Hermite polynomial based expansion of European option prices

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ABSTRACT

We seek a closed-form series approximation of European option prices under a variety of diffusion models. The proposed convergent series are derived using the Hermite polynomial approach. Departing from the usual option pricing routine in the literature, our model assumptions have no requirements for affine dynamics or explicit characteristic functions. Moreover, convergent expansions provide a distinct insight into how and on which order the model parameters affect option prices, in contrast with small-time asymptotic expansions in the literature. With closed-form expansions, we explicitly translate model features into option prices, such as mean-reverting drift and self-exciting or skewed jumps. Numerical examples illustrate the accuracy of this approach and its advantage over alternative expansion methods.

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1. Introduction

How are the drift, volatility and jump components of underlying risk-neutral dynamics translated into option prices? On which order does each component matter as options approach expiration? Is it possible to separate their first-order contributions to option prices? Are there economic insights behind the impact of each component? These questions are important not only for financial engineers who are trying to construct effective models that better capture market movement, but are also relevant to market participants who are hedging the risk of their portfolios with contingent claims. This paper addresses these questions by seeking closed-form series expansions of option prices, which fill the gap between closed-form solutions and numerical methods of option valuation.

Even for one-factor continuous-time dynamic models, closed-form option pricing formulae are very rare. Most models with closed-form solutions are either confined to the log-normal class (including the famous Black–Scholes–Merton formula by Black and Scholes (1973) and Merton (1976)), or the Bessel process class (e.g. CIR and CEV models discussed in Cox (1975), Cox et al. (1985) and Cox and Ross (1976)), or can be reduced to these classes

after transformation (see Goldenberg, 1991). These formulae, most of which were derived decades ago, are still playing important roles in the financial industry today. Recently, the Fourier-transform-based approach has evolved rapidly, substantially enlarging the class of models that have closed-form option pricing formulae. Explicit formulae have been derived via Fourier Transform for particular diffusions with stochastic volatility (Heston, 1993) and jumps (Bates, 1996), along with stochastic interest rates (Scott, 1997). Nevertheless, the Fourier-transform-based approach relies on explicit characteristic functions, naturally arising from Lévy–Khintchine representations, or on closed-form solutions to ordinary differential systems for affine jump diffusion models (Duffie et al., 2000). In spite of efficient algorithm developed by Carr and Madan (1998), such an approach could be computationally intensive if the ordinary differential equations have no closed-form solutions. To tackle option-valuation problems that do not have closed-form solutions, numerical methods of solving partial differential equations or simulation-based methods are often developed. Unfortunately, when trying to estimate or calibrate these models, numerical methods are computationally expensive because optimization on top of these pricing algorithms is extremely time-consuming. In such circumstances, a closed-form expansion formula may simplify the task to a great extent.

In this paper, we complement the literature on option pricing by providing a closed-form expansion for European-type option prices. Unlike numerical methods, the expansion approach offers a number of advantages. Instead of simply calculating a number, it gives more insight into how parameters influence prices and to

what extent, which even closed-form solutions cannot offer. Furthermore, expansion formulae are smooth, so that differentiation becomes equally tractable, hence permitting real-time calibration and hedging. Comparative statics results are also straightforward to derive with closed-form formulae. Moreover, the expansions significantly improve the efficiency in computations when bringing models to data. The proposed method works for one-factor diffusion models, potentially inhomogeneous with jumps, and certain multivariate models, and allows nonlinear dynamics, which may not have explicit characteristic functions.

In the same spirit, Kimmel (2009) obtains explicit analytic series for bond prices under diffusion models. Nevertheless, such an explicit expansion formula for option prices is more cumbersome to obtain, due to the non-smoothness of the payoff function at maturity. Recently, Kristensen and Mele (2011) propose an approach to approximating the option price by expanding the difference between the true model price and the Black–Scholes price. Their approach avoids the singularity with the help of an auxiliary closed-form pricing formula. Nevertheless, this expansion is less informative regarding how option prices are determined near expiration. In contrast, we propose a series expansion with a well-designed initial term that suffices to capture this singularity and the remaining part is approximated by power series. As a result, the structure of option prices becomes transparent. A similar strategy has been successfully applied to transition density or likelihood expansions for homogeneous diffusions (Aït-Sahalia, 2002, 2008, 1999), for jump diffusions (Yu, 2007), and inhomogeneous diffusions (Egorov et al., 2003). The advantage of series expansion over alternative simulation-based approaches has been documented in Jensen and Poulsen (2002) and Hurn et al. (2007). Related work also includes Aït-Sahalia and Kimmel (2010, 2007), Bakshi and Ju (2005), and Bakshi et al. (2006).

Parallel to these convergent series, for one-factor models Henry-Labordere (2005); Gatheral et al. (2012) obtain asymptotic expansions for short-term option prices. This type of expansion may offer decent approximations with only one or two terms (see e.g., Olver, 1974). However, their approximations deteriorate rapidly as time-to-maturity drifts away from zero or as more terms are included into the asymptotic expansions. These asymptotic expansions' leading term equals exactly the option payoff function, which is non-smooth. The remaining terms are also non-smooth. Hence, the smooth approximation for option prices and greeks cannot be provided by a finite number of terms. Partly for this reason, most literature targets implied volatility or transition densities, which permit smooth expansions, such as those discussed in Hagan and Woodward (1999), Hagan et al. (2002), Fouque et al. (2000), Lewis (2000), Berestycki et al. (2002), Medvedev and Scaillet (2007), Henry-Labordere (2008), Takahashi et al. (2009), Gatheral et al. (2012), Li (2013a, 2013b), Lorig et al. (2013), and Forde et al. (2013). In contrast, this paper proposes smooth and uniform convergent series approximations for option prices, which can offer insight and intuitions pertaining to some option pricing models. These series are also applicable to statistical inference, as the consistency of parametric estimation requires uniform approximations.

To demonstrate the economic value of this approach, we provide closed-form expansions for a variety of models in asset pricing. These examples are designed to study some specific features in the underlying dynamics, such as mean-reverting drift, and self-exciting or skewed jumps. The closed-form expansions make the structure of options written on these models transparent, shedding light on how these characteristics are translated into option prices and what the magnitude of their effects are. This kind of intuition cannot be obtained via general asymptotic expansions, as the latter fail to converge when more terms are added. This particular advantage distinguishes the current approach from the existing short-time asymptotic expansions.

The paper is organized as follows. Section 2 introduces two approaches to closed-form expansions for binary options. Section 3 discusses vanilla option pricing based on jump diffusion models and certain multivariate models. Section 4 provides several examples to show how to translate some features of underlying dynamics into option prices. Section 5 benchmarks this approach against alternative expansion methods for implied volatility. Section 6 concludes. The appendix contains extensions and the mathematical proofs.

2. Closed-form expansion of option prices

A typical European-style claim offers the holder the right but not the obligation to either buy (Call) or sell (Put) a predetermined contingent payoff at maturity. For example, vanilla call options allow the holder to purchase the underlying security at the strike price if the exercise value of the underlying settles above it at expiration. As recent financial innovation is spanning the market, more exotic options are created and flourished, such as binary call options, whose exercise values are either some fixed amount of cash (cash-or-nothing) or the value of the underlying security itself (asset-or-nothing), if the option expires in-the-money. Unlike vanilla options, most binary options were only traded over-the-counter before June 2008, when the CBOE started offering continuous quotations of the standardized exchange-traded cash-or-nothing binary options on the S&P 500 index and its implied volatility (VIX). In the following, we discuss explicit expansions for both vanilla and binary options.

We start with a simple case in which the underlying security X_t follows a scalar diffusion process and the interest rate is a fixed constant r. Suppose there exists an equivalent martingale measure Q under which the dynamics of X_t satisfies:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t^Q. \tag{1}$$

If X_t itself is a tradable asset, then $\mu(X_t) = rX_t$, but we do not need to impose this constraint here.

As is well known, European option price $\Psi(\Delta, x)$ with maturity T and strike K satisfies the Feynman–Kac partial differential equation,

$$-\frac{\partial\Psi(\Delta, x)}{\partial\Delta} + \mu(x)\frac{\partial\Psi(\Delta, x)}{\partial x}$$
$$+\frac{1}{2}\sigma^2(x)\frac{\partial^2\Psi(\Delta, x)}{\partial x^2} - r\Psi(\Delta, x) = 0, \tag{2}$$

with the initial condition determined by the contingent payoff $f(x)$ at expiration:

$$\Psi(0, x) = f(x)1_{\{x>K\}}, \tag{3}$$

where $\Delta = T - t$ denotes the time to maturity and x is the price level at time t.

Equivalently, the option price $\Psi(\Delta, x)$ can also be written as the discounted expectation of future payoff under a risk-neutral measure:

$$\Psi(\Delta, x) = e^{-r\Delta}E^Q(f(X_T)1_{\{X_T>K\}}|X_t = x)$$
$$= e^{-r\Delta}\int_K^\infty f(z)p_X(\Delta, z|x)dz, \tag{4}$$

where $p_X(\Delta, z|x)$ denotes the state price density or transition density, which also satisfies (2).

Our goal is to develop a closed-form series expansion of the option prices in the time variable. Such expansions are able to demonstrate which factors are more important to option prices as time approaches maturity. A straightforward approach is to

directly postulate an appropriate series and verify it by plugging it into the equation. For example, a Taylor series around $\Delta = 0$ is able to solve a similar equation for zero-coupon bond, as shown by Kimmel (2009). Unlike pricing a zero-coupon bond, the "hockey stick" option payoff introduces singularity in the initial condition, which precludes the possibility of analytic expansion for the option price. Nevertheless, it is possible to propose an irregular leading term that suffices to "absorb" the singularity, leaving the remaining part approximated by power series. We can imagine that there may be too many candidates for the leading term. Without any theoretical guidance, seeking the right one (provided it is unique) amounts to finding a needle in a haystack. However, it turns out that we can adopt a bottom-up strategy to build the entire expansion from scratch using the Hermite polynomials.

2.1. The Hermite polynomials approach

To introduce this method, we start with expanding the price of a cash-or-nothing binary call option with $f(x) = 1$. In fact, it is clear from Eq. (4) that, barring from the time discounting, the option price is equal to the probability of expiring in-the-money. Hence expanding it is equivalent to approximating the cumulative transition distribution function. Aït-Sahalia (2002) discusses the transition density expansion using the Hermite polynomials, which is relevant to the current problem, as these two problems amount to solving the same partial differential equation with different initial conditions. The initial value of the transition density at $\Delta = 0$ is a δ-function, or Dirac mass, whereas for the distribution function, the initial value is a step function, or Heaviside function. Our strategy is to begin with expanding the transition density, and then integrate it to obtain the expansion for the distribution function.

We follow steps similar to Aït-Sahalia (2002). First, we transform the underlying process X to Y, whose transition density becomes "closer" to the density of normal distribution. Next, we perform another transformation from Y to Z, which is sufficiently "close" to a $N(0, 1)$ variable. We then find a convergent density expansion for Z with standard normal density serving as a leading term. Further, we obtain the density expansion for X by applying the Jacobian formula, and finally integrate it over the region $x > K$ to calculate the binary option price.

The above strategy does not require X to be positive. For example, X can be interpreted as the log price, in which case K should be regarded as the log strike. We denote the domain of X_t as D_X. Most financial models in practice have $D_X = (0, \infty)$, or $(-\infty, \infty)$.

To fix ideas, we change the variable X to Y so that the diffusion term of Y is standardized:

$$Y_t = \gamma(X_t) = \int^{X_t} \frac{1}{\sigma(s)} ds.$$

The lower bound of the integration does not play a role, and hence is omitted here. It is straightforward to find that Y_t satisfies

$$dY_t = \mu_Y(Y_t)dt + dW_t^Q,$$

where $\mu_Y(y) = \frac{\mu(\gamma^{-1}(y))}{\sigma(\gamma^{-1}(y))} - \frac{1}{2}\frac{d\sigma(\gamma^{-1}(y))}{dx}$.

For any given $\Delta > 0$ and any given y in D_Y, the domain of Y, we make the second transformation:

$$Z_{t+\Delta} = \frac{Y_{t+\Delta} - y}{\sqrt{\Delta}}.$$

The above transformation makes it possible to approximate the density of Z given $Y_t = y$, to the Jth order in the following way:

$$p_Z^{(J)}(\Delta, z|y) = \phi(z)\sum_{j=0}^{J} \eta_j(\Delta, y)H_j(z),$$

where $\phi(z)$ is the standard normal density, and $H_j(z)$s are the Hermite polynomials satisfying: $H_0(z) = 1$, and for any $j \geq 1$,

$$H_j(z) = \phi(z)^{-1}\frac{d^j}{dz^j}\phi(z).$$

It is natural to come up with Hermite polynomials, as they are constructed using the standard normal density function, orthogonal to each other, and hence qualified to be a potential basis in the function space. Due to the orthogonality of $H_j(z)$, the coefficients $\eta_j(\Delta, y)$ are given by

$$\eta_j(\Delta, y) = \frac{1}{j!}\int_{-\infty}^{\infty} H_j(z)p_Z(\Delta, z|y)dz.$$

Let $p_Y(\Delta, \omega|y)$ denote the transition density of $Y_{t+\Delta}|Y_t$, therefore the density of $Z_{t+\Delta} = z$ given $Y_t = y$ can also be expressed as:

$$p_Z(\Delta, z|y) = \Delta^{\frac{1}{2}}p_Y(\Delta, \Delta^{\frac{1}{2}}z + y|y).$$

Further, Jacobian formula yields the transition density of $X_{t+\Delta} = s$ given $X_t = x$:

$$p_X(\Delta, s|x) = \sigma(s)^{-1} \times \Delta^{-\frac{1}{2}}p_Z(\Delta, \Delta^{-\frac{1}{2}}(\gamma(s) - \gamma(x))|\gamma(x)).$$

Since we have established the Jth order density expansion for $p_Z(\Delta, z|y)$, the above equation immediately provides the Jth order expansion for $p_X(\Delta, s|x)$, denoted as $p_X^{(J)}(\Delta, s|x)$.

Given transition density expansion, it is straightforward to construct the cumulative distribution function by integration, which also yields the binary option price $\Psi(\Delta, x)$ using the probability representation of the solution to (2). Loosely speaking, allowing the interchange of integral and infinite summation immediately yields the expansion of $\Psi(\Delta, x)$:

$$\Psi(\Delta, x) = e^{-r\Delta}\int_K^{\infty} p_X^{(\infty)}(\Delta, s|x)ds \quad \text{(by Eq. (4))}$$

$$= e^{-r\Delta}\int_{\frac{\gamma(K)-\gamma(x)}{\sqrt{\Delta}}}^{\frac{\gamma(\infty)-\gamma(x)}{\sqrt{\Delta}}} p_Z^{(\infty)}(\Delta, z|\gamma(x))dz$$

(changing variable from X to Z)

$$\sim e^{-r\Delta}\sum_{j=0}^{\infty} \eta_j(\Delta, \gamma(x))\int_{\frac{\gamma(K)-\gamma(x)}{\sqrt{\Delta}}}^{\frac{\gamma(\infty)-\gamma(x)}{\sqrt{\Delta}}} \phi(z)H_j(z)dz$$

(interchanging integration and summation)

$$\sim -e^{-r\Delta}\left(\Phi(z) + \phi(z)\sum_{j=0}^{\infty} \eta_{j+1}(\Delta, \gamma(x))H_j(z)\right)\Bigg|_{z=\frac{\gamma(K)-\gamma(x)}{\sqrt{\Delta}}},$$

where $\Phi(z)$ is the standard normal cumulative distribution function. The last step utilizes the fact that for $j \geq 1$,

$$\int \phi(z)H_j(z) = \frac{1}{\sqrt{2\pi}}\int \frac{d^j}{dz^j}[e^{-\frac{z^2}{2}}]dz = \frac{1}{\sqrt{2\pi}}\frac{d^{j-1}}{dz^{j-1}}[e^{-\frac{z^2}{2}}]$$

$$= \phi(z)H_{j-1}(z), \tag{5}$$

and the conjecture that the integral evaluated at the upper limit converges to zero. The following theorem guarantees that the proposed series is a legitimate candidate to approximating $\Psi(\Delta, x)$.

Theorem 1. *Under Assumptions 1–3 given in Appendix A.1, there exists $\bar{\Delta} > 0$ (could be ∞), such that for every $\Delta \in (0, \bar{\Delta})$, the following sequence*

$$\Psi^{(J)}(\Delta, x) = e^{-r\Delta}\left(\Phi\left(\frac{\gamma(x) - \gamma(K)}{\sqrt{\Delta}}\right) + \phi\left(\frac{\gamma(x) - \gamma(K)}{\sqrt{\Delta}}\right)\right.$$

$$\left. \times \sum_{j=0}^{J}(-1)^{j+1}\eta_{j+1}(\Delta, \gamma(x))H_j\left(\frac{\gamma(x) - \gamma(K)}{\sqrt{\Delta}}\right)\right)$$

converges to $\Psi(\Delta, x)$ uniformly in x over any compact set in $(0, \infty)$, where $\Psi(\Delta, x)$ solves the Feynman–Kac partial differential equation (2) with the initial condition $\Psi(0, x) = 1_{\{x > K\}}$, for any $K \in (0, \infty)$.

