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EXPLICIT IMPLIED VOLATILITIES FOR MULTIFACTOR LOCAL-STOCHASTIC VOLATILITY MODELS

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We consider an asset whose risk-neutral dynamics are described by a general class of local-stochastic volatility models and derive a family of asymptotic expansions for European-style option prices and implied volatilities. We also establish rigorous error estimates for these quantities. Our implied volatility expansions are explicit; they do not require any special functions nor do they require numerical integration. To illustrate the accuracy and versatility of our method, we implement it under four different model dynamics: constant elasticity of variance local volatility, Heston stochastic volatility, three-halves stochastic volatility, and SABR local-stochastic volatility.

KEY WORDS: implied volatility, local-stochastic volatility, CEV, Heston, SABR.

1. INTRODUCTION

Local-stochastic volatility (LSV) models combine the features of local volatility (LV) and SV models by describing the instantaneous volatility of an underlying S by a function $\sigma(t, S_t, Y_t)$ where Y is some auxiliary, possibly multidimensional, stochastic process (see, for instance, Lipton 2002; Alexander and Nogueira 2004; Ewald 2005; Henry-Labordère 2009b; and Clark 2010). Typically, unobservable LSV (or SV or LV) model parameters are obtained by calibrating to implied volatilities that are observed on the market. For this reason, closed-form approximations for model-induced implied volatilities are useful. A number of different approaches have been taken for computing approximate implied volatilities in LV, SV, and LSV models. We review some of these approaches below.

Concerning LV models, perhaps, the earliest and most well-known implied volatility result is due to Hagan and Woodward (1999), who use singular perturbation methods to obtain an implied volatility expansion for general LV models. For certain models (e.g., constant elasticity of variance [CEV]), they obtain closed-form approximations. More recently, Lorig (2013) uses regular perturbation methods to obtain an implied volatility

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expansion when an LV model can be written as a regular perturbation around Black–Scholes. Jacquier and Lorig (2013) extend and refine the results of Lorig (2013) to find closed-form approximations of implied volatility for local Lévy-type models with jumps. Gatheral et al. (2012) examine the small-time asymptotics of implied volatility for LV models using heat kernel methods.

There is no shortage of implied volatility results for SV models either. Fouque, Lorig, and Sircar (2012) (see also Fouque et al. 2011) derive an asymptotic expansion for general multiscale SV models using combined singular and regular perturbation theory. Forde and Jacquier (2011) use the Freidlin–Wentzell theory of large deviations for SDEs to obtain the small-time behavior of implied volatility for general SV models with zero correlation. Their work adds mathematical rigor to previous work by Lewis (2007). Forde and Jacquier (2009) use large deviation techniques to obtain the small-time behavior of implied volatility in the Heston model (with correlation). They further refine these results in Forde, Jacquier, and Lee (2012). More recently, Jacquier and Lorig (2015) provide an explicit implied volatility approximation for any model with an analytically tractable characteristic function, which includes both affine SV and exponential Lévy models.

Concerning LSV models, perhaps, the most well-known implied volatility result is due to Hagan et al. (2002), who use Wentzel-Kramers-Brillouin approximation methods to obtain implied volatility asymptotics in an LSV model with a CEV-like factor of LV and a GBM-like auxiliary factor of volatility (i.e., the SABR model). Another important contribution is due to Berestycki, Busca, and Florent (2004), who show that implied volatility in an LSV setting can be obtained by solving a quasi-linear parabolic partial differential equation. More recently, Henry-Labordère (2005) uses a heat kernel expansion on a Riemann manifold to derive first-order asymptotics for implied volatility for any LSV model. As an example, he introduces the λ-SABR model, which is an LSV model with a mean reverting auxiliary factor of volatility, and obtains closed-form asymptotic formulas for implied volatility in this setting. See also Henry-Labordère (2009a). Finally, we mention Watanabe (1987) and the recent work of Benhamou, Gobet, and Miri (2010) and Bompis and Gobet (2012) who use Malliavin calculus techniques to derive closed-form approximations for implied volatility in an LSV setting. There are also some model-free results concerning the extreme-strike behavior of implied volatility. Most notably, we mention the work of Lee (2004) and Gao and Lee (2014).

In this paper, we provide closed-form approximations for implied volatility for a very general class of LSV models. We show (through a series of numerical experiments) that our approximation performs favorably when compared to other well-known implied volatility approximations (e.g., Hagan and Woodward 1999 for CEV; Forde et al. 2012 for Heston; and Hagan et al. 2002 for SABR). Additionally, we prove that our implied volatility expansion satisfies some rigorous error bounds for short maturities. As a by-product of the implied volatility analysis, we obtain some results concerning the short-maturity behavior of the Black–Scholes price, which appear to be new and of some independent interest. All of our results are consistent with the previously derived short-maturity asymptotic results appearing in Berestycki et al. (2002), Berestycki, Busca, and Florent (2004), and Bompis and Gobet (2012). The methodology presented in this paper builds upon the asymptotic pricing formulas first derived in Pagliarani and Pascucci (2012) for scalar diffusions and later extended in Pagliarani, Pascucci, and Riga (2013) and Lorig, Pagliarani, and Pascucci (2015b) for scalar Lévy-type processes.

The rest of this paper proceeds as follows: In Section 2, we introduce a general class of LSV models. We also derive a family of asymptotic expansions for European option prices and, under certain assumptions, provide rigorous error bounds for our pricing approximations. In Section 3, we translate our asymptotic price expansion into an asymptotic expansion of implied volatility. In Section 4, we establish rigorous error estimates for both our pricing and implied volatility expansions. Finally, in Section 5, we test our implied volatility approximation on four well-known models: CEV LV, Heston SV, three-halves SV, and SABR LSV.

2. ASYMPTOTIC PRICING FOR A GENERAL CLASS OF LSV MODELS

For simplicity, we assume a frictionless market, no arbitrage, zero interest rates, and no dividends. We take, as given, an equivalent martingale measure $\mathbb{P}$, chosen by the market on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$. The filtration $\{\mathcal{F}_t, t \geq 0\}$ represents the history of the market. All stochastic processes defined below live on this probability space and all expectations are taken with respect to $\mathbb{P}$. We consider a strictly positive asset S whose risk-neutral dynamics are given by $S = \exp(X)$ where $X = Z^{(1)}$ is the first component of a d-dimensional diffusion $Z = (X, Y)$, which solves

$$\left.\begin{aligned} \mathrm{d}Z_t^{(i)} &= \mu_i(t, Z_t)\mathrm{d}t + \sigma_i(t, Z_t)\mathrm{d}W_t^{(i)}, \quad Z_0 = z \in \mathbb{R}^d, \\ \mathrm{d}\langle W^{(i)}, W^{(j)}\rangle_t &= \rho_{ij}(t, Z_t)\,\mathrm{d}t, \quad |\rho_{ij}| < 1. \end{aligned}\right\} \tag{2.1}$$

We assume that SDE (2.1) has a unique strong solution. Sufficient conditions for the existence of a unique strong solution can be found, for example, in Ikeda and Watanabe (1989) or Pascucci (2011). We also assume that the coefficients are such that $\mathbb{E}[S_t] < \infty$ for all $t \in [0, T_0]$ for some positive T_0.

Let V_t be the time t value of a European derivative, expiring at time $T > t$ with payoff $\varphi(X_T)$. Using risk-neutral pricing, to value a European-style option, we must compute functions of the form

$$u(t, x, y) := \mathbb{E}[\varphi(X_T)|X_t = x, Y_t = y].$$

It is well known that under mild assumptions, the function u satisfies the Kolmogorov backward equation

$$(\partial_t + \mathcal{A}(t))u(t, x, y) = 0, \qquad u(T, x, y) = \varphi(x), \tag{2.2}$$

where the operator $\mathcal{A}(t)$ is given explicitly by

$$\mathcal{A}(t) = \frac{1}{2}\sum_{i,j=1}^{d} \rho_{ij}(t, z)\sigma_i(t, z)\sigma_j(t, z)\partial_{z_i z_j} + \sum_{i=1}^{d} \mu_i(t, z)\partial_{z_i}. \tag{2.3}$$

As a standing assumption, we impose $\mu_1 = -\frac{1}{2}\sigma_1^2$ so as to ensure that $S = \mathrm{e}^X$ is a martingale. For many models in finance, the dimension of the diffusion is $d = 1$ (e.g., CEV) or $d = 2$ (e.g., Heston and SABR). For the special cases $d = 1, 2$, we write $\mathcal{A}(t)$ as

$$\mathcal{A}(t) = a(t, x, y)(\partial_x^2 - \partial_x) + f(t, x, y)\partial_y + b(t, x, y)\partial_y^2 + c(t, x, y)\partial_x\partial_y, \quad (x, y) \in \mathbb{R}^2, \tag{2.4}$$

where

$$a := \frac{\sigma_1^2}{2}, \qquad f := \mu_2, \qquad b := \frac{\sigma_2^2}{2}, \qquad c := \rho\sigma_1\sigma_2.$$

When $d = 1$ (i.e., LV models), only a appears.

REMARK 2.1. (Deterministic and stochastic interest rates). For deterministic interest rates $r(t)$, one must compute expectations of the form

$$\widetilde{u}(t, \widetilde{x}, y) := \mathbb{E}\left[\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}\varphi(\widetilde{X}_T)|\widetilde{X}_t = \widetilde{x},\ Y_t = y\right], \qquad \text{where} \qquad \mathrm{d}\widetilde{X}_t = \mathrm{d}X_t + r(t)\mathrm{d}t.$$

In this case, a simple change of variables

$$u(t, x(t, \widetilde{x}), y) := \mathrm{e}^{\int_t^T r(s)}\widetilde{u}(t, \widetilde{x}, y), \qquad x(t, \widetilde{x}) := \widetilde{x} + \int_t^T r(s)\mathrm{d}s, \tag{2.5}$$

reveals that the function u, as defined as in (2.5), satisfies (2.2). On the other hand, the case of stochastic interest rates r_t cannot be covered by a simple change of variables. Our techniques seem to be flexible enough to incorporate a stochastic short rate (similar to Benhamou, Gobet, and Miri 2012). However, such an extension is not trivial. As such, we leave it for future study.

2.1. Polynomial Expansions of $\mathcal{A}(t)$

We note that (2.3) is a special case of the more general d-dimensional second-order differential operator

$$\mathcal{A}(t) = \sum_{i,j=1}^{d} a_{ij}(t, z)\partial_{z_i z_j} + \sum_{i=1}^{d} a_i(t, z)\partial_{z_i}, \qquad t \in \mathbb{R}_+,\ z \in \mathbb{R}^d. \tag{2.6}$$

Equivalently, we can also write the operator $\mathcal{A}(t)$ in a more compact form, i.e.,

$$\mathcal{A}(t) := \sum_{|\alpha| \leq 2} a_\alpha(t, z) D_z^\alpha, \qquad t \in \mathbb{R}_+, z \in \mathbb{R}^d, \tag{2.7}$$

where using standard multi-index notation, we have

$$\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}_0^d, \qquad |\alpha| = \sum_{i=1}^{d} \alpha_i, \qquad D_z^\alpha = \partial_{z_1}^{\alpha_1} \cdots \partial_{z_d}^{\alpha_d}.$$

In this section, we introduce a family of expansion schemes for $\mathcal{A}(t)$, which we shall use to construct closed-form approximate solutions (one for each family) of the Cauchy problem (2.2).

DEFINITION 2.2. Let $N \in \mathbb{N}_0$. We say that $(\mathcal{A}_n(t))_{0 \leq n \leq N}$ is an Nth-order polynomial expansion if

$$\mathcal{A}_n(t, z) \equiv \mathcal{A}_n(t) := \sum_{|\alpha| \leq 2} a_{\alpha,n}(t, z) D_z^\alpha, \tag{2.8}$$

where

(i) for any $t \in [0, T]$, the functions $a_{\alpha,n}(t, \cdot)$ are polynomials, and for any $z \in \mathbb{R}^d$, the functions $a_{\alpha,n}(\cdot, z)$ belong to $L^\infty([0, T])$,
(ii) for any $t \in [0, T]$, we have $a_{\alpha,0}(t, \cdot) = a_{\alpha,0}(t)$, and the constant-in-space coefficients' second-order operator $\mathcal{A}_0(t)$ is elliptic.

The idea behind our approximation method is to choose a polynomial expansion such that the sequences of partial sums $\sum_{n=0}^{N} a_{\alpha,n}(t)$ approximate the coefficients $a_\alpha(t,z)$, either pointwise or in some norm. Below, we present some examples.

EXAMPLE 2.3 (Taylor polynomial expansion). Assume the coefficients $a_\alpha(t,\cdot) \in C^N(\mathbb{R}^d)$. Then, for any fixed $\bar{z} \in \mathbb{R}^d$, $n \le N$, we define $a_{\alpha,n}$ as the nth-order term of the Taylor expansion of a_α in the spatial variables around $\bar{z}$. That is, we set

$$a_{\alpha,n}(\cdot,z) = \sum_{|\beta|=n} \frac{D_z^\beta a_\alpha(\cdot,\bar{z})}{\beta!}(z-\bar{z})^\beta, \qquad n \le N, \qquad |\alpha| \le 2,$$

where as usual $\beta! = \beta_1! \cdots \beta_d!$ and $z^\beta = z_1^{\beta_1} \cdots z_d^{\beta_d}$.

EXAMPLE 2.4 (Time-dependent Taylor polynomial expansion). Assume the coefficients $a_\alpha(t,\cdot) \in C^N(\mathbb{R}^d)$. Then, for any fixed $\bar{z} : \mathbb{R}_+ \to \mathbb{R}^d$, we define $a_{\alpha,n}$ as the nth-order term of the Taylor expansion of a_α in the spatial variables around $\bar{z}(\cdot)$. That is, we set

$$a_{\alpha,n}(\cdot,z) = \sum_{|\beta|=n} \frac{D_z^\beta a_\alpha(\cdot,\bar{z}(\cdot))}{\beta!}(z-\bar{z}(\cdot))^\beta, \qquad n \le N, \qquad |\alpha| \le 2.$$

EXAMPLE 2.5. (Hermite polynomial expansion). Hermite expansions can be useful when the diffusion coefficients are not smooth. A remarkable example in financial mathematics is given by the Dupire's LV formula for models with jumps (see Friz, Gerhold, and Yor 2013). In some cases, e.g., the well-known variance-gamma model, the fundamental solution (i.e., the transition density of the underlying stochastic model) has singularities. In such cases, it is natural to approximate it in some L^p norm rather than in the pointwise sense. For the Hermite expansion centered at $\bar{z}$, one sets

$$a_{\alpha,n}(t,z) = \sum_{|\beta|=n} \langle \mathbf{H}_\beta(\cdot-\bar{z}), a_\alpha(t,\cdot)\rangle_\Gamma \mathbf{H}_\beta(z-\bar{z}), \qquad n \ge 0, \qquad |\alpha| \le 2,$$

where the inner product $\langle\cdot,\cdot\rangle_\Gamma$ is an integral over $\mathbb{R}^d$ with a Gaussian weighting centered at $\bar{z}$ and the functions $\mathbf{H}_\beta(z) = H_{\beta_1}(z_1)\cdots H_{\beta_d}(z_d)$ where H_n is the nth one-dimensional Hermite polynomial (properly normalized so that $\langle \mathbf{H}_\alpha, \mathbf{H}_\beta\rangle_\Gamma = \delta_{\alpha,\beta}$ with $\delta_{\alpha,\beta}$ being the Kronecker delta function).

2.2. Formal Solution

In this section, we introduce a heuristic procedure to construct an approximate solution of the backward Cauchy problem (2.2). Hereafter, we will explicitly indicate t-dependence in all operators. On the other hand, we will generally hide z-dependence, except where it is needed for clarity.

