I.1 Multiplication $A\boldsymbol{x}$ Using Columns of A

We hope you already know some linear algebra. It is a beautiful subject—more useful to more people than calculus (in our quiet opinion). But even old-style linear algebra courses miss basic and important facts. This first section of the book is about *matrix-vector* multiplication $A\boldsymbol{x}$ and the column space of a matrix and the rank.

We always use examples to make our point clear.

Example 1 Multiply A times $\boldsymbol{x}$ using the three rows of A. Then use the two columns:

这种方式可利于计算,但不便理解 ←

By rows
$$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ 2x_1 + 4x_2 \\ 3x_1 + 7x_2 \end{bmatrix} = \begin{matrix} \text{inner products} \\ \text{of the rows} \\ \text{with } \boldsymbol{x} = (x_1, x_2) \end{matrix}$$

将矩阵A的行向量与x作点乘

By columns
$$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} = \begin{matrix} \text{combination} \\ \text{of the columns} \\ \boldsymbol{a}_1 \text{ and } \boldsymbol{a}_2 \end{matrix}$$

矩阵A的列向量的线性组合

You see that both ways give the same result. The first way (a row at a time) produces three inner products. Those are also known as "dot products" because of the dot notation:

$$\textbf{row} \cdot \textbf{column} = (\mathbf{2},\ \mathbf{3}) \cdot (\boldsymbol{x}_1,\ \boldsymbol{x}_2) = \mathbf{2}\boldsymbol{x}_1 + \mathbf{3}\boldsymbol{x}_2 \qquad (1)$$

This is the way to find the three separate components of $A\boldsymbol{x}$. We use this for computing—but not for understanding. It is low level. Understanding is higher level, using vectors.

The vector approach sees $A\boldsymbol{x}$ as a "linear combination" of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. This is the fundamental operation of linear algebra! A linear combination of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ includes two steps:

(1) Multiply the columns $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ by "scalars" x_1 and x_2

(2) Add vectors $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 = A\boldsymbol{x}$.

Thus $A\boldsymbol{x}$ is a linear combination of the columns of A. This is fundamental. ←矩阵乘以向量的意义

column space of A →

This thinking leads us to the **column space** of A. The key idea is to take **all combinations** of the columns. All real numbers x_1 and x_2 are allowed—the space includes $A\boldsymbol{x}$ for all vectors $\boldsymbol{x}$. In this way we get infinitely many output vectors $A\boldsymbol{x}$. And we can see those outputs geometrically.

a. 什么是矩阵A的column space?

记为C(A),它是矩阵A "独立"列向量的线性组合.

In our example, each $A\boldsymbol{x}$ is a vector in 3-dimensional space. That 3D space is called $\mathbf{R}^3$. (The $\mathbf{R}$ indicates real numbers. Vectors with three complex components lie in the space $\mathbf{C}^3$.) We stay with real vectors and we ask this key question:

All combinations $A\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2$ produce what part of the full 3D space?

Answer: Those vectors produce a **plane**. The plane contains the complete line in the direction of $\boldsymbol{a}_1 = (2, 2, 3)$, since every vector $x_1\boldsymbol{a}_1$ is included. The plane also includes the line of all vectors $x_2\boldsymbol{a}_2$ in the direction of $\boldsymbol{a}_2$. And it includes the *sum* of any vector on one line plus any vector on the other line. **This addition fills out an infinite plane containing the two lines**. But it does not fill out the whole 3-dimensional space $\mathbf{R}^3$.

b. C(A)是什么?是一个3维空间吗?

C(A)构成3D中的一个平面.

Definition **The combinations of the columns** fill out the **column space** of A.

Here the column space is a plane. That plane includes the zero point $(0,0,0)$ which is produced when $x_1 = x_2 = 0$. The plane includes $(5,6,10) = \boldsymbol{a}_1 + \boldsymbol{a}_2$ and $(-1,-2,-4) = \boldsymbol{a}_1 - \boldsymbol{a}_2$. Every combination $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2$ is in this column space. With probability 1 it does **not** include the random point **rand**$(3,1)$! Which points are in the plane ?

