1. 素数 prime $= \{ \overset{n \geq 2}{\cancel{n}} : 0 < p \mid n \Rightarrow p = 1 \text{ or } p = n \}$

2. $\pi(n) = \# \{ p \text{ is prime and } p \leq n \}$

$x > 0$, $\pi(x) = \# \{ p \text{ is prime and } p \leq x \}$

3. Gauss function. $[x]$, $= n$, $x-1 < n \leq x$, $n \in \mathbb{Z}$, $\Rightarrow$ by defn $x-1 < [x] \leq x$, $x \in \mathbb{R}$.

4. 自然对数函数 $\ln^x$, $x > 0$, 记做 $\log^x$.

~~重要结~~ 素数定理. prime number theorem:

$$\lim_{x \to +\infty} \frac{\pi(x)}{x} \log^x = 1.$$

5. • 定义 von Mangoldt 函数:

$$\Lambda(n) = \begin{cases} \log p, \text{ 如果 } n = p^m, \text{ 其中 } p \text{ is prime and } m \geq 1 \\ 0. \end{cases}$$

$\Lambda(0) = 0$,
$\Lambda(1) = 0$, $\Lambda(2) = \Lambda(2^2 = 4) = \log^2$, $\Lambda(3) = \Lambda(9) = \log^3$,
$\Lambda(5) = \Lambda(25) = \log^5$, $\Lambda(6) = 0$, $\Lambda(7) = \log^7$, $\Lambda(8) = \log^2$, $\cdots$.

• 定义 Chebyshev function $\psi(x)$: $x > 0$

$$\psi(x) \triangleq \sum_{0 \leq n \leq x} \Lambda(n)$$

易知 $\psi(x) = \sum_{\substack{p^k \leq x,\ k \geq 1 \\ p \text{ is prime}}} \log p = \sum_{\substack{p \leq x \\ p \text{ is prime}}} \left[\frac{\log x}{\log p} \right] \cdot \log p$

Exercises: $\psi(0.5) = \overset{\Lambda(0) =}{0}$, $\psi(1) = \Lambda(0) + \Lambda(1) = 0 + 0 = 0$,
$\psi(1.9) = 0$, $\psi(2) = \Lambda(0) + \Lambda(1) + \Lambda(2) = \log^2$, $\psi(3) = \Lambda(2) + \Lambda(3) = \log^2 + \log^3$

• 定义 $F(x) = \sum_{m=1}^{\infty} \psi\left(\frac{x}{m}\right)$, $x > 0$

Ex: $F(1) = \psi(1) = 0$, $F(2) = \psi(1) + \psi(2) = \Lambda(2) = \log^2$,

$F(3.5) = \underline{\psi(3.5)} + \psi(1.75) + \psi\left(\frac{3.5}{3}\right) + \cdots = \underline{\Lambda(2) + \Lambda(3)} + 0$
$= \log^2 + \log^3$, $\cdots$

6. 引理：(1) $\frac{\psi(x)}{x} \le \frac{\pi(x)}{x}\log x \le \frac{1}{\log x} + \frac{\psi(x)}{x}\cdot\frac{\log x}{\log\left(\frac{x}{\log^2 x}\right)}$, $x>e$

(2) $F(x) = x\log x - x + b(x)\log x$，其中 $b(x)$ 在 x 充分大时有界

即 $\exists\, x_0>0$ 以及常数 $C>0$ 使得

$|b(x)| \le C$ 对所有 $x \ge x_0$ 成立.

证明：

(1) · ~~for prime p,~~ $\cancel{k = \left[\frac{\log x}{\log p}\right]}$，~~即满足 $p^k \le x < p^{k+1}$~~.

$$\psi(x) = \sum_{p\le x}\left[\frac{\log x}{\log p}\right]\log p$$

$$\le \sum_{p\le x}\frac{\log x}{\log p}\cdot\log p \le \sum_{p\le x}\log x = \pi(x)\cdot\log x \Rightarrow$$

$$\frac{\psi(x)}{x} \le \frac{\pi(x)}{x}\log x$$

· Let $1<y<x$, $\Rightarrow \pi(x)-\pi(y) = \sum_{y<p\le x}1 \le \sum_{y<p\le x}\frac{\log p}{\log y} \le \frac{\psi(x)}{\log y}$