Let us consider a polynomial expansion $(\mathcal{A}_n(t))_{n\ge0}$, and assume that the operator $\mathcal{A}(t)$ in (2.7) can be formally written as

$$\mathcal{A}(t) = \mathcal{A}_0(t) + \mathcal{B}(t), \qquad \mathcal{B}(t) = \sum_{n=1}^{\infty} \mathcal{A}_n(t). \tag{2.9}$$

Inserting expansion (2.9) for $\mathcal{A}(t)$ into Cauchy problem (2.2), we find

$$(\partial_t + \mathcal{A}_0(t))u(t) = -\mathcal{B}(t)u(t), \qquad u(T) = \varphi.$$

By Duhamel's principle, we have

$$u(t) = \mathcal{P}_0(t, T)\varphi + \int_t^T \mathrm{d}t_1\, \mathcal{P}_0(t, t_1)\mathcal{B}(t_1)u(t_1), \tag{2.10}$$

where $\mathcal{P}_0(t, T)$ is the *semigroup* of operators generated by $\mathcal{A}_0(t)$; we will explicitly define $\mathcal{P}_0(t, T)$ in (2.16). Inserting expression (2.10) for u into the right-hand side of (2.10) and iterating, we obtain

$$\begin{aligned}
u(t_0) &= \mathcal{P}_0(t_0, T)\varphi + \int_{t_0}^T \mathrm{d}t_1\, \mathcal{P}_0(t_0, t_1)\mathcal{B}(t_1)\mathcal{P}_0(t_1, T)\varphi \\
&\quad + \int_{t_0}^T \mathrm{d}t_1 \int_{t_1}^T \mathrm{d}t_2\, \mathcal{P}_0(t_0, t_1)\mathcal{B}(t_1)\mathcal{P}_0(t_1, t_2)\mathcal{B}(t_2)u(t_2) = \cdots \\
&= \mathcal{P}_0(t_0, T)\varphi + \sum_{k=1}^{\infty} \int_{t_0}^T \mathrm{d}t_1 \int_{t_1}^T \mathrm{d}t_2 \cdots \int_{t_{k-1}}^T \mathrm{d}t_k \\
&\qquad \mathcal{P}_0(t_0, t_1)\mathcal{B}(t_1)\mathcal{P}_0(t_1, t_2)\mathcal{B}(t_2)\cdots\mathcal{P}_0(t_{k-1}, t_k)\mathcal{B}(t_k)\mathcal{P}_0(t_k, T)\varphi \qquad (2.11)\\
&= \mathcal{P}_0(t_0, T)\varphi + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \int_{t_0}^T \mathrm{d}t_1 \int_{t_1}^T \mathrm{d}t_2 \cdots \int_{t_{k-1}}^T \mathrm{d}t_k
\end{aligned}$$

$$\sum_{i\in I_{n,k}} \mathcal{P}_0(t_0, t_1)\mathcal{A}_{i_1}(t_1)\mathcal{P}_0(t_1, t_2)\mathcal{A}_{i_2}(t_2)\cdots\mathcal{P}_0(t_{k-1}, t_k)\mathcal{A}_{i_k}(t_k)\mathcal{P}_0(t_k, T)\varphi, \tag{2.12}$$

$$I_{n,k} = \{i = (i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1 + i_2 + \cdots + i_k = n\}. \tag{2.13}$$

To obtain (2.12) from (2.11), we have used the fact that from (2.9), the operator $\mathcal{B}(t)$ is an infinite sum, and we have partitioned on the sum $(i_1 + i_2 + \cdots + i_k)$ of the subscripts of the $(\mathcal{A}_{i_k}(t))$. In light of expansion (2.12), we set

$$u = \sum_{n=0}^{\infty} u_n,$$

where we have defined

$$\begin{aligned}
u_0(t_0) &:= \mathcal{P}_0(t_0, T)\varphi, \\
u_n(t_0) &:= \sum_{k=1}^{n} \int_{t_0}^T \mathrm{d}t_1 \int_{t_1}^T \mathrm{d}t_2 \cdots \int_{t_{k-1}}^T \mathrm{d}t_k
\end{aligned}$$

$$\sum_{i\in I_{n,k}} \mathcal{P}_0(t_0, t_1)\mathcal{A}_{i_1}(t_1)\mathcal{P}_0(t_1, t_2)\mathcal{A}_{i_2}(t_2)\cdots\mathcal{P}_0(t_{k-1}, t_k)\mathcal{A}_{i_k}(t_k)\mathcal{P}_0(t_k, T)\varphi. \tag{2.14}$$

2.3. Expression for u_0

By assumption, the functions $a_{\alpha,0}$ depend only on t. Therefore, the operator $\mathcal{A}_0(t)$ is the generator of a diffusion with time-dependent parameters. It will be useful to write the operator $\mathcal{A}_0(t)$ in the following form:

$$\mathcal{A}_0(t) = \frac{1}{2}\sum_{i,j=1}^{d} C_{ij}(t)\partial_{z_i z_j} + \langle m(t), \nabla_z\rangle, \qquad \langle m(t), \nabla_z\rangle = \sum_{i=1}^{d} m_i(t)\partial_{z_i}. \tag{2.15}$$

Here, the $d \times d$-matrix $C(t)$ is positive definite, for any $t \in [0, T]$, and m is a d-dimensional vector. The action of the semigroup of operators $\mathcal{P}_0(t_0, T)$ generated by

$\mathcal{A}_0(t)$ is well known. For any measurable function φ that is at most exponentially growing, we have

$$u_0(t_0) := \mathcal{P}_0(t_0, T)\varphi = \int_{\mathbb{R}^d} \Gamma_0(t, \cdot; T, \zeta)\varphi(\zeta)\,\mathrm{d}\zeta, \tag{2.16}$$

where $\Gamma_0(t, z; T, \zeta)$ is the d-dimensional Gaussian density

$$\Gamma_0(t, z; T, \zeta) = \frac{1}{\sqrt{(2\pi)^d|\mathbf{C}(t, T)|}} \exp\left(-\frac{1}{2}\langle \mathbf{C}^{-1}(t, T)(\zeta - z - \mathbf{m}(t, T)), (\zeta - z - \mathbf{m}(t, T))\rangle\right),$$

with covariance matrix $\mathbf{C}(t, T)$ and mean vector $z + \mathbf{m}(t, T)$ given by

$$\mathbf{C}(t, T) = \int_t^T \mathrm{d}s\, C(s), \qquad \mathbf{m}(t, T) = \int_t^T \mathrm{d}s\, m(s).$$

Note that the function u_0 as it is defined in (2.16) is the unique nonrapidly increasing solution of the homogeneous backward Cauchy problem $(\partial_t + \mathcal{A}_0(t))u_0 = 0$ with terminal condition $u_0(T) = \varphi$.

2.4. Expression for u_n

Remarkably, as the following theorem shows, every $u_n(t)$ can be expressed as a differential operator $\mathcal{L}_n(t, T)$ acting on $u_0(t)$.

THEOREM 2.6. *Assume $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$. Then, the function u_n defined in* (2.14) *is given explicitly by*

$$u_n(t_0) = \mathcal{L}_n(t_0, T)u_0(t_0), \tag{2.17}$$

where u_0 is given by (2.16) *and*

$$\mathcal{L}_n(t_0, T) = \sum_{k=1}^{n} \int_{t_0}^{T} \mathrm{d}t_1 \int_{t_1}^{T} \mathrm{d}t_2 \cdots \int_{t_{k-1}}^{T} \mathrm{d}t_k \sum_{i \in I_{n,k}} \mathcal{G}_{i_1}(t_0, t_1)\mathcal{G}_{i_2}(t_0, t_2)\cdots\mathcal{G}_{i_k}(t_0, t_k), \tag{2.18}$$

with $I_{n,k}$ as defined in (2.13)

$$\mathcal{G}_i(t_0, t_k) := \mathcal{A}_i(t_k, \mathcal{M}(t_0, t_k)) = \sum_{|\alpha| \le 2} a_{\alpha,i}(t_k, \mathcal{M}(t_0, t_k))D_z^{\alpha}, \tag{2.19}$$

with $\mathcal{A}_i(t, z)$ as in (2.8) *and*

$$\mathcal{M}(t, s) := z + \mathbf{m}(t, s) + \mathbf{C}(t, s)\nabla_z. \tag{2.20}$$

Proof. The main idea of the proof is to show that the operator $\mathcal{G}_i(t_0, t_k)$ in (2.19) satisfies

$$\mathcal{P}_0(t_0, t_k)\mathcal{A}_i(t_k) = \mathcal{G}_i(t_0, t_k)\mathcal{P}_0(t_0, t_k). \tag{2.21}$$

Assuming (2.21) holds, we can use the fact that $\mathcal{P}_0(t_k, t_{k+1})$ is a semigroup

$$\mathcal{P}_0(t_0, T) = \mathcal{P}_0(t_0, t_1)\mathcal{P}_0(t_1, t_2)\cdots\mathcal{P}_0(t_{k-1}, t_k)\mathcal{P}_0(t_k, T),$$

and we can rewrite (2.14) as

$$u_n(t_0) = \sum_{k=1}^{n} \int_{t_0}^{T} \mathrm{d}t_1 \int_{t_1}^{T} \mathrm{d}t_2 \cdots \int_{t_{k-1}}^{T} \mathrm{d}t_k \sum_{i \in I_{n,k}} \mathcal{G}_{i_1}(t_0, t_1)\mathcal{G}_{i_2}(t_0, t_2) \cdots \mathcal{G}_{i_k}(t_0, t_k)\mathcal{P}_0(t_0, T)\varphi,$$

from which (2.17) and (2.18) follow directly. Thus, we only need to show that $\mathcal{G}_i(t_0, t_k)$ satisfies (2.21). The condition $\varphi \in \mathcal{S}(\mathbb{R}^d)$ guarantees that $u_0(t, \cdot)$ belongs to the Schwartz class of rapidly decaying functions for all $t < T$. Therefore, any function of the form $p(z) D_z^{\beta} u_0(t, z)$, where p is a polynomial, has a Fourier representation. Thus, without loss of generality, we can investigate how the operator $\mathcal{P}_0(t_0, t_k)\mathcal{A}_i(t_k)$ acts on the oscillating exponential $\mathrm{e}_\lambda(x) := \mathrm{e}^{\mathtt{i}\langle\lambda, x\rangle}$. We note that

$$\mathcal{P}_0(t_0, t_k)\mathrm{e}_\lambda(z) = \mathrm{e}^{\Phi_0(t_0,t_k,\lambda)}\mathrm{e}_\lambda(z), \quad \text{where} \quad \Phi_0(t_0, t_k, \lambda) = \sum_{|\alpha|\leq 2} (\mathtt{i}\lambda)^\alpha \int_{t_0}^{t_k} \mathrm{d}t\, a_{\alpha,0}(t). \tag{2.22}$$

Next, we observe that the operator $\mathcal{M}_i(t_0, t_k)$, the ith component of $\mathcal{M}(t_0, t_k)$ in (2.20), can be written

$$\mathcal{M}_i(t_0, t_k) = M_i(t_0, t_k, -\mathtt{i}\nabla_z), \qquad M_i(t_0, t_k, \lambda) = -\mathtt{i}\partial_{\lambda_i}\left(\Phi_0(t_0, t_k, \lambda) + \mathtt{i}\langle\lambda, z\rangle\right). \tag{2.23}$$

Using (2.23), we observe that for any natural number n, we have

$$\begin{aligned}
(-\mathtt{i}\partial_{\lambda_i})^n \mathrm{e}^{\Phi_0(t_0,t_k,\lambda)}\mathrm{e}_\lambda(z) &= (-\mathtt{i}\partial_{\lambda_i})^{n-1} M_i(t_0, t_k, \lambda)\mathrm{e}^{\Phi_0(t_0,t_k,\lambda)}\mathrm{e}_\lambda(z) \\
&= \mathcal{M}_i(t_0, t_k)(-\mathtt{i}\partial_{\lambda_i})^{n-1}\mathrm{e}^{\Phi_0(t_0,t_k,\lambda)}\mathrm{e}_\lambda(z) = \cdots \\
&= [\mathcal{M}_i(t_0, t_k)]^n \mathrm{e}^{\Phi_0(t_0,t_k,\lambda)}\mathrm{e}_\lambda(z).
\end{aligned}$$

Noting the ∂_{λ_i} and ∂_{λ_j} commute, it is clear that $\mathcal{M}_i(t_0, t_k)$ and $\mathcal{M}_j(t_0, t_k)$ also commute. Thus, for any multi-index β, we have

$$(-\mathtt{i}\nabla_\lambda)^\beta \mathrm{e}^{\Phi_0(t_0,t_k,\lambda)}\mathrm{e}_\lambda(z) = (\mathcal{M}(t_0, t_k))^\beta \mathrm{e}^{\Phi_0(t_0,t_k,\lambda)}\mathrm{e}_\lambda(z). \tag{2.24}$$

Finally, we compute

$$\begin{aligned}
\mathcal{P}_0(t_0, t_k)\mathcal{A}_i(t_k)\mathrm{e}_\lambda(z) &= \sum_{|\alpha|\leq 2} \mathcal{P}_0(t_0, t_k) a_{\alpha,i}(t_k, z) D_z^\alpha \mathrm{e}_\lambda(z) && \text{(by (2.9))} \\
&= \sum_{|\alpha|\leq 2} (\mathtt{i}\lambda)^\alpha \mathcal{P}_0(t_0, t_k) a_{\alpha,i}(t_k, z)\mathrm{e}_\lambda(z) \\
&= \sum_{|\alpha|\leq 2} (\mathtt{i}\lambda)^\alpha a_{\alpha,i}(t_k, -\mathtt{i}\nabla_\lambda)\mathcal{P}_0(t_0, t_k)\mathrm{e}_\lambda(z) \\
&= \sum_{|\alpha|\leq 2} (\mathtt{i}\lambda)^\alpha a_{\alpha,i}(t_k, -\mathtt{i}\nabla_\lambda)\mathrm{e}^{\Phi_0(t_0,t_k,\lambda)}\mathrm{e}_\lambda(z) && \text{(by (2.22))} \\
&= \sum_{|\alpha|\leq 2} (\mathtt{i}\lambda)^\alpha a_{\alpha,i}(t_k, \mathcal{M}(t_0, t_k))\mathrm{e}^{\Phi_0(t_0,t_k,\lambda)}\mathrm{e}_\lambda(z) && \text{(by (2.24))} \\
&= \sum_{|\alpha|\leq 2} a_{\alpha,i}(t_k, \mathcal{M}(t_0, t_k)) D_z^\alpha \mathrm{e}^{\Phi_0(t_0,t_k,\lambda)}\mathrm{e}_\lambda(z) \\
&= \sum_{|\alpha|\leq 2} a_{\alpha,i}(t_k, \mathcal{M}(t_0, t_k)) D_z^\alpha \mathcal{P}_0(t_0, t_k)\mathrm{e}_\lambda(z) && \text{(by (2.22))} \\
&= \mathcal{G}_i(t_0, t_k)\mathcal{P}_0(t_0, t_k)\mathrm{e}_\lambda(z), && \text{(by (2.8) and (2.19))}
\end{aligned}$$

which concludes the proof. □

REMARK 2.7. (Call payoffs). As we will show in Section 4.1, the functions (u_n) can be alternatively characterized as solutions of a nested sequence of Cauchy problems (see equation (4.14) for the case when $(\mathcal{A}_n(t))$ is expanded in a Taylor series as in Example 2.3). One can check directly that when $\varphi(x) = (\mathrm{e}^x - \mathrm{e}^k)$, the functions (u_n) with each u_n given by $u_n(t) = \mathcal{L}_n(t, T)u_0(t)$ satisfy the nested Cauchy problems. Thus, Theorem 2.6 also holds for Call option payoffs. This is true for any expansion $(\mathcal{A}_n(t))$ satisfying Definition 2.2.

REMARK 2.8. The number of terms in $\mathcal{L}_n(t, T)$ grows faster than $n!$, which presents a computational challenge for large n. Nevertheless, we shall see in the numerical example provided in Section 5 that excellent approximations can be achieved with $n = 3$.

REMARK 2.9. When $d = 1, 2$, the operator $\mathcal{A}(t)$ is given by (2.4). In this case, we write $\mathcal{A}_i(t)$ as

$$\mathcal{A}_i(t) := a_i(t, x, y)\big(\partial_x^2 - \partial_x\big) + f_i(t, x, y)\partial_y + b_i(t, x, y)\partial_y^2 + c_i(t, x, y)\partial_x\partial_y,$$

and we have explicitly

$$\begin{aligned}\mathcal{G}_i(t, s) &:= a_i\left(s, \mathcal{M}_x(t, s), \mathcal{M}_y(t, s)\right)\left(\partial_x^2 - \partial_x\right) + f_i\left(s, \mathcal{M}_x(t, s), \mathcal{M}_y(t, s)\right)\partial_y \\ &\quad + b_i\left(s, \mathcal{M}_x(t, s), \mathcal{M}_y(t, s)\right)\partial_y^2 + c_i\left(s, \mathcal{M}_x(t, s), \mathcal{M}_y(t, s)\right)\partial_x\partial_y,\end{aligned} \tag{2.25}$$

$$\mathcal{M}_x(t, s) = x - \int_t^s \mathrm{d}q\, a_0(q) + 2\int_t^s \mathrm{d}q\, a_0(q)\partial_x + \int_t^s \mathrm{d}q\, c_0(q)\partial_y,$$

$$\mathcal{M}_y(t, s) = y + \int_t^s \mathrm{d}q\, f_0(q) + 2\int_t^s \mathrm{d}q\, b_0(q)\partial_y + \int_t^s \mathrm{d}q\, c_0(q)\partial_x.$$

3. IMPLIED VOLATILITY EXPANSION

In this section, we derive an explicit implied volatility approximation from the asymptotic pricing expansion developed in the previous section. To begin our analysis, we fix a multifactor LSV model for $X = \log S$ as in (2.1), a time t, a maturity date $T > t$, the initial values $(X_t, Y_t) = (x, y) \in \mathbb{R} \times \mathbb{R}^{d-1}$, and a Call option payoff $\varphi(X_T) = (\mathrm{e}^{X_T} - \mathrm{e}^k)^+$. Our goal is to find the implied volatility for *this particular Call option*. To ease notation, we will sometimes suppress the dependence on (t, T, x, y, k). However, the reader should keep in mind that the implied volatility of the option under consideration *does* depend on (t, T, x, y, k), even if this is not explicitly indicated. Below, we provide definitions of the *Black–Scholes price* and *implied volatility*, which will be fundamental throughout this section.

DEFINITION 3.1. The Black–Scholes price u^{BS} is given by

$$u^{\mathrm{BS}}(\sigma; \tau, x, k) := \mathrm{e}^x\mathcal{N}(d_+) - \mathrm{e}^k\mathcal{N}(d_-), \quad d_\pm := \frac{1}{\sigma\sqrt{\tau}}\left(x - k \pm \frac{\sigma^2\tau}{2}\right), \quad \tau := T - t, \tag{3.1}$$

where $\mathcal{N}$ is the cumulative distribution function (CDF) of a standard normal random variable.

REMARK 3.2. It follows from (2.16) that when $\varphi(x) = (\mathrm{e}^x - \mathrm{e}^k)^+$, we have

$$u_0(t, x) = u^{\mathrm{BS}}(\sigma_0; T - t, x, k), \quad \text{where} \quad \sigma_0 = \sqrt{\frac{2}{T - t}\int_t^T a_0(s)\mathrm{d}s}, \tag{3.2}$$

where $a_0 = C_{1,1}$ as in (2.15), or according to the multi-index notation, $a_0 = a_{(2,0,\dots,0),0}$.

