Journal of Financial Econometrics, 2019, Vol. 17, No. 2, 284–337
doi: 10.1093/jjfinec/nby031
Advance Access Publication Date: 30 November 2018
Article

FARVaR: Functional Autoregressive Value-at-Risk*

Charlie X. Cai[1], Minjoo Kim [1], Yongcheol Shin[2] and Qi Zhang[3]

[1]The University of Liverpool Management School, [2]University of York and [3]Durham University Business School

Address correspondence to Minjoo Kim, Department of Finance and Accounting, The University of Liverpool Management School, Liverpool L69 3BX, UK, or e-mail: Minjoo.Kim@liverpool.ac.uk

Received May 23, 2016; revised October 15, 2018; editorial decision October 18, 2018; accepted October 19, 2018

Abstract

Motivated by the stylized fact that intraday returns can provide additional information on the tail behavior of daily returns, we propose a functional autoregressive value-at-risk (VaR) approach which can directly incorporate such informational advantage into the daily VaR forecast. Our approach leads to greater flexibility in modeling the dynamic evolution of the density function of intraday returns and the ability to capture substantial swings in the tails following major events. We comprehensively evaluate our proposed model using intraday transaction data and demonstrate that it can improve coverage ability, reduce economic cost, and enhance statistical reliability in market risk management.

Key words: density forecasts, functional autoregressive model covariance, market risk management

JEL classification: C53, G17

* The authors thank Allan Timmermann (the editor), the associate editor, and two anonymous referees, as well as Heather Anderson, Amir Armanious, Kausik Chaudhuri, Jin-Chuan Duan, Robert Faff, Ana-Maria Fuertes, Matthew Greenwood-Nimmo, Bonsoo Koo, Shuddhasattwa Rafiq, Joon Park, Vasilis Sarafidis, Peter Spencer, Param Sylvapulle; seminar participants at the Universities of Brisbane, Bristol, Deakin, Durham, Essex, Glasgow, Leeds, Macquarie, Monash, and York, and the National University of Singapore; and conference delegates at the 2nd International Conference of the Financial Engineering and Banking Society in London, June 2012, the Econometric Society Australasian Meeting (ESAM) conference in Hobart, July 2014, the 2nd Vienna Workshop on High-Dimensional Time Series in Macroeconomics and Finance, May 2015, and the 8th Annual Society for Financial Econometrics in Aarhus, June 2015. M.K. acknowledges that this paper is an extension of the second chapter of his PhD thesis, submitted to the University of Leeds in 2011. The usual disclaimer applies. The data are posted on the journal's web site and MATLAB codes are available from Kim's personal homepage https://sites.google.com/site/drminjookim/.

© The Author(s) 2018. Published by Oxford University Press. All rights reserved.
For permissions, please email: journals.permissions@oup.com

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

In the aftermath of the 2008 global financial crisis, the effective role of value-at-risk (VaR) in risk management has been subjected to intense debate.[1] Although the crisis revealed that the main weakness of VaR lies in ignoring the liquidity risk and underestimating the correlation risk, the general consensus is that VaR is still one of the most important risk management tools available (e.g., Croft, 2011). The search for more accurate VaR modeling will therefore continue (e.g., Adams, Füss, and Gropp, 2014).

Recently, we have been confronted with new challenges. The ever-increasing prominence of computer-based trading in the financial market makes it more important to carefully examine risks arising from the automated intraday activities typically related to high-frequency trading (HFT). While the market microstructure literature is still debating whether and how HFT will influence market quality, intraday dynamics are expected to become an increasingly important input to price discovery.[2] Even for daily risk management, we expect to develop a more accurate VaR model by capturing information generated from the higher frequency data. The usefulness of intraday information has been demonstrated extensively in the realized volatility literature (Andersen et al., 2001; Engle and Giampiero, 2006), though the direct incorporation of intraday return information in risk management modeling is still at an early stage (e.g., Fuertes and Olmo, 2013; Hallam and Olmo, 2014).

It is well-established that intraday returns follow a non-normal and time-varying distribution (e.g., Andersen et al., 2001). Moreover, it would be extremely challenging to model such complex intraday return dynamics, as their random character may be further complicated by market microstructure noise. To address this challenge, we propose a novel semi-parametric approach, called functional autoregressive value-at-risk (FARVaR), for forecasting the daily VaR using intraday information.[3] First, we estimate the density of intraday returns nonparametrically for each day and apply the FAR model[4] to the sequence of intraday densities constructed over T days to obtain h-day-ahead forecasts of intraday density. In general, it would be non-trivial to generate the density function of daily returns directly from that of intraday returns, even if the density function of intraday returns were known or given. To address this challenging issue, we suggest two approaches. The first is parametric and relies on the use of the normal inverse Gaussian (NIG) distribution[5] for the

1 New York Times reporter J. Nocera wrote an extensive piece, “Risk Mismanagement,” on January 2, 2009, discussing the role VaR played in the financial crisis of 2007–2008. Having interviewed risk managers, she suggests that VaR was very useful to risk experts but nevertheless exacerbated the crisis by giving false security to bank regulators. In 2009, Professor N. N. Taleb testified in Congress asking for the banning of VaR because “tail risks are non-measurable.”

2 Brogaard, Hendershott, and Riordan (2014) show that HFT facilitates price efficiency by trading in the direction of permanent price changes and in the opposite direction to transitory pricing errors, suggesting that HFT is beneficial to price discovery and market quality; see also Hasbrouck and Saarb (2013). On the other hand, Zhang (2010) argues that HFT is harmful to price discovery, as HFT is positively correlated with stock price volatility, and this correlation is stronger during periods of high market uncertainty.

3 Andersen et al. (2003) document that daily realized volatility estimates based on intraday returns provide volatility forecasts that are superior to forecasts constructed parametrically from daily returns only.

4 Bosq (2000) introduces the statistical foundation of FAR modeling. See also Park and Qian (2012).

5 The NIG distribution is one of the most popular parametric distributions for describing the return distribution, and enables matching the first four moments of the observed data (Barndorff-Nielsen, 1997).

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

returns. The second is based on a nonparametric sampling scheme, which can produce a bootstrap density function of the daily return. Since we are more interested in modeling the VaR of portfolios containing multiple assets in many finance applications, we extend FARVaR for a single asset into one for multiple assets by incorporating copula (mFARVaR).

FARVaR provides a few advances over the existing literature. First, it avoids any (distributional) uncertainty associated with misspecified parametric models, by estimating the intraday density nonparametrically. Furthermore, FAR can easily overcome the shortcomings of nonparametric models by capturing such complex dynamic structure via a FAR operator, which can represent all contemporaneous and time-dependent associations among all the moments or quantiles. Second, high-frequency financial data are often characterized as extremely dispersed and non-normally distributed (Hasbrouck, 2007). Through a detailed exploratory analysis of the time-varying intraday moments, we establish two stylized facts: (i) volatility and skewness of intraday returns are rather persistent; (ii) there exist complex (potentially nonlinear) and time-varying associations among the moments. In this regard, we expect that neither the parametric nor the nonparametric approach will have the capacity to unravel an exact relationship among intraday moments or quantiles. By contrast, FARVaR is designed to utilize evolutions of intraday return density in directly forecasting daily return density and daily VaR in a flexible manner. Third, while parametric models can reduce the economic cost but tend to underestimate VaR (Netftci, 2000), nonparametric models can provide conservative coverage ability but fail to reduce the economic cost. This fundamental trade-off may reflect their respective forecasting inaccuracies. As confirmed by our empirical evaluation, FARVaR can simultaneously improve the coverage ability and reduce the economic cost. Last, the VaR forecast of FARVaR is less sensitive to the undelying market regime than that of existing semiparametric VaR models. This makes FARVaR superior to other models in terms of coverage ability.

We conduct various evaluation schemes using real data from 30 stocks listed in the Dow Jones Industrial Average (DJIA) index over the period 2000–2008. All the evaluations are based on out-of-sample forecasting. First, we evaluate the performance of intraday return density forecasting for different functional models. We find that FAR is the best predictor for the density of intraday returns as it produces the smallest divergence among the functional models, including a functional martingale process and a functional i.i.d. process.

Next, we employ a wide range of backtesting tools to evaluate and compare the performance of FARVaR against the existing VaR models. A backtest is not only a formal framework for verifying whether the actual loss is in line with the projected loss, but also an essential procedure for selecting suitable internal VaR models for capital requirements as recommended by the Basel Committee on Banking Supervision (BCBS). We conduct three broad types of test. We examine the coverage ability and the economic cost using conventional quantitative measures such as the empirical coverage probability (ECP) and the predictive quantile loss (PQL) (Koenker and Bassett, 1978). We also assess the Basel penalty zone (BPZ) (Basel Committee on Banking and Supervision, 1996) and the market risk capital requirement (MRCR) (Basel Committee on Banking and Supervision, 1996, 2005). In addition, we evaluate the statistical adequacy of the VaR models using the conditional coverage (CC) test (Christoffersen, 1998) and the dynamic quantile (DQ) test (Engle and Manganelli, 2004).

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

These backtesting results demonstrate that FARVaR is the most reliable VaR model. The Basel II Accord was designed to encourage sensible risk-taking using appropriate (internal) models of risk to forecast daily VaR and capital charges. Within the Basel II rules banks may prefer to report high VaRs to avoid the possibility of regulatory intrusion.[6] This conservative risk reporting suggests that efficiency gains may be feasible (McAleer, Jimïœnez-Martïœn, and Pïœrez-Amaral, 2013). We demonstrate that FARVaR can achieve such a non-trivial goal by simultaneously improving coverage ability, reducing economic cost, and enhancing statistical reliability.

1 FARVaR

The aim of our study is to develop a novel risk-management model which forecasts a daily VaR using intraday information. In this context, an important issue is how best to develop an appropriate econometric methodology for estimating and forecasting the density of daily returns using intraday returns. The most popular approach is to construct realized volatilities from intraday returns and forecast a daily volatility by applying a parametric model to the realized volatilities.[7] However, this approach uses only the realized volatilities, so it is not sufficiently broad to generate the entire density of daily returns. In practice, this approach is bound to suffer from potentially misspecified parametric assumptions imposed on the dynamics of realized volatilities and the distribution of daily returns.

We therefore develop the semiparametric FARVaR, which generates the density forecast of daily returns directly from the density forecast of intraday returns. It is a two-step procedure, as illustrated in Figure 1. First, we estimate the density of intraday returns nonparametrically by a kernel density estimator and forecast an intraday density by means of FAR. Second, we propose two approaches to constructing the density forecast of daily returns directly from the density forecast of intraday returns: the parametric approximation based on the general class of the NIG distribution, and a nonparametric bootstrap scheme. Given the density forecast of daily returns, we can then easily compute a daily VaR.

1.1 Forecasting Density Function of Intraday Returns by FAR

Bosq (2000) introduces the statistical foundation of FAR modeling. Its asymptotic theory is refined by Cardot, Mas, and Sarda (2007) and Mas (2007). Park and Qian (2012) develop the functional regression of a continuous state distribution which is more general than FAR. FAR has been applied to forecasting a climate pattern with regard to temperature (Besse, Cardot, and Stephenson, 2000) and ozone (Damon and Guillas, 2002; Aneiros-Perez et al., 2004). Recently, the FAR approach has been adopted in economics and finance. Laukaitis (2008) applies it to forecasting an intraday cash flow and a transaction intensity in the credit card payment system. Bowsher and Meeks (2008) and Kargin and Onatski (2008) employ it to forecast a yield curve as a function of maturity. Chaudhuri, Kim, and Shin (2016) apply it to modeling the cross-sectional distribution of sectoral

6 Although market risk management is in transition from VaR to ES under the Basel III Accord, the Basel II rules are still crucial requirements for the robust ES model.

7 Giot and Laurent (2004) use an ARCH-type model; Clements, Galvïœo, and Kim (2008) consider a mixed data sampling and a heterogeneous autoregressive model; and Andersen et al. (2003) employ a vector autoregressive model.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

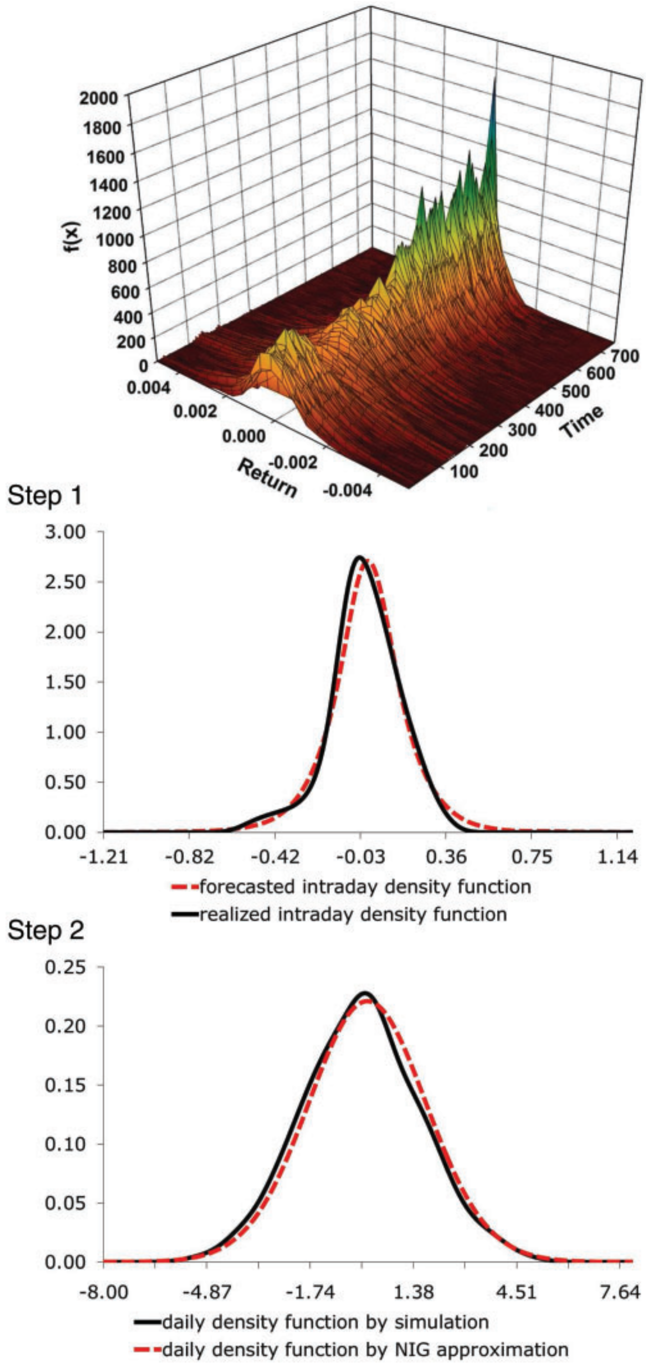


Figure 1 FARVaR algorithm.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Figure 1 Continued

Notes: The first figure presents the time-series plot of the density estimates of intraday returns over time. Time denotes days and Return denotes the intraday returns. We estimate the density of intraday returns by Equations (A.2) and (A.3) and plot Equation (A.5): $(\widehat{f}_1, \widehat{f}_2, \dots, \widehat{f}_T)$.

Step **1**: This figure presents the 1-day-ahead density forecast of intraday returns and the realized density estimate of intraday returns at time $T+1$. We model the densities of intraday returns by FAR and forecast the density by Equation (10). The realized density of intraday returns is estimated by Equations (A.2) and (A.3).

Step **2**: This figure presents the density of daily returns generated from the density forecast of intraday returns in Step 1. There are two densities of daily returns: one generated by the NIG approximation using the first four moments from the density forecast of intraday returns described in Section 1.2.1; the other by simulating intraday returns randomly drawn from the density forecast of intraday returns decribed in Section 1.2.2. This figure illustrates the two-step algorithm of FARVaR described in Section 1 and Appendix Section A.1. We use the intraday returns on the equal-weighted portfolio of 30 stocks to estimate the density of intraday returns.

inflation rates nonparametrically, and develop the flexible framework for forecasting the density of national inflation rates based on FAR.

The foundation of forecasting an intraday density function relies heavily on Park and Qian (2012). Thus, we refer readers to the asymptotic theory and its proofs in that paper. In the current paper, we provide detailed descriptions in Appendix Section A.1 for the estimation of FAR and the forecasting of the intraday density function, to facilitate replication.

Using the series of asset prices observed at a fixed time interval (e.g., 5 min),[8] we calculate an intraday return by the first difference of logged prices, $r_{t,i} = \ln P_{t,i} - \ln P_{t,i-1}$ for $t = 1, \dots, T$ and $i = 1, \dots, m$. The index t denotes the day t and the index i denotes the ith observation over the fixed interval after a market opens on day t. Thus, $P_{t,0}$ denotes the opening price and $P_{t,m}$ the closing price on day t.[9] For example, given a 5-min interval, $r_{2,1}$ and $r_{2,2}$ denote the realized stock returns at 9:35 A.M. and 9:40 A.M. on the second day (see Endnote 8). A daily return is then calculated as $r_t = \ln P_{t,m} - \ln P_{t,0}$. As a result, we have the following sequences of time series:

$$X = (X_t)_{t=1}^{T} \text{ and } X_t = (r_{t,i})_{i=1}^{m}, \tag{1}$$

where X_t is an intraday return path on day t and X is the set of intraday return paths over T days.

First, we make an assumption on intraday returns that is crucial for applying FAR to their density function.

8 In practice, a trading action and its frequency are irregular in time, so we filter the trading data and extract prices at fixed time intervals (e.g., 9:35 A.M., 9:40 A.M., 9:45 A.M., ... , 4:00 P.M.). In the empirical application below, we follow Andersen et al. (2001), and collect intraday returns over 5-min intervals, as this is short enough to secure the accuracy of continuous record asymptotic, but long enough to weaken cumulative noises from microstructure frictions.

9 If we include an overnight jump in price, $P_{t,0}$ should be the closing price from day $t-1$. This suggests that the overnight return, $r_{t,1}$, may have a different data-generating process from other intraday returns (e.g., Taylor, 2007; Ahoniemi, Fuertes, and Olmo, 2016). The modeling of the overnight jump is beyond the scope of the current study. Notice, however, that we have obtained qualitatively similar evaluation results both with and without including the overnight returns.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Assumption 1 (Piecewise Stationarity). *Intraday returns are strictly stationary in short time intervals, so that the density functions of intraday returns can be consistently estimated for each day. However, this local stationarity does not carry over to longer horizons.*

Let the density function of intraday returns on day t be f_t, such that the sequence of intraday densities (f_t) can be well-defined in $\mathcal{H} = L^2(\mathcal{C})$, the Hilbert spaces of square integrable functions defined on the compact subset $\mathcal{C}$ of $\mathbb{R}$. Thus, we can treat f_t as a functional-valued random variable. Under Assumption 1, it would be very reasonable to model the time-varying density function of intraday returns by the FAR of order 1:

$$w_t = Aw_{t-1} + \epsilon_t,\ t = 1, \dots, T, \tag{2}$$

where $w_t = f_t - \mathbb{E}[f]$ is the fluctuation of the density function from the well-defined common expectation of the density function, $\mathbb{E}[f]$,[10] and A is the so-called autoregressive operator.

Assumption 2 *We assume that:*

(a) A is a compact linear operator in $\mathcal{H}$, satisfying $||A^{\kappa}|| < 1$ for some $\kappa > 1$[11];

(b) (ϵ_t) is a sequence of the functional white noise such that $\mathbb{E}[\epsilon_t] = 0$ and $\mathbb{E}||\epsilon_t||^4 < \infty$, and is independent of w_0. Further, its distribution is a probability measure on $\mathcal{H}$ such that $\epsilon_t : (\Omega, \mathcal{F}, \mathbb{P}) \mapsto \mathcal{H}$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying probability space.

The one-step-ahead forecast of a density function can be evaluated by the conditional expectation of the density function on a past information set $(\mathcal{F}_{t-1})$:

$$\mathbb{E}[f_t|\mathcal{F}_{t-1}] = \mathbb{E}[f] + Aw_{t-1}. \tag{3}$$

If A is the zero operator, the best predictor would be the unconditional expectation of density function (AVE). On the other hand, if A is the identity operator, the best predictor would be the last observation (LAST).

Since (f_t) cannot be directly observed in practice, we estimate the density function of intraday returns at each point in time, t. We estimate it nonparametrically using the kernel density estimator[12]:

$$\widehat{f}_t(x_j) = \frac{1}{nh_t} \sum_{i=1}^{m} K\left(\frac{x_j - r_{t,i}}{h_t}\right),\ t = 1, \dots, T;\ j = 1, \dots, n, \tag{4}$$

where K is a kernel, m is the number of intraday returns, h_t a bandwidth, and n the number of discrete grids.[13] One important issue is the selection of kernel and bandwidth value. The optimal value of bandwidth is derived by minimizing a loss function and applying a cross-

10 The expectation denotes the point-wise functional mean of a density function throughout the paper. See also Equation (A.6) in Appendix Section A.1.

11 The role of A is to transform one function into another in the infinite dimensional space.

12 Under the piecewise stationarity of intraday returns, we still allow them to be autocorrelated or dependent. In such a situation, unless we are certain about a true parametric distribution, the nonparametric approach is more reliable in constructing an intraday density (see Hall, Lahiri, and Truong, 1995; Wu, 1997).

13 The grid set covers the range of sample values for $x_1 < \cdots < x_n$. For simplicity, we use an equal interval, $\delta = x_j - x_{j-1}$, for all $j = 1, \dots, n$.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

validation (CV) selector. We employ the popular Gaussian kernel and follow Silverman's (1986) rule of thumb such that the optimal bandwidth is given by $1.06\widehat{\sigma}_t m^{-1/5}$, with $\widehat{\sigma}_t$ being the sample standard deviation of $r_{t,i}$. See also Härdle and Linton (1994) for more detail on the choice of kernel and bandwidth value.