$\Rightarrow \pi(x) \le y + \frac{\psi(x)}{\log y}$,

取 $y = \frac{x}{\log^2 x}$, $x>e$ $\Rightarrow \pi(x) \le \frac{x}{\log^2 x} + \frac{\psi(x)}{\log\left(\frac{x}{\log^2 x}\right)}$

$$\Rightarrow \frac{\pi(x)}{x}\cdot\log x \le \frac{1}{\log x} + \frac{\psi(x)}{x}\cdot\frac{\log x}{\log\left(\frac{x}{\log^2 x}\right)}.$$

Note that $\lim\limits_{x\to\infty}\frac{1}{\log x} = 0$

$$\lim_{x\to\infty}\frac{\log x}{\log\left(\frac{x}{\log^2 x}\right)} = \lim_{x\to\infty}\frac{\log x}{\log x - 2\log(\log x)} = 1$$

所以 $\lim\limits_{x\to+\infty}\frac{\pi(x)}{x}\log x = 1 \iff \lim\limits_{x\to+\infty}\frac{\psi(x)}{x} = 1$.

引理. (2) 证明.

$$\log n = F(n) - F(n-1) = \sum_{m=1}^{\infty}\Big[\underbrace{\psi\left(\tfrac{n}{m}\right) - \psi\left(\tfrac{n-1}{m}\right)}_{\shortparallel\ a}\Big]$$, finite summation.

$$a \Rightarrow a = \begin{cases} 0, & \text{如果 } \tfrac{n}{m} \text{ 不是整数.} \\ \Lambda\left(\tfrac{n}{m}\right) \end{cases}$$

$$\boxed{\text{区间 } \left(\tfrac{n-1}{m}, \tfrac{n}{m}\right] \text{ 中至多有一个整数: } \tfrac{n}{m}}$$

$$= \sum_{m \mid n} \Lambda\left(\tfrac{n}{m}\right)$$

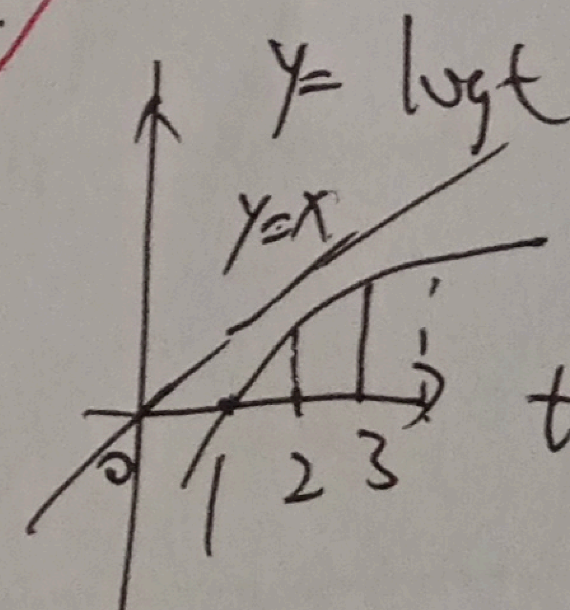
$$\xlongequal[\text{对称性.}]{m\mid n,\ \overset{\leftrightarrow}{}\ \left(\frac{n}{m}\right)\mid n} \sum_{d\mid n} \Lambda(d) \xlongequal[j_1\log p_1 + j_2 \log p_2 + \cdots + j_k \log p_k]{n = p_1^{j_1}\cdots p_k^{j_k},\ \Lambda(p^k) = \log p} \log n$$

Recall $F(1) = 0$, $F(2) = \log 2$,

$$\Rightarrow F(n) = F(n) - F(n-1) + F(n-1) + F(n-2) + \cdots + F(2) - F(1) + F(1)$$
$$= \log n + \log n-1 + \cdots + \log 2 + \log 1$$
$$= \log n!$$

our goal is $F(x) = x\log x - x + b(x)\log x$,

定义 $J(x) = \int_1^x \log t \, dt \nearrow, \xlongequal[\text{close to } F(x)]{} x\log x - x + 1$ b/c $\begin{cases} \frac{d}{dx}[x\log x - x + 1] \\ = \log x + 1 - 1 = \log x. \\ -1|_{x=1} = 0. \end{cases}$ $x \gg 1$



