

## THE ASYMPTOTIC EXPANSION APPROACH TO THE VALUATION OF INTEREST RATE CONTINGENT CLAIMS

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We propose a new methodology for the valuation problem of financial contingent claims when the underlying asset prices follow a general class of continuous Itô processes. Our method can be applied to a wide range of valuation problems including complicated contingent claims associated with the term structure of interest rates. We illustrate our method by giving two examples: the valuation problems of swaptions and average (Asian) options for interest rates. Our method gives some explicit formulas for solutions, which are sufficiently numerically accurate for practical purposes in most cases. The continuous stochastic processes for spot interest rates and forward interest rates are not necessarily Markovian nor diffusion processes in the usual sense; nevertheless our approach can be rigorously justified by the Malliavin–Watanabe Calculus in stochastic analysis.

KEY WORDS: derivatives, term structure of interest rates, asymptotic expansion, small disturbance asymptotics, Malliavin–Watanabe Calculus

### 1. INTRODUCTION

In the past two decades various contingent claims have been introduced and actively traded in financial markets. In particular, various types of interest rates based contingent claims appeared and attracted attention in financial markets. This paper presents a new methodology which is applicable to the valuation problem of financial contingent claims such as options, swaps, and other derivative securities when the underlying asset prices follow the general class of continuous Itô processes. Our method is especially useful for the pricing problem of interest rate based derivatives when the underlying forward rates follow a general class of continuous Itô processes.

In the valuation problem of financial contingent claims, it has been known that we can rarely obtain explicit formulas on solutions when the underlying assets follow the general class of continuous Itô processes. This is particularly evident for contingent claims based on the term structure of interest rates because their payoff functions are usually complicated functionals of the underlying asset prices and the term structure of interest rates must satisfy the strong restrictions implied by fundamental economic theory. In order to cope with these problems, two methods, called the partial differential equation (PDE) approach and the Monte Carlo (MC) approach have been widely known and used for practical valuation problems (see Duffie 1992 or Hull 1993 for the details

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of these methods). The asymptotic expansion approach we are proposing in this paper is different from these conventional methods. As we shall show later, our method has several advantages compared to those existing methods.

The asymptotic expansion approach is based on the key empirical observation on many asset prices, including interest rates, that the observed and estimated volatilities for financial asset prices may vary over time, but they are not very large in comparison with the observed levels of asset prices. This observation has even been true for the stock prices whose volatilities are relatively large in comparison with other financial prices. It was using this key observation that we (Kunitomo and Takahashi 1992) developed the asymptotic theory called *small disturbance asymptotics*<sup>1</sup> for solving the valuation problem of average (or Asian) options for foreign exchange rates when the volatility parameter goes to zero. We proposed to use the limiting distribution as the first-order approximation to the exact distribution of the payoff function of average options when the underlying asset prices follow geometric Brownian motion. Although the approximations we proposed have given relatively accurate numerical values, they are not completely satisfactory in some cases for practical purposes. For the same valuation problem in Kunitomo and Takahashi (1992), Yoshida (1992a) has obtained further results on average options when the underlying asset prices follow geometric Brownian motion by using an asymptotic expansion technique originally developed for an application in statistics.

The main purpose of the present paper is to show that the asymptotic expansion method in small disturbance asymptotics can be effectively applicable to various valuation problems of contingent claims appearing when the underlying asset prices follow the general class of continuous Itô processes. In particular, we shall show that the asymptotic expansion approach is very simple, yet it gives a unified approach to the problem of interest rate based contingent claims valuation. However, we shall point out that some economic considerations of theoretical restrictions on the structure of stochastic processes should be indispensable when we apply the asymptotic expansion method. In the term structure of interest rates, for instance, we require strong conditions on the form of their drift functions because of the no-arbitrage theory. This implies that the continuous stochastic processes for spot interest rates and forward rates are not necessarily Markovian nor diffusion processes in the usual sense.

In a companion of the present paper, Takahashi (1999)<sup>2</sup> systematically investigated the valuation problem of various contingent claims when the asset price  $S(t)$  follows a diffusion process:

$$(1.1) \quad S(t) = S(0) + \int_0^t \mu(S(v), v) dv + \int_0^t \sigma^*(S(v), v) dB(v),$$

where  $\mu(S(v), v)$  and  $\sigma^*(S(v), v)$  are the instantaneous mean and volatility functions, and  $B(v)$  is the standard Brownian motion. It is evident that the Black–Scholes economy and the Cox–Ingersol–Ross model on the spot interest rate are special cases of this framework. In the simplest Black–Scholes economy, for instance, let  $S^{(\epsilon)}(t)$  satisfy the

<sup>1</sup> The small disturbance asymptotic theory had originally been developed for the analysis of simultaneous equation systems in econometrics. See Anderson (1977) or Kunitomo et al. (1983) for the details.

<sup>2</sup> This is essentially based on Chapter 1 of an unpublished Ph.D. dissertation (Takahashi 1995) at Berkeley. It gives many numerical examples and discusses the validity of the small disturbance asymptotic expansion method when the asset price follows a (time) nonhomogeneous diffusion process.

integral equation of a diffusion process:

$$(1.2) \quad S^{(\epsilon)}(t) = S(0) + \int_0^t r S^{(\epsilon)}(v) dv + \epsilon \int_0^t \sigma(S^{(\epsilon)}(v), v) dB(v),$$

where  $\sigma(S^{(\epsilon)}(v), v)$  is the volatility term with  $0 < \epsilon \leq 1$  and  $r$  is the risk-free (constant) rate. Then a small disturbance asymptotic theory can be constructed by considering the situation when  $\epsilon \rightarrow 0$  and we can develop the valuation method of contingent claims based on  $\{S^{(\epsilon)}(t)\}$  with the no-arbitrage theory.

As we explain in the Appendix, our method is not an ad hoc approximation method because it can be rigorously justified by the Malliavin–Watanabe theory in stochastic analysis. The Malliavin–Watanabe Calculus has been developed as an infinite dimensional analysis of the Wiener functional by several probabilists in the last two decades. We intend to apply this powerful calculus on continuous stochastic processes to the valuation problem of financial contingent claims along the line developed by Watanabe (1987) and Yoshida (1992a). However, we should mention that the spot and forward interest rates in the term structure model are not necessarily Markovian in the usual sense but the existing asymptotic expansion methods initiated by Watanabe (1987) and refined by Yoshida (1992a, 1992b) in stochastic analysis have been developed for the time homogeneous Markovian processes. Hence, we need to extend the existing results on the validity of the asymptotic expansion approach to the solution of certain stochastic integral equations for the interest rate processes. Since this extension to a class of non-Markovian continuous processes is not a trivial task, the asymptotic expansion approach developed in this paper also would be interesting to researchers in stochastic analysis.

Furthermore, as we shall illustrate in Section 4, the resulting formulas we derive for complicated contingent claims are numerically accurate in many practical situations. Thus the asymptotic expansion approach would be not only theoretically interesting, but also quite useful for researchers in financial economics and practitioners in financial markets.

In Section 2, we formulate the valuation problem of contingent claims based on the term structure of interest rates. In Section 3, we explain the asymptotic expansion approach for this problem and give some theoretical results. Then, in Section 4, we show some numerical results on interest rate derivatives as illustrative examples. Section 5 summarizes our results and provides concluding comments. Some mathematical details including useful formulas and the mathematical validity of our method via the Malliavin–Watanabe theory are presented in the Appendix.

## 2. THE VALUATION PROBLEM OF INTEREST RATE BASED CONTINGENT CLAIMS

We consider a continuous time economy with a trading interval  $[0, \bar{T}]$ , where  $\bar{T} < +\infty$ ; it is also complete in the proper economic sense. Let  $P(s, t)$  denote the price of the discount bond at  $s$  with maturity date  $t$  ( $0 \leq s \leq t \leq \bar{T} < +\infty$ ) and we use the notational convention that  $P(t, t) = 1$  at maturity date  $s = t$  for normalization. Let also  $P(s, t)$  be continuously differentiable with respect to  $t$  and  $P(s, t) > 0$  for  $0 \leq s \leq t \leq \bar{T}$ . Then the instantaneous forward rate at  $s$  for future date  $t$  ( $0 \leq s \leq t \leq \bar{T}$ ) is defined by

$$(2.1) \quad f(s, t) = -\frac{\partial \log P(s, t)}{\partial t}.$$

In the term structure model of interest rates we assume that a family of forward rate processes<sup>3</sup>  $\{f(s, t)\}$  for  $0 \leq s \leq t \leq \bar{T}$  follows the stochastic integral equation

$$(2.2) \quad f(s, t) = f(0, t) + \int_0^s \left[ \sum_{i=1}^n \sigma_i^*(f(v, t), v, t) \int_v^t \sigma_i^*(f(v, y), v, y) dy \right] dv \\ + \int_0^s \sum_{i=1}^n \sigma_i^*(f(v, t), v, t) dB_i(v),$$

where  $f(0, t)$  are nonrandom initial forward rates,  $\{B_i(v), i = 1, \dots, n\}$  are  $n$  independent Brownian motions, and  $\{\sigma_i^*(f(v, t), v, t), i = 1, \dots, n\}$  are the volatility functions. We assume that the initial forward rates are observable and fixed. When  $f(s, t)$  is continuous at  $s = t$  for  $0 \leq s \leq t \leq \bar{T}$ , the spot interest rate at  $s$  can be defined by  $r(s) = f(s, s) (= \lim_{t \downarrow s} f(s, t))$ .

Consider the contingent claims based on the term structure of interest rates. There have been many interest rate based contingent claims developed and traded in financial markets. Most of those contingent claims can be regarded as functionals of bond prices with different maturities. Let  $\{c_j, j = 1, \dots, m\}$  be a sequence of positive payments and  $\{T_j, j = 1, \dots, m\}$  be a sequence of payment periods satisfying the condition  $0 \leq s \leq T_1 \leq \dots \leq T_m \leq \bar{T}$ . Then the price of the coupon-bearing bond with payments  $\{c_j, j = 1, \dots, m\}$  at  $s$  should be given by<sup>4</sup>

$$(2.3) \quad P_{m, \{T_j\}, \{c_j\}}(s) = \sum_{j=1}^m c_j P(s, T_j),$$

where  $\{P(s, T_j), j = 1, \dots, m\}$  are the prices of zero-coupon bonds with different maturities. For illustrations we give two examples of interest rate based contingent claims, which are important for practice in financial markets.

**EXAMPLE 2.1.** The payoff functions of options on the coupon-bearing bond with payments  $\{c_j, j = 1, \dots, m\}$  at  $\{T_j, j = 1, \dots, m\}$  can be written as

$$(2.4) \quad V^{(1)}(T) = [P_{m, \{T_j\}, \{c_j\}}(T) - K]^+$$

and

$$(2.5) \quad V^{(2)}(T) = [K - P_{m, \{T_j\}, \{c_j\}}(T)]^+,$$

where  $K$  is a fixed strike price and the max function is defined by  $[X]^+ = \max(X, 0)$ .  $V^{(1)}(T)$  and  $V^{(2)}(T)$  are the payoffs of the call options and put options on the coupon bond, respectively. The swaptions expiring on date  $T$  ( $0 < T \leq T_1$ ) also can be written in the form of (2.4) or (2.5). (See the second numerical example in Section 4.)

<sup>3</sup> The restrictions in (2.2) we are imposing in this arbitrage-free formulation were derived by Heath, Jarrow, and Morton (1992). There have been other approaches to the problem of the term structure of interest rates as discussed by Duffie (1992) or Hull (1993).

<sup>4</sup> We implicitly assume that there does not exist any default risk associated with bonds or any transaction costs.

EXAMPLE 2.2. The yield of a zero coupon bond at  $t$  with time to maturity of  $\tau$  ( $0 < t < t + \tau < T + \tau$ ) years is given by

$$(2.6) \quad L^\tau(t) = \left[ \frac{1}{P(t, t + \tau)} - 1 \right] \frac{1}{\tau}.$$

The payoffs of the options on average interest rates can be written as

$$(2.7) \quad V^{(3)}(T) = \left[ \frac{1}{T} \int_0^T L^\tau(t) dt - K \right]^+$$

and

$$(2.8) \quad V^{(4)}(T) = \left[ K - \frac{1}{T} \int_0^T L^\tau(t) dt \right]^+,$$

where  $K$  is a fixed strike price.  $V^{(3)}(T)$  and  $V^{(4)}(T)$  are the payoffs of the call options and put options of average options on interest rates, respectively. There exist some options on the average of  $L^\tau$ , where  $\tau$  is a constant duration, for instance, six-month LIBOR rate.

The valuation problem of a contingent claim in a complete market can be simply defined as the determination of its “fair” value at financial markets. Let  $V(T)$  be the payoff of a contingent claim at the terminal period  $T$ . Then the standard martingale theory in financial economics predicts that the fair price of  $V(T)$  at time  $t$  ( $0 \leq t < T$ ) should be given by

$$(2.9) \quad V_t(T) = \mathbf{E}_t \left[ e^{-\int_t^T r(s) ds} V(T) \right],$$

where  $\mathbf{E}_t[\cdot]$  is the conditional expectation operator given the information available at  $t$  with respect to the equivalent martingale measure. When we do not impose the drift restrictions given by (2.2) for forward rate processes, we have to change the underlying probability measure into the equivalent martingale measure by the no-arbitrage theory and we can obtain the same results reported in this paper. Since this complicates our notations as well as our explanations, we have directly imposed the restrictions given by (2.2).

### 3. THE ASYMPTOTIC EXPANSION APPROACH

There are mainly two difficulties in the valuation problem of interest rate based contingent claims. First, the payoff functions and the discount factors are usually nonlinear functionals of bonds with different maturities and the spot interest rate. More importantly, the coupon-bearing bond prices are also complicated functionals of instantaneous forward rate processes. Therefore, except in some special cases we cannot obtain explicit formulas for the solution in the valuation problems of interest rate based contingent claims.