DEFINITION 3.3. For fixed (τ, x, k), the implied volatility corresponding to a Call price $u \in ((e^x - e^k)^+, e^x)$ is defined as the unique strictly positive real solution σ of the equation

$$u^{\mathrm{BS}}(\sigma; \tau, x, k) = u. \tag{3.3}$$

3.1. Formal Derivation

We present here a formal derivation of our implied volatility expansion, which is based on the price expansion presented in Section 2. Throughout this section, (t, T, x, k) *are fixed* and thus we use the short notation

$$u^{\mathrm{BS}}(\sigma) = u^{\mathrm{BS}}(\sigma; T - t, x, k)$$

for the Black–Scholes price. Consider the family of approximate Call prices indexed by δ

$$u(\delta) = \sum_{n=0}^{N} \delta^n u_n = u^{\mathrm{BS}}(\sigma_0) + \sum_{n=1}^{N} \delta^n u_n + \delta^{N+1}\left(u - \sum_{n=0}^{N} u_n\right), \qquad \delta \in [0, 1], \tag{3.4}$$

with σ_0 as in (3.2) and the functions $u_n(t) = \mathcal{L}_n(t, T)u_0(t)$ as given in Theorem 2.6. Note that setting $\delta = 1$ yields our price expansion. Defining

$$g(\delta) := (u^{\mathrm{BS}})^{-1}(u(\delta)), \qquad \delta \in [0, 1], \tag{3.5}$$

we seek the implied volatility $\sigma = g(1)$. We will show in Section 4.2, Lemma 4.14, that under suitable assumptions, $u(\delta) \in ((e^x - e^k)^+, e^x)$ for any $\delta \in [0, 1]$. This guarantees that $g(\delta)$ in (3.5) exists. By expanding both sides of (3.5) as a Taylor series in δ, we see that σ admits an expansion of the form

$$\sigma = g(1) = \sigma_0 + \sum_{n=1}^{\infty} \sigma_n, \qquad \sigma_n = \frac{1}{n!}\partial_\delta^n g(\delta)|_{\delta=0}. \tag{3.6}$$

Note that by (3.4), we also have

$$u_n = \frac{1}{n!}\partial_\delta^n u^{\mathrm{BS}}(g(\delta))|_{\delta=0}, \qquad 1 \le n \le N. \tag{3.7}$$

The right-hand side of (3.7) can be computed by applying the Bell polynomial version of the Faa di Bruno's formula, which is given in Appendix A. We have

$$u_n = \frac{1}{n!}\sum_{h=1}^{n} \partial_\sigma^h u^{\mathrm{BS}}(\sigma_0)\mathbf{B}_{n,h}\left(\partial_\delta g(\delta), \partial_\delta^2 g(\delta), \dots, \partial_\delta^{n-h+1} g(\delta)\right)|_{\delta=0}, \quad 1 \le n \le N. \tag{3.8}$$

Combining (3.8) with (3.6), one can solve for σ_n explicitly in terms of $(\sigma_k)_{0\le k\le n-1}$, which yields

$$\sigma_n = \frac{u_n}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)} - \frac{1}{n!}\sum_{h=2}^{n} \mathbf{B}_{n,h}\left(1!\,\sigma_1, 2!\,\sigma_2, \dots, (n-h+1)!\,\sigma_{n-h+1}\right)\frac{\partial_\sigma^h u^{\mathrm{BS}}(\sigma_0)}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)}, \qquad 1 \le n \le N. \tag{3.9}$$

Note that expression (3.9) for σ_n involves two sorts of terms: $u_n/\partial_\sigma u^{\mathrm{BS}}(\sigma_0)$ and $\partial_\sigma^n u^{\mathrm{BS}}(\sigma_0)/\partial_\sigma u^{\mathrm{BS}}(\sigma_0)$. We will prove that these terms can be computed explicitly without any numerical integration or special functions. The proof will rely on the following lemma.

LEMMA 3.4. *Let $m \geq 0$ and fix (t, T, k, σ_0). Then,*

$$\frac{\partial_x^m(\partial_x^2 - \partial_x)u^{\mathrm{BS}}(\sigma_0)}{(\partial_x^2 - \partial_x)u^{\mathrm{BS}}(\sigma_0)} = \left(-\frac{1}{\sigma_0\sqrt{2\tau}}\right)^m H_m(\zeta), \zeta := \frac{x - k - \frac{1}{2}\sigma_0^2\tau}{\sigma_0\sqrt{2\tau}}, \qquad \tau := T - t, \tag{3.10}$$

where $H_n(\zeta) := (-1)^n \mathrm{e}^{\zeta^2}\partial_\zeta^n \mathrm{e}^{-\zeta^2}$ is the nth Hermite polynomial.

Proof. Using the Black–Scholes formula (3.1), a direct computation shows

$$\left(\partial_x^2 - \partial_x\right)u^{\mathrm{BS}}(\sigma_0) = \frac{1}{\sigma_0\sqrt{2\pi\tau}}\mathrm{e}^{-\zeta^2+k},$$

with $\zeta = \zeta(x)$ as above. Hence,

$$\frac{\partial_x^m(\partial_x^2 - \partial_x)u^{\mathrm{BS}}(\sigma_0)}{(\partial_x^2 - \partial_x)u^{\mathrm{BS}}(\sigma_0)} = \mathrm{e}^{\zeta^2}\partial_x^m\mathrm{e}^{-\zeta^2} = \left(\frac{1}{\sigma_0\sqrt{2\tau}}\right)^m \mathrm{e}^{z^2}\partial_\zeta^m\mathrm{e}^{-\zeta^2} = \left(\frac{-1}{\sigma_0\sqrt{2\tau}}\right)^m H_m(\zeta),$$

where in the last equality, we have used the definition of the mth Hermite polynomial, recalled above. □

PROPOSITION 3.5. *Fix (t, T, k, σ_0) and let ζ and τ be as in Lemma 3.4. Then, for any $n \geq 2$, we have*

$$\frac{\partial_\sigma^n u^{\mathrm{BS}}(\sigma_0)}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)} = \sum_{q=0}^{\lfloor n/2\rfloor}\sum_{p=0}^{n-q-1} c_{n,n-2q}\sigma_0^{n-2q-1}\tau^{n-q-1}\binom{n-q-1}{p}\left(\frac{1}{\sigma_0\sqrt{2\tau}}\right)^{p+n-q-1} H_{p+n-q-1}(\zeta),$$

where the coefficients $(c_{n,n-2k})$ are defined recursively by

$$c_{n,n} = 1, \text{ and } c_{n,n-2q} = (n-2q+1)c_{n-1,n-2q+1} + c_{n-1,n-2q-1}, \qquad q \in \{1, 2, \cdots, \lfloor n/2\rfloor\}.$$

Proof. Define the operator $\mathcal{J} := \tau(\partial_x^2 - \partial_x)$. It is classical that $\partial_\sigma u^{\mathrm{BS}}(\sigma_0) = \sigma_0\mathcal{J}u^{\mathrm{BS}}(\sigma_0)$. We claim that the following identity holds for any $n \in \mathbb{N}$:

$$\partial_\sigma^n u^{\mathrm{BS}}(\sigma_0) = \sum_{q=0}^{\lfloor n/2\rfloor} c_{n,n-2q}\sigma_0^{n-2q}\mathcal{J}^{n-q}u^{\mathrm{BS}}(\sigma_0), \tag{3.11}$$

where $c_{n,n} = 1$ and $c_{n,n-2q} = (n-2q+1)c_{n-1,n-2q+1} + c_{n-1,n-2q-1}$ for any integer $q \in \{1, 2, \cdots, \lfloor n/2\rfloor\}$. The proof of (3.11) is a simple yet tedious recursion relation, which we omit for brevity. Now, we compute

$$\begin{aligned}
\frac{\partial_\sigma^n u^{\mathrm{BS}}(\sigma_0)}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)} &= \sum_{q=0}^{\lfloor n/2\rfloor} c_{n,n-2q}\sigma_0^{n-2q}\frac{\mathcal{J}^{n-q}u^{\mathrm{BS}}(\sigma_0)}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)} = \sum_{q=0}^{\lfloor n/2\rfloor} c_{n,n-2q}\sigma_0^{n-2q}\tau^{n-q}\frac{(\partial_x^2 - \partial_x)^{n-q}u^{\mathrm{BS}}(\sigma_0)}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)} \\
&= \sum_{q=0}^{\lfloor n/2\rfloor} c_{n,n-2q}\sigma_0^{n-2q}\tau^{n-q}\frac{(\partial_x^2 - \partial_x)^{n-q-1}(\partial_x^2 - \partial_x)u^{\mathrm{BS}}(\sigma_0)}{\tau\sigma_0(\partial_x^2 - \partial_x)u^{\mathrm{BS}}(\sigma_0)} \\
&= \sum_{q=0}^{\lfloor n/2\rfloor}\sum_{p=0}^{n-q-1} c_{n,n-2q}\sigma_0^{n-2q-1}\tau^{n-q-1}\binom{n-q-1}{p}(-1)^{n-q-1-p}\frac{\partial_x^{p+n-q-1}(\partial_x^2 - \partial_x)u^{\mathrm{BS}}(\sigma_0)}{(\partial_x^2 - \partial_x)u^{\mathrm{BS}}(\sigma_0)} \\
&= \sum_{q=0}^{\lfloor n/2\rfloor}\sum_{p=0}^{n-q-1} c_{n,n-2q}\sigma_0^{n-2q-1}\tau^{n-q-1}\binom{n-q-1}{p}\left(\frac{1}{\sigma_0\sqrt{2\tau}}\right)^{p+n-q-1} H_{p+n-q-1}(\zeta),
\end{aligned}$$

where to obtain the last equality, we have used (3.10).

PROPOSITION 3.6. *Fix (t, T, x, y). For every polynomial expansion $(\mathcal{A}_n(t))$ satisfying Definition 2.2 and for every $n \in \mathbb{N}$, the ratio $u_n/\partial_\sigma u^{\mathrm{BS}}(\sigma_0)$ is a finite sum of the form*

$$\frac{u_n}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)} = \sum_m \chi_m^{(n)}(t, T, x, y)\left(-\frac{1}{\sigma_0\sqrt{2\tau}}\right)^m H_m(\zeta), \tag{3.12}$$

where ζ and τ are as in Lemma 3.4. The coefficients $\chi_m^{(n)}(t, T, x, y)$ are explicit functions of x and y and contain iterated integrals in the time variable. If the iterated time integrals can be computed explicitly, then $\chi_m^{(n)}(t, T, x, y)$ is explicit in all variables.

Proof. From equation (2.25) and Remark 3.2, we observe that for the case $d = 1, 2$

$$\mathcal{G}_i(t, s)u_0 := a_i\left(s, \mathcal{M}_x(t, s), \mathcal{M}_y(t, s)\right)(\partial_x^2 - \partial_x)u^{\mathrm{BS}}(\sigma_0).$$

For a general LSV model with $d - 1$ factors of volatility, we have

$$\mathcal{M}_y(t, s) = (\mathcal{M}_{y_1}(t, s), \mathcal{M}_{y_2}(t, s), \ldots, \mathcal{M}_{y_{d-1}}(t, s)).$$

Therefore, using Theorem 2.6, we have

$$\frac{u_n(t)}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)} = \frac{\mathcal{L}_n(t, T)u_0(t)}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)} = \frac{\widetilde{\mathcal{L}}_n(t, T)(\partial_x^2 - \partial_x)u^{\mathrm{BS}}(\sigma_0)}{\tau\sigma_0(\partial_x^2 - \partial_x)u^{\mathrm{BS}}(\sigma_0)}, \tag{3.13}$$

where

$$\widetilde{\mathcal{L}}_n(t, T) = \sum_{k=1}^{n}\int_t^T \mathrm{d}t_1 \int_{t_1}^T \mathrm{d}t_2 \cdots \int_{t_{k-1}}^T \mathrm{d}t_k \sum_{i \in I_{n,k}}$$
$$\mathcal{G}_{i_1}(t, t_1)\cdots\mathcal{G}_{i_{k-1}}(t, t_{k-1})a_{i_k}\left(s, \mathcal{M}_x(t, t_k), \mathcal{M}_y(t, t_k)\right). \tag{3.14}$$

It is clear that $\widetilde{\mathcal{L}}_n(t, T)$ is a differential operator that takes derivatives with respect to x and y and has coefficients that depend on (t, T, x, y). Noting that $\partial_y^m u^{\mathrm{BS}}(\sigma_0) = 0$ for all $m \geq 1$, it is clear from (3.13) that $u_n/\partial_\sigma u^{\mathrm{BS}}(\sigma_0)$ is of the form

$$\frac{u_n(t)}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)} = \sum_m \chi_m^{(n)}(t, T, x, y)\frac{\partial_x^m(\partial_x^2 - \partial_x)u^{\mathrm{BS}}(\sigma_0)}{(\partial_x^2 - \partial_x)u^{\mathrm{BS}}(\sigma_0)}. \tag{3.15}$$

Equation (3.12) follows from equation (3.15) and Lemma 3.4. The sequence of coefficients $(\chi_m^{(n)})$ must be computed on a case-by-case basis because the $(\chi_m^{(n)})$ depends on the coefficients of the generator $\mathcal{A}(t)$ and the choice of polynomial expansion $(\mathcal{A}_n(t))$. □

From Propositions 3.5 and 3.6, it is apparent that as long as the iterated time integrals in (3.14) can be computed explicitly (which is always the case when the coefficients in the polynomial expansion $(\mathcal{A}_n(t))$ are piecewise polynomial in time), every term in (3.9) can be computed without the need for numerical integration or special functions.

Explicit expressions for each σ_n in the sequence $(\sigma_n)_{n\geq 1}$ can be computed by hand. However, as the number of terms grows quickly with n, it is helpful to use a computer algebra program such as Wolfram's Mathematica. In Appendix B, we provide explicit expressions for σ_n ($n \leq 2$) for the case where the coefficients of $\mathcal{A}(t)$ are expanded as Taylor series (Example 2.3). The authors' web site also provides Mathematica notebooks, which contain the expressions for σ_n ($n \leq 4$) in the LSV models described in Section 5.

REMARK 3.7. When the risk-free rate of interest is a deterministic function of time $r(t)$, the implied volatility results above hold with $k \to k - \int_t^T r(s)\mathrm{d}s$.

4. ASYMPTOTIC ERROR ESTIMATES FOR TAYLOR EXPANSIONS

In this section, we provide pointwise short-time error estimates for the approximate solution of the Cauchy problem (2.2) discussed in Section 2, as well as for the approximate implied volatility presented in Section 3. Throughout this section, we shall assume that $T_0 > 0$ and $N \in \mathbb{N}_0$ are fixed and the coefficients of the operator $\mathcal{A}(t)$ in (2.6) satisfy the following assumption:

ASSUMPTION 4.1. *There exists a positive constant M such that:*

i) Uniform ellipticity:

$$M^{-1}|\xi|^2 < \sum_{i,j=1}^{d} a_{ij}(t,z)\xi_i\xi_j < M|\xi|^2, \qquad t \in [0, T_0],\ z, \xi \in \mathbb{R}^d.$$

ii) Regularity and boundedness: the coefficients $a_{ij}, a_i \in C([0, T_0] \times \mathbb{R}^d)$ *and* $a_{ij}(t,\cdot), a_i(t,\cdot) \in C^{N+1}(\mathbb{R}^d)$*, with their partial derivatives of all orders bounded by M, uniformly with respect to* $t \in [0, T_0]$.

Under Assumption 4.1, it is well known that $\mathcal{A}(t)$ admits a *fundamental solution* $\Gamma(t, z; T, \zeta)$, which is the solution of the Cauchy problem (2.2) with $\varphi = \delta_\zeta$. Equivalently, for any $T \in]0, T_0[$ and for any measurable function φ with at most exponential growth, the backward parabolic Cauchy problem (2.2) admits a unique classical solution u, which is given by

$$u(t,z) = \int_{\mathbb{R}^d} \Gamma(t, z; T, \zeta)\varphi(\zeta)\mathrm{d}\zeta, \qquad t \in [0, T[,\ z \in \mathbb{R}^d. \tag{4.1}$$

Furthermore, by the Feynman–Kac representation theorem, the function $\Gamma(t, z; T, \zeta)$ is also the transition density of the stochastic process generated by $\mathcal{A}(t)$.

REMARK 4.2. Assumption 4.1 can be considerably relaxed. The main results (Theorem 4.5 and Corollary 4.6 below) have recently been extended in Pagliarani and Pascucci (2014), to include the majority of popular models in mathematical finance (e.g., CEV, Heston, SABR, three-halves, etc.).

Consider now the Taylor polynomial expansion discussed in Example 2.3. It will be helpful to explicitly indicate the dependence on the expansion point $\bar{z}$. In particular, for any $\bar{z} \in \mathbb{R}^d$, we consider the polynomial expansion $(\mathcal{A}_n^{(\bar{z})}(t))_{0\le n\le N}$, given by

$$\mathcal{A}_n^{(\bar{z})}(t,z) \equiv \mathcal{A}_n^{(\bar{z})}(t) := \sum_{|\alpha|\le 2} a_{\alpha,n}^{(\bar{z})}(t,z)D_z^\alpha$$

$$a_{\alpha,n}^{(\bar{z})}(\cdot,z) = \sum_{|\beta|=n} \frac{D_z^\beta a_\alpha(\cdot,\bar{z})}{\beta!}(z-\bar{z})^\beta, \qquad n \le N. \tag{4.2}$$

Now, fix a maturity date T. We define the Nth-order Taylor approximations centered at $\bar{z} \in \mathbb{R}^d$ of Γ and u, respectively, as

$$\bar{u}_N^{(\bar{z})}(t, z) := \sum_{n=0}^{N} u_n^{(\bar{z})}(t, z), \qquad \bar{\Gamma}_N^{(\bar{z})}(t, z, T, \zeta) := \sum_{n=0}^{N} \Gamma_n^{(\bar{z})}(t, z, T, \zeta),$$
$$t < T, \qquad z, \zeta \in \mathbb{R}^d, \tag{4.3}$$

where the functions

$$u_n^{(\bar{z})}(t, \cdot) = \mathcal{L}_n^{(\bar{z})}(t, T) u_0^{(\bar{z})}(t, \cdot), \qquad \Gamma_n^{(\bar{z})}(t, \cdot; T, \zeta) = \mathcal{L}_n^{(\bar{z})}(t, T) \Gamma_0^{(\bar{z})}(t, \cdot; T, \zeta), \tag{4.4}$$

are as given in Theorem 2.6. Note that $\bar{u}_N^{(\bar{z})}$ is defined for a fixed T, as indicated by (4.4). Also, note that we have once again used the superscript $\bar{z}$ above to emphasize the dependence on the initial point of the Taylor expansion. For the particular choice $\bar{z} = z$, we give the following definition:

DEFINITION 4.3. For a fixed maturity date T, we define the Nth-order Taylor approximations of u and Γ, respectively, as

$$\bar{u}_N(t, z) := \bar{u}_N^{(z)}(t, z), \qquad \bar{\Gamma}_N(t, z; T, \zeta) := \bar{\Gamma}_N^{(z)}(t, z; T, \zeta), \tag{4.5}$$

where $\bar{u}_N^{(z)}(t, z)$ and $\bar{\Gamma}_N^{(z)}(t, z; T, \zeta)$ are as defined in (4.3) and (4.4).