如果 $n \in \mathbb{N}$ 使得 $n \le x < n+1 \le x+1$, i.e. $n = [x]$

则有 $J(x) = \int_1^2 \log t \, dt + \cdots + \int_{n-1}^{n} \log t \, dt + \int_n^x \log t \, dt$

$$J(x) \ge \log 1 + \log 2 + \cdots + \log n-1 = F(n-1)$$
$$J(x) \le \log 2 + \log 3 + \cdots + \log n + \log n+1 = F(n+1)$$
$$J(n) \le J(x) \le J(n+1)$$
$$\Rightarrow F(n-1) \le J(n) \le J(x) \le F(n+1)$$
$$F(n) \le F(x) \le F(n+1)$$

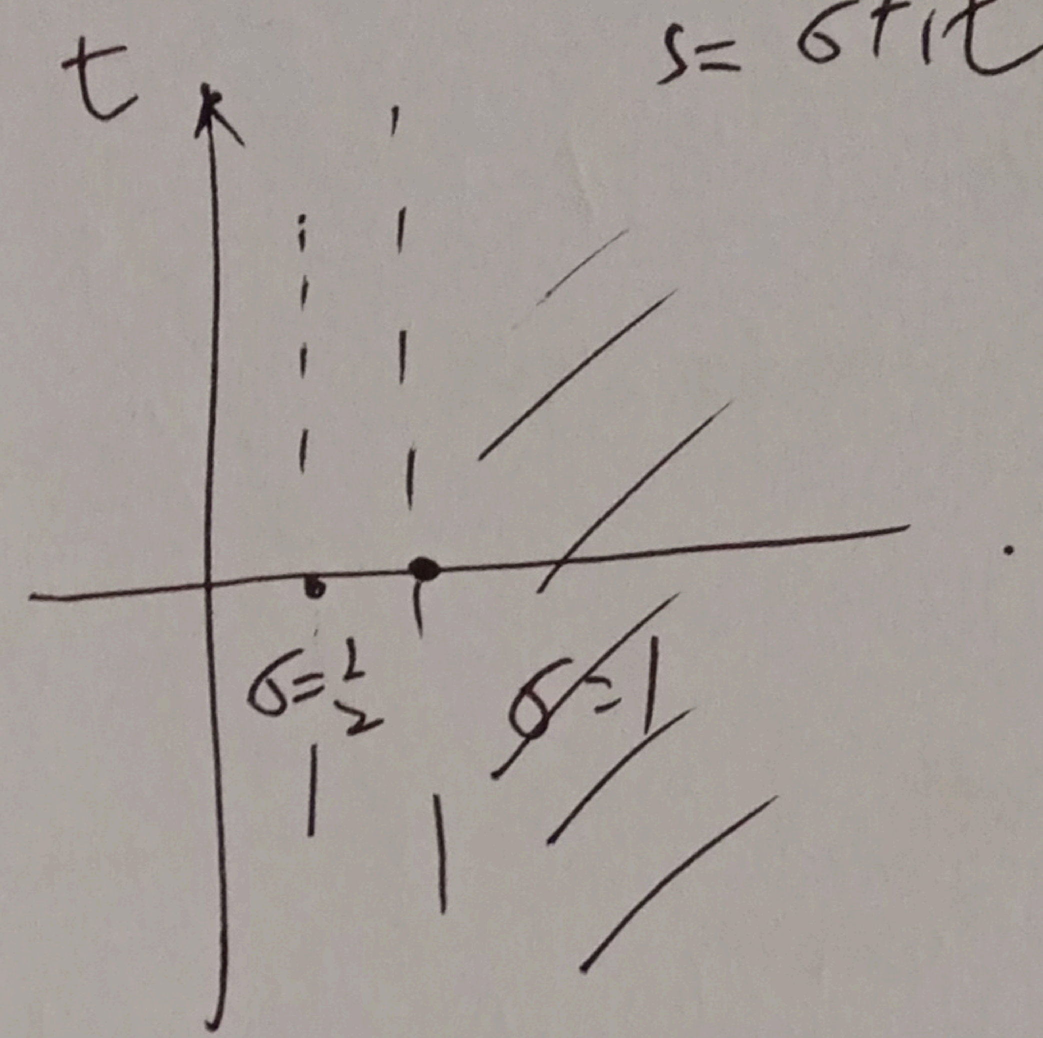
$$\Rightarrow |F(x) - \underbrace{J(x)}_{\substack{\shortparallel \\ x\log x - x + 1}}| \le F(n+1) - F(n-1) = \log n\cdot(n+1) \le \log x\cdot(x+1) \le 2\log(x+1)$$

$$\Rightarrow F(x) = x\log x - x + b(x)\log x$$, b is bdd as $x \to \infty$.