如何判断1个向量是否在A的列空间中?

$$\boxed{\boldsymbol{b} = (\boldsymbol{b_1}, \boldsymbol{b_2}, \boldsymbol{b_3}) \text{ is in the column space of } \boldsymbol{A} \textbf{ exactly when } \boldsymbol{Ax} = \boldsymbol{b} \text{ has a solution } (\boldsymbol{x_1}, \boldsymbol{x_2})}$$

When you see that truth, you understand the column space $\mathbf{C}(A)$: The solution $\boldsymbol{x}$ shows how to express the right side $\boldsymbol{b}$ as a combination $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2$ of the columns. For some $\boldsymbol{b}$ this is impossible—they are not in the column space.

若 $Ax=b$ 无解 则 b 不在 $C(A)$ 中

Example 2 $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not in $\mathbf{C}(A)$. $A\boldsymbol{x} = \begin{bmatrix} 2x_1 + 3x_2 \\ 2x_1 + 4x_2 \\ 3x_1 + 7x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is unsolvable.

The first two equations force $x_1 = \frac{1}{2}$ and $x_2 = 0$. Then equation 3 fails : $3\left(\frac{1}{2}\right) + 7(0) = \mathbf{1.5}$ (**not 1**). This means that $\boldsymbol{b} = (1,1,1)$ is not in the column space—the plane of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$.

$C(A_2) = x_1 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$

Example 3 What are the column spaces of $A_2 = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 7 & 10 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 1 \end{bmatrix}$?

A_2 的第3列也可由第1,2列表示

Solution. The column space of A_2 is the same plane as before. The new column $(5,6,10)$ is the sum of column 1 + column 2. So $\boldsymbol{a}_3$ = column 3 is already in the plane and adds nothing new. By including this "*dependent*" column we don't go beyond the original plane.

$C(A_3)$ 构成了整个 R^3.

The column space of A_3 is the whole 3D space $\mathbf{R}^3$. Example 2 showed us that the new third column $(1,1,1)$ is not in the plane $\mathbf{C}(A)$. Our column space $\mathbf{C}(A_3)$ has grown bigger. But there is nowhere to stop between a plane and the full 3D space. Visualize the $x-y$ plane and a third vector (x_3, y_3, z_3) out of the plane (meaning that $z_3 \neq 0$). They combine to give **every vector in R**^{3}.

Here is a total list of all possible column spaces inside $\mathbf{R}^3$. Dimensions $0,1,2,3$:

R^3 的子空间.

Subspaces of R^3
- The **zero vector** $(0,0,0)$ by itself
- A **line** of all vectors $x_1\boldsymbol{a}_1$
- A **plane** of all vectors $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2$
- The **whole R**^{3} with all vectors $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3$

什么是向量的 "independent"

In that list we need the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ to be "**independent**". The only combination that gives the zero vector is $0\boldsymbol{a}_1 + 0\boldsymbol{a}_2 + 0\boldsymbol{a}_3$. So $\boldsymbol{a}_1$ by itself gives a line, $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ give a plane, $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ give every vector $\boldsymbol{b}$ in $\mathbf{R}^3$. The zero vector is in every subspace ! In linear algebra language :

- Three independent columns in $\mathbf{R}^3$ produce an **invertible matrix :** $AA^{-1} = A^{-1}A = I$.
- $A\boldsymbol{x} = \mathbf{0}$ requires $\boldsymbol{x} = (0,0,0)$. Then $A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution $\boldsymbol{x} = \boldsymbol{A^{-1}b}$.

