Verifying Quantum Error Correction Codes with SAT Solvers

Finding Bugs in Surface Code Implementations

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Outline

Why Quantum Error Correction is Different

Classical Error Correction:

- Errors mainly occur during transmission/storage
- Can copy data freely for redundancy
- Can measure directly without disturbing data

Quantum Error Correction:

- **Errors occur continuously during computation**
- No-cloning theorem: cannot copy quantum states
- Measurement inevitably destroys quantum states

Detectors: Measuring Errors Indirectly

- Through measurements, we convert quantum states into classical bits.
- A linear combination of measurements that is deterministic under no error is a detector.
- The decoder, which is a classical algorithm, rely on the detectors to infer the error pattern.

Key Idea

Different error patterns $\rightarrow$ different detectors $\rightarrow$ error correction possible

Key question: How do we evaluate whether a quantum error correction code is good?

- What is the maximum number of correctable errors?

The Distance d of a Code

A code with distance d can correct up to $\lfloor \frac{d-1}{2} \rfloor$ errors.

- Distance 3 code: corrects 1 error
- Distance 5 code: corrects 2 errors

Our Approach: Assuming Optimal Decoder

What does optimal mean?

- Decoder uses detector detectors to infer which errors occurred
- **Optimal decoder:** Always chooses the minimum-weight error pattern consistent with the detector

For linear codes:

- If distance is d, then any two error patterns with $< d$ errors produce different detectors OR same logical outcome
- **Simplified verification:** Just check if errors with *zero detector* (no detectors fired) can corrupt the logical qubit!

Why Zero-detector Check is Sufficient

General problem: Two different error patterns E_1 and E_2 might produce the same detector but different logical outcomes

- This would fool the decoder!
- Checking all pairs would be exponential

Our simplification (for linear codes):

- Due to linearity: $E_1 \oplus E_2$ produces zero detector
- If E_1 and E_2 differ in logical outcome, then $E_1 \oplus E_2$ is a zero-detector logical error
- **So we only need to find zero-detector errors that flip the logical!**

Result

Distance d = minimum weight of zero-detector logical error $\Rightarrow$ can correct $\lfloor \frac{d-1}{2} \rfloor$ errors

SAT Encoding: The Big Picture

Goal: Encode quantum error correction verification as a SAT problem

What we need to encode:

1. Each possible error mechanism: boolean variable e_i
2. Detector constraints: each detector measures XOR of certain errors
3. Observable constraints: logical qubits are corrupted by XOR of errors
4. Cardinality constraint: at most k errors occur

Challenge: SAT solvers work with AND/OR/NOT, but quantum error correction uses XOR extensively!

The Input: Detector Error Model (DEM)

Input format: Stim's DEM file describes error mechanisms

Example DEM Entries

```
error D0 D2 L0
error D1 D3
error D0 D1
```

Interpretation:

- Each `error` line is one error mechanism
- `D#`: This error triggers detector #
- `L#`: This error flips logical observable #

Encoding Step 1: Boolean Variables

Create one boolean variable per error mechanism

- Parse DEM file to count n error mechanisms
- Create variables: $e_1, e_2, \ldots, e_n$
- $e_i =$ True means error i occurs
- $e_i =$ False means error i does not occur

Example

If DEM has 100 error lines, we create variables $e_1, \ldots, e_{100}$

Encoding Step 2: XOR Constraints (The Hard Part)

Problem: Detectors compute XOR, but SAT uses AND/OR/NOT

Example: Detector D0 fires iff $e_1 \oplus e_3 \oplus e_7 = 1$

For verification, we want detectors to NOT fire:

$$e_1 \oplus e_3 \oplus e_7 = 0$$

Solution: Tseitin Transformation

- Introduce auxiliary variables
- Convert XOR into CNF clauses using helper variables
- Two methods: *chain* and *tree* encoding

XOR Encoding: Chain Method

Encode $e_1 \oplus e_2 \oplus e_3 \oplus e_4 = 0$

Chain approach:

1. Create auxiliary variable $a_1 = e_1 \oplus e_2$
2. Create auxiliary variable $a_2 = a_1 \oplus e_3$
3. Create auxiliary variable $a_3 = a_2 \oplus e_4$
4. Assert $a_3 = 0$

Binary XOR encoding: $c = a \oplus b$ becomes 4 CNF clauses:

- $\neg a \vee \neg b \vee \neg c$
- $a \vee b \vee \neg c$
- $a \vee \neg b \vee c$
- $\neg a \vee b \vee c$

XOR Encoding: Tree Method

Encode $e_1 \oplus e_2 \oplus e_3 \oplus e_4 = 0$

Tree approach (more parallel):

1. Level 1: $a_1 = e_1 \oplus e_2, \quad a_2 = e_3 \oplus e_4$
2. Level 2: $a_3 = a_1 \oplus a_2$
3. Assert $a_3 = 0$

Trade-off

Chain: Linear depth, can be slow for SAT propagation
Tree: Logarithmic depth, better propagation, more variables

Encoding Step 3: Cardinality Constraints

Constraint: At most k errors occur

$$\sum_{i=1}^{n} e_i \leq k$$

Naive encoding: Forbid all $\binom{n}{k+1}$ combinations $\rightarrow$ exponential!