To obtain explicit forms of $\eta_j(\Delta, y)$, we may recycle those coefficients calculated in Aït-Sahalia (2002) (e.g. (4.4)–(4.9) on page 238). In fact,

$$\eta_j(\Delta, y) = \frac{1}{j!} E\left(H_j(\Delta^{-\frac{1}{2}}(Y_{t+\Delta} - y)) | Y_t = y \right),$$

and for any polynomial $g(Y_{t+\Delta}, y)$, apply the Taylor series expansion in Δ:

$$E\left(g(Y_{t+\Delta}, y) | Y_t = y \right) = \sum_{i=0}^{l} \mathcal{L}_Y^i \cdot g(y, y) \frac{\Delta^i}{i!}$$
$$+ E\left(\mathcal{L}_Y^{l+1} \cdot g(Y_{t+\Delta}, y) | Y_t = y \right) \frac{\Delta^{l+1}}{(l+1)!},$$

where $\mathcal{L}_Y$ is the infinitesimal generator of the process Y_t. Therefore, the Taylor series of $\eta_j(\Delta, x)$ up to the lth order can be calculated explicitly. If we gather all terms with respect to increasing orders of Δ, and let J go to ∞, we may obtain an alternative expression of the expansion:

$$\Psi(\Delta, x) = e^{-r\Delta} \left(\Phi\left(\frac{\gamma(x) - \gamma(K)}{\sqrt{\Delta}} \right) + \sqrt{\Delta} \phi\left(\frac{\gamma(x) - \gamma(K)}{\sqrt{\Delta}} \right) \right.$$
$$\left. \times \exp\left(-\int_{\gamma(K)}^{\gamma(x)} \mu_Y(s) ds \right) \sum_{j=0}^{\infty} \chi_j(\gamma(x), \gamma(K)) \frac{\Delta^j}{j!} \right), \quad (6)$$

where

$$\chi_0(y, \omega) = (y - \omega)^{-1} \left(\exp\left(\int_{\omega}^{y} \mu_Y(s) ds \right) - 1 \right),$$

and for $j \geq 1$,

$$\chi_j(y, \omega) = j(y - \omega)^{-j-1} \int_{\omega}^{y} (u - \omega)^j$$
$$\times \left(\lambda_Y(u) \chi_{j-1}(u, \omega) + \frac{1}{2} \frac{\partial^2 \chi_{j-1}(u, \omega)}{\partial u^2} \right) du,$$
$$\lambda_Y(y) = -\frac{1}{2} \left(\mu_Y^2(y) + \frac{\partial \mu_Y(y)}{\partial y} \right),$$

which can be calculated explicitly once and for all, using Mathematica.

The expansion formula immediately uncovers that the volatility function specified in the model plays the leading role in determining the option prices especially in the short run, and that the normal cumulative and density functions are sufficient to capture the entire singularity from the contingent payoff.

Theorem 1 establishes a convergent series expansion of the cash-or-nothing binary call option prices. Binary put options with payoff $1_{\{x < K\}}$ can be expanded similarly. In regard to vanilla options with payoff $(x - K)^+$, we may reduce the pricing problem to the one that has already been solved above. More specifically, the drift term in most financial models is of the affine form $\mu(x) = \kappa(\alpha - x)$. In such cases, we can decompose the contingent payoff $x - K$ at expiration to be $\alpha - K$ and $x - \alpha$. The former payoff can be replicated trivially using cash-or-nothing binary options. For the latter part, we can perform the transformation: $\Psi(\Delta, x) = (x - \alpha)\hat{\Psi}(\Delta, x)$. The Feynman–Kac equation (2) is simplified to a similar one for $\hat{\Psi}(\Delta, x)$ with constant "interest rate" $\kappa + r$:

$$-\frac{\partial \hat{\Psi}(\Delta, x)}{\partial \Delta} + \left(\kappa(\alpha - x) + \frac{\sigma^2(x)}{x - \alpha} \right) \frac{\partial \hat{\Psi}(\Delta, x)}{\partial x}$$
$$+ \frac{1}{2} \sigma^2(x) \frac{\partial^2 \hat{\Psi}(\Delta, x)}{\partial x^2} - (\kappa + r) \hat{\Psi}(\Delta, x) = 0 \quad (7)$$

with the desired initial condition $\hat{\Psi}(0, x) = 1_{\{x > K\}}$.

To fully explore the potential of the expansion method and pricing vanilla options under more complex models, we propose a more convenient approach, which we call the lucky guess approach or the method of undetermined coefficients.

2.2. The lucky guess approach

Alternatively, inspired by the Hermite polynomials approach, we can postulate an appropriate series expansion form, plug it into the equation and solve for the coefficients directly. This top-down approach may produce the same convergent sequence, and is sometimes more effective and convenient for general models and payoff functions. It would become a bold guess and most likely an unlucky one if the leading term is not derived from the first method. Therefore, the two proposed methods are always combined together.

To introduce this approach, we once again solve the partial differential equation (2) with the initial condition $\Psi(0, x) = 1_{\{x > K\}}$ first, leaving general cases aside for the moment. We postulate that the solution has the following form:

$$\Psi(\Delta, x) = e^{-r\Delta} \left(\Phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}} \right) + \frac{\sqrt{\Delta}}{\sqrt{2\pi}} \right.$$
$$\left. \times \exp\left(-\frac{C^{(-2)}(x)}{\Delta} \right) \sum_{k=0}^{\infty} C^{(k)}(x) \Delta^k \right). \quad (8)$$

Even without insight from the Hermite polynomial approach, this expression is reasonable to come up with for this simple case. The discount factor $e^{-r\Delta}$ is added to account for the $-r\Psi(\Delta, x)$ term in (2), and this trick works whenever the coefficient in front of $\Psi(\Delta, x)$ in (2) is constant. The first term in brackets plays the role of smoothing out the singularity in the initial condition in that $\Phi(\cdot)$ is an infinitely smooth approximation of the Heaviside function, and its value depends on the sign of $C^{(-1)}(x)$, when Δ approaches zero. The second term in brackets resembles the transition density expansions given by Aït-Sahalia (2002).

Matching coefficients for terms with the same order of Δ, we have the following recursive equations:

$$C^{(-1)}(x) = \int_K^x \frac{1}{\sigma(s)} ds, \qquad C^{(-2)}(x) = \frac{1}{2} \left(\int_K^x \frac{1}{\sigma(s)} ds \right)^2. \quad (9)$$

For $k \geq -1$,

$$C^{(k+1)}(x) \left(\frac{1}{2} + (k+1) + \mathcal{L}C^{(-2)}(x) \right)$$
$$+ \sigma^2(x) \frac{dC^{(k+1)}(x)}{dx} \frac{dC^{(-2)}(x)}{dx} = \mathcal{L}C^{(k)}(x), \quad (10)$$

and the boundary condition at $x = K$ is embedded in (10) due to $\frac{dC^{(-2)}(x)}{dx}|_{x=K} = 0$. $\mathcal{L}$ is the infinitesimal generator of the process X_t.

Solving the system of equations (10), we find that the coefficients in the expansion (8) for $\Psi(\Delta, x)$ are given explicitly by (9) and for $k \geq -1$,

$$C^{(k+1)}(x) = \int_K^x \left\{ \left(\sigma(s) C^{(-1)}(s) \right)^{-1} \mathcal{L}C^{(k)}(s) \right.$$
$$\cdot \exp\left(\int_x^s \left(\sigma(u) C^{(-1)}(u) \right)^{-1} \left((k+2) \right. \right.$$
$$\left. \left. \left. + C^{-1}(u) \mathcal{L}C^{-1}(u) \right) du \right) \right\} ds. \quad (11)$$

It is easy to verify that the two expansion formulae (6) and (8) agree with each other. For more complicated models, we often

use the first method to find the leading term, then postulate an appropriate expression and plug it into the equation. The next section provides extensions showing how to derive vanilla option price expansion formulae for more general models such as jump diffusions and certain multivariate models. Expansions for time-inhomogeneous diffusions can be similarly derived using leading terms provided by Egorov et al. (2003), hence are omitted here.

3. Extensions

3.1. Jump diffusion models

We investigate jump diffusion models in this section, in which jumps are supposed to be large and infrequent with daily observations. In practice, such models are employed to explain abnormally large daily returns due to economic catastrophes, unexpected news and other rare events. We assume that the underlying state variable in the risk-neutral world follows a jump diffusion:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t^Q + dZ_t, \tag{12}$$

where Z_t is a finite activity pure jump process with jump intensity $\lambda(X_t)$, and jump size density $\nu(z) : \mathbb{R} \to \mathbb{R}_+$.

A European call option with payoff $f(X_T)1_{\{X_T > K\}}$ at maturity has price $\Psi(\Delta, x) : [0, T] \times \mathbb{R} \to \mathbb{R}$, which satisfies the following Feynman–Kac integro-differential equation:

$$
\begin{aligned}
0 = {} & -\frac{\partial \Psi(\Delta, x)}{\partial \Delta} + \mu(x)\frac{\partial \Psi(\Delta, x)}{\partial x} \\
& + \frac{1}{2}\sigma^2(x)\frac{\partial^2 \Psi(\Delta, x)}{\partial x^2} - r(x)\Psi(\Delta, x) \\
& + \lambda(x)\int_{-\infty}^{\infty}\Big(\Psi(\Delta, x+z) - \Psi(\Delta, x)\Big)\nu(z)dz \\
= {} & \Big(-\frac{\partial}{\partial \Delta} + \mathcal{A} - r(x)\Big)\Psi(\Delta, x),
\end{aligned}
\tag{13}
$$

with the initial condition

$$\Psi(0, x) = f(x)1_{\{x > K\}}, \tag{14}$$

where the operator $\mathcal{A}$ is given by

$$
\begin{aligned}
\mathcal{A}g(x) = {} & \mu(x)\frac{\partial g(x)}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 g(x)}{\partial x^2} \\
& + \lambda(x)\int_{-\infty}^{\infty}(g(x+z) - g(x))\nu(z)dz.
\end{aligned}
$$

Hereafter, we denote the operator without the integral part as $\mathcal{L}$, and y^{inv} as the inverse of function y.

In this case, taking conditional expectation on the number of jumps can help find an appropriate form. Note that

$$p(N_\Delta = 1 | X_t = x) = O(\Delta), \qquad p(N_\Delta > 1 | X_t = x) = o(\Delta),$$

and by conditional expectation,

$$
\begin{aligned}
& E(f(X_T)1_{\{X_T > K\}} | X_t = x) \\
& = \sum_{n=0}^{\infty} E(f(X_T)1_{\{X_T > K\}} | X_t = x, N_\Delta = n)P(N_\Delta = n | X_t = x).
\end{aligned}
$$

Therefore, as the option approaches expiration, jumps in the underlying variable occur on the order of $O(\Delta)$, so that the dominating term remains to be the part contributed by the conditional risk-neutral density on no jumps. The following form is hereby postulated naturally:

$$
\begin{aligned}
\Psi(\Delta, x) = {} & \Phi\Big(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\Big)\sum_{k=0}^{\infty} B^{(k)}(x)\Delta^k \\
& + \sqrt{\Delta}\phi\Big(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\Big)\sum_{k=0}^{\infty} C^{(k)}(x)\Delta^k + \sum_{k=1}^{\infty} D^{(k)}(x)\Delta^k.
\end{aligned}
\tag{15}
$$

The first term resembles the previous case, where an additional series is attached for non-constant $r(x)$ and stochastic intensity $\lambda(x)$. The last two terms have been used for the transition density approximation by Yu (2007). Close scrutiny of the derivations reveals that the proposed expansion series serve good approximation for jump diffusions with jump size ranging over the entire $(-\infty, +\infty)$. We also provide new formulae for jumps with $(0, \infty)$ size in the appendix.

Taking the proposed formula into Eq. (13), and matching coefficients in terms of Δ and $\phi(\frac{C^{(-1)}(x)}{\sqrt{\Delta}})$, we can derive Theorem 2.

Theorem 2. *Assume $\Psi(\Delta, x)$ satisfies the option pricing Eqs. (13) and (14). The coefficients in the expansion (15) satisfy the following restrictions:*

$$0 = B^{(0)} - f(x), \tag{16}$$

$$0 = (k+1)B^{(k+1)} + (r(x) + \lambda(x))B^{(k)} - \mathcal{L}B^{(k)}, \tag{17}$$

$$
\begin{aligned}
0 = {} & -C^{(0)}\Big(1 + C^{(-1)}\mathcal{L}C^{(-1)}\Big) - \sigma(x)C^{(-1)}\frac{dC^{(0)}(x)}{dx} \\
& + B^{(0)}\mathcal{L}C^{(-1)} + \sigma(x)\frac{dB^{(0)}(x)}{dx},
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
0 = {} & -C^{(k+1)}\Big(k + 2 + C^{(-1)}\mathcal{L}C^{(-1)}\Big) \\
& - \sigma(x)C^{(-1)}\frac{dC^{(k+1)}(x)}{dx} + \sigma(x)\frac{dB^{(k+1)}(x)}{dx} \\
& + (\mathcal{L} - r(x) - \lambda(x))C^{(k)} + B^{(k+1)}\mathcal{L}C^{(-1)},
\end{aligned}
\tag{19}
$$

$$0 = -D^{(1)} + \lambda(x)h_0(x, 0), \tag{20}$$

$$
\begin{aligned}
0 = {} & -(k+2)D^{(k+2)} + (\mathcal{A} - r(x))D^{(k+1)} \\
& + \lambda(x)\sum_{r=0}^{k+1}\frac{1}{(2r)!}\frac{\partial^{2r}h_{k+1-r}(x, \omega)}{\partial\omega^{2r}}\bigg|_{\omega=0} M_{2r} \\
& + \lambda(x)\sum_{r=0}^{k}\frac{1}{(2r)!}M_{2r}\frac{\partial^{2r}g_{k-r}(x, \omega)}{\partial\omega^{2r}}\bigg|_{\omega=0},
\end{aligned}
\tag{21}
$$

where

$$h_k(x, \omega) = \int_{(C^{(-1)})^{inv}(\omega)-x}^{\infty} B^{(k)}(x+z)\nu(z)dz,$$

$$
\begin{aligned}
g_k(x, \omega) = {} & C^{(k)}\Big((C^{(-1)})^{inv}(\omega)\Big)\nu\Big((C^{(-1)})^{inv}(\omega) - x\Big) \\
& \times \left(\left|\frac{dC^{(-1)}(x)}{dx}\bigg|_{(C^{(-1)})^{inv}(\omega)}\right|\right)^{-1},
\end{aligned}
$$

and M_s is the sth moment of the standard normal.

Eqs. (18) and (19) are ordinary differential equations, which have similar closed-form solutions as in (11), and therefore omitted here. The other equations can be solved one after another by induction. For jump size distribution supported on the interval $[0, \infty)$, e.g. exponential distribution, the derivation is similar, but the formulae are more complicated as given in the Appendix A.4.

From (15) and Theorem 2, we can distinguish the order of the impact by drift, volatility and jump components on the price of close-to-maturity vanilla options respectively:

Remark 1. For vanilla call options with $f(x) = x - K$ under such a jump diffusion model, we can deduce that the approximation up to the order $\Delta^{\frac{3}{2}}$ is

$$
\begin{aligned}
\Psi(\Delta, x) = {} & \Phi\left(\Delta^{-\frac{1}{2}}\int_K^x \frac{1}{\sigma(s)}ds\right)\Big((x - K) + B^{(1)}(x)\Delta\Big) \\
& + (x - K)\left(\int_K^x \frac{1}{\sigma(s)}ds\right)^{-1}\phi\left(\Delta^{-\frac{1}{2}}\int_K^x \frac{1}{\sigma(s)}ds\right)\Delta^{\frac{1}{2}} \\
& + D^{(1)}(x)\Delta + O(\Delta^{\frac{3}{2}}),
\end{aligned}
$$

where the drift and jumps are relevant to $B^{(1)}(x)$ and $D^{(1)}(x)$. Therefore, volatility determines the leading terms, followed by jumps and drift part which affect first order terms.

It is also straightforward to tell from the above formula that when $x \ll K$, i.e., the option is deep out of the money, the first two terms are close to zeros, so that it is the jump relevant term $D^{(1)}$ that plays a leading role, which agrees with the intuition that such options will expire worthless unless jumps are expected to occur before expiration.