如果 3x3矩阵A 由3个"独立"的3维向量组成, 则 A是可逆的, 并且定义 $AA^{-1} = A^{-1}A = I$;

You see the picture for the columns of an n by n invertible matrix. Their combinations fill its column space : **all of R**^{n}. We needed those ideas and that language to go further.

若 $Ax=0$ 仅有零解, 则 A可逆, $Ax=b$ 仅有唯一解.

Independent Columns and the Rank of A

什么是 basis of $C(A)$?

After writing those words, I thought this short section was complete. *Wrong*. With just a small effort, we can find a **basis** for the column space of A, we can **factor** A into C times R, and we can prove the **first great theorem** in linear algebra. You will see the rank of a matrix and the dimension of a subspace.

All this comes with an understanding of **independence**. The goal is to create a matrix C whose columns come directly from A—but not to include any column that is a combination of previous columns. The columns of C (as many as possible) will be "independent". Here is a natural construction of C from the n columns of A:

$A = C \times R$

C 如何构建

If column 1 of A is not all zero, put it into the matrix C.

If column 2 of A is not a multiple of column 1, put it into C.

If column 3 of A is not a combination of columns 1 and 2, put it into C. *Continue*.

At the end C will have r columns ($\boldsymbol{r} \leq \boldsymbol{n}$).

$C \rightarrow$ They will be a "basis" for the column space of A.

The left out columns are combinations of those basic columns in C.

A **basis** for a subspace is a full set of independent vectors: **All vectors in the space are combinations of the basis vectors**. Examples will make the point.

Example 4 If $A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix}$ then $C = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\quad n = 3$ columns in A, $\boldsymbol{r} = \mathbf{2}$ columns in C

Column 3 of A is 2 (column 1) + 2 (column 2). Leave it out of the basis in C.

Example 5 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ then $C = A$. $\quad n = 3$ columns in A, $\boldsymbol{r} = \mathbf{3}$ columns in C

This matrix A is invertible. Its column space is all of $\mathbf{R}^3$. Keep all 3 columns.

Example 6 If $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix}$ then $C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad n = 3$ columns in A, $\boldsymbol{r} = \mathbf{1}$ column in C

rank(A) 是矩阵C列向量的个数

The number $\boldsymbol{r}$ is the **"rank"** of A. It is also the rank of C. **It counts independent columns.** Admittedly we could have moved from right to left in A, starting with its *last* column. This would not change the final count r. *Different basis, but always the same number of vectors*. That number r is the **"dimension"** of the column space of A and C (same space).

The rank of a matrix is the dimension of its column space.

The matrix C connects to A by a third matrix R: $\boldsymbol{A} = \boldsymbol{CR}$. Their shapes are $(m \text{ by } n) = (m \text{ by } r)\,(r \text{ by } n)$. I can show this "factorization of A" in Example 4 above:

$$\boxed{\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \boldsymbol{CR}} \tag{2}$$

When C multiplies the first column $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ of R, this produces column 1 of C and A.

When C multiplies the second column $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ of R, we get column 2 of C and A.

When C multiplies the third column $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ of R, we get $2(\text{column } 1) + 2(\text{column } 2)$.

This matches column 3 of A. All we are doing is to put the right numbers in R. Combinations of the columns of C produce the columns of A. Then $A = CR$ stores this information as a matrix multiplication. Actually R is a famous matrix in linear algebra:

$$\boxed{\boldsymbol{R} = \textbf{rref}(\boldsymbol{A}) = \textbf{row-reduced echelon form of } \boldsymbol{A} \text{ (without zero rows).}}$$

Example 5 has $C = A$ and then $R = I$ (identity matrix). Example 6 has only *one* column in C, so it has one row in R:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \end{bmatrix} = CR$$

All three matrices have rank $r = 1$
Column Rank = Row Rank

"独立"的列向量个数
=
"独立"的行向量个数

$$\boxed{\text{The number of } \textit{independent columns} \text{ equals the number of } \textit{independent rows}}$$

This rank theorem is true for every matrix. Always columns and rows in linear algebra! The m rows contain the same numbers a_{ij} as the n columns. But different vectors.