The autoregressive operator in Equation (2) is defined as $A = C_1 C_0^{-1}$, where $C_0 = \mathbb{E}[w_t \otimes w_t]$ and $C_1 = \mathbb{E}[w_t \otimes w_{t-1}]$ are the autocovariance operators of orders 0 and 1, respectively. However, since the autocovariance operators are defined in an infinite dimension, the inverse of C_0 is not well-defined; that is, it presents an ill-posed inverse problem. To avoid this, we project C_0 onto a finite L-dimensional subspace in $\mathcal{H}$. Let us denote the inverse of C_0 on the L-dimensional subspace by $C_{0,L}^{+}$. The autoregressive operator restricted to the L-dimensional subspace is $A_L = C_1 C_{0,L}^{+}$. The choice of L is guided by a functional principal component analysis (FPCA) and a CV (see Ramsey and Silverman, 1997). Then the estimator of A in the L-dimensional subspace is given by $\widehat{A}_L = \widehat{C}_1 \widehat{C}_{0,L}^{+}$. Therefore, the one-step-ahead forecast conditional on the information set ($\mathcal{F}_T$) is evaluated by

$$\widehat{f}_{T+1} = \bar{f} + \widehat{A}_L \widehat{w}_T, \tag{5}$$

where $\bar{f} = T^{-1}\sum_{t=1}^{T} \widehat{f}_t$ and $\widehat{w}_T = \widehat{f}_T - \bar{f}$. See Theorem 5 of Park and Qian (2012) for the consistency of $\widehat{A}_L$ and our Appendix Section A.1 for the estimation of FAR and the forecasting of the intraday density in detail.

In general, the FAR algorithm involves non-negligible computation time. Hence, we introduce two popular transforms of $\widehat{w}_t$, by the fast Fourier transform and the wavelet transform. Both transforms can reduce the computation time dramatically (making it 30 times faster) by shrinking the dimension of the function. Once we obtain the forecast of the transformed $\widehat{w}_{T+1}$, we then recover the original forecast by the inverse Fourier or wavelet transform, and evaluate the density forecast in Equation (5) (see Besse, Cardot, and Stephenson, 2000; Antoniadis and Sapatinas, 2003, for details). We call these computationally efficient transforms, respectively, FAR-fft and FAR-wv.

1.2 Forecasting Density Function of Daily Returns and VaR from Density Forecasts of Intraday Returns

We aim to explicitly employ information on intraday returns to improve the forecasting precision of the tail behavior of daily returns. This requires us to generate the density forecast of the daily returns directly from the density forecast of intraday returns. Let the probability and cumulative distribution functions (CDFs) of the daily returns on day t be g_t and G_t, respectively. In general, this task would be nontrivial even if we successfully obtain the h-day-ahead density forecast of the intraday returns, f_{t+h}, via FAR, as in the previous section. To address this challenging issue, we therefore suggest two approaches. We first consider a parametric approximation on the basis of the NIG distribution, regarded as one of the most flexible distributions for describing the distribution of returns (Barndorff-Nielsen, 1997). We then propose a more general nonparametric bootstrap approximation.

1.2.1 NIG approximation approach (FARVaR-nig)

This approach employs the NIG parametric approximation and constructs the density of daily returns through a flexible match of the first four moments from the density of intraday returns. Let μ_t, v_t, s_t, and k_t be, respectively, the mean, variance, skewness, and kurtosis

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

of the intraday returns on day t. Then, we calculate the four parameters $(\alpha_t, \beta_t, \gamma_t, \delta_t)$ that determine the shape of the NIG distribution as follows:

$$\alpha_t = v_t^{-\frac{1}{2}}(3k_t - 4s_t^2 - 9)^{\frac{1}{2}}\left(k_t - \tfrac{5}{3}s_t^2 - 3\right)^{-1}, \beta_t = s_t v_t^{-\frac{1}{2}}\left(k_t - \tfrac{5}{3}s_t^2 - 3\right)^{-1},$$
$$\gamma_t = \mu_t - 3s_t v_t^{\frac{1}{2}}(3k_t - 4s_t^2 - 9)^{-1}, \delta_t = 3^{\frac{3}{2}}\left\{v_t\left(k_t - \tfrac{5}{3}s_t^2 - 3\right)\right\}^{\frac{1}{2}}(3k_t - 4s_t^2 - 9)^{-1}, \tag{6}$$

where α_t determines the tail heaviness, β_t the asymmetry, γ_t the location, and δ_t the scale of the distribution. Note that the kurtosis should satisfy $k_t > 3 + (5/3)s_t^2$. The density of the intraday returns is then approximated by the NIG density:

$$f_t(x) = \left[\frac{\alpha_t \delta_t J_1\left(\alpha_t\sqrt{\delta_t^2 + (x-\gamma_t)^2}\right)}{\pi\sqrt{\delta_t^2 + (x-\gamma_t)^2}}\right] e^{\delta_t \lambda_t + \beta_t(x-\gamma_t)}, \tag{7}$$

where $\lambda_t = \sqrt{\alpha_t^2 - \beta_t^2}$ and J_1 denote the modified Bessel function of the second kind. The NIG distribution is closed under the convolution of independent random variables X_1 and X_2:

$$X_1 \sim \text{NIG}(\alpha, \beta, \gamma_1, \delta_1) \text{ and } X_2 \sim \text{NIG}(\alpha, \beta, \gamma_2, \delta_2)$$
$$\Rightarrow X_1 + X_2 \sim \text{NIG}(\alpha, \beta, \gamma_1 + \gamma_2, \delta_1 + \delta_2). \tag{8}$$

Hence, the density function of the daily returns on day t $\left(r_t = \sum_{i=1}^{m} r_{t,i}\right)$, denoted $g_t(x)$, can be approximated by the following NIG density:

$$g_t(x) = \left[\frac{m\alpha_t \delta_t J_1\left(\alpha_t\sqrt{m^2\delta_t^2 + (x-m\gamma_t)^2}\right)}{\pi\sqrt{m^2\delta_t^2 + (x-m\gamma_t)^2}}\right] e^{m\delta_t \lambda_t + \beta_t(x-m\gamma_t)}, \tag{9}$$

where m is the number of intraday observations on day t. Finally, we can evaluate a daily VaR from the cumulative NIG density function of the daily returns. Notice, however, that the analytic form of the cumulative NIG density function does not exist. Hence, we approximate it by numerically evaluating the integral of the NIG density in Equation (9) [see also Equation (A.25) in Appendix Section A.1].

1.2.2 Simulation approach (FARVaR-sim)

The simulation approach is based on repeated random sampling as follows. It is straightforward to construct the CDF of the intraday returns from the density forecast of the intraday returns by

$$\widehat{F}_{T+1}(x) = \int_{-\infty}^{x} \widehat{f}_{T+1}(s)\mathrm{d}s. \tag{10}$$

As $\widehat{f}_{T+1}$ is the discrete approximation of the continuous density function, we also approximate $\widehat{F}_{T+1}$ numerically by the following middle Riemann sum:

$$\widehat{F}_{T+1}(z_j) = \frac{1}{2}\left[\sum_{i=1}^{j}\widehat{f}_{T+1}(x_{i+1})\Delta x_{i+1} + \sum_{i=1}^{j}\widehat{f}_{T+1}(x_i)\Delta x_{i+1}\right], j = 1, \ldots, n-1, \tag{11}$$

where $z_j = (x_{j+1} + x_j)/2$ and $\Delta x_{i+1} = x_{i+1} - x_i$.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Next, given Assumption 1, we can randomly draw intraday returns from the CDF, $\widehat{F}_{T+1}$, in Equation (11), in the following way. First, we draw m real numbers randomly from the uniform distribution over $(0,1)$, and denote them by $\{p_1^{(b)}, p_2^{(b)}, \ldots, p_m^{(b)}\}$, where $p_i^{(b)} \sim U(0,1)$ for each bootstrap iteration, $b = 1, \ldots, B$. $\widehat{F}_{T+1}$ is discretized over the grid set, $\{z_1, z_2, \ldots, z_{n-1}\}$, while $p_i^{(b)}$ is drawn from a continuous domain. In this case, an equality in $\widehat{F}_{T+1}(z_j) = p_i^{(b)}$ does not hold for some i. Then we select a z_j such that $\widehat{F}_{T+1}(z_j)$ is the nearest to $p_i^{(b)}$. We present this grid selection conveniently by

$$z_i^{(b)} \equiv \underset{z_j \in \{z_1, \ldots, z_{n-1}\}}{\operatorname{argmin}} |\widehat{F}_{T+1}(z_j) - p_i^{(b)}|. \tag{12}$$

We collect a set of m simulated intraday returns in $\{z_1^{(b)}, z_2^{(b)}, \ldots, z_m^{(b)}\}$ and then construct a daily return by $r^{(b)} = \sum_{i=1}^{m} z_i^{(b)}$ for each iteration, $b = 1, \ldots, B$. Finally, we can approximate the (empirical) CDF of the daily returns by

$$\widehat{G}_{T+1}(\omega) = \frac{1}{B}\sum_{b=1}^{B} 1\{r^{(b)} \leq \omega\}. \tag{13}$$

See also Figure 2 for a graphical representation of the simulation approach.

It is then straightforward to evaluate a daily VaR forecast given a nominal probability, α by

$$\widehat{\mathrm{VaR}}_{T+1}(\alpha) = \sup\Big(\omega | \widehat{G}_{T+1}(\omega) \leq \alpha\Big). \tag{14}$$

With regard to our proposed FARVaR approach, we note and address the following issues. First, the randomness of density forecasts is generated by two elements, as shown in Equation (5). The first is associated with the estimation of the autoregressive operator. Since a true density function is unobservable, we have to estimate it. Thus, the second is associated with the estimation of the density function. Although it is not straightforward to construct the asymptotic confidence interval of a VaR forecast, we can obtain it by randomly sampling in-sample prediction errors, $\{\widehat{\epsilon}_1, \ldots, \widehat{\epsilon}_T\}$. See Appendix Section A.3 for the bootstrap-based construction of the confidence interval. Figure 3 presents the empirical distributions of VaR forecasts by FARVaR-nig and FARVaR-sim, along with their bootstrapped 95% confidence intervals.

Second, the simulation approach is a more general and robust approach than the NIG approximation, because the latter requires strong distributional assumptions: that (i) an intraday return density is characterized by the first four moments such that it follows the NIG distribution, and (ii) intraday returns are independently and identically distributed. In particular, the validity of the second assumption is questionable since many studies have documented evidence that the intraday returns exhibit dependency or autocorrelation (e.g., Andersen and Bollerslev, 1997). However, in the case where such dependency, say at the 5-min interval, is relatively negligible, we may expect that the NIG approximation works reasonably well.

Third, our proposed FARVaR approach is closely related to the recent study by Hallam and Olmo (2014), who propose a method for estimating the density of daily returns directly from intraday returns by rescaling them through multiplying the scaling factor, under the assumption that the intraday returns are self-affine or unifractal. They estimate a daily density by applying either the location-scale t-distribution or the kernel density estimator to

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

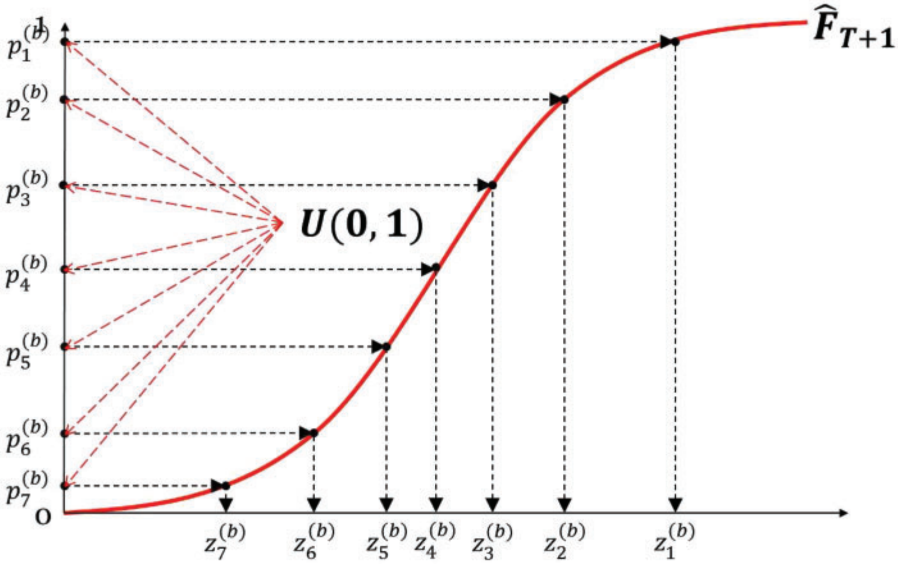


Figure 2 Simulating intraday returns from CDF of intraday returns.

Notes: This figure depicts the idea of simulating intraday returns from the empirical CDF of intraday returns in Section 1.2.2. Given the piecewise stationarity assumption of intraday returns, we randomly draw m real numbers from the uniform distribution over $(0,1)$, and denote them by $\{p_1^{(b)}, p_2^{(b)}, \ldots, p_m^{(b)}\}$, where $p_i^{(b)} \sim U(0,1)$ for each iteration, $b = 1, \ldots, B$. We then select a $z_i^{(b)}$ corresponding to $p_i^{(b)}$, following the algorithm described in Equation (12) and construct a set of m simulated intraday returns, $\{z_1^{(b)}, z_2^{(b)}, \ldots, z_m^{(b)}\}$.

transformed daily returns, and forecast the density of the daily returns by forecasting distributional parameters via a simple autoregressive model or as the weighted sum of past densities. Although their approach is more flexible than the previous studies, it is still restrictive in two ways. First, the higher-order moments such as skewness and kurtosis of both transformed daily and intraday returns are equivalent under the common rescaling. This would violate the market microstructure stylized evidence that intraday returns are more skewed and leptokurtic than daily returns. Second, their proposed density forecasting approach is more restrictive than our proposed FAR modeling in the sense that the dependence structure allowed in their approach is substantially simpler.[14]

2 Multivariate FARVaR

So far we have developed FARVaR for a single asset. However, we are more interested in modeling VaR for a portfolio containing multiple assets in many finance applications, such

14 For example, Chaudhuri, Kim, and Shin (2016) show that the average of past densities is inferior to FAR in forecasting the density of inflation rates.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

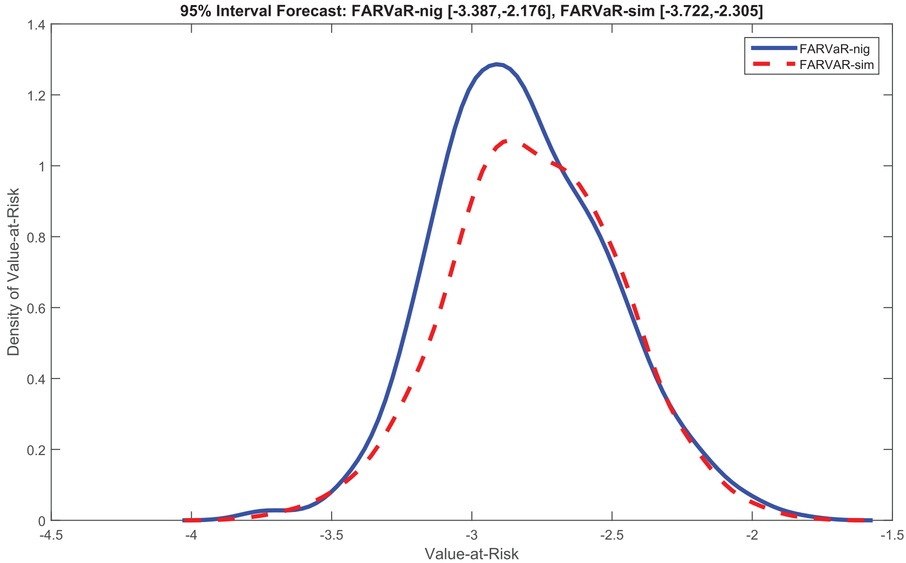


Figure 3 Bootstrap-based confidence interval of VaR forecast.

Notes: This figure presents the empirical distribution of VaR as forecast by FARVaR-nig and FARVaR-sim, along with their 95% confidence intervals. We obtain the empirical distribution by bootstrapping the intraday density from the in-sample prediction errors. Appendix Section A.3 provides the detailed construction of the empirical distribution.

as a portfolio containing basket options like those for the maximum performance of several stocks. In this case, the dependence structure (or correlation) of the multiple assets is of particular importance for computing a portfolio VaR.

However, there are limitations in directly applying FAR to a multivariate density function. Let us take a bivariate density function as an example. For a univariate density function, $f(x) : \mathbb{R} \to \mathbb{R}_+$, the autoregressive operator is defined in the form $A := \{a(i,j)\}$ for $i,j \in \mathbb{R}$. For the bivariate density function, $f(x,y) : \mathbb{R}^2 \to \mathbb{R}_+$, the autoregressive operator is defined, however, in the form $A := \{a(i,j,k)\}$ for $i,j,k \in \mathbb{R}$. Thus, we cannot apply estimations and inferences developed for the univariate density function directly to the multivariate function. For this reason, we develop FARVaR for multiple assets in an indirect way using a copula (hereafter mFARVaR) in this paper.

Let us consider a portfolio containing K assets:

$$r_{p,t,i} = \sum_{k=1}^{K} w_k r_{k,t,i},\ i = 1,\dots,m; t = 1,\dots,T. \tag{15}$$

According to Sklar's theorem, every K marginal distribution function $\{F_{1,t},\dots,F_{K,t}\}$ can be written as

$$F_t(r_{1,t,i},\dots,r_{K,t,i}) = C_t\big(F_{1,t}(r_{1,t,i}),\dots,F_{K,t}(r_{K,t,i})\big),\ t = 1,\dots,T;\ i = 1,\dots,m \tag{16}$$

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

for some copula C_t which is uniquely determined on $[0,1]^K$ for the multivariate distribution function with absolutely continuous margins. Conversely, any copula C_t of their joint distribution function may be extracted from Equation (16) by evaluating

$$C_t(\mathrm{u}_{t,i}) := C_t(u_{1,t,i},\dots,u_{K,t,i}) = F_t\left(F_{1,t}^{-1}(u_{1,t,i}),\dots,F_{K,t}^{-1}(u_{K,t,i})\right), \tag{17}$$

where $F_{k,t}^{-1}$ is the quantile function of the margin. In our exercise, we can compute $\mathrm{u}_{t,i} = (u_{1,t,i},\dots,u_{K,t,i})'$ from the nonparametric density function, f_k, and estimate the copula C_t given the K marginal density functions for each day.

Next, we assume that C_t follows VAR(1), which is reasonably consistent with the FAR of order 1. For simplicity, we assume that C_t has the form of a multivariate Student's $t(\nu_t, 0, \mathrm{P}_t)$-distribution, where P_t is the correlation matrix and ν_t the degree of freedom. The unique copula is given by

$$C_{\nu_t,\mathrm{P}_t}(\mathrm{u}_{t,i}) = \int_{-\infty}^{t_{\nu_t}^{-1}(u_{1,t,i})} \cdots \int_{-\infty}^{t_{\nu_t}^{-1}(u_{K,t,i})} \frac{\Gamma\left(\frac{\nu_t+K}{2}\right)}{\Gamma\left(\frac{\nu_t}{2}\right)\sqrt{(\pi\nu_t)^K|\mathrm{P}_t|}}\left(1+\frac{\mathrm{x}'\mathrm{P}_t^{-1}\mathrm{x}}{\nu_t}\right)^{-\frac{\nu_t+K}{2}} \mathrm{d}x_K\cdots\mathrm{d}x_1, \tag{18}$$

where $t_{\nu_t}^{-1}$ denotes the quantile function of a standard univariate t_{ν_t} distribution. Then we model $\mathrm{vech}(\mathrm{P}_t)$ and ν_t by VAR(1) and AR(1), as

$$\mathrm{vech}(\mathrm{P}_t) = \mathbf{c} + \Gamma\mathrm{vech}(\mathrm{P}_{t-1}) + \mathbf{e}_t, \tag{19}$$

$$\nu_t = \alpha + \varphi\nu_{t-1} + \zeta_t, \tag{20}$$

and forecast both, which are then used to construct a copula forecast, $\widehat{C}_{T+1} \equiv C_{\widehat{\nu}_{T+1},\widehat{\mathrm{P}}_{T+1}}$.

Hence, we can forecast the K marginal intraday density functions by FAR and their copula by VAR(1). Then, we randomly draw $\boldsymbol{u}$ from the copula forecast,

$$\mathrm{u}_{T+1,i}^{(b)} = \left(u_{1,T+1,i}^{(b)},\dots,u_{K,T+1,i}^{(b)}\right)', \tag{21}$$

and convert these into intraday returns using the K marginal density forecasts, $\widehat{f}_{1,T+1},\dots,\widehat{f}_{K,T+1}$:

$$r_{1,T+1,i}^{(b)} = \widehat{F}_{1,T+1}^{-1}\left(u_{1,T+1,i}^{(b)}\right),\dots,r_K^{(b)} = \widehat{F}_{K,T+1}^{-1}\left(u_{K,T+1,i}^{(b)}\right),\ i=1,\dots,m. \tag{22}$$

See Equations (10) and (11) for the numerical computation of the CDF from the density forecast.

We update the intraday returns of the portfolio using the simulated intraday returns,

$$r_{p,T+1,i}^{(b)} = \sum_{k=1}^{K} w_k r_{k,T+1,i}^{(b)},\ i=1,\dots,m, \tag{23}$$

and generate daily portfolio returns by summing the intraday portfolio returns,

$$r_{p,T+1}^{(b)} = \sum_{i=1}^{m} r_{p,T+1,i}^{(b)}, \tag{24}$$

for each iteration $b = 1,2,\dots,B$.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Finally, we approximate the (empirical) CDF of the daily portfolio returns by

$$\widehat{G}_{T+1}(\omega) = \frac{1}{B}\sum_{b=1}^{B} 1\left\{r_{p,T+1}^{(b)} \leq \omega\right\}. \tag{25}$$

We then evaluate the daily VaR forecast for the nominal probability, α, by

$$\widehat{\mathrm{VaR}}_{p,T+1}(\alpha) = \sup\left(\omega|\widehat{G}_{T+1}(\omega) \leq \alpha\right). \tag{26}$$

Appendix Section A.2 describes the practical algorithm of mFARVaR in detail.

To recap, we have developed the multivariate FARVaR approach together with associated econometric techniques. This clearly shows that the FAR modeling can be extended to analyzing the dependence structure among multiple assets. In practice, we can apply the multivariate FARVaR to the model with the large number of assets (say, 100 or more) so far as we can collect sufficient intraday samples and/or we can employ any factor-based copula.

3 Intraday Data

We consider the constituents of the DJIA index over the period 2000–2008.[15] Two market downturns were experienced during the sample period: one caused by the dot-com bubble burst (2001–2002) and the other by the subprime mortgage crisis (2007–2008). In order to guarantee the reliability of backtesting, recommended practice is to include both normal and market downturn periods. The more market events in a sample period, the more scenarios for backtesting; nevertheless, our sample period meets a minimum requirement for the reliability of backtesting.