In order to develop a new asymptotic expansion approach, we first reformulate (2.2) and we assume that a family of instantaneous forward rate processes obeys the stochastic integral equation:

$$(3.1) \quad f^{(\varepsilon)}(s, t) = f(0, t) + \varepsilon^2 \int_0^s \left[ \sum_{i=1}^n \sigma_i(f^{(\varepsilon)}(v, t), v, t) \int_v^t \sigma_i(f^{(\varepsilon)}(v, y), v, y) dy \right] dv \\ + \varepsilon \int_0^s \sum_{i=1}^n \sigma_i(f^{(\varepsilon)}(v, t), v, t) dB_i(v),$$

where  $0 < \varepsilon \leq 1$  and  $0 \leq s \leq t \leq \bar{T}$ . The volatility functions  $\sigma_i(f^{(\varepsilon)}(s, t), s, t)$  depend not only on  $s$  and  $t$ , but also on  $f^{(\varepsilon)}(s, t)$  in the general case. Let  $f^{(\varepsilon)}(s, t)$  be continuous at  $s = t$  for  $0 \leq s \leq t \leq \bar{T}$ . Then the spot interest rate process can be defined by

$$(3.2) \quad r^{(\varepsilon)}(s) = f^{(\varepsilon)}(s, s).$$

We note that these equations for  $\{r^{(\varepsilon)}(s)\}$  and  $\{f^{(\varepsilon)}(s, t)\}$  can be obtained simply by substituting  $\varepsilon \sigma_i(f^{(\varepsilon)}(v, t), v, t)$  for  $\sigma_i^*(f(v, t), v, t)$  in (2.2).

The asymptotic expansion approach we propose in this paper consists of the following three steps. First, given the forward rate process, we consider a functional of the entire forward rate process  $\{f^{(\varepsilon)}(s, t)\}$  with a parameter  $\varepsilon$  ( $0 < \varepsilon \leq 1$ ) as

$$(3.3) \quad U^{(\varepsilon)} = U(\{f^{(\varepsilon)}(s, t)\}),$$

where  $U(\cdot)$  is a smooth functional of the process  $\{f^{(\varepsilon)}(s, t)\}$  as we shall illustrate later in this section. Since we do not know the distribution of (3.3), we consider its stochastic expansion around the deterministic process

$$(3.4) \quad U^{(0)} = U(\{f^{(0)}(s, t)\})$$

when the volatility parameter  $\varepsilon$  goes to zero. Second, the formal stochastic expansions of the discounted payoff function can be taken with respect to the polynomial order of the volatility coefficients  $\varepsilon^k$  ( $k = 1, 2, \dots$ ). Then the asymptotic expansion for the corresponding density function can be derived with respect to the parameter  $\varepsilon$ . Finally, we truncate the resulting stochastic expansion and take the expectation in (2.9) with respect to the density function given the information available at time  $t$ .

In order to implement this procedure, we first need to obtain the stochastic expansions of the stochastic processes  $\{f^{(\varepsilon)}(s, t)\}$  and  $\{r^{(\varepsilon)}(s)\}$ . We shall make the following assumptions.<sup>5</sup>

**ASSUMPTION I.** (i) Given  $\varepsilon$  ( $0 < \varepsilon \leq 1$ ), the volatility functions  $\{\sigma_i(f^{(\varepsilon)}(s, t), s, t)\}$  are continuous real-valued functions defined in  $\{0 < s \leq t \leq \bar{T}\}$ . They are nonnegative, bounded, and smooth in their first argument, and all derivatives in their first argument are bounded uniformly in  $\varepsilon$ . (ii) The initial (nonrandom) forward rates  $f(0, t)$  are Lipschitz continuous with respect to  $t$ .

<sup>5</sup> We also implicitly assume the measurability conditions on functions in Assumption I, which are necessary for rigorous mathematical arguments in Appendix A.3.

ASSUMPTION II. For any  $0 < s \leq t \leq \bar{T}$ ,

$$(3.5) \quad \Sigma(s, t) = \int_0^s \sum_{i=1}^n \sigma_i^{(0)}(v, t)^2 dv > 0,$$

where  $\{\sigma_i(f^{(0)}(s, t), s, t)\}$  are real-valued functions defined in  $\{0 < s \leq t \leq \bar{T}\}$  and  $\{\sigma_i^{(0)}(v, t)\}$  are given by

$$(3.6) \quad \sigma_i^{(0)}(v, t) = \sigma_i(f^{(\varepsilon)}(v, t), v, t)|_{\varepsilon=0}.$$

The conditions we made in the first part of Assumption I exclude the possible explosion of the solution for (3.1). They are quite strong and could be relaxed considerably, which may be interesting from the view of stochastic analysis. For practical purposes, however, we can often use the truncation argument as in an example given by Heath et al. (1992). (See (4.1) in Sect. 4 of this paper for an example.) Assumption II ensures the key conditions for the truncated version for the nondegeneracy of the Malliavin covariance for the forward rate process, which is essential for the validity of the asymptotic expansion of its density and distribution function. (See Theorem A.3 in Appendix A.3 on the associated technical problem.) Then we can derive the stochastic expansions of the forward rate and spot interest rate processes and the expansions of the distributions of some functionals. The outline of the derivations and their mathematical validity are given in the Appendix.

THEOREM 3.1. Under Assumption I, the stochastic expansion of the instantaneous forward rate  $\{f^{(\varepsilon)}(s, t)\}$  in (3.1) is given by

$$(3.7) \quad f^{(\varepsilon)}(s, t) = f(0, t) + \varepsilon A(s, t) + \varepsilon^2 C(s, t) + o_p(\varepsilon^2)$$

as  $\varepsilon \rightarrow 0$ . In particular, the spot rate process can be expanded as

$$(3.8) \quad r^{(\varepsilon)}(t) = f(0, t) + \varepsilon A(t, t) + \varepsilon^2 C(t, t) + o_p(\varepsilon^2).$$

The coefficients  $A(s, t)$  and  $C(s, t)$  in (3.7) and (3.8) are defined by

$$(3.9) \quad A(s, t) = \int_0^s \sum_{i=1}^n \sigma_i^{(0)}(v, t) dB_i(v),$$

$$(3.10) \quad C(s, t) = \int_0^s b^{(0)}(v, t) dv + \int_0^s \sum_{i=0}^n A(v, t) \partial \sigma_i^{(0)}(v, t) dB_i(v),$$

where

$$(3.11) \quad b^{(0)}(v, t) = b(f^{(\varepsilon)}(v, t), v, t)|_{\varepsilon=0},$$

$$(3.12) \quad \partial \sigma_i^{(0)}(v, t) = \frac{\partial \sigma_i(f^{(\varepsilon)}(v, t), v, t)}{\partial f^{(\varepsilon)}(v, t)}|_{\varepsilon=0},$$

and

$$(3.13) \quad b(f^{(\varepsilon)}(v, t), v, t) = \sum_{i=1}^n \sigma_i(f^{(\varepsilon)}(v, t), v, t) \int_v^t \sigma_i(f^{(\varepsilon)}(v, y), v, y) dy.$$

In the representations above, the first terms of (3.7) and (3.8) are deterministic functions. The second term  $A(s, t)$  in (3.7) follows the Gaussian distribution with zero mean and variance  $\Sigma(s, t)$ , which corresponds to the limit of the Malliavin covariance for the normalized forward rate processes in the theory of Malliavin–Watanabe Calculus when  $\varepsilon \rightarrow 0$ . The stochastic expansion method around the Gaussian distribution is standard in statistical asymptotic theory.

The next step in the asymptotic expansion approach is to obtain the stochastic expansions of the bond price process and the discount factor. For this purpose, we utilize the relationship between the zero-coupon bond price and the forward rates:

$$(3.14) \quad P^{(\varepsilon)}(s, t) = \exp \left[ - \int_s^t f^{(\varepsilon)}(s, u) du \right].$$

Using (3.7), we immediately have a stochastic expansion of the bond price process  $\{P^{(\varepsilon)}(s, t)\}$  as

$$(3.15) \quad P^{(\varepsilon)}(s, t) = \frac{P(0, t)}{P(0, s)} \exp \left[ -\varepsilon \int_s^t A(s, u) du - \varepsilon^2 \int_s^t C(s, u) du + o_p(\varepsilon^2) \right],$$

where  $P(0, t)$  and  $P(0, s)$  are the observable initial discount bond prices. Because the coupon bond price  $\{P_{m, \{T_j\}, \{c_j\}}^{(\varepsilon)}(T)\}$  defined by (2.3) with  $P^{(\varepsilon)}(T, T_j)$  instead of  $P(T, T_j)$  is a linear combination of zero-coupon bond prices, it has a stochastic expansion given by

$$(3.16) \quad \begin{aligned} P_{m, \{T_j\}, \{c_j\}}^{(\varepsilon)}(T) &= \sum_{j=1}^m c_j \frac{P(0, T_j)}{P(0, T)} \exp \left[ -\varepsilon \int_T^{T_j} A(T, u) du - \varepsilon^2 \right. \\ &\quad \left. \times \int_T^{T_j} C(T, u) du + o_p(\varepsilon^2) \right] \\ &= \sum_{j=1}^m c_j \frac{P(0, T_j)}{P(0, T)} \left[ 1 - \varepsilon \int_T^{T_j} A(T, u) du - \varepsilon^2 \int_T^{T_j} C(T, u) du \right. \\ &\quad \left. + \varepsilon^2 \frac{1}{2} \left( \int_T^{T_j} A(T, u) du \right)^2 + o_p(\varepsilon^2) \right]. \end{aligned}$$

By using a Fubini-type result,<sup>6</sup> we can write

$$(3.17) \quad \begin{aligned} \int_T^{T_j} A(T, u) du &= \int_t^T \left[ \int_0^T \sum_{i=1}^n \sigma_i^{(0)}(v, u) dB_i(v) \right] du \\ &= \int_0^T \sigma_{T, T_j}^{(0)}(v) d\mathbf{B}(v), \end{aligned}$$

where  $\mathbf{B}(v) = (B_i(v))$  is an  $n \times 1$  vector of standard (i.e., mutually independent) Brownian motions and a  $1 \times n$  vector  $\sigma_{T, T_j}^{(0)}(v)$  with the  $i$ th component is given by

$$(3.18) \quad \sigma_{T, T_j}^{(0)}(v) = \left[ \int_T^{T_j} \sigma_i^{(0)}(v, u) du \right].$$

<sup>6</sup> We can use Lemma 4.1 of Ikeda and Watanabe (1989) under Assumption I as a generalized Fubini-type theorem.



Since (3.17) is a linear combination of  $\{B_i(v)\}$  with deterministic coefficients, it follows a Gaussian distribution. Also we have

$$(3.19) \quad \int_T^{T_j} C(T, u) du = k_1(T, T_j) + \int_T^{T_j} \left[ \int_0^T \left[ \int_0^s \sigma^{(0)}(v, u) d\mathbf{B}(v) \right] \partial \sigma^{(0)}(s, u) d\mathbf{B}(s) \right] du,$$

where  $\sigma^{(0)}(v, u) = (\sigma_i^{(0)}(v, u))$  and  $\partial \sigma^{(0)}(s, u) = (\partial \sigma_i^{(0)}(s, u))$  are  $1 \times n$  vectors of deterministic functions, and

$$(3.20) \quad k_1(T, T_j) = \int_0^T \left[ \int_T^{T_j} b^{(0)}(v, u) du \right] dv.$$

Hence we notice that (3.19) is a quadratic functional of  $n$  standard Brownian motions. Similarly, by making use of (3.8), the stochastic expansion of the discount factor process is given by

$$(3.21) \quad e^{-\int_0^T r^{(e)}(s) ds} = P(0, T) \exp \left[ -\varepsilon \int_0^T A(s, s) ds - \varepsilon^2 \int_0^T C(s, s) ds + o_p(\varepsilon^2) \right] \\ = P(0, T) \left[ 1 - \varepsilon \int_0^T A(s, s) ds - \varepsilon^2 \int_0^T C(s, s) ds + \varepsilon^2 \frac{1}{2} \left( \int_0^T A(s, s) ds \right)^2 \right] + o_p(\varepsilon^2).$$

The second term of the discount factor process in (3.21) can be expressed as

$$(3.22) \quad \int_0^T A(t, t) dt = \int_0^T \int_v^T \sigma_i^{(0)}(v, t) dt d\mathbf{B}_i(v) \\ = \int_0^T \sigma_T^{(0)}(v) d\mathbf{B}(v),$$

where a  $1 \times n$  vector  $\sigma_T^{(0)}(v)$  with the  $i$ th component is given by

$$(3.23) \quad \sigma_T^{(0)}(v) = \left[ \int_v^T \sigma_i^{(0)}(v, t) dt \right].$$