The theorem is proved by $A = CR$. Look at that differently—by rows instead of columns. The matrix R has r rows. **Multiplying by C takes combinations of those rows.** Since $A = CR$, we get every row of A from the r rows of R. And those r rows are independent, so they are a **basis for the row space of $\boldsymbol{A}$**. The column space and row space of A both have dimension r, with r basis vectors—columns of C and rows of R.

C是一个 basis of C(A).
R是一个 basis of C(A^T)

One minute: Why does R have independent rows? Look again at Example 4.

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{array}{l} \leftarrow \text{independent} \\ \leftarrow \text{rows of } R \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \end{array} \\ \text{ones and zeros} \end{array}$$

It is those ones and zeros in R that tell me: No row is a combination of the other rows.

The big factorization for data science is the "SVD" of A—when the first factor C has r *orthogonal* columns and the second factor R has r *orthogonal* rows.

之后会提到
SVD

1. $1\times\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + 1\times\begin{bmatrix}-1\\-1\\0\\1\end{bmatrix} + 1\times\begin{bmatrix}0\\0\\-1\\-2\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \Leftrightarrow \begin{bmatrix}1&-1&0\\1&-1&0\\1&0&-1\\1&1&-2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

The shape of A is 4×3. And the shape of x and 0 is 4×1.

2. $\because Ax = Ay$
$\therefore A(x-y) = 0$
and $A0 = 0$

Problem Set I.1

1 Give an example where a combination of three nonzero vectors in $\mathbf{R}^4$ is the zero vector. Then write your example in the form $A\boldsymbol{x} = \mathbf{0}$. What are the shapes of A and $\boldsymbol{x}$ and $\mathbf{0}$?

3. (1)
$\begin{bmatrix}a_1 & a_2 & a_3 \cdots a_n\end{bmatrix}\begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}$

2 Suppose a combination of the columns of A equals a different combination of those columns. Write that as $A\boldsymbol{x} = A\boldsymbol{y}$. Find two combinations of the columns of A that equal the zero vector (in matrix language, find two solutions to $A\boldsymbol{z} = \mathbf{0}$).

(2) $\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}\begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix} = \begin{bmatrix}c_1a_{11} + c_2a_{12} + \cdots + c_na_{1n}\\ \vdots\\ c_1a_{m1} + c_2a_{m2} + \cdots + c_na_{mn}\end{bmatrix}$

3 (Practice with subscripts) The vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ are in m-dimensional space $\mathbf{R}^m$, and a combination $c_1\boldsymbol{a}_1 + \cdots + c_n\boldsymbol{a}_n$ is the zero vector. That statement is at the vector level.

(1) Write that statement at the matrix level. Use the matrix A with the $\boldsymbol{a}$'s in its columns and use the column vector $\boldsymbol{c} = (c_1, \ldots, c_n)$.

(2) Write that statement at the scalar level. Use subscripts and sigma notation to add up numbers. The column vector $\boldsymbol{a}_j$ has components $a_{1j}, a_{2j}, \ldots, a_{mj}$.

4. $A = \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$

$x = \begin{bmatrix}2\\-1\\-1\end{bmatrix}$ $y = \begin{bmatrix}1\\-2\\1\end{bmatrix}$

There only exists two independent vector because other vectors can be the combination of x and y.

4 Suppose A is the 3 by 3 matrix **ones**$(3,3)$ of all ones. Find two independent vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ that solve $A\boldsymbol{x} = \mathbf{0}$ and $A\boldsymbol{y} = \mathbf{0}$. Write that first equation $A\boldsymbol{x} = \mathbf{0}$ (with numbers) as a combination of the columns of A. Why don't I ask for a third independent vector with $A\boldsymbol{z} = \mathbf{0}$?