Three companies experienced crucial financial problems in 2008: bailouts for American International (AIG) and Citigroup were announced by the Federal Reserve Board of Governors; in the case of General Motors (GM), a government bridge loan was given to the auto manufacturers by the US government, which is classed as a *de facto* bailout. It is clearly important for regulators to understand the risk profiles of such companies, so as to monitor and safeguard the US financial system. At the same time, these companies are actively traded and their transactions are thus likely to generate huge amounts of information, especially at the intraday level. Hence, we aim to address the important empirical question of whether and how the use of intraday information can improve risk management modeling in practice.

There were five times of change in the components of the DJIA over the sample period: January 27, 2003, April 8, 2004, November 21, 2005, February 19, 2008, and September 22, 2008.[16] We choose the benchmark list of constituents as at November 21, 2005, which

15 Note that the data for HP are collected from May 6, 2002 to December 31, 2008 and the data for Verizon Communications are collected from July 3, 2000 to December 31, 2008.

16 January 27, 2003: Name changes only: Allied Signal Incorporated merged with and changed its name to Honeywell International; Exxon Corporation and Mobil merged and changed to Exxon Mobil Corporation; J.P. Morgan & Company changed to JPMorgan Chase & Co.; Minnesota Mining & Manufacturing changed to 3M Company; Philip Morris Companies Inc. changed to Altria Group Incorporated. April 8, 2004: AT&T, Eastman Kodak, and International Paper were replaced by American International, Pfizer, and Verizon. November 21, 2005: SBC Communications Inc. was renamed AT&T Inc. after it acquired the original AT&T. February 19, 2008: Altria Group and

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

corresponds to the middle of our sample period and includes the three financially distressed companies (AIG, Citigroup, and GM) which were removed from the list in 2008. See Table 1 for details of the 30 components. Intraday transaction data are collected from the Trade and Quote (TAQ) database. Following existing studies, such as Lee and Ready (1991) and Hvidkjaer (2006), we have utilized a filtering procedure to exclude any data likely to be erroneous. Specifically, all the trades (quotes) with condition codes A, C, D, G, L, N, O, R, X, Z, 8, 9 (4, 5, 7–9, 11, 13–17, 19, 20) are eliminated. The trades with a correction code greater than 2 are also removed. (Refer to the TAQ manual for definition of the codes.) Quotes are excluded if the ask is equal to or less than the bid, if the ask spread is above 75% of mid-quote, or if the ask (bid) is more than double or less than half of the previous ask (bid). We only consider trades reported from 9:30 A.M. to 4:00 P.M. According to Lee and Ready (1991), trades occur 5 s earlier than the reported time. Thus, we calculate trade time as the reported time minus 5 s. Trades are deleted if the trade price is more than double or less than half of the previous trade. After filtering, we calculate the closing TAQ prices in every 5-min interval (see Footnote 8). Closing TAQ prices are the price of the last trade in each interval and the corresponding quote price when the trade occurs. If there is no trade, the price for the current interval is replaced by the closest TAQ prices from the previous interval.

3.1 Descriptive Statistics

Table 2 presents the descriptive statistics of the intraday and daily returns of the 30 stocks over the sample period, 2000–2008. Panel A reports results for the intraday returns. The mean returns are very close to zero, a consistent finding in the market microstructure literature. The standard deviation ranges widely between 0.17 (Intel) and 0.54 (AIG). There is significant evidence of asymmetry as skewness is non-zero and mainly negative. As expected, kurtosis is significantly higher than that of the normal distribution, highlighting the typical fat tails of financial data.

Panel B reports the descriptive statistics of the daily returns. The mean returns are small and around a few basis points. The average returns of AIG (–0.17%) and Citigroup (–0.08%) are relatively low, reflecting that they experienced bailouts in 2008. The maximum (minimum) and the standard deviation of these companies are also much larger (smaller). The asymmetry of the return distribution is not as severe as in the intraday returns, though AIG (–6.49) and Procter & Gamble (–5.17) are substantially negatively skewed. Fat tails are observed for all the companies. In particular, the kurtosis of AIG (158.5), Citigroup (47.82), and Procter & Gamble (120.3) are significantly higher. Overall, these findings not only confirm the typical characteristics of financial time series but also display some extreme statistics, especially for companies bailed out during the global financial crisis in 2008.

3.2 Time-Varying Moments of Intraday Returns

To enhance our understanding of complex intraday return dynamics, we provide a time-varying descriptive analysis of the four moments (mean, standard deviation, skewness, and kurtosis) of the intraday returns on the value-weighted portfolio of 30 stocks.

Honeywell were replaced by Bank of America and Chevron. September 22, 2008: American International was replaced by Kraft Foods Inc.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Table 1 Components of the DJIA

Ticker: Company	Exchange	Industry	Data period
AA: Alcoa Corporation	NYSE	Aluminum	January 3, 2000–December 31, 2008
AIG: American International Group, Inc.	NYSE	Multiline insurance and brokers	January 3, 2000–December 31, 2008
AXP: American Express Company	NYSE	Consumer credit card services	January 3, 2000–December 31, 2008
BA: The Boeing Company	NYSE	Commercial aircraft manufacturing	January 3, 2000–December 31, 2008
C: Citigroup Inc.	NYSE	Banking	January 3, 2000–December 31, 2008
CAT: Caterpillar Inc.	NYSE	Construction machinery	January 3, 2000–December 31, 2008
DD: E. I. du Pont de Nemours & Company	NYSE	Industrial conglomerate	January 3, 2000–December 31, 2008
DIS: The Walt Disney Company	NYSE	Broadcasting	January 3, 2000–December 31, 2008
GE: General Electric	NYSE	Industrial conglomerate	January 3, 2000–December 31, 2008
GM: General Motors	NYSE	Auto and truck manufacturing	January 3, 2000–December 31, 2008
HD: The Home Depot, Inc.	NYSE	Home improvement products and services	January 3, 2000–December 31, 2008
HON: Honeywell International Inc.	NYSE	Industrial conglomerate	May 6, 2002–December 31, 2008
HPQ: HP Inc.	NYSE	Computer hardware	January 3, 2000–December 31, 2008
IBM: International Business Machines Corp.	NYSE	Computers and technology	January 3, 2000–December 31, 2008
INTC: Intel Corporation	NASDAQ	Semiconductors	January 3, 2000–December 31, 2008
JNJ: Johnson & Johnson	NYSE	Pharmaceuticals	January 3, 2000–December 31, 2008
JPM: JPMorgan Chase & Co.	NYSE	Banking	January 3, 2000–December 31, 2008
KO: The Coca-Cola Company	NYSE	Non-alcoholic beverages	January 3, 2000–December 31, 2008
MCD: McDonald's Corporation	NYSE	Quick service restaurants	January 3, 2000–December 31, 2008
MMM: 3M Company	NYSE	Industrial conglomerate	January 3, 2000–December 31, 2008
MO: Altria Group, Inc.	NYSE	Cigars and cigarette manufacturing	January 3, 2000–December 31, 2008
MRK: Merck & Co., Inc.	NYSE	Pharmaceuticals	January 3, 2000–December 31, 2008
MSFT: Microsoft Corporation	NASDAQ	Software	January 3, 2000–December 31, 2008
PFE: Pfizer Inc.	NYSE	Pharmaceuticals	January 3, 2000–December 31, 2008
PG: Procter & Gamble Co.	NYSE	Personal products	January 3, 2000–December 31, 2008
T: AT&T Inc.	NYSE	Telecommunications services	January 3, 2000–December 31, 2008
UTX: United Technologies Corporation	NYSE	Aerospace and defence	January 3, 2000–December 31, 2008
VZ: Verizon Communications Inc.	NYSE	Telecommunications services	July 3, 2000–December 31, 2008
WMT: Wal-Mart Stores, Inc.	NYSE	Supermarkets and convenience stores	January 3, 2000–December 31, 2008
XOM: Exxon Mobil Corporation	NYSE	Oil and gas refining and marketing	January 3, 2000–December 31, 2008

Note: From November 21, 2005, after the close, the DJIA consisted of the above 30 major companies.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Table 2 Descriptive statistics of intraday returns and daily returns

	Panel A: Intraday returns				Panel B: Daily returns			
Ticker: Company	Mean	StDev	SK	K	Mean	StDev	SK	K
AA: Alcoa Corporation	–0.0007	0.31	0.18	176.72	–0.0573	2.71	–0.06	10.77
AIG: American International Group, Inc.	–0.0024	0.54	–50.37	10,690.99	–0.1698	3.99	–6.49	158.52
AXP: American Express Company	–0.0006	0.29	–2.39	207.30	–0.0423	2.41	–0.37	9.26
BA: The Boeing Company	0.0000	0.25	–3.75	324.89	0.0011	2.10	–0.33	9.37
C: Citigroup Inc.	–0.0010	0.35	15.90	2240.65	–0.0794	2.97	0.58	47.82
CAT: Caterpillar Inc.	0.0003	0.26	–2.11	173.43	0.0267	2.08	–0.21	7.28
DD: E. I. du Pont de Nemours & Company	–0.0005	0.23	–0.80	58.49	–0.0425	1.88	–0.22	8.96
DIS: The Walt Disney Company	–0.0001	0.27	–2.27	434.35	–0.0113	2.22	–0.05	11.87
GE: General Electric	–0.0007	0.24	–1.03	150.30	–0.0508	2.03	–0.22	9.47
GM: General Motors	–0.0006	0.28	–6.38	663.38	0.0300	1.79	0.15	14.63
HD: The Home Depot, Inc.	–0.0003	0.29	0.46	371.78	–0.0485	2.40	–1.11	22.44
HON: Honeywell International Inc.	0.0006	0.28	–0.31	238.78	–0.0253	2.38	–0.48	13.30
HPQ: HP Inc.	–0.0002	0.23	–2.61	477.37	0.0437	2.25	0.13	9.91
IBM: International Business Machines Corp.	–0.0010	0.37	–37.02	6336.19	–0.0127	1.94	–0.05	10.35
INTC: Intel Corporation	0.0001	0.17	–6.78	721.42	–0.0472	2.95	–0.49	9.49
JNJ: Johnson & Johnson	–0.0008	0.44	–148.97	44,269.37	0.0110	1.43	–0.84	18.52
JPM: JPMorgan Chase & Co.	–0.0001	0.19	–0.91	76.73	–0.0032	2.64	0.03	11.91
KO: The Coca-Cola Company	0.0003	0.23	–2.18	140.36	–0.0106	1.54	0.08	9.12
MCD: McDonald's Corporation	0.0001	0.20	0.32	80.99	0.0196	1.78	–0.19	7.61
MMM: 3M Company	–0.0003	0.36	–193.24	61,862.39	0.0081	1.59	0.19	7.99
MO: Altria Group, Inc.	–0.0005	0.24	–15.69	1810.63	0.0431	1.88	–0.02	12.94
MRK: Merck & Co., Inc.	–0.0006	0.25	–1.39	258.30	–0.0358	2.05	–1.83	31.32
MSFT: Microsoft Corporation	–0.0003	0.23	–3.98	326.59	–0.0488	2.25	–0.08	11.07
PFE: Pfizer Inc.	0.0001	0.20	–44.87	8810.92	–0.0261	1.81	–0.24	7.34
PG: Procter & Gamble Co.	–0.0003	0.32	45.67	9311.84	0.0061	1.66	–5.19	120.31
T: AT&T Inc.	0.0003	0.23	–2.10	229.93	–0.0539	2.44	0.28	14.90
UTX: United Technologies Corporation	–0.0002	0.23	–1.01	68.85	0.0295	1.91	–0.16	9.55
VZ: Verizon Communications Inc.	–0.0001	0.22	–0.14	61.41	–0.0186	1.88	0.17	8.28
WMT: Wal-Mart Stores, Inc.	0.0004	0.20	–0.69	63.75	–0.0089	1.79	0.26	6.88
XOM: Exxon Mobil Corporation	–0.0018	0.39	–1.94	549.39	0.0306	1.79	0.15	14.63

Notes: This table presents the descriptive statistics of the intraday and daily returns of the 30 stocks listed in the DJIA index. We calculate the mean, standard deviation (StDev), skewness (SK), and kurtosis (K) of each individual company over the sample period. Panel A reports the descriptive statistics of the intraday returns. Panel B presents the descriptive statistics of the daily returns.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

In Figure 4, we display the autocorrelation function (ACF) of volatility (standard deviation) and skewness, respectively. As expected, volatility exhibits high persistence with its first-order autoregressive [AR(1)] coefficient of around 0.76; this is a consistent finding with those documented in Andersen et al. (2001). Skewness is weakly persistent with its AR(1) coefficient slightly negative at –0.074, but statistically significant. On the other hand, the ACFs of mean and kurtosis are statistically insignificant (and not reported here).

We turn to analyzing time-varying patterns of contemporaneous correlation among the moments. Figure 5 displays scatter plots between moment pairs.[17] First, from plot (a), we find a U-shaped relationship between return and volatility, suggesting that the correlation is regime-dependent and measured at 0.67 when the mean is positive (bull market) and –0.63 when it is negative (bear market). This finding is qualitatively consistent with the recent literature documenting a non-monotonic relationship between return and volatility (e.g., Rossi and Timmermann, 2010; Chiang and Li, 2012). The U-shaped risk–return relationship observed at the intraday frequency is also in line with the market microstructure model (e.g., Hasbrouck, 2007, chap. 2), where the unconditional mean reverts to zero at high frequency. The HFT strategy should be a directional bet with information advantage (e.g., knowledge about order flows), since bearing additional volatility risk will not be necessarily compensated by higher returns. This evidence may suggest that the imposition of a time-invariant and linear risk–return trade-off would lead to inaccurate and misleading forecasts.

Second, plot (b) shows that mean and skewness are positively associated, with a correlation coefficient of 0.51. There has been mixed evidence on the mean–skewness relationship, mostly at lower frequencies (weekly or monthly; e.g., Rehmany and Vilkovz, 2012; Conrad et al., 2013). To the best of our knowledge, there is no documented evidence at the intraday level. Intuitively, our finding suggests that extreme values or outliers may play a significant role in generating intraday returns, as a few large positive or negative movements are likely to render the average return moving in the same direction.

Third, we find from plot (c) that there also exists a U-shaped relationship between volatility and skewness, with the correlation measured at –0.37 when skewness is negative (downside risk) and 0.23 when skewness is positive (upside uncertainty). Given that we obtain the U-shaped risk–return and the linear return–skewness trade-off, this finding is consistent with transitivity. It confirms that extreme large price movements (positive or negative skewness) are associated with higher volatility. The steeper slope observed under downside risk also confirms that the volatility–skewness relationship is stronger when the market is hit by bad news.

Fourth, plot (d) displays a strong quadratic U-shaped relationship between skewness and kurtosis. The correlation is measured at –0.96 when skewness is negative and 0.98

17 These graphs are produced with intraday returns. In particular, while it is tempting to see the relationship between mean and volatility in (a) as a test of the mean–variance relationship, it is not the objective we have here. More formal tests and consideration should be undertaken to draw such a conclusion. These graphs should be considered as revealing empirical regularities which demonstrate the regime-dependent nonlinear relationship between the moments of intraday returns. We thank the referee for pointing out, in particular, the potential for estimation errors and the relevance of intraday returns to the general mean–variance setting.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

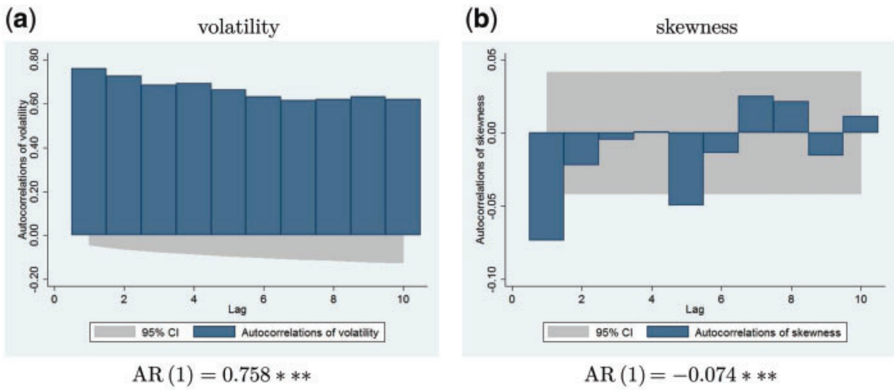


Figure 4 Autocorrelation function of moments of intraday returns.

Notes: This figure displays the ACFs of the volatility and skewness of intraday returns, constructed by the equal-weighted portfolio of 30 stocks. 95% CI denotes the 95% confidence interval. *, **, and *** indicate that the null of a zero AR(1) coefficient is rejected at the 10%, 5%, and 1% significance level, respectively.

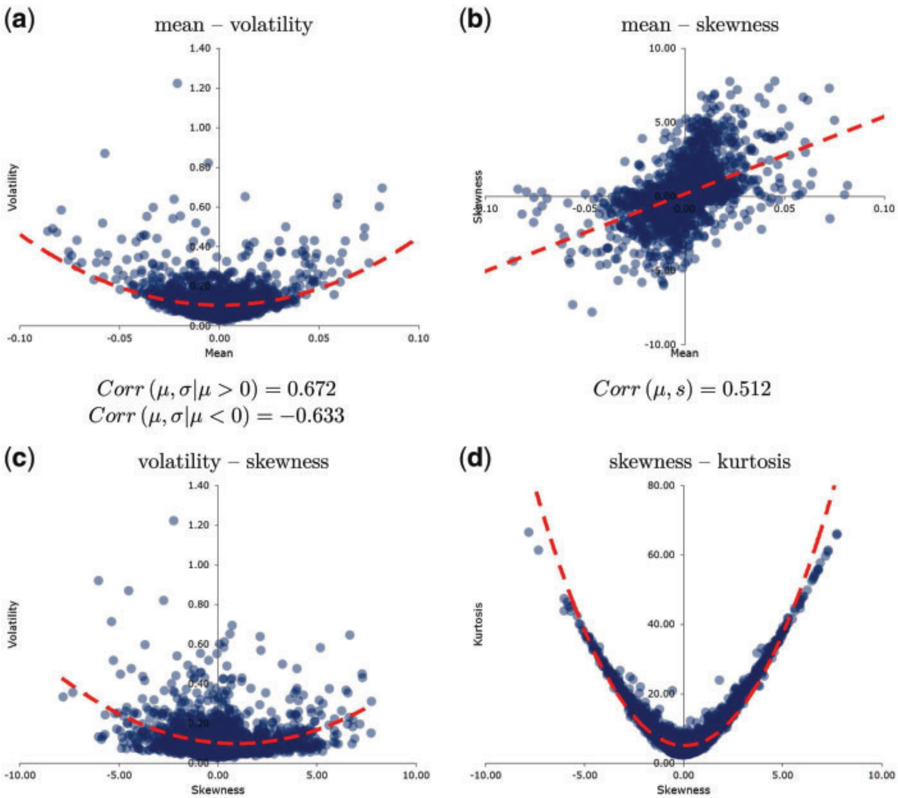
when skewness is positive. This quadratic U-shaped relationship is a pleasing finding mathematically.

Overall, our time-varying moment-based analysis reveals two stylized facts. First, the second and third moments of the intraday return density are persistent. Second, contemporaneous associations among the four moments are all complex and time-varying. These results provide support for the numerous lower-frequency studies showing that the time-varying characteristics of the higher-order moments should be carefully modeled for the purposes of portfolio allocation and asset pricing (Harvey and Siddique, 2000; Cenesizoglu and Timmermann, 2012). This highlights the challenging issue of modeling intraday return dynamics. A nonparametric approach tends to ignore the time-varying nature of the density function, which leads to an inaccurate forecast. The fully parametric approach, on the other hand, suffering from misspecification, is ill-equipped to unravel an exact relationship among the higher-order moments, resulting in the erroneous measurement of risk. In this regard, we expect FARVaR to take into account the relative advantages of both approaches and thus deal in a robust manner with the complex characteristics of intraday returns when forecasting their density.

3.3 Time-Varying Dependence Structure of Multiple Intraday Returns

The correlation structure of individual assets constituting a portfolio is an essential input to portfolio risk management. Recently, more attention has been focused on dependence structure, which is a broader concept than linear correlation (see McNeil, Frey, and Embrechts, 2005). We too have developed mFARVaR on the basis of a copula by taking into account the dependence structure of the individual assets making up the portfolio. In this section, we estimate the copula correlation coefficients between individual intraday returns and analyze their dynamics.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024



$Corr\,(\mu,\sigma|\mu>0)=0.672$
$Corr\,(\mu,\sigma|\mu<0)=-0.633$

$Corr\,(\mu,s)=0.512$

$Corr\,(\sigma,s|s>0)=0.227$
$Corr\,(\sigma,s|s<0)=-0.366$

$Corr\,(s,k|s>0)=0.977$
$Corr\,(s,k|s<0)=-0.963$

Figure 5 Contemporaneous relationships between moments of intraday returns.

Notes: This figure displays scatter plots between moment pairs of intraday returns. Corr$(x,y|z)$ denotes a correlation coefficient between x and y, given a condition z. μ, σ, and s denote the mean, volatility, and skewness of intraday returns constructed by an equal-weighted portfolio of 30 stocks.

Given the limited number of intraday returns for each day, we construct 10 portfolios, each consisting of three stocks. To avoid the selection bias, we randomly draw the 10 sets of stocks from 30 stocks without replacement. In each set, we construct the equal-weighted portfolio of the three stocks. We will also use these portfolios for the backtesting of mFARVaR in Section 6.