We now give analogous definitions for the implied volatility expansion. As in Section 3, we use the notation $(x, y) \in \mathbb{R} \times \mathbb{R}^{d-1}$ to indicate a point in $\mathbb{R}^d$, where we separate x from all other components in order to distinguish the log-price from all the other variables (e.g., variance process, vol–vol process, etc.). For a Call option with maturity date T and log strike k, we define the Nth-order Taylor approximation centered at $(\bar{x}, \bar{y}) \in \mathbb{R} \times \mathbb{R}^{d-1}$ of the implied volatility σ as

$$\bar{\sigma}_N^{(\bar{x},\bar{y})}(t, x, y, k) : = \sigma_0^{(\bar{x},\bar{y})}(t) + \sum_{n=0}^{N} \sigma_n^{(\bar{x},\bar{y})}(t, x, y, k),$$
$$t < T, \quad (x, y) \in \mathbb{R} \times \mathbb{R}^{d-1}, \tag{4.6}$$

where we recall

$$\sigma_0^{(\bar{x},\bar{y})}(t) = \sqrt{\frac{2}{T-t} \int_t^T a_{(2,0,\ldots,0)}(s, \bar{x}, \bar{y})\, \mathrm{d}s},$$

$$\sigma_n^{(\bar{x},\bar{y})}(t, x, y, k) = \frac{u_n^{(\bar{x},\bar{y})}(t, x, y, k)}{\partial_\sigma u^{\mathrm{BS}}\big(\sigma_0^{(\bar{x},\bar{y})}(t); T-t, x, k\big)} - \frac{1}{n!}$$
$$\sum_{h=2}^{n} \mathbf{B}_{n,h} \left(1!\, \sigma_1^{(\bar{x},\bar{y})}, 2!\, \sigma_2^{(\bar{x},\bar{y})}, \ldots, (n-h+1)!\, \sigma_{n-h+1}^{(\bar{x},\bar{y})}\right)$$
$$\times \frac{\partial_\sigma^h u^{\mathrm{BS}}\big(\sigma_0^{(\bar{x},\bar{y})}(t); T-t, x, k\big)}{\partial_\sigma u^{\mathrm{BS}}\big(\sigma_0^{(\bar{x},\bar{y})}(t); T-t, x, k\big)}, \qquad n \geq 1,$$

$$u_n^{(\bar{x},\bar{y})}(t, x, y, k) = \mathcal{L}_n^{(\bar{x},\bar{y})}(t, T) u_0^{(\bar{x},\bar{y})}(t, x, k) = \mathcal{L}_n^{(\bar{x},\bar{y})}(t, T) u^{\mathrm{BS}}(\sigma_0^{(\bar{x},\bar{y})}(t); T-t, x, k). \tag{4.7}$$

A few notes are in order. First, we have added the argument k to the function $u_n^{(\bar{x},\bar{y})}$ to indicate its dependence on the log strike. Second, the function $u_n^{(\bar{x},\bar{y})}$ depends on the maturity date T, as indicated by (4.7). Third, each $\sigma_n^{(\bar{x},\bar{y})}$ in the sequence $(\sigma_n^{(\bar{x},\bar{y})})_{n\geq 1}$ depends on (t, x, y, k). Though, for clarity, we have not written all of these arguments in $\mathbf{B}_{n,h}(1!\,\sigma_1^{(\bar{x},\bar{y})}, 2!\,\sigma_2^{(\bar{x},\bar{y})}, \ldots, (n-h+1)!\,\sigma_{n-h+1}^{(\bar{x},\bar{y})})$. Fourth, we have once again explicitly indicated, with a superscript $(\bar{x}, \bar{y})$, the dependence on the initial point of the Taylor expansion. For the particular choice $\bar{x} = x$ and $\bar{y} = y$, define the following concept.

DEFINITION 4.4. For a Call option with log strike k and maturity T, we define the *Nth-order Taylor approximation of the implied volatility* σ as

$$\bar{\sigma}_N(t, x, y, k) := \bar{\sigma}_N^{(x,y)}(t, x, y, k), \tag{4.8}$$

where $\bar{\sigma}_N^{(x,y)}(t, x, y, k)$ is as defined in (4.6)–(4.7).

4.1. Error Estimates for the Transition Density and Prices

The following theorem provides an asymptotic pointwise estimate as $t \to T^-$ for the error introduced by replacing the exact transition density Γ with the Nth-order approximation $\bar{\Gamma}_N$.

THEOREM 4.5. *Let Assumption 4.1 hold and let* $0 < T \leq T_0$. *Then, for any* $\varepsilon > 0$, *we have*

$$\left|\Gamma(t, z; T, \zeta) - \bar{\Gamma}_N(t, z; T, \zeta)\right| \leq C(T-t)^{\frac{N+1}{2}}\Gamma^{M+\varepsilon}(t, z; T, \zeta), \qquad 0 \leq t < T,\ z, \zeta \in \mathbb{R}^d, \tag{4.9}$$

where $\bar{\Gamma}_N(t, z; T, \zeta)$ *is as defined in* (4.5) *and* $\Gamma^{M+\varepsilon}(t, z; T, \zeta)$ *is the fundamental solution of the heat operator*

$$H^{M+\varepsilon} = (M+\varepsilon)\sum_{i=1}^{d} \partial_{z_i}^2 + \partial_t, \tag{4.10}$$

and C is a positive constant that depends only on M, N, T_0, and ε.

Combining Theorem 4.5 with (4.1), we obtain an asymptotic estimate for $|u(t, z) - \bar{u}_N(t, z)|$, the pricing error.

COROLLARY 4.6. *Under the assumptions of Theorem 4.5, for any* $0 < T \leq T_0$, $\varepsilon > 0$, *we have*

$$|u(t, z) - \bar{u}_N(t, z)| \leq C(T-t)^{\frac{N+1}{2}} \int_{\mathbb{R}^d} \Gamma^{M+\varepsilon}(t, z; T, \zeta)\varphi(\zeta)\mathrm{d}\zeta, \quad 0 \leq t < T,\ z \in \mathbb{R}^d, \tag{4.11}$$

where $\bar{u}_N(t, z)$ *is as defined in* (4.5).

The proof of Theorem 4.5 relies on the following Gaussian estimates (see Friedman 1964, chapter 1).

LEMMA 4.7. *Let $\mathcal{A}(t)$ be a differential operator satisfying Assumption 4.1 and let $\Gamma = \Gamma(t, z; T, \zeta)$ be the fundamental solution corresponding to $\mathcal{A}(t)$. Then, for any $\varepsilon > 0$ and $\beta, \gamma \in \mathbb{N}_0^d$ with $|\gamma| \leq N+3$, we have*

$$\begin{aligned} &|(z-\zeta)^{\beta}\, D_z^{\gamma}\,\Gamma(t, z; T, \zeta)| \leq C(T-t)^{\frac{|\beta|-|\gamma|}{2}}\Gamma^{M+\varepsilon}(t, z; T, \zeta), \\ &0 \leq t < T \leq T_0, \quad z, \zeta \in \mathbb{R}^d, \end{aligned} \tag{4.12}$$

where $\Gamma^{M+\varepsilon}$ *is the fundamental solution of the heat operator* (4.10) *and C is a positive constant, which depends only on* M, N, T_0, ε, *and* $|\beta|$.

We also need the following preliminary estimates (see Lorig et al. 2015a, lemma 6.23).

LEMMA 4.8. *Under the assumptions of Theorem 4.5, for any* $n \in \mathbb{N}$ *with* $n \leq N$, $\epsilon > 0$, *and for any* $\beta \in \mathbb{N}_0^d$, *we have*

$$\left| D_z^{\beta} \Gamma_n^{(\bar{z})}(t, z) \right| \leq C (T - t)^{\frac{n-|\beta|}{2}} \left(1 + |z - \bar{z}|^n (T - t)^{-\frac{n}{2}} \right) \Gamma^{M+\varepsilon}(t, z; T, \zeta), \tag{4.13}$$

which holds for $0 \leq t < T \leq T_0$, $z, \zeta, \bar{z} \in \mathbb{R}^d$. *Here, the function* $\Gamma^{M+\varepsilon}$ *is the fundamental solution of the heat operator* (4.10) *and C is a positive constant, which depends only on* M, N, T_0, ε, *and* $|\beta|$.

Proof of Theorem 4.5. From Lorig et al. (2015a, theorem 3.8), for any given $T \leq T_0$, the functions $(u_n^{(\bar{z})})_{n \geq 1}$ given by (2.17) and (2.18) can be equivalently defined as the unique nonrapidly increasing solutions of the following sequence of nested heat-type Cauchy problems:

$$\begin{cases} \left(\partial_t + \mathcal{A}_0^{(\bar{z})}(t) \right) u_n^{(\bar{z})}(t, z) = - \sum\limits_{h=1}^{n} \mathcal{A}_h^{(\bar{z})}(t) u_{n-h}^{(\bar{z})}(t, z), & t < T, \ z \in \mathbb{R}^d, \\ u_n^{(\bar{z})}(T, z) = 0, & z \in \mathbb{R}^d. \end{cases} \tag{4.14}$$

The result then follows directly from Lorig et al. (2015a, theorem 3.10). For completeness, we provide here a sketch of the proof given in Lorig et al. (2015a). By (4.14), it is easy to prove that $v^{(\bar{z})} := u - \bar{u}_N^{(\bar{z})}$ solves

$$\begin{cases} (\partial_t + \mathcal{A}(t)) v^{(\bar{z})}(t, z) = - \sum\limits_{n=0}^{N} (\mathcal{A}(t) - \bar{\mathcal{A}}_n^{(\bar{z})}(t)) u_{N-n}^{(\bar{z})}(t, z), & t < T, \ z \in \mathbb{R}^d, \\ v^{(\bar{z})}(T, z) = 0, & z \in \mathbb{R}^d, \end{cases}$$

where we have defined $\bar{\mathcal{A}}_n^{(\bar{z})}(t) = \sum_{i=0}^{n} \mathcal{A}_i^{(\bar{z})}(t)$. Thus, by Duhamel's principle,

$$u(t, z) - \bar{u}_N(t, z) = \int_t^T \int_{\mathbb{R}^d} \Gamma(t, z; s, \xi) \sum_{n=0}^{N} \left(\mathcal{A}(s) - \bar{\mathcal{A}}_n^{(z)}(s) \right) u_{N-n}^{(z)}(s, \xi) \, \mathrm{d}\xi \, \mathrm{d}s, \qquad t < T, \ z \in \mathbb{R}^d.$$

Now, by (4.2), we have

$$\begin{aligned} |(\mathcal{A}(s) - \bar{\mathcal{A}}_n^{(z)}(s)) u_{N-n}^{(z)}(s, \xi)| \leq \sum_{|\alpha|} \leq 2 \left| a_{\alpha}^{(z)}(s, \xi) - \sum_{i=0}^{n} a_{\alpha,n}^{(z)}(s, \xi) \right| |D_{\xi}^{\alpha} u_{N-n}^{(z)}(s, \xi)| \\ = \sum_{|\alpha| \leq 2} \left| a_{\alpha}(s, \xi) - \sum_{i=0}^{n} \sum_{|\beta|=n} \frac{D_z^{\beta} a_{\alpha}(s, z)}{\beta!} (\xi - z)^{\beta} \right| \\ \left| D_{\xi}^{\alpha} u_{N-n}^{(z)}(s, \xi) \right| \leq M |\xi - z|^{n+1} \sum_{|\alpha| \leq 2} \left| D_{\xi}^{\alpha} u_{N-n}^{(z)}(s, \xi) \right|, \end{aligned}$$

where the last line follows by the hypothesis (ii) in Assumption 4.1 on the coefficients $(a_{\alpha})_{|\alpha| \leq 2}$. Finally, by considering $u_n^{(z)}(t, z) = \Gamma_n^{(z)}(t, z, ; T, \zeta) = \mathcal{L}_n^{(z)}(t, T) \Gamma_0^{(z)}(t, z; T, \zeta)$, we obtain

$$|\Gamma(t,z;T,\zeta) - \bar{\Gamma}_N(t,z;T,\zeta)| \le M \sum_{n=0}^{N} \sum_{|\alpha|\le 2} \int_t^T \int_{\mathbb{R}^d} \Gamma(t,z;s,\xi)|\xi - z|^{n+1} |D_\xi^\alpha \Gamma_{N-n}^{(z)}(s,\xi;T,\zeta)| \,\mathrm{d}\xi \mathrm{d}s.$$

The result now follows by repeatedly applying the Gaussian estimates (4.12) and (4.13), along with the semigroup property

$$\int_{\mathbb{R}^d} \Gamma^{M+\varepsilon}(t,z;s,\xi)\Gamma^{M+\varepsilon}(s,\xi;T,\zeta)\mathrm{d}\xi \mathrm{d}s = \Gamma^{M+\varepsilon}(t,z;T,\zeta) \qquad t < s < T, \quad z,\zeta \in \mathbb{R}^d.$$

□

4.2. Short-Time Asymptotics for the Implied Volatility

We provide error estimates for the Nth-order implied volatility approximation $\bar{\sigma}_N$, defined in (4.8), on the subset $|x-k| \le \lambda\sqrt{T-t}$ where λ is an arbitrary, but fixed, positive constant.

THEOREM 4.9. *Let Assumption 4.1 hold and let $\lambda > 0$. Denote by $\sigma(t,x,y,k)$ the exact implied volatility of a Call option, with* log *strike k and maturity T. That is, $\sigma(t,x,y,k)$ is the unique positive solution of $u^{\mathrm{BS}}(\sigma; T-t, x, k) = u(t,x,y,k)$, where u is the classical solution of* (2.2) *with time T terminal condition $\varphi(x) = (\mathrm{e}^x - \mathrm{e}^k)^+$. Then, the Nth-order implied volatility approximation $\bar{\sigma}_N(t,x,y,k)$, defined in* (4.8)*, satisfies*

$$\begin{aligned} &|\sigma(t,x,y,k) - \bar{\sigma}_N(t,x,y,k)| \le C(T-t)^{\frac{N+1}{2}}, \\ &0 \le t < T \le T_0, \ y \in \mathbb{R}^{d-1}, \ |x-k| \le \lambda\sqrt{T-t}, \end{aligned} \tag{4.15}$$

where C is a positive constant that depends only on M, N, T_0, and λ.

REMARK 4.10. In the particular case $d = 1$, the above result is consistent with Bompis and Gobet (2012, theorem 22) where an implied volatility approximation for LV models has been derived. A direct computation shows that such an expansion is equivalent to our $\bar{\sigma}_2$. Although Theorem 4.9 holds true for any order $N \in \mathbb{N}_0$ and any dimension $d \in \mathbb{N}$, the estimate in Bompis and Gobet (2012) was proved by the authors under milder assumptions for the generator $\mathcal{A}(t)$, and for three different choices of the initial point $\bar{x}$ of the Taylor expansion: $\bar{x} = x$, $\bar{x} = k$, and $\bar{x} = \frac{x+k}{2}$.

REMARK 4.11. Theorem 4.9 also gives an explicit representation for the nth-order derivative with respect to T of the implied volatility surface at $x = k$ and $T = t$. More precisely, as a corollary of (4.15), we have:

$$\partial_t^n \sigma(t,x,y,k)|_{t=T,k=x} = \partial_t^n \bar{\sigma}_N(t,x,y,k)|_{t=T,k=x}, \qquad \forall N \ge 2n. \tag{4.16}$$

A direct computation shows that for $n = 0$, the representation (4.16) is consistent with the well-known results by Berestycki et al. (2002) and Berestycki et al. (2004). It is also easy to check that our expansion gives the correct slope of the implied volatility at the money in the limit as $t \to T$. For the special case $d = 1$, we recover the practitioners' 1/2 slope rule, which gives the at-the-money slope of implied volatility as one-half the slope of the LV function.

In what follows, the maturity date $T \in (0, T_0]$ is fixed. We recall the Black–Scholes price

$$u^{\mathrm{BS}}(\sigma) = u^{\mathrm{BS}}(\sigma; T-t, x, k),$$

as it is in Definition 3.1 and we denote by $(u^{\mathrm{BS}})^{-1}(u; T-t, x, k) = (u^{\mathrm{BS}})^{-1}(u)$ its inverse with respect to the σ variable. We also introduce the following function:

$$\begin{aligned} u(\delta) = u(\delta; t, x, y, k) &:= \sum_{n=0}^{N} \delta^n u_n^{(x,y)}(t, x, y, k) + \delta^{N+1}\big(u - \bar{u}_N\big)(t, x, y, k) \\ &= u^{\mathrm{BS}}\big(\sigma_0^{(x,y)}(t); T-t, x, k\big) + \sum_{n=1}^{N} \delta^n u_n^{(x,y)}(t, x, y, k) + \delta^{N+1}\big(u - \bar{u}_N\big) \\ &\quad \delta\ (t, x, y, k), \qquad \delta \in [0, 1], \end{aligned} \tag{4.17}$$

where we have used Remark 3.2. Note that the function $u_n^{(x,y)}(t, x, y, k)$ is defined for a fixed maturity date T, as indicated by (4.7).

REMARK 4.12. It is possible to prove (see Lorig et al. 2015a) that in the case of a Call option, the estimate (4.11), as well as (4.13), can be improved by exploiting the local Lipschitz continuity of the payoff $\varphi(x) = (e^x - e^k)^+$. More precisely, it is possible to prove that for any log-strike $k \in \mathbb{R}$, we have

$$|u(t, x, y, k) - \bar{u}_N(t, x, y, k)| \le C(T-t)^{\frac{N+2}{2}} e^x, \quad 0 \le t < T,\ (x, y) \in \mathbb{R} \times \mathbb{R}^{d-1},\ k \in \mathbb{R},$$

and that for any $n \in \mathbb{N}$ with $n \le N$, we also have

$$\left|u_n^{(x,y)}(t, x, y, k)\right| \le C(T-t)^{\frac{n+1}{2}} e^x, \qquad 0 \le t < T,\ (x, y) \in \mathbb{R} \times \mathbb{R}^{d-1},\ k \in \mathbb{R},$$

where, as in Theorem 4.9, C is a positive constant that only depends on M, N, and T_0.