3.2. Multivariate diffusion models

In this section, we discuss the possibility of extending the previous procedure to multivariate models. Assume that underlying factors follow

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t^Q, \tag{22}$$

where X_t is an n-dimensional vector of state variables and W_t is a n-dimensional independent Brownian motion under Q measure. The potential correlation between state variables is therefore embedded in the volatility matrix. Denote $V(x) = \sigma(x)\sigma(x)^\tau = (v_{ij}(x))$ to be the covariance matrix. The Feynman–Kac partial differential equation is:

$$-\frac{\partial \Psi(\Delta, x)}{\partial \Delta} + \sum_{i=1}^{n} \mu_i(x)\frac{\partial \Psi(\Delta, x)}{\partial x_i}$$

$$+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} v_{ij}(x)\frac{\partial^2 \Psi(\Delta, x)}{\partial x_i x_j} - r(x)\Psi(\Delta, x) = 0. \tag{23}$$

The first obstacle of extending the previous procedure is the existence of $\gamma(X)$ in the multivariate case, which is guaranteed for univariate models. Aït-Sahalia (2008) defines the concept of reducibility: a diffusion X is said to be reducible if and only if there exists an invertible mapping γ, such that $Y_t = \gamma(X_t)$ satisfies a diffusion with unit covariance matrix. It turns out that reducibility alone is insufficient for obtaining closed-form formulae for option prices as in the previous cases. In fact, if we follow the Hermite polynomials approach, the leading term of the density of Z should be standard multivariate normal, hence the leading term of binary cash-or-nothing option price is

$$\Psi^{(0)}(\Delta, x) = (2\pi\Delta)^{-\frac{n}{2}} \int dy_2 \ldots dy_n \int_D dy_1$$

$$\times \exp\left(-\frac{\|\gamma(y) - \gamma(x)\|^2}{2\Delta}\right) \cdot |\det(\sigma^{-1}(y))|,$$

where D denotes the region in which the option expires in the money. Unlike scalar cases, the multivariate integral may not be simplified to a one-dimensional integral by changing variables, owing to the fact that the domain of integration is rather complicated once transformed by $\gamma(\cdot)$. If x_1 is the price of the underlying that determines the option price, namely $D = 1_{\{y_1 > K\}}$, then degenerate cases that can be solved explicitly require $V(x)$ to satisfy the following condition in addition to reducibility:

$$V(x) = \begin{pmatrix} v_{11}(x_1) & V_{12}(x) \\ V_{12}(x)^\tau & V_{22}(x) \end{pmatrix}, \tag{24}$$

where $v_{11}(x_1)$ is a scalar that depends on x_1 alone. In such a case, an appropriate solution has the following form:

$$\Psi(\Delta, x) = \Phi\left(\frac{1}{\sqrt{\Delta}}\int_K^{x_1} \frac{1}{\sqrt{v_{11}(u)}}du\right)\sum_{j=0}^{\infty} B^{(j)}(x)\Delta^j$$

$$+ \sqrt{\Delta}\phi\left(\frac{1}{\sqrt{\Delta}}\int_K^{x_1} \frac{1}{\sqrt{v_{11}(u)}}du\right)\sum_{j=0}^{\infty} C^{(j)}(x)\Delta^j.$$

The formulae that determine the coefficients are similar to those given in Theorem 2, but are in vector form, see Appendix A.5.

In regard to irreducible cases, the leading terms may not be the normal cumulative distribution function and its density. In fact, these leading terms are closely related to the concept of geodesics and Riemannian manifolds in differential geometry. In general multivariate cases, the partial differential equation may correspond to nontrivial manifolds such as Poincaré hyperbolic surface, whose geodesics could have rather complex structure, see e.g. Henry-Labordere (2008). It may be possible to further expand the option price around state variables, as developed in Aït-Sahalia (2008) for density expansion. However, such expansions may not be able to offer any intuition. We leave these cases for future work.

4. Translating model structure into option prices

This section provides several examples of series expansions for option prices, in order to shed light on more insight that the expansion formulae offer. We treat different features of option dynamics as building blocks, and focus on one block per example to compare and explain the intuition behind. There is no difficulty combining different blocks to generate new pricing formulae. The figures in the appendix illustrate the accuracy of approximation by comparing the expansion formulae with closed-form solutions or Monte Carlo simulations. With parameters that are calibrated from the market prices, the expansion formulae approximate the true prices very well.

4.1. The benchmark model

We first provide the expansions for the benchmark Black–Scholes model.

Example 1 (*Black–Scholes Model*). In the Black–Scholes model, the stock price follows:

$$dX_t = (r - \delta)X_t dt + \sigma X_t dW_t^Q.$$

The vanilla option price has a closed-form expression given by Black and Scholes (1973). (See Box I.)

4.2. The elasticity of variance

A direct generalization of the Black–Scholes model is the following Constant Elasticity of Variance (CEV) model which introduces one additional parameter γ. The elasticity of variance with respect to the state variable X is $2\gamma - 2$. When $\gamma \to 1$, the expansion formulae match the Black–Scholes case.

Example 2 (*CEV Model*). The CEV model assumes that

$$dX_t = (r - \delta)X_t dt + \sigma X_t^\gamma dW_t^Q. \tag{25}$$

We consider the case with $\gamma > 1$, so that the probability of hitting the boundary is 0. The closed-form pricing formula is available in Cox (1975). (See Box II.)

The elasticity parameter γ appears in every order of the expansion, as the volatility exerts an inherent influence on option prices: no matter how small the time to maturity is, volatility plays a leading role in determining the option prices, as it controls the short term fluctuation of the underlying dynamics. Schroder (1989) develops an algorithm to approximate the option prices based on this CEV model, whereas the above closed-form formulae offer an alternative method, which is much simpler.

The following closed-form expansions are plotted in Fig. 1.

$$\Psi(\Delta, x) = \Phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} B^{(k)}(x)\Delta^k + \sqrt{\Delta}\phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} C^{(k)}(x)\Delta^k$$

$$B^{(k)}(x) = \frac{(-1)^k}{k!}(x\delta^k - Kr^k), \quad k \geq 0, \qquad C^{(-1)}(x) = \frac{\log\left(\frac{x}{K}\right)}{\sigma}, \qquad C^{(0)}(x) = \frac{\sigma(x-K)}{\log\left(\frac{x}{K}\right)}, \quad \text{if } x \neq K; \text{ or } K\sigma, \text{ if } x = K.$$

$$C^{(1)}(x) = \frac{\sigma\left((K-x)\sigma^2 + K^{\frac{1}{2}+\frac{r}{\sigma^2}-\frac{\delta}{\sigma^2}}x^{\frac{1}{2}-\frac{r}{\sigma^2}+\frac{\delta}{\sigma^2}}\sigma^2\log\left(\frac{x}{K}\right) + (Kr - x\delta)\log\left(\frac{x}{K}\right)^2\right)}{\log\left(\frac{x}{K}\right)^3}, \quad \text{if } x \neq K; \quad \text{or}$$

$$K\left(\frac{(r-\delta)^2}{2\sigma} - \frac{(r+\delta)\sigma}{2} - \frac{\sigma^3}{24}\right), \quad \text{if } x = K.$$
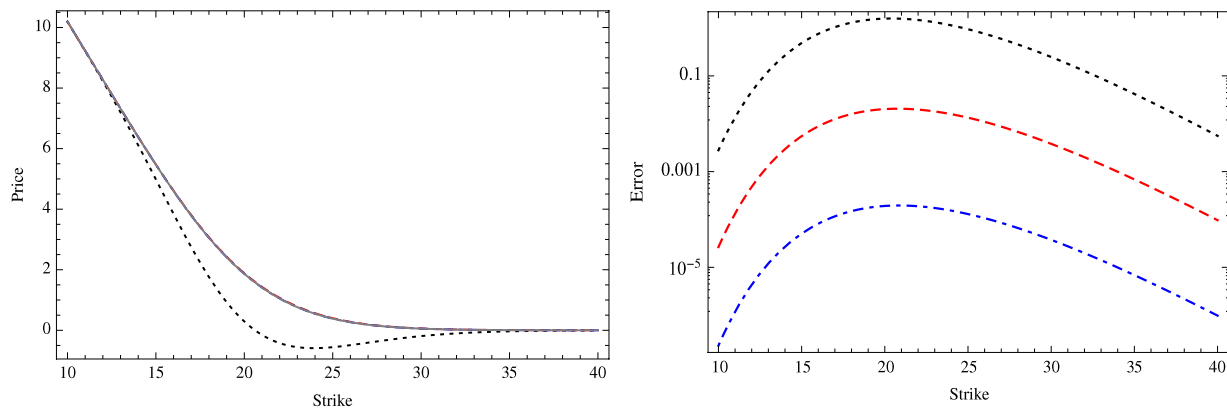
Box I.

Fig. 1. Black–Scholes model. Note: The black dotted line, red dashed line and blue dotted-dash line illustrate the $O(\Delta^{1/2})$, $O(\Delta^{3/2})$ and $O(\Delta^{5/2})$ order approximations respectively. The grey line denotes the true prices. Y-axis of the right panel is on a logarithmic scale. The parameters are: $\sigma = 0.2$, $r = 4\%$, $\delta = 0.01$, $x = 20$, and $\Delta = 1$.

The following closed-form expansions are plotted in Fig. 2.

$$\Psi(\Delta, x) = \Phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} B^{(k)}(x)\Delta^k + \sqrt{\Delta}\phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} C^{(k)}(x)\Delta^k$$

$$B^{(k)}(x) = \frac{(-1)^k}{k!}(x\delta^k - Kr^k), \quad k \geq 0$$

$$C^{(-1)}(x) = -\frac{K^{1-\gamma} - x^{1-\gamma}}{\sigma - \gamma\sigma}, \qquad C^{(0)}(x) = \frac{K^\gamma(K-x)x^\gamma(-1+\gamma)\sigma}{K^\gamma x - Kx^\gamma}, \quad \text{if } x \neq K; \text{ or } K^\gamma\sigma, \text{ if } x = K.$$

$$C^{(1)}(x) = \frac{(Kx)^\gamma(-1+\gamma)\sigma}{(-K^\gamma x + Kx^\gamma)^3}\left(K^{1+2\gamma}rx^2 + K^3rx^{2\gamma} - K^{2\gamma}x^3\delta - K^2x(2r(Kx)^\gamma + x^{2\gamma}\delta)\right.$$

$$+ e^{\frac{(Kx)^{-2\gamma}(K^{2\gamma}x^2 - K^2x^{2\gamma})(r-\delta)}{2(-1+\gamma)\sigma^2}} K^{1+\frac{3\gamma}{2}}x^{5\gamma/2}(-1+\gamma)\sigma^2 - e^{\frac{(Kx)^{-2\gamma}(K^{2\gamma}x^2 - K^2x^{2\gamma})(r-\delta)}{2(-1+\gamma)\sigma^2}} K^{5\gamma/2}x^{1+\frac{3\gamma}{2}}(-1+\gamma)\sigma^2$$

$$\left. - x(Kx)^{2\gamma}(-1+\gamma)^2\sigma^2 + K(Kx)^\gamma(2x^2\delta + (Kx)^\gamma(-1+\gamma)^2\sigma^2)\right), \quad \text{if } x \neq K; \quad \text{or}$$

$$\left(\frac{K^{2-\gamma}(r-\delta)^2}{2\sigma} - \frac{K^\gamma(r+\delta)\sigma}{2} + \frac{K^{-2+3\gamma}(-2+\gamma)\gamma\sigma^3}{24}\right), \quad \text{if } x = K.$$

Box II.

4.3. The barriers

In an alternative specification of Quadratic Volatility (QV) models, the underlying of the option is bounded within an interval (l, u).

Example 3 (*QV Model*). The QV model assumes that

$$dX_t = \sigma\frac{(u-X_t)(X_t-l)}{u-l}dW_t^Q,$$

where $l < x < u$ and $l < K < u$, so that X_t is a bounded martingale. Ingersoll (1997) proposes such a model for foreign exchange futures. The closed-form option pricing formula is available there, see also Rady (1997) and Andersen (2011) for alternative derivations. (See Box III.)

Notice that for the ATM option, its price up to $O(\Delta)$ is given by:

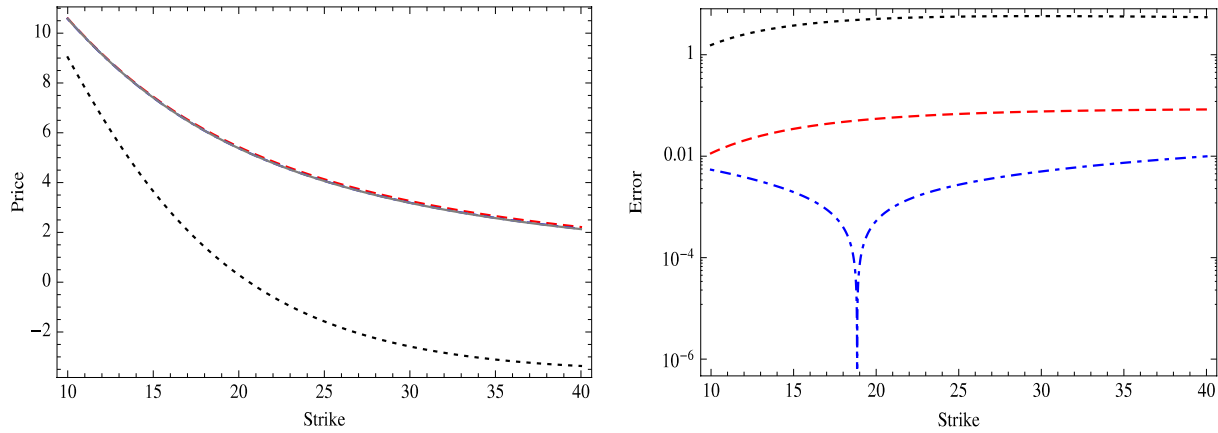$$\Psi(\Delta, x) = \frac{1}{2}(x-K) + \sigma\sqrt{\Delta}\frac{(K-l)(u-K)}{u-l}.$$

Fig. 2. CEV model. Note: The black dotted line, red dashed line and blue dotted-dash line illustrate the $O(\Delta^{1/2})$, $O(\Delta^{3/2})$ and $O(\Delta^{5/2})$ order approximations respectively. The grey line denotes the true prices. Y-axis of the right panel is on a logarithmic scale. The parameters are: $\sigma = 0.2$, $x = 20$, $\Delta = 1$, and $\gamma = 1.4$.

The following closed-form expansions are plotted in Fig. 3.

$$\Psi(\Delta, x) = \Phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} B^{(k)}(x)\Delta^k + \sqrt{\Delta}\phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} C^{(k)}(x)\Delta^k$$

$$B^{(0)}(x) = x - K, \qquad B^{(k)}(x) = 0, \quad \text{for } k \geq 1.$$

$$C^{(-1)}(x) = \frac{1}{\sigma} \log\left(\frac{(x-l)(u-K)}{(K-l)(u-x)}\right), \qquad C^{(0)}(x) = \frac{\sigma(x-K)}{\log\left(\frac{(x-l)(u-K)}{(K-l)(u-x)}\right)}, \quad \text{if } x \neq K; \quad \text{or} \quad \frac{(K-l)(u-K)}{u-l}\sigma, \quad \text{if } x = K.$$

$$C^{(1)}(x) = \frac{\sigma\left((K-x)\sigma^2 + \frac{\sqrt{(K-l)(u-x)(x-l)(u-K)}}{u-l}\sigma^2 \log\left(\frac{(x-l)(u-K)}{(K-l)(u-x)}\right)\right)}{\log\left(\frac{(x-l)(u-K)}{(K-l)(u-x)}\right)^3}, \quad \text{if } x \neq K; \quad \text{or} \quad -\frac{(K-l)(u-K)\sigma^3}{24(u-l)}, \quad \text{if } x = K.$$

Box III.

When the upper barrier u increases, the option becomes more expensive, as the volatility rises and the chance of hitting in-the-money region increases. When the lower barrier decreases, the ATM price still rises, as the volatility effect dominates. When barriers go to 0 and $+\infty$, the prices converge to the Black–Scholes prices. In practice, this model is usually specified as

$$dX_t = \beta\left(\psi X_t + (1 - \psi)x + \frac{\gamma}{2x^2}(X_t - x)^2\right)dW_t^Q, \tag{26}$$

where β, ψ, and γ proxy the volatility level, skew, and convexity. We will revisit this model and the CEV model in Section 5, where we compare the accuracy of different expansion methods in terms of implied volatility, see Gatheral et al. (2012).

4.4. The effect of mean-reversion

Mean-reversion is a common phenomenon in financial markets. For example, empirical evidence in the literature has pointed out that volatility and short-rate dynamics often exhibit such a feature. When it comes to pricing options written on volatility or short rate, the mean-reversion feature may be carried over to their risk-neutral dynamics, since neither volatility nor short rates are directly tradable assets.

Does mean-reversion in the underlying state variable affect short-term options written on it? Consider a call option written on the volatility dynamics. Intuitively, as volatility drifts away above its long-run mean, it has a stronger tendency of moving downwards, decreasing its potential future higher value. Therefore, out-of-the-money options in such cases may be traded at a discount. When volatility is below its long-run mean level, in-the-money options may be traded at a premium.