5 The linear combinations of $\boldsymbol{v} = (1,1,0)$ and $\boldsymbol{w} = (0,1,1)$ fill a plane in $\mathbf{R}^3$.

(a) Find a vector $\boldsymbol{z}$ that is perpendicular (→垂直) to $\boldsymbol{v}$ and $\boldsymbol{w}$. Then $\boldsymbol{z}$ is perpendicular to every vector $c\boldsymbol{v} + d\boldsymbol{w}$ on the plane: $(c\boldsymbol{v} + d\boldsymbol{w})^{\mathrm{T}}z = c\boldsymbol{v}^{\mathrm{T}}\boldsymbol{z} + d\boldsymbol{w}^{\mathrm{T}}z = 0 + 0$.

(b) Find a vector $\boldsymbol{u}$ that is not on the plane. Check that $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{z} \neq 0$.

5. (a) $v^Tz = 0$, $w^Tz = 0$ $\begin{bmatrix}v^T\\w^T\end{bmatrix}z = \begin{bmatrix}0\\0\end{bmatrix}$

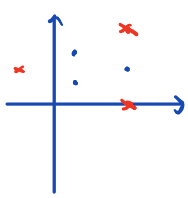
$\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}z = \begin{bmatrix}0\\0\end{bmatrix}$

z can be $\begin{bmatrix}1\\-2\\1\end{bmatrix}$.

(b) if u is not on the plane. So it isn't the combination of v and w. So $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is a choice.

6 If three corners of a parallelogram are $(1,1)$, $(4,2)$, and $(1,3)$, what are all three of the possible fourth corners? Draw two of them.

6.



7 Describe the column space of $A = [\boldsymbol{v} \;\; \boldsymbol{w} \;\; \boldsymbol{v} + 2\boldsymbol{w}]$. Describe the nullspace of A: all vectors $\boldsymbol{x} = (x_1, x_2, x_3)$ that solve $A\boldsymbol{x} = \mathbf{0}$. Add the "dimensions" of that plane (the column space of A) and that line (the nullspace of A):

dimension of column space + dimension of nullspace = number of columns

7. one basis of $C(A)$ is $[v \; w]$. and $Null(A)$'s basis is $\begin{bmatrix}-1\\-2\\1\end{bmatrix}$.

The dimesion of $C(A)$ is 2, and the dimesion of $Null(A)$ is 1. so $2+1=3$.

8 $\boldsymbol{A} = \boldsymbol{CR}$ is a representation of the columns of A in the basis formed by the columns of C with coefficients in R. If $A_{ij} = j^2$ is 3 by 3, write down A and C and R.

9 Suppose the column space of an m by n matrix is all of $\mathbf{R}^3$. What can you say about m? What can you say about n? What can you say about the rank r?

8. $A = \begin{bmatrix}1&1&1\\4&4&4\\9&9&9\end{bmatrix}$, so $A = CR = \begin{bmatrix}1\\4\\9\end{bmatrix}\begin{bmatrix}1&1&1\end{bmatrix}$

9. $m = n = r = 3$

10. $C_1 = \begin{bmatrix}1\\3\\2\end{bmatrix}$ $C_2 = \begin{bmatrix}1&2\\4&5\\7&8\end{bmatrix}$

11. $A_1 = C_1 \cdot R_1 = \begin{bmatrix}1\\3\\2\end{bmatrix}\begin{bmatrix}1&2&-2\end{bmatrix}$ $A_2 = \begin{bmatrix}1&2\\4&5\\7&8\end{bmatrix}\begin{bmatrix}1&0&-1\\0&1&2\end{bmatrix}$

10 Find the matrices C_1 and C_2 containing independent columns of A_1 and A_2:

$$A_1 = \begin{bmatrix}1&3&-2\\3&9&-6\\2&6&-4\end{bmatrix} \qquad A_2 = \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$$

11 Factor each of those matrices into $A = CR$. The matrix R will contain the numbers that multiply columns of C to recover columns of A.

