Orthonormal Columns in Q Give $Q^TQ = I$

$\Leftarrow$ identity matrix

符号Q一般用于表述一种矩阵，即满足性质$Q^TQ=I$.

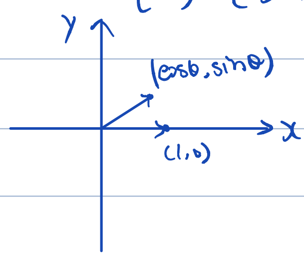
Q中的列向量 q_i，有 $q_i^Tq_i=1$，且 $q_i^Tq_j=0$，$i\neq j$

① rotation matrix

Square: $Q=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$，Orthonormal matrix — 一般是一个旋转矩阵.

It doesn't change length.

比如 $Q\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$，旋转之后，向量的模不改变，仅是转动了一定的角度.



证明:

已知$Q^TQ=I$，对于任意的向量x，

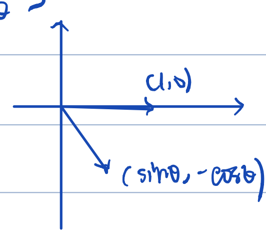
$$\|Qx\|^2 = (Qx)^T(Qx) = x^TQ^TQx = x^Tx = \|x\|^2$$

$$\therefore \|Qx\|=\|x\|$$

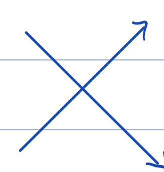
② reflection matrix

$Q=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$，这是一个对称矩阵

$Q\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\sin\theta\\ -\cos\theta\end{bmatrix}$



而原本的坐标系会变为 $\longrightarrow$



[Householder reflections] → 一类 reflection matrix

① start with $u^Tu=1$

② $H = I - 2uu^T$

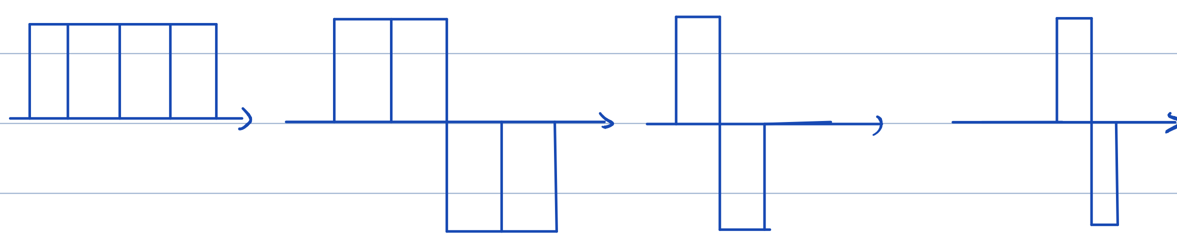
CHECK $H^TH = I$

$$\Rightarrow (I-2uu^T)^T(I-2uu^T)$$

$$= I - 4uu^T + 4uu^Tuu^T$$

$$= I$$

Wavelets matrix



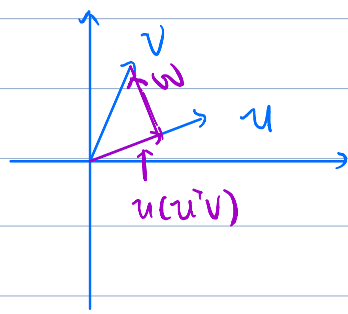
$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}$$ ← Haar wavelets

$$W_8 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

The eigenvectors of symmetric matrix are orthogonal.

Problem for Lecture 3

2. Draw unit vectors u and v that are not orthogonal. Show that $w = v - u(u^T v)$ is orthogonal to u (and add w to your picture).



① u and v are unit vectors, and they are not orthogonal.
$\|u\| = \|v\| = 1$

② cause u and v are unit vectors, $u(u^T v)$ is the projection from v to u.

③ so $w = v - u(u^T v)$ is orthogonal to u.

4. Key property of every orthogonal matrix: $\|Qx\|^2 = \|x\|^2$ for every vector x. More than this, show that $(Qx)^T(Qy) = x^T y$ for every vector x and y. So lengths and angles are not changed by Q. Computations with Q never overflow!

6. A permutation matrix has the same columns as the identity matrix (in some order). Explain why this permutation matrix and every permutation matrix is orthogonal:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$ has orthonormal columns so $P^T P = I$

and $P^{-1} = P^T$.

When a matrix is symmetric or orthogonal, it will have orthogonal eigenvectors. This is the most important source of orthogonal vectors in applied mathematics.