Totalizer encoding: Efficient polynomial-size encoding

- Builds a circuit that counts the number of true variables
- Uses auxiliary variables to represent partial sums
- Results in $O(nk)$ clauses and variables
- Provided by PySAT

Encoding Step 4: Observable Constraints

Goal: Find errors that corrupt the logical qubit

Constraint: At least one logical observable is flipped

1. For each observable L_j, encode: $r_j = \bigoplus_i e_i$ where error i affects L_j
2. Assert: $r_1 \vee r_2 \vee \cdots \vee r_m$ (at least one observable flipped)

Verification Strategy

Two types of problems:

Can the code correct k errors? (UNSAT Problem)

- Try to find a counterexample with $\leq k$ errors that cannot be corrected
- If UNSAT, the code can correct k errors

Can the code fail with k errors? (SAT Problem)

- Try to find an example with $\leq k$ errors that leads to failure
- If SAT, the code cannot correct all k-error cases

Bug Discovery in Nature Paper

Major Achievement: We successfully identified and verified a bug in a recently published Nature paper [1]!

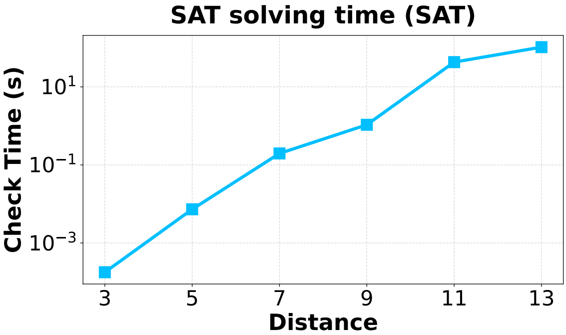
Distance	Actual Correctable Errors	Claimed
3	0	0
5	1	1
7	2	2
9	3	3
11	**3**	4
13	**4**	5

The code fails to correct the claimed number of errors for distances 11 and 13!

Performance Results

Bug Detection: Pretty Fast! ✓

- SAT solver quickly finds counterexamples
- Verification of bug completed in reasonable time
- Demonstrates effectiveness of SAT-based approach



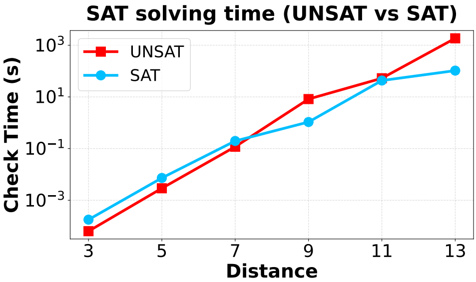
The Verification Challenge

The Problem

We can propose a fix for the bug, but we **cannot verify** whether it works using SAT solvers alone.

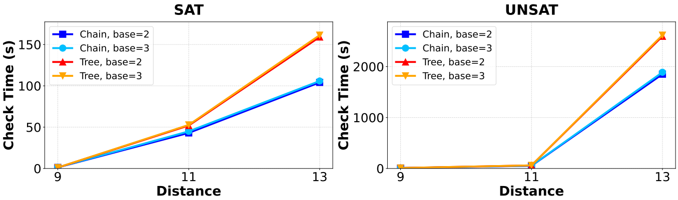
Verification is very slow...

- Proving correctness requires solving UNSAT problems
- Much harder than finding bugs (SAT problems)



Comparison of XOR Encoding Strategies

1. **Chain Encoding**: sequentially combine variables using XOR
2. **Tree Encoding**: parallel, balanced reduction of XOR terms
3. **Base-2 Encoding**: use $c = a \oplus b$ as the atomic building block
   $\Rightarrow$ introduces more auxiliary variables but simpler constraints
4. **Base-3 Encoding**: use $d = a \oplus b \oplus c$ as the atomic block
   $\Rightarrow$ reduces auxiliary variables but increases clause complexity



Why is This So Hard?

Combination of SAT solver weaknesses:

1. **UNSAT problems**
   - Proving non-existence is inherently harder than finding examples
2. **XOR encodings**
   - SAT solvers struggle with parity constraints [3]
3. **Cardinality constraints**
   - "At most k errors", is similar to Pigeonhole principle, which is challenging [2]

Each alone is challenging; together they are formidable!

A Hybrid Approach

Leverage the strengths of different tools:

SAT Solvers: Fast Pruning

- Quickly find bugs and counterexamples
- Prune the search space efficiently
- Identify promising candidates

Lean Theorem Prover: Formal Verification

- Formally verify correctness of proposed fixes
- Provide mathematical proof of error correction properties
- Guarantee correctness where SAT solvers struggle

Summary

- **Problem:** Verifying quantum error correction codes
- **Approach:** SAT solver with specialized encodings
- **Success:** Found bugs in published Nature paper
- **Challenge:** Verifying fixes is hard
- **Future:** Hybrid SAT + Lean approach

Thank you!

Questions?

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