An Introduction to Singular Value Decomposition

Principle and Applications

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Review of basics

Review - Eigenvalues

If A is a symmetric real $n \times n$ matrix, x is a n-dimensional vector, λ is a real number, and

$$Ax = \lambda x$$

, then we have:

- x is one of eigenvectors of matrix A
- λ is one of eigenvalues of matrix A

The eigenvectors and eigenvalues are used in Matrix Decomposition, which can be expressed as follows:

$$A = W \Sigma W^{-1}$$

where W and Σ consist of eigenvectors and eigenvalues.

Review - Eigenvalues

How can we utilize the eigenvalues and eigenvectors?

It's fundamental of many theory. Include but not limit to:

- Matrix diagonalize
- PCA dimension reduction
- Solution to Markov process

Here we dive into its function in **SVD (Singular Value Decomposition)**.

Singular Value Decomposition

SVD - Definition

If A is $m \times n$ matrix, the SVD of matrix A is:

$$A = U\Sigma V^T$$

where $U^T U = I$, $V^T V = I$.

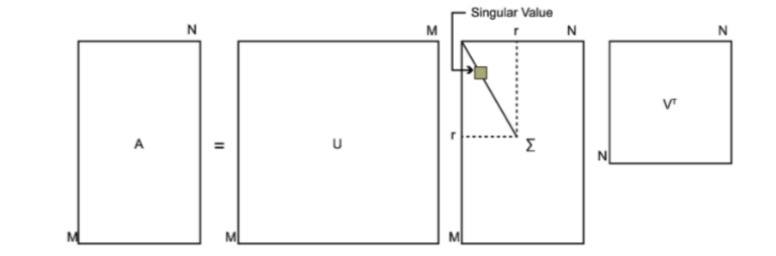

Figure 1: Definition of SVD

SVD - Solution

How to obtain the value of U, V, Σ ?

With the help of eigenvalues and eigenvectors, it is easy for us to construct matrixs $A^T A$ and AA^T to get the solution.

Examples

Examples of SVD

If

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$$

We can construct matrixs $A^T A$ and AA^T by

$$\mathbf{A^T A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and

$$\mathbf{AA^T} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

then we find the eigenvectors and eigenvalues of $\mathbf{A^T A}$:

$$\lambda_1 = 3; v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}; \lambda_2 = 1; v_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

Examples of SVD

the eigenvectors and eigenvalues of $\mathbf{AA^T}$:

$$\lambda_1 = 3; u_1 = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}; \lambda_2 = 1; u_2 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}; \lambda_3 = 0; u_3 = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

Then we can find the singular values using the equation $\sigma_i = \sqrt{\lambda_i}$, so the singular values are expressed by:

$$\sigma_1 = \sqrt{3}, \sigma_2 = 1$$

As a result, **Singular Value Decomposition** of matrix A can be summarized as:

$$U\Sigma V^T = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

Applications

Applications - Important feature extraction

We first to arrange the singular values in descending order, then extract the top-k the singular values:

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}V^T_{n\times n} \approx U_{m\times k}\Sigma_{k\times k}V^T_{k\times n}$$

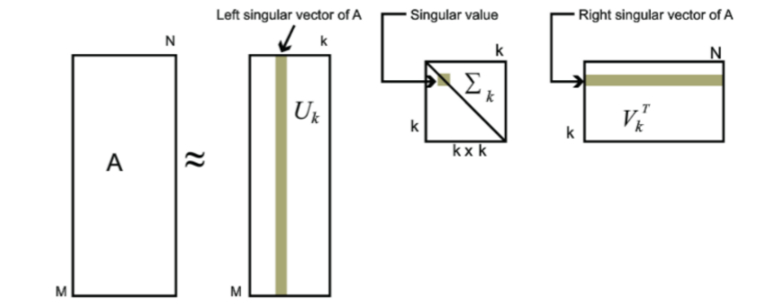

Figure 2: Applications of SVD

Applications - Important feature extraction

One of application scenario is:

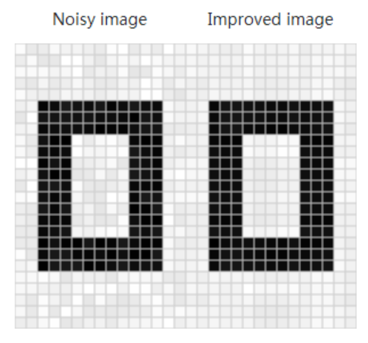

Figure 3: Important feature extraction by SVD

Applications - More

- Data compression
- Noise reduction (or feature extraction)
- PCA dimension reduction

Questions?

References

References:

- PCA - Tutorial
- We Recommend a Singular Value Decomposition
- Cnblog - tutorial of SVD