Identifying this effect is not an easy task, not to mention measuring the exact amount of premium. It is possible, however, to characterize it using closed-form expansions. Intuitively, since it takes time for volatility to return to its long-run mean level, (in other words, the mean-reversion effect needs time to manifest), it is likely that the mean-reversion would not impact the leading order of the option prices. In fact, this is evident from the expansion for cash-or-nothing binary option prices, which also reflects the probability of moving above the pre-specified strike levels.

4.4.1. Affine mean-reversion

We expand the cash-or-nothing binary volatility option with a square-root (SQR) model to demonstrate the affine mean-reversion effect. This type of volatility model has been considered in Heston (1993), Pan (2002) and Mencia and Sentana (2009).

Example 4 (*SQR Model*). The SQR model for volatility assumes that

$$dV_t = \beta(\alpha - V_t)dt + \sigma\sqrt{V_t}dW_t^Q.$$

The true transition density is given by Cox et al. (1985), from which we can calculate the binary option price by integration:

$$\Psi(\Delta, v) = \int_K^{\infty} p(u, \Delta|v)du,$$

where

$$p(u, \Delta|v) = ce^{-y-x}\left(\frac{x}{y}\right)^{\frac{q}{2}} I_q\left(2\sqrt{xy}\right), \quad c = \frac{2\beta}{\sigma^2(1 - e^{-\beta\Delta})},$$

$$x = cu, \quad y = cve^{-\beta\Delta}$$

and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q. (See Box IV.)
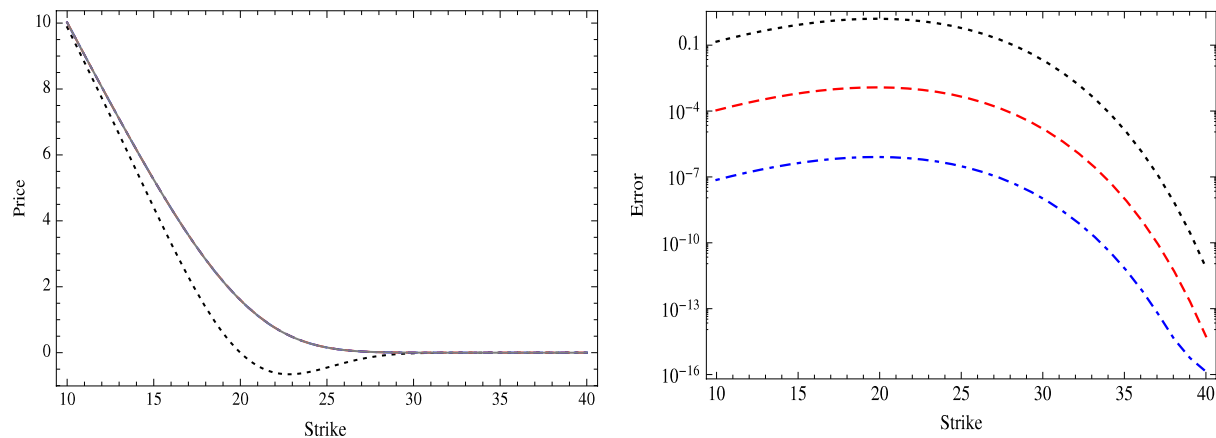
Fig. 3. QV model. Note: The black dotted line, red dashed line and blue dotted-dash line illustrate the $O(\Delta^{1/2})$, $O(\Delta^{3/2})$ and $O(\Delta^{5/2})$ order approximations respectively. The grey line denotes the true prices. Y-axis of the right panel is on a logarithmic scale. The parameters are: $\sigma = 0.2$, $\psi = -0.5$, $\gamma = -0.1$, $x = 20$, and $\Delta = 1$.

Closed-form expansions for a binary call option with payoff $1_{\{V_t > K\}}$ are given below. Fig. 4 illustrates the accuracy.

$$\Psi(\Delta, v) = \Phi\left(\frac{C^{(-1)}(v)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} B^{(k)}(v)\Delta^k + \sqrt{\Delta}\phi\left(\frac{C^{(-1)}(v)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} C^{(k)}(v)\Delta^k$$

$$B^{(k)}(v) = \frac{(-1)^k r^k}{k!}, \qquad C^{(-1)}(v) = \frac{2(\sqrt{v} - \sqrt{K})}{\sigma}$$

$$C^{(0)}(v) = -\frac{e^{-\frac{K\beta}{\sigma^2}} v^{-\frac{\alpha\beta}{\sigma^2}}\left(-e^{\frac{v\beta}{\sigma^2}} K^{\frac{\alpha\beta}{\sigma^2}} v^{1/4} + e^{\frac{K\beta}{\sigma^2}} K^{1/4} v^{\frac{\alpha\beta}{\sigma^2}}\right)\sigma}{2K^{1/4}\left(\sqrt{K} - \sqrt{v}\right)}, \quad \text{if } v \neq K; \quad \text{or} \quad \frac{-4K\beta + 4\alpha\beta - \sigma^2}{4\sqrt{K}\sigma}, \quad \text{if } v = K.$$

$$C^{(1)}(v) = \frac{e^{\frac{v\beta}{\sigma^2}} v^{\frac{1}{4} - \frac{\alpha\beta}{\sigma^2}}}{192(\sqrt{K} - \sqrt{v})^2\sigma}\left(e^{-\frac{K\beta}{\sigma^2}} K^{-\frac{3}{4} + \frac{\alpha\beta}{\sigma^2}}\left(-16K^2\beta^2 + 48\alpha^2\beta^2 - 3\sigma^4 + 48K(2\alpha\beta^2 - (2r + \beta)\sigma^2)\right) - \frac{1}{K^{1/4}(-\sqrt{K}v + v^{3/2})}\right.$$

$$\times \left(24e^{-\frac{v\beta}{\sigma^2}} K^{1/4} v^{\frac{3}{4} + \frac{\alpha\beta}{\sigma^2}}\sigma^2(4Kr - 8\sqrt{K}r\sqrt{v} + 4rv + \sigma^2) - e^{-\frac{K\beta}{\sigma^2}} K^{\frac{\alpha\beta}{\sigma^2}}\sqrt{v}(\sqrt{v}(16v^2\beta^2 - 96v\alpha\beta^2 - 48\alpha^2\beta^2 + 96rv\sigma^2)\right.$$

$$\left.\left. + 48\alpha\beta\sigma^2 + 15\sigma^4) + \sqrt{K}(-16v^2\beta^2 + 48\alpha^2\beta^2 - 48\alpha\beta\sigma^2 + 9\sigma^4 + 96v(\alpha\beta^2 - r\sigma^2)))\right)\right), \quad \text{if } v \neq K; \quad \text{or}$$

$$\frac{64K^3\beta^3 - 64\alpha^3\beta^3 + 48\alpha^2\beta^2\sigma^2 + 4\alpha\beta\sigma^4 - 3\sigma^6 - 48K^2\beta(4\alpha\beta^2 - (8r + \beta)\sigma^2) + 12K(4\alpha\beta - \sigma^2)(4\alpha\beta^2 - (8r + \beta)\sigma^2)}{384K^{3/2}\sigma^3}$$
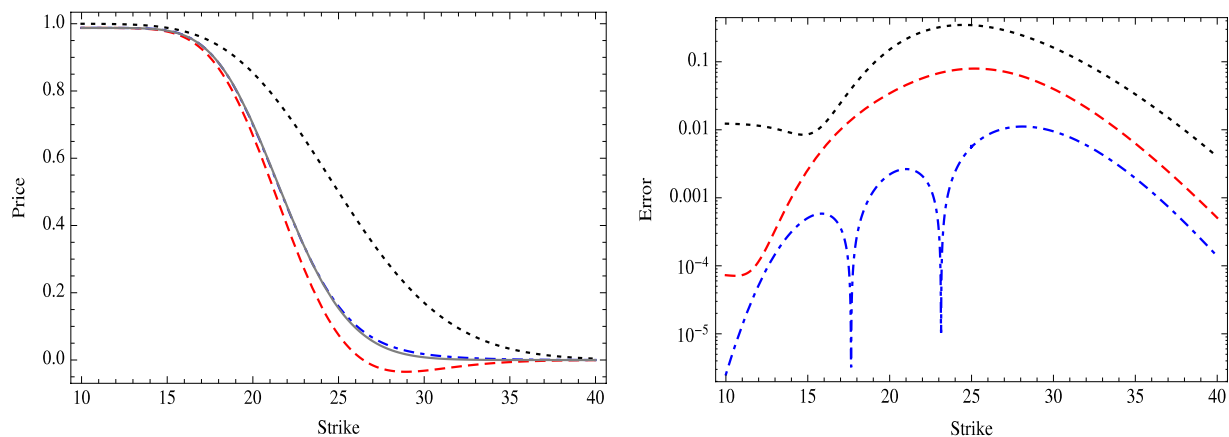
if $v = K$.

Box IV.

Fig. 4. Square-root model. Note: The black dotted line, red dashed line and blue dotted-dash line illustrate the $O(\Delta^{1/2})$, $O(\Delta^{3/2})$ and $O(\Delta^{5/2})$ order approximations respectively. The grey line denotes the true prices. Y-axis of the right panel is on a logarithmic scale. The parameters are: $\sigma = 0.2$, $\alpha = 0.2$, $\beta = 4$, $r = 5\%$, $v = 0.25$, and $\Delta = 90/360$. The current level of the VIX is equal to $100 \cdot v$.

The $O(1)$ order term $\Phi\left(\frac{2\left(\sqrt{v}-\sqrt{K}\right)}{\sigma\sqrt{\Delta}}\right)$ in the expansion reflects the effect of moneyness. If the current variance level v is higher than the strike price K, the chance of expiring in-the-money is dominant, otherwise the probability of expiring out-of-the-money is larger. When the option is at-the-money, the chance of moving upward or downward is equal up to the $O(1)$ order. This term therefore represents the probability as if the underlying volatility process were not mean-reverting. The Δ appeared in the denominator controls the first order temporal effect, that is, the shorter the time-to-maturity is, the more weight the moneyness effect accounts for. As Δ decreases to zero, the leading term approaches to the indicator function.

The effect of mean-reversion is manifested at the $O(\Delta^{1/2})$ order for cash-or-nothing binary options. The effect is more transparent regarding at-the-money options. Fixing K at the current variance level v, the probability becomes increasingly large as α increases due to the pulling effect of mean-reversion when $\beta > 0$. The term $-\sigma/(4K^{1/2})$ is a higher order adjustment to the $O(1)$ term that may have nothing to do with mean-reversion effect. Excluding this adjustment term, simple calculation derives that as long as $\alpha > (v-K)/(\log v - \log K)$, the option may be traded at a premium due to mean-reversion, otherwise, the option may be traded at a discount. Moreover, the tradeoff level for α is always between v and K regardless of whether the option is in-the-money or out-of-the-money.

Similarly, for asset-or-nothing binary options, the mean-reversion effect also appears in $C^{(0)}$ on the order of $O(\Delta^{1/2})$, although the effect is more intricate since the payoff at expiration is also involved. More interestingly, when it comes to vanilla options, the mean-reversion effect by each component cancels out on the order of $O(\Delta^{1/2})$.

4.4.2. Double mean-reversion

The same is true for the Double Mean-Reversion (DMR) model, which includes an additional stochastic factor for the mean level in the SQR model, as discussed in Amengual (2008), Mencia and Sentana (2009), and Egloff et al. (2010).

Example 5 (*DMR Model*)**.** The DMR model for volatility assumes that

$$dV_t = \beta(y_t - V_t)dt + \sigma\sqrt{V_t}dW_t^Q,$$
$$dy_t = \xi(\alpha - y_t)dt + \kappa\sqrt{y_t}dB_t^Q,$$

where W_t and B_t are independent. (See Box V.)

The parameter β governs the speed of mean-reversion towards the short-run stochastic mean level y. Both β and y appear on the order of $O(\Delta)$, whereas the long-run mean-reversion rate, ξ, and the long-run mean level α influence the option prices on the order of $O(\Delta^2)$.

4.4.3. Nonlinear mean-reversion

The following model is designed to capture nonlinear mean-reversion speed. Such a model has been discussed in Aït-Sahalia (1999, 1996), and Gallant and Tauchen (1998) for modelling interest rates. Recently, Eraker and Wang (2012) have proposed a similar model for VIX options, which can also be nested within this framework.

Example 6 (*Nonlinear Mean Reversion (NMR) Model*)**.** The NMR model is specified as

$$dV_t = \left(\frac{a}{V_t} + b + cV_t + dV_t^2\right)dt + \sigma V_t^\gamma dW_t^Q,$$

where $\gamma > 1$, and a, b, c, and d control the speed of mean-reversion. (See Box VI.)

4.5. The impact of jumps

What is the price impact on options contributed by underlying jumps? Intuitively, the presence of jumps effectively influences the option prices by altering the volatility, skewness, and kurtosis of the underlying returns. The mean of the jump size distribution seems irrelevant, as the jump compensator always zeroes it out so that the mean return of the risk-neutral dynamics is always equal to the risk free rate. We try to understand the role of jumps by investigating the option price expansion among alternative models, including Merton's, Kou's and Hawkes' jumps. For this purpose, we simplify the models to cases with constant volatility, but the proposed expansions work for more general setup.

4.5.1. Benchmark jumps

The paradigm that incorporates discontinuous returns is Merton's jump diffusion discussed in Merton (1976).

Example 7 (*Merton's Jump-Diffusion Model*)**.** Assume that

$$\frac{dX_t}{X_t} = (r - (m-1)\lambda)dt + \sigma dW_t^Q + (e^J - 1)dN_t,$$

where J has normal distribution $N\left(\log m - \frac{v^2}{2}, v^2\right)$ and N_t is a Poisson process with intensity λ. The closed-form vanilla option pricing formula is given by Merton (1976):

$$\Psi(\Delta, x) = \sum_{k=0}^{\infty} \frac{(\lambda\Delta)^k}{k!}\Bigg(xm^k e^{-\lambda m\Delta}\Phi$$
$$\times\left(\frac{\log(\frac{x}{K}) + (r - \lambda(m-1) + \sigma^2/2)\Delta + (\log m + v^2/2)k}{\sqrt{\sigma^2\Delta + kv^2}}\right)$$
$$- Ke^{-(\lambda+r)\Delta}\Phi$$
$$\times\left(\frac{\log(\frac{x}{K}) + (r - \lambda(m-1) - \sigma^2/2)\Delta + (\log m - v^2/2)k}{\sqrt{\sigma^2\Delta + kv^2}}\right)\Bigg).$$

(See Box VII.)