7. Riemann Zeta function.

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\sum_{n=1}^{\infty}n^{-s},\quad s=\overset{\mathrm{Re}\,s}{\overset{\shortparallel}{\sigma}}+it,\quad \left|\frac{1}{n^s}\right|=\left|\frac{1}{n^{\sigma}\cdot n^{it}}\right|=\frac{1}{n^{\sigma}}$$

$\Rightarrow$ $\zeta(s)$ 绝对收敛 $\Leftrightarrow$ $\sigma>1$. i.e. $\mathrm{Re}\,s>1$.



· 如果 $\mathrm{Re}\,s>1$，则有

$$s\int_1^{N+1}[x]\,x^{-1-s}\,dx = s\sum_{n=1}^{N}n\cdot\underbrace{\int_n^{n+1}x^{-1-s}\,dx}_{=\frac{1}{s}\cdot\left[n^{-s}-(n+1)^{-s}\right]}$$

$$=1\cdot\left[1^{-s}-2^{-s}\right]+2\cdot\left[2^{-s}-3^{-s}\right]+\cdots+N\cdot\left[N^{-s}-(N+1)^{-s}\right]$$

$$=\sum_{n=1}^{N}n^{-s}-\underbrace{N\cdot(N+1)^{-s}\longrightarrow 0\quad \text{as } N\to+\infty}$$

take limit as $N\to\infty$

$$\Rightarrow\quad \zeta(s)=\sum_{n=1}^{\infty}n^{-s}=s\int_1^{\infty}[x]\,x^{-1-s}\,dx\quad \text{For all } s=\sigma+it \text{ s.t. } \sigma>1.$$

进一步可以证明. $\zeta(1+it)\neq 0,\ t\neq 0,\ t\in\mathbb{R}$.

且. $$\lim_{t\to 0}\frac{it\cdot\zeta(1+it)}{}=1$$

$$\boxed{\lim_{z\to a}(z-a)\,\zeta(z)=1.\qquad a=1.}$$

$$\lim_{z\to 1}(z-1)\,\zeta(z)=1$$

$\Downarrow$ $z=1+it$.

$\Downarrow$

$$\lim_{t\to 0}it\,\zeta(1+it)=1$$

8. Ingham's Tauberian Thm.

假设 $\begin{cases} g(x) \text{ is } \nearrow \text{ on } (0,\infty) \\ g(x) \equiv 0 \text{ 如果 } x<1 \\ G(x) \hat{=} \sum_{n=1}^{\infty} g\left(\frac{x}{n}\right) \text{ 且 } G(x) = ax\log x + bx + x\cdot\varepsilon(x) \end{cases}$

其中 $\lim_{x\to\infty} \varepsilon(x) = 0$.

则有 $\lim_{x\to+\infty} \frac{g(x)}{x} = a$.

类比. g 与 chybyshev 函数 $\psi(x) \hat{=} \sum_{0\le n\le x} \Lambda(n)$

$= \sum_{\substack{p\le x \\ p \text{ prime}}} \left[\frac{\log x}{\log p}\right]\log p$

$F(x) = x\log x - x + b(x)\log x$

bdd. $x \gg 1$.

$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$

证明: • $\frac{g(x)}{x}$ 有界

$g(x) - g\left(\frac{x}{2}\right) \le \sum_{n=1}^{\infty} (-1)^{n+1} g\left(\frac{x}{n}\right)$

$= g(x) - g\left(\frac{x}{2}\right) + \underbrace{g\left(\frac{x}{3}\right) - g\left(\frac{x}{4}\right)}_{\ge 0} + \underbrace{g\left(\frac{x}{5}\right) - g\left(\frac{x}{6}\right)}_{\ge 0} + \cdots$

$= g(x) + g\left(\frac{x}{2}\right) + g\left(\frac{x}{3}\right) + \cdots - 2\left[g\left(\frac{x}{2}\right) + g\left(\frac{x}{4}\right) + g\left(\frac{x}{6}\right) + \cdots\right]$

$= G(x) - 2G\left(\frac{x}{2}\right)$

$= ax\log x + bx + x\varepsilon(x) - 2\left(a\cdot\frac{x}{2}\cdot\log\frac{x}{2} + b\cdot\frac{x}{2} + \frac{x}{2}\varepsilon\left(\frac{x}{2}\right)\right)$