The proof of Theorem 4.9 is based on the previous remark and the following results concerning the short-maturity behavior of the Black–Scholes price. In the following lemma, τ denotes the time to maturity.

LEMMA 4.13. *For any positive constants $C, \lambda, \sigma_1, \sigma_2$ with $\sigma_1 < \sigma_2$, there exists a positive τ_0 that depends only on $C, \lambda, \sigma_1, \sigma_2$ such that*

$$u^{\mathrm{BS}}(\sigma_1; \tau, x, k) + Ce^x\tau \le u^{\mathrm{BS}}(\sigma_2; \tau, x, k),$$

for any $\tau \in [0, \tau_0]$ and $|x-k| \le \lambda\sqrt{\tau}$.

Proof. We recall the following expression for the Black–Scholes price given by Roper and Rutkowski (2009):

$$u^{\mathrm{BS}}(\sigma; \tau, x, k) = \left(e^x - e^k\right)^+ + e^x\sqrt{\frac{\tau}{2\pi}} \int_0^{\sigma} e^{-\frac{1}{2}\left(\frac{x-k}{w\sqrt{\tau}} + \frac{w\sqrt{\tau}}{2}\right)^2} dw.$$

Thus, assuming that $|x-k| \le \lambda\sqrt{\tau}$, we have

$$\begin{aligned} u^{\mathrm{BS}}(\sigma_2; \tau, x, k) - u^{\mathrm{BS}}(\sigma_1; \tau, x, k) &= e^x\sqrt{\frac{\tau}{2\pi}} \int_{\sigma_1}^{\sigma_2} e^{-\frac{1}{2}\left(\frac{x-k}{w\sqrt{\tau}} + \frac{w\sqrt{\tau}}{2}\right)^2} dw \\ &\ge e^x\sqrt{\frac{\tau}{2\pi}} e^{-\frac{1}{2}\left(\frac{\lambda}{\sigma_1} + \frac{\sigma_2\sqrt{\tau}}{2}\right)^2}(\sigma_2 - \sigma_1) \ge Ce^x\tau, \end{aligned}$$

for any $\tau \in [0, \tau_0]$ where τ_0 is positive and suitably small, depending only on λ, C, σ_1, and σ_2. □

LEMMA 4.14. *Let $u(\delta)$ be as in (4.17). Under the assumptions of Theorem 4.9, for any $\varepsilon > 0$, there exists $\tau_0 > 0$, only dependent on M, N, T_0, λ, and ε, such that*

$$u^{\mathrm{BS}}\big(\sqrt{2/(M+\varepsilon)}; T-t, x, k\big) \le u(\delta) \le u^{\mathrm{BS}}\big(\sqrt{2(M+\varepsilon)}; T-t, x, k\big),$$

or equivalently

$$\sqrt{2/(M+\varepsilon)} \le \big(u^{\mathrm{BS}}\big)^{-1}(u(\delta); T-t, x, k) \le \sqrt{2(M+\varepsilon)},$$

for any $t \in [T-\tau_0, T)$, $|x-k| \le \lambda\sqrt{T-t}$, $y \in \mathbb{R}^{d-1}$ and $\delta \in [0, 1]$.

Proof. Throughout this proof, C will always denote a positive constant that depends only on M, N, T_0, and λ. Let

$$v(\delta) = \sum_{n=1}^{N} \delta^n u_n^{(x,y)}(t, x, y, k) + \delta^{N+1}\big(u - \bar{u}_N\big)(t, x, y, k).$$

By Remark 4.12, and as $\delta \in [0, 1]$, we obtain

$$|v(\delta)| \le C(T-t)e^x,$$

and because $u(\delta) = u^{\mathrm{BS}}\big(\sigma_0^{(x,y)}(t); T-t, x, k\big) + v(\delta)$, we then have

$$u^{\mathrm{BS}}\big(\sigma_0^{(x,y)}(t); T-t, x, k\big) - Ce^x(T-t) \le u(\delta) \le u^{\mathrm{BS}}\big(\sigma_0^{(x,y)}(t); T-t, x, k\big) + Ce^x(T-t).$$

The result then follows by applying Lemma 4.13 to estimate the far left- and right-hand sides of the previous inequalities from below and above, respectively. □

LEMMA 4.15. *Under the assumptions of Theorem 4.9, for any $N \in \mathbb{N}$, there exist positive constants C and τ_0, only dependent on M, N, T_0, and λ, such that*

$$\left|\partial_u^n \big(u^{\mathrm{BS}}\big)^{-1}\big(u(\delta; t, x, y, k); T-t, x, k\big)\right| \le C\big(e^k\sqrt{T-t}\,\big)^{-n}, \tag{4.18}$$

for any $n \in \mathbb{N}$, $t \in [T-\tau_0, T)$, $|x-k| \le \lambda\sqrt{T-t}$, $y \in \mathbb{R}^{d-1}$ and $\delta \in [0, 1]$.

Proof. Throughout this proof, C will denote a positive constant only dependent on M, N, T_0, and λ. Note that for any $\sigma > 0$, we have

$$\partial_\sigma u^{\mathrm{BS}}(\sigma) \equiv \partial_\sigma u^{\mathrm{BS}}(\sigma; T-t, x, k) = \frac{e^k\sqrt{T-t}}{\sqrt{2\pi}} \exp\left(-\frac{\big(\sigma^2(T-t) - 2(x-k)\big)^2}{8\sigma^2(T-t)}\right),$$

and thus

$$\frac{e^k\sqrt{T-t}}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2 T_0}{8} - \frac{\lambda^2}{2\sigma^2} - \frac{\lambda\sqrt{T_0}}{2}\right) \le \partial_\sigma u^{\mathrm{BS}}(\sigma) \le \frac{e^k\sqrt{T-t}}{\sqrt{2\pi}}, 0 \le t < T,\ |x-k| \le \lambda\sqrt{T-t}.$$

Therefore, applying Lemma 4.14 with $\varepsilon = 1$, there exists a positive τ_0, only dependent on M, N, T_0, and λ, such that

$$C\frac{e^k\sqrt{T-t}}{\sqrt{2\pi}} \le \partial_\sigma u^{\mathrm{BS}}\Big(\big(u^{\mathrm{BS}}\big)^{-1}(u(\delta))\Big) \le \frac{e^k\sqrt{T-t}}{\sqrt{2\pi}}, \tag{4.19}$$

for any $y \in \mathbb{R}^{d-1}$, $t \in [T-\tau_0, T)$, $|x-k| \leq \lambda\sqrt{T-t}$, and $\delta \in [0,1]$, where C is the positive constant

$$C = \min_{\sigma \in \left[\sqrt{2/(M+1)}, \sqrt{2(M+1)}\right]} \exp\Big(-\frac{\sigma^2 T_0}{8} - \frac{\lambda^2}{2\sigma^2} - \frac{\lambda\sqrt{T_0}}{2}\Big).$$

Furthermore, by combining the second inequality in (4.19) with Proposition 3.5, we also obtain

$$\left|\partial_\sigma^n u^{\mathrm{BS}}\Big(\big(u^{\mathrm{BS}}\big)^{-1}(u(\delta))\Big)\right| \leq Ce^k\sqrt{T-t}. \tag{4.20}$$

We now proceed by induction on n. The case $n=1$ clearly follows from the first inequality in (4.19). We have

$$\left|\partial_u \big(u^{\mathrm{BS}}\big)^{-1}(u(\delta))\right| = \frac{1}{\partial_\sigma u^{\mathrm{BS}}\big(\big(u^{\mathrm{BS}}\big)^{-1}(u(\delta))\big)} \leq \frac{Ce^{-k}}{\sqrt{T-t}}.$$

Let us now assume that (4.18) holds for any $m \leq n$, and prove that it holds for $n+1$. By Faà di Bruno's formula (see Appendix A, equation (A.1)), we have

$$\partial_u^{n+1}\Big(u^{\mathrm{BS}}\Big)^{-1}(u) = \frac{\sum_{h=2}^{n+1} \partial_\sigma^h u^{\mathrm{BS}}\Big(\Big(u^{\mathrm{BS}}\Big)^{-1}(u)\Big)\mathbf{B}_{n+1,h}\Big(\partial_u\Big(u^{\mathrm{BS}}\Big)^{-1}(u), \cdots, \partial_u^{n-h+2}\Big(u^{\mathrm{BS}}\Big)^{-1}(u)\Big)}{\partial_\sigma u^{\mathrm{BS}}\Big(\Big(u^{\mathrm{BS}}\Big)^{-1}(u)\Big)},$$

and thus, by (4.19) and (4.20), we obtain

$$\begin{aligned}
&\left|\partial_u^{n+1}\Big(u^{\mathrm{BS}}\Big)^{-1}(u(\delta))\right| \leq C\sum_{h=2}^{n+1}\left|\mathbf{B}_{n+1,h}\Big(\partial_u\Big(u^{\mathrm{BS}}\Big)^{-1}(u(\delta)), \cdots, \partial_u^{n-h+2}\Big(u^{\mathrm{BS}}\Big)^{-1}(u(\delta))\Big)\right| \\
&\leq C\sum_{h=2}^{n+1}\sum_{j_1,\cdots,j_{n-h+2}}\left|\partial_u\Big(u^{\mathrm{BS}}\Big)^{-1}(u(\delta))\right|^{j_1}\cdots\left|\partial_u^{n-h+2}\Big(u^{\mathrm{BS}}\Big)^{-1}(u(\delta))\right|^{j_{n-h+2}} \quad \text{(by (A.2) in Appendix (A)} \\
&\leq C\sum_{h=2}^{n+1}\sum_{j_1,\cdots,j_{n-h+2}}\Big(e^k\sqrt{T-t}\Big)^{-j_1}\cdots\Big(e^k\sqrt{T-t}\Big)^{-(n-h+2)j_{n-h+2}} \quad \text{(by inductive hypothesis)} \\
&\leq C\sum_{h=2}^{n+1}\sum_{j_1,\cdots,j_{n-h+2}}\Big(e^k\sqrt{T-t}\Big)^{-(n+1)} = C\Big(e^k\sqrt{T-t}\Big)^{-(n+1)},
\end{aligned}$$

where the last inequality follows from the second identity of (A.3) in Appendix A. This concludes the proof. □

Proof of Theorem 4.9. Throughout this proof, C will indicate a positive constant only dependent on M, N, T_0, and λ. It suffices to prove (4.15) for small $T-t$. We recall the function

$$g(\delta) = g(\delta; t, x, y, k) := \big(u^{\mathrm{BS}}\big)^{-1}\big(u(\delta; t, x, y, k); T-t, x, k\big), \qquad \delta \in [0,1],$$

which we defined in (3.5). By definition of $u(\delta)$ in (4.17), it is clear that

$$\sigma(t, x, y, k) = g(1; t, x, y, k). \tag{4.21}$$

Furthermore, with $\bar{\sigma}_N(t, x, y, k)$ as defined in (4.8), we have

$$\bar{\sigma}_N(t, x, y, k) = \sigma_0^{(x,y)}(t) + \sum_{n=0}^{N} \sigma_n^{(x,y)}(t, x, y, k) = \sum_{n=0}^{N} \frac{1}{n!} \partial_\delta^n g(\delta; t, x, y, k)\big|_{\delta=0}, \tag{4.22}$$

as by (4.17) and (3.6), $g(\delta)|_{\delta=0} = \sigma_0^{(x,y)}$, and $\partial_\delta^n g(\delta)\big|_{\delta=0} = \sigma_n^{(x,y)}$ for $1 \leq n \leq N$. Now, by (4.21) and (4.22), and by the Taylor theorem with Lagrange remainder, there exists $\bar{\delta} \in [0, 1]$ such that

$$\begin{aligned} \sigma - \bar{\sigma}_N &= g(1) - \sum_{n=0}^{N} \frac{1}{n!} \partial_\delta^n g(0) = \frac{1}{(N+1)!} \partial_\delta^{N+1} g(\bar{\delta}) \\ &= \frac{1}{(N+1)!} \sum_{h=1}^{N+1} \partial_u^h \left(u^{\mathrm{BS}}\right)^{-1} \left(u(\bar{\delta})\right) \mathbf{B}_{N+1,h} \left(\partial_\delta u(\bar{\delta}), \partial_\delta^2 u(\bar{\delta}), \cdots, \partial_\delta^{N-h+2} u(\bar{\delta})\right), \end{aligned} \tag{4.23}$$

by (A.1) in Appendix A. Now, by (4.17) and Remark 4.12, we obtain

$$|\partial_\delta^n u(\bar{\delta})| \leq C\left(\sum_{h=n}^{N} |u_n^{(x,y)}| + |u - \bar{u}_N| \right) \leq C(T-t)^{\frac{n+1}{2}} e^x. \tag{4.24}$$

Therefore, for any $1 \leq h \leq N+1$, by (A.2) in Appendix A, we have

$$\begin{aligned} &\left|\mathbf{B}_{N+1,h}\left(\partial_\delta u(\bar{\delta}), \partial_\delta^2 u(\bar{\delta}), \cdots, \partial_\delta^{N-h+2} u(\bar{\delta})\right)\right| \\ &\leq C \sum_{j_1, \cdots, j_{N-h+2}} \left|\partial_\delta u(\bar{\delta})\right|^{j_1} \left|\partial_\delta^2 u(\bar{\delta})\right|^{j_2} \cdots \left|\partial_\delta^{N-h+2} u(\bar{\delta})\right|^{j_{N-h+2}} \\ &\leq C\left(\sqrt{T-t}\right)^{N+h+1} e^{hx}, \end{aligned} \tag{4.25}$$

where in the last inequality, we have used (4.24) and both the identities from (A.3) in Appendix A. Combining (4.18) and (4.25) with (4.23), we obtain

$$|\sigma - \bar{\sigma}_N| \leq C(T-t)^{\frac{N+1}{2}} \sum_{h=1}^{N+1} e^{h(x-k)}.$$

The result follows because $e^{h(x-k)} \leq e^{h\lambda\sqrt{T-t}}$ for $|x-k| \leq \lambda\sqrt{T-t}$. □

5. IMPLIED VOLATILITY EXAMPLES

This section uses the results of Section 3 to compute approximate model-induced implied volatilities under four different model dynamics for which European option prices can be computed explicitly:

- Section 5.1: CEV LV model,
- Section 5.2: Heston SV model,
- Section 5.3: Three-halves SV model, and
- Section 5.4: SABR LSV model.

Note that all of these models fail to satisfy the rigorous assumptions required in Theorems 4.5 and 4.9 to prove the error bounds (4.9). However, as mentioned in Remark 4.2, Theorem 4.5 and Corollary 4.6 have recently been extended in Pagliarani and Pascucci (2014), to include all of the examples presented here.

In three of the four examples that follow, we use a Taylor series polynomial expansion of $\mathcal{A}(t)$ as in Example 2.3. In these three cases, approximate-implied volatilities can be computed using the formulas given in Appendix B. For the Heston model, we use the time-dependent Taylor expansion of $\mathcal{A}(t)$ as in Example 2.4. In all cases, Mathematica notebooks containing the implied volatility formulas are available free of charge on the authors' web site.

5.1. CEV LV Model

In the CEV LV model of Cox (1975), the dynamics of the underlying S are given by

$$\mathrm{d}S_t = \delta S_t^{\beta-1} S_t \mathrm{d}W_t, \qquad S_0 = s > 0.$$

The parameter β controls the relationship between volatility and price. When $\beta < 1$, volatility increases as $S \to 0^+$. This feature, referred to as the *leverage effect*, is commonly observed in equity markets. When $\beta < 1$, one also observes a negative at-the-money skew in the model-induced implied volatility surface. Like the leverage effect, a negative at-the-money skew is commonly observed in equity options markets. The origin is attainable when $\beta < 1$. In order to prevent the process S from taking negative values, one typically specifies zero as an absorbing boundary. Hence, the state space of S is $[0, \infty)$. In log notation $X := \log S$, we have the following dynamics:[1]

$$\mathrm{d}X_t = -\frac{1}{2}\delta^2 \mathrm{e}^{2(\beta-1)X_t}\mathrm{d}t + \delta\, \mathrm{e}^{(\beta-1)X_t}\mathrm{d}W_t, \qquad X_0 = x := \log s. \tag{5.1}$$

The generator of X is given by

$$\mathcal{A} = \frac{1}{2}\delta^2 \mathrm{e}^{2(\beta-1)x}(\partial_x^2 - \partial_x).$$

Thus, from (2.4), we identify

$$a(x, y) = \frac{1}{2}\delta^2 \mathrm{e}^{2(\beta-1)x}, \qquad b(x, y) = 0, \qquad c(x, y) = 0, \qquad f(x, y) = 0.$$

We fix a time to maturity t and log-strike k. Using the formulas from Appendix B as well as the Mathematica notebook provided on the authors' web site, we compute explicitly

$$\begin{aligned}
\sigma_0 &= \delta\, \mathrm{e}^{(\beta-1)x},\\
\sigma_1 &= \frac{1}{2}(\beta-1)\sigma_0(k-x),\\
\sigma_2 &= \frac{t}{24}(\beta-1)^2\sigma_0^3 - \frac{t^2}{96}(\beta-1)^2\sigma_0^5 + \frac{1}{12}(\beta-1)^2\sigma_0(k-x)^2,\\
\sigma_3 &= \frac{t}{16}(\beta-1)^3\sigma_0^3(k-x) + \frac{-5t^2}{192}(\beta-1)^3\sigma_0^5(k-x).
\end{aligned} \tag{5.2}$$

In the CEV setting, the exact price of a Call option is derived in Cox (1975):

[1] Here, we define $\log 0 := \lim_{x \searrow 0} \log x = -\infty$.