These expansions are derived using Theorem 2 by considering an equivalent option written on the log price, so that the jump size is independent of the price level X. The same expansion formulae apply with x replaced by $\log(x)$ and K by $\log(K)$. Apparently, up to the order of $O(\Delta^{1/2})$, the option prices are indistinguishable under Black–Scholes and Merton's Jump diffusion models. This is intuitive, as in expectation Poisson jumps occur with probability of the order $O(\Delta)$. The impact of jumps on option prices is therefore on the order of $O(\Delta)$. More specifically, we can separate the first-order impact by jumps from the option price explicitly:

$$\underbrace{mx\lambda\left(\Phi\left(\frac{\log(\frac{x}{K}) + \log(m) + \frac{1}{2}v^2}{v}\right) - \Phi\left(\frac{\log(\frac{x}{K})}{\sigma\sqrt{\Delta}}\right)\right)}_{\text{asset-or-nothing portion}}$$
$$\underbrace{- K\lambda\left(\Phi\left(\frac{\log(\frac{x}{K}) + \log(m) - \frac{v^2}{2}}{v}\right) - \Phi\left(\frac{\log(\frac{x}{K})}{\sigma\sqrt{\Delta}}\right)\right)}_{\text{cash-or-nothing portion}}.$$

The expansion implies that as jump intensity λ rises, the impact of jump becomes increasingly important. For at-the-money options, if the jump size has a positive mean level that is $\log(m) > v^2/2$, then the probability of expiring in the money is boosted. So is the price of a cash-or-nothing binary portion, which agrees with

The following closed-form expansions for a vanilla call option are plotted in Fig. 5.

$$\Psi(\Delta, v, y) = \Phi\left(\frac{C^{(-1)}(v, y)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} B^{(k)}(v, y) \Delta^k + \sqrt{\Delta} \phi\left(\frac{C^{(-1)}(v, y)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} C^{(k)}(v, y) \Delta^k$$

$$B^{(0)}(v, y) = v - K, \qquad B^{(1)}(v, y) = -r(v - K) - (v - y)\beta,$$

$$B^{(2)}(v, y) = -\frac{1}{2}\left(Kr^2 - r^2 v + 2r(-v + y)\beta + \beta(-\alpha\xi - v\beta + y(\beta + \xi))\right)$$

$$C^{(-1)}(v, y) = \frac{2(\sqrt{v} - \sqrt{K})}{\sigma}, \qquad C^{(0)}(v, y) = \frac{\sigma(\sqrt{v} + \sqrt{K})}{2}$$

$$C^{(1)}(v, y) = -\frac{1}{8(\sqrt{K} - \sqrt{v})^2} e^{-\frac{K\beta}{\sigma^2}} v^{-\frac{y\beta}{\sigma^2}} \sigma \left(-2e^{\frac{v\beta}{\sigma^2}} K^{\frac{1}{4} + \frac{y\beta}{\sigma^2}} v^{1/4} \sigma^2 + e^{\frac{K\beta}{\sigma^2}} v^{\frac{y\beta}{\sigma^2}} \left(4K^{3/2} r - 4Kr\sqrt{v} + \sqrt{v}\left(4rv + 4v\beta - 4y\beta + \sigma^2\right)\right.\right.$$

$$\left.\left. + \sqrt{K}\left(-4rv - 4v\beta + 4y\beta + \sigma^2\right)\right)\right), \quad \text{if } v \neq K; \quad \text{or}$$

$$\frac{16K^2\beta^2 + 16y^2\beta^2 - \sigma^4 - 16K\left(2y\beta^2 + (2r + \beta)\sigma^2\right)}{32\sqrt{K}\sigma}, \quad \text{if } v = K.$$
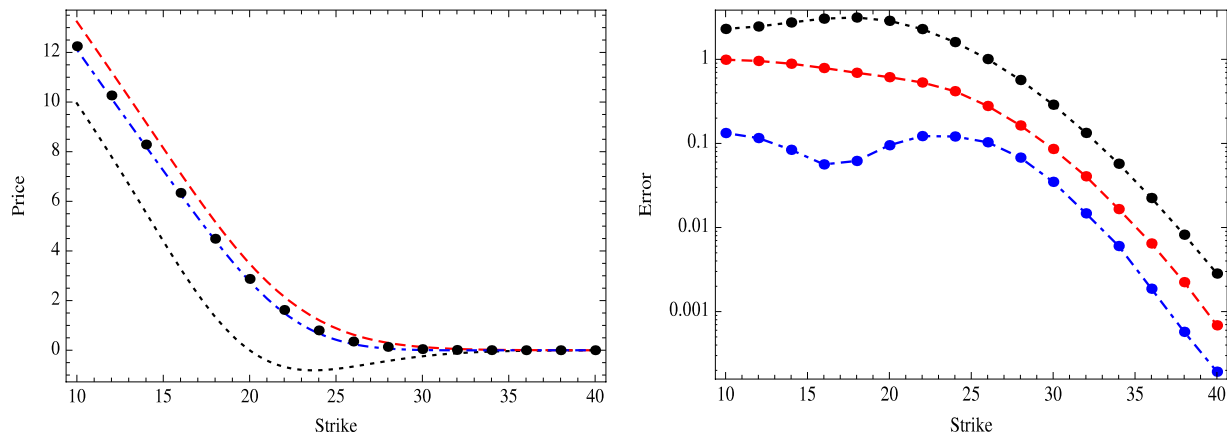
Box V.

Fig. 5. Double mean-reversion model. Note: The black dotted line, red dashed line and blue dotted-dash line illustrate the $O(\Delta^{1/2})$, $O(\Delta^{3/2})$ and $O(\Delta^{5/2})$ order approximations respectively. The black dots denote the true prices. Y-axis of the right panel is on a logarithmic scale. The parameters are: $\sigma = 0.25, \alpha = 0.25, \beta = 2.5, r = 5\%, v = 0.2, y = 0.25, \xi = 4, \kappa = 0.2$, and $\Delta = 60/360$. The current level of the VIX is equal to $100 \cdot v$.

the intuition. More generally, whether the jump impact on this portion is positive or not depends on the relative magnitude of $(\log(x/K) + \log(m) - v^2/2)/v$ and $\log(x/K)/(\sigma\sqrt{\Delta})$. When the variance uncertainty parameter σ diminishes to zero, the price of jump impact is negative for in-the-money binary options, since if there were no jumps in reality the option would for sure expire in-the-money. When it comes to a vanilla option, it can be shown that this first order jump impact is always positive. Again, this is in agreement with our intuition that the presence of jump simply augments the volatility of the underlying returns, leading to an increase in the value of any options with convex payoffs.

4.5.2. One-sided and skewed jumps

To better capture the volatility smile and asymmetric leptokurtic feature of asset returns, Kou (2002) introduces an alternative jump diffusion model where the jumps J in log returns have asymmetric double exponential distribution with the density:

$$v(z) = p \cdot \eta_1 e^{-\eta_1 z} 1_{\{z \geq 0\}} + q \cdot \eta_2 e^{\eta_2 z} 1_{\{z < 0\}},$$

where $\eta_1 > 1, \eta_2 > 0, p + q = 1$, and $0 \leq p, q \leq 1$. The mean, variance, and skewness of the jump size in log returns are:

$$\gamma_1 = \frac{p}{\eta_1} - \frac{q}{\eta_2}, \qquad \gamma_2 = pq\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)^2 + \frac{p}{\eta_1^2} + \frac{q}{\eta_2^2},$$

$$\gamma_3 = \frac{2(p^3 - 1)\eta_1^3 - 2(q^3 - 1)\eta_2^3 + 6pq\eta_1\eta_2(q\eta_2 - p\eta_1)}{(p\eta_2^2 + q\eta_1^2 + pq(\eta_1 + \eta_2)^2)^{\frac{3}{2}}}.$$

By changing p, η_1, and η_2, this model is able to capture many empirical features of jumps, which may have disparate impacts on option prices. We analyse the first order impact through closed-form expansions:

Example 8 (*Kou's Jump-Diffusion Model*). Assume that

$$d \log(X_t) = \mu dt + \sigma dW_t^Q + J dN_t,$$

where J has an asymmetric double exponential distribution with parameters p, η_1, and η_2, and N_t is a Poisson process with intensity λ. In this case, we have

$$\mu = r - \frac{1}{2}\sigma^2 - \lambda\left(\frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1\right).$$

The closed-form vanilla option pricing formula is given by Theorem 2 in Kou (2002), and omitted here due to its complexity. We expand this model using expansion formulae given in the Appendix A.4. (See Box VIII.)

The following closed-form expansions for a vanilla call option are plotted in Fig. 6.

$$\Psi(\Delta, v) = \Phi\left(\frac{C^{(-1)}(v)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} B^{(k)}(v)\Delta^k + \sqrt{\Delta}\phi\left(\frac{C^{(-1)}(v)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} C^{(k)}(v)\Delta^k$$

$$B^{(0)}(v) = v - K, \qquad B^{(1)}(v) = \frac{a}{v} + b + cv + dv^2$$

$$B^{(2)}(v) = \frac{1}{2}\left(\left(c - \frac{a}{v^2} + 2dv\right)\left(b + \frac{a}{v} + v(c + dv)\right) + v^{-3+2\gamma}\left(a + dv^3\right)\sigma^2\right)$$

$$C^{(-1)}(v) = -\frac{K^{1-\gamma} - v^{1-\gamma}}{\sigma - \gamma\sigma}, \qquad C^{(0)}(v) = \frac{(K-v)(Kv)^\gamma(-1+\gamma)\sigma}{K^\gamma v - Kv^\gamma}, \quad \text{if } v \neq K; \text{ or } K^\gamma\sigma, \text{ if } v = K.$$

$$C^{(1)}(v) = \frac{1}{\left(K - \left(\frac{K}{v}\right)^\gamma v\right)^2} e^{\frac{v^{-2\gamma}\left(\frac{cv^2}{-1+\gamma} + \frac{a}{\gamma} + \frac{2dv^3}{-3+2\gamma} + \frac{2bv}{-1+2\gamma}\right)}{2\sigma^2}} v^{\gamma/2} \int_K^v -\frac{1}{\left(2\left(K^\gamma s - Ks^\gamma\right)^2 \sigma\right)} e^{-\frac{s^{-2\gamma}\left(\frac{cs^2}{-1+\gamma} + \frac{a}{\gamma} + \frac{2ds^3}{-3+2\gamma} + \frac{2bs}{-1+2\gamma}\right)}{2\sigma^2}}$$

$$K^{2\gamma}s^{-1-\frac{3\gamma}{2}}(\sigma - \gamma\sigma)^2 \left(2\left(K^\gamma s - Ks^\gamma\right)(a + s(b + s(c + ds)))\left(Ks^\gamma + K^{1+\gamma}(-1+\gamma) - K^\gamma s\gamma\right) + K^\gamma s^{2\gamma}(-1+\gamma)\right.$$

$$\left. \left(K^{1+\gamma}s(-2+\gamma) - Ks^{1+\gamma}(-2+\gamma) - K^\gamma s^2\gamma + K^2s^\gamma\gamma\right)\sigma^2\right) ds, \quad \text{if } v \neq K; \quad \text{or}$$

$$\frac{1}{24}K^{-2-\gamma}\left(K^2\right)^\gamma \gamma\sigma\left(6a + 6bK + 6cK^2 + 6dK^3 - 2K^{2\gamma}\sigma^2 + K^{2\gamma}\gamma\sigma^2\right), \quad \text{if } v = K.$$
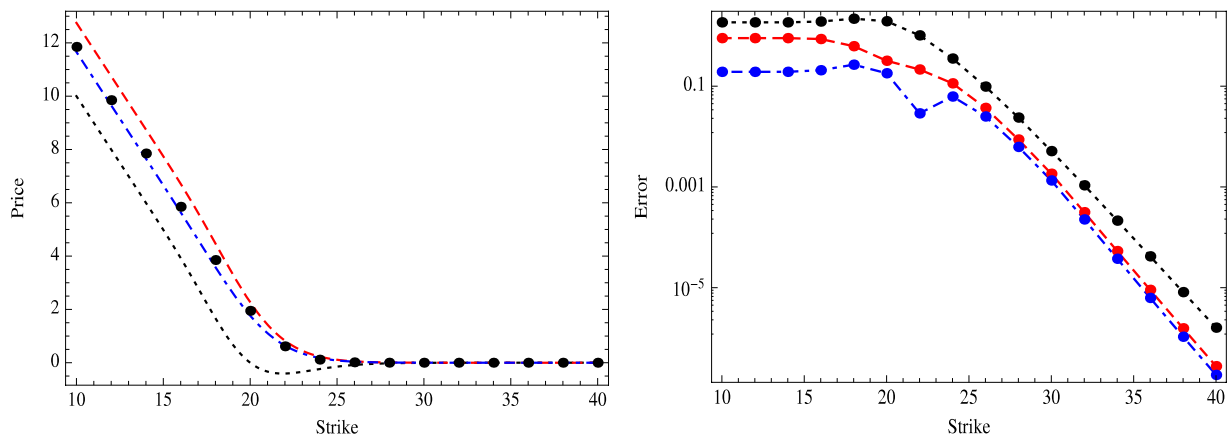
Box VI.

Fig. 6. Nonlinear mean-reversion model. Note: The black dotted line, red dashed line and blue dotted-dash line illustrate the $O(\Delta^{1/2})$, $O(\Delta^{3/2})$ and $O(\Delta^{5/2})$ order approximations respectively. The black dots denote the true prices. Y-axis of the right panel is on a logarithmic scale. The parameters are: $\sigma = 0.05$, $a = 500$, $b = 5$, $c = 0.05$, $d = -0.05$, $\sigma = 0.05$, $\gamma = 1.5$, and $\Delta = 90/360$. The current level of the VIX is equal to 20.

The first-order jump contribution can be separated out as well:

$$\lambda K\left(\frac{q}{1+\eta_2}\left(\frac{K}{x}\right)^{\eta_2}\Phi\left(\frac{\log(\frac{x}{K})}{\sigma\sqrt{\Delta}}\right) + \frac{p}{-1+\eta_1}\left(\frac{x}{K}\right)^{\eta_1}\right.$$

$$\left. \times\left(1 - \Phi\left(\frac{\log(\frac{x}{K})}{\sigma\sqrt{\Delta}}\right)\right)\right).$$

Similarly, no matter how much the proportion of positive jumps is, the jump impact on a vanilla option is always positive, as the primary contribution by jumps is to amplify the volatility. Nevertheless, always-positive-jumps ($q = 0$) tend to have a larger impact on in-the-money call options, whereas always-negative-jumps ($p = 0$) mainly target out-of-the-money call options. Therefore, we can expect that such jumps would turn an implied volatility smile into a volatility smirk. As λ decreases to zero, the model degenerates to the Black–Scholes case and the jump influence fades away.

4.5.3. Self-exciting jumps

To make the dynamics of jumps more reflective of the reality of the markets, we consider the following Hawkes jump-diffusion example where the jump intensity itself is another stochastic process driven by the same Poisson process. This jump process features contagion, or self-excitation. That is, the occurrence of a jump raises the odd of another jump in near future. A similar model was originally proposed by Hawkes (1971), and has recently been employed to model contagion phenomenon in finance by Aït-Sahalia et al. (2010). We compare this model with Merton's jump diffusion by examining the first several terms in the expansion.

Example 9 (*Hawkes' Jump-Diffusion Model*). Assume that the log price follows:

$$d\log X_t = \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma\,dW_t^Q + J_x dN_t - (m-1)\lambda_t dt,$$

$$d\lambda_t = \alpha(\lambda_\infty - \lambda_t)dt + J_\lambda dN_t,$$

where J_x has normal distribution $N\left(\log m - \frac{v^2}{2}, v^2\right)$, J_λ is exponentially distributed with mean β, and N_t is a self-exciting pure jump process with intensity λ_t following another stationary process with average intensity $E(\lambda_t) = \alpha\lambda_\infty/(\alpha - \beta)$. When $\beta = 0$ and $\lambda_0 = \lambda_\infty$, we have $\lambda_t = \lambda_\infty$, hence the model degenerates to

The following Closed-form expansions are plotted in Fig. 7.

$$\Psi(\Delta, x) = \Phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right)\sum_{k=0}^{\infty} B^{(k)}(x)\Delta^k + \sqrt{\Delta}\phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right)\sum_{k=0}^{\infty} C^{(k)}(x)\Delta^k + \sum_{k=1}^{\infty} D^{(k)}(x)\Delta^k$$

$$B^{(k)}(x) = \frac{(-1)^k\left(x(m\lambda)^k - K(r+\lambda)^k\right)}{k!}, \quad k \geq 0, \quad C^{(-1)}(x) = \frac{\log\left(\frac{x}{K}\right)}{\sigma}, \quad C^{(0)}(x) = \frac{\sigma(x-K)}{\log\left(\frac{x}{K}\right)}, \quad \text{if } x \neq K; \text{ or } K\sigma, \text{ if } x = K.$$

$$C^{(1)}(x) = \frac{\sigma}{\log\left(\frac{x}{K}\right)^3}\left((K-x)\sigma^2 + K\left(\frac{x}{K}\right)^{\frac{1}{2}-\frac{r}{\sigma^2}+\frac{(-1+m)\lambda}{\sigma^2}}\sigma^2\log\left(\frac{x}{K}\right) + (-mx\lambda + K(r+\lambda))\log\left(\frac{x}{K}\right)^2\right), \quad \text{if } x \neq K; \text{ or}$$

$$K\left(\frac{r^2}{2\sigma} + \frac{(-1+m)^2\lambda^2}{2\sigma} - \frac{(1+m)\lambda\sigma}{2} - \frac{\sigma^3}{24} - \frac{r\left(2(-1+m)\lambda + \sigma^2\right)}{2\sigma}\right), \quad \text{if } x = K.$$

$$D^{(1)}(x) = mx\lambda\Phi\left(\frac{\log(\frac{x}{K}) + \log(m) + \frac{1}{2}v^2}{v}\right) - K\lambda\Phi\left(\frac{\log(\frac{x}{K}) + \log(m) - \frac{v^2}{2}}{v}\right)$$

$$D^{(2)}(x) = \frac{1}{4}\lambda\left(e^{-\frac{v^4+4\log(\frac{mx}{K})^2}{8v^2}}\sqrt{\frac{2mxK}{\pi}}\frac{\sigma^2}{v} - 2K\lambda\Phi\left(\frac{-v^2 + 2\log(m) + \log(x) - \log(K)}{\sqrt{2}v}\right)\right.$$
$$- 4m^2x\lambda\Phi\left(\frac{v^2 + 2\log(mx) - 2\log(K)}{2v}\right) + 2m^2x\lambda\Phi\left(\frac{v^2 + 2\log(m) + \log(x) - \log(K)}{\sqrt{2}v}\right)$$
$$\left. + 4K(r+\lambda)\Phi\left(\frac{-v^2 + 2\log(mx) - 2\log(K)}{2v}\right)\right).$$
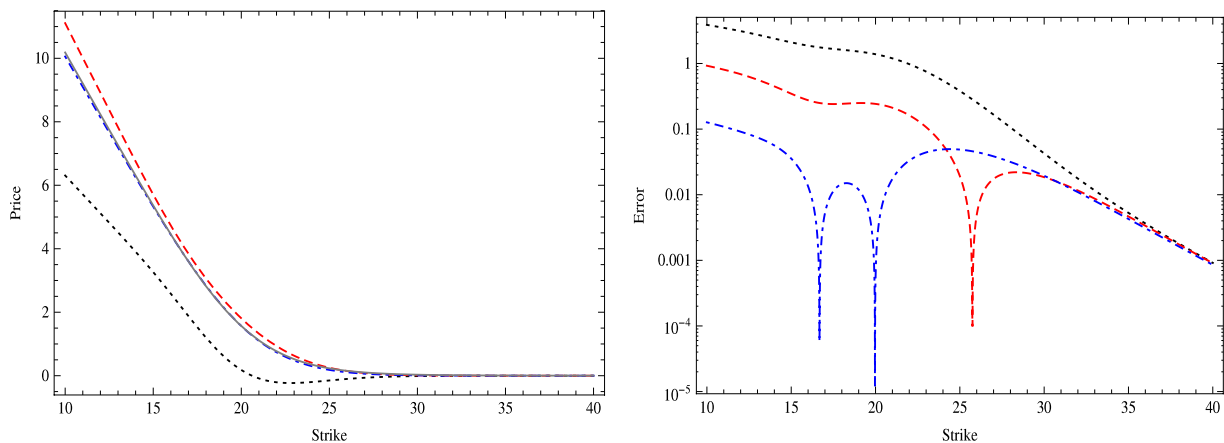
Box VII.