Since (3.22) is also a linear combination of  $\{B_i(v)\}$  with deterministic coefficients, the second term of (3.21) follows a Gaussian distribution. The third term of (3.21) can be expressed as

$$(3.24) \quad \int_0^T C(t, t) dt = k_2(T) + \int_0^T \left[ \int_0^t \left[ \int_0^s \sigma^{(0)}(v, t) d\mathbf{B}(v) \right] \partial \sigma^{(0)}(s, t) d\mathbf{B}(s) \right] dt,$$

where

$$(3.25) \quad k_2(T) = \int_0^T \left[ \int_v^T b^{(0)}(v, t) dt \right] dv.$$

The third step in our approach is to obtain the asymptotic expansion of the discounted payoff functional on the expiration date. We shall illustrate this procedure by using the

two examples given in Section 2. By using (3.16) and (3.21), the asymptotic expansion of the discounted coupon bond price minus the discounted strike price is given by

$$(3.26) \quad \begin{aligned} g^{(\varepsilon)} &= e^{-\int_0^T r^{(\varepsilon)}(s) ds} \left[ P_{m, \{T_j\}, \{c_j\}}(T) - K \right] \\ &= g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + o_p(\varepsilon^2), \end{aligned}$$

where the coefficients  $g_i$  ( $i = 0, 1, 2$ ) are given by

$$(3.27) \quad g_0 = \sum_{j=1}^m c_j P(0, T_j) - K P(0, T),$$

$$(3.28) \quad g_1 = \int_0^T \sigma_{g_1}^*(v) dB(v),$$

$$(3.29) \quad \sigma_{g_1}^*(v) = -g_0 \sigma_T^{(0)}(v) - \sum_{j=1}^m c_j P(0, T_j) \sigma_{T, T_j}^{(0)}(v)$$

and

$$(3.30) \quad \begin{aligned} g_2 &= \frac{1}{2} g_0 \left\{ \int_0^T A(s, s) ds \right\}^2 + \left\{ \int_0^T A(s, s) ds \right\} \\ &\quad \times \sum_{j=1}^m c_j P(0, T_j) \left\{ \int_T^{T_j} A(T, u) du \right\} \\ &\quad + \frac{1}{2} \sum_{j=1}^m c_j P(0, T_j) \left\{ \int_T^{T_j} A(T, u) du \right\}^2 - g_0 \int_0^T C(s, s) ds \\ &\quad - \sum_{j=1}^m c_j P(0, T_j) \left\{ \int_T^{T_j} C(T, u) du \right\}. \end{aligned}$$

What we really need is not to derive the stochastic expansion of the random variable  $g^{(\varepsilon)}$ , but to obtain the asymptotic expansion of its density function. We note that  $g^{(\varepsilon)} \rightarrow g_0$  as  $\varepsilon \rightarrow 0$  in the sense of probability. Thus we consider the normalized random variable

$$(3.31) \quad X_T^{(\varepsilon)} = \frac{1}{\varepsilon} (g^{(\varepsilon)} - g_0).$$

The leading term of (3.31) is  $g_1$  in (3.28) which is a simple function of Brownian motions. Hence the random variable  $(g^{(\varepsilon)} - g_0)/\varepsilon = g_1 + \varepsilon g_2 + o_p(\varepsilon)$  converges to  $g_1$ , which has a nondegenerate Gaussian distribution, as  $\varepsilon \rightarrow 0$ . Notice that we need a condition for the nondegeneracy for the limiting distribution of (3.31). We expect that the exact distribution of (3.31) can be represented as the limiting Gaussian distribution plus some adjustment terms with smaller orders with respect to  $\varepsilon$  when  $\varepsilon$  is small. In order to find the explicit formula of its asymptotic expansion, we formally expand the characteristic function of  $X_T^{(\varepsilon)}$  with respect to  $\varepsilon$  as

$$(3.32) \quad \begin{aligned} \varphi_X(t) &= \mathbf{E} \left[ e^{itX_T^{(\varepsilon)}} \right] \\ &= \mathbf{E} \left[ e^{itg_1} (1 + \varepsilon it \mathbf{E}[g_2 | g_1]) \right] + o(\varepsilon), \end{aligned}$$

where  $\mathbf{E}[g_2 | g_1]$  is the conditional expectation operator.

Here it is possible to show the existence of  $p$ th order moments for (3.31) for any  $p > 1$  under Assumption I by using Lemma A.4 and a similar argument used in Lemma A.3 in Appendix A.3. Then by applying the conditional expectation formulas in Lemma A.1 in Appendix A.1 with  $k = 1$  and  $g_1 = x$  to each term of  $g_2$  in (3.30), we can evaluate  $\mathbf{E}[g_2|g_1 = x]$  and find that it is a quadratic function of  $x$  by using Lemma A.1. Applying the inversion formula for the characteristic function given in Lemma A.2 for the second term of the second equation in (3.32), and noting  $g(x)$  is a quadratic function of  $x$  with  $h(-it) = \varepsilon it$ , we obtain the next result.

**THEOREM 3.2.** *In addition to Assumption I, we assume that*

$$(3.33) \quad \Sigma = \int_0^T \sigma_{g_1}^*(t) \sigma_{g_1}'(t) dt > 0.$$

*Then the asymptotic expansion for the density function of  $X_T^{(\varepsilon)}$  for (3.26) as  $\varepsilon \rightarrow 0$  can be given by*

$$(3.34) \quad f_X^{(\varepsilon)}(x) = \phi_\Sigma(x) + \varepsilon \left[ \frac{c}{\Sigma} x^3 + \left( \frac{f}{\Sigma} - 2c \right) x \right] \phi_\Sigma(x) + O(\varepsilon^2),$$

*where  $\phi_\Sigma(x)$  stands for the Gaussian density function with zero mean and the variance  $\Sigma$ . The coefficients  $c$  and  $f$  in (3.34) are determined by the integral equation*

$$(3.35) \quad \mathbf{E}[g_2|g_1 = x] = cx^2 + f.$$

We note that the characteristic function method<sup>7</sup> we used above is a formal one from a rigorous mathematical view because we need the regularity conditions to ensure the asymptotic expansion of the continuous stochastic processes with respect to the parameter  $\varepsilon$  and the inversion procedure for the characteristic function of the random variable  $X_T^{(\varepsilon)}$ . It is not easy to show these conditions directly since  $X_T^{(\varepsilon)}$  is a complicated functional of the solution of the stochastic differential equation given by (3.1). However, we can rigorously prove the validity of the procedure along the lines developed by Watanabe (1987) using the Malliavin Calculus, which will be discussed in the Appendix (Sec. A.3).

The asymptotic variance  $\Sigma$  is the limit of the Malliavin covariance when  $\varepsilon \rightarrow 0$  for the call options of the coupon bond and the swaptions whose payoff function is given by (2.4). The explicit formulas of the coefficients in (3.35) are quite complicated in this problem. By using Lemma A.1 from the Appendix with  $k = 1$  and  $x = g_1$ , we can show that  $c$  and  $f$  for the call options of the coupon bond and swaptions in Example 2.1 are

<sup>7</sup> See Bhattacharya and Rao (1976) on the related mathematical problems and techniques of the standard characteristic function approach for the simple discrete i.i.d. case, for instance.

given by

$$\begin{aligned}
 (3.36) \quad c = & \frac{1}{2} \frac{g_0}{\Sigma^2} \left[ \int_0^T \sigma_T^{(0)}(v) \sigma_{g_1}^*(v)' dv \right]^2 \\
 & + \frac{1}{\Sigma^2} \left[ \int_0^T \sigma_T^{(0)}(v) \sigma_{g_1}^*(v)' dv \right] \sum_{j=1}^m c_j P(0, T_j) \left[ \int_0^T \sigma_{T, T_j}^{(0)}(v) \sigma_{g_1}^*(v)' dv \right] \\
 & + \frac{1}{2} \frac{1}{\Sigma^2} \sum_{j=1}^m c_j P(0, T_j) \left[ \int_0^T \sigma_{T, T_j}^{(0)}(v) \sigma_{g_1}^*(v)' dv \right]^2 \\
 & - \frac{g_0}{\Sigma^2} \left[ \int_0^T \left[ \int_0^t \sigma_{g_1}^*(s) \partial \sigma^{(0)}(s, t)' \left( \int_0^s \sigma^{(0)}(v, t) \sigma_{g_1}^*(v)' dv \right) ds \right] dt \right] \\
 & - \frac{1}{\Sigma^2} \sum_{j=1}^m c_j P(0, T_j) \left[ \int_T^{T_j} \left[ \int_0^T \sigma_{g_1}^*(s) \partial \sigma^{(0)}(s, u)' \right. \right. \\
 & \quad \left. \left. \times \left( \int_0^s \sigma^{(0)}(v, u) \sigma_{g_1}^*(v)' dv \right) ds \right] du \right]
 \end{aligned}$$

and

$$\begin{aligned}
 (3.37) \quad f = & -g_0 k_2(T) - \sum_{j=1}^m c_j P(0, T_j) k_1(T, T_j) \\
 & - \frac{1}{2} \frac{g_0}{\Sigma} \left[ \int_0^T \sigma_T^{(0)}(v) \sigma_{g_1}^*(v)' dv \right]^2 + \frac{g_0}{2} \left[ \int_0^T \sigma_T^{(0)}(v) \sigma_T^{(0)}(v)' dv \right] \\
 & + \sum_{j=1}^m c_j P(0, T_j) \int_0^T \sigma_T^{(0)}(v) \sigma_{T, T_j}^{(0)}(v)' dv \\
 & - \frac{1}{\Sigma} \left[ \int_0^T \sigma_T^{(0)}(v) \sigma_{g_1}^*(v)' dv \right] \sum_{j=1}^m c_j P(0, T_j) \left[ \int_0^T \sigma_{T, T_j}^{(0)}(v) \sigma_{g_1}^*(v)' dv \right] \\
 & - \frac{1}{2} \frac{1}{\Sigma} \sum_{j=1}^m c_j P(0, T_j) \left[ \int_0^T \sigma_{T, T_j}^{(0)}(v) \sigma_{g_1}^*(v)' dv \right]^2 \\
 & + \frac{1}{2} \sum_{j=1}^m c_j P(0, T_j) \left[ \int_0^T \sigma_{T, T_j}^{(0)}(v) \sigma_{T, T_j}^{(0)}(v)' dv \right] \\
 & + \frac{g_0}{\Sigma} \left[ \int_0^T \left[ \int_0^t \sigma_{g_1}^*(s) \partial \sigma^{(0)}(s, t)' \left( \int_0^s \sigma^{(0)}(v, t) \sigma_{g_1}^*(v)' dv \right) ds \right] dt \right] \\
 & + \frac{1}{\Sigma} \sum_{j=1}^m c_j P(0, T_j) \left[ \int_T^{T_j} \left[ \int_0^T \sigma_{g_1}^*(s) \partial \sigma^{(0)}(s, u)' \right. \right. \\
 & \quad \left. \left. \times \left( \int_0^s \sigma^{(0)}(v, u) \sigma_{g_1}^*(v)' dv \right) ds \right] du \right].
 \end{aligned}$$

We can also treat Example 2.2 using the same method. After some tedious calculations for the call options of the average interest rates whose payoff function is given by (2.7),

we can determine the stochastic expansion;

$$(3.38) \quad \begin{aligned} g^{(\varepsilon)} &= e^{-\int_0^T r^{(\varepsilon)}(s) ds} \left( \frac{1}{T\tau} \right) \left[ \int_0^T \frac{1}{P(t, t+\tau)} dt - k \right] \\ &= g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + o_p(\varepsilon^2), \end{aligned}$$

where we use the notation  $k = (1 + K\tau)T$ . In this expression, the coefficients  $g_i$  ( $i = 0, 1, 2$ ) are given by

$$(3.39) \quad g_0 = \frac{P(0, T)}{T\tau} \left[ \int_0^T \frac{P(0, t)}{P(0, t+\tau)} dt - k \right],$$

$$(3.40) \quad g_1 = \int_0^T \sigma_{g_1}^*(v) dB(v),$$

and

$$(3.41) \quad \begin{aligned} g_2 &= \frac{1}{2} \frac{P(0, T)}{T\tau} \int_0^T \frac{P(0, t)}{P(0, t+\tau)} \left[ \int_0^t \sigma_{t, t+\tau}^{(0)}(v) dB(v) \right]^2 dt \\ &\quad - \frac{P(0, T)}{T\tau} \int_0^T \frac{P(0, t)}{P(0, t+\tau)} \left[ \int_0^t \sigma_{t, t+\tau}^{(0)}(v) dB(v) \right] \\ &\quad \times \left[ \int_0^T \sigma_T^{(0)}(v) dB(v) \right] dt + \frac{1}{2} g_0 \left[ \int_0^T \sigma_T^{(0)}(v) dB(v) \right]^2 \\ &\quad + \frac{P(0, T)}{T\tau} \int_0^T \frac{P(0, t)}{P(0, t+\tau)} \left[ \int_t^{t+\tau} B(t, u) du \right] dt \\ &\quad - g_0 \left[ \int_0^T B(t, t) dt \right], \end{aligned}$$

where we use the notations

$$(3.42) \quad \begin{aligned} \sigma_{g_1}^*(v) &= \frac{P(0, T)}{T\tau} \int_v^T \frac{P(0, t)}{P(0, t+\tau)} \sigma_{t, t+\tau}^{(0)}(v) dt - \frac{P(0, T)}{T\tau} \\ &\quad \times \left[ \int_0^T \frac{P(0, t)}{P(0, t+\tau)} dt - k \right] \sigma_T^{(0)}(v), \\ \sigma_{t, t+\tau}^{(0)}(v) &= \left[ \int_t^{t+\tau} \sigma_i^{(0)}(v, u) du \right]_i, \end{aligned}$$

and

$$\sigma_T^{(0)}(v) = \left[ \int_v^T \sigma_i^{(0)}(v, u) du \right]_i.$$

Then, the asymptotic variance  $\Sigma$  in (3.34) is given by the formula of (3.33) for Example 2.2, where we use (3.42) instead of (3.29). By a tedious but straightforward