This is one way to look at matrix multiplication: **$\boldsymbol{C}$ times each column of $\boldsymbol{R}$.**

12. The basis of $C(A_1)$ is C_1, the shape is 3×1;
The basis of $C(A_2)$ is C_2, the shape is 3×2.

12 Produce a basis for the column spaces of A_1 and A_2. What are the *dimensions* of those column spaces—the number of independent vectors? What are the *ranks* of A_1 and A_2? How many independent rows in A_1 and A_2?

13 Create a 4 by 4 matrix A of rank 2. What shapes are C and R?

13. The shape of C is 4×2, and the shape of R is 2×4.

14 Suppose two matrices A and B have the same column space.

(a) Show that their row spaces can be different.

(b) Show that the matrices C (basic columns) can be different.

(c) What number will be the same for A and B? α

$\overset{A}{[\,\text{—}\,]} = \overset{C}{[\,\uparrow\,]}\overset{R}{[\,=\,]}$

15 If $A = CR$, the first row of A is a combination of the rows of R. Which part of which matrix holds the coefficients in that combination—the numbers that multiply the rows of R to produce row 1 of A? → Apparently, it's the first row of C that contains the coefficients.

The rows of R are independent. And the combination of these rows is the row space of A.

16 The rows of R are a basis for the row space of A. *What does that sentence mean?*

17 For these matrices with square blocks, find $A = CR$. What ranks?

$$A_1 = \begin{bmatrix}\text{zeros}&\text{ones}\\\text{ones}&\text{ones}\end{bmatrix}_{\mathbf{4\times4}} \qquad A_2 = \begin{bmatrix}A_1\\A_1\end{bmatrix}_{\mathbf{8\times4}} \qquad A_3 = \begin{bmatrix}A_1&A_1\\A_1&A_1\end{bmatrix}_{\mathbf{8\times8}}$$

18. $\begin{bmatrix}0&A\\0&A\end{bmatrix} = \begin{bmatrix}C\\C\end{bmatrix}\begin{bmatrix}0&R\end{bmatrix}$

18 If $A = CR$, what are the CR factors of the matrix $\begin{bmatrix}0&A\\0&A\end{bmatrix}$?

19 "Elimination" subtracts a number ℓ_{ij} times row j from row i: a "row operation." Show how those steps can reduce the matrix A in Example 4 to R (except that this row echelon form R has a row of zeros). The rank won't change!

$$A = \begin{bmatrix}1&3&8\\1&2&6\\0&1&2\end{bmatrix} \quad \rightarrow \quad \rightarrow \quad R = \begin{bmatrix}1&0&2\\0&1&2\\0&0&0\end{bmatrix} = \textbf{rref}\,(A).$$

⇒行最简形

17. $A_1 = \begin{bmatrix}0&0&1&1\\0&0&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix} = \begin{bmatrix}0&1\\0&1\\1&1\\1&1\end{bmatrix}\begin{bmatrix}1&1&0&0\\0&0&1&1\end{bmatrix}$

$A_3 = \begin{bmatrix}0&1\\0&1\\1&1\\1&1\\0&1\\0&1\\1&1\\1&1\end{bmatrix}\begin{bmatrix}1&1&0&0&1&1&0&0\\0&0&1&1&0&0&1&1\end{bmatrix}$

$A_2 = \begin{bmatrix}A_1\\A_1\end{bmatrix} = \begin{bmatrix}0&1\\0&1\\1&1\\1&1\\0&1\\0&1\\1&1\\1&1\end{bmatrix}\begin{bmatrix}1&1&0&0\\0&0&1&1\end{bmatrix}$

For $A = CMR$, C and R some colums or rows come from A.