Fig. 7. Merton's jump diffusion model. Note: The black dotted line, red dashed line and blue dotted-dash line illustrate the $O(\Delta^{1/2})$, $O(\Delta^{3/2})$ and $O(\Delta^{5/2})$ approximations respectively. The grey line denotes the true prices. Y-axis of the right panel is on a logarithmic scale. The parameters are: $\sigma = 0.2, r = 4\%, x = 20, \Delta = 1, m = 0.5, v = 0.2$, and $\lambda = 0.25$.

Merton's case. The partial differential equation corresponding to this model is

$$0 = -\frac{\partial\Psi(\Delta, x, \lambda)}{\partial\Delta} + (r - (m-1)\lambda)x\frac{\partial\Psi(\Delta, x, \lambda)}{\partial x}$$
$$+ \frac{1}{2}\sigma^2 x^2\frac{\partial^2\Psi(\Delta, x, \lambda)}{\partial x^2} + \alpha(\lambda_\infty - \lambda)\frac{\partial\Psi(\Delta, x, \lambda)}{\partial\lambda}$$
$$- r\Psi(\Delta, x, \lambda) + \lambda\iint_{\mathbb{R}^2}\left(\Psi(\Delta, xe^z, \lambda + y)\right.$$
$$\left. - \Psi(\Delta, x, \lambda)\right)v_x(z)v_\lambda(y)dzdy.$$

(See Box IX.)

Note that the contagion parameter ($\beta \neq 0$) is revealed in $D^{(2)}(x, \lambda)$. Therefore, it influences the second order ($O(\Delta^2)$) of jump impact on the option value. This seems reasonable, as the contagion is expected to be exhibited after the first jump occurs. Although this model could potentially generate a cluster of large jumps, such jumps would not be able to replace the Brownian

leading term in the option expansions. When $\beta = 0$, and $\lambda = \lambda_\infty$, the expansions degenerate to those in Merton's case.

5. Numerical comparisons with asymptotic expansions

In this section, we benchmark our approach against alternative implied volatility expansions proposed in Gatheral et al. (2012) and Henry-Labordere (2008). Similar or related work also includes Hagan and Woodward (1999), Hagan et al. (2002), Henry-Labordere (2005), and Andersen and Brotherton-Ratcliffe (2001). Following the numerical setting in Gatheral et al. (2012), we consider the CEV and QV models discussed in Examples 2 and 3.

The underlying dynamics are driven by (25) and (26) respectively. The parameters in the CEV model are $\sigma = 0.25$ and $\gamma = 1.4$. For the QV model, we set $\sigma = 0.25$, $\psi = -0.5$, and $\gamma = -0.1$. Interest rates and dividends are set to 0s. We obtain the implied volatility by numerically inverting the Black–Scholes formula of price expansions. The time-to-maturities are chosen from 2 months to 1 year, and the moneyness from 0.5 to 2. We

The following Closed-form expansions are plotted in Fig. 8.

$$\Psi(\Delta, x) = \Phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} B^{(k)}(x)\Delta^k + \sqrt{\Delta}\phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} C^{(k)}(x)\Delta^k + \left(1 - \Phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right)\right) \sum_{k=1}^{\infty} D^{(k)}(x)\Delta^k$$

$$B^{(0)}(x) = x - K, \qquad B^{(1)}(x) = \frac{K(r(1+\eta_2) - (-1+p)(\frac{K}{x})^{\eta_2}\lambda)}{1+\eta_2},$$

$$B^{(2)}(x) = -\frac{1}{2(-1+\eta_1)(1+\eta_2)^2(\eta_1+\eta_2)}\left(K(-1+p)\left(\frac{K}{x}\right)^{\eta_2}\lambda^2\left((-1+p)\eta_1^2(1+2\eta_2) + (-1+p)\eta_1(-1-\eta_2+2\eta_2^2)\right.\right.$$
$$\left.+ \eta_2(1+p+2\eta_2+2p\eta_2+2p\eta_2^2) + (-1+p)(-1+\eta_1)\eta_2(1+\eta_2)(\eta_1+\eta_2)\log\left(\frac{K}{x}\right)\right) - \frac{Kr^2}{2}\right)$$
$$+ \frac{1}{2}K(-1+p)\left(\frac{K}{x}\right)^{\eta_2}\lambda(2r - \eta_2\sigma^2)$$

$$C^{(-1)}(x) = \frac{\log\left(\frac{x}{K}\right)}{\sigma}, \qquad C^{(0)}(x) = \frac{\sigma(x-K)}{\log\left(\frac{x}{K}\right)}, \quad \text{if } x \neq K; \text{ or } K\sigma, \text{ if } x = K.$$

$$C^{(1)}(x) = -\frac{\sigma}{(-1+\eta_1)(1+\eta_2)\log(\frac{K}{x})^3}\left((K-x)(-1+\eta_1)(1+\eta_2)\sigma^2 - \left(\frac{K}{x}\right)^{\frac{1}{2}+\frac{r}{\sigma^2} - \frac{(1+(-1+p)\eta_1+p\eta_2)\lambda}{(-1+\eta_1)(1+\eta_2)\sigma^2}}x(-1+\eta_1)(1+\eta_2)\right.$$
$$\sigma^2\log\left(\frac{K}{x}\right) - K\left(-r(-1+\eta_1)(1+\eta_2) + \left(-\left(\frac{K}{x}\right)^{\eta_2}(-1+\eta_1) + p\left(\frac{x}{K}\right)^{\eta_1}\left(1 - \left(\frac{K}{x}\right)^{\eta_1+\eta_2} + \left(\frac{K}{x}\right)^{\eta_1+\eta_2}\eta_1\right.\right.\right.$$
$$\left.\left.\left.+ \eta_2\right)\right)\lambda\right)\log\left(\frac{K}{x}\right)^2\right), \quad \text{if } x \neq K; \text{ or}$$
$$- \frac{K\lambda\left(2r(1+(-1+p)\eta_1+p\eta_2) + (-1+(-2+p)\eta_2+\eta_1(1+p+2\eta_2))\sigma^2\right)}{2(-1+\eta_1)(1+\eta_2)\sigma} + \frac{K(1+(-1+p)\eta_1+p\eta_2)^2\lambda^2}{2(-1+\eta_1)^2(1+\eta_2)^2\sigma}$$
$$- \frac{K\left(-12r^2 + 12r\sigma^2 + \sigma^4\right)}{24\sigma}$$

$$D^{(1)}(x) = \frac{Kp\left(\frac{x}{K}\right)^{\eta_1}\lambda}{-1+\eta_1}$$

$$D^{(2)}(x) = \frac{Kp\left(\frac{x}{K}\right)^{\eta_1}\lambda^2}{2(-1+\eta_1)^2(1+\eta_2)(\eta_1+\eta_2)}\left(2\eta_1 - p\eta_1 - 4\eta_1^2 + 2p\eta_1^2 + 2\eta_1^3 - 2p\eta_1^3 + p\eta_2\right.$$
$$\left.- p\eta_1\eta_2 - 2p\eta_1^2\eta_2 + p\eta_2^2 - 2p\eta_1\eta_2^2 + p(-1+\eta_1)\eta_1(1+\eta_2)(\eta_1+\eta_2)\log\left(\frac{K}{x}\right)\right) + \frac{1}{2}Kp\left(\frac{x}{K}\right)^{\eta_1}\lambda\left(2r + \eta_1\sigma^2\right)$$
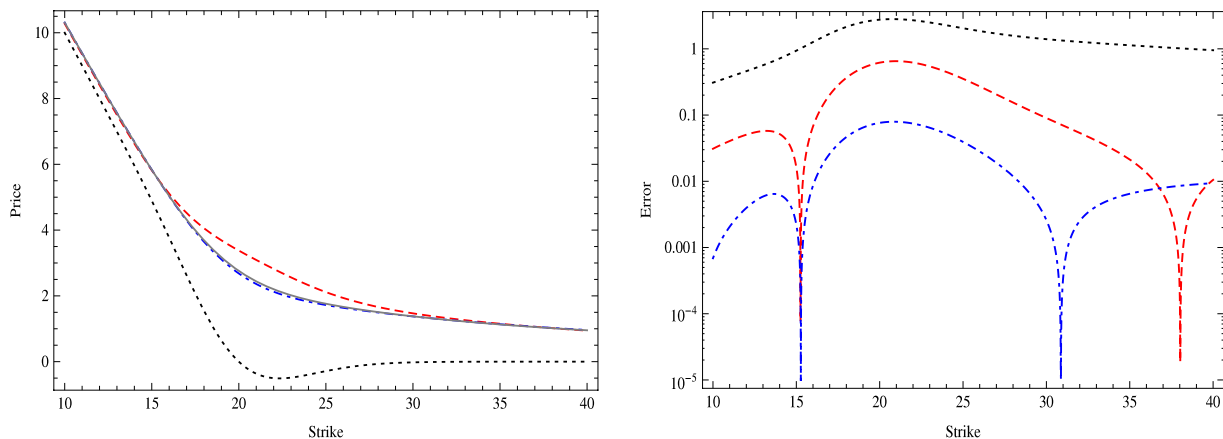
Box VIII.

Fig. 8. Kou's jump diffusion model. Note: The black dotted line, red dashed line and blue dotted-dash line illustrate the $O(\Delta^{1/2})$, $O(\Delta^{3/2})$ and $O(\Delta^{5/2})$ approximations respectively. The grey line denotes the true prices. Y-axis of the right panel is on a logarithmic scale. The parameters are: $\sigma = 0.2$, $r = 4\%$, $x = 20$, $\Delta = 0.5$, $\eta_1 = 2.5$, $\eta_2 = 3.2$, $p = 0.4$, and $\lambda = 1.0$.

plot in Fig. 10 the uniform absolute error throughout the moneyness dimension for each maturity. The results show that Hermite

Polynomial based expansions dominate the other methods with only 2 or 3 terms.

The following closed-form expansions are plotted in Fig. 9.

$$\Psi(\Delta, x, \lambda) = \Phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right)\sum_{k=0}^{\infty}B^{(k)}(x,\lambda)\Delta^k + \sqrt{\Delta}\phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right)\sum_{k=0}^{\infty}C^{(k)}(x,\lambda)\Delta^k + \sum_{k=1}^{\infty}D^{(k)}(x,\lambda)\Delta^k$$

$$B^{(0)}(x,\lambda) = x - K, \qquad B^{(1)}(x,\lambda) = -mx\lambda + K(r+\lambda)$$

$$B^{(2)}(x,\lambda) = \frac{1}{2}\Big(mx\big(m\lambda^2 + \alpha(\lambda - \lambda_\infty)\big) - K\big((r+\lambda)^2 + \alpha(\lambda - \lambda_\infty)\big)\Big)$$

$$C^{(-1)}(x,\lambda) = \frac{\log\left(\frac{x}{K}\right)}{\sigma}, \qquad C^{(0)}(x,\lambda) = \frac{\sigma(x-K)}{\log\left(\frac{x}{K}\right)}, \quad \text{if } x \neq K; \text{ or } K\sigma, \text{ if } x = K.$$

$$C^{(1)}(x,\lambda) = \frac{1}{\log(\frac{x}{K})^3}\sigma\left((K-x)\sigma^2 + K\left(\frac{x}{K}\right)^{\frac{1}{2} - \frac{r}{\sigma^2} + \frac{(-1+m)\lambda}{\sigma^2}}\sigma^2\log\left(\frac{x}{K}\right) + \left(K(r+\lambda) - xm\lambda\right)\log\left(\frac{x}{K}\right)^2\right), \quad \text{if } x \neq K; \quad \text{or}$$

$$K\left(\frac{r^2}{2\sigma} + \frac{(-1+m)^2\lambda^2}{2\sigma} - \frac{(1+m)\lambda\sigma}{2} - \frac{\sigma^3}{24} - \frac{r\left(2(-1+m)\lambda + \sigma^2\right)}{2\sigma}\right), \quad \text{if } x = K.$$

$$D^{(1)}(x,\lambda) = mx\lambda\Phi\left(\frac{\log(\frac{x}{K}) + \log(m) + \frac{1}{2}v^2}{v}\right) - K\lambda\Phi\left(\frac{\log(\frac{x}{K}) + \log(m) - \frac{v^2}{2}}{v}\right)$$

$$D^{(2)}(x,\lambda) = \frac{1}{4}\left(e^{-\frac{v^4 + 4\log(\frac{mx}{K})^2}{8v^2}}\sqrt{\frac{2Kmx}{\pi}}\frac{\lambda\sigma^2}{v} + 2K(2r\lambda + \lambda(\beta + 2\lambda) + \alpha(\lambda - \lambda_\infty))\Phi\left(\frac{-v^2 - 2\log(K) + 2\log(mx)}{2v}\right)\right.$$

$$+ 2m^2x\lambda(\beta + \lambda)\Phi\left(\frac{v^2 - \log(K) + 2\log(m) + \log(x)}{\sqrt{2}v}\right) - 2K\lambda(\beta + \lambda)\Phi\left(\frac{-v^2 + 2\log(m) - \log(K) + \log(x)}{\sqrt{2}v}\right)$$

$$\left. - 2mx(m\lambda(\beta + 2\lambda) + \alpha(\lambda - \lambda_\infty))\Phi\left(\frac{v^2 - 2\log(K) + 2\log(mx)}{2v}\right)\right)$$
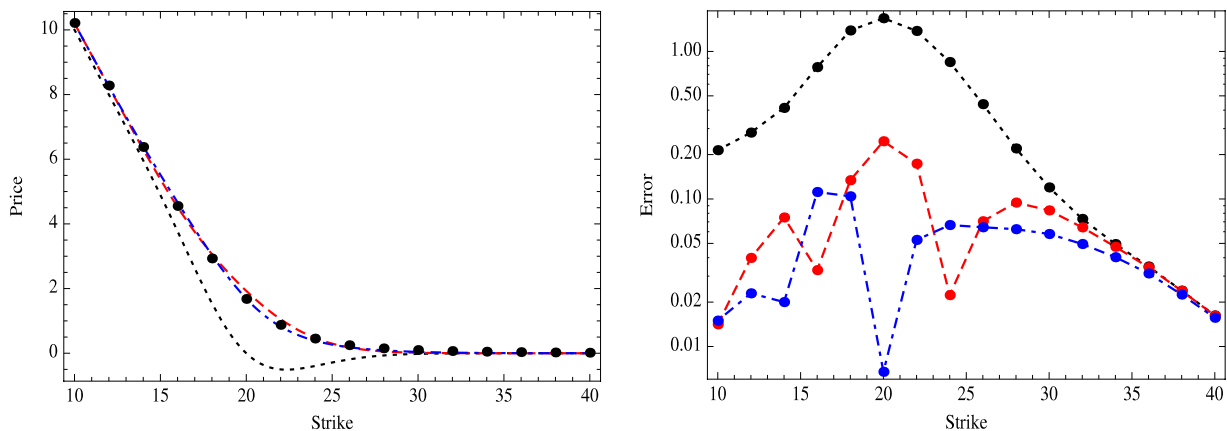
Box IX.

Fig. 9. Hawkes' jump diffusion model. Note: The black dotted line, red dashed line and blue dotted-dash line illustrate the $O(\Delta^{1/2})$, $O(\Delta^{3/2})$ and $O(\Delta^{5/2})$ order approximations respectively. The black dots denote the true prices. Y-axis of the right panel is on a logarithmic scale. The parameters are: $\sigma = 0.2$, $\alpha = 15$, $\beta = 10$, $m = 0.98$, $v = 0.15$, $r = 4\%$, $\lambda = 1.0$, $\lambda_\infty = 0.8$, $x = 20$, and $\Delta = 0.5$. The number of sample path is 5000 in Monte Carlo, each of which is simulated using Euler's method with 2520 time intervals.