Table 3 presents the descriptive statistics and autocorrelation tests of copula correlation coefficients between individual intraday returns. We estimate the copula correlation coefficients of Student's *t*-copula for each day. (See Appendix Section A.2 for estimating a copula in detail.) Panel A reports the minimum, maximum, and mean values of the copula correlation coefficients for each pair of individual stocks. The mean value is between 0.2 and 0.4 and the range between –0.4 and 0.9, which shows the large variation of the copula correlation coefficients over time. In Panel B, we compute the sample ACF of order 1 and conduct the Ljung–Box *Q*-test for the autocorrelations of the copula correlation

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Table 3 Descriptive statistics and autocorrelation of copula correlation coefficients

Portfolio	Panel A. Min/Mean/Max δ_{12}	δ_{13}	δ_{23}	Panel B. Autocorrelation Statistics	δ_{12}	δ_{13}	δ_{23}
P(AA, CAT, INTC)	−0.30/0.28/0.83	−0.26/0.28/0.80	−0.27/0.33/0.82	ACF(1)	0.484***	0.365***	0.463***
				Q(5)	2161***	1233***	1976***
P(AIG, C, GM)	−0.18/0.36/0.81	−0.30/0.22/0.72	−0.26/0.27/0.72	ACF(1)	0.537***	0.447***	0.455***
				Q(5)	2691***	1874***	2083***
P(AXP, DD, PG)	−0.27/0.32/0.80	−0.50/0.28/0.73	−0.31/0.28/0.80	ACF(1)	0.477***	0.393***	0.435***
				Q(5)	2080***	1338***	1687***
P(BA, GE, XOM)	−0.26/0.30/0.78	−0.31/0.26/0.82	−0.31/0.32/0.81	ACF(1)	0.448***	0.473***	0.567***
				Q(5)	1943***	2233***	3059***
P(DIS, HPQ, MMM)	−0.29/0.28/0.81	−0.22/0.31/0.85	−0.27/0.31/0.79	ACF(1)	0.436***	0.521***	0.402***
				Q(5)	1204***	1915***	957***
P(HD, HON, IBM)	−0.28/0.28/0.86	−0.23/0.31/0.83	−0.39/0.31/0.84	ACF(1)	0.486***	0.452***	0.476***
				Q(5)	2215***	1788***	2043***
P(JNJ, JPM, MCD)	−0.25/0.26/0.75	−0.29/0.22/0.77	−0.33/0.26/0.81	ACF(1)	0.348***	0.376***	0.451***
				Q(5)	1031***	1387***	1773***
P(KO, MSFT, VZ)	−0.34/0.29/0.77	−0.29/0.27/0.79	−0.27/0.30/0.79	ACF(1)	0.422***	0.424***	0.469***
				Q(5)	1420***	1347***	2087***
P(MO, MRK, PFE)	−0.33/0.20/0.74	−0.31/0.21/0.73	−0.45/0.31/0.85	ACF(1)	0.373***	0.382***	0.479***
				Q(5)	1227***	1418***	2257***
P(T, UTX, WMT)	−0.25/0.24/0.83	−0.28/0.25/0.74	−0.27/0.30/0.79	ACF(1)	0.512***	0.417***	0.479***
				Q(5)	2556***	1736***	2046***

Notes: This table presents the descriptive statistics and autocorrelation of copula correlation coefficients between the individual intraday returns constituting the equal-weighted portfolio of three stocks. In Panel A, we calculate the minimum/mean/maximum of copula correlation coefficients for each pair of individual stocks. In Panel B, we compute the sample ACF of order 1 and conduct the Ljung–Box Q-test for autocorrelation of copula correlation coefficients at a lag of 5. Note that *, **, and *** indicate that the null of zero autocorrelations is rejected at the 10%, 5%, and 1% significance level, respectively.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

coefficients at a lag of 5. As the test results show, there are significant autocorrelations in all portfolios.

We confirm that there is a strong dependence structure of intraday returns constituting a portfolio, beyond their linear correlation, and that it follows the autoregressive process. The overall results therefore support mFARVaR as developed in Section 2.

4 Pseudo Out-of-Sample Forecasting

We conduct pseudo out-of-sample forecasting exercises to evaluate the forecasting performance of a number of functional models as proposed above (FAR, FAR-fft, FAR-wv, AVE, and LAST; see Section 1.1 for details). In particular, we choose the continuous version of the fast Fourier transform algorithm, with 40 pairs of coefficients for FAR-fft, and three-level Daubechies wavelets for FAR-wv, respectively. This practice of holding out a sample is called "pseudo real-time" experiment (e.g., Elliot and Timmermann, 2008). In this way, we evaluate the performance in forecasting the density of intraday returns for each functional model.

To examine which of the five functional models described above can produce the most accurate density forecasts of intraday returns, we evaluate the three divergence criteria measuring the distance between the forecast and true density functions: the Hilbert norm (D_{H}), the uniform norm (D_{U}), and the generalized entropy (D_{E}):

$$D_{\mathrm{H}} = \frac{\int \left(\widehat{f}(x) - f(x)\right)^2 \mathrm{d}x}{\int \widehat{f}(x)^2 \mathrm{d}x + \int f(x)^2 \mathrm{d}x},\ D_{\mathrm{U}} = \frac{\sup_x |\widehat{f}_t(x) - f_t(x)|}{\sup_x f_t(x)},\ D_{\mathrm{E}} = \int f(x)\pi\left(\frac{\widehat{f}(x)}{f(x)}\right)\mathrm{d}x, \quad (27)$$

where $\widehat{f}$ (f) denotes the forecast (realized) density function of intraday returns and $\pi(y) = (\gamma - 1)^{-1}(y^\gamma - 1)$, with $\gamma > 0$ and $\gamma \neq 1$. Since the true density function is unobservable, we proxy it by the kernel density estimator in Equation (4). If π is the natural log, this becomes the Kullback–Leibler divergence measure. We use the generalized version to avoid the log of zero, which would happen when the density estimate or forecast has zero points. We set $\gamma = 1/2$ in our study. All three quantities (called the global error) are non-negative and produce a zero value if $\widehat{f} = f$. D_{H} is useful for evaluating the model's goodness-of-fit, D_{U} for comparing the function shapes, and D_{E} for assessing the difference in information content.

Throughout the paper, we employ the rolling sample approach, with a window size of 250 business days to accommodate any time-varying patterns. To begin with, we estimate the five functional models using intraday data over the period January 3, 2000–December 27, 2000 and compute the 1-day-ahead density forecast of intraday returns for December 28, 2000. We repeat this procedure by moving forward a day at a time in a rolling manner, ending with the density forecast for December 31, 2008. This generates 2013 density forecasts of intraday returns for each firm, except for HP, AT&T, and Verizon, with 1428, 2006, and 1887 density forecasts, respectively.

We evaluate the mean divergence for each firm and report the average across 30 firms by

$$\bar{D}_{\mathrm{H}} = \frac{1}{30}\sum_{k=1}^{30}\left(\frac{\sum_{t=1}^{N_k} D_{\mathrm{H},k,t}}{N_k}\right), \bar{D}_{\mathrm{U}} = \frac{1}{30}\sum_{k=1}^{30}\left(\frac{\sum_{t=1}^{N_k} D_{\mathrm{U},k,t}}{N_k}\right), \bar{D}_{\mathrm{E}} = \frac{1}{30}\sum_{k=1}^{30}\left(\frac{\sum_{t=1}^{N_k} D_{\mathrm{E},k,t}}{N_k}\right), \quad (28)$$

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Table 4 Performance of density forecast of intraday returns

Model	D_H		D_U		D_E	
FAR	0.02852	(9)	0.26648	(13)	0.04124	(4)
FAR-fft	0.02837	(14)	0.26625	(15)	0.04048	(19)
FAR-wv	0.02841	(7)	0.26781	(2)	0.04063	(7)
AVE	0.04568	(0)	0.36825	(0)	0.05762	(0)
LAST	0.03415	(0)	0.27786	(0)	0.05386	(0)

Notes: This table presents the performance of the density forecast of intraday returns for different functional models: FAR, FAR-fft, FAR-wv, AVE, and LAST. For the 30 stocks, one-step-ahead rolling forecasting is performed based on the 250-day window size. To begin with, we estimate the five functional models using intraday data over the period January 3, 2000–December 27, 2000 and compute the 1-day-ahead density forecast of intraday returns for December 28, 2000. We repeat this procedure by moving forward a day at a time in a rolling manner, ending with the density forecast for December 31, 2008. The Hilbert norm (D_H), the uniform norm (D_U), and the generalized entropy (D_E) are employed as divergence criteria [see Equation (27) for the definition], which are evaluated for the mean value of the 2013 intraday return density forecasts (except for HP, AT&T, and Verizon, where 1418, 2006, and 1887 forecasts, respectively, are used for the calculation). The main figure reports the average of each divergence measure for 30 stocks evaluated by Equation (28), and $(\cdot)$ indicates the number of stocks for which a given model achieves the smallest value. The best forecasting performance for each measure is highlighted by a red color.

where N_k is the number of daily density forecasts for a firm, k. Overall, from Table 4, we find that the FAR models outperform the other functional models, LAST and AVE. In particular, FAR-fft turns out to be the best predictor, producing the smallest divergence. Therefore, we will employ the FAR-fft approach as our FAR model in the further analysis of VaR.

5 Backtesting: Single Asset

We examine the performance of FARVaR against a number of existing VaR models popularly employed by both academics and practitioners: historical simulation (HS), filtered historical simulation (FHS), RiskMetrics (RM; RiskMetrics, 1996), GARCH, the filtered extreme value theory (FEVT) models, conditional autoregressive VaR by regression quantiles (CAViaR), and CAViaR–GARCH. HS is a static nonparametric model and most popular for simplicity (Perignon, Deng, and Wang, 2008).[18] RM and GARCH are typical dynamic parametric models; RM assumes the normal distribution of asset returns while GARCH can also allow for the fat-tailed Student's t-distribution. FHS is a hybrid approach, applying HS to returns filtered by GARCH. FEVT is suggested to control for time-varying volatility (Diebold, Schuermann, and Stroughair, 1998; McNeil and Frey, 2000); similarly to FHS, it applies the EVT procedure to returns filtered by GARCH. Here, we consider the filtered generalized extreme value (FGEV) distribution and the filtered generalized Pareto distribution (FGPD). Finally, we examine CAViaR and CAViAR–GARCH, which incorporates GARCH in CAViaR (see Engle and Manganelli, 2004). In the literature, FHS has been one of the most successful VaR models (e.g., Barone-Adesi, Giannopoulos, and Vosper, 2002; Kuester, Mittnik, and Paolella, 2006; Pritsker, 2006). More details of the

18 A McKinsey report in May 2012 estimated that 85% of large banks were using HS.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

models can be found in Appendix Section A.5. Notice that all the existing VaR models use daily returns as defined in Section 1.1 (i.e., $r_t = \ln P_{t,m} - \ln P_{t,0}$).

We evaluate the performance of the VaR models in terms of coverage ability, economic cost, and statistical validity. As strongly recommended by the BCBS, backtesting is a key part of the internal VaR model development for market risk management. We employ a number of backtesting tools: ECP, the BPZ (Basel Committee on Banking and Supervision, 1996), MRCR (Basel Committee on Banking and Supervision, 1996), PQL (Koenker and Bassett, 1978), the CC test (Christoffersen, 1998), and the DQ test (Engle and Manganelli, 2004).

Backtesting evaluates coverage ability and the economic cost arising from failure to cover the realized extreme event. Hence, all backtesting tools have been developed on the basis of the failure of a model. The failure is defined by an indicator function which takes unity when a realized return is not covered by the VaR forecast:

$$H_s = 1\{r_s < \widehat{\mathrm{VaR}}_s(\alpha)\},\ s = 1, \ldots, N, \tag{29}$$

where $\widehat{\mathrm{VaR}}_s(\alpha)$ is the VaR forecast given the information set available at s–1 with the nominal coverage probability α.

First, ECP and BPZ evaluate the coverage ability as follows: ECP is calculated by the sample average of H_s, that is,

$$\mathrm{ECP} = \frac{1}{N}\sum_{s=1}^{N} H_s. \tag{30}$$

BPZ describes the strength of an internal VaR model through evaluating its failure rate, which is the number of daily violations of the 99% VaR over the previous 250 business days; we expect, on average, 2.5 violations under a correctly forecasting model. The Basel Committee rules that up to four violations are acceptable, and defines this range as the "Green" zone. If there are five or more violations, banks fall into the "Yellow" (5–9) or "Red" (10+) zones. The penalty is cumulatively imposed by the multiplicative factor (κ); this factor is determined according to the number of violations: 3 (for 0–4 violations), 3.4 (5), 3.5 (6), 3.65 (7), 3.75 (8), 3.85 (9), and 4 (10+). In the Yellow zone, the supervisor will decide the penalty according to the reason for the violation.[19] In the Red zone, the penalty will be automatically generated. Hence, the regulator prefers the VaR model producing ECP close to the nominal probability and BPZ indicating the Green zone.

Second, MRCR and PQL evaluate the economic costs of the VaR model. Providing that a bank has a sound risk management system, an independent risk-control unit and external audits, MRCR is summarized by the following four factors: (i) the quantitative parameters, (ii) the treatment of correlations, (iii) the market risk charge, and (iv) the plus factor [see Basel Committee on Banking and Supervision (1996, 2005) for details and Jorion (2006) for a compact summary]. Then MRCR can be formulated by

$$\mathrm{MRCR}_s = \max\left(\kappa \frac{1}{60}\sum_{i=1}^{60} \widehat{\mathrm{VaR}}_{s-i}(\alpha), \widehat{\mathrm{VaR}}_{s-1}(\alpha)\right) + \mathrm{SRC}_s,\ s = 251, \ldots, N, \tag{31}$$

19 See Jorion (2006, p. 149) for detailed descriptions of the reasons suggested by the Basel Committee: (i) basic integrity of the model, (ii) model accuracy could be improved, (iii) intraday trading, and (iv) bad luck.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

where SRC is the additional capital charge for the specific risk (Basel Committee on Banking and Supervision, 1996, 2005) and κ is the multiplicative factor from BPZ.[20] We report the average of MRCR as

$$\text{MRCR} = \frac{1}{N-250}\sum_{s=251}^{N} \text{MRCR}_s. \tag{32}$$

PQL measures the expected economic cost of the VaR model using the "check" function (Koenker and Bassett, 1978). It is consistently estimated by

$$\text{PQL} = \frac{1}{N}\sum_{s=1}^{N} (\alpha - H_s)\left[r_s - \widehat{\text{VaR}}_s(\alpha)\right]. \tag{33}$$

When the VaR forecast fails to cover a realized return, $|r_s - \widehat{\text{VaR}}_s(\alpha)|$ is the economic loss and we impose a harsh penalty on it. Alternatively, when it does cover the realized return, $|r_s - \widehat{\text{VaR}}_s(\alpha)|$ is the opportunity cost and we impose a mild penalty on it as compensation for its success. Thus, we can rewrite PQL, by following our economic intuition, as

$$\text{PQL} = (1-\alpha)\left(\underbrace{\frac{1}{N}\sum_{s=1}^{N} |r_s - \widehat{\text{VaR}}_s(\alpha)| H_s}_{\text{Expected Economic Loss}}\right) + \alpha\left(\underbrace{\frac{1}{N}\sum_{s=1}^{N} |r_s - \widehat{\text{VaR}}_s(\alpha)|(1-H_s)}_{\text{Expected Opportunity Cost}}\right). \tag{34}$$

It is a reasonable measure of the expected economic cost of the VaR model, considering both the economic loss and the opportunity cost.

Finally, the CC and DQ tests evaluate the statistical validity of the VaR model. The CC test verifies whether the conditional expectation of H_s is equal to the coverage probability. Christoffersen (1998) shows that it is equivalent to testing whether H_s is an identically and independently distributed Bernoulli process with probability α. Hence, the likelihood ratio statistic simultaneously tests whether the unconditional coverage probability is α (unconditional coverage test) and whether the binary random variable is independent (independence test). It follows the chi-squared distribution with two degrees of freedom under the null hypothesis. The DQ test extends the CC test by allowing for more time-dependent information such as lagged realized violations and the VaR forecast. Specifically, we regress the demeaned binary variable on (constant, lagged demeaned binary variable, and the VaR forecast), and test the null, $R^2 = 0$, by the Wald test statistic given by

$$\text{DQ} = \frac{(\tilde{\mathbf{H}}'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\tilde{\mathbf{H}})}{\alpha(1-\alpha)} = \frac{\hat{\beta}'\mathbf{Z}'\mathbf{Z}\hat{\beta}}{\alpha(1-\alpha)} \overset{a}{\sim} \chi^2_{p+2}, \tag{35}$$

where $\tilde{\mathbf{H}} = (\tilde{H}_{p+1}, \dots, \tilde{H}_N)'$, $\tilde{H}_s = H_s - \alpha$, $\mathbf{Z} = (\mathbf{z}_{p+1}, \dots, \mathbf{z}_N)'$, and $\mathbf{z}_s = (1, \tilde{H}_{s-1}, \dots, \tilde{H}_{s-p}, \widehat{\text{VaR}}_s)'$. The regulator prefers the VaR model which is not rejected. We use the first four lags, that is, $\mathbf{z}_s = (1, \tilde{H}_{s-1}, \dots, \tilde{H}_{s-4}, \widehat{\text{VaR}}_s)'$, where the DQ statistic follows the chi-squared distribution with six degrees of freedom under the null.

20 It is difficult to identify the specific risk and its capital charge for an individual company. Hence, we only include the VaR part of Equation (31) for the purpose of comparison. Since we are comparing alternative VaR models for each stock or portfolio, SRC is fixed across models, and should not therefore affect our overall conclusion.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

In the next subsections, we repeat the rolling estimation procedure as described in Section 4 and apply the backtesting to VaR forecasts. We estimate VaR models using the window size of 250 days over the period January 3, 2000–December 27, 2000 and compute the 1-day-ahead 99% VaR forecasts for December 28, 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008. This generates 2013 density forecasts of intraday returns for each firm except for HP, AT&T, and Verizon, which generate 1428, 2006, and 1887 forecasts, respectively. This is our main analysis. We also apply the procedure to the value-weighted portfolio of 30 stocks.

We further compare FARVaR with FHS to see how differently these two hybrid approaches use information for the VaR forecast. As a robustness check, we run various exercises in the following subsections. First, we apply the backtesting to subperiods divided by the market regimes: normal and market downturn periods. Second, we compare our probability density function (PDF)-based FARVaR with a CDF-based one. Third, we check the asymmetry of the return distribution. We investigate how FARVaR performs at the right tail by taking a short position. Finally, we consider a longer window size (500 days) for investigating any effect of the window size on our evaluations. We estimate VaR models using the longer window size of 500 days over the period January 3, 2000–December 31, 2001 and compute the 1-day-ahead 99% VaR forecast for January 2, 2001. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008. This generates 1763 daily density forecasts of intraday returns for each firm except for HP, AT&T, and Verizon, which generate 1178, 1756, and 1637 forecasts, respectively.

5.1 Main Analysis

For our main analysis, we apply the VaR models to the left-tail behavior of return distribution with a window size of 250 days; that is, we hold a long position. We report the average of ECP, MRCR, and PQL for the 30 stocks,

$$\overline{\mathrm{ECP}} = \frac{1}{30}\sum_{k=1}^{30} \mathrm{ECP}_k,\ \overline{\mathrm{MRCR}} = \frac{1}{30}\sum_{k=1}^{30} \mathrm{MRCR}_k,\ \overline{\mathrm{PQL}} = \frac{1}{30}\sum_{k=1}^{30} \mathrm{PQL}_k, \tag{36}$$

where k indicates a stock. We report BPZ based on the average violation of the 30 stocks. For the CC and DQ tests, we count the frequency of an individual model being rejected at the 5% significance level out of the 30 asset returns.

Table 5 presents the backtesting results for the long position. In terms of ECP, FARVaR-sim (FARVaR-nig) slightly over-forecasts (under-forecasts) VaR, though both outcomes stay in the BPZ Green zone. Overall, we find that the coverage ability of FARVaR is quite reliable. RM severely under-forecasts VaR, receiving the warning Yellow zone, a finding consistent with Johansson, Seiler, and Michael (1999) and Netftci (2000). This implies that the coverage ability of RM is unreliable, rendering it inappropriate for use as an internal VaR model. Despite the fact that GARCH employs the (fat-tailed) Student's t-distribution, its results are not significantly better than those of RM. As expected, the FEVT models substantially over-forecast VaR, though their outcomes stay in the Green zone due to their conservative forecasting. (Under BPZ, over-forecasts of VaR tend to receive better scores, given the Basel Committee's prudential principle.) Finally, the CAViaR

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Table 5 Backtesting VaR models: 250-day window size and long position

Model	ECP (%)	BPZ	MRCR (%)	PQL (%)	CC	DQ
FARVaR-sim	0.87	Green	42.65	6.79	6	12
FARVaR-nig	1.07	Green	40.80	6.76	4	11
HS	1.19	Green	45.28	7.44	12	25
FHS	1.10	Green	45.29	6.87	1	12
RM	1.82	Yellow	39.59	6.89	25	29
GARCH	1.60	Green	39.78	6.80	19	25
FGEV	0.49	Green	58.22	7.61	19	11
FGPD	0.51	Green	60.21	7.82	20	12
CAViaR	2.83	Yellow	47.43	8.25	30	30
CAViaR–GARCH	4.54	Red	47.25	9.74	30	30

Notes: This table presents the backtesting results for the long position. We estimate VaR models using the window size of 250 days over the period January 3, 2000–December 27, 2000 and compute the 1-day-ahead 99% VaR forecast for December 28, 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008. This generates 2013 daily forecasts per company except for HP, AT&T, and Verizon, which generate 1428, 2006, and 1887 forecasts, respectively. We report the average of the ECP, MRCR, and PQL for the 30 stocks evaluated by Equation (36). The BPZ is evaluated based on the average violation of the 30 stocks. For the CC and DQ tests, we count the frequency of an individual model being rejected at the 5% significance level for the 30 stocks.

models considerably under-forecast VaR, receiving the warning Yellow (CAViaR) and Red (CAViaR–GARCH) zones, respectively. They turn out to be the worst models in terms of coverage ability.

Overall results reveal a number of stylized facts. First, the parametric models tend to under- or over-forecast the tail behavior of the return distribution. Hence with sufficient sample observations, the nonparametric return distribution could generally improve coverage ability, as demonstrated by the performance of FARVaR and FHS. The coverage ability of the GARCH model can be significantly improved by simulating the tail behavior from the nonparametric empirical distribution. Second, we can estimate the tail behavior more precisely when we model the complex dynamic relationships directly among moments or quantiles rather than when we focus on modeling the specific quantile, say at 1% or 5%, like CAViaR. As FARVaR is designed to fully address these two important issues, it can therefore produce more reliable coverage ability than the existing parametric models, RM, GARCH, and CAViaR.

Next, we turn to assessing the economic costs accompanying the different VaR models. RM requires the smallest MRCR, followed by GARCH and the FARVaR models. Conversely, the FEVT models incur the highest costs as they considerably over-forecast VaR, and thus necessitate the maintenance of high levels of capital, entailing high opportunity costs.[21] The CAViaR models also incur high economic costs to compensate for their

21 For example, suppose that the CFO of a bank manages $1 bn and reports the economic cost of the internal VaR model using MRCR and PQL on a daily basis. When the bank uses FGPD, then the CFO should allocate $602.1 m against the maximum loss over a 10-day horizon, or endure a $78.2 m daily loss. If the CFO switches to employ FARVaR-nig, then FARVaR-nig could reduce the capital

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

poor coverage ability. FARVaR is able to reduce the economic costs significantly by producing reliable coverage ability as well as by explicitly addressing the dynamics of the return distribution.