calculation in the present case, we arrive at

$$\begin{aligned}
 (3.43) \quad c = & \frac{1}{2} \frac{1}{\Sigma^2} \frac{P(0, T)}{T\tau} \int_0^T \frac{P(0, t)}{P(0, t + \tau)} \left[ \int_0^t \sigma_{t, t+\tau}^{(0)}(v) \sigma_{g_1}^*(v)' dv \right]^2 dt \\
 & - \frac{P(0, T)}{T\tau} \frac{1}{\Sigma^2} \left[ \int_0^T \frac{P(0, t)}{P(0, t + \tau)} \left( \int_0^t \sigma_{t, t+\tau}^{(0)}(v) \sigma_{g_1}^*(v)' dv \right) dt \right] \\
 & \times \left[ \int_0^T \sigma_T^{(0)}(v) \sigma_{g_1}^*(v)' dv \right] + \frac{1}{2} \frac{1}{\Sigma^2} g_0 \left[ \int_0^T \sigma_T^{(0)}(v) \sigma_{g_1}^*(v)' dv \right]^2 \\
 & + \frac{P(0, T)}{T\tau \Sigma^2} \int_0^T \frac{P(0, t)}{P(0, t + \tau)} \left[ \int_t^{t+\tau} \left[ \int_0^t \sigma_{g_1}^*(s) \partial \sigma^{(0)}(s, u)' \right. \right. \\
 & \quad \left. \left. \times \left( \int_0^s \sigma^{(0)}(v, u) \sigma_{g_1}^*(v)' dv \right) ds \right] du \right] dt \\
 & - \frac{g_0}{\Sigma^2} \int_0^T \left[ \int_0^t \sigma_{g_1}^*(s) \partial \sigma^{(0)}(s, t)' \left( \int_0^s \sigma^{(0)}(v, t) \sigma_{g_1}^*(v)' dv \right) ds \right] dt
 \end{aligned}$$

and

$$\begin{aligned}
 (3.44) \quad f = & -\frac{1}{2} \frac{P(0, T)}{T\tau} \frac{1}{\Sigma} \int_0^T \frac{P(0, t)}{P(0, t + \tau)} \left[ \int_0^t \sigma_{t, t+\tau}^{(0)}(v) \sigma_{g_1}^*(v)' dv \right]^2 dt \\
 & + \frac{1}{2} \frac{P(0, T)}{T\tau} \int_0^T \frac{P(0, t)}{P(0, t + \tau)} \left[ \int_0^t \sigma_{t, t+\tau}^{(0)}(v) \sigma_{t, t+\tau}^{(0)}(v)' dv \right] dt \\
 & + \frac{P(0, T)}{T\tau} \frac{1}{\Sigma} \int_0^T \frac{P(0, t)}{P(0, t + \tau)} \left[ \int_0^t \sigma_{t, t+\tau}^{(0)}(v) \sigma_{g_1}^*(v)' dv \right] dt \\
 & \times \left[ \int_0^T \sigma_T^{(0)}(v) \sigma_{g_1}^*(v)' dv \right] - \frac{P(0, T)}{T\tau} \int_0^T \frac{P(0, t)}{P(0, t + \tau)} \\
 & \times \left[ \int_0^t \sigma_{t, t+\tau}^{(0)}(v) \sigma_T^{(0)}(v)' dv \right] dt - \frac{1}{2} \frac{1}{\Sigma} g_0 \left[ \int_0^T \sigma_T^{(0)}(v) \sigma_{g_1}^*(v)' dv \right]^2 \\
 & + \frac{1}{2} g_0 \left[ \int_0^T \sigma_T^{(0)}(v) \sigma_T^{(0)}(v)' dv \right] + \frac{P(0, T)}{T\tau} \\
 & \times \left[ \int_0^T k_1(t, t + \tau) \frac{P(0, t)}{P(0, t + \tau)} dt \right] - g_0 k_2(T) - \frac{P(0, T)}{T\tau \Sigma} \\
 & \times \int_0^T \frac{P(0, t)}{P(0, t + \tau)} \left[ \int_t^{t+\tau} \left[ \int_0^t \sigma_{g_1}^*(s) \partial \sigma^{(0)}(s, u)' \right. \right. \\
 & \quad \left. \left. \times \left( \int_0^s \sigma^{(0)}(v, u) \sigma_{g_1}^*(v)' dv \right) ds \right] du \right] dt \\
 & + \frac{g_0}{\Sigma} \int_0^T \left[ \int_0^t \sigma_{g_1}^*(s) \partial \sigma^{(0)}(s, t)' \left( \int_0^s \sigma^{(0)}(v, t) \sigma_{g_1}^*(v)' dv \right) ds \right] dt.
 \end{aligned}$$

The last step is to derive the asymptotic expansion of the expectation of the discounted terminal payoff based on the asymptotic expansion of the exact density function. By taking the mathematical expectation of (2.9) for (3.26) and (3.38) with respect to the

asymptotic expansions of the density functions given in Theorem 3.2, we have

$$\begin{aligned}
 (3.45) \quad E[g_0 + \varepsilon X_T^{(\varepsilon)}]^+ &= \int_{g_0 + \varepsilon x \geq 0} (g_0 + \varepsilon x) f_X^{(\varepsilon)}(x) dx \\
 &= \int_{g_0 + \varepsilon x \geq 0} (g_0 + \varepsilon x) \phi_\Sigma(x) \left\{ 1 + \varepsilon \left[ \frac{c}{\Sigma} x^3 + \left( \frac{f}{\Sigma} - 2c \right) x \right] \right. \\
 &\quad \left. + \varepsilon^2 h(x) + o(\varepsilon^2) \right\} dx,
 \end{aligned}$$

where  $h(x)$  is a function of the polynomial order and  $\phi_\Sigma(x)$  is the Gaussian density function.

Then by applying the integration-by-parts formula, we immediately obtain the following result.

**THEOREM 3.3.** *In addition to Assumption I, we assume that  $\Sigma > 0$ . Then the asymptotic expansions of  $V_0(T)$  in (2.9) for (2.4) and (2.7) as  $\varepsilon \rightarrow 0$  are given by*

$$\begin{aligned}
 (3.46) \quad V_0(T) &= g_0 \int_{-y_\varepsilon}^{+\infty} \phi_\Sigma(x) dx + \varepsilon \int_{-y_\varepsilon}^{+\infty} x \phi_\Sigma(x) dx \\
 &\quad + \varepsilon^2 \int_{-y_\varepsilon}^{+\infty} (cx^2 + f) \phi_\Sigma(x) dx + o(\varepsilon^2),
 \end{aligned}$$

where  $y_\varepsilon = (1/\varepsilon)g_0$ . The coefficients  $c$  and  $f$  are the same as in Theorem 3.2, and given by (3.36)–(3.37) and (3.43)–(3.44), respectively.

Note that each term of the right-hand side of (3.46) depends on  $y_\varepsilon$  with the parameter  $\varepsilon$ , but such expression can be rigorously justified. The mathematical validity of our method in this section is discussed in the Appendix, Section A.3. It is also possible to obtain an asymptotic expansion with respect to polynomials of  $\varepsilon$  under an additional assumption.<sup>8</sup>

All terms on the right-hand side of (3.46) are known functions of the distribution function and the density function of  $N(0, \Sigma)$ . For instance, take the relation

$$(3.47) \quad \int_{-y}^{\infty} x^2 \phi_\Sigma(x) dx = \Sigma \Phi(\Sigma^{-1/2}y) - y \Sigma \phi_\Sigma(y),$$

where  $\Phi(\cdot)$  is the distribution function of the standard normal distribution. This and similar formulas are useful for the numerical implementation of our procedure.

#### 4. NUMERICAL EXAMPLES

In this section, we present some numerical results to illustrate the method introduced in Section 3. For this purpose, we use the pricing problem of swaptions and average options of interest rates in the term structure model explained in Section 2. For the simplicity of

<sup>8</sup> Originally Kunitomo and Takahashi (1995) presented such an expression. However, the resulting approximations are the same as (3.46). We owe the referee for clarifying the related problem.

exposition, we assume that the instantaneous forward rate processes  $\{f^{(\varepsilon)}(s, t)\}$  have a one-factor volatility function<sup>9</sup> with  $n = 1$ . For  $0 \leq \beta \leq 1$ , we take

$$(4.1) \quad \sigma_1(f^{(\varepsilon)}(s, t), s, t) = f^{(\varepsilon)}(s, t)^\beta h_1(f^{(\varepsilon)}(s, t), M) + (M + 1)^\beta h_2(f^{(\varepsilon)}(s, t), M),$$

where  $M = 10,000$  and  $f^{(\varepsilon)}(s, t)$  is a real-valued function given  $\varepsilon$ . The smooth functions are taken as  $h_1(\xi, M) = h(M + 1 - \xi)/[h(\xi - M) + h(M + 1 - \xi)]$ ,  $h_2(\xi, M) = h(\xi - M)/[h(\xi - M) + h(M + 1 - \xi)]$ , and

$$h(\xi) = \begin{cases} e^{-1/\varepsilon} & (\xi > 0) \\ 0 & (\xi \leq 0). \end{cases}$$

When  $\beta = 1$ , (4.1) corresponds to the geometric Brownian motion case for the forward rates approximately and we have used its truncated version with smoothness because there can be some explosive solutions as we discussed in Assumption I in Section 3. When  $\beta = 0$ ,  $\sigma_1(f^{(\varepsilon)}(s, t), s, t) = 1$  in (4.1) and it corresponds to a continuous analog of the discrete model by Ho and Lee (1986).

For the first numerical example, Table 4.1 shows the numerical values of call options on average interest rates for the case when  $\beta = 0$  and  $T$  is 0.25 year or 1 year. The time to maturity of the underlying interest rates is one year and the average is taken over interest rates whose maturity is one year ( $\tau = 1$ ). For simplicity, the present term structure at  $t = 0$  is assumed to be a flat 5 percent per year and the volatility parameter is assumed to be 150 basis points per year ( $\varepsilon = 0.015$ ). We can use the approximations based on the asymptotic expansions in Section 3 and measure their accuracy by the Monte Carlo results. We have given the results for the out-of-the money case ( $K = 5.5$  percent or 6 percent), the at-the-money case ( $K = 5$  percent), and the in-the-money case ( $K = 4.5$  percent or 4 percent).

For comparative purposes, the numerical values obtained by Monte Carlo simulations and by the finite difference method via the PDE approach have been calculated. The number of simulations in our Monte Carlo calculations is 500,000 and we expect that the results obtained by this method are very accurate. The finite difference values in Table 4.1 are based on a numerical solution of the PDE for the average options of interest rate processes under the assumption that they follow (4.1) when  $\beta = 0$ . This method has been recently developed by He and Takahashi (2000). From Table 4.1 we can find that the differences between the option values obtained by the asymptotic expansion approach and the Monte Carlo approach are very small—less than 1 percent of the underlying price levels.

For the second numerical example, Table 4.2, Part A, shows the results for the call options of a swap contract (the swaption) in the case when  $\beta = 0$  and  $\varepsilon = 0.01$  (100 basis points). Table 4.2, Parts B–D contain the numerical values for the case when  $\beta = 1$  and  $\varepsilon = 0.2$  (20 percent). In Table 4.2, Parts A and B the terms of the underlying interest rate swap is 5 years and the time to expiration is also 5 years;  $T$  is 3 years in Part C and 1 year in Part D. We set  $\tau = 1$  year,  $T_1 = T + 1, \dots, T_5 = T + 5$ ,  $m = 5$ , and the

<sup>9</sup> This case has been used by Amin and Bodurtha (1995) as a numerical example on the forward interest rates.



TABLE 4.1  
Average Options on Interest Rates

A.  $T = 3$  months

Strike rate %	5.50	5.00	4.50
(1) Stochastic Expansion	5.36	25.12	63.98
(1-1) Difference (bp)	-0.034	0.002	0.080
(1-2) Diff. rate %	-0.56	0.01	0.13
(2) Finite difference	5.36	24.99	63.81
(2-1) Difference (bp)	-0.026	-0.13	-0.09
(2-2) Diff. rate %	-0.48	-0.52	-0.14
(3) Monte Carlo	5.39	25.12	63.90
(4) European call	16.30	38.05	71.76

B.  $T = 1$  year

Strike rate %	6.00	5.00	4.00
(1) Stochastic Expansion	8.16	41.37	112.30
(1-1) Difference (bp)	0.067	0.030	-0.010
(1-2) Diff. rate %	0.83	0.07	-0.01
(2) Finite difference	8.06	41.32	112.25
(2-1) Difference (bp)	-0.03	-0.017	-0.060
(2-2) Diff. rate %	-0.37	-0.04	-0.05
(3) Monte Carlo	8.09	41.34	112.31
(4) European call	28.14	67.26	129.60

Notes: The values are calculated in terms of basis points except for the percentage differences (%). The figures in (1), (2), and (3) are the results evaluated by the asymptotic expansion method, the finite difference method, and the Monte Carlo method, respectively. Difference (bp) and Diff. (difference) rate (%) in (1) and (2) are calculated as the deviations from the Monte Carlo results in (3). For instance, -0.034 in (1-1) of Part A is the difference between 5.36 in (1) and 5.39 in (3) and -0.56% in (1-2) is  $-0.034/5.39 \times 100\%$ . The numbers in (4) stand for the exact values of the corresponding European call options.

present term structure at  $t = 0$  is assumed to be a flat 5 percent per year. Because in this case we have  $c_j = S\tau$  ( $j = 1, \dots, m-1$ ),  $c_m = 1 + S\tau$ ,  $K = 1$ , and

$$(4.2) \quad g_0 = S\tau \sum_{j=1}^m P(0, T_j) + P(0, T_m) - P(0, T),$$

we have set

$$(4.3) \quad S = \frac{P(0, T) - P(0, T_5)}{\tau \sum_{j=1}^5 P(0, T_j)} = 0.051271$$

for the at-the-money case. We have computed the results for the out-of-the money case ( $S = 5.1271\% \times 0.8$ ,  $5.1271\% \times 0.6$ , and  $5.1271\% \times 0.4$ ), the at-the-money case