6. Concluding remarks

This paper introduces a novel approach to expanding European option prices using a closed-form convergent sequence in the time variable. The proposed approach works with general dynamics, without any requirement on affine structure or closed-form characteristics functions, bridging the gap between sparse closed-form solutions and less informative numerical methods. In addition to the computational benefits from closed-form expansion formulae, this approach provides insight into how model parameters affect option prices and their relative importance as the option contract approaches expiration, which even closed-form solutions may not offer. Numerical comparisons highlight the accuracy of our approach as opposed to alternative expansion methods in the literature.
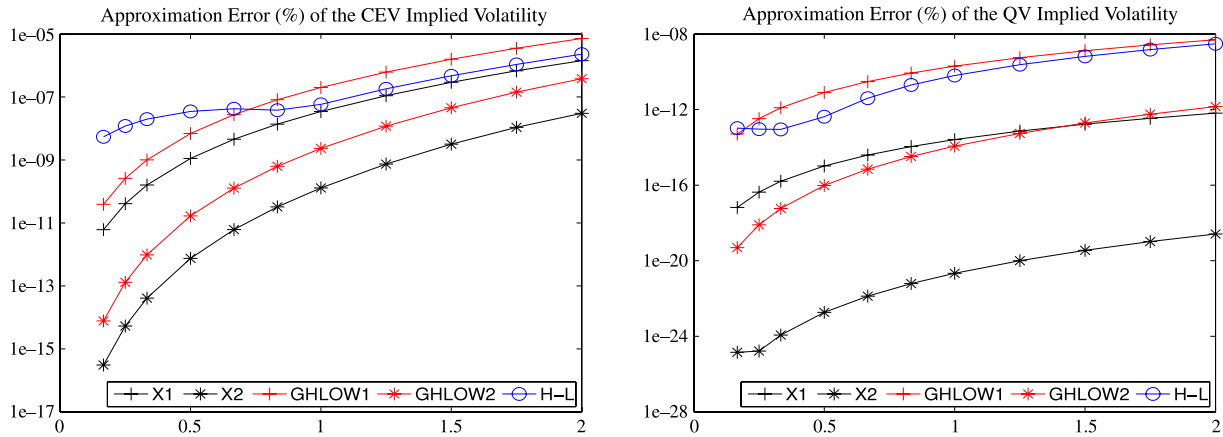
Acknowledgements

Fig. 10. Comparison of implied volatility approximations. Note: In this figure, we compare three different approximations of implied volatility for the CEV and the QV model respectively. "X1" and "X2" are obtained by inverting Hermite Polynomial based expansions of option prices up to the orders of $O(\Delta^{5/2})$ and $O(\Delta^{7/2})$, "GHLOW1" and "GHLOW2" are the first-order and second-order asymptotic approximations by Gatheral et al. (2012), and "H–L" is the approximation proposed by Henry-Labordere (2008).

Appendix

A.1. Assumptions

Assumption 1 (*Smoothness of the Coefficients*)**.** The functions $\mu(x)$ and $\sigma(x)$ are infinitely differentiable in $x \in D_X$.

Assumption 2 (*Nondegeneracy of the Diffusion*)**.** 1. $D_X = (-\infty, +\infty)$: there exists a constant $c > 0$, such that $\sigma(x) > c$ for all $x \in D_X$.

2. $D_X = (0, +\infty)$: near 0, if $\lim_{x \to 0+} \sigma(x) = 0$, there exist constant ξ_0, $\omega \geq 0$ and $\rho \geq 0$ such that $\sigma(x) \geq \omega x^\rho$ for all $0 < x < \xi_0$; away from 0, for each $\xi > 0$, there exists a constant $c_\xi > 0$ such that $\sigma(x) \geq c_\xi$ for all $x \in [\xi, \infty]$.

The constraints on the boundary behaviour are directly given to the transform diffusion Y_t for convenience. The domain of Y is given by $D_Y = (\underline{y}, \bar{y})$.

Assumption 3 (*Boundary Behaviour*)**.** Let $\lambda_Y(y) = -\frac{1}{2}(\mu_Y^2(y) + \frac{\partial \mu_Y(y)}{\partial y})$. μ_Y and $\frac{\partial \mu_Y(y)}{\partial y}$ have at most polynomial growth near the boundaries and $\lim_{y \to \partial D_Y} \lambda(y) < +\infty$. Near $\bar{y} = +\infty$, $\mu_Y(y) \leq -Ky^\beta$ for some $\beta > 1$; near $\underline{y} = -\infty$, $\mu_Y(y) \geq K|y|^\beta$ for some $\beta > 1$; near $\underline{y} = 0$, $\mu_Y(y) \geq \kappa \underline{y}^{-\alpha}$ for some $\alpha > 1$ and $\kappa > 0$; and near $\bar{y} = 0$, $\mu_Y(y) \leq -\kappa|y|^{-\alpha}$ for some $\alpha > 1$ and $\kappa > 0$.

A.2. Proof of Theorem 1

The proof is separated into two steps:

Step 1: Convergence of $\Psi^{(J)}(\Delta, x)$.

Choose $\bar{\Delta}$ as in Proposition 2 of Aït-Sahalia (2002). It suffices to show that $\sum_{j=0}^{J}(-1)^{j+1}\eta_{j+1}(\Delta, y)H_j(z)$ absolutely converges, uniformly for y in any compact set of D_y and for $z \in \mathbb{R}$. Let $z = \Delta^{-\frac{1}{2}}(y - y_0)$. Define

$$\nu_j(\Delta, y) = \frac{1}{j!}\int_{-\infty}^{\infty} H_j(z)\frac{\partial p_Z(\Delta, z|y)}{\partial z}dz.$$

As argued in Aït-Sahalia (2002) (page 252–253) that $\eta_j(\Delta, y)$ is well defined and satisfies that $\eta_j(\Delta, y) = \nu_{j+1}(\Delta, y)$, and by Theorem II in Stone (1927), there exists a constant K, such that for all $z \in \mathbb{R}$ and every integer j, $|H_j(z)| \leq K(j!)^{\frac{1}{2}}j^{-\frac{1}{4}}(1+2^{-\frac{5}{4}}|z|^{\frac{5}{2}})e^{\frac{z^2}{4}}$, therefore, we

have

$$|\eta_{j+1}(\Delta, y)H_j(z)| = |\nu_{j+2}(\Delta, y)||H_j(z)|$$
$$\leq \frac{1}{2}K(1 + 2^{-\frac{5}{4}}|z|^{\frac{5}{2}})e^{\frac{z^2}{4}}$$
$$\times \left((j!)j^{-\frac{1}{2}}((j+2)!)^{-1} + (j+2)!\nu_{j+2}^2(\Delta, y)\right).$$

Notice that $\sum(j!)j^{-\frac{1}{2}}((j+2)!)^{-1}$ converges. Also, Aït-Sahalia (2002) (page 253) shows that

$$\sum_{j=0}^{J} j!\nu_j^2(\Delta, y) \leq (2\pi)^{\frac{1}{2}}\int_{-\infty}^{+\infty} e^{\frac{z^2}{2}}\left(\frac{\partial p_Z(\Delta, z|y)}{\partial z}\right)^2 dz < \infty$$

uniformly for y over any compact set of D_Y. Hence, $\Psi^{(J)}(\Delta, x)$ converges, and $\Psi(\Delta, x)$ is well defined.

Step 2: $\lim_{\Delta \to 0}\Psi(\Delta, x) = 1$, if $x > K$, or 0 if $x < K$.

In fact, we have already shown that

$$\phi(z)\sum_{j=0}^{\infty}|\eta_{j+1}(\Delta, y)H_j(z)|$$
$$\leq \frac{1}{2}K\phi(z)(1 + 2^{-\frac{5}{4}}|z|^{\frac{5}{2}})e^{\frac{z^2}{4}}\left(\sum_{j=0}^{\infty}(j!)j^{-\frac{1}{2}}((j+2)!)^{-1}\right.$$
$$\left. + (2\pi)^{\frac{1}{2}}\int_{-\infty}^{+\infty} e^{\frac{z^2}{2}}\left(\frac{\partial p_Z(\Delta, z|y)}{\partial z}\right)^2 dz\right).$$

Further, it is shown by Aït-Sahalia (2002) (page 253) that for some constant b_i, $i = 0, \ldots, 4$,

$$\left|\frac{\partial p_Z(\Delta, z|y)}{\partial z}\right| \leq b_0 e^{-\frac{3z^2}{8}} R(|z|, |y|)e^{b_1|z||y| + b_2|z| + b_3|y| + b_4 y^2}, \quad (27)$$

where $R(|z|, |y|)$ is a polynomial of finite order in $|z|$ and y.

Plugging in $y = \gamma(x)$ and $z = \frac{\gamma(x) - \gamma(K)}{\sqrt{\Delta}}$, and let $\Delta \to 0$, it follows that

$$\phi\left(\frac{\gamma(x) - \gamma(K)}{\sqrt{\Delta}}\right)\sum_{j=0}^{\infty}\left|\eta_{j+1}\left(\Delta, \gamma(x)\right)H_j\left(\frac{\gamma(x) - \gamma(K)}{\sqrt{\Delta}}\right)\right| \longrightarrow 0.$$

Also, because $\gamma(x)$ is increasing in x, $\lim_{\Delta \to 0}\Phi\left(\frac{\gamma(x) - \gamma(K)}{\sqrt{\Delta}}\right) = 1_{\{x > K\}}$, which concludes the second step.

Step 3: $\Psi(\Delta, x)$ solves the Feynman–Kac PDE (2).

For convenience, we change variable $y = \gamma(x)$. Let $\tilde{\Psi}(\Delta, y) = \Psi(\Delta, x)$, and $\tilde{\Psi}^{(J)}(\Delta, y) = \Psi^{(J)}(\Delta, x)$. The Feynman–Kac PDE becomes

$$\left(-\frac{\partial}{\partial \Delta} + \mathcal{L}_Y\right)\tilde{\Psi}(\Delta, y) = 0, \tag{28}$$

where $\mathcal{L}_Y f = \mu_Y(y)\frac{\partial f}{\partial y} + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}$. Taking derivative of $\eta_j(\Delta, y)$ with respect to Δ and y, and using the fact that $H_j'(z) = -jH_{j-1}(z)$, we have

$$\frac{\partial \eta_j(\Delta, y)}{\partial \Delta} = \frac{1}{j!}\int_{-\infty}^{+\infty}\left(H_j'\left(\frac{\omega - y}{\sqrt{\Delta}}\right)\left(-\frac{\omega - y}{2\Delta^{\frac{3}{2}}}\right)p_Y(\Delta, \omega|y)\right.$$
$$\left. + H_j\left(\frac{\omega - y}{\sqrt{\Delta}}\right)\frac{\partial p_Y(\Delta, \omega|y)}{\partial \Delta}\right)d\omega,$$

$$\frac{\partial \eta_j(\Delta, y)}{\partial y} = \frac{1}{j!}\int_{-\infty}^{+\infty}\left(\left(-\frac{1}{\sqrt{\Delta}}\right)H_j'\left(\frac{\omega - y}{\sqrt{\Delta}}\right)p_Y(\Delta, \omega|y)\right.$$
$$\left. + H_j\left(\frac{\omega - y}{\sqrt{\Delta}}\right)\frac{\partial p_Y(\Delta, \omega|y)}{\partial y}\right)d\omega,$$

$$\frac{\partial^2 \eta_j(\Delta, y)}{\partial y^2} = \frac{1}{j!}\int_{-\infty}^{+\infty}\left(H_j'\left(\frac{\omega - y}{\sqrt{\Delta}}\right)\left(-\frac{1}{\sqrt{\Delta}}\right)\frac{\partial p_Y(\Delta, \omega|y)}{\partial y}\right.$$
$$\left. + H_j\left(\frac{\omega - y}{\sqrt{\Delta}}\right)\frac{\partial^2 p_Y(\Delta, \omega|y)}{\partial y^2}\right)d\omega$$
$$+ \frac{1}{(j-1)!}\int_{-\infty}^{+\infty}\left(H_{j-1}'\left(\frac{\omega - y}{\sqrt{\Delta}}\right)\left(-\frac{1}{\Delta}\right)p_Y(\Delta, \omega|y)\right.$$
$$\left. + \frac{1}{\sqrt{\Delta}}H_{j-1}\left(\frac{\omega - y}{\sqrt{\Delta}}\right)\frac{\partial p_Y(\Delta, \omega|y)}{\partial y}\right)d\omega.$$

By tedious calculation using the fact that $H_{j+1}(z) = H_j'(z) - zH_j(z)$, we have

$$\left(-\frac{\partial}{\partial \Delta} + \mathcal{L}_Y\right)\tilde{\Psi}^{(J)}(\Delta, y)$$
$$= \frac{1}{\sqrt{\Delta}}\phi\left(\frac{y - \gamma(K)}{\sqrt{\Delta}}\right)(-1)^{J+1}\left(\mu_Y(y)\eta_{J+1}(\Delta, y)\right.$$
$$\left. + \frac{\partial \eta_{J+1}(\Delta, y)}{\partial y}\right)H_{J+1}\left(\frac{y - \gamma(K)}{\sqrt{\Delta}}\right). \tag{29}$$

We have shown that as $J \to 0$, $\left|\eta_{J+1}(\Delta, y)H_{J+1}\left(\frac{y - \gamma(K)}{\sqrt{\Delta}}\right)\right| \longrightarrow 0$, uniformly in y over any compact set of D_Y. Similarly, as above, notice that

$$\frac{\partial \eta_{J+1}(\Delta, y)}{\partial y} = \frac{1}{(J+2)!}\int_{-\infty}^{+\infty}H_{J+2}(\omega)\frac{\partial^2 p_Z(\Delta, \omega|y)}{\partial \omega \partial y}d\omega,$$

it is then sufficient to show that $\int_{-\infty}^{+\infty}e^{\frac{\omega^2}{2}}\left(\frac{\partial^2 p_Z(\Delta, \omega|y)}{\partial \omega \partial y}\right)^2 d\omega < \infty$, which can be implied from a similar bound as in (27). Because of (29), and uniform convergence of $\tilde{\Psi}^{(J)}(\Delta, z)$ to $\tilde{\Psi}(\Delta, z)$, it follows that $\tilde{\Psi}(\Delta, z)$ satisfies the PDE (28), which concludes the proof.