Finally, we discuss the statistical adequacy of the different VaR models by applying the CC test and the DQ test to each of the 30 asset returns at the 5% significance level. The relevant columns report the number of individual models rejected out of 30 returns. The null hypothesis of the CC test is rejected for six and four cases, respectively, in the FARVaR-sim and FARVaR-nig models. These rejection frequencies are well below those of other models, such as GARCH (19), CAViaR (30), and CAViaR–GARCH (30); except for FHS displaying only one rejection. The rejection frequencies for the DQ test are significantly higher for all the models.[22] The lowest rejection frequencies are reported at 11 out of 30 for FARVaR-nig and FGEV, followed by FARVaR-sim and FHS at 12. Again, the null hypothesis of the DQ test is rejected for all stocks when using CAViaR. Combining these results, we may conclude that the semiparametric approach, such as FARVaR and FSH, is statistically more adequate than either parametric or nonparametric models.

In sum, the backtesting performance of VaR modeling can be greatly improved by combining the relative advantages of parametric and nonparametric models. In this regard, we recommend FARVaR as it turns out to be a more reliable approach, substantially improving the coverage ability through avoiding severe misspecification errors, while considerably reducing economic costs through explicitly modeling the dynamics of intraday density in a functional space.

5.2 Portfolio Analysis

The main analysis shows that FARVaR performs well for individual stocks. Furthermore, it is important to analyze how well FARVaR performs relative to other VaR models when applied to portfolios. To this end, we evaluate the performance of the VaR models by applying them to the value-weighted portfolio of 30 stocks. We investigate them to the left-tail behavior of the return distribution with a window size of 250 days.

Table 6 presents the backtesting results. The FARVaR models slightly over-forecast VaR but maintain relatively low economic cost, which is the highly desirable property for a VaR model. Further, their VaR forecasts are not rejected by the CC and DQ tests. FHS shows performance similar to FARVaR for all backtesting except for ECP. Its ECP is slightly higher than 1%; that is, it slightly under-forecasts VaR.

The nonparametric and parametric models, HS, GARCH, and RM, under-forecast VaR. Their ECPs are noticeably higher than 1%. HS stays in a safe zone with relatively low economic cost; further, it is not rejected by the CC and DQ tests. However, both GARCH and RM suffer from weak coverage ability and statistical inadequacy due to their considerable under-forecast; that is, their ECPs are close to 2%. The FEVT models show performance similar to the hybrid semiparametric models but their economic costs are higher than those of the latter models. The CAViaR models perform poorly for all aspects: they largely under-forecast VaR, stay in the Yellow and Red zones, and suffer from poor statistical power.

requirement to \$408 m and the daily loss to \$67.6 m, saving \$194.1 m on the capital requirement and \$10.6 m on the daily loss by.

22 Berkowitz, Christoffersen, and Pelletier (2011) show that the CC test is less powerful against inaccurate VaR models than the DQ test.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Table 6 Backtesting VaR models with value-weighted portfolio of 30 stocks: 250-day window size and long position

Models	ECP (%)	BPZ	MRCR (%)	PQL (%)	CC	DQ
FARVaR-sim	0.65	Green	31.21	4.51	3.065	10.387
FARVaR-nig	0.80	Green	28.84	4.44	1.165	1.379
HS	1.24	Green	27.35	4.61	1.750	8.236
FHS	1.14	Green	25.89	4.48	1.649	9.408
RM	1.79	Yellow	24.64	4.38	10.463***	41.198***
GARCH	1.84	Yellow	25.00	4.42	13.267***	38.099***
FGEV	0.55	Green	35.47	5.09	5.100*	5.490
FGPD	0.65	Green	35.56	5.18	3.065	4.592
CAViaR	2.09	Yellow	26.66	4.59	21.70***	107.515***
CAViaR-GARCH	3.68	Red	27.56	5.52	88.239***	333.923***

Notes: This table presents the backtesting results of VaR models applied to the value-weighted portfolio of 30 stocks. The portfolio takes a long position. We estimate VaR models using the window size of 250 days over the period January 3, 2000–December 27, 2000 and compute the 1-day-ahead 99% VaR forecast for December 28, 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008. This generates 2013 daily forecasts per company except for HP, AT&T, and Verizon, which generate 1428, 2006, and 1887 forecasts, respectively. We report the ECP, BPZ, MRCR, PQL, CC-test statistic, and DQ-test statistic. *, **, and *** denote that the CC (DQ)-test statistic is rejected at the 10%, 5%, and 1% significance level, respectively.

Both FARVaR and FHS perform reasonably well with the portfolio. We find that FARVaR tends to slightly over-forecast VaR while FHS tends to under-forecast it. However, it is difficult to judge which model performs better from the backtesting only. In order to better understand the differences between the two models, in the next section we further analyze distinctive aspects of how FARVaR and FHS use information and forecast VaR.

5.3 FARVaR vs. FHS

The hybrid approaches, FARVaR and FHS, greatly improve the VaR forecast. However, the two models use information about the distribution of daily stock returns, and forecast the daily VaR, in different ways. FARVaR forecasts VaR relying on information from realized intraday densities, while FHS relies on observed daily returns. So are the two models able to obtain the same information, about the distribution of daily stock returns, in different ways? To answer this question, we investigate how closely the VaR forecasts from the two approaches track each other. This kind of analysis would also help deepen our understanding of whether both FARVaR and FHS capture the same information about stock returns, or whether there are important differences.

Figure 6 plots the VaR forecasts of FARVaR and FHS applied to the value-weighted portfolio of 30 stocks, as described in Section 5.2. The upper panel compares FARVaR-nig with FHS and the lower one FARVaR-sim with FHS. The time-series patterns of the VaR forecasts by FARVaR and FHS appear to track each other over the sample period, but we find that their responses to the underlying market regime are different. FHS responds sensitively to realized stock returns because it standardizes those by the GARCH filter. It

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

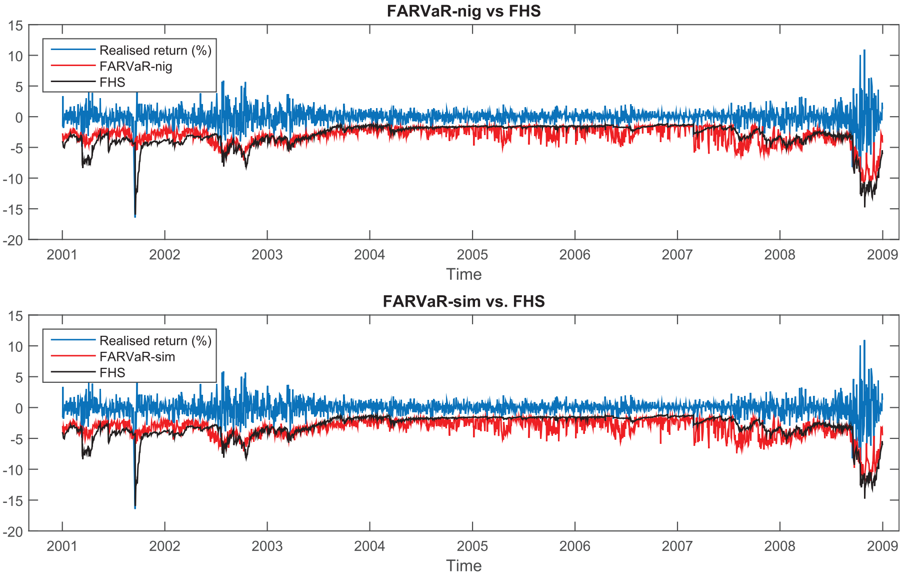


Figure 6 VaR forecasts by FARVaR and FHS.

Notes: This figure presents the VaR forecasts by FARVaR-nig, FARVaR-sim, and FHS applied to the value-weighted portfolio of 30 stocks. The portfolio takes a long position. We estimate the VaR models using the window size of 250 days over the period January 3, 2000–December 27, 2000 and compute the 1-day-ahead 99% VaR forecast for December 28, 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008.

therefore tends to relatively over-forecast VaR in periods of market downturn, such as the dot-com bubble burst (2001–2002) or subprime mortgage crisis (2007–2008), while under-forecasting in a normal period, such as (2003–2006).[23] By contrast, FARVaR depends on the realized density rather than the observed stock returns. Thus, its VAR forecast is relatively less sensitive to the underlying market regime than that of FHS.

For a more specific analysis, we investigate the failures of the VaR forecasts in Table 7. Failure is defined as when a realized return is not covered by the VaR forecast. In times of market downturn (dot-com bubble burst and subprime mortgage crisis), the FARVaR models and FHS show a similar number of failures, and the dates of their failures also mostly coincide. In other words, FARVaR relatively under-forecasts VaR to a slightly greater degree than FHS, but there is no notable difference between the two models in terms of coverage ability. However, in normal market periods, FHS under-forecasts VaR to a relatively greater degree and its failures are overwhelmingly more numerous than those of FARVaR.

Overall, the two models offer acceptable VaR forecasts and we cannot find any significant differences. The subtle difference is that the VaR forecast of FARVaR is less sensitive to the underlying market regime than that of FHS. This is due to the way that the two

23 Care is needed to select a right filter for a given underlying regime to improve this tendency (see Gurrola-Perez and Murphy, 2015).

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Table 7 FARVaR and FHS: 250-day window size and long position

Date	Return	FARVaR-sim	FARVaR-nig	FHS
March 12, 2001	−4.896	−3.159	−2.985	−3.544
June 13, 2001	−3.610	−2.807	−2.749	−2.673
September 17, 2001	−16.453	−3.996	−4.050	−4.257
February 4, 2002	−2.519	−2.764	−2.471	−4.312
June 3, 2002	−2.450	−2.772	−2.433	−3.150
July 10, 2002	−3.365	−4.242	−4.165	−3.269
July 19, 2002	−4.792	−4.895	−4.788	−3.222
September 3, 2002	−4.520	−4.414	−4.598	−3.609
March 24, 2003	−4.363	−4.039	−4.171	−5.116
May 19, 2003	−2.703	−2.917	−2.405	−3.176
October 22, 2003	−1.664	−3.148	−2.867	−1.613
January 28, 2004	−1.367	−2.014	−1.779	−1.327
March 10, 2004	−1.701	−2.757	−2.666	−1.366
March 11, 2004	−1.688	−2.221	−1.993	−1.600
February 22, 2005	−1.645	−1.943	−1.695	−1.489
April 15, 2005	−1.738	−3.941	−3.532	−1.621
June 23, 2005	−1.463	−2.025	−1.565	−1.437
January 20, 2006	−1.830	−2.006	−2.020	−1.623
May 17, 2006	−1.809	−3.398	−3.293	−1.388
May 30, 2006	−1.660	−1.904	−1.505	−1.634
November 27, 2006	−1.364	−1.775	−1.710	−1.134
February 27, 2007	−3.313	−1.982	−1.547	−1.252
October 19, 2007	−2.381	−4.274	−4.014	−1.837
December 11, 2007	−2.007	−1.865	−1.699	−3.651
June 6, 2008	−3.423	−3.171	−2.800	−2.800
June 26, 2008	−3.247	−3.085	−2.681	−2.926
September 15, 2008	−8.228	−5.889	−4.859	−3.168
September 17, 2008	−6.688	−6.565	−6.844	−4.769
September 29, 2008	−6.342	−5.701	−5.644	−6.584
December 1, 2008	−8.625	−4.598	−4.163	−11.463

Notes: This table presents the failures of VaR forecasts by FARVaR-sim, FARVaR-nig, and FHS applied to the value-weighted portfolio of 30 stocks. The portfolio takes a long position. We estimate VaR models using the window size of 250 days over the period January 3, 2000–December 27, 2000 and compute the 1-day-ahead 99% VaR forecasts for December 28, 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008. Note that a colored cell indicates the failure of a VaR forecast.

models use information; that is, FARVaR utilizes the entire probability distribution given by the realized density, while FHS relies largely on information provided by the realized returns.

5.4 Subperiod Analysis

As seen in the comparison between FARVaR and FHS, VaR models may have different backtesting results according to the underlying market regime. To investigate this point, we divide the full sample period into three subperiods. First, there are two notable market downturns in our sample: one caused by the dot-com bubble burst (2001–2002) and the

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Table 8 Backtesting VaR models excluding periods of market downturn: 250-day window size and long position

Models	ECP (%)	BPZ	MRCR (%)	PQL (%)	CC	DQ
FARVaR-sim	0.57	Green	32.87	5.02	9	5
FARVaR-nig	0.74	Green	30.81	4.93	1	4
HS	0.66	Green	33.63	5.39	2	10
FHS	0.85	Green	31.98	5.19	0	9
RM	1.45	Green	27.96	4.95	5	16
GARCH	1.14	Green	27.94	4.92	3	7
FGEV	0.43	Green	42.06	5.94	11	4
FGPD	0.44	Green	44.24	6.14	9	4
CAViaR	2.59	Yellow	33.54	6.22	26	30
CAViaR-GARCH	4.12	Red	33.16	7.26	29	30

Notes: This table presents the backtesting results for the long position excluding the periods of market downturn (the dot-com bubble burst of 2001–2002 and the subprime mortgage crisis of 2007–2008). We estimate VaR models using the window size of 250 days over the period January 4, 2002–December 31, 2002 and compute the 1-day-ahead 99% VaR forecast for January 2, 2003. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2006. This generates 1009 daily forecasts per company except for HP, which generates 924 forecasts. See the note of Table 5 for ECP, BPZ, MRCR, PQL, CC, and DQ.

other by the subprime mortgage crisis (2007–2008). The remaining subperiod (2003–2006) can be regarded as relatively normal. We apply backtesting to each subperiod in the same way as in our main analysis (long position and 250-day window size).

Table 8 presents the backtesting results for the normal subperiod. The FARVaR models and FHS show reasonable coverage ability and economic cost. They are also statistically reliable. Other models, except for CAViaR, also work reasonably well. However, the FEVT models entail very high economic cost even during the normal subperiod.

Table 9 presents the backtesting results for the subperiods of market downturn. In the nature of FEVT, it can provide the most prudential VaR forecast during a market downturn. Thus, despite the high economic cost of the FEVT models, they successfully reduce the number of failures when the market is in deep recession. In the case of the FARVaR models and FHS, their failures increase but they still remain in the Green zone, at a lower economic cost than for FEVT. In particular, the FARVaR models show statistically reasonable results during the subperiod of the dot-com bubble burst.

In sum, the difference in how the VaR models use information accounts for the difference in their VaR forecasts as the market regime changes. Although FEVT works more robustly than others during periods of market downturn, FARVaR and FHS also work reasonably well. In particular, FARVaR is statistically more reliable than FHS during periods of market downturn. On balance, FARVaR thus gives the best performance overall across different market conditions.

5.5 PDF vs. CDF

In Section 1.2.2, the numerical approximation of the CDF in our proposed simulation approach uses the PDF forecast. Since this numerical approximation affects the accuracy of the CDF, we may consider to model CDFs directly instead of PDFs.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Table 9 Backtesting VaR models during periods of market downturn: 250-day window size and long position

Models	ECP (%)	BPZ	MRCR (%)	PQL (%)	CC	DQ
Panel A. Dot-com bubble burst (2001–2002)						
FARVaR-sim	0.87	Green	58.86	8.43	1	8
FARVaR-nig	1.03	Green	56.79	8.34	1	9
HS	0.97	Green	63.01	9.34	2	16
FHS	1.11	Green	65.75	9.19	1	17
RM	1.86	Yellow	57.26	9.17	7	23
GARCH	1.63	Yellow	55.88	9.01	2	19
FGEV	0.46	Green	84.98	10.26	1	4
FGPD	0.48	Green	89.70	10.63	1	5
CAViaR	2.60	Yellow	66.91	10.68	14	24
CAViaR-GARCH	5.12	Red	67.88	13.48	25	29
Panel B. Subprime mortgage crisis (2007–2008)						
FARVaR-sim	1.47	Green	59.90	8.77	5	15
FARVaR-nig	1.79	Green	59.20	8.94	7	16
HS	2.44	Yellow	60.04	9.75	19	26
FHS	1.60	Green	69.94	8.00	3	14
RM	2.51	Yellow	65.28	8.60	21	27
GARCH	2.51	Yellow	64.66	8.43	20	29
FGEV	0.63	Green	84.22	8.38	0	2
FGPD	0.67	Green	83.61	8.49	0	2
CAViaR	3.48	Yellow	78.19	9.90	27	30
CAViaR-GARCH	4.77	Red	78.09	11.05	29	30

Notes: This table presents the backtesting results for the long position during the periods of market downturn (the dot-com bubble burst of 2001–2002 and the subprime mortgage crisis of 2007–2008). In Panel A (the dot-com bubble burst), we estimate VaR models using the window size of 250 days over the period January 5, 2000–December 31, 2000 and compute the 1-day-ahead 99% VaR forecast for January 3, 2001. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2002. This generates 500 daily forecasts per company except for HP and Verizon, which generate 0 (VaR forecast is available from 2003) and 374 forecasts, respectively. In Panel B (the subprime mortgage crisis), we estimate VaR models using the window size of 250 days over the period January 4, 2006–December 29, 2006 and compute the 1-day-ahead 99% VaR forecast for January 3, 2007. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008. This generates 504 daily forecasts per company except for AT&T, which generate 497 forecasts. See the note of Table 5 for ECP, BPZ, MRCR, PQL, CC, and DQ.

The fact that the PDF follows the FAR process means that the PDF tends to revert toward its unconditional mean. We can therefore deduce that the CDF does the same. We model the (centered) CDF, $W(x) = F(x) - \bar{F}(x)$, as the FAR of order 1:

$$W_t = BW_{t-1} + \xi_t, \tag{37}$$

where B denotes the autoregressive operator.

We apply FARVaR-sim in Section 1.2.2 to the CDFs of the value-weighted portfolio of 30 stocks. We use FARVaR-cdf-sim to indicate the CDF-based FARVaR-sim. The empirical

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

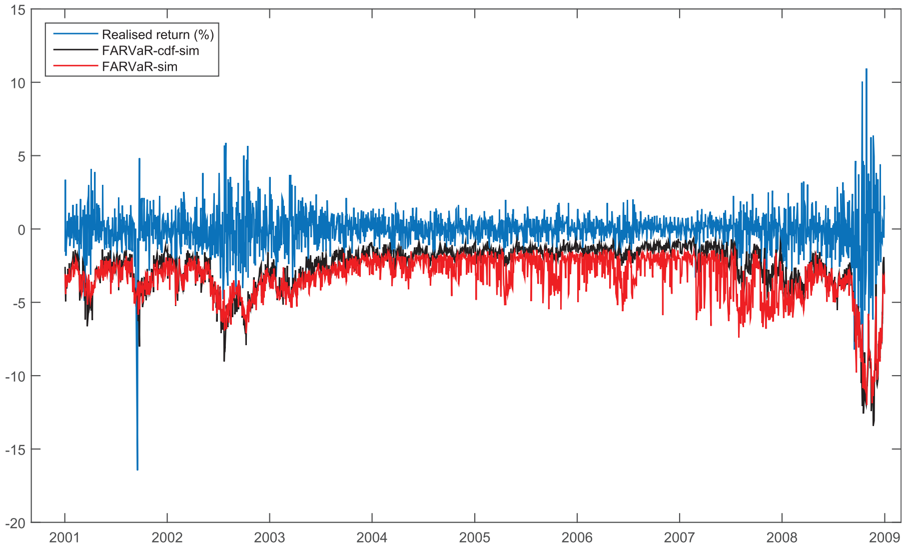


Figure 7 VaR forecasts by FARVaR-cdf-sim and FARVaR-sim.

Notes: This figure presents the VaR forecasts by FARVaR-cdf-sim applied to the value-weighted portfolio of 30 stocks. The portfolio takes a long position. We estimate the VaR model using the window size of 250 days over the period January 3, 2000–December 27, 2000 and compute the 1-day-ahead 99% VaR forecast for December 28, 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008. For comparison, we also plot the VaR forecasts of our PDF-based FARVaR-sim.

CDFs can be consistently estimated by

$$\widehat{F}_t(x_j) = \frac{1}{m}\sum_{i=1}^{m} 1(r_{t,i} \leq x_j),\ t = 1, \ldots, T;\ j = 1, \ldots, n \tag{38}$$

with a large number of intraday observations. However, 78 intraday observations are too small to accurately measure probabilities in tails by Equation (38). Furthermore, we need a smooth empirical CDF to randomly generate intraday returns. We therefore estimate them using the kernel density estimator.

We plot the rolling VaR forecasts for FARVaR-cdf-sim in Figure 7 and show the backtesting results in Table 10. We also present the results of our PDF-based FARVaR-sim for comparison. In Figure 7, FARVaR-cdf-sim under-forecasts VaR relative to FARVaR-sim; this is clearly observable during the normal subperiod. Table 10 shows that FARVaR-cdf-sim under-performs FARVaR-sim for all backtests, given its weak coverage ability due to under-forecasting. Furthermore, the model is rejected for all statistical tests while FARVaR-sim is not rejected for any tests at the 5% significance level.

