.133$$
$$P_{ddd} = P_0/1 = 0.100$$
$$P_{uu} = P_{uud} + P_{uuu} = 0.100 + 0.200 = 0.300$$
$$P_{ud} = P_{du} = P_{udd} + P_{udu} = P_{dud} + P_{duu} = 0.133 + 0.100 = 0.233$$
$$P_{dd} = P_{ddd} + P_{ddu} = 0.100 + 0.133 = 0.233$$
$$P_u = P_{ud} + P_{uu} = 0.233 + 0.300 = 0.533$$
$$P_d = P_{dd} + P_{du} = 0.233 + 0.233 = 0.467$$
$$P = P_d + P_u = 0.467 + 0.533 = 1$$

Move probabilities:

$$p[uu] = P_{uuu}/P_{uu} = 0.200/0.300 = 0.667$$
$$p[ud] = p[du] = P_{udu}/P_{ud} = P_{duu}/P_{du} = 0.100/0.233 = 0.429$$
$$p[dd] = P_{ddu}/P_{dd} = 0.133/0.233 = 0.571$$
$$p[u] = P_{uu}/P_u = 0.300/0.533 = 0.563$$
$$p[d] = P_{du}/P_d = 0.233/0.467 = 0.500$$
$$p = P_u/P = 0.533/1 = 0.533$$

Interior nodal values:

$$R[uu] = ((1 - p[uu])R_2 + p[uu]R_3)/(r/\delta)$$
$$= ((1 - 0.667) \times 1.0851 + 0.667 \times 1.2776)/1.0089 = 1.2023$$
$$R[ud] = ((1 - p[ud])R_1 + p[ud]R_2)/(r/\delta)$$
$$= ((1 - 0.429) \times 0.9216 + 0.429 \times 1.0851)/1.0089 = 0.9826$$
$$R[du] = ((1 - p[du])R_1 + p[du]R_2)/(r/\delta)$$
$$= ((1 - 0.429) \times 0.9216 + 0.429 \times 1.0851)/1.0089 = 0.9826$$

$$
\begin{aligned}
R[dd] &= ((1 - p[dd])R_0 + p[dd]R_1)/(r/\delta) \\
&= ((1 - 0.531) \times 0.7827 + 0.531 \times 0.9216)/1.0089 = 0.8542 \\
R[u] &= ((1 - p[u])R[ud] + p[u]R[uu])/(r/\delta) \\
&= 1.008 \times ((1 - 0.563) \times 0.9826 + 0.563 \times 1.2023)/1.0173 = 1.0961 \\
R[d] &= ((1 - p[d])R[dd] + p[d]R[du])/(r/\delta) \\
&= 1.008 \times ((1 - 0.500) \times 0.8542 + 0.500 \times 0.9826)/1.0173 = 0.9100 \\
R &= ((1 - p)R[d] + pR[u])/(r/\delta) \\
&= 1.008 \times ((1 - 0.533) \times 0.9100 + 0.533 \times 1.0961)/1.0173 = 1
\end{aligned}
$$

Optional Calculations

Interior nodal probabilities:

$$
\begin{aligned}
P[uu] &= P_{uud} + P_{uuu} = 0.100 + 0.200 = 0.300 \\
P[ud] &= P[du] = P_{udd} + P_{udu} + P_{dud} + P_{duu} \\
&= 0.133 + 0.100 + 0.133 + 0.100 = 0.467 \\
P[dd] &= P_{ddd} + P_{ddu} = 0.100 + 0.133 = 0.233 \\
P[u] &= P_{uu} + P_{ud} = 0.300 + 0.233 = 0.533 \\
P[d] &= P_{du} + P_{dd} = 0.233 + 0.233 = 0.467
\end{aligned}
$$

Move sizes:

$$
\begin{aligned}
u[uu] &= R_3/R[uu] = 1.2776/1.2023 = 1.0626 \\
u[ud] &= u[du] = R_2/R[ud] = R_2/R[du] = 1.0851/0.9826 = 1.1043 \\
d[uu] &= R_2/R[uu] = 1.0851/1.2023 = 0.9025 \\
u[dd] &= R_1/R[dd] = 0.9216/0.8542 = 1.0789 \\
d[du] &= d[ud] = R_1/R[du] = R_1/R[ud] = 0.9216/0.9826 = 0.9379 \\
d[dd] &= R_0/R[dd] = 0.7827/0.8542 = 0.9163 \\
u[u] &= R[uu]/R[u] = 1.2023/1.0961 = 1.0969 \\
u[d] &= R[du]/R[d] = 0.9826/0.9100 = 1.0798 \\
d[u] &= R[ud]/R[u] = 0.9826/1.0961 = 0.8965 \\
d[d] &= R[dd]/R[d] = 0.8542/0.9100 = 0.9387 \\
u &= R[u]/R = 1.0961/1 = 1.0961 \\
d &= R[d]/R = 0.9100/1 = 0.9100
\end{aligned}
$$

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