$= ax\underbrace{\left[\log x - \log x/2\right]}_{=\log 2} + x\cdot\left[\varepsilon(x) - \varepsilon\left(\frac{x}{2}\right)\right]$

$\le Ax, \quad A>0$

$\Rightarrow g(x) = g(x) - g\left(\frac{x}{2}\right) + g\left(\frac{x}{2}\right) - g\left(\frac{x}{4}\right) + \cdots$

$\le A\cdot x + A\cdot\left(\frac{1}{2}x\right) + A\cdot\left(\frac{1}{4}x\right) + \cdots$

$= Ax\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) = 2Ax \Rightarrow \frac{g(x)}{x} \le 2A.$

• 定义 $h(x) = g(e^x)$, 则 Domain$(h) = \mathbb{R}$, $H(x) \hat{=} G(e^x)$

$\Rightarrow h(x) = 0$ 若 $x<0$

$H(x) = G(e^x) = \cancel{\sum_{n=1}^{\infty} g\left(\frac{e^x}{n}\right)} = \cancel{\sum_{n=1}^{\infty} h(x - \log n)}$

$\cancel{e^{x-\log n} = e^x\cdot\frac{1}{e^{\log n}} = \frac{e^x}{n}}$

$= ae^x\cdot\log e^x + be^x + e^x\cdot\varepsilon(e^x) = e^x(ax + b + \varepsilon_1(x))$, $\varepsilon_1(x) \to 0$ as $x\to+\infty$

定义 $\varphi(x) = e^{-x} h(x) = \frac{g(e^x)}{e^x} \Rightarrow \varphi(x)$ is 有界. ~~$x>0$~~ $x\in\mathbb{R}$.

下证 $\lim_{x\to\infty}\varphi(x) = a$.

$k(x) \triangleq [e^x]e^{-x}$, 选择 λ 使得 $\frac{\lambda}{\pi}\notin\mathbb{Q}$. $\Rightarrow k(x)=0$ if $x<0$ b/c $0<e^x<1$

$K(x) \triangleq 2k(x) - k(x-1) - k(x-\lambda)$, $K(x)=0$ if $x<\min\{0,1,\lambda\}$.

- $K\in L^1(\mathbb{R})$, i.e. $\int_{-\infty}^{\infty}|K(x)|\,dx<\infty$.

pf. ① $e^xK(x)$ is bdd.

$$K(x) = 2[e^x]e^{-x} - [e^{x-1}]e^{-(x-1)} - [e^{x-\lambda}]e^{-(x-\lambda)}$$

$$\Rightarrow e^xK(x) = 2[e^x] - [e^{x-1}]\cdot e - [e^{x-\lambda}]e^{\lambda}$$

$$= [e^x]-[e^{x-1}]\cdot e + [e^x] - [e^{x-\lambda}]\cdot e^{\lambda} \quad \text{bdd.}$$

② $\int_0^{+\infty} e^{-x}\,dx = e^{-x}\Big|_{\infty}^{0} = 1$

$\int_1^{\infty} e^{-x}\,dx = e^{-x}\Big|_{\infty}^{1} = e^{-1}$

$\int_\lambda^{\infty} e^{-x}\,dx = e^{-x}\Big|_{\infty}^{\lambda} = e^{-\lambda}$.

③ $\int_{-\infty}^{\infty}|K(x)|\,dx \le \max\Big[\int_{0,1,\lambda}^{\infty} e^{-x}\cdot\underbrace{|K(x)\cdot e^x|}_{\le C}\,dx \le C\cdot\max(1,e^{-\lambda})$

- Recall $\zeta(s) = s\int_1^\infty [x]x^{-1-s}\,dx$, Re $s>1$. No zero on $s=1+it$, $t\neq0$.

考虑 (consider) $\int_{-\infty}^\infty k(x)e^{-xs}\,dx = \int_0^\infty [e^x]e^{-x}\cdot e^{-xs}\,dx = \int_0^\infty [e^x]e^{-x(s+1)}\,dx$

$e^x = y$, $dy = e^x\,dx$, $dx = e^{-x}\,dy = y^{-1}\,dy$

$$= \int_1^\infty [y]\cdot(y^{-1})^{s+1}\cdot y^{-1}\,dy$$

$$= \int_1^\infty [y]\,y^{-2-s}\,dy$$

$\zeta(1+s) = (1+s)\int_1^\infty [y]\cdot y^{-2-s}\,dy$ No zero on $s\neq0$

$$= \frac{\zeta(1+s)}{1+s} \quad (\neq0 \text{ if } s=it,\ t\neq0)$$

$s=it$; ~~$s=1+it$~~.