A.3. Proof of Theorem 2

Taking the postulated terms of (15) into Eq. (13) respectively, we obtain

$$\left(-\frac{\partial}{\partial \Delta} + \mathcal{A} - r(x)\right)\left(\Phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right)\sum_{k=0}^{\infty}B^{(k)}(x)\Delta^k\right)$$
$$= \Phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right)\sum_{k=0}^{\infty}\left(-(k+1)B^{(k+1)}(x) - (r(x) + \lambda(x))B^{(k)}(x)\right.$$

$$\left. + \mathcal{L}B^{(k)}(x)\right)\Delta^k + \Delta^{-\frac{1}{2}}\phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right)\sum_{k=0}^{\infty}\left(B^{(k)}(x)\mathcal{L}C^{(-1)}(x)\right.$$
$$\left. + \left(\frac{dB^{(k)}(x)}{dx}\right)\sigma^2(x)\left(\frac{dC^{(-1)}(x)}{dx}\right)\right)\Delta^k$$
$$+ \lambda(x)\sum_{k=0}^{\infty}\int_{-\infty}^{\infty}\Phi\left(\frac{C^{(-1)}(x+z)}{\sqrt{\Delta}}\right)B^{(k)}(x+z)\nu(z)dz\Delta^k, \tag{30}$$

and

$$\left(-\frac{\partial}{\partial \Delta} + \mathcal{A} - r(x)\right)\left(\Delta^{\frac{1}{2}}\phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right)\sum_{k=0}^{\infty}C^{(k)}(x)\Delta^k\right)$$
$$= \phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right)\left(\psi_1(\Delta, x) + \psi_2(\Delta, x) + \psi_3(\Delta, x)\right)$$
$$+ \psi_4(\Delta, x),$$

where

$$\psi_1(\Delta, x) = \Delta^{-\frac{3}{2}}\sum_{k=0}^{\infty}\frac{1}{2}(C^{(-1)}(x))^2 C^{(k)}(x)$$
$$\times \left(\left(\frac{dC^{(-1)}(x)}{dx}\right)^2\sigma^2(x) - 1\right)\Delta^k,$$

$$\psi_2(\Delta, x) = \Delta^{\frac{1}{2}}\sum_{k=0}^{\infty}\left(-C^{(k+1)}(x)\left(1 + C^{(-1)}(x)\mathcal{L}C^{(-1)}(x)\right)\right.$$
$$- (k+1)C^{(k+1)}(x) - C^{(-1)}(x)\left(\frac{dC^{(k+1)}(x)}{dx}\right)$$
$$\left. \times \sigma^2(x)\left(\frac{dC^{(-1)}(x)}{dx}\right) + (\mathcal{L} - r(x) - \lambda(x))C^{(k)}(x)\right)\Delta^k,$$

$$\psi_3(\Delta, x) = \Delta^{-\frac{1}{2}}\left(-C^{(0)}(x)\left(1 + C^{(-1)}(x)\mathcal{L}C^{(-1)}(x)\right)\right.$$
$$\left. - C^{(-1)}(x)\left(\frac{dC^{(0)}(x)}{dx}\right)\sigma^2(x)\left(\frac{dC^{(-1)}(x)}{dx}\right)\right),$$

$$\psi_4(\Delta, x) = \lambda(x)\frac{\Delta^{\frac{1}{2}}}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{(C^{(-1)}(x+z))^2}{2\Delta}\right)$$
$$\times C^{(k)}(x+z)\nu(z)dz\Delta^k.$$

Let $D^{(0)}(x) = 0$. The contribution of the last term is

$$\left(-\frac{\partial}{\partial \Delta} + \mathcal{A} - r\right)\left(\sum_{k=1}^{\infty}D^{(k)}(x)\Delta^k\right)$$
$$= \sum_{k=0}^{\infty}\left(-(k+1)D^{(k+1)}(x) + (\mathcal{A} - r(x))D^{(k)}(x)\right)\Delta^k.$$

We only need to simplify the integral terms in $\psi_4(\Delta, x)$. Imposing that $\frac{dC^{(-1)}(x)}{dx} > 0$, and by the inverse mapping theorem, we assume that $C^{(-1)}(x)$ is invertible with the other variables fixed. By a similar argument as in Yu (2007), we change variable such that $\omega = C^{(-1)}(x+z)$, which is a function of z. Let M_r be the rth moment of the standard normal variable.

$$\int_{-\infty}^{\infty}\exp\left(-\frac{(C^{(-1)}(x+z))^2}{2\Delta}\right)C^{(k)}(x+z)\nu(z)dz$$
$$= \int_{-\infty}^{\infty}\exp\left(-\frac{\omega^2}{2\Delta}\right)C^{(k)}\left(((C^{(-1)})^{inv}(\omega))\right)$$
$$\times \nu((C^{(-1)})^{inv}(\omega) - x)\bigg/\left|\frac{dC^{(-1)}}{dx}\right|_{(C^{(-1)})^{inv}(\omega)}d\omega$$

$$= \int_{-\infty}^{\infty} \exp\left(-\frac{\omega^2}{2\Delta}\right) g_k(x, \omega) d\omega$$

$$= \int_{-\infty}^{\infty} \exp\left(-\frac{\omega^2}{2\Delta}\right) \sum_{r=0}^{\infty} \frac{1}{r!} \frac{\partial^r g_k}{\partial \omega^r}\bigg|_{\omega=0} \omega^r d\omega$$

$$= \sum_{r=0}^{\infty} \frac{1}{r!} \frac{\partial^r g_k}{\partial \omega^r}\bigg|_{\omega=0} \int_{-\infty}^{\infty} \exp\left(-\frac{\omega^2}{2\Delta}\right) \omega^r d\omega$$

$$= \sqrt{2\pi} \sum_{r=0}^{\infty} \frac{1}{r!} M_r \frac{\partial^r g_k}{\partial \omega^r}\bigg|_{\omega=0} \Delta^{\frac{r+1}{2}}$$

$$= \sqrt{2\pi} \sum_{r=0}^{\infty} \frac{1}{(2r)!} M_{2r} \frac{\partial^{2r} g_k}{\partial \omega^{2r}}\bigg|_{\omega=0} \Delta^{r+\frac{1}{2}}$$

Therefore, we have

$$\psi_4(\Delta, x) = \Delta \lambda(x) \sum_{k=0}^{\infty} \Delta^k \sum_{r=0}^{k} \frac{1}{(2r)!} M_{2r} \frac{\partial^{2r} g_{k-r}}{\partial \omega^{2r}}\bigg|_{\omega=0},$$

and

$$g_k(x, \omega) = C^{(k)}\left((C^{(-1)})^{inv}(\omega)\right) v\left(((C^{(-1)})^{inv}(\omega) - x)\right)$$

$$\times \left(\left|\frac{dC^{(-1)}}{dx}\right|_{(C^{(-1)})^{inv}(\omega)}\right)^{-1}.$$

Similarly, we can deduce that

$$\sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \Phi\left(\frac{C^{(-1)}(x+z)}{\sqrt{\Delta}}\right) B^{(k)}(x+z) v(z) dz \Delta^k$$

$$= \sum_{k=0}^{\infty} \frac{\Delta^{-\frac{1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{\omega^2}{2\Delta}\right) d\omega$$

$$\times \int_{(C^{(-1)})^{inv}(\omega)-x}^{\infty} B^{(k)}(x+z) v(z) dz \Delta^k$$

$$= \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{1}{(2r)!} \frac{\partial^{2r} h_{k-r}(x, \omega)}{\partial \omega^{2r}}\bigg|_{\omega=0} M_{2r} \Delta^k,$$

where

$$h_k(x, \omega) = \int_{(C^{(-1)})^{inv}(\omega)-x}^{\infty} B^{(k)}(x+z) v(z) dz.$$

Matching the coefficients yields the desired equations.

A.4. Expansion formulae for jump diffusions with positive jump sizes

When the jump size density is supported on $[0, \infty)$, or is non-smooth at origin, e.g. double exponential distributions, the expansion formulae are more involved. In fact, we may rewrite the postulated formula in the following way:

$$\Psi(\Delta, x) = \Phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} B^{(k)}(x) \Delta^k$$

$$+ \sqrt{\Delta} \phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} C^{(k)}(x) \Delta^k$$

$$+ \left(1 - \Phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right)\right) \sum_{k=1}^{\infty} D^{(k)}(x) \Delta^k.$$

Without loss of generality, we consider the $[0, \infty)$ case and focus on terms that are different from the previous case:

$$\psi_4(\Delta, x) = \lambda(x) \frac{\Delta^{\frac{1}{2}}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \int_0^{\infty} \exp\left(-\frac{(C^{(-1)}(x+z))^2}{2\Delta}\right)$$

$$\times C^{(k)}(x+z) v(z) dz \Delta^k$$

$$= \lambda(x) \phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} G_0(k, x) \Delta^{k+\frac{3}{2}}$$

$$+ \lambda(x) \left(1 - \Phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right)\right) \sum_{k=0}^{\infty} G_1(k, x) \Delta^{k+1},$$

where

$$G_0(k, x) = \sum_{j=0}^{k} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^l (C^{(-1)}(x))^{l+s}}{(s+2j+1)!!(s)!!(l)!}$$

$$\times \frac{\partial^{s+2j+1} g_{k-j}}{\partial \omega^{s+2j+1}}\bigg|_{\omega=C^{(-1)}(x)},$$

$$G_1(k, x) = \sum_{j=0}^{k} \sum_{r=0}^{\infty} \frac{(-1)^r C^{(-1)}(x)^r}{(r)!(2j)!!} \frac{\partial^{r+2j} g_{k-j}}{\partial \omega^{r+2j}}\bigg|_{\omega=C^{(-1)}(x)}.$$

Also,

$$\sum_{k=0}^{\infty} \int_0^{\infty} \Phi\left(\frac{C^{(-1)}(x+z)}{\sqrt{\Delta}}\right) B^{(k)}(x+z) v(z) dz \Delta^k$$

$$= \sum_{k=0}^{\infty} \int_0^{\infty} B^{(k)}(x+z) v(z) dz \Delta^k$$

$$- \sum_{k=0}^{\infty} \int_{C^{(-1)}(x)}^{\infty} \left(1 - \Phi\left(\frac{\omega}{\sqrt{\Delta}}\right)\right) \left(-\frac{\partial h_k(x, \omega)}{\partial \omega}\right) d\omega \Delta^k$$

$$= \sum_{k=0}^{\infty} \int_0^{\infty} B^{(k)}(x+z) v(z) dz \Delta^k$$

$$+ \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{\partial^{r+1} h_k(x, \omega)}{\partial \omega^{r+1}}\bigg|_{\omega=0} \int_{C^{-1}(x)}^{\infty} \left(1 - \Phi\left(\frac{\omega}{\sqrt{\Delta}}\right)\right) \frac{\omega^r}{r!} d\omega \Delta^k$$

$$= \phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} H_0(k, x) \Delta^{k+\frac{1}{2}}$$

$$+ \Phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right) \sum_{k=0}^{\infty} (H_2(k, x) - H_1(k-1, x)) \Delta^k$$

$$+ \sum_{k=0}^{\infty} \left(h_k(x, C^{(-1)}(x)) - H_2(k, x) + H_1(k-1, x)\right) \Delta^k,$$

where

$$H_0(k, x) = \sum_{j=0}^{k} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^l (C^{(-1)}(x))^{l+s}}{(s+2j+1)!!(s)!!(l)!}$$

$$\times \frac{\partial^{s+2j+1} h_{k-j}}{\partial \omega^{s+2j+1}}\bigg|_{\omega=C^{(-1)}(x)},$$

$$H_1(k, x) = \sum_{j=0}^{k} \sum_{r=0}^{\infty} \frac{(-1)^r (C^{(-1)}(x))^r}{(r)!(2j+2)!!} \frac{\partial^{r+2j+2} h_{k-j}}{\partial \omega^{r+2j+2}}\bigg|_{\omega=C^{(-1)}(x)},$$

$$H_2(k, x) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s (C^{(-1)}(x))^{r+1+s}}{(r+1)!s!} \frac{\partial^{r+1+s} h_k}{\partial \omega^{r+1+s}}\bigg|_{\omega=C^{(-1)}(x)}.$$

The above calculations rely on the following formula: for any integer $k \geq 0$,

$$
\int_A^\infty \omega^r e^{-\frac{\omega^2}{2\Delta}} d\omega
$$

$$
= \begin{cases}
2^k \Delta^{k+1} e^{-\frac{A^2}{2\Delta}} \sum_{l=0}^{k} \frac{k!}{l!}\left(\frac{A^{2l}}{2^l \Delta^l}\right) & r = 2k+1, \\[2ex]
2^{k+\frac{1}{2}} \Delta^{k+\frac{3}{2}} \left\{ \frac{(2k+1)!!}{2^k} \sqrt{\pi} \Phi\left(-\frac{A}{\sqrt{\Delta}}\right) \right. & \\[1ex]
\left. + e^{-\frac{A^2}{2\Delta}} \sum_{l=1}^{k+1} \frac{(2k+1)!!}{2^{k+\frac{1}{2}}(2l-1)!!}\left(\frac{A^{2l-1}}{\Delta^{l-\frac{1}{2}}}\right) \right\} & r = 2k+2.
\end{cases}
$$

and

$$
\int_A^\infty \omega^r \left(1 - \Phi\left(\frac{\omega}{\sqrt{\Delta}}\right)\right) d\omega
$$

$$
= -\left(1 - \Phi\left(\frac{A}{\sqrt{\Delta}}\right)\right)\frac{A^{r+1}}{r+1} + \frac{1}{\sqrt{\Delta}} \int_A^\infty \frac{\omega^{r+1}}{r+1} \phi\left(\frac{\omega}{\sqrt{\Delta}}\right) d\omega.
$$

Finally, we denote $D^{(0)} = 0$ and consider

$$
\left(-\frac{\partial}{\partial \Delta} + \mathcal{A} - r(x)\right)\left(\left(1 - \Phi\left(\frac{C^{(-1)}(x)}{\sqrt{\Delta}}\right)\right) \sum_{k=0}^\infty D^{(k)}(x)\Delta^k\right).
$$

Notice that we may reuse formulae (30) by replacing $C^{(-1)}$ and $B^{(k)}$ there with $-C^{(-1)}$ and $D^{(k)}$. The rest derivation is exactly the same, hence is omitted.

The new restrictions on $B^{(k)}(x)$, $C^{(k)}(x)$ and $D^{(k)}(x)$ given below can be derived by matching the coefficients before $\Phi(\cdot)$, $\phi(\cdot)$ and Δ^k. For any $k \geq -1$, we have

$$
0 = B^{(0)} - f(x),
$$

$$
0 = (k+2)B^{(k+2)} + (r(x) + \lambda(x))B^{(k+1)} \\
- \mathcal{L}B^{(k+1)} - \lambda(x)h_k(x, C^{(-1)}),
$$

$$
0 = -C^{(0)}\left(1 + C^{(-1)}\mathcal{L}C^{(-1)}\right) - \sigma(x)C^{(-1)}\frac{dC^{(0)}(x)}{dx} \\
+ B^{(0)}\mathcal{L}C^{(-1)} + \sigma(x)\frac{dB^{(0)}(x)}{dx},
$$

$$
0 = -C^{(k+2)}\left(k+3 + C^{(-1)}\mathcal{L}C^{(-1)}\right) - \sigma(x)C^{(-1)}\frac{dC^{(k+2)}}{dx} \\
+ \lambda(x)(H_0(k+1, x) - J_0(k+1, x) + G_0(k, x)) \\
+ \sigma(x)\left(\frac{dB^{(k+2)}}{dx} - \frac{dD^{(k+2)}}{dx}\right) + (\mathcal{L} - r(x) - \lambda(x))C^{(k+1)} \\
+ (B^{(k+2)} - D^{(k+2)})\mathcal{L}C^{(-1)},
$$

$$
0 = -(k+2)D^{(k+2)} + (\mathcal{L} - r(x) - \lambda(x))D^{(k+1)} \\
+ \lambda(x)\Big(H_1(k, x) - J_1(k, x) + G_1(k, x) + h_{k+1}(x, C^{-1}) \\
- H_2(k+1, x) + J_2(k+1, x)\Big),
$$

where

$$
J_0(k, x) = \sum_{i=0}^{k} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^l (C^{(-1)}(x))^{l+s}}{(s+2i+1)!!(s)!!(l)!} \frac{\partial^{s+2i+l} j_{k-i}}{\partial \omega^{s+2i+l}}\Bigg|_{\omega = C^{(-1)}(x)},
$$

$$
J_1(k, x) = \sum_{i=0}^{k} \sum_{r=0}^{\infty} \frac{(-1)^r (C^{(-1)}(x))^r}{(r)!(2i+2)!!} \frac{\partial^{r+2i+1} j_{k-i}}{\partial \omega^{r+2i+1}}\Bigg|_{\omega = C^{(-1)}(x)},
$$

$$
J_2(k, x) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s (C^{(-1)}(x))^{r+1+s}}{(r+1)!s!} \frac{\partial^{r+s} j_k}{\partial \omega^{r+s}}\Bigg|_{\omega = C^{(-1)}(x)},
$$

and

$$
j_k(x, \omega) = -D^{(k)}\left((C^{(-1)})^{inv}(\omega)\right)v\left((C^{(-1)})^{inv}(\omega) - x\right) \\
\times \left(\left|\frac{dC^{(-1)}}{dx}\right|_{(C^{(-1)})^{inv}(\omega)}\right)^{-1}.
$$

A.5. Expansion formulae for multivariate diffusions

For multivariate models with covariance matrices of the form (24) and option payoff at expiration $f(x)1_{\{x_1 > K\}}$, the coefficients for option price expansion satisfy the following restrictions:

$$
0 = B^{(0)} - f(x),
$$

$$
0 = (k+1)B^{(k+1)} + r(x)B^{(k)} - \mathcal{L}B^{(k)},
$$

$$
0 = -C^{(0)}\left(1 + C^{(-1)}\mathcal{L}C^{(-1)}\right) - C^{(-1)}\frac{\partial C^{(-1)}}{\partial x^\tau} \cdot V(x) \cdot \frac{\partial C^{(0)}}{\partial x} \\
+ B^{(0)}(x)\mathcal{L}C^{(-1)} + \frac{\partial C^{(-1)}}{\partial x^\tau} \cdot V(x) \cdot \frac{\partial B^{(0)}}{\partial x},
$$

$$
0 = -C^{(k+1)}\left(k+2 + C^{(-1)}\mathcal{L}C^{(-1)}\right) \\
- C^{(-1)}\frac{\partial C^{(-1)}}{\partial x^\tau} \cdot V(x) \cdot \frac{\partial C^{(k+1)}}{\partial x} + B^{(k+1)}\mathcal{L}C^{(-1)} \\
+ \frac{\partial C^{(-1)}}{\partial x^\tau} \cdot V(x) \cdot \frac{\partial B^{(k+1)}}{\partial x} + (\mathcal{L} - r(x))C^{(k)},
$$

where $C^{(-1)}(x) = \Delta^{-1/2}\int_K^{x_1} v_{11}(s)^{-1/2}ds$, $\mathcal{L} = \mu(x)^\tau \cdot \frac{\partial}{\partial x} + \frac{1}{2}\text{Trace}(V(x) \cdot \frac{\partial^2}{\partial x \partial x^\tau})$, and $\cdot$ denotes matrix multiplication.

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