TABLE 4.2  
Swaption

Fixed rate %	8.20	7.18	6.16	5.13	4.10	3.08	2.05
A. Gaussian Case: 5 years to 5 years							
(1) Stochastic Expansion	1070.1	774.8	518.5	315.1	171.2	81.2	31.6
(1-1) Difference (bp)	-0.9	0.3	0.4	0.2	0.4	0.0	-1.7
(1-2) Diff. rate %	-0.1	0.0	0.1	0.1	0.2	0.0	-5.0
(2) Monte Carlo	1071.0	774.6	518.2	315.0	170.9	81.3	33.1
B. Log-Normal Case: 5 years to 5 years							
(1) Stochastic Expansion	1118.0	819.1	546.5	315.1	143.3	42.3	3.2
(1-1) Difference (bp)	5.0	5.0	3.9	2.8	3.1	2.7	-0.8
(1-2) Diff. rate %	0.4	0.6	0.7	0.9	2.2	6.8	-20.0
(2) Monte Carlo	1113.0	814.1	542.6	312.3	140.2	39.6	4.0
C. Log-Normal Case: 3 years to 5 years							
(1) Stochastic Expansion	1180.7	837.3	525.7	269.7	97.8	17.5	-1.4
(1-1) Difference (bp)	2.4	3.4	2.8	1.4	0.9	-0.1	-2.1
(1-2) Diff. rate %	0.2	0.4	0.5	0.5	1.0	-0.7	**
(2) Monte Carlo	1178.3	834.0	522.9	268.4	96.8	17.6	0.7
D. Log-Normal Case: 1 year to 5 years							
(1) Stochastic Expansion	1264.1	852.2	469.7	172.1	27.0	0.0	-0.1
(1-1) Difference (bp)	0.2	0.4	1.0	0.5	0.1	-0.7	-0.1
(1-2) Diff. rate %	0.0	0.0	0.2	0.3	0.2	**	**
(2) Monte Carlo	1263.9	851.9	468.7	171.7	27.0	0.7	0.0

Notes: The values in Table 4.2 are also calculated in terms of basis points except for the percentage differences(%). The figures in (1) and (2) are the results evaluated by the asymptotic expansion method and the Monte Carlo method, respectively. Difference (bp) and Diff. (difference) rate (%) in (1) are calculated as the deviations from the Monte Carlo results in (2). For instance, -0.9 in (1-1) of part A is the difference between 1070.1 in (1) and 1071.0 in (2) and -0.1% in (1-2) is  $-0.9/1071.0 \times 100\%$  in Part A. The values marked with \*\* in Parts C and D are not meaningful.

( $S = 5.1271\%$ ), and the in-the-money case ( $S = 5.1271\% \times 1.2, 5.1271\% \times 1.4$ , and  $5.1271\% \times 1.6$ ). The approximations based on the asymptotic expansions in Section 3 and the Monte Carlo results are presented for the cases when  $\beta = 0$  and  $\beta = 1$ .

From Table 4.2, Part A, we also find that the differences in the swaption values between the two approaches using Gaussian forward rates are very small and negligible in most cases. From Parts B–D of the table we can see that the differences in the option values become larger under the geometric Brownian forward rates case relative to Part A; this is due to the non-Gaussianity of the underlying forward rates and the spot rate. By examining these results with respect to different terms to expiration, we note that the approximations become more accurate when  $T$  is shorter. For practical purposes, the *difference percentage rate* can be ignored even if it is large when the option value is quite close to zero. On the whole, in the swaption example, the approximations are sufficiently accurate in absolute terms and the differences between the approximations and the corresponding Monte Carlo results are within three basis points.

As these two examples show, the values of our approximations are reliable to at least two digits. Thus, we can tentatively conclude that the approximation formulas we obtained in Section 3 are accurate and useful for practical purposes.

## 5. CONCLUDING REMARKS

This paper proposes a new methodology for the valuation problem of financial contingent claims when the underlying forward rates follow a general class of continuous Itô processes, which are not necessarily in a class of continuous diffusion processes. Our method, called the *small disturbance asymptotic expansion approach*, can be applicable to a wide range of valuation problems including complicated contingent claims associated with the term structure of interest rates. We have illustrated our methodology by deriving some useful formulas for swaptions and average (or Asian) options for interest rates. We have also given some evidence that the resulting formulas are numerically accurate enough for practical applications. Since the asymptotic expansion approach can be rigorously justified by the Malliavin–Watanabe Calculus in stochastic analysis, it is not an ad hoc method to give numerical approximations. The asymptotic expansions explained in Section 3 can be made up to any order of precision  $O(\varepsilon^k)$  ( $k = 1, 2, \dots$ ), in principle.

There are several advantageous aspects in our method over the PDE and the Monte Carlo methods, which have been extensively used in practical applications. First, our method is applicable in a unified manner to the pricing problem of various types of functionals of asset prices in the economy governed by the general class of continuous Itô processes, which are not necessarily Markovian in the usual sense. The problem has been known to be difficult using the existing methods. Second, our method is computationally efficient in comparison with other methods since it is very fast to obtain numerical results with a computer. Third, the distributions of the underlying assets and their functionals at any date can be evaluated by our method. This aspect is quite useful in various kinds of simulations. For instance, the pricing formulas derived by our method can be used as control variates to improve the efficiency of Monte Carlo simulations. The PDE method, on the other hand, is difficult to implement, especially when the underlying assets or term structure of interest rates follow multifactor processes. Monte Carlo simulations are often quite time consuming in this case. Takahashi (1995) has discussed some extensions of our method to the pricing problem of derivatives in more complicated multicurrency and multifactor situations when the forward rate processes are not necessarily Markovian in the usual sense.

Finally, we should mention that the asymptotic expansion approach in this paper can give a powerful and useful tool not only to the valuation problem of contingent claims associated with the term structure of interest rates, but also to other problems in financial economics (for an example, see Kim and Kunitomo 1999). Our method usually gives some explicit formulas that may shed some new light on the solution of the problem under consideration when the underlying asset prices follow a general class of continuous Itô processes. Hence, we do not need to use simple stochastic processes among the class of diffusion or Markovian processes in the usual sense only because the resulting solutions are manageable. We suspect that there have been some works on interest rates which used simple but unreasonable stochastic processes mainly because the resulting analyses are mathematically convenient.

## APPENDIX

Here, we gather some mathematical details that were omitted in the previous sections. We also briefly discuss the validity of our method by the use of the Malliavin–Watanabe theory in stochastic analysis.

## A.1. Two Useful Lemmas

We first give some formulas on the conditional expectation operations as Lemma A.1, which is a slight generalization of Lemma 5.7 of Yoshida (1992a). The proof is a direct result of calculations by making use of the Gaussianity of continuous processes involved.

LEMMA A.1. *Let  $\mathbf{B}(t)$  be an  $n \times 1$  vector of independent Brownian motions and  $\mathbf{x}$  be a  $k$ -dimensional vector. Let also  $\mathbf{q}_1(t)$  be an  $R^1 \mapsto R^{k \times n}$  nonstochastic function and*

$$(A.1) \quad \Sigma = \int_0^T \mathbf{q}_1(t) \mathbf{q}_1'(t) dt$$

*a positive definite matrix. (i) Suppose  $\mathbf{q}_2(u)$  and  $\mathbf{q}_3(u)$  are  $R^1 \mapsto R^{m \times n}$  nonstochastic functions. Then for  $0 \leq s \leq t \leq T$*

$$(A.2) \quad \mathbf{E} \left[ \int_0^t \left[ \int_0^s \mathbf{q}_2(u) d\mathbf{B}(u) \right]' \mathbf{q}_3(s) d\mathbf{B}(s) \middle| \int_0^T \mathbf{q}_1(u) d\mathbf{B}(u) = \mathbf{x} \right] \\ = \text{trace} \int_0^t \mathbf{q}_1(s) \mathbf{q}_3(s)' \left( \int_0^s \mathbf{q}_2(u) \mathbf{q}_1(u)' du \right) ds \Sigma^{-1} [\mathbf{x} \mathbf{x}' - \Sigma] \Sigma^{-1}.$$

*(ii) Suppose  $\mathbf{q}_2(u)$  and  $\mathbf{q}_3(u)$  are  $R^1 \mapsto R^n$  nonstochastic functions. Then for  $0 \leq s \leq t \leq T$*

$$(A.3) \quad \mathbf{E} \left[ \left[ \int_0^s \mathbf{q}_2(u) d\mathbf{B}(u) \right] \left[ \int_0^t \mathbf{q}_3(v) d\mathbf{B}(v) \right]' \middle| \int_0^T \mathbf{q}_1(u) d\mathbf{B}(u) = \mathbf{x} \right] \\ = \int_0^s \mathbf{q}_2(u) \mathbf{q}_3(u)' du + \left[ \int_0^s \mathbf{q}_2(u) \mathbf{q}_1(u)' du \right] \\ \Sigma^{-1} [\mathbf{x} \mathbf{x}' - \Sigma] \Sigma^{-1} \left[ \int_0^t \mathbf{q}_1(v) \mathbf{q}_3(v)' dv \right].$$

The second lemma is on the inversion formulas of the characteristic functions of some random variables. The proof is also a direct result of calculation, which has been given in Fujikoshi et al. (1982).

LEMMA A.2 (Fujikoshi et al. 1982). *Suppose that  $\mathbf{x}$  follows an  $n$ -dimensional Gaussian distribution with mean  $\mathbf{0}$  and variance-covariance matrix  $\Sigma$ . The density function of  $\mathbf{x}$  is denoted by  $\phi_\Sigma(\cdot)$ . Then for any polynomial functions  $g(\cdot)$  and  $h(\cdot)$ ,*

$$(A.4) \quad \mathcal{F}^{-1} \left[ h(-it) \mathbf{E} \left[ g(\mathbf{x}) e^{it' \mathbf{x}} \right] \right]_{(\xi)} = h \left[ \frac{\partial}{\partial \xi} \right] g(\xi) \phi_\Sigma(\xi),$$

where  $i = \sqrt{-1}$ ,

$$(A.5) \quad \mathcal{F}^{-1} \left[ h(-it) \mathbf{E} \left[ g(\mathbf{x}) e^{it' \mathbf{x}} \right] \right]_{(\xi)} = \left( \frac{1}{2\pi} \right)^n \int_{\mathbf{R}^n} e^{-it' \xi} h(-it) \mathbf{E} \left[ g(\mathbf{x}) e^{it' \mathbf{x}} \right] d\mathbf{x},$$

and the expectation operation  $\mathbf{E}[\cdot]$  is taken over  $\mathbf{x} = (x_i) \in \mathbf{R}^n$ . Also  $\mathcal{F}^{-1}[\cdot]_{(\xi)}$  denotes the  $n$ -dimensional Fourier inversion  $\mathcal{F}^{-1}[\cdot]$  being evaluated at  $\xi$  and  $t' \mathbf{x} = \sum_{i=1}^n t_i x_i$  for  $\mathbf{t} = (t_i) \in \mathbf{R}^n$ .

## A.2. A Sketch of Derivations of Asymptotic Expansions

We give a brief sketch on our derivations of the stochastic expansions of some random variables used in Section 3. The following derivations are purely formal, but the mathematical validity of our method is discussed in the next subsection. From (3.1), the deterministic process for  $\{f^{(\varepsilon)}(s, t)\}$  follows when  $\varepsilon \rightarrow 0$  is given by

$$(A.6) \quad f^{(0)}(s, t) = \lim_{\varepsilon \rightarrow 0} f^{(\varepsilon)}(s, t) = f(0, t).$$

Then we define the random variables  $A(s, t)$  and  $C(s, t)$  by

$$(A.7) \quad A(s, t) = \frac{\partial f^{(\varepsilon)}(s, t)}{\partial \varepsilon} \Big|_{\varepsilon=0},$$

and

$$(A.8) \quad C(s, t) = \frac{1}{2} \frac{\partial^2 f^{(\varepsilon)}(s, t)}{\partial^2 \varepsilon} \Big|_{\varepsilon=0}.$$

By a direct calculation of differentiation, we have

$$(A.9) \quad \begin{aligned} A(s, t) &= \int_0^s \left[ 2\varepsilon b(f^{(\varepsilon)}(v, t), v, t) + \varepsilon^2 \frac{\partial b(f^{(\varepsilon)}(v, t), v, t)}{\partial \varepsilon} \right]_{\varepsilon=0} dv \\ &\quad + \int_0^s \sum_{i=1}^n \left[ \sigma_i(f^{(\varepsilon)}(v, t), v, t) + \varepsilon \frac{\partial \sigma_i(f^{(\varepsilon)}(v, t), v, t)}{\partial \varepsilon} \right]_{\varepsilon=0} dB_i(v) \\ &= \int_0^s \sum_{i=1}^n \sigma_i^{(0)}(v, t) dB_i(v). \end{aligned}$$

Similarly, we have

$$(A.10) \quad \begin{aligned} C(s, t) &= \int_0^s \left[ b(f^{(\varepsilon)}(v, t), v, t) + 2\varepsilon \frac{\partial b(f^{(\varepsilon)}(v, t), v, t)}{\partial \varepsilon} \right. \\ &\quad \left. + \frac{\varepsilon^2}{2} \frac{\partial^2 b(f^{(\varepsilon)}(v, t), v, t)}{\partial^2 \varepsilon} \right]_{\varepsilon=0} dv \\ &\quad + \int_0^s \sum_{i=1}^n \left[ \frac{\partial \sigma_i(f^{(\varepsilon)}(v, t), v, t)}{\partial \varepsilon} \right. \\ &\quad \left. + \frac{1}{2} \varepsilon \frac{\partial^2 \sigma_i(f^{(\varepsilon)}(v, t), v, t)}{\partial^2 \varepsilon} \right]_{\varepsilon=0} dB_i(v) \\ &= \int_0^s b^{(0)}(v, t) dv + \int_0^s \sum_{i=1}^n \partial \sigma_i^{(0)}(v, t) A(v, t) dB_i(v). \end{aligned}$$

Hence we have obtained the stochastic differential equations which  $\{A(s, t)\}$  and  $\{C(s, t)\}$  must satisfy.