$K\in L^1(\mathbb{R})$

$\Rightarrow \hat{k}(t) \triangleq \int_{-\infty}^\infty k(x)e^{-itx}\,dx \Rightarrow \begin{cases} k(x-1)^\wedge(t) = e^{-it}\hat{k}(t) \\ k(x-\lambda)^\wedge(t) = e^{-i\lambda t}\hat{k}(t) \end{cases}$, $\hat{k}$ is cont.

$= \hat{K}(t) = (2-e^{-it}-e^{-i\lambda t})\cdot\frac{\zeta(1+it)}{1+it} \xRightarrow[\lambda\notin\mathbb{Q}]{} \hat{K}(t)\neq0$ 如果 $t\neq0$.

$$\hat{k}(t)=\int_{-\infty}^{\infty}k(x)e^{-itx}\,dx=(2-e^{-it}-e^{-i\lambda t})\cdot\frac{f(1+it)}{1+it}$$

$|e^{-itx}|\le 1$, $\lim_{t\to 0}e^{-itx}=1$ 对所有 x 成立.

Recall 实分析中 有界收敛定理：

$$\begin{cases} f_n:\mathbb{R}\to\mathbb{K}\ \in L^1,\ f\in L^1 \\ |f_n|\le|f|, \\ \lim\limits_{n\to\infty}f_n(x)=0 \text{ 则 } \lim\limits_{n\to\infty}\int f_n(x)\,dx=0. \end{cases}$$

$$\Rightarrow \hat{k}(0)=\lim_{t\to 0}\hat{k}(t)$$
$$=\lim_{t\to 0}(2-e^{-it}-e^{-i\lambda t})\cdot\frac{f(1+it)}{1+it}\quad 1+it\to 1$$
$$=\lim_{t\to 0} it\cdot f(1+it)\cdot\frac{(1-e^{-it})+(1-e^{-i\lambda t})}{it}$$

Recall $\lim\limits_{x\to 0}\frac{e^x-1}{x}=\lim\limits_{x\to 0}\frac{e^x}{1}=1$

$$\Rightarrow \lim_{t\to 0}\frac{1-e^{-it}}{it}=1$$
$$\lim_{t\to 0}\frac{1-e^{-i\lambda t}}{it\cdot\lambda}\cdot\lambda=\lambda$$

$\int_{-\infty}^{\infty}k(x)\,dx$
$\|$
$$\Rightarrow \hat{k}(0)=1\cdot(1+\lambda)=1+\lambda$$

Recall $\varphi(x)=e^{-x}h(x)\ \underline{\underline{x\in\mathbb{R}}}\ e^{-x}g(e^{x})$

考虑. $\mu=\sum_{n=1}^{\infty}\delta_{\log n}$, 其中 δ 为 Dirac 测度 s.t.

$$\begin{cases} \int_{-\infty}^{\infty}f(x)\,d\delta=f(0) \\ \int_{-\infty}^{\infty}f(x)\,d\delta_t=f(t) \\ f \text{ 任意函数 } \boxed{\text{可测}} \end{cases}$$

$v\triangleq\chi_{[0,\infty)}$,
$$=\begin{cases}0, & x<0\\ 1, & x\geq 0.\end{cases}$$

$u(x)\triangleq[e^x]$, $x\in\mathbb{R}$.

$x\geq\log n$
$e^x\geq n$.