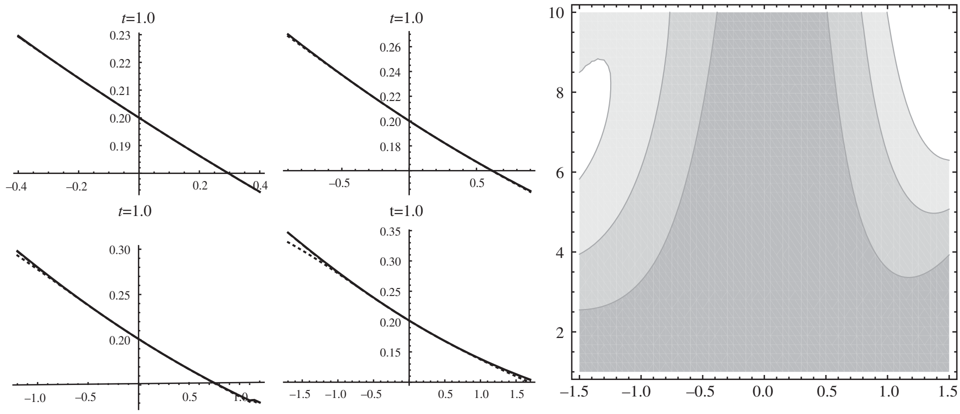

FIGURE 5.1. Left: Implied volatility in the CEV model (5.1) is plotted as a function of log-moneyness $(k - x)$ for four different maturities t. The solid line corresponds to the implied volatility σ obtained by computing the exact price u using (5.3) and then by solving (3.3) numerically. The dashed line (which is nearly indistinguishable from the solid line) corresponds to our third-order implied volatility approximation $\bar{\sigma}_3$, which we compute by summing the terms in (5.2). The dotted line corresponds to the implied volatility expansion σ^{HW} of Hagan and Woodward (1999), which is computed using (5.4). Right: We plot the absolute value of the relative error $|\bar{\sigma}_3 - \sigma|/\sigma$ of our third-order implied volatility approximation as a function of log-moneyness $(k - x)$ and maturity t. The horizontal axis represents log-moneyness $(k - x)$ and the vertical axis represents maturity t. Ranging from darkest to lightest, the regions above represent relative errors of $< 0.3\%$, 0.3–0.6%, 0.6–0.9%, and $> 0.9\%$. We use the following parameters: $\beta = 0.3$, $\delta = 0.2$, and $x = 0.0$.

$$
\begin{aligned}
u(t, x) &= e^x Q\left(\kappa, 2 + \frac{2}{2-\beta}, 2\chi\right) - \mathrm{e}^k \left(1 - Q\left(2\chi, \frac{2}{2-\beta}, 2\kappa\right)\right), \\
Q(w, v, \mu) &= \sum_{n=0}^{\infty} \left(\frac{(\mu/2)^n \mathrm{e}^{-\mu/2}}{n!} \frac{\Gamma(v/2 + n, w/2)}{\Gamma(v/2 + n)} \right), \\
\chi &= \frac{2\mathrm{e}^{(2-\beta)x}}{\delta^2 (2-\beta)^2 t}, \\
\kappa &= \frac{2\mathrm{e}^{(2-\beta)k}}{\delta^2 (2-\beta)^2 t},
\end{aligned}
\tag{5.3}
$$

where $\Gamma(a)$ and $\Gamma(a, b)$ denote the complete and incomplete Gamma functions, respectively. Thus, the implied volatility σ can be obtained numerically by solving (3.3). In Figure 5.1, we plot our third-order implied volatility approximation $\bar{\sigma}_3$ and the numerically obtained implied volatility σ. For comparison, we also plot the implied volatility expansion of Hagan and Woodward (1999)

$$
\begin{aligned}
\sigma^{\mathrm{HW}} &= \frac{\delta}{f^{1-\beta}} \left(1 + \frac{(1-\beta)(2+\beta)}{24} \left(\frac{e^x - e^k}{f} \right)^2 + \frac{(1-\beta)^2}{24} \frac{\delta^2 t}{f^{2(1-\beta)}} + \cdots \right), \\
f &= \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^k).
\end{aligned}
\tag{5.4}
$$

5.2. Heston SV Model

Perhaps, the most well-known stochastic volatility model is that of Heston (1993). In the Heston model, the dynamics of the underlying S are given by

$$\begin{aligned}
\mathrm{d}S_t &= \sqrt{Z_t}\, S_t \mathrm{d}W_t, && S_0 = s > 0,\\
\mathrm{d}Z_t &= \kappa(\theta - Z_t)\mathrm{d}t + \delta\sqrt{Z_t}\mathrm{d}B_t, && Z_0 = z > 0,\\
\mathrm{d}\langle W, B\rangle_t &= \rho\,\mathrm{d}t.
\end{aligned}$$

As pointed out in Andersen and Piterbarg (2007), one must set $\rho < 0$ in order to prevent a moment explosion. In order to improve the efficacy of our approximation, it is convenient to perform the following change of variable $(X_t, V_t) := (\log S, e^{\kappa t} Z_t)$. Changing from Z to V removes the geometric part of the drift (see also Bompis and Gobet 2012). By Ito's formula, we obtain

$$\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}e^{-\kappa t} V_t \mathrm{d}t + \sqrt{e^{-\kappa t} V_t}\mathrm{d}W_t, && X_0 = x := \log s,\\
\mathrm{d}V_t &= \theta\kappa\, e^{\kappa t}\mathrm{d}t + \delta\sqrt{e^{\kappa t} V_t}\mathrm{d}B_t, && V_0 = v := z > 0,\\
\mathrm{d}\langle W, B\rangle_t &= \rho\,\mathrm{d}t.
\end{aligned} \tag{5.5}$$

The generator of (X, V) is given by

$$\mathcal{A}(t) = \frac{1}{2}e^{-\kappa t} v\left(\partial_x^2 - \partial_x\right) + \theta\kappa\, e^{\kappa t}\,\partial_v + \frac{1}{2}\delta^2\delta e^{\kappa t} v\,\partial_v^2 + \delta\rho v\,\partial_x\partial_v.$$

Thus, using (2.4), we identify

$$a(x, v) = \frac{1}{2}e^{-\kappa t} v, \qquad b(x, v) = \delta^2\delta e^{\kappa t} v, \quad c(x, v) = \delta\rho v, \qquad f(x, v) = \theta\kappa\, e^{\kappa t}.$$

We fix a time to maturity t, a log-strike k, and we consider the time-dependent Taylor series expansion of $\mathcal{A}(t)$, as described in Example 2.4. with $(\bar{x}(t), \bar{v}(t)) = (X_0, \mathbb{E}[V_t]) := (x, \theta(e^{\kappa t} - 1))$. Using the Mathematica notebook provided on the authors' web site, we compute explicitly

$$\begin{aligned}
\sigma_0 &= \sqrt{\frac{-\theta + \theta\kappa t + e^{-\kappa t}(\theta - v) + v}{\kappa t}},\\
\sigma_1 &= \frac{\delta\rho z e^{-\kappa t}\left(-2\theta - \theta\kappa t - e^{\kappa t}(\theta(\kappa t - 2) + v) + \kappa t v + v\right)}{\sqrt{2}\kappa^2\sigma_0^2 t^{3/2}},\\
\sigma_2 &= \frac{\delta^2 e^{-2\kappa t}}{32\kappa^4\sigma_0^5 t^3}\Bigg(-2\sqrt{2}\kappa\sigma_0^3 t^{3/2} z\left(-\theta - 4e^{\kappa t}(\theta + \kappa t(\theta - v)) + e^{2\kappa t}(\theta(5 - 2\kappa t) - 2v) + 2v\right)\\
&\quad + \kappa\sigma_0^2 t\left(4z^2 - 2\right)\left(\theta + e^{2\kappa t}\left(-5\theta + 2\theta\kappa t + 8\rho^2(\theta(\kappa t - 3) + v) + 2v\right)\right)\\
&\quad + \kappa\sigma_0^2 t\left(4z^2 - 2\right)\left(4e^{\kappa t}\left(\theta + \theta\kappa t + \rho^2(\theta(\kappa t(\kappa t + 4) + 6) - v(\kappa t(\kappa t + 2) + 2)) - \kappa t v\right) - 2v\right)\\
&\quad + 4\sqrt{2}\rho^2\sigma_0\sqrt{t} z\left(2z^2 - 3\right)\left(-2\theta - \theta\kappa t - e^{\kappa t}(\theta(\kappa t - 2) + v) + \kappa t v + v\right)^2\\
&\quad + 4\rho^2\left(4\left(z^2 - 3\right)z^2 + 3\right)\left(-2\theta - \theta\kappa t - e^{\kappa t}(\theta(\kappa t - 2) + v) + \kappa t v + v\right)^2\Bigg)\\
&\quad - \frac{\sigma_1^2\left(4(x - k)^2 - \sigma_0^4 t^2\right)}{8\sigma_0^3 t}, \quad z = \frac{x - k - \frac{\sigma_0^2 t}{2}}{\sigma_0\sqrt{2t}}.
\end{aligned} \tag{5.6}$$

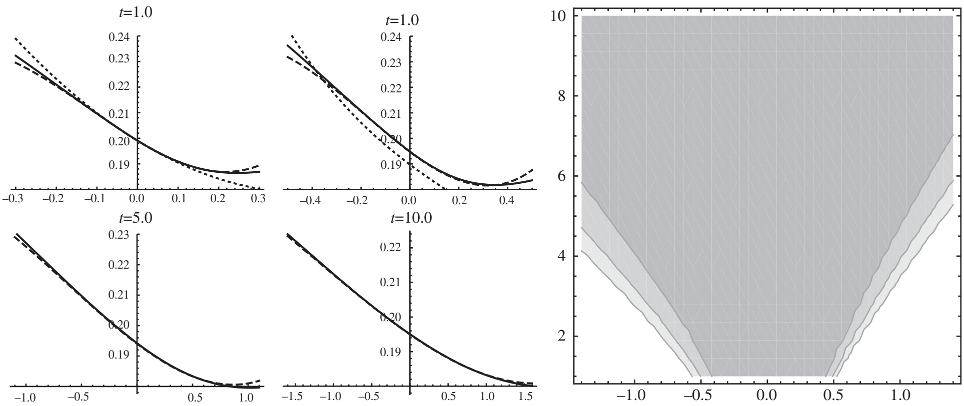

FIGURE 5.2. Left: Implied volatility in the Heston model (5.5) is plotted as a function of log-moneyness $(k - x)$ for four different maturities t. The solid line corresponds to the implied volatility σ, obtained by computing the exact price u using (5.7) and then by solving (3.3) numerically. The dashed line corresponds to our third-order implied volatility approximation $\bar{\sigma}_3$, which we compute by summing the terms in (5.6) (note: σ_3 does not appear in the text). The dotted line (which only appears for the shortest two maturities) corresponds to the implied volatility expansion σ^{FJL} of Forde et al. (2012); it is computed using (5.8). Note that the dotted line does not appear in the plots for the two largest maturities. Right: We plot the absolute value of the relative error $|\bar{\sigma}_3 - \sigma|/\sigma$ of our third-order implied volatility approximation as a function of log-moneyness $(k - x)$ and maturity t. The horizontal axis represents log-moneyness $(k - x)$ and the vertical axis represents maturity t. Ranging from darkest to lightest, the regions above represent relative errors of $< 1\%$, 1–2%, 2–3%, and $> 3\%$. We use the parameters given in Forde et al. (2012): $\kappa = 1.15$, $\theta = 0.04$, $\delta = 0.2$, $\rho = -0.40$ $x = 0.0$, and $y = \log\theta$.

The expression for σ_3 is too long to be reported. However, the explicit form of σ_3 is provided in the Mathematica notebook on the authors' web site.

The characteristic function of X_t is computed explicitly in Heston (1993)

$$
\begin{aligned}
\eta(t, x, y, \lambda) := \log \mathbb{E}_{x,y} e^{\mathtt{i}\lambda X_t} &= \mathtt{i}\lambda x + C(t, \lambda) + D(t, \lambda) e^{y}, \\
C(t, \lambda) &= \frac{\kappa\theta}{\delta^2}\left((\kappa - \rho\delta\mathtt{i}\lambda + d(\lambda))t - 2\log\left[\frac{1 - f(\lambda)e^{d(\lambda)t}}{1 - f(\lambda)}\right]\right), \\
D(t, \lambda) &= \frac{\kappa - \rho\delta\mathtt{i}\lambda + d(\lambda)}{\delta^2}\,\frac{1 - e^{d(\lambda)t}}{1 - f(\lambda)e^{d(\lambda)t}}, \\
f(\lambda) &= \frac{\kappa - \rho\delta\mathtt{i}\lambda + d(\lambda)}{\kappa - \rho\delta\mathtt{i}\lambda - d(\lambda)}, \\
d(\lambda) &= \sqrt{\delta^2(\lambda^2 + \mathtt{i}\lambda) + (\kappa - \rho\mathtt{i}\lambda\delta)^2}.
\end{aligned}
$$

Thus, the price of a European Call option can be computed using standard Fourier methods

$$
u(t, x, y) = \frac{1}{2\pi}\int_{\mathbb{R}} d\lambda_r\, e^{\eta(t,x,y,\lambda)}\widehat{\varphi}(\lambda),
$$

$$\widehat{\varphi}(\lambda) = \frac{-\mathrm{e}^{k-\mathtt{i}k\lambda}}{\mathtt{i}\lambda + \lambda^2}, \quad \lambda = \lambda_r + \mathtt{i}\lambda_i, \quad \lambda_i < -1. \tag{5.7}$$

Because the Call option payoff $\varphi(x) = (\mathrm{e}^x - \mathrm{e}^k)^+$ is not in $L^1(\mathbb{R})$, its Fourier transform $\widehat{\varphi}(\lambda)$ must be computed in a generalized sense by fixing an imaginary component of the Fourier variable $\lambda_i < -1$. Using (5.7), the implied volatility σ can be computed to solving (3.3) numerically. In Figure 5.2, we plot our third-order implied volatility approximation $\bar{\sigma}_3$ and the numerically obtained implied volatility σ. For comparison, we also plot the small-time near-the-money implied volatility expansion of Forde et al. (2012) (see theorem 3.2 and corollary 4.3)

$$\begin{aligned}
\sigma^{\mathrm{FJL}} &= \left(g_0^2 + g_1\, t + o(t)\right)^{1/2}, g_0 \\
&= \mathrm{e}^{y/2}\left(1 + \frac{1}{4}\rho\delta(k-x)\mathrm{e}^{-y} + \frac{1}{24}\left(1 - \frac{5\rho^2}{2}\right)\delta^2(k-x)^2\mathrm{e}^{-2y}\right) + \mathcal{O}((k-x)^3), \\
g_1 &= -\frac{\delta^2}{12}\left(1 - \frac{\rho^2}{4}\right) + \frac{\mathrm{e}^y\rho\delta}{4} + \frac{\kappa}{2}(\theta - \mathrm{e}^y) + \frac{1}{24}\rho\delta\mathrm{e}^{-y}(\delta^2\overline{\rho}^2 - 2\kappa(\theta + \mathrm{e}^y) + \rho\delta\mathrm{e}^y)(k-x) \\
&\quad + \frac{\delta^2\mathrm{e}^{-2y}}{7680}\left(176\delta^2 - 480\kappa\theta - 712\rho^2\delta^2 + 521\rho^4\delta^2 + 40\rho^3\delta\mathrm{e}^y + 1040\kappa\theta\rho^2 - 80\kappa\rho^2\mathrm{e}^y\right)(k-x)^2 \\
&\quad + \mathcal{O}((k-x)^3), \qquad \overline{\rho} = \sqrt{1-\rho^2}.
\end{aligned} \tag{5.8}$$

5.3. Three-Halves SV Model

We consider now the three-halves stochastic volatility model. The risk-neutral dynamics of the underlying S in this setting are given by

$$\begin{aligned}
\mathrm{d}S_t &= \sqrt{Z_t}S_t\mathrm{d}W_t, & S_0 &= s > 0, \\
\mathrm{d}Z_t &= Z_t\left(\kappa(\theta - Z_t)\mathrm{d}t + \delta\sqrt{Z_t}\mathrm{d}B_t\right), & Z_0 &= z > 0, \\
\mathrm{d}\langle W, B\rangle_t &= \rho\,\mathrm{d}t.
\end{aligned}$$

As in all stochastic volatility models, one typically sets $\rho < 0$ in order to capture the leverage effect. The three-halves model is noteworthy in that it does not fall into the affine class of Duffie, Pan, and Singleton (2000), and yet it still allows for European option prices to be computed in semi-closed form (as a Fourier integral). Notice, however, that the characteristic function (given in (5.11) below) involves special functions such as the Gamma and the confluent hypergeometric functions. Therefore, Fourier pricing methods are not an efficient means of computing prices. The importance of the three-halves model in the pricing of options on realized variance is well documented by Drimus (2012). In particular, the three-halves model allows for upward-sloping implied volatility of variance smiles, whereas Heston's model leads to downward-sloping volatility of variance smiles, in disagreement with observed skews in variance markets.