Next, we substitute  $\{f^{(\varepsilon)}(s, u)\}$  in (3.14) and use the fact that  $\{P^{(\varepsilon)}(s, t)\}$  are nonstochastic functions at  $s = 0$  which lead to (3.15) and (3.16). The stochastic expansion of the discount factor can be obtained by using  $\{r^{(\varepsilon)}(s)\}$  instead of  $\{f^{(\varepsilon)}(s, t)\}$ . In (3.17), (3.19), (3.22), and (3.24), we can utilize the Fubini-type theorem on the exchanges of integration operations. By expanding the exponential functions with respect to  $\varepsilon$ , we have

$$(A.11) \quad P^{(\varepsilon)}(s, t) = \frac{P(0, t)}{P(0, s)} \left[ 1 - \varepsilon \int_s^t A(s, u) du - \varepsilon^2 \int_s^t C(s, u) du + \varepsilon^2 \frac{1}{2} \left\{ \int_s^t A(s, u) du \right\}^2 \right] + o_p(\varepsilon^2)$$

and

$$(A.12) \quad e^{-\int_0^T r^{(\varepsilon)}(s) ds} = P(0, T) \left[ 1 - \varepsilon \int_0^T A(t, t) dt - \varepsilon^2 \int_0^T C(t, t) dt + \varepsilon^2 \frac{1}{2} \left\{ \int_0^T A(t, t) dt \right\}^2 \right] + o_p(\varepsilon^2),$$

respectively.

Finally, we multiply the stochastic expansions of the discount factor and the terminal payoff function. Then by rearranging each term in the resulting stochastic expansions, we can obtain the form of (3.26) and (3.38) in Examples 2.1 and 2.2, respectively.

### A.3. Validity of the Asymptotic Expansion Approach

The mathematical validity of the asymptotic expansion approach in this paper can be given along the line based on the remarkable work by Watanabe (1987) on the Malliavin Calculus in stochastic analysis. Yoshida (1992a, 1992b) has utilized the results and method originally developed by Watanabe and given some useful results on the validity of the asymptotic expansions of some functionals on continuous-time homogeneous diffusion processes. The validity of our method can be obtained by similar arguments to those used by Yoshida (1992a, 1992b) and Ikeda and Watanabe (1989, Chap. V), but with substantial modifications. This is mainly because the continuous stochastic processes defined by (3.1) for the spot interest rate and forward rates are not necessarily Markovian in the usual sense.

Since the rigorous proofs of our claims in this section can be quite lengthy, but for the most part are some extensions of the existing results in stochastic analysis, we shall only give a rough sketch below. Our arguments on the validity of the asymptotic expansion approach for interest rate based contingent claims consist of four steps. The main aim in the following steps will be to check the truncated version of the nondegeneracy condition for the Malliavin covariance in our situation.

**[Step 1]:** First, we shall prepare some notations. For this purpose, we freely use the notations of Ikeda and Watanabe (1989) as a standard textbook. We discuss only the validity of the asymptotic expansion approach based on a one-dimensional Wiener space

without loss of generality because we only need more complicated notations in the general case. Let  $(W, P)$  be the one-dimensional Wiener space and let  $\mathbf{H}$  be the Cameron–Martin subspace of  $W$  endowed with the norm

$$(A.13) \quad |h|_H^2 = \int_0^{\bar{T}} |\dot{h}(t)|^2 dt$$

for  $h \in \mathbf{H}$ . The norm of  $\mathbf{R}$ -valued Wiener functional  $g$  for any  $s \in \mathbf{R}$  and  $p \in (1, \infty)$  is defined by

$$(A.14) \quad \|g\|_{p,s} = \|(I - \mathcal{L})^{s/2} g\|_p,$$

where  $\mathcal{L}$  is the Ornstein–Uhlenbeck operator and  $\|\cdot\|_p$  is the  $L_p$ -norm in the standard stochastic analysis. An  $\mathbf{R}$ -valued function  $g: W \mapsto \mathbf{R}$  is called an  $\mathbf{R}$ -valued polynomial functional if  $g = p([h_1](B), \dots, [h_n](B))$ , where  $n \in \mathbf{Z}^+$ ,  $h_i \in \mathbf{H}$ ,  $p(x_1, \dots, x_n)$  is a polynomial, and

$$[h](B) = \int_0^{\bar{T}} \dot{h}(t) dB(t)$$

for  $h \in \mathbf{H}$  are defined in the sense of stochastic integrals.

Let  $\mathbf{P}(\mathbf{R})$  denote the totality of  $\mathbf{R}$ -valued polynomials on the Wiener space  $(W, P)$ . Then  $\mathbf{P}(\mathbf{R})$  is dense in  $L_p(\mathbf{R})$ . The Banach space  $\mathbf{D}_p^s(\mathbf{R})$  is the completion of  $\mathbf{P}(\mathbf{R})$  with respect to  $\|\cdot\|_{p,s}$ . The dual space of  $\mathbf{D}_p^s(\mathbf{R})$  is  $\mathbf{D}_q^{-s}(\mathbf{R})$ , where  $s \in \mathbf{R}$ ,  $p > 1$ , and  $1/p + 1/q = 1$ . The space  $\mathbf{D}^\infty(\mathbf{R}) = \bigcap_{s>0} \bigcap_{1<p<+\infty} \mathbf{D}_p^s(\mathbf{R})$  is the set of Wiener functionals and  $\tilde{\mathbf{D}}^{-\infty}(\mathbf{R}) = \bigcup_{s>0} \bigcap_{1<p<+\infty} \mathbf{D}_p^{-s}(\mathbf{R})$  is a space of generalized Wiener functionals. For  $F \in \mathbf{P}(\mathbf{R})$  and  $h \in \mathbf{H}$ , the derivative of  $F$  in the direction of  $h$  is defined by

$$(A.15) \quad D_h F(B) = \lim_{\delta \rightarrow 0} \frac{1}{\delta} \{F(B + \delta h) - F(B)\}.$$

Then for  $F \in \mathbf{P}(\mathbf{R})$  and  $h \in \mathbf{H}$  there exists  $DF \in \mathbf{P}(\mathbf{H} \otimes \mathbf{R})$  such that  $D_h F(B) = \langle DF(B), h \rangle_H$ , where  $\langle \cdot \rangle_H$  is the inner product of  $\mathbf{H}$  and  $DF$  is called the  $H$ -derivative of  $F$ . It is known that the norm  $\|\cdot\|_{p,s}$  is equivalent to the norm  $\sum_{k=0}^s \|D^k \cdot\|_p$ . For  $F \in \mathbf{D}^\infty(\mathbf{R})$ , we can define the Malliavin covariance by

$$(A.16) \quad \sigma(F) = \langle DF(B), DF(B) \rangle_H,$$

where  $\langle \cdot \rangle_H$  is the inner product of  $\mathbf{H}$ . It is known that the operator  $D$  can be well defined in  $\mathbf{D}^\infty(\mathbf{R})$ . (For the details see Ikeda and Watanabe 1989, Chap. V.)

**[Step 2]:** We set  $n = 1$  and  $\sigma(f^{(\varepsilon)}(v, t), v, t) = \sigma_1(f^{(\varepsilon)}(v, t), v, t)$  in (3.1) without the loss of generality. (We shall use  $n$  and  $m$  for different purposes in this subsection.) We further set  $\varepsilon = 1$  in Step 2. The starting point of our discussion is the result by Morton (1989) on the existence and uniqueness of the solution of the stochastic integral equation (3.1) for forward rate processes.

**THEOREM A.1:** *Assume the conditions in Assumption I in Section 3 except those on the smoothness of second- and higher order derivatives. Then there exists a jointly continuous process  $\{f^{(\varepsilon)}(s, t), 0 \leq s \leq t \leq \bar{T}\}$  satisfying (3.1) with  $\varepsilon = 1$  and the solution for (3.1) is unique.*

We shall consider the  $H$ -derivatives of the forward rate processes  $\{f^{(1)}(s, t)\}$ . For any  $h \in H$ , we successively define a sequence of random variables  $\{\xi^{(n)}(s, t)\}$  using the integral equation:

$$(A.17) \quad \begin{aligned} \xi^{(n+1)}(s, t) = & \int_0^s \left[ \partial \sigma(f^{(1)}(v, t), v, t) \int_v^t \sigma(f^{(1)}(v, y), v, y) dy \xi^{(n)}(v, t) \right] dv \\ & + \int_0^s \left[ \sigma(f^{(1)}(v, t), v, t) \int_v^t \partial \sigma(f^{(1)}(v, y), v, y) \xi^{(n)}(v, y) dy \right] dv \\ & + \int_0^s \partial \sigma(f^{(1)}(v, t), v, t) \xi^{(n)}(v, t) dB(v) \\ & + \int_0^s \sigma(f^{(1)}(v, t), v, t) \dot{h}(v) dv, \end{aligned}$$

where the initial condition is given by  $\xi^{(0)}(s, t) = 0$ . Then we have the next result by using the standard method in stochastic analysis.

LEMMA A.3. For any  $p > 1$  and  $0 \leq s \leq t \leq \bar{T}$ ,

$$(A.18) \quad E[|\xi^{(n)}(s, t)|^p] < \infty$$

and, as  $n \rightarrow \infty$ ,

$$(A.19) \quad \sup_{0 \leq s \leq t} E \left[ \sup_{0 \leq u \leq s} |\xi^{(n+1)}(u, t) - \xi^{(n)}(u, t)|^2 \right] \rightarrow 0.$$

*Proof of Lemma A.3.* (i) We use the induction argument for  $n$ . We have (A.18) when  $n = 1$  because  $\sigma(\cdot)$  is bounded and  $\dot{h}(v)$  is a square-integrable function in (A.17). Suppose (A.18) hold for  $n = m$ . Then there exist positive constants  $M_i (i = 1, \dots, 4)$  such that

$$(A.20) \quad \begin{aligned} |\xi^{(m+1)}(s, t)|^p \leq & M_1 \int_0^s |\xi^{(m)}(v, t)|^p dv \\ & + M_2 \left[ \sup_{0 \leq u \leq s} \left| \int_0^u \xi^{(m)}(v, t) dB(v) \right| \right]^p \\ & + M_3 \int_0^s \int_v^t |\xi^{(m)}(v, y)|^p dy dv + M_4 \left[ \int_0^s |\dot{h}(v)|^2 dv \right]^{p/2}. \end{aligned}$$

By a martingale inequality (Theorem III-3.1 of Ikeda and Watanabe 1989), the expectation of the second term on the right-hand side of (A.20) is less than

$$(A.21) \quad M_2' E \left[ \left( \int_0^s |\xi^{(m)}(v, t)|^2 dv \right)^{p/2} \right] \leq M_2'' \int_0^s E[|\xi^{(m)}(v, t)|^p] dv,$$

where  $M_2'$  and  $M_2''$  are positive constants. Because  $\dot{h}(v)$  is square-integrable, we have (A.18) when  $n = m + 1$ .



(ii) From (6.17), there exist positive constants  $M_i (i = 5, 6, 7)$  such that, for  $0 \leq s \leq t$ ,

$$\begin{aligned}
 (A.22) \quad |\xi^{(n+1)}(s, t) - \xi^{(n)}(s, t)|^2 &\leq M_5 \left[ \int_0^s |\xi^{(n)}(v, t) - \xi^{(n-1)}(v, t)| dv \right]^2 \\
 &\quad + M_6 \left[ \int_0^s \int_v^t |\xi^{(n)}(v, y) - \xi^{(n-1)}(v, y)| dy dv \right]^2 \\
 &\quad + M_7 \left[ \int_0^s \partial \sigma(f^{(1)}(v, t), v, t) |\xi^{(n)}(v, t) \right. \\
 &\quad \left. - \xi^{(n-1)}(v, t)| dB(v) \right]^2 \\
 &\equiv \sum_{i=1}^3 I_i^{(n)}(s, t),
 \end{aligned}$$

where we have defined  $I_i^{(n)}(s, t)$  by the last equality. Using the Cauchy–Schwartz inequality,

$$(A.23) \quad E \left[ \sup_{0 \leq u \leq s} I_1^{(n)}(u, t) \right] \leq M_5 s \int_0^s E \left[ |\xi^{(n)}(v, t) - \xi^{(n-1)}(v, t)|^2 \right] dv.$$

By repeating the above argument to the second term of (A.22), we have

$$\begin{aligned}
 (A.24) \quad I_2^{(n)}(u, t) &\leq M_6 u \int_0^u \left[ \int_v^t |\xi^{(n)}(v, y) - \xi^{(n-1)}(v, y)| dy \right]^2 dv \\
 &\leq M_6 u t \int_0^u \int_v^t |\xi^{(n)}(v, y) - \xi^{(n-1)}(v, y)|^2 dy dv.
 \end{aligned}$$

Then

$$(A.25) \quad E \left[ \sup_{0 \leq u \leq s} I_2^{(n)}(u, t) \right] \leq M_6 s t \int_0^s \int_v^t E \left[ |\xi^{(n)}(v, y) - \xi^{(n-1)}(v, y)|^2 \right] dy dv.$$