This page is about the factorization $A = CR$ and its close relative $A = \boldsymbol{CMR}$. As before, $\boldsymbol{C}$ has r independent columns taken from A. *The new matrix* $\boldsymbol{R}$ *has* r *independent rows, also taken directly from* A. The r by r "**mixing matrix**" is $\boldsymbol{M}$. This invertible matrix makes $A = \boldsymbol{CMR}$ a true equation.

The rows of R (not bold) were chosen to produce $A = CR$, but those rows of R *did not come directly from* A. We will see that R has the form $\boldsymbol{MR}$ (bold $\boldsymbol{R}$).

Rank-1 example
$A = CR = \boldsymbol{CMR}$

$$\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} \mathbf{2} \\ \mathbf{3} \end{bmatrix} \begin{bmatrix} \tfrac{\mathbf{1}}{\mathbf{2}} \end{bmatrix} \begin{bmatrix} \mathbf{2} & \mathbf{4} \end{bmatrix}$$

→ it comes from A.

→ it comes from A

In this case $\boldsymbol{M}$ is just 1 by 1. How do we find $\boldsymbol{M}$ in other examples of $A = \boldsymbol{CMR}$? $\boldsymbol{C}$ and $\boldsymbol{R}$ are not square. They have *one-sided* inverses. *We invert* $\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}$ and $\boldsymbol{RR}^{\mathrm{T}}$.

$$\boxed{A = \boldsymbol{CMR}} \quad \boldsymbol{C}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}\,\boldsymbol{M}\,\boldsymbol{RR}^{\mathrm{T}} \quad \boxed{\boldsymbol{M} = (\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C})^{-1}(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{R}^{\mathrm{T}})(\boldsymbol{RR}^{\mathrm{T}})^{-1}} \quad (*)$$

Here are extra problems to give practice with all these rectangular matrices of rank r. $\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}$ and $\boldsymbol{RR}^{\mathrm{T}}$ have rank r so they are invertible (see the last page of Section I.3).

20 Show that equation $(*)$ produces $\boldsymbol{M} = \begin{bmatrix} \frac{1}{2} \end{bmatrix}$ in the small example above.

21 The rank-2 example in the text produced $A = CR$ in equation (2):

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} = CR$$

Frankly Speaking, it's too Complicated by (*).

Choose rows 1 and 2 directly from A to go into $\boldsymbol{R}$. Then from equation $(*)$, find the 2 by 2 matrix $\boldsymbol{M}$ that produces $A = \boldsymbol{CMR}$. Fractions enter the inverse of matrices:

23.

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Then use (*) to get M.

Inverse of a 2 by 2 matrix
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad (**)$$

22 Show that this formula $(**)$ breaks down if $\begin{bmatrix} b \\ d \end{bmatrix} = m \begin{bmatrix} a \\ c \end{bmatrix}$: *dependent columns*.

→ $ad - bc = 0$, so (**) break down.

23 Create a 3 by 2 matrix A with rank 1. Factor A into $A = CR$ and $A = \boldsymbol{CMR}$.

24 Create a 3 by 2 matrix A with rank 2. Factor A into $A = \boldsymbol{CMR}$.

The reason for this page is that the factorizations $A = CR$ and $A = \boldsymbol{CMR}$ have jumped forward in importance for large matrices. When $\boldsymbol{C}$ takes columns directly from A, and $\boldsymbol{R}$ takes rows directly from A, those matrices preserve properties that are lost in the more famous QR and SVD factorizations. Where $A = QR$ and $A = U\Sigma V^{\mathrm{T}}$ involve orthogonalizing the vectors, $\boldsymbol{C}$ and $\boldsymbol{R}$ keep the original data:

If A is nonnegative, so are $\boldsymbol{C}$ and $\boldsymbol{R}$. If A is sparse, so are $\boldsymbol{C}$ and $\boldsymbol{R}$.

A=CMR 获得的C与R能较好保存A的某些原有信息，所以可能与QR分解或SVD分解来得更有用.