In log notation $(X, Y) := (\log S, \log Z)$, we have the following dynamics:

$$\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}\mathrm{e}^{Y_t}\mathrm{d}t + \mathrm{e}^{\frac{1}{2}Y_t}\mathrm{d}W_t, & X_0 &= x := \log s, \\
\mathrm{d}Y_t &= \left(\kappa(\theta - \mathrm{e}^{Y_t}) - \frac{1}{2}\delta^2\mathrm{e}^{Y_t}\right)\mathrm{d}t + \delta\,\mathrm{e}^{\frac{1}{2}Y_t}\mathrm{d}B_t, & Y_0 &= y := \log z, \\
\mathrm{d}\langle W, B\rangle_t &= \rho\,\mathrm{d}t.
\end{aligned} \tag{5.9}$$

The generator of (X, Y) is given by

$$\mathcal{A} = \frac{1}{2}\mathrm{e}^{y}\left(\partial_x^2 - \partial_x\right) + \left(\kappa(\theta - \mathrm{e}^{y}) - \frac{1}{2}\delta^2\mathrm{e}^{y}\right)\partial_y + \frac{1}{2}\delta^2\mathrm{e}^{y}\partial_y^2 + \rho\,\delta\,\mathrm{e}^{y}\partial_x\partial_y.$$

Thus, using (2.4), we identify

$$a(x, y) = \frac{1}{2}\mathrm{e}^{y}, \quad b(x, y) = \frac{1}{2}\delta^2\mathrm{e}^{y}, \quad c(x, y) = \rho\,\delta\,\mathrm{e}^{y}, \quad f(x, y) = \kappa(\theta - \mathrm{e}^{y}) - \frac{1}{2}\delta^2\mathrm{e}^{y}.$$

We fix a time to maturity t and log-strike k. Using the formulas from Appendix B as well as the Mathematica notebook provided on the authors' web site, we compute explicitly

$$\begin{aligned}
\sigma_0 &= \mathrm{e}^{y/2}, \sigma_1 = \frac{t}{8}(2\theta\kappa\sigma_0 - \sigma_0^3(\delta^2 - \delta\rho + 2\kappa)) + \frac{1}{4}\delta\rho\sigma_0(k - x),\\
\sigma_2 &= \frac{t}{96}\delta^2(8 - 7\rho^2)\sigma_0^3 + \frac{t^2}{384}(-36\theta\kappa\sigma_0^3(\delta^2 - \delta\rho + 2\kappa)\\
&\quad + \sigma_0^5(13\delta^4 - 26\delta^3\rho + 4\delta^2(13\kappa + 4\rho^2 - 1) - 52\delta\kappa\rho + 52\kappa^2) + 20\theta^2\kappa^2\sigma_0)\\
&\quad + \frac{t}{96}\delta\rho\sigma_0(6\theta\kappa - 7\sigma_0^2(\delta^2 - \delta\rho + 2\kappa))(k - x) - \frac{1}{48}\delta^2(\rho^2 - 2)\sigma_0(k - x)^2,\\
\sigma_3 &= \frac{t^2}{256}\delta^2\sigma_0^3(5(3\rho^2 - 4)\sigma_0^2(\delta^2 - \delta\rho + 2\kappa) + 2\theta\kappa(8 - 7\rho^2))\\
&\quad + \frac{t^3}{3072}\Big(- 132\theta^2\kappa^2\sigma_0^3(\delta^2 - \delta\rho + 2\kappa) + 10\theta\kappa\sigma_0^5(13\delta^4 - 26\delta^3\rho\\
&\quad + 4\delta^2(13\kappa + 4\rho^2 - 1) - 52\delta\kappa\rho + 52\kappa^2)\\
&\quad + 24\theta^3\kappa^3\sigma_0 - \sigma_0^7(\delta^2 - \delta\rho + 2\kappa)(35\delta^4 - 70\delta^3\rho + 2\delta^2(70\kappa + 29\rho^2 - 16)\\
&\quad - 140\delta\kappa\rho + 140\kappa^2)\Big) + \frac{t}{128}\delta^3\rho(4 - 3\rho^2)\sigma_0^3(k - x)\\
&\quad + \frac{t^2\delta\rho\sigma_0}{1536}\Big(- 84\theta\kappa\sigma_0^2(\delta^2 - \delta\rho + 2\kappa)\Big)(k - x)\\
&\quad + \frac{t^2\delta\rho\sigma_0}{1536}\Big(+ \sigma_0^4(45\delta^4 - 90\delta^3\rho + 4\delta^2(45\kappa + 14\rho^2 - 4)\\
&\quad -180\delta\kappa\rho + 180\kappa^2) + 20\theta^2\kappa^2\Big)(k - x)\\
&\quad + \frac{t}{384}\delta^2\sigma_0((\rho^2 - 8)\sigma_0^2(\delta^2 - \delta\rho + 2\kappa) - 2\theta\kappa(\rho^2 - 2))(k - x)^2.
\end{aligned} \tag{5.10}$$

To the best of our knowledge, the above formula is the first explicit implied volatility expansion for the three-halves model. The characteristic function of X_t is given, for example, in proposition 3.2 of Baldeaux and Badran (2012). We have

$$\begin{aligned}
\mathbb{E}_{x,y}\mathrm{e}^{\mathrm{i}\lambda X_t} &= \mathrm{e}^{\mathrm{i}\lambda x}\frac{\Gamma(\gamma - f)}{\Gamma(\gamma)}\left(\frac{2}{\delta^2 z}\right)^{f} M\left(f, \gamma, \frac{-2}{\delta^2 z}\right),\\
z &= \frac{\mathrm{e}^{y}}{\kappa\theta}(\mathrm{e}^{\kappa\theta t} - 1), \qquad \gamma = 2\left(f + 1 - \frac{p}{\delta^2}\right),\\
f &= -\left(\frac{1}{2} - \frac{p}{\delta^2}\right) + \left(\left(\frac{1}{2} - \frac{p}{\delta^2}\right)^2 + 2\frac{q}{\delta^2}\right)^{1/2},\\
p &= -\kappa + \mathrm{i}\delta\rho\lambda, \quad q = \frac{1}{2}(\mathrm{i}\lambda + \lambda^2),
\end{aligned} \tag{5.11}$$

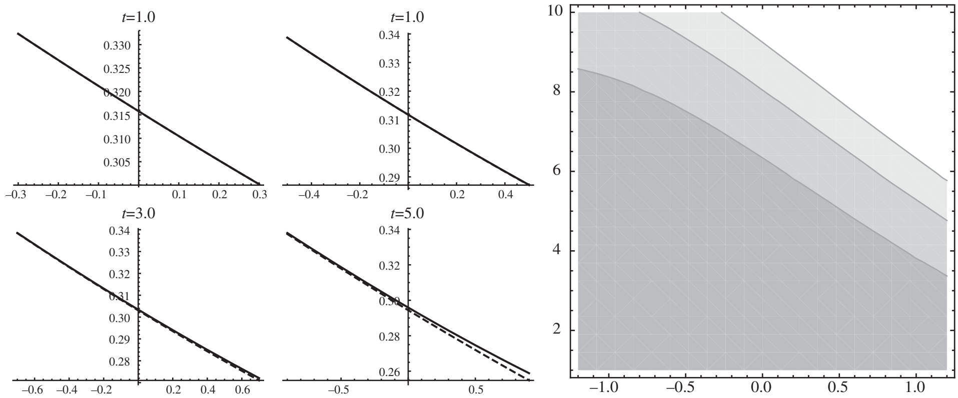

FIGURE 5.3. Left: Implied volatility in the three-halves stochastic volatility model (5.10) is plotted as a function of log-moneyness $(k - x)$ for four different maturities t. The solid line corresponds to the implied volatility σ, obtained by computing the exact price u using (5.12) and then by solving (3.3) numerically. The dashed line corresponds to our third-order implied volatility approximation $\bar{\sigma}_3$, which we compute by summing the terms in (5.10). Right: We plot the absolute value of the relative error $|\bar{\sigma}_3 - \sigma|/\sigma$ of our third-order implied volatility approximation as a function of log-moneyness $(k - x)$ and maturity t. The horizontal axis represents log-moneyness $(k - x)$ and the vertical axis represents maturity t. Ranging from darkest to lightest, the regions above represent relative errors of $< 1\%$, 1–2%, 2–3%, and $> 3\%$. We use the following parameters: $\kappa = 0.25$, $\theta = 0.1$, $\delta = 0.8$, $\rho = -0.85$ $x = 0.0$, and $y = \log\theta$.

where Γ is a Gamma function and M is a confluent hypergeometric function. Thus, the price of a European Call option can be computed using standard Fourier methods

$$u(t, x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}\lambda_r \, \widehat{\varphi}(\lambda) \mathbb{E}_{x,y} \mathrm{e}^{\mathrm{i}\lambda X_t}, \quad \lambda = \lambda_r + \mathrm{i}\lambda_i, \quad \lambda_i < -1, \tag{5.12}$$

where $\widehat{\varphi}(\lambda)$ is given in (5.7). Using (5.12), the implied volatility σ can be computed to solving (3.3) numerically. In Figure 5.3, we plot our third-order implied volatility approximation $\bar{\sigma}_3$ and the numerically obtained implied volatility σ.

5.4. SABR LSV

The SABR model of Hagan et al. (2002) is a LSV model in which the risk-neutral dynamics of S are given by

$$\begin{aligned} \mathrm{d}S_t &= Z_t S_t^{\beta} \mathrm{d}W_t, & S_0 &= s > 0, \\ \mathrm{d}Z_t &= \delta Z_t \mathrm{d}B_t, & Z_0 &= z > 0, \\ \mathrm{d}\langle W, B\rangle_t &= \rho \, \mathrm{d}t. \end{aligned}$$

Modeling Z as a geometric Brownian motion results in a true implied volatility smile (i.e., upward sloping implied volatility for high strikes); this is in contrast to the CEV model, for which the model-induced implied volatility is monotone decreasing (for $\beta < 1$). In log notation $(X, Y) := (\log S, \log Z)$, we have the following dynamics:

$$
\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}\mathrm{e}^{2Y_t+2(\beta-1)X_t}\mathrm{d}t + \mathrm{e}^{Y_t+(\beta-1)X_t}\mathrm{d}W_t, && X_0 = x := \log s, \\
\mathrm{d}Y_t &= -\frac{1}{2}\delta^2\mathrm{d}t + \delta\,\mathrm{d}B_t, && Y_0 = y := \log z, \\
\mathrm{d}\langle W, B\rangle_t &= \rho\,\mathrm{d}t.
\end{aligned}
\tag{5.13}
$$

The generator of (X, Y) is given by

$$
\mathcal{A} = \frac{1}{2}\mathrm{e}^{2y+2(\beta-1)x}(\partial_x^2 - \partial_x) - \frac{1}{2}\delta^2\partial_y + \frac{1}{2}\delta^2\partial_y^2 + \rho\,\delta\,\mathrm{e}^{y+(\beta-1)x}\partial_x\partial_y.
$$

Thus, using (2.4), we identify

$$
a(x, y) = \frac{1}{2}\mathrm{e}^{2y+2(\beta-1)x}, \quad b(x, y) = \frac{1}{2}\delta^2, \quad c(x, y) = \rho\,\delta\,\mathrm{e}^{y+(\beta-1)x}, \quad f(x, y) = -\frac{1}{2}\delta^2.
$$

We fix a time to maturity t and log-strike k. Using the formulas from Appendix A.3 as well as the Mathematica notebook provided on the authors' web site, we compute explicitly

$$
\begin{aligned}
&\sigma_0 = \mathrm{e}^{y+(\beta-1)x}, \quad \sigma_1 = \sigma_{1,0} + \sigma_{0,1}, \quad \sigma_2 = \sigma_{2,0} + \sigma_{1,1} + \sigma_{0,2}, \\
&\sigma_3 = \sigma_{3,0} + \sigma_{2,1} + \sigma_{1,2} + \sigma_{0,3},
\end{aligned}
\tag{5.14}
$$

where

$$
\begin{aligned}
\sigma_{1,0} &= \frac{1}{2}(k-x)(-1+\beta)\sigma_0, \\
\sigma_{0,1} &= \frac{1}{4}\delta\left(2(k-x)\rho + t\sigma_0\left(-\delta + \rho\sigma_0\right)\right), \\
\sigma_{2,0} &= \frac{t}{24}(\beta-1)^2\sigma_0^3 - \frac{t^2}{96}(\beta-1)^2\sigma_0^5 + \frac{1}{12}(\beta-1)^2\sigma_0(k-x)^2, \\
\sigma_{1,1} &= \frac{t}{4}(\beta-1)\delta\rho\sigma_0^2 - \frac{t^2}{48}(\beta-1)\delta\rho\sigma_0^4 - \frac{t}{24}(\beta-1)\delta\sigma_0\left(3\delta - 5\rho\sigma_0\right)(k-x), \\
\sigma_{0,2} &= \frac{t}{24}\delta^2\left(8-3\rho^2\right)\sigma_0 + \frac{t^2}{96}\delta^2\sigma_0\left(5\delta^2 + 2\sigma_0\left(\left(6\rho^2-2\right)\sigma_0 - 7\delta\rho\right)\right) \\
&\quad - \frac{t}{24}\delta^2\rho\left(\delta - 3\rho\sigma_0\right)(k-x) + \frac{\delta^2\left(2-3\rho^2\right)}{12\sigma_0}(k-x)^2, \\
\sigma_{3,0} &= t\left(\frac{1}{16}(\beta-1)^3\sigma_0^3\right)(k-x) + t^2\left(\frac{1}{192}(-5)(\beta-1)^3\sigma_0^5\right)(k-x), \\
\sigma_{2,1} &= t^2\left(\frac{1}{96}(\beta-1)^2\delta\sigma_0^3\left(7\rho\sigma_0 - 3\delta\right)\right) + t^3\left(\frac{1}{384}(\beta-1)^2\delta\sigma_0^5\left(5\delta - 7\rho\sigma_0\right)\right) \\
&\quad + t\left(\frac{13}{48}(\beta-1)^2\delta\rho\sigma_0^2\right)(k-x) + t^2\left(\frac{1}{192}(-13)(\beta-1)^2\delta\rho\sigma_0^4\right)(k-x) \\
&\quad + t\left(-\frac{1}{48}(\beta-1)^2\delta\sigma_0\left(\delta - 3\rho\sigma_0\right)\right)(k-x)^2
\end{aligned}
$$

$$\sigma_{1,2} = t^2\left(\frac{1}{48}(\beta-1)\delta^2\rho\sigma_0^2\left(13\rho\sigma_0-7\delta\right)\right)+t^3\left(\frac{1}{192}(\beta-1)\delta^2\rho\sigma_0^4\left(5\delta-7\rho\sigma_0\right)\right)$$
$$+t\left(\frac{1}{48}(\beta-1)\delta^2\left(3\rho^2+8\right)\sigma_0\right)(k-x)$$
$$+t^2\left(\frac{1}{192}(\beta-1)\delta^2\sigma_0\left(5\delta^2-22\delta\rho\sigma_0+4\left(5\rho^2-3\right)\sigma_0^2\right)\right)(k-x)$$
$$+t\left(\frac{1}{24}(\beta-1)\delta^2\rho^2\sigma_0\right)(k-x)^2+\left(\frac{(\beta-1)\delta^2\left(3\rho^2-2\right)}{24\sigma_0}\right)(k-x)^3,$$

$$\sigma_{0,3} = t^2\left(\frac{1}{96}\delta^3\sigma_0\left(3\delta\left(\rho^2-4\right)+\rho\left(26-9\rho^2\right)\sigma_0\right)\right)$$
$$+t^3\left(\frac{1}{384}\delta^3\sigma_0\left(\sigma_0\left(19\delta^2\rho+2\sigma_0\left(\delta\left(8-21\rho^2\right)+\rho\left(15\rho^2-11\right)\sigma_0\right)\right)-3\delta^3\right)\right)$$
$$+t\left(\frac{1}{48}\delta^3\rho\left(3\rho^2-2\right)\right)(k-x)+t^2\left(-\frac{1}{192}\delta^3\rho\left(\delta^2+6\sigma_0\left(\delta\rho+\left(1-2\rho^2\right)\sigma_0\right)\right)\right)(k-x)$$
$$+t\left(-\frac{1}{16}\delta^3\rho\left(\rho^2-1\right)\right)(k-x)^2+\left(\frac{\delta^3\rho\left(6\rho^2-5\right)}{24\sigma_0^2}\right)(k-x)^3.$$

There is no formula for European option prices in the general SABR setting. However, for the special zero-correlation case $\rho = 0$, the exact price of a European Call is computed in Antonov and Spector (2012):

$$u(t,x) = e^{(x+k)/2}\frac{e^{-\delta^2 t/8}}{\sqrt{2\pi\delta^2 t}}\left\{\frac{1}{\pi}\int_0^\infty dV\int_0^\pi d\phi\frac{1}{V}\left(\frac{V}{V_0}\right)^{-1/2}\frac{\sin\phi\sin(|\nu|\phi)}{b-\cos\phi}\exp\left(\frac{\xi_\phi^2}{2\delta^2 t}\right)\right.$$
$$\left.+\frac{\sin(|\nu|\pi)}{\pi}\int_0^\infty dV\int_0^\infty d\psi\frac{1}{V}\left(\frac{V}{V_0}\right)^{-1/2}\frac{\sinh\psi}{b-\cosh\psi}e^{-|\nu|\psi}\exp\left(\frac{\xi_\psi^2}{2\delta^2 t}\right)\right\}$$
$$+\left(e^x-e^k\right)^+,\ \xi_\phi = \arccos\left(\frac{q_h^2+q_x^2+V^2+V_0^2}{2VV_0}-\frac{q_hq_x}{VV_0}\cos\phi\right), \tag{5.15}$$
$$\xi_\psi = \arccos\left(\frac{q_h^2+q_x^2+V^2+V_0^2}{2VV_0}+\frac{q_hq_x}{VV_0}\cosh\psi\right),$$
$$b=\frac{q_h^2+q_x^2}{2q_hq_x},\qquad q_h=\frac{e^{(1-\beta)k}}{1-\beta},\qquad q_x=\frac{e^{(1-\beta)x}}{1-\beta},\qquad \nu=\frac{-1}{2(1-\beta)},\qquad V_0=\frac{e^y}{\delta}.$$

Thus, in the zero-correlation setting, the implied volatility σ can be obtained by using the above formula and then by solving (3.3) numerically. In Figure 5.4, we plot our third-order implied volatility approximation $\bar{\sigma}_3$ and the numerically obtained implied volatility σ. For comparison, we also plot the implied volatility expansion of Hagan et al. (2002)

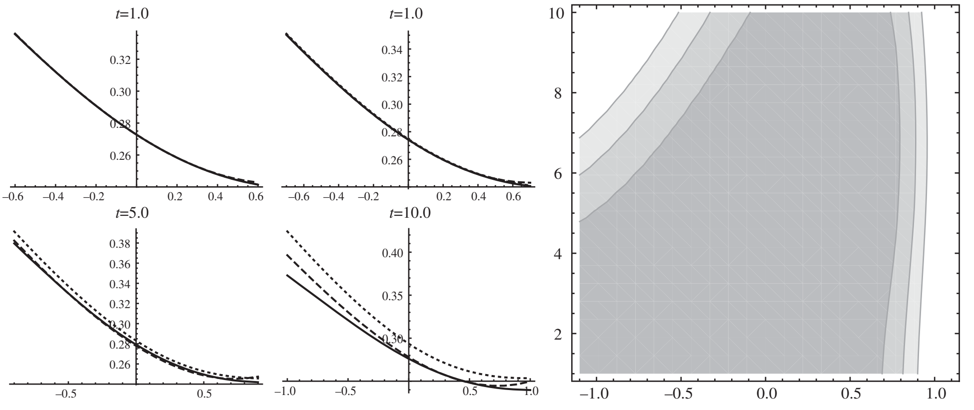

FIGURE 5.4. Left: Implied volatility in the SABR model (5.13) is plotted as a function of log-moneyness $(k - x)$ for four different maturities t. The solid line corresponds to the implied volatility σ, obtained by computing the exact price u using (5.15) and then by solving (3.3) numerically. The dashed line corresponds to our third-order implied volatility approximation $\bar{\sigma}_3$, which we compute using (5.14). The dotted line corresponds to the implied volatility expansion σ^{HKLW} of Hagan et al. (2002), which is computed using (5.16). For the two shortest maturities, both implied volatility expansions $\bar{\sigma}_3$ and σ^{HKLW} provide an excellent approximation of the true implied volatility σ. However, for the two longest maturities, it is clear that our third-order expansion $\bar{\sigma}_3$ provides a better approximation to the true implied volatility σ than does the implied volatility expansion σ^{HKLW} of Hagan et al. (2002). Right: We plot the absolute value of the relative error $|\bar{\sigma}_3 - \sigma|/\sigma$ of our third-order implied volatility approximation as a function of log-moneyness $(k - x)$ and maturity t. The horizontal axis represents log-moneyness $(k - x)$ and the vertical axis represents maturity t. Ranging from darkest to lightest, the regions above represent relative errors of $< 1\%$, 1–2%, 2–3%, and $> 3\%$. We use the following parameters: $\beta = 0.4$, $\delta = 0.25$, $\rho = 0.0$, $x = 0.0$, and $y = -1.3$.

$$
\begin{aligned}
\sigma^{\text{HKLW}} &= \delta \frac{x-k}{D(\zeta)} \left\{ 1 + t\delta^2 \left[\frac{2\gamma_2 - \gamma_1^2 + 1/f^2}{24} \left(\frac{\mathrm{e}^{y+\beta f}}{\delta} \right)^2 + \frac{\rho \gamma_1 \mathrm{e}^{y+\beta f}}{4\delta} + \frac{2 - 3\rho^2}{24} \right] \right\}, \\
f &= \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^k), \; \zeta = \frac{\delta\, \mathrm{e}^{-y}}{\beta - 1} \left(\mathrm{e}^{(1-\beta)k} - \mathrm{e}^{(1-\beta)x} \right), \; \gamma_1 = \beta/f, \\
\gamma_2 &= \beta(\beta-1)/f^2, \; D(\zeta) = \log\left(\frac{\sqrt{1 - 2\rho\zeta + \zeta^2} + \zeta - \rho}{1 - \rho} \right).
\end{aligned}
\tag{5.16}
$$

Note that we use the "corrected" SABR formula, which appears in Obloj (2008).