For the third term of (A.22), we have

$$(A.26) \quad E \left[ \sup_{0 \leq u \leq s} I_3^{(n)}(u, t) \right] \leq M_7' \int_0^s E \left[ |\xi^{(n)}(v, t) - \xi^{(n-1)}(v, t)|^2 \right] dv$$

because of the boundedness of  $\partial \sigma(\cdot)$ , where  $M_7'$  is a positive constant. By using (A.24), (A.25), and (A.26), we have

$$\begin{aligned}
 (A.27) \quad E \left[ \sup_{0 \leq u \leq s} |\xi^{(n+1)}(u, t) - \xi^{(n)}(u, t)|^2 \right] \\
 \leq M_8 \left( \int_0^s E \left[ \sup_{0 \leq v \leq u} |\xi^{(n)}(v, t) - \xi^{(n-1)}(v, t)|^2 \right] du \right. \\
 \left. + \int_0^s \int_u^t E \left[ \sup_{0 \leq v \leq u} |\xi^{(n)}(v, y) - \xi^{(n-1)}(v, y)|^2 \right] dy du \right),
 \end{aligned}$$

where  $M_8$  is a positive constant. By defining a sequence of  $\{u^{(n)}(s, t)\}$  as

$$u^{(n+1)}(s, t) = E \left[ \sup_{0 \leq u \leq s} |\xi^{(n+1)}(u, t) - \xi^{(n)}(u, t)|^2 \right],$$

we have the relation

$$(A.28) \quad u^{(n+1)}(s, t) \leq M_8 \int_0^s \left[ \int_u^t u^{(n)}(u, y) dy + u^{(n)}(u, t) \right] du.$$

If we have an inequality

$$(A.29) \quad u^{(n+1)}(s, t) \leq \frac{1}{(n+1)!} [M_8(t+1)s]^{n+1}$$

for a positive constant  $M_8$ , we can show (A.19) as  $n \rightarrow +\infty$ . For this purpose, we use the induction argument for  $n \geq 1$ . When  $n = 1$ , there exists a positive constant  $M_8$  such that

$$(A.30) \quad \begin{aligned} u^{(1)}(s, t) &= E \left[ \sup_{0 \leq u \leq s} |\xi^{(1)}(u, t) - \xi^{(0)}(u, t)|^2 \right] \\ &= E \left[ \sup_{0 \leq u \leq s} \left| \int_0^u \sigma(f^{(1)}(v, t), v, t) \dot{h}(v) dv \right|^2 \right] \\ &\leq M_8(1+t)s \end{aligned}$$

because  $\xi^{(0)}(s, t) = 0$ ,  $\sigma(\cdot)$  is bounded, and  $\dot{h}(v)$  is square-integrable. Suppose (A.29) holds for  $m = n + 1$ . Then

$$(A.31) \quad \begin{aligned} u^{(m+1)}(s, t) &\leq M_8 \int_0^s \left[ \int_u^t u^{(m)}(u, y) dy + u^{(m)}(u, t) \right] du \\ &\leq M_8 \int_0^s \left[ \int_u^t M_8^m (y+1)^m \frac{u^m}{m!} dy + M_8^m (t+1)^m \frac{u^m}{m!} \right] du \\ &\leq M_8^{m+1} (t+1)^{m+1} \frac{s^{m+1}}{(m+1)!}. \end{aligned} \quad \square$$

By using (A.29) and the Chebyshev's inequality, we have

$$(A.32) \quad \begin{aligned} \sum_{n=1}^{\infty} P \left\{ \sup_{0 \leq u \leq s \leq t} |\xi^{(n+1)}(u, t) - \xi^{(n)}(u, t)| > \frac{1}{2^n} \right\} \\ \leq \sum_{n=1}^{\infty} \frac{1}{n!} [4M_8(\bar{T}+1)\bar{T}]^n < +\infty \end{aligned}$$

uniformly for any  $0 \leq s \leq t \leq \bar{T}$ . Then by the Borel–Cantelli lemma, the sequence of random variables  $\{\xi^{(n)}(u, t)\}$  converges uniformly on  $0 \leq u \leq s \leq t (\leq \bar{T})$ . Hence

we can establish the existence of the  $H$ -derivative of  $f^{(1)}(s, t)$ , which is given by the solution of the stochastic integral equation:

$$(A.33) \quad D_h f^{(1)}(s, t) = \int_0^s \left[ \partial \sigma(f^{(1)}(v, t), v, t) \int_v^t \sigma(f^{(1)}(v, y), v, y) dy D_h f^{(1)}(v, t) \right] dv \\ + \int_0^s \left[ \sigma(f^{(1)}(v, t), v, t) \int_v^t \partial \sigma(f^{(1)}(v, y), v, y) D_h f^{(1)}(v, y) dy \right] dv \\ + \int_0^s \partial \sigma(f^{(1)}(v, t), v, t) D_h f^{(1)}(v, t) dB(v) \\ + \int_0^s \sigma(f^{(1)}(v, t), v, t) \dot{h}(v) dv.$$

We note that for the spot rate process the  $H$ -derivative can be well defined by

$$(A.34) \quad D_h r^{(1)}(t) = \lim_{s \rightarrow t} D_h f^{(1)}(s, t).$$

Now we define the random variables  $\{\xi_{s,t}(u)\}$  for  $0 \leq u \leq s \leq t$  by

$$(A.35) \quad \xi_{s,t}(u) = \int_u^s \left[ \partial \sigma(f^{(1)}(v, t), v, t) \int_v^t \sigma(f^{(1)}(v, y), v, y) dy \xi_{v,t}(u) \right] dv \\ + \int_u^s \left[ \sigma(f^{(1)}(v, t), v, t) \int_v^t \partial \sigma(f^{(1)}(v, y), v, y) \xi_{v,y}(u) dy \right] dv \\ + \int_u^s \partial \sigma(f^{(1)}(v, t), v, t) \xi_{v,t}(u) dB(v) \\ + \sigma(f^{(1)}(u, t), u, t).$$

Then we can show that

$$(A.36) \quad \int_0^s \xi_{s,t}(u) \dot{h}(u) du = D_h f^{(1)}(s, t).$$

The rigorous proof for the existence of random variables  $\{D_h f^{(1)}(s, t)\}$  and  $\{\xi_{s,t}(u)\}$  can be given by extending the recursive method used in the proof of Proposition 10.1 in Ikeda and Watanabe (1989). Here we need to approximate the solution of the stochastic integral equations (A.33) and (A.35) in two arguments  $s$  and  $t$  by a sequence of solutions with step coefficients. Since it is straightforward but quite lengthy to do this, we omit the details.

Next, we examine the existence of higher order moments of  $\{\xi_{s,t}(u)\}$  satisfying (A.35). To do this, we prepare the following inequality.

**LEMMA A.4.** Suppose for  $k_0 \geq 0, k_1 > 0, A_N > 0$ , and  $0 < u \leq s \leq t \leq \bar{T}$ , a function  $w_N(u, s, t)$  satisfies (i)  $0 \leq w_N(u, s, t) \leq A_N$  and (ii)

$$(A.37) \quad w_N(u, s, t) \leq k_0 + k_1 \left[ \int_u^s w_N(u, v, t) dv + \int_u^s \int_v^t w_N(u, v, y) dy dv \right].$$

Then

$$(A.38) \quad w_N(u, s, t) \leq k_0 e^{k_1(1+t)s}.$$

*Proof of Lemma A.4.* By substituting (i) into the right-hand side of (A.37), we have

$$(A.39) \quad \begin{aligned} w_N(u, s, t) &\leq k_0 + A_N k_1 \left[ \int_u^s dv + \int_u^s \int_v^t dy dv \right] \\ &\leq k_0 + A_N k_1 (1+t)s. \end{aligned}$$

By repeating the substitution of (A.39) into the right-hand side of (A.37), we have

$$(A.40) \quad w_N(u, s, t) \leq k_0 \sum_{k=0}^n \frac{1}{k!} [k_1(1+t)s]^k + \frac{1}{(n+1)!} A_N [k_1(1+t)s]^{n+1}.$$

Then we have (A.38) by taking  $n \rightarrow +\infty$ .  $\square$

In order to use Lemma A.4, we consider the truncated random variable

$$(A.41) \quad \zeta_{s,t}^N(u) = [\xi_{s,t}(u)] I_N(s, t),$$

where  $I_N(s, t) = 1$  if

$$\sup_{0 \leq u \leq v \leq s, v \leq y \leq t, s \leq t} |\xi_{v,y}(u)| \leq N$$

and  $I_N(s, t) = 0$  otherwise. By using the boundedness conditions in Assumption I and the fact that  $\dot{h}(s)$  is square integrable, we can show that there exist positive constants  $M_i (i = 9, \dots, 12)$  such that for any  $p > 1$

$$(A.42) \quad \begin{aligned} |\zeta_{s,t}^N(u)|^p &\leq M_9 \int_u^s |\zeta_{v,t}^N(u)|^p dv + M_{10} \left| \int_u^s \zeta_{v,t}^N(u) dB(v) \right|^p \\ &\quad + M_{11} \int_u^s \int_v^t |\zeta_{v,y}^N(u)|^p dy dv + M_{12} |\sigma(f^{(1)}(u, t), u, t)|^p \\ &\equiv \sum_{i=1}^4 J_i^N(u, s, t), \end{aligned}$$

where we have defined  $J_i^N(u, s, t) (i = 1, \dots, 4)$  by the last equality. By using a martingale inequality, we have

$$(A.43) \quad \begin{aligned} E[J_2^N(u, s, t)] &\leq M'_{10} E \left[ \int_u^s |\zeta_{v,t}^N(u)|^2 dv \right]^{p/2} \\ &\leq M''_{10} E \left[ \int_u^s |\zeta_{v,t}^N(u)|^p dv \right], \end{aligned}$$

where  $M'_{10}$  and  $M''_{10}$  are positive constants. Also  $J_4^N(u, s, t)$  is bounded because  $\sigma(\cdot)$  is bounded. If we set  $w_N(u, s, t) = E[|\zeta_{s,t}^N(u)|^p]$ , then we can directly apply Lemma A.4. By taking the limit of the expectation function  $w_N(u, s, t)$  as  $N \rightarrow \infty$ , we have the following result.

**LEMMA A.5.** For any  $p > 1$  and  $0 \leq u \leq s \leq t \leq \bar{T}$ ,

$$(A.44) \quad E[|\xi_{s,t}(u)|^p] < +\infty.$$

Through this lemma and the equivalence of two norms stated in Step 1, we can establish that

$$(A.45) \quad f^{(1)}(s, t) \in \cap_{1 < p < +\infty} D_p^1(\mathbf{R}).$$

Then, by repeating the above procedure,<sup>10</sup> we can derive the higher order  $H$ -derivatives of  $f^{(1)}(s, t)$ . Hence we can obtain the following result.

**THEOREM A.2.** *Suppose Assumption I in Section 3 holds for the forward rate processes. Then for  $0 < s \leq t \leq \bar{T}$ ,*

$$(A.46) \quad f^{(1)}(s, t) \in D^\infty(\mathbf{R}).$$

Obviously, by a method similar to (A.33) and (A.34) we can establish the existence of the  $H$ -derivative of  $\{f^{(\varepsilon)}(s, t)\}$  and  $\{r^{(\varepsilon)}(t)\}$  with  $0 < \varepsilon \leq 1$ . Then we have the corresponding results on  $\{f^{(\varepsilon)}(s, t)\}$ . By applying the successive substitution and using the boundedness and smoothness conditions in Assumption I, we have the mathematical validity on our formal derivation of Theorem 3.1 in Section 3.

**[Step 3]:** Let a stochastic process  $\{Y^{(\varepsilon)}(s, t), 0 \leq s \leq t \leq T\}$  be the solution of the stochastic integral equation:

$$(A.47) \quad Y^{(\varepsilon)}(s, t) = 1 + \varepsilon^2 \int_0^s \left[ \partial \sigma(f^{(\varepsilon)}(v, t), v, t) \int_v^t \sigma(f^{(\varepsilon)}(v, y), v, y) dy \right] \\ \times Y^{(\varepsilon)}(v, t) dv + \varepsilon \int_0^s \partial \sigma(f^{(\varepsilon)}(v, t), v, t) Y^{(\varepsilon)}(v, t) dB(v).$$

Since the coefficients of  $Y^{(\varepsilon)}(s, t)$  on the right-hand side of (A.47) are bounded by Assumption I, we can obtain the following lemma.