Although it is beyond the scope of this paper to investigate the theoretical basis for its under-performance, we do have some doubts about FAR modeling of the CDF: First, with

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Table 10 FARVaR-cdf-sim and FARVaR-sim: 250-day window size and long position

Panel A. Backtesting

Model	ECP	BPZ	MRCR	PQL	CC	DQ
CDF	1.89%	Yellow	26.06%	4.30%	14.228***	33.280***
PDF	0.65%	Green	31.21%	4.51%	3.065	10.387

Panel B. Failures

Date	Return	CDF	PDF	Date	Return	CDF	PDF
March 12, 2001	–4.896	–3.535	–3.159	May 17, 2006	–1.809	–1.578	–3.398
June 13, 2001	–3.610	–2.727	–2.807	May 30, 2006	–1.660	–1.173	–1.904
July 6, 2001	–1.963	–1.713	–2.420	November 27, 2006	–1.364	–1.322	–1.775
August 8, 2001	–1.487	–1.453	–1.849	January 25, 2007	–1.005	–0.946	–1.861
September 17, 2001	–16.453	–4.671	–3.996	February 27, 2007	–3.313	–1.015	–1.982
January 29, 2002	–2.491	–2.154	–2.706	March 13, 2007	–1.924	–1.537	–3.386
February 4, 2002	–2.519	–2.149	–2.764	May 10, 2007	–1.048	–0.899	–1.885
June 3, 2002	–2.450	–1.945	–2.772	July 10, 2007	–1.313	–1.064	–1.963
September 3, 2002	–4.520	–4.729	–4.414	July 24, 2007	–1.508	–1.406	–5.302
March 24, 2003	–4.363	–4.172	–4.039	August 9, 2007	–2.636	–2.530	–5.908
May 19, 2003	–2.703	–2.217	–2.917	August 28, 2007	–2.058	–1.537	–3.257
September 24, 2003	–1.722	–1.666	–3.308	December 11, 2007	–2.007	–1.406	–1.865
April 13, 2004	–1.323	–1.173	–1.857	December 27, 2007	–1.370	–1.003	–2.479
April 20, 2004	–1.442	–1.406	–1.865	February 5, 2008	–2.964	–2.884	–3.484
September 22, 2004	–1.377	–1.249	–2.326	June 6, 2008	–3.423	–2.567	–3.171
February 22, 2005	–1.645	–1.388	–1.943	June 26, 2008	–3.247	–2.706	–3.085
May 31, 2005	–0.768	–0.766	–1.916	September 15, 2008	–8.228	–5.067	–5.889
December 27, 2005	–0.965	–0.761	–1.912	September 17, 2008	–6.688	–7.691	–6.565
January 20, 2006	–1.830	–1.316	–2.006	September 29, 2008	–6.342	–5.261	–5.701
May 11, 2006	–1.172	–1.169	–1.670	December 1, 2008	–8.625	–3.769	–4.598

Notes: This table presents the results of VaR forecasts by FARVaR-cdf-sim and (PDF-based) FARVaR-sim applied to the value-weighted portfolio of 30 stocks. The portfolio takes a long position. We estimate VaR models using the window size of 250 days over the period January 3, 2000–December 27, 2000 and compute the 1-day-ahead 99% VaR forecast for December 28, 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008. Panel A reports the backtesting results. We report the ECP, BPZ, MRCR, PQL, CC-test statistic, and DQ-test statistic. *, **, and *** denote that the CC (DQ)-test statistic is rejected at the 10%, 5%, and 1% significance level, respectively. Panel B reports the failures of VaR forecasts by FARVaR-cdf-sim (CDF) and FARVaR-sim (PDF). The failure is highlighted by a red color.

regard to the definition of the error term in Equation (37), based on Equation (2), the error term at a point x is defined by

$$\xi_t(x) = \int_{-\infty}^{x} \epsilon_t(s)\mathrm{d}s. \tag{39}$$

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Thus, it has zero mean and variance,

$$\mathrm{Var}(\xi_t(x)) = \mathrm{Var}(\int_{-\infty}^{x} \epsilon_t(s)\mathrm{d}s). \tag{40}$$

It is not clear, however, whether it has a finite variance, because it depends on the covariance of ϵ_t, but it must be a function of x. This heteroscedasticity can affect the estimation of the autoregressive operator. Second, the CDF is more restrictive than the PDF. It is bounded in the range of $[0, 1]$ and is the non-decreasing function. However, the forecast of the CDF can violate these restrictions unless we impose a certain functional transformation. Note that the only restriction on the PDF is that it has a non-negative value. The CDF-based approach is attractive in many respects but it needs more theoretical study with regard to estimations and forecasts. We leave it for future study.

5.6 Short Position

It is not unusual to observe that the return distribution is asymmetric. Hence, it is worth investigating how FARVaR performs at the right tail by taking a short position. Table 11 presents the backtesting results for the short position. Overall results are qualitatively similar to those for the long position. More specifically, the coverage ability of FARVaR is somewhat weakened, while its economic cost is slightly reduced. The overall performance of FARVaR remains better than that of other models except for FHS.

Next, we find that the FARVaR models are equally or slightly less rejected than other models by both CC and DQ tests, although, for both FGEV and FGPD, the null hypothesis of the CC test is rejected more while the null hypothesis of the DQ test is less rejected than FARVaR. For CAViaR, the null hypotheses of both tests are rejected for all 30 stocks. Once again, we find that the rejection frequencies of parametric or nonparametric models remain substantially high. This confirms that the hybrid models, FARVaR and FHS, are statistically the most reliable VaR models, irrespective of the asymmetry of the asset return distribution.

5.7 Longer Window Size

In practice, the selection of an optimal window size is a nontrivial issue. As the window size increases, estimation and forecasting precision generally improve. However, there is correspondingly greater uncertainty about the latent market regimes caused by a sequence of rare or extreme shocks hitting the market; for this reason, it would be more desirable to select shorter and homogeneous samples rather than longer and heterogeneous ones. With this caveat, we have also conducted backtesting exercises using the longer window size of 500 days, and find qualitatively similar results to those in Tables 12 and 13.

6 Backtesting: Multiple Assets

In this section, we examine the performance of mFARVaR against the existing multivariate VaR models: multivariate GARCH-based FHS [multivariate FHS (mFHS)-MGARCH] and copula-based FHS (mFHS-copula).[24] mFHS-MGARCH standardizes individual returns by the multivariate GARCH (MGARCH) filter and applies the Monte-Carlo (MC) simulation

24 Note that our proposed multivariate FARVaR shares the almost same idea with the multivariate FHS.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Table 11 Backtesting VaR models: 250-day window size and short position

Model	ECP (%)	BPZ	MRCR (%)	PQL (%)	CC	DQ
FARVaR-sim	1.22	Green	39.74	5.98	5	12
FARVaR-nig	0.96	Green	41.20	5.93	4	7
HS	1.20	Green	45.13	7.06	7	20
FHS	1.03	Green	42.88	6.05	0	3
RM	1.66	Yellow	38.64	6.09	19	20
GARCH	1.46	Green	39.04	5.97	12	11
FGEV	0.48	Green	52.53	6.60	20	6
FGPD	0.48	Green	53.77	6.73	24	4
CAViaR	2.31	Yellow	44.13	7.10	30	30
CAViaR-GARCH	2.27	Yellow	42.70	7.30	30	30

Notes: This table presents the backtesting results for the short position. We estimate VaR models using the window size of 250 days over the period January 3, 2000–December 27, 2000 and compute the 1-day-ahead 99% VaR forecast for December 28, 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008. This generates 2013 daily forecasts per company except for HP, AT&T, and Verizon, which generate 1428, 2006, and 1887 forecasts, respectively. See the note of Table 5 for ECP, BPZ, MRCR, PQL, CC, and DQ.

Table 12 Backtesting VaR models: 500-day window size and long position

Model	ECP (%)	BPZ	MRCR (%)	PQL (%)	CC	DQ
FARVaR-sim	0.89	Green	40.24	6.58	8	10
FARVaR-nig	1.05	Green	38.77	6.59	7	13
HS	1.55	Green	42.09	7.85	22	27
FHS	1.13	Green	40.88	6.37	2	10
RM	1.82	Yellow	36.84	6.57	27	28
GARCH	1.61	Green	37.25	6.46	19	21
FGEV	0.20	Green	70.93	8.75	29	22
FGPD	0.21	Green	72.42	8.92	30	23
CAViaR	1.76	Yellow	42.12	6.88	28	30
CAViaR-GARCH	3.54	Yellow	45.22	8.34	30	30

Notes: This table presents the backtesting results for the long position. We estimate VaR models using the window size of 500 days over the period January 3, 2000–December 31, 2001 and compute the 1-day-ahead 99% VaR forecast for January 2, 2002. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008. This generates 1763 daily forecasts per company except for HP, AT&T, and Verizon, which generate 1178, 1756, and 1637 forecasts, respectively. See the note of Table 5 for ECP, BPZ, MRCR, PQL, CC, and DQ.

to those returns as FHS does. For the MGARCH specification, we model the time-varying conditional correlations by the constant conditional correlation (CCC) model of Bollerslev (1990).[25] Analogously, mFHS-copula standardizes individual returns by the copula filter

25 We also employ the BEKK model (Engle and Kroner, 1995) and the dynamic conditional correlation (DCC) model (Engle, 2002). These models produce qualitatively similar results to those from CCC.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Table 13 Backtesting VaR models: 500-day window size and short position

Models	ECP (%)	BPZ	MRCR (%)	PQL (%)	CC	DQ
FARVaR-sim	0.84	Green	40.12	5.74	5	7
FARVaR-nig	0.96	Green	41.20	5.93	3	7
HS	1.20	Green	45.13	7.06	13	29
FHS	1.03	Green	42.88	6.05	2	1
RM	1.66	Yellow	38.64	6.09	24	22
GARCH	1.46	Green	39.04	5.97	13	11
FGEV	0.48	Green	52.53	6.60	30	21
FGPD	0.48	Green	53.77	6.73	30	19
CAViaR	2.31	Yellow	44.13	7.10	19	26
CAViaR-GARCH	1.98	Yellow	41.02	6.59	25	28

Notes: This table presents the backtesting results for the short position. We estimate VaR models using the window size of 500 days over the period January 3, 2000–December 31, 2001 and compute the 1-day-ahead 99% VaR forecast for January 2, 2002. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008. This generates 1763 daily forecasts per company except for HP, AT&T, and Verizon, which generate 1178, 1756, and 1637 forecasts, respectively. See the note of Table 5 for ECP, BPZ, MRCR, PQL, CC, and DQ.

and applies the MC simulation to those returns. We model the dependence structure of the standardized returns by the Student's t-copula.[26] In addition, we evaluate the performance of mFARVaR by imposing no correlations on the multiple intraday returns (mFARVaR-naïve).

We use the 10 equal-weighted portfolios of three stocks constructed in Section 3.3. For each portfolio, we estimate the multivariate VaR models, applied to the left-tail behavior of the return distribution, with the window size of 250 days, and compute a 1-day-ahead 99% VaR forecast. This forecasting exercise generates 2013 daily forecasts for each portfolio except for three: P(DIS, HPQ, MMM), P(KO, MSFT, VS), and P(T, UTX, WMT), which generate 1428, 1887, and 2006 daily forecasts, respectively. (See Table 1 for the details of company tickers.)

Tables 14–17 present the backtesting results of mFARVaR, mFARVaR-naïve, mFHS-MGARCH, and mFHS-copula, respectively. First, we compare mFARVaR with mFARVaR-naïve. This comparison demonstrates the economic gains obtained by modeling the dependence structure in mFARVaR. For all portfolios, the ECP of mFARVaR-naïve is much higher than 1% and only three portfolios remain in the Green zone. This weak coverage ability is mainly caused by its considerable under-forecast of VaR. For example, Figure 8 plots the VaR forecasts by mFARVaR and mFARVaR-naïve for the portfolio P(T, UTX, WMT). It clearly shows that mFARVaR-naïve systematically under-forecasts VaR compared with mFARVaR.[27] Furthermore, the CC and DQ tests are rejected for all

26 We model the dependence structure relying on the constant Student's t-copula, but the performance of mFHS-copula could be improved by modeling the dependence structure using a time-varying copula such as the generalized autoregressive score model (Creal, Koopman, and Lucas, 2013).

27 We find near-identical patterns in other portfolios.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Table 14 Backtesting multivariate FARVaR (mFARVaR)

Portfolio	ECP (%)	BPZ	MRCR (%)	PQL (%)	CC	DQ
P(AA, CAT, INTC)	1.54	Green	38.91	5.79	6.094**	16.621**
P(AIG, C, GM)	2.19	Green	40.47	8.99	22.333***	93.375***
P(AXP, DD, PG)	0.99	Green	28.96	4.16	0.403	5.709
P(BA, GE, XOM)	1.24	Green	31.89	4.45	1.776	8.753
P(DIS, HPQ, MMM)	0.91	Green	26.74	4.06	0.353	0.642
P(HD, HON, IBM)	0.60	Green	35.90	4.70	3.998	18.556***
P(JNJ, JPM, MCD)	0.60	Green	29.29	3.80	3.998	5.265
P(KO, MSFT, VZ)	0.74	Green	27.76	3.62	1.588	1.679
P(MO, MRK, PFE)	1.69	Green	29.73	4.96	9.211**	18.443***
P(T, UTX, WMT)	0.85	Green	29.10	3.91	0.779	0.993
Overall	1.14	Green	31.87	4.85	3	4

Notes: This table presents the backtesting results of mFARVaR applied to the 10 portfolios. Each portfolio takes a long position and is the equal-weighted portfolio of three stocks randomly drawn without replacement from the 30 stocks. We estimate mFARVaR using the window size of 250 days over the period January 3, 2000–December 27, 2000 and compute the 1-day-ahead 99% VaR forecast for December 28, 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008. This generates 2013 daily forecasts per company except for P(DIS, HPQ, MMM), P(KO, MSFT, VZ), and P(T, UTX, WMT), which generate 1428, 1887, and 2006 forecasts, respectively. Note that we select the *t*-copula for modeling the tail dependence. We report the ECP, BPZ, MRCR, PQL, CC-test statistic, and DQ-test statistic for each portfolio. *, **, and *** denote that the CC (DQ)-test is rejected at the 10%, 5%, and 1% significance level, respectively. The "Overall" row reports the average of the ECP, MRCR, and PQL for the 10 portfolios. The BPZ is evaluated based on the average violation of the 10 portfolios. For the CC and DQ tests, we count the number of portfolios being rejected at the 5% significance level.

portfolios. By contrast, mFARVaR achieves more accurate VaR forecasts by taking into account the dependence structure of the individual intraday returns. On average, its ECP is slightly higher than 1%, but this is due to more frequent failures in the portfolios that include stocks such as AIG or GM which have experienced serious financial problems. Moreover, all portfolios remain in the Green zone and are less frequently rejected than other models by the CC and DQ tests. From this comparison, we can see how important is the dependence structure of intraday returns constituting a portfolio, and can confirm that the implemented modeling of dependence structure in mFARVaR robustly works well.

Next, we compare mFARVaR with mFHS, which is the rival model for a single asset. In mFHS-MGARCH, there is no significant difference in economic cost. As seen in the case of a single asset, the CC test is slightly less frequently rejected than for mFARVaR, while the DQ test is slightly more frequently rejected. However, the crucial difference is found in its coverage ability: the ECP for mFHS-MGARCH is higher than 1% for all portfolios. This tendency is not desirable in risk management. For example, Figure 9 plots the VaR forecasts by mFARVaR and mFHS-MGARCH for the portfolio P(T, UTX, WMT). Analogously to the case of a single asset in Figure 6, mFHS-MGARCH tends to over-forecast VaR in a period of market downturn such as the dot-com bubble burst (2001–2002) or the subprime mortgage crisis (2007–2008), while under-forecasting it in a normal period (2003–2006); thereby more failures are observed during the normal period. mFHS-copula shows even

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Table 15 Backtesting naïve multivariate FARVaR (mFARVaR-naïve)

Portfolio	ECP (%)	BPZ	MRCR (%)	PQL (%)	CC	DQ
P(AA, CAT, INTC)	3.43	Yellow	33.42	7.32	74.630***	176.327***
P(AIG, C, GM)	4.03	Yellow	35.29	11.83	113.537***	527.076***
P(AXP, DD, PG)	2.19	Yellow	24.82	4.65	23.403***	60.938***
P(BA, GE, XOM)	2.69	Yellow	26.80	5.28	37.741***	119.653***
P(DIS, HPQ, MMM)	2.66	Yellow	24.67	4.55	29.528***	94.404***
P(HD, HON, IBM)	1.59	Green	30.53	4.80	7.068**	36.896***
P(JNJ, JPM, MCD)	1.59	Green	26.17	3.98	6.416**	35.877***
P(KO, MSFT, VZ)	2.02	Yellow	25.07	3.92	15.267***	27.620***
P(MO, MRK, PFE)	2.88	Yellow	27.19	5.79	47.918***	99.015***
P(T, UTX, WMT)	1.95	Green	25.64	4.20	14.288***	31.111***
Overall	2.50	Yellow	27.96	5.63	10	10

Notes: This table presents the backtesting results of naïve mFARVaR applied to the 10 portfolios. Each portfolio takes a long position and is the equal-weighted portfolio of three stocks randomly drawn without replacement from the 30 stocks. We estimate mFARVaR-naïve using the window size of 250 days over the period January 3, 2000–December 27, 2000 and compute the 1-day-ahead 99% VaR forecasts for December 28, 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008. This generates 2013 daily forecasts per company except for P(DIS, HPQ, MMM), P(KO, MSFT, VZ), and P(T, UTX, WMT), which generate 1428, 1887, and 2006 forecasts, respectively. Note that we select the *t*-copula for modeling the tail dependence. See the note of Table 14 for ECP, BPZ, MRCR, PQL, CC, DQ, and Overall.

Table 16 Backtesting MGARCH-based FHS (mFHS-MGARCH)

Portfolio	ECP (%)	BPZ	MRCR (%)	PQL (%)	CC	DQ
P(AA, CAT, INTC)	1.49	Green	39.66	5.87	5.187*	25.394***
P(AIG, C, GM)	1.79	Green	49.83	7.33	11.602***	16.211**
P(AXP, DD, PG)	1.49	Green	27.30	4.38	5.187*	16.864***
P(BA, GE, XOM)	1.44	Green	30.48	4.76	3.614	16.530**
P(DIS, HPQ, MMM)	1.26	Green	26.94	4.37	2.406	11.297*
P(HD, HON, IBM)	1.39	Green	31.82	4.51	3.586	19.192***
P(JNJ, JPM, MCD)	1.39	Green	29.21	4.01	3.586	6.211
P(KO, MSFT, VZ)	1.43	Green	28.04	3.84	3.849	9.003
P(MO, MRK, PFE)	1.34	Green	34.22	5.09	2.896	5.910
P(T, UTX, WMT)	1.55	Green	28.39	4.28	5.646*	12.664**
Overall	1.46	Green	32.59	4.84	1	6

Notes: This table presents the backtesting results of mFHS-MGARCH applied to the 10 portfolios. Each portfolio takes a long position and is the equal-weighted portfolio of three stocks randomly drawn without replacement from the 30 stocks. We estimate mFHS-MGARCH using the window size of 250 days over the period January 3, 2000–December 27, 2000 and compute the 1-day-ahead 99% VaR forecast for December 28, 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008. This generates 2013 daily forecasts per company except for P(DIS, HPQ, MMM), P(KO, MSFT, VZ), and P(T, UTX, WMT), which generate 1428, 1887, and 2006 forecasts, respectively. Note that we select the *t*-copula for modeling the tail dependence. See the note of Table 14 for ECP, BPZ, MRCR, PQL, CC, DQ, and Overall.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Table 17 Backtesting copula-based mFHS (mFHS-copula)

Portfolio	ECP (%)	BPZ	MRCR (%)	PQL (%)	CC	DQ
P(AA, CAT, INTC)	1.24	Green	42.85	5.82	1.750	17.381***
P(AIG, C, GM)	2.04	Yellow	50.17	7.82	20.451***	53.293***
P(AXP, DD, PG)	1.24	Green	27.92	4.21	1.750	16.142**
P(BA, GE, XOM)	1.34	Green	31.13	4.50	5.494*	25.374***
P(DIS, HPQ, MMM)	1.89	Yellow	28.25	4.82	9.508***	61.765***
P(HD, HON, IBM)	0.94	Green	34.05	4.80	0.425	16.371**
P(JNJ, JPM, MCD)	1.69	Green	29.41	4.28	13.466***	44.084***
P(KO, MSFT, VZ)	1.64	Green	28.58	4.14	12.708***	65.711***
P(MO, MRK, PFE)	2.14	Yellow	34.68	5.68	25.931***	147.051***
P(T, UTX, WMT)	1.90	Green	30.25	4.48	24.731***	106.869***
Overall	1.61	Green	33.73	5.06	6	10

Notes: This table presents the backtesting results of mFHS-copula applied to the 10 portfolios. Each portfolio takes a long position and is the equal-weighted portfolio of three stocks randomly drawn without replacement from the 30 stocks. We estimate mFHS-copula using the window size of 250 days over the period January 3, 2000–December 27, 2000 and compute the 1-day-ahead 99% VaR forecast for December 28, 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008. This generates 2013 daily forecasts per company except for P(DIS, HPQ, MMM), P(KO, MSFT, VZ), and P(T, UTX, WMT), which generate 1428, 1887, and 2006 forecasts, respectively. Note that we select the *t*-copula for modeling the tail dependence. See the note of Table 14 for ECP, BPZ, MRCR, PQL, CC, DQ, and Overall.

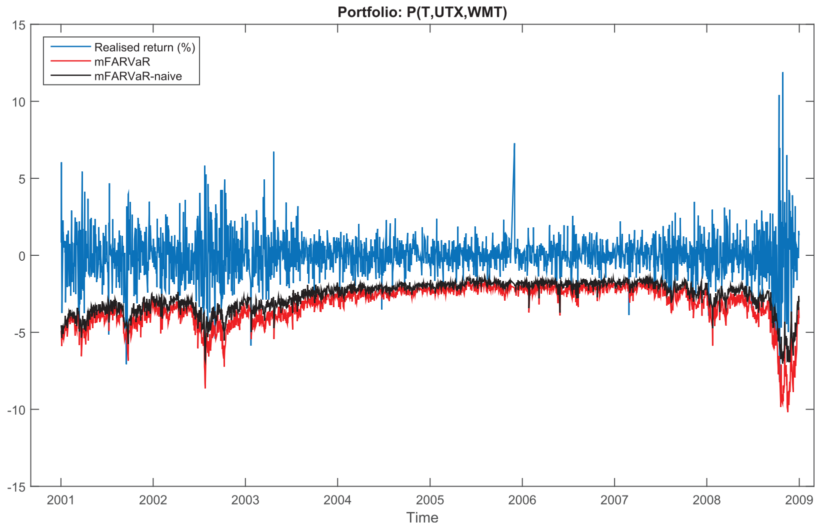


Figure 8 VaR forecasts by mFARVaR and mFARVaR-naïve.

Notes: This figure presents the VaR forecasts by mFARVaR and mFARVaR-naïve applied to the portfolio P(T, UTX, WMT). The portfolio takes a long position. We estimate the VaR models using the window size of 250 days over the period January 3, 2000–December 27, 2000 and compute the 1-day-ahead 99% VaR forecast for December 28, 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

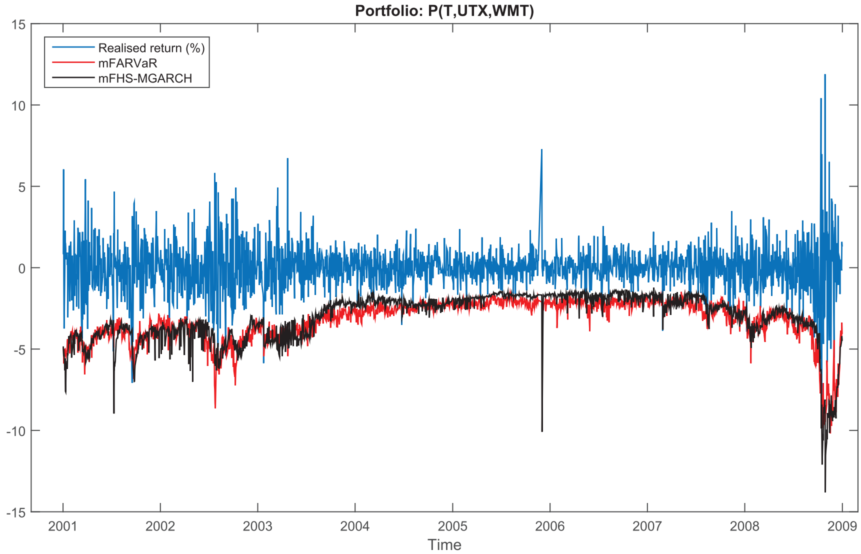


Figure 9 VaR forecasts by mFARVaR and mFHS.