6. CONCLUSIONS AND FUTURE WORK

In this paper, we consider a general class of parametric LSV models. In this setting, we provide a family of approximations—one for each polynomial expansion of $\mathcal{A}(t)$—for (i) European-style option prices and (ii) implied volatilities. The terms in our option price expansions are expressed as differential operators acting on the Black–Scholes price.

Thus, to compute approximate prices, one requires only a normal CDF. Our implied volatility expansions are explicit, requiring neither special functions nor any numerical integration. Thus, approximate implied volatilities can be computed even faster than option prices.

We carry out extensive computations using the Taylor series expansion of $\mathcal{A}(t)$. In particular, we establish the rigorous error bounds of our pricing and implied volatility approximations. We also implement our implied volatility expansion in four separate models: CEV LV, Heston SV, three-halves SV, and SABR LSV. In each setting, we demonstrate that our implied volatility expansion provides an excellent approximation of the true implied volatility over a large range of strikes and maturities.

APPENDIX A: FAÀ DI BRUNO'S FORMULA AND BELL POLYNOMIALS

Here, we briefly recall the well-known Faà di Bruno's formula (see Riordan 1946 and Johnson 2002), more precisely, its Bell polynomial version. Let f and g be two C^∞ real-valued functions on $\mathbb{R}$. The following representation holds:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(g(x)) = \sum_{h=1}^{n} f^{(h)}(g(x)) \cdot \mathbf{B}_{n,h}\left(\frac{\mathrm{d}}{\mathrm{d}x}g(x), \frac{\mathrm{d}^2}{\mathrm{d}x^2}g(x), \cdots, \frac{\mathrm{d}^{n-h+1}}{\mathrm{d}x^{n-h+1}}g(x)\right), \qquad n \geq 1, \tag{A.1}$$

with $\mathbf{B}_{n,h}$ being the family of the Bell polynomials defined as

$$\mathbf{B}_{n,h}(z) = \sum \frac{n!}{j_1! j_2! \cdots j_{n-h+1}!} \left(\frac{z_1}{1!}\right)^{j_1} \left(\frac{z_2}{2!}\right)^{j_2} \cdots \left(\frac{z_{n-h+1}}{(n-h+1)!}\right)^{j_{n-h+1}}, \qquad 1 \leq h \leq n, \tag{A.2}$$

where the sum is taken over all sequences $j_1, j_2, \cdots, j_{n-h+1}$ of nonnegative integers such that

$$j_1 + j_2 + \cdots + j_{n-h+1} = h, \quad \text{and} \quad j_1 + 2j_2 + \cdots + (n-h+1)j_{n-h+1} = n. \tag{A.3}$$

APPENDIX B: IMPLIED VOLATILITY EXPRESSIONS

In this appendix, we assume a time-homogeneous diffusion and use the Taylor series expansion of $\mathcal{A}$ as in Example 2.3 with $(\bar{x}, \bar{y}) = (X_0, Y_0) := (x, y)$. With $\mathcal{A}$ given by (2.4), we introduce the notation

$$\eta_{i,j} = \frac{\partial_x^i \partial_y^j \eta(\bar{x}, \bar{y})}{i!j!}, \eta \in \{a, b, c, f\},$$

and we compute explicitly (below τ is time to maturity)

$$\sigma_0 = \sqrt{2a_{0,0}}, \qquad \sigma_1 = \sigma_{1,0} + \sigma_{0,1}, \qquad \sigma_2 = \sigma_{2,0} + \sigma_{1,1} + \sigma_{0,2},$$

where

$$\sigma_{1,0} = \left(\frac{a_{1,0}}{2\sigma_0}\right)(k-x), \qquad \sigma_{0,1} = \tau\left(\frac{a_{0,1}\left(c_{0,0}+2f_{0,0}\right)}{4\sigma_0}\right) + \left(\frac{a_{0,1}c_{0,0}}{2\sigma_0^3}\right)(k-x),$$

and

$$\begin{aligned}
\sigma_{2,0} &= \tau\left(\frac{1}{12}\sigma_0 a_{2,0} - \frac{a_{1,0}^2}{8\sigma_0}\right) + \tau^2\left(-\frac{1}{96}\sigma_0 a_{1,0}^2\right) + \left(\frac{2\sigma_0^2 a_{2,0} - 3a_{1,0}^2}{12\sigma_0^3}\right)(k-x)^2,\\
\sigma_{1,1} &= \frac{\tau}{12\sigma_0^3}\left(2\sigma_0^2 a_{1,1}c_{0,0} + a_{0,1}\left(a_{1,0}c_{0,0} - 2\sigma_0^2 c_{1,0}\right)\right) + \frac{\tau^2}{48\sigma_0}\left(-a_{0,1}a_{1,0}c_{0,0}\right)\\
&\quad + \frac{\tau}{24\sigma_0^3}\left(4\sigma_0^2 a_{1,1}\left(c_{0,0}+2f_{0,0}\right) + a_{0,1}\left(2\sigma_0^2\left(c_{1,0}+2f_{1,0}\right) - 5a_{1,0}\left(c_{0,0}+2f_{0,0}\right)\right)\right)(k-x)\\
&\quad + \frac{1}{6\sigma_0^5}\left(2\sigma_0^2 a_{1,1}c_{0,0} + a_{0,1}\left(\sigma_0^2 c_{1,0} - 5a_{1,0}c_{0,0}\right)\right)(k-x)^2,\\
\sigma_{0,2} &= \frac{\tau}{24\sigma_0^5}\left(4\sigma_0^2 a_{0,2}\left(3\sigma_0^2 b_{0,0} - c_{0,0}^2\right) + a_{0,1}\left(a_{0,1}\left(9c_{0,0}^2 - 8\sigma_0^2 b_{0,0}\right) - 4\sigma_0^2 c_{0,0}c_{0,1}\right)\right)\\
&\quad + \frac{\tau^2}{24\sigma_0^3}\Big(a_{0,1}\left(-2\sigma_0^2 a_{0,1}b_{0,0} + c_{0,0}\left(\sigma_0^2\left(c_{0,1}+2f_{0,1}\right) - 3a_{0,1}f_{0,0}\right)\right)\\
&\quad + a_{0,1}f_{0,0}\left(2\sigma_0^2\left(c_{0,1}+2f_{0,1}\right) - 3a_{0,1}f_{0,0}\right) + \sigma_0^2 a_{0,2}\left(c_{0,0}+2f_{0,0}\right)^2\Big)\\
&\quad + \frac{\tau}{24\sigma_0^5}\Big(a_{0,1}\left(c_{0,0}\left(4\sigma_0^2\left(c_{0,1}+f_{0,1}\right) - 18a_{0,1}f_{0,0}\right) - 9a_{0,1}c_{0,0}^2 + 4\sigma_0^2 c_{0,1}f_{0,0}\right)\\
&\quad + 4\sigma_0^2 a_{0,2}c_{0,0}\left(c_{0,0}+2f_{0,0}\right)\Big)(k-x)\\
&\quad + \frac{1}{12\sigma_0^7}\left(a_{0,1}\left(a_{0,1}\left(4\sigma_0^2 b_{0,0} - 9c_{0,0}^2\right) + 2\sigma_0^2 c_{0,0}c_{0,1}\right) + 2\sigma_0^2 a_{0,2}c_{0,0}^2\right)(k-x)^2.
\end{aligned}$$

Higher order terms are too long to be reported. However, σ_3 and (for LV models) σ_4 can be computed easily using the Mathematica code provided on the authors' web site: http://explicitsolutions.wordpress.com

REFERENCES

ALEXANDER, C., and L. NOGUEIRA (2004): Stochastic Local Volatility, *Proceedings of the Second IASTED International Conference on Financial Engineering and Applications*, Cambridge MA, USA, pp. 136–141.

ANDERSEN, L. B. G., and V. V. PITERBARG (2007): Moment Explosions in Stochastic Volatility models, *Finance Stoch.* 11(1), 29–50.

ANTONOV, A., and M. SPECTOR (2012): Advanced Analytics for the Sabr Model. SSRN.

BALDEAUX, J., and A. BADRAN (2012): Consistent Modeling of Vix and Equity Derivatives Using a 3/2 Plus Jumps Model, arXiv preprint arXiv:1203.5903.

BENHAMOU, E., E. GOBET, and M. MIRI (2010): Time Dependent Heston Model, *SIAM J. Financ. Math.* 1(1), 289–325.

BENHAMOU, E., E. GOBET, and M. MIRI (2012): Analytical Formulas for a Local Volatility Model with Stochastic Rates, *Quant. Finance* 12(2), 185–198.

BERESTYCKI, H., J. BUSCA, and I. FLORENT (2002): Asymptotics and Calibration of Local Volatility Models, *Quant. Finance* 2(1), 61–69.

BERESTYCKI, H., J. BUSCA, and I. FLORENT (2004): Computing the Implied Volatility in Stochastic Volatility Models, *Commun. Pure Appl. Math.* 57(10), 1352–1373.

BOMPIS, R., and E. GOBET (2012): Asymptotic and Non Asymptotic Approximations for Option Valuation, in *Recent Developments in Computational Finance: Foundations, Algorithms and Applications*, T. Gerstner and P. Kloeden, eds., Singapore: World Scientific Publishing Company, pp. 1–80.

CLARK, I. (2010): *Foreign Exchange Option Pricing: A Practitioner's Guide*, Chichester: Wiley.

COX, J. (1975): Notes on Option Pricing I: Constant Elasticity of Diffusions, Unpublished draft, Stanford University. A revised version of the paper was published by the *Journal of Portfolio Management* in 1996.

DRIMUS, G. G. (2012): Options on Realized Variance by Transform Methods: A Non-Affine Stochastic Volatility Model, *Quant. Finance* 12(11), 1679–1694.

DUFFIE, D., J. PAN, and K. SINGLETON (2000): Transform Analysis and Asset Pricing for Affine Jump-Diffusions, *Econometrica* 68(6), 1343–1376.

EWALD, C., -O., (2005): Local Volatility in the Heston Model: A Malliavin Calculus Approach, *J. Appl. Math. Stoch. Anal.* 2005(3), 307–322.

FORDE, M., and A. JACQUIER (2009): Small-Time Asymptotics for Implied Volatility under the Heston Model, *Int. J. Theor. Appl. Finance* 12(6), 861–876.

FORDE, M., and A. JACQUIER (2011): Small-Time Asymptotics for an Uncorrelated Local-Stochastic Volatility Model, *Appl. Math. Finance* 18(6), 517–535.

FORDE, M., A. JACQUIER, and R. LEE (2012): The Small-Time Smile and Term Structure of Implied Volatility under the Heston Model, *SIAM J. Financ. Math.* 3(1), 690–708.

FOUQUE, J.-P., M. LORIG, and R. SIRCAR (2012): Second Order Multiscale Stochastic Volatility Asymptotics: Stochastic Terminal Layer Analysis and Calibration, arXiv preprint arXiv:1209.0697.

FOUQUE, J.-P., G. PAPANICOLAOU, R. SIRCAR, and K. SOLNA (2011): *Multiscale Stochastic Volatility for Equity, Interest Rate, and Credit Derivatives*, Cambridge: Cambridge University Press.

FRIEDMAN, A. (1964): *Partial Differential Equations of Parabolic Type*, Englewood Cliffs, NJ: Prentice-Hall Inc.

FRIZ, P. K., S. GERHOLD, and M. YOR (2013): How to Make Dupire's Local Volatility Work with Jumps, arXiv preprint1302.5548.

GAO, K., and R. LEE (2014): Asymptotics of Implied Volatility to Arbitrary Order, *Finance Stoch.* 18(2), 349–392.

GATHERAL, J., E. P. HSU, P. LAURENCE, C. OUYANG, and T.-H. WANG (2012): Asymptotics of Implied Volatility in Local Volatility Models, *Math. Finance* 22(4), 591–620.

HAGAN, P., D. KUMAR, A. LESNIEWSKI, and D. WOODWARD (2002): Managing Smile Risk, *Wilmott Mag.* 1000, 84–108.

HAGAN, P., and D. WOODWARD (1999): Equivalent Black Volatilities, *Appl. Math. Finance* 6(3), 147–157.

HENRY-LABORDÈRE, P. (2005): A General Asymptotic Implied Volatility for Stochastic Volatility Models, eprint arXiv:cond-mat/0504317.

HENRY-LABORDÈRE, P. (2009a): *Analysis, Geometry, and Modeling in Finance: Advanced Methods in Option Pricing*, Volume 13, Boca Raton, FL: Chapman & Hall.

HENRY-LABORDÈRE, P. (2009b): Calibration of Local Stochastic Volatility Models to Market Smiles: A Monte-Carlo Approach, *RISK, September,* 112–117.

HESTON, S. (1993): A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Rev. Financ. Stud.* 6(2), 327–343.

IKEDA, N., and S. WATANABE (1989): *Stochastic Differential Equations and Diffusion Processes*, 2nd ed., Volume 24 of North-Holland Mathematical Library, Amsterdam: North-Holland Publishing Co.

JACQUIER, A., and M. LORIG (2013): The Smile of Certain Lévy-Type Models, *SIAM J. Financ. Math.* 4(1), 804–830.

JACQUIER, A., and M. LORIG (2015): From Characteristic Functions to Implied Volatility Expansions, *Adv. Appl. Probab.* 47(3), 837–857.

JOHNSON, W. P. (2002): The Curious History of Faà di Bruno's Formula, *Amer. Math. Monthly* 109(3), 217–234.

LEE, R. W. (2004): The Moment Formula for Implied Volatility at Extreme Strikes, *Math. Finance* 14(3), 469–480.

LEWIS, A. (2007): Geometries and Smile Asymptotics for a Class of Stochastic Volatility Models. Available from author's personal webpage at: http://www.optioncity.net/pubs/UCSB2007Talk.pdf. Accessed date August 20, 2015.

LIPTON, A. (2002): The Vol Smile Problem, *Risk* 15(2), 61–66.

LORIG, M. (2013): The Exact Smile of Certain Local Volatility Models, *Quant. Finance* 13(6), 897–905.

LORIG, M., S. PAGLIARANI, and A. PASCUCCI (2015a): Analytical Expansions for Parabolic Equations, *SIAM J. Appl. Math.* 75, 468–491.

LORIG, M., S. PAGLIARANI, and A. PASCUCCI (2015b): A Family of Density Expansions for Lévy-Type Processes with Default, *Ann. Appl. Probab.* 25(1), 235–267.

OBLOJ, J. (2008): Fine-Tune Your Smile: Correction to Hagan et al., *Wilmott Mag*. 35, 102–104

PAGLIARANI, S., and A. PASCUCCI (2012): Analytical Approximation of the Transition Density in a Local Volatility Model, *Central Eur. J. Math.* 10(1), 250–270.

PAGLIARANI, S., and A. PASCUCCI (2014): Asymptotic Expansions for Degenerate Parabolic Equations, *Comptes Rendus Math.* 352(12), 1011–1016.

PAGLIARANI, S., A. PASCUCCI, and C. RIGA (2013): Adjoint Expansions in Local Lévy Models, *SIAM J. Financ. Math.* 4, 265–296.

PASCUCCI, A. (2011): *PDE and Martingale Methods in Option Pricing*, Volume 2 of Bocconi & Springer Series, Milan: Springer.

RIORDAN, J. (1946): Derivatives of Composite Functions, *Bull. Amer. Math. Soc.* 52, 664–667.

ROPER, M., and M. RUTKOWSKI (2009): On the Relationship between the Call Price Surface and the Implied Volatility Surface Close to Expiry, *Int. J. Theor. Appl. Finance* 12(4), 427–441.

WATANABE, S. (1987): Analysis of Wiener Functionals (Malliavin Calculus) and Its Applications to Heat Kernels, *Ann. Probab.* 15(1), 1–39.