**LEMMA A.6.** *For any  $1 < p < +\infty$ ,  $0 < \varepsilon \leq 1$ , and  $0 < s \leq t \leq \bar{T}$ ,*

$$(A.48) \quad E[|Y^{(\varepsilon)}(s, t)|^p] + E[|Y^{(\varepsilon)-1}(s, t)|^p] < +\infty.$$

*Proof of Lemma A.6.* We define a sequence of random variables  $\{Y_n^{(\varepsilon)}(s, t)\}$  as

$$(A.49) \quad Y_{n+1}^{(\varepsilon)}(s, t) = 1 + \varepsilon^2 \int_0^s \left[ \partial \sigma(f^{(\varepsilon)}(v, t), v, t) \int_v^t \sigma(f^{(\varepsilon)}(v, y), v, y) dy \right] \\ \times Y_n^{(\varepsilon)}(v, t) dv + \varepsilon \int_0^s \partial \sigma(f^{(\varepsilon)}(v, t), v, t) Y_n^{(\varepsilon)}(v, t) dB(v),$$

where the initial condition is given by  $Y_0^{(\varepsilon)}(s, t) = 1$ . Then, by the same argument used in the proof of Lemma A.3, we have

$$(A.50) \quad E[|Y_n^{(\varepsilon)}(s, t)|^p] < \infty$$

<sup>10</sup> To be more rigorous mathematically, we have to use Lemma 2.1 of Kusuoka and Stroock (1982) and Lemma 7 of Yoshida (1997) for obtaining higher order derivatives. See Kunitomo and Takahashi (1998) for the details.

and, as  $n \rightarrow \infty$ ,

$$(A.51) \quad \sup_{0 \leq s \leq t \leq \bar{T}} E \left[ \sup_{0 \leq u \leq s} |Y_{n+1}^{(\varepsilon)}(u, t) - Y_n^{(\varepsilon)}(u, t)|^2 \right] \rightarrow 0.$$

Hence we can establish the existence of the random variables  $\{Y^{(\varepsilon)}(s, t)\}$  satisfying (A.47). Then by the same argument as in (A.41)–(A.44), we have

$$(A.52) \quad E[|Y^{(\varepsilon)}(s, t)|^p] < \infty$$

for any  $p > 1$ . Let  $Z^{(\varepsilon)}(s, t) = Y^{(\varepsilon)-1}(s, t)$ . Then we can show that

$$(A.53) \quad d[Z^{(\varepsilon)}(s, t)Y^{(\varepsilon)}(s, t)] = 0$$

and

$$(A.54) \quad Z^{(\varepsilon)}(s, t) = 1 - \varepsilon^2 \int_0^s \left[ \partial \sigma(f^{(\varepsilon)}(v, t), v, t) \int_v^t \sigma(f^{(\varepsilon)}(v, y), v, y) dy \right] \\ \times Z^{(\varepsilon)}(v, t) dv - \varepsilon \int_0^s \partial \sigma(f^{(\varepsilon)}(v, t), v, t) Z^{(\varepsilon)}(v, t) dB(v)$$

by using Itô's Lemma and  $Z^{(\varepsilon)}(0, t) = 1$ . Hence, by an argument similar to  $Y^{(\varepsilon)}(s, t)$ , we can establish that

$$(A.55) \quad E[|Z^{(\varepsilon)}(s, t)|^p] < \infty$$

for any  $p > 1$ . □

Now consider the asymptotic behavior of a functional

$$(A.56) \quad F^{(\varepsilon)}(s, t) = \frac{1}{\varepsilon} [f^{(\varepsilon)}(s, t) - f^{(0)}(0, t)]$$

as  $\varepsilon \rightarrow 0$ . By using the stochastic process  $\{Y^{(\varepsilon)}(s, t)\}$ , the  $H$ -derivative of  $F^{(\varepsilon)}(s, t)$  can be represented as

$$(A.57) \quad D_h F^{(\varepsilon)}(s, t) = \int_0^s Y^{(\varepsilon)}(s, t) Y^{(\varepsilon)-1}(v, t) C^{(\varepsilon)}(v, t) dv,$$

where

$$(A.58) \quad C^{(\varepsilon)}(v, t) = \sigma(f^{(\varepsilon)}(v, t), v, t) \dot{h}(v) + \varepsilon \sigma(f^{(\varepsilon)}(v, t), v, t) \\ \times \int_v^t \partial \sigma(f^{(\varepsilon)}(v, y), v, y) D_h f^{(\varepsilon)}(v, y) dy.$$

Then by rearranging each term in the integrands of (A.57), we have the representation

$$(A.59) \quad D_h F^{(\varepsilon)}(s, t) = \int_0^s v_{s,t}^{(\varepsilon)}(u) \dot{h}(u) du,$$

where

$$(A.60) \quad \begin{aligned} v_{s,t}^{(\varepsilon)}(u) &= Y^{(\varepsilon)}(s, t) Y^{(\varepsilon)-1}(u, t) \sigma(f^{(\varepsilon)}(u, t), u, t) \\ &\quad + \varepsilon \int_u^s Y^{(\varepsilon)}(s, t) Y^{(\varepsilon)-1}(v, t) \sigma(f^{(\varepsilon)}(v, t), v, t) \\ &\quad \times \left( \int_v^t \partial \sigma(f^{(\varepsilon)}(v, y), v, y) \xi_{v,y}^{(\varepsilon)}(u) dy \right) dv, \end{aligned}$$

and the random variable  $\{\xi_{v,y}^{(\varepsilon)}(u)\}$  is defined as (A.35) with  $\varepsilon$ . Then the Malliavin covariance for  $F^{(\varepsilon)}(s, t)$  is given by

$$(A.61) \quad \begin{aligned} \sigma(F^{(\varepsilon)}(s, t)) &= |DF^{(\varepsilon)}(s, t)|_{H \otimes R}^2 \\ &= \int_0^s [v_{s,t}^{(\varepsilon)}(u)]^2 du. \end{aligned}$$

Let

$$(A.62) \quad \begin{aligned} \eta_c^{(\varepsilon)}(s, t) &= c \int_0^s |\varepsilon \left( \int_u^s Y^{(\varepsilon)}(s, t) Y^{(\varepsilon)-1}(v, t) \sigma(f^{(\varepsilon)}(v, t), v, t) \right. \\ &\quad \times \left. \int_v^t \partial \sigma(f^{(\varepsilon)}(v, y), v, y) \xi_{v,y}^{(\varepsilon)}(u) dy dv \right)|^2 du \\ &\quad + c \int_0^s |Y^{(\varepsilon)}(s, t) Y^{(\varepsilon)-1}(u, t) \sigma(f^{(\varepsilon)}(u, t), u, t) \\ &\quad - \sigma(f^{(0)}(u, t), u, t)|^2 du \end{aligned}$$

for a positive constant  $c$ . Then the condition in Assumption II in Section 3 is equivalent to the nondegeneracy condition:

$$(A.63) \quad \Sigma(s, t) = \int_0^s \sigma(f^{(0)}(v, t), v, t)^2 dv > 0$$

because  $Y^{(0)}(v, t) = 1$  for  $0 < v \leq s \leq t$ . The next lemma shows that the truncation by  $\eta_c^{(\varepsilon)}(s, t)$  is negligible in probability.

LEMMA A.7. For  $0 < s \leq t \leq \bar{T}$  and any  $n \geq 1$ ,

$$(A.64) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{-n} P\{|\eta_c^{(\varepsilon)}(s, t)| > \frac{1}{2}\} = 0.$$

*Proof of Lemma A.7.* We rewrite (A.62) as  $\eta_c^{(\varepsilon)}(s, t) = \eta_1^{(\varepsilon)} + \eta_2^{(\varepsilon)}$ . By using Assumption I, Lemma A.5, Lemma A.6, and the Markov inequality, it is straightforward to show that for any  $p > 1$  and  $c_1 > 0$  there exists a positive constant  $c_2$  such that

$$(A.65) \quad P\{|\eta_1^{(\varepsilon)}| > c_1\} \leq c_2 \varepsilon^{2p}.$$

By the continuity of the volatility function  $\sigma(\cdot)$ , there exist positive constants  $M_{13}$  and  $M_{14}$  such that

$$(A.66) \quad |\eta_2^{(\varepsilon)}| \leq M_{13} |f^{(\varepsilon)}(s, t) - f^{(0)}(0, t)| + M_{14} |Y^{(\varepsilon)}(s, t) Y^{(\varepsilon)-1}(v, t) - 1|.$$

Then by Lemma 10.5 of Ikeda and Watanabe (1989), for a positive  $c_3$  and sufficiently small  $\varepsilon > 0$ , there exist positive constants  $c_4$  and  $c_5$  such that

$$(A.67) \quad P \left\{ \sup_{0 \leq s \leq t \leq T} |f^{(\varepsilon)}(s, t) - f^{(0)}(0, t)| > c_3 \right\} \leq c_4 \exp(-c_5 \varepsilon^{-2}).$$

We rewrite the second term on the right-hand side of (A.66) for  $\eta_2^{(\varepsilon)}$ ,

$$(A.68) \quad \eta_{22}^{(\varepsilon)} = M_{14} Y^{(\varepsilon)}(v, t)^{-1} |Y^{(\varepsilon)}(s, t) - Y^{(\varepsilon)}(v, t)|,$$

where

$$(A.69) \quad \begin{aligned} Y^{(\varepsilon)}(s, t) - Y^{(\varepsilon)}(v, t) &= \varepsilon^2 \int_v^s \left[ \partial \sigma(f^{(\varepsilon)}(u, t), u, t) \int_u^t \sigma(f^{(\varepsilon)}(u, y), u, y) dy \right] Y^{(\varepsilon)}(u, t) du \\ &\quad + \varepsilon \int_v^s \partial \sigma(f^{(\varepsilon)}(u, t), u, t) Y^{(\varepsilon)}(u, t) dB(u). \end{aligned}$$

Then by Lemma A.6, for any  $p > 1$  and  $c_6 > 0$  there exists a positive constant  $c_7$  such that

$$(A.70) \quad P\{|\eta_{22}^{(\varepsilon)}| > c_6\} \leq c_7 \varepsilon^{2p}.$$

By using (A.65), (A.67), and (A.70), we have (A.64).  $\square$

Thus by modifying the method developed by Yoshida (1992a) for the present case, we have the key result on the validity of the asymptotic expansion approach in this paper.

**THEOREM A.3.** *Under Assumptions I and II in Section 3, the Malliavin covariance  $\sigma(F^{(\varepsilon)})$  of  $F^{(\varepsilon)}$  is uniformly nondegenerate in the sense that there exists  $c_0 > 0$  such that for any  $c > c_0$  and any  $p > 1$ ,*

$$(A.71) \quad \sup_{\varepsilon} E \left[ I(|\eta_c^{(\varepsilon)}| \leq 1) \sigma(F^{(\varepsilon)})^{-p} \right] < +\infty,$$

where  $I(\cdot)$  is the indicator function.

Hence we have obtained a truncated version of the nondegeneracy condition on the Malliavin covariance for the spot interest rate and forward rates processes, which are the solutions of the stochastic integral equation (3.1).

It is also straightforward to obtain the truncated version of the nondegeneracy condition on the Malliavin covariance under the assumption  $\Sigma > 0$  for the discount coupon bond price process and the average interest rate process as we stated in Theorem 3.2. This is because under Assumption I in Section 3, we can show that for any  $p > 1$  and  $0 \leq s \leq t \leq T$ ,

$$(A.72) \quad E[|P^{(\varepsilon)}(s, t)|^p] < +\infty.$$

Then by applying the same methods to the bond processes as for the forward and spot interest rate processes, we can obtain the corresponding results under Assumption I and



$\Sigma > 0$  in (3.33). Then we have the results as Theorem A.2 and Theorem A.3 for the coupon bond process and the average interest rate.

The rest of our arguments for the asymptotic expansion approach is based on Theorem 4.1 of Yoshida (1992a), which is an extension of Theorem 2.3 of Watanabe (1987), because it gives the validity of the asymptotic expansion of the distribution function of functionals with truncation under the nondegeneracy condition on the Malliavin covariance given by (A.71). Let  $\psi : \mathbf{R} \rightarrow \mathbf{R}$  be a smooth function such that  $0 \leq \psi(x) \leq 1$ ,  $\psi(x) = 1$  for  $|x| \leq \frac{1}{2}$ , and  $\psi(x) = 0$  for  $|x| \geq 1$ . Then the composite functional  $\psi(\eta^{(\varepsilon)})I_A(F^{(\varepsilon)})$  is well defined for any  $A \in \mathbf{B}$  in the sense that it is in  $\tilde{\mathbf{D}}^{-\infty}$ , where  $\mathbf{B}$  is the Borel  $\sigma$ -field in  $\mathbf{R}$  and  $I_A(\cdot)$  is the indicator function. By using Theorem A.3, lemmas in this Appendix and Theorem 4.1 of Yoshida (1992a) it has a proper asymptotic expansion as  $\varepsilon \rightarrow 0$  uniformly in  $\tilde{\mathbf{D}}^{-\infty}$ . Then we have a proper asymptotic expansion for the density function of our interest by taking the expectation operations. By applying the same arguments to the bond processes, we finally have Theorems 3.2 and 3.3. The details of our derivations and proofs omitted here are given in Kunitomo and Takahashi (1998).

At the end of this step, we should mention that the integrals in (3.46) depend on the parameter  $\varepsilon$ . Since Yoshida (1992b) has given Theorem 2.2 and Lemma 2.1 for the validity of asymptotic expansions when the bounded integral operators depend on a parameter, his arguments cover the situation of our Theorem 3.3. Thus our formal derivations in Section 3 can be rigorously justified by applying the results of Yoshida (1992a, 1992b).

**[Step 4]:** The inversion technique we use is different from that used by Yoshida (1992a, 1992b). He used the Schwartz's type distribution theory for the generalized Wiener functionals. Our method is based on the simple inversion technique for the characteristic functions of random variables, which has been standard in the statistical asymptotic theory. Hence what we need to show is that the formulas resulting from our method are equivalent to his final formulas. In the notations of Yoshida (1992a), we take  $\varphi(x) = 1$  in his Lemma 5.6. Then he used

$$(A.73) \quad p'_1(x) = (-1)^k \frac{\partial^k}{\partial x_1 \cdots \partial x_k} \mathbf{E}[\varphi(f_0) f_1^i \partial_i I_A(f_0)]$$

and

$$(A.74) \quad p''_1(x) = (-1)^k \frac{\partial^k}{\partial x_1 \cdots \partial x_k} \mathbf{E}[\{f_1^i \partial_i \varphi(f_0)\} I_A(f_0)],$$

where  $I_A(f_0)$  is the indicator function and  $f_0$  corresponds to the random variable of the order  $O_p(1)$ , which is similar to  $g_1$  in our notation. The differentiation of the indicator function has some proper mathematical meaning in the sense of differentiation on a generalized Wiener functionals. By the use of the pullback operation of the generalized Wiener functionals, Yoshida (1992a) obtained the explicit expansion form of the density function for a particular functional in his problem as

$$(A.75) \quad p_1(x) = p'_1(x) + p''_1(x).$$

In our framework it is straightforward to show that

$$(A.76) \quad p'_1(x) = (-1) \frac{d}{dx} [\mathbf{E}(g_2 | g_1 = x) \phi_\Sigma(x)]$$

and  $p_1''(x) = 0$  since  $\partial_i \varphi(\cdot) = 0$  when  $k = 1$  by using our notations in this paper. Then we notice that (A.76) is exactly what the inversion formula (Lemma A.2) gives as the second-order term in the asymptotic expansion of the density function of the normalized random variable  $X_T^{(e)}$  in (3.31). The formal proof for this equivalence of two methods in more general cases is given in Kunitomo and Takahashi (1998, Sec. 3).

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