Notes: This figure presents the VaR forecasts by mFARVaR and mFHS-MGARCH applied to the portfolio P(T, UTX, WMT). The portfolio takes a long position. We estimate the VaR models using the window size of 250 days over the period January 3, 2000–December 27, 2000 and compute the 1-day-ahead 99% VaR forecast for December 28, 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for December 31, 2008.

worse results: its ECP is much higher than 1% and only three portfolios remain in the Green zone. Moreover, many portfolios are statistically rejected.

Overall results reveal the following stylized facts. First, the dependence structure of intraday returns constituting a portfolio is essential for enhancing the performance of VaR forecasting in applications considering multiple assets. The copula framework is well equipped in our proposed mFARVaR and contributes to improving its VaR forecasts. Second, there is no dominance between mFARVaR and mFHS-MGARCH in terms of economic cost or statistical adequacy, just as in the case of a single asset. However, we find the crucial differences between the two models in their coverage ability: the ECP of mFHS-MGARCH is higher than that of mFARVaR, but not overwhelmingly so. The problem for mFHS-MGARCH is that it has a tendency to under-forecast VaR, and thus suffers from more failures during a normal market period; this tendency needs to be improved. Consequently, we find that mFARVaR has potential as the most efficient model to utilize information from multiple intraday returns and has competitive advantage over the existing multivariate VaR models.

7 Concluding Remarks

With the growing importance of intraday activity in the financial market, such as high-frequency and algorithmic trading, we have proposed the FARVaR approach in order to improve daily VaR evaluation by explicitly incorporating intraday returns information.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

FARVaR is a semiparametric approach which applies FAR to forecasting the nonparametric density of intraday returns. Furthermore, we have provided a practical algorithm for constructing the density forecast of daily returns directly from the density forecast of intraday returns, using a parametric approximation based on the NIG distribution and a nonparametric bootstrap approximation. More importantly, we have extended the FARVaR approach to multiple assets for more flexible application.

We have conducted comprehensive evaluation exercises using actual data from 30 stocks listed in the DJIA index over the period 2000–2008. First, we find that FAR turns out to be the best functional model predictor for the density of intraday returns for all 30 stocks. Second, we have conducted a number of backtests, which confirm that FARVaR outperforms other nonparametric and parametric VaR models. Third, we find that the overall performance of FARVaR-sim is slightly more favorable than that of FARVaR-nig. Given that the former is a more efficient approach, we recommend the use of FARVaR-sim in practice. Finally, the backtesting for multiple assets demonstrates that mFARVaR has competitive advantage over existing multivariate VaR models.

Overall, this study enhances our understanding of contemporary VaR analysis in several ways. First, as intraday information becomes more helpful in forecasting the daily risk, a robust nonparametric modeling of the density of intraday returns is likely to be a key input to improving daily risk management. Second, FARVaR can accommodate the complex dynamics of intraday return density in a flexible manner because FAR is a generalization of all classes of autoregressive models. Furthermore, FARVaR can be flexibly combined with a framework for multivariate distribution functions like copula. Third, we demonstrate that the hybrid approach can simultaneously improve coverage ability and reduce economic cost. Specifically, the robust nonparametric approach helps to improve coverage ability while the dynamic parametric FAR modeling can reduce economic cost. Finally, Basel III suggests, but does not require, that banks and other financial institutions move away from VaR and toward a coherent measure of tail risk known as "expected shortfall (ES)." Once the daily density has been estimated by the FARVaR approach, it is rather straightforward to calculate the expected value of the tail, where the tail is defined as the density beyond the VaR quantile.

Supplementary Data

Supplementary data are available at *Journal of Financial Econometrics* online.

Appendix

A.1 FARVaR

We describe a detailed FARVaR estimation and forecasting procedure. Given a series of equity prices observed at a fixed time interval, we calculate intraday returns by the first difference of the logged prices. Suppose that we observe m intraday returns, $(r_{t,i})_{i=1}^{m}$, at each day t over T days. For simplicity, we consider the following balanced panel data:

$$\mathbf{R} = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ r_{T,1} & r_{T,2} & \cdots & r_{T,m} \end{bmatrix}_{T\times m}, \tag{A.1}$$

which consists of T row vectors with each row vector containing m intraday returns.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Step 1: Forecasting Density of Intraday Returns by FAR

A. Kernel density estimation of intraday return density

We set an $n \times 1$ vector of discrete grids, $\mathbf{x} = (x_1, \ldots, x_n)'$, which covers the range of intraday samples with $n = 1024$ as used in the empirical analysis. We then estimate the density of intraday returns by the following kernel density estimator:

$$\tilde{f}_t(x_j) = \frac{1}{nh_t}\sum_{i=1}^{m} K\left(\frac{x_j - r_{t,i}}{h_t}\right),\ t = 1, ..., T,\ j = 1, \ldots, n, \tag{A.2}$$

where K is a kernel and h_t is a bandwidth. Following Silverman (1986), we employ the popular Gaussian kernel and the rule of thumb that the optimal bandwidth is given by $h_t = 1.06\widehat{\sigma}_t m^{-1/5}$, where $\widehat{\sigma}_t$ is the sample standard deviation of $\{r_{t,1}, \ldots, r_{t,m}\}$.

Since the density function must satisfy $\int f(x)\mathrm{d}x = 1$, we normalize the estimated density function by

$$\widehat{f}_t(x_j) = \frac{\tilde{f}_t(x_j)}{\mathrm{RSUM}(\tilde{\mathbf{f}}_t)}, \tag{A.3}$$

where $\mathrm{RSUM}(\tilde{\mathbf{f}}_t)$ approximates $\int \tilde{f}_t(x)\mathrm{d}x$ by the following numerical middle Riemann sum of $\tilde{\mathbf{f}}_t = \left(\tilde{f}_t(x_1), \ldots, \tilde{f}_t(x_n)\right)'$:

$$\begin{aligned}\mathrm{RSUM}(\tilde{\mathbf{f}}_t) &= \frac{1}{2}\left[\sum_{j=1}^{n-1}\tilde{f}_t(x_{j+1})(x_{j+1} - x_j) + \sum_{j=1}^{n-1}\tilde{f}_t(x_j)(x_{j+1} - x_j)\right] \\ &= \frac{1}{2}\left[\sum_{j=1}^{n-1}\tilde{f}_t(x_{j+1}) + \sum_{j=1}^{n-1}\tilde{f}_t(x_j)\right]\Delta,\end{aligned} \tag{A.4}$$

where $\Delta = x_{j+1} - x_j$ is an equally partitioned interval for all $j = 1, \ldots, n-1$. Next, we construct the matrix of densities by

$$\widehat{\mathbf{F}} = \left[\widehat{\mathbf{f}}_1 \quad \widehat{\mathbf{f}}_2 \quad \cdots \quad \widehat{\mathbf{f}}_T\right] = \begin{bmatrix} \widehat{f}_1(x_1) & \widehat{f}_2(x_1) & \cdots & \widehat{f}_T(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{f}_1(x_n) & \widehat{f}_2(x_n) & \cdots & \widehat{f}_T(x_n) \end{bmatrix}_{n \times T}. \tag{A.5}$$

Then we estimate a point-wise unconditional functional mean by

$$\bar{\mathbf{f}} = \begin{bmatrix} \bar{f}(x_1) \\ \vdots \\ \bar{f}(x_n) \end{bmatrix} = \begin{bmatrix} T^{-1}\sum_{t=1}^{T}\widehat{f}_t(x_1) \\ \vdots \\ T^{-1}\sum_{t=1}^{T}\widehat{f}_t(x_n) \end{bmatrix}_{n \times 1}, \tag{A.6}$$

and build the matrix of fluctuations around the unconditional functional mean by

$$\widehat{\mathbf{W}} = [\widehat{\mathbf{w}}_1, \widehat{\mathbf{w}}_2, \ldots, \widehat{\mathbf{w}}_T] = \begin{bmatrix} \widehat{w}_1(x_1) & \widehat{w}_2(x_1) & \cdots & \widehat{w}_T(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{w}_1(x_n) & \widehat{w}_2(x_n) & \cdots & \widehat{w}_T(x_n) \end{bmatrix}_{n \times T}, \tag{A.7}$$

where $\widehat{\mathbf{w}}_t = \widehat{\mathbf{f}}_t - \bar{\mathbf{f}}$.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

B. Estimation of FAR model

We estimate the autocovariance operators of order 0 and 1 by

$$\widehat{C}_0 = \frac{1}{T}\sum_{t=1}^{T}\widehat{\mathbf{w}}_t\widehat{\mathbf{w}}'_t,\ \widehat{C}_1 = \frac{1}{T-1}\sum_{t=2}^{T}\widehat{\mathbf{w}}_t\widehat{\mathbf{w}}'_{t-1}, \tag{A.8}$$

and then obtain the eigenvalues (λ_j) of $\widehat{C}_0$ and its corresponding eigenfunctions $(\mathbf{v}_j)$ by

$$\lambda = \begin{bmatrix}\lambda_1\\ \vdots \\ \lambda_n\end{bmatrix}_{n\times 1}, \mathbf{V} = \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} & \cdots & v_{n1}\\ \vdots & \vdots & \ddots & \vdots\\ v_{1n} & v_{2n} & \cdots & v_{nn}\end{bmatrix}_{n\times n}. \tag{A.9}$$

Next, we choose $L(<n)$ eigenvalues, $(\lambda_1, \lambda_2, \ldots, \lambda_L)$, and the corresponding eigenfunctions, $(\mathbf{v}_1, \ldots, \mathbf{v}_L)$. Hence, we approximate the inverse of $\widehat{C}_0$ in the L-dimensional subspace by

$$\widehat{C}^{+}_{0,L} = \sum_{k=1}^{L}\lambda_k^{-1}\mathbf{v}_k\mathbf{v}'_k. \tag{A.10}$$

The choice of L is guided by applying a FPCA and a CV (see Ramsey and Silverman, 1997). FPCA explains the variation of the fluctuation and CV selects the optimal dimension $L(\leq L_{\max})$ by minimizing the following criterion:

$$\sum_{i=1}^{N_{\text{cv}}}\text{RSUM}\left(\left[\widehat{\mathbf{w}}^{L}_{T-i+1} - \widehat{\mathbf{w}}_{T-i+1}\right]^2\right), \tag{A.11}$$

where N_{cv} is the number of the last observations used in CV and $\widehat{\mathbf{w}}^{L}_{T-i+1}$ represents the in-sample forecasts of $\mathbf{w}_{T-i+1}$ on the L-dimensional subspace. We set $L_{\max}$ = 20 in the empirical analysis, and find that CV selects the optimal value of L ranging between 5 and 10. Finally, we estimate the autoregressive operator A in the L-dimensional subspace, consistently, by

$$\widehat{A}_L = \widehat{C}_1\widehat{C}^{+}_{0,L}, \tag{A.12}$$

where

$$\widehat{A}_L = \begin{bmatrix}\widehat{a}_{11} & \widehat{a}_{12} & \cdots & \widehat{a}_{1n}\\ \widehat{a}_{21} & \widehat{a}_{22} & \cdots & \widehat{a}_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \widehat{a}_{n1} & \widehat{a}_{n2} & \cdots & \widehat{a}_{nn}\end{bmatrix}. \tag{A.13}$$

C. Density forecast of intraday returns

A one-step-ahead conditional density forecast is evaluated by

$$\widehat{\mathbf{f}}_{T+1} = \bar{\mathbf{f}} + \widehat{A}_L\widehat{\mathbf{w}}_T, \tag{A.14}$$

or equivalently,

$$\begin{bmatrix}\widehat{f}_{T+1}(x_1)\\ \widehat{f}_{T+1}(x_2)\\ \vdots\\ \widehat{f}_{T+1}(x_n)\end{bmatrix} = \begin{bmatrix}\bar{f}(x_1)\\ \bar{f}(x_2)\\ \vdots\\ \bar{f}(x_n)\end{bmatrix} + \begin{bmatrix}\widehat{a}_{11}\widehat{w}_T(x_1) + \widehat{a}_{12}\widehat{w}_T(x_2) + \cdots + \widehat{a}_{1n}\widehat{w}_T(x_n)\\ \widehat{a}_{21}\widehat{w}_T(x_1) + \widehat{a}_{22}\widehat{w}_T(x_2) + \cdots + \widehat{a}_{2n}\widehat{w}_T(x_n)\\ \vdots\\ \widehat{a}_{n1}\widehat{w}_T(x_1) + \widehat{a}_{n2}\widehat{w}_T(x_2) + \cdots + \widehat{a}_{nn}\widehat{w}_T(x_n)\end{bmatrix}. \tag{A.15}$$

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Step 2: Forecasting Daily VaR

A. NIG approximation approach

We numerically evaluate the first four moments from the density forecast of intraday returns in Equation (A.14) [i.e., mean (μ_{T+1}), variance (v_{T+1}), skewness (s_{T+1}), kurtosis (k_{T+1})], by

$$\widehat{\mu}_{T+1} = \mathrm{RSUM}(\mathbf{x} \odot \widehat{\mathbf{f}}_{T+1}), \tag{A.16}$$

$$\widehat{v}_{T+1} = \mathrm{RSUM}((\mathbf{x} - \widehat{\mu}_{T+1})^2 \odot \widehat{\mathbf{f}}_{T+1}), \tag{A.17}$$

$$\widehat{s}_{T+1} = \mathrm{RSUM}((\mathbf{x} - \widehat{\mu}_{T+1})^3 \odot \widehat{\mathbf{f}}_{T+1})/\widehat{v}_{T+1}^{3/2}, \tag{A.18}$$

$$\widehat{k}_{T+1} = \mathrm{RSUM}((\mathbf{x} - \widehat{\mu}_{T+1})^4 \odot \widehat{\mathbf{f}}_{T+1})/\widehat{v}_{T+1}^{2}, \tag{A.19}$$

where $\odot$ stands for an element-by-element multiplication operator and $\mathrm{RSUM}(\mathbf{y}^k \odot \mathbf{f})$ approximates $\int y^k f(y)\mathrm{d}y$ for $k \geq 1$. We then calculate the four parameters, $(\widehat{\alpha}_{T+1}, \widehat{\beta}_{T+1}, \widehat{\gamma}_{T+1}, \widehat{\delta}_{T+1})$ from the four moments in Equation (A.16)–(A.19) by

$$\widehat{\alpha}_{T+1} = \widehat{v}_{T+1}^{-\frac{1}{2}}\left(3\widehat{k}_{T+1} - 4\widehat{s}_{T+1}^2 - 9\right)^{\frac{1}{2}}\left(\widehat{k}_{T+1} - \frac{5}{3}\widehat{s}_{T+1}^2 - 3\right)^{-1}, \tag{A.20}$$

$$\widehat{\beta}_{T+1} = \widehat{s}_{T+1}\widehat{v}_{T+1}^{-\frac{1}{2}}\left(\widehat{k}_{T+1} - \frac{5}{3}\widehat{s}_{T+1}^2 - 3\right)^{-1}, \tag{A.21}$$

$$\widehat{\gamma}_{T+1} = \widehat{\mu}_{T+1} - 3\widehat{s}_{T+1}\widehat{v}_{T+1}^{\frac{1}{2}}\left(3\widehat{k}_{T+1} - 4\widehat{s}_{T+1}^2 - 9\right)^{-1}, \tag{A.22}$$

$$\widehat{\delta}_{T+1} = 3^{\frac{3}{2}}\left[\widehat{v}_{T+1}\left(\widehat{k}_{T+1} - \frac{5}{3}\widehat{s}_{T+1}^2 - 3\right)\right]^{\frac{1}{2}}\left(3\widehat{k}_{T+1} - 4\widehat{s}_{T+1}^2 - 9\right)^{-1}. \tag{A.23}$$

We can approximate the density of daily returns by the following NIG density formula:

$$\widehat{g}_{T+1}(x) = \left[\frac{m\widehat{\alpha}_{T+1}\widehat{\delta}_{T+1}J_1\left(\widehat{\alpha}_{T+1}\sqrt{m^2\widehat{\delta}_{T+1}^2 + (x - m\widehat{\gamma}_{T+1})^2}\right)}{\pi\sqrt{m^2\widehat{\delta}_{T+1}^2 + (x - m\widehat{\gamma}_{T+1})^2}}\right]\mathrm{e}^{m\widehat{\delta}_{T+1}\widehat{\lambda}_{T+1} + \widehat{\beta}_{T+1}(x - m\widehat{\gamma}_{T+1})}. \tag{A.24}$$

Since the analytic form of the cumulative NIG density function does not exist, we approximate the cumulative NIG density $\widehat{G}_{T+1}$ by evaluating the integral of $\widehat{g}_{T+1}$:

$$\widehat{G}_{T+1}(\omega) = \int_{-\infty}^{\omega} \widehat{g}_{T+1}(x)\mathrm{d}x. \tag{A.25}$$

Since there is no analytic formula for $\widehat{G}_{T+1}(\omega)$, the numerical integration method is employed.[28]

28 We use the MATLAB package for the NIG distribution provided by Dr. Ralf Werner. The package uses a Gaussian integration rule for computing the CDF. http://uk.mathworks.com/matlabcentral/fileexchange/6050-normal-inverse-gaussion-distribution.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

B. Simulation approach

To construct the CDF of daily returns, $\widehat{F}_{T+1}$, from the density forecast of intraday returns, $\widehat{f}_{T+1}$, in Equation (A.14), we approximate an empirical CDF using the middle Riemann sum:

$$\widehat{F}_{T+1}(z_j) = \mathrm{RSUM}(\widehat{\mathbf{f}}_{T+1}[1:j+1]),\ j = 1, \ldots, n-1, \tag{A.26}$$

where $\widehat{\mathbf{f}}_{T+1}[1:j+1] := (\widehat{f}_{T+1}(x_1), \ldots, \widehat{f}_{T+1}(x_{j+1}))'$. We express the CDF of intraday returns as an $(n-1) \times 1$ vector:

$$\widehat{F}_{T+1} = \begin{bmatrix} \widehat{F}_{T+1}(z_1) \\ \widehat{F}_{T+1}(z_2) \\ \vdots \\ \widehat{F}_{T+1}(z_{n-1}) \end{bmatrix}_{(n-1)\times 1}. \tag{A.27}$$

Then we simulate intraday and daily return forecasts and the daily VaR forecasts from $\widehat{F}_{T+1}$, as described in Section 1.2.2 and depicted in Figure 2.

A.2 Multivariate FARVaR

We describe the detailed estimation and forecasting procedure of mFARVaR. Consider K assets constituting a portfolio in Equation (15).

Step 1. Forecasting K Marginal Densities of Intraday Returns by FAR

We forecast K marginal densities individually by FAR [see Equation (A.14)]. We then numerically approximate a CDF by applying the middle Riemann sum in Equation (A.26) to each density forecast. We express the CDF of intraday returns as an $(n-1) \times K$ matrix:

$$\widehat{\mathfrak{F}}_{T+1} = \begin{bmatrix} \widehat{F}_{1,T+1}(z_1) & \widehat{F}_{2,T+1}(z_1) & \cdots & \widehat{F}_{K,T+1}(z_1) \\ \widehat{F}_{1,T+1}(z_2) & \widehat{F}_{2,T+1}(z_2) & \cdots & \widehat{F}_{K,T+1}(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{F}_{1,T+1}(z_{n-1}) & \widehat{F}_{2,T+1}(z_{n-1}) & \cdots & \widehat{F}_{K,T+1}(z_{n-1}) \end{bmatrix}_{(n-1)\times K}. \tag{A.28}$$

Step 2. Forecasting Copula of K Intraday Return Series by VAR(1)

For each day, we construct the matrix of the probability integral transform (PIT) of intraday returns. To this end, we first approximate K CDFs by applying the middle Riemann sum in Equation (A.26) to the estimated density functions:

$$\widehat{\mathfrak{F}}_{t} = \begin{bmatrix} \widehat{F}_{1,t}(z_1) & \widehat{F}_{2,t}(z_1) & \cdots & \widehat{F}_{K,t}(z_1) \\ \widehat{F}_{1,t}(z_2) & \widehat{F}_{2,t}(z_2) & \cdots & \widehat{F}_{K,t}(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{F}_{1,t}(z_{n-1}) & \widehat{F}_{2,t}(z_{n-1}) & \cdots & \widehat{F}_{K,t}(z_{n-1}) \end{bmatrix}_{(n-1)\times K} \quad \text{for } t = 1, \ldots, T. \tag{A.29}$$

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Then we compute the PIT of intraday returns by $\widehat{u}_{k,t,i} \approx \widehat{F}_{k,t}(r_{k,t,i})$ and construct a PIT matrix:

$$\widehat{\mathbf{U}}_t = \begin{bmatrix} \widehat{u}_{1,t,1} & \widehat{u}_{2,t,1} & \cdots & \widehat{u}_{K,t,1} \\ \widehat{u}_{1,t,2} & \widehat{u}_{2,t,2} & \cdots & \widehat{u}_{K,t,2} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{u}_{1,t,m} & \widehat{u}_{2,t,m} & \cdots & \widehat{u}_{K,t,m} \end{bmatrix}_{m\times K} \quad \text{for } t = 1,\ldots,T. \tag{A.30}$$

Next, we estimate a copula using the PIT series for each day. Let us assume that the copula has the form of the multivariate Student $t(\nu_t, 0, \mathrm{P}_t)$-distribution, where P_t is the correlation matrix and ν_t the degree of freedom. We estimate the copula parameters, $\widehat{\mathrm{P}}_t$ and $\widehat{\nu}_t$, using the quasi-maximum-likelihood estimator:

$$\left\{\widehat{\nu}_t, \widehat{\mathrm{P}}_t\right\} = \underset{\{\nu_t,\mathrm{P}_t\}\in\Theta}{\operatorname{argmax}} \sum_{i=1}^{m} \ln c_t(u_{1,t,i},\ldots,u_{K,t,i};\nu_t,\mathrm{P}_t). \tag{A.31}$$

Then we model the dynamics of copula parameters by VAR(1) in Equations (19)–(20) and forecast them:

$$\left\{\widehat{\nu}_{T+1}, \widehat{\mathrm{P}}_{T+1}\right\} \Rightarrow C_{\widehat{\nu}_{T+1},\hat{\mathrm{P}}_{T+1}}. \tag{A.32}$$

Step 3. Simulating Daily Portfolio VaR

We randomly generate $\mathbf{U}_{T+1}$ from the copula forecast, $C_{\hat{\nu}_{T+1},\hat{\mathrm{P}}_{T+1}}$,

$$\mathbf{U}_{T+1}^{(b)} = \begin{bmatrix} u_{1,T+1,1}^{(b)} & u_{2,T+1,1}^{(b)} & \cdots & u_{K,T+1,1}^{(b)} \\ u_{1,T+1,2}^{(b)} & u_{2,T+1,2}^{(b)} & \cdots & u_{K,T+1,2}^{(b)} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1,T+1,m}^{(b)} & u_{2,T+1,m}^{(b)} & \cdots & u_{K,T+1,m}^{(b)} \end{bmatrix}_{m\times K} \quad \text{for } b = 1,\ldots,B, \tag{A.33}$$

and convert those into intraday returns using the inverse CDF of intraday returns,

$$\mathbf{R}_{T+1}^{(b)} = \begin{bmatrix} r_{1,T+1,1}^{(b)} & r_{2,T+1,1}^{(b)} & \cdots & r_{K,T+1,1}^{(b)} \\ r_{1,T+1,2}^{(b)} & r_{2,T+1,2}^{(b)} & \cdots & r_{K,T+1,2}^{(b)} \\ \vdots & \vdots & \ddots & \vdots \\ r_{1,T+1,m}^{(b)} & r_{2,T+1,m}^{(b)} & \cdots & r_{K,T+1,m}^{(b)} \end{bmatrix}_{m\times K} \quad \text{for } b = 1,\ldots,B, \tag{A.34}$$

where we obtain $r_{k,T+1,i} = \widehat{F}_{k,T+1}^{-1}(u_{k,T+1,i}^{(b)})$ following the algorithm depicted in Figure 2. Note that we randomly generate $\mathbf{U}_{T+1}$ from the independent multivariate uniform distribution for mFARVaR-naïve.

Then we update the intraday returns of the portfolio, generate the daily portfolio returns, and evaluate the daily VAR following Equations (23)–(26).

A.3 Confidence Interval of VaR Forecast

In order to construct the confidence interval of the VaR forecast, we employ the nonparametric bootstrap approach. First, we draw the prediction errors, $\{\epsilon_1^{(b)},\ldots,\epsilon_T^{(b)}\}$ for $b = 1,\ldots,B$, with replacements from the pool of sample residuals, $\{\widehat{\epsilon}_1,\ldots,\widehat{\epsilon}_1\}$, where

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

$\widehat{\epsilon}_t = \widehat{w} - \widehat{A}_L \widehat{w}_{t-1}$. Next, we update the density function based on FAR, using the prediction errors:

$$\left(f_t^{(b)} - \bar{f}\right) = \widehat{A}_L \left(f_{t-1}^{(b)} - \bar{f}\right) + \epsilon_t^{(b)},\ t = 1, \ldots, T, \tag{A.35}$$

where we assume that $f_1^{(b)}$ is given by the initial density function estimate. We finally forecast a daily VaR using the simulated density functions, $\{f_1^{(b)}, \ldots, f_T^{(b)}\}$. We repeat the bootstrapping many times to construct the empirical distribution of daily VaR forecasts by

$$\widehat{D}_{T+1}(\omega) = \frac{1}{B} \sum_{b=1}^{B} 1\left\{\widehat{\mathrm{VaR}}_{T+1}^{(b)} \leq \omega\right\}. \tag{A.36}$$

Finally, it is straightforward to construct a confidence interval given a probability from the empirical distribution.

A.4 Computational Burden

We provide detail on the computational burden that is involved in computing FARVaR for both single and multiple assets in our empirical analysis. As benchmarks we also compare our models with FHS, mFHS-MGARCH, and mFHS-copula. We repeat the forecast exercise 100 times and use the average value. The PC used has Intel(R) Core(TM) i7-2640M CPU 2.8 GHz (dual cores and four logical processors) and 8 GB RAM, running 64-bit Windows 10 Pro OS. We use MATLAB 2014b (64-bit). We use Econometrics Toolbox for (AR-GARCH-based) FHS and mFHS-copula, and the Oxford MFE Toolbox for mFHS-MGARCH. We use a 250-day window to estimate models and 1000 iterations for the simulation-based models.

Single asset			Multiple assets (three assets)		
FARVaR-sim	FARVaR-nig	FHS	mFARVaR	mFHS-MGARCH	mFHS-copula
2.344s	1.368s	0.329s	16.793s	1.537s	1.476s

A.5 VaR Models

Consider the sequence of daily returns (r_t) generated by a probability law

$$\mathbb{P}\{r_t \leq x | \mathcal{F}_{t-1}\} \equiv F_t(r) \tag{A.37}$$

conditional on the available information at t–1 ($\mathcal{F}_{t-1}$). Then VaR with a given tail probability α is defined as the conditional quantile from F_t such that

$$\mathrm{VaR}_t(\alpha) = F_t^{-1}(\alpha) \ \text{ or } \ F_t(\mathrm{VaR}_t(\alpha)) = \alpha. \tag{A.38}$$

Daily asset returns are usually represented with mean and volatility,

$$r_t = \mu_t + \sigma_t z_t, \tag{A.39}$$

where $\mu_t = \mathbb{E}[r_t | \mathcal{F}_{t-1}]$ and $\sigma_t^2 = \mathrm{Var}(r_t | \mathcal{F}_{t-1})$. The standardized return $z_t = (r_t - \mu_t)/\sigma_t$ has

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

a conditional distribution function $F_{t*} \equiv \mathbb{P}\{z_t \leq z | \mathcal{F}_{t-1}\}$. Thus, VaR in Equation (A.38) can be alternatively defined as

$$\mathrm{VaR}_t(\alpha) = \mu_t + \sigma_t F_{t*}^{-1}(\alpha). \tag{A.40}$$

Equations (A.38) and (A.40) are often categorized as unfiltered VaR and filtered VaR, respectively. For practical use, we estimate VaR by

$$\widehat{\mathrm{VaR}}_t(\alpha) = \begin{cases} F_t^{-1}(\alpha) & \text{(unfiltered)} \\ \widehat{\mu}_t + \widehat{\sigma}_t F_{t*}^{-1}(\alpha) & \text{(filtered).} \end{cases} \tag{A.41}$$

A. Historical simulation (HS)

HS is a static nonparametric model and is most popular for its simplicity (Perignon, Deng, and Wang, 2008). It does not make any strong asumption about the probability distribution of asset returns. It employs a window of observations generally ranging from 6 months to 2 years. Asset returns within this window are sorted in ascending order and the α-quantile of interest is given by the return that leaves α% of the observation on its left side and $(1-\alpha)$% on its right side. Hence, it uses the unfiltered VaR, and F_T is an empirical probability distribution function of historical asset returns.

B. RiskMetrics

RM uses the filtered VaR. It estimates a conditional mean and volatility by $\widehat{\mu}_{T+1} = T^{-1}\sum_{t=1}^{T} r_t$ and $\widehat{\sigma}^2_{T+1} = 0.94\widehat{\sigma}^2_T + 0.06(r_T - \widehat{\mu}_T)^2$. Further, it assumes that z_t follows the standard normal distribution.

C. GARCH

GARCH uses the filtered VaR. The conditional mean is estimated by AR(1) and the conditional volatility is estimated by GARCH(1, 1). It assumes that z_t follows the Student's t-distribution.

D. Filtered historical simulation

FHS is the hybrid approach, applying HS to returns filtered by AR(1)–GARCH(1, 1). We randomly draw filtered returns $\{z_1^{(b)}, \ldots, z_T^{(b)}\}$ from the empirical distribution $\{z_1, \ldots, z_T\}$ and simulate a sample path of (r_t) by the AR(1)–GARCH(1, 1) process. We iterate this procedure many times and construct the simulated distribution of r_T to get an unfiltered VaR.

E. Filtered extreme value theory

FEVT is suggested to control for time-varying volatility (Diebold, Schuermann, and Stroughair, 1998; McNeil and Frey, 2000). Similarly to FHS, it applies the EVT procedure to returns filtered by AR(1)–GARCH(1, 1). Here, we consider the FGEV distribution and the FGPD.

In the fixed time interval, consider collection of n filtered returns $\{z_1, \ldots, z_n\}$. Let $z_{(1)} = \min_{1 \leq j \leq n}\{z_j\}$ as the minimum return and $z_{(n)} = \max_{1 \leq j \leq n}\{r_j\}$ as the maximum return. Assume that (z_t) is the sequence of i.i.d random variables. Then the CDF of $z_{(1)}$ is given by

$$\mathrm{GEV}: \; F(z_{(1)}) = \begin{cases} \exp\left[-(1+\xi z)^{-1/\xi}\right] & \text{if } \xi \neq 0 \\ \exp\left[-\exp(z)^{-1/\xi}\right] & \text{if } \xi = 0 \end{cases} \tag{A.42}$$

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

$$\text{GPD}: \ F(z_{(1)}) = \begin{cases} 1-(1+\xi z)^{-1/\xi} & \text{for } \xi \neq 0 \\ 1-\exp(-z) & \text{for } \xi = 0, \end{cases} \tag{A.43}$$

where $z_{(1)} \geq 0$ for $\xi \geq 0$ and $0 \leq z_{(1)} \leq -1/\xi$ for $\xi < 0$.

F. Conditional autoregressive VaR by regression quantile

CAViaR and CAViAR–GARCH, which incorporates GARCH in CAViaR (Engle and Manganelli, 2004):

$$\text{CAViaR}: \ \text{VaR}_t(\alpha) = \beta_0 + \beta_1 q_{t-1}(\alpha) + \beta_2 |r_{t-1}|, \tag{A.44}$$

$$\text{CAViaR–GARCH}: \ \text{VaR}_t(\alpha) = \left(\beta_0 + \beta_1 q^2_{t-1}(\alpha) + \beta_2 r^2_{t-1}\right)^{1/2}. \tag{A.45}$$

G. Multivariate filtered historical simulation

mFHS is the hybrid approach, applying HS to multiple returns filtered by AR(1)–MGARCH(1, 1) (mFHS-MGARCH) or AR(1)-copula (mFHS-copula). We randomly draw multiple filtered returns $\{\mathbf{z}_1^{(b)}, \ldots, \mathbf{z}_T^{(s)}\}$, where $\mathbf{z}_t^{(b)} = (z_{1,t}^{(b)}, \ldots, z_{K,t}^{(b)})'$, from its empirical distribution $\{\mathbf{z}_1, \ldots, \mathbf{z}_T\}$ and simulate a sample path of $(r_{1,t}, \ldots, r_{k,t})'$ by the AR(1)–MGARCH(1, 1) or AR(1)-copula process. Then we get a simulated portfolio return, $r_{p,T+1}^{(b)} = \sum_{k=1}^K w_k r_{k,T+1}^{(b)}$. We iterate this procedure many times and construct the simulated distribution of portfolio returns to get an unfiltered VaR.

References

Adams, Z., R. Füss, and R. Gropp. 2014. Spillover Effects among Financial Institutions: A State-Dependent Sensitivity Value-at-Risk Approach. *Journal of Financial and Quantitative Analysis* 49: 575–598.

Ahoniemi, K., A.-M. Fuertes, and J. Olmo. 2016. Overnight News and Daily Equity Trading Risk Limits. *Journal of Financial Econometrics* 14: 525–551.

Andersen, T. G., and T. Bollerslev. 1997. Intraday Periodicity and Volatility Persistence in Financial Markets. *Journal of Empirical Finance* 4: 115–158.

Andersen, T. G., T. Bollerslev, F. X. Diebold, and H. Ebens. 2001. The Distribution of Realized Stock Return Volatility. *Journal of Financial Economics* 61: 43–76.

Andersen, T. G., T. Bollerslev, F. X. Diebold, and P. Labys. 2003. Modeling and Forecasting Realized Volatility. *Econometrica* 71: 579–625.

Aneiros-Perez, G., H. Cardot, G. Estevez-Perez, and P. Vieu. 2004. Maximum Ozone Concentration Forecasting by Functional Non-parametric Approaches. *Environmetrics* 15: 675–685.

Antoniadis, A., and T. Sapatinas. 2003. Wavelet Methods for Continuous-Time Prediction Using Hilbert-Valued Autoregressive Processes. *Journal of Multivariate Analysis* 87: 133–158.

Barndorff-Nielsen, O. E. 1997. Normal Inverse Gaussian Distribution and Stochastic Volatility Modelling. *Scandinavian Journal of Statistics* 24: 1–13.

Barone-Adesi, G., K. Giannopoulos, and L. Vosper. 2002. Backtesting Derivative Portfolios with Filtered Historical Simulation FHS. *European Financial Management* 8: 31–58.

Basel Committee on Banking and Supervision. 1996. *Amendment to the Base Capital Accord to Incorporate Market Risk*. Basel, Switzerland: Bank for International Settlements.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Basel Committee on Banking and Supervision. 2005. *The Application of Base II to Trading Activity and the Treatment of Double Default Effect*. Basel, Switzerland: Bank for International Settlements.

Berkowitz, J., P. Christoffersen, and D. Pelletier. 2011. Evaluating Value-at-Risk Models with Desk-Level Data. *Management Science* 57: 2213–2227.

Besse, P. C., H. Cardot, and D. B. Stephenson. 2000. Autoregressive Forecasting of Some Functional Climatic Variations. *Scandinavian Journal of Statistics* 27: 673–688.

Bollerslev, T. 1990. Modelling the Coherence in Short-Run Nominal Exchange Rates: A Multivariate Generalised ARCH Model. *Review of Economics and Statistics* 72: 498–505.

Bosq, D. 2000. *Linear Processes in Function Spaces*. New York: Springer.

Bowsher, C., and G. R. Meeks. 2008. The Dynamics of Economic Functions: Modeling and Forecasting the Yield Curve. *Journal of American Statistical Association* 103: 1419–1437.

Brogaard, J., T. Hendershott, and R. Riordan. 2014. High-Frequency Trading and Price Discovery. *Review of Financial Studies* 27: 2267–2306.

Cardot, H., A. Mas, and P. Sarda. 2007. CLT in Functional Linear Regression Models. *Probability Theory and Related Fields* 138: 325–361.

Cenesizoglu, T., and A. Timmermann. 2012. Do Return Prediction Models Add Economic Value? *Journal of Banking and Finance* 36: 2974–2987.

Chaudhuri, K., M. Kim, and Y. Shin. 2016. Forecasting the Distribution of Inflation Rates: A Functional Autoregressive Approach. *Journal of the Royal Statistical Society: Series A (Statistics in Society)* 179: 65–102.

Chiang, T. C., and J. Li. 2012. Stock Returns and Risk: Evidence from Quantile Regression Analysis. *Journal of Risk and Financial Management* 5: 20–58.

Christoffersen, P. F. 1998. Evaluating Interval Forecasts. *International Economic Review* 39: 841–862.

Clements, M. P., A. Galvïœo, and J. Kim. 2008. Quantile Forecasts of Daily Exchange Rate Returns from Forecasts of Realized Volatility. *Journal of Empirical Finance* 15: 729–750.

Conrad, J., R. F. Dittmar, and E. Ghysels. 2013. Ex Ante Skewness and Expected Stock Returns. *The Journal of Finance* 68: 85–124.

Creal, D., S. J. Koopman, and A. Lucas. 2013. Generalized Autoregressive Score Models with Applications. *Journal of Applied Econometrics* 28: 777–795.

Croft, J. 2011. Value at Risk: The Danger of Relying Too Much on Only One Tool. *Financial Times*: 21 March.

Damon, J., and S. Guillas. 2002. The Inclusion of Exogenous Variables in Functional Autoregressive Ozone Forecasting. *Environmetrics* 13: 759–774.

Diebold, F. X., T. Schuermann, and J. Stroughair. 1998. "Pitfalls and Opportunities in the Use of Extreme Value Theory in Risk Management." In A. P. N. Refenes, A. N. Burgess and J. D. Moody (eds.), *Advances in Computational Finance*. Amsterdam, The Netherlands: Kluwer Academic Publishers.

Elliot, G., and A. Timmermann. 2008. Economic Forecasting. *Journal of Economic Literature* 46: 3–56.

Engle, R. E. 2002. Dynamic Conditional Correlation—A Simple Calss of Multivariate GARCH Models. *Journal of Business & Economic Statistics* 20: 339–350.

Engle, R. E., and G. Giampiero. 2006. A Multiple Indicators Model for Volatility Using Intra-daily Data. *Journal of Econometrics* 131: 3–27.

Engle, R. E., and K. F. Kroner. 1995. Multivariate Simultaneous Generalised GARCH. *Econometric Theory* 11: 122–150.

Engle, R. F., and S. Manganelli. 2004. CaViaR: Conditional Autoregressive Value at Risk by Regression Quantiles. *Journal of Business & Economic Statistics* 22: 367–381.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Fuertes, A. M., and J. Olmo. 2013. Optimally Harnessing Inter-day and Intra-day Information for Daily Value-at-Risk Prediction. *International Journal of Forecasting* 29: 28–42.

Giot, P., and S. Laurent. 2004. Modelling Daily Value-at-Risk Using Realized Volatility and ARCH Type Models. *Journal of Empirical Finance* 11: 379–398.

Gurrola-Perez, P., and D. Murphy. 2015. *"Filtered Historical Simulation Value-at-Risk Models and Their Competitors."* Working paper No. 525, Bank of England.

Hall, P., S. N. Lahiri, and Y. K. Truong. 1995. On Bandwidth Choice for Density Estimation with Dependent Data. *The Annals of Statistics* 23: 2241–2263.

Hallam, M., and J. Olmo. 2014. Semiparametric Density Forecasts of Daily Financial Returns from Intraday Data. *Journal of Financial Econometrics* 12: 408–432.

Härdle, W. K., and O. Linton. 1994. "Applied Nonparametric Methods." In R. F. Engle and D. McFadden (eds.), *Handbook of Econometrics*. 1st edn, Vol. 4, Chapter 38, pp. 2295–339. Elsevier.

Harvey, C. R., and A. Siddique. 2000. Conditional Skewness in Asset Pricing Tests. *Journal of Finance* 55: 1263–1295.

Hasbrouck, J. 2007. *Empirical Market Microstructure: The Intuitions, Economics, and Econometrics of Securities Trading*. New York: Oxford University Press.

Hasbrouck, J., and G. Saarb. 2013. Low-Latency Trading. *Journal of Financial Markets* 16: 646–679.

Hvidkjaer, S. 2006. A Trade-Based Analysis of Momentum. *Review of Financial Studies* 19: 457–491.

Johansson, F., M. Seiler, and J. Michael. 1999. Measuring Downside Portfolio Risk. *The Journal of Portfolio Management* 26: 96–107.

Jorion, P. 2006. *Value-at-Risk: The New Benchmark for Managing Financial Risk*, 3rd edn. Chicago (IL): McGraw-Hill.

Kargin, V., and A. Onatski. 2008. Curve Forecasting by Functional Autoregression. *Journal of Multivariate Analysis* 99: 2508–2526.

Koenker, R., and G. Bassett. 1978. Regression Quantiles. *Econometrica* 46: 33–50.

Kuester, K., S. Mittnik, and M. S. Paolella. 2006. Value-at-Risk Prediction: A Comparison of Alternative Strategies. *Journal of Financial Econometrics* 4: 53–89.

Laukaitis, A. 2008. Functional Data Analysis for Cash Flow and Transactions Intensity Continuous-Time Prediction Using Hilbert-Valued Autoregressive Processes. *European Journal of Operational Research* 185: 1607–1614.

Lee, M. C., and M. J. Ready. 1991. Inferring Trade Direction from Intraday Data. *Journal of Finance* 46: 733–746.

Mas, A. 2007. Weak Convergence in the Functional Autoregressive Model. *Journal of Multivariate Analysis* 98: 1231–1261.

McAleer, M.,. J. Jimïœnez-Martïœn, and T. Pïœrez-Amaral. 2013. Has the Basel Accord Improved Risk Management during the Global Financial Crisis? *The North American Journal of Economics and Finance* 26: 250–265.

McNeil, A., and R. Frey. 2000. Estimation of Tail-Related Risk Measures for Heteroscedastic Financial Time Series: An Extreme Value Approach. *Journal of Empirical Finance* 7: 271–300.

McNeil, A., R. Frey, and P. Embrechts. 2005. *Quantitative Risk Management: Cocepts, Techniques and Tools*. New Jersey: Princeton University Press.

Netftci, S. 2000. Value at Risk Calculations, Extreme Events, and Tail Estimation. *Journal of Derivatives* 7: 23–38.

Park, J. Y., and J. Qian. 2012. Functional Regression of Continuous State Distributions. *Journal of Econometrics* 167: 397–412.

Perignon, C., Z. Y. Deng, and Z. J. Wang. 2008. Do Banks Overstate Their at-Risk? *Journal of Banking and Finance* 32: 783–794.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024

Pritsker, M. 2006. The Hidden Dangers of Historical Simulation. *Journal of Banking and Finance* 30: 561–582.

Ramsey, J. O., and B. W. Silverman. 1997. *Functional Data Analysis*. New York: Springer.

Rehmany, Z., and G. Vilkovz. 2012. *Risk-Neutral Skewness: Return Predictability and Its Sources*. Available at SSRN: https://ssrn.com/abstract=1301648.

RiskMetrics. 1996. *"RiskMetrics Technical Document,"* 4th edn. Discussion Paper, J. P. Morgan, New York.

Rossi, A., and A. Timmermann. 2010. *What Is the Shape of the Risk-Return Relation?* Available at SSRN: https://ssrn.com/abstract=1364750.

Silverman, B. W. 1986. *Density Estimation for Statistics and Data Analysis*. London, UK: Chapman and Hall.

Taylor, N. 2007. A Note on the Importance of Overnight Information in Risk Management Models. *Journal of Banking and Finance* 31: 161–180.

Wu, B. 1997. Kernel Density Estimation under Weak Dependence with Sampled Data. *Journal of Statistical Planning and Inference* 61: 141–154.

Zhang, F. 2010. *High-Frequency Trading, Stock Volatility, and Price Discovery*. Available at SSRN: https://ssrn.com/abstract=1691679.

Downloaded from https://academic.oup.com/jfec/article/17/2/284/5225005 by Erasmus University Rotterdam user on 21 October 2024