Recall $H(x)=\sum_{n=1}^{\infty}h(x-\log n)=\int_{-\infty}^{\infty}h(x-y)\,d\mu(y)=h*\mu=\mu*h.$

- $(v*\mu)(x)=\sum_{n=1}^{\infty}v(x-\log n)=\sum_{1\le n\le e^x}1=[e^x].$

so far. $u = [e^x] = v * \mu$,

$$\Rightarrow (h*u)(x) = (h*v*\mu)(x) = (h*\mu*v)(x)$$
$$= (H*v)(x) = \int_{-\infty}^{\infty} H(y)\,v(x-y)\,dy$$
$$= \int_0^x H(y)\,dy.$$

- $(\varphi * k)(x) = \int_{-\infty}^{\infty} \varphi(x-y)\,k(y)\,dy$

$$\underset{k(x)=[e^x]e^{-x}}{\overset{\varphi(x)=e^{-x}h(x)}{=\!=}} \int_{-\infty}^{\infty} e^{y-x} h(x-y)\,\overset{u(y)}{[e^y]}\,e^{-y}\,dy$$

$$= \int_{-\infty}^{\infty} e^{-x} h(x-y)[e^y]\,dy = e^{-x}\cdot(h*u)(x)$$
$$= \underline{e^{-x}\int_0^x H(y)\,dy}$$

Recall $H(x) = e^x(ax+b+\varepsilon_1(x))$
$\varepsilon_1 \to 0$ as $x\to+\infty$

$$\Rightarrow (\varphi*k)(x) = e^{-x}\int_0^x e^y(ay+b+\varepsilon_1(y))\,dy$$

$\int_0^x ye^y\,dy = (ye^y - e^y)\Big|_0^x$
$= (xe^x - e^x) - (0-1)$
$= xe^x - e^x + 1$

$x>0$

$\int_0^x e^y\,dy = e^x - 1$

$= ax$
$+ b - a$
$+ \varepsilon_2(x)$
as $x\to\infty$.

- if $h(x) \hat{=} (f*g)(x)$
$\int f(y)\,g(x-x_0-y)\,dy = h(x-x_0)$.

$$\Rightarrow \lim_{x\to\infty}(\varphi*k)(x) = \lim_{x\to\infty} 2(\varphi*k)(x) - (\varphi*k(\cdot-1))(x) - (\varphi*k(\cdot-\lambda))(x).$$
$$= \lim_{x\to\infty} 2(\varphi*k)(x) - (\varphi*k)(x-1) - (\varphi*k)(x-\lambda)$$
$$= 2\cancel{ax} + 2(\cancel{b}-a) - [\cancel{a(x}-1) + \cancel{b}-a] - [\cancel{a(x}-\lambda) + \cancel{b}-a]$$
$$= a + \lambda a = (1+\lambda)a \overset{P7.}{=} a\cdot\int_{-\infty}^{\infty} k(y)\,dy, \text{ note } k\in L^1(\mathbb{R}).$$

· Wiener - Tauberian 定理.

Note. $\hat{k}(t) = (2 - e^{-it} - e^{-i\lambda t}) \cdot \frac{f(1+it)}{1+it} \neq 0$ if $t \neq 0$

$\hat{k}(0) = 1+\lambda \neq 0$

$\frac{\lambda}{\pi} \notin \mathbb{Q}$.

$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$e^{2\pi i} = 1$

· $\varphi \in L^\infty(\mathbb{R})$, $k \in L^1(\mathbb{R})$.

$t = 2k\pi$,

$\lambda t = 2k'\pi$. $\lambda = \frac{k'}{k}$

$\Rightarrow \lim_{x\to\infty} (\varphi * f)(x) = a\int_{-\infty}^{\infty} f(y)\,dy$, for all $f \in L^1(\mathbb{R})$.

· We still need to show $\lim_{x\to\infty} \varphi(x) = a$ to conclude the P.N.T.

$\forall \varepsilon > 0$.

choose f_1, f_2 s.t.

$$\begin{cases} \operatorname{supp} f_1 \subseteq [0,\varepsilon], \ \operatorname{supp} f_2 \subseteq [-\varepsilon, 0] \\ f_1, f_2 \geq 0. \quad \int f_1\,dx = \int f_2\,dx = 1 \end{cases}$$

Recall $\varphi(x) \triangleq e^{-x} h(x)$, $h(x) = g(e^x) \uparrow$.

so $\varphi(x)\cdot e^x$ is $\uparrow$ $\Rightarrow$ $\varphi(y)\cdot e^{y} \leq \varphi(x)\cdot e^{x}$ for $x-\varepsilon \leq y \leq x$

$$\Rightarrow \begin{cases} \varphi(y) \leq \varphi(x)\cdot e^{x-y} \\ \quad \leq \varphi(x)\cdot e^{\varepsilon}, \quad y \in [x-\varepsilon, x]. \quad \cdots ① \\ \\ \varphi(y)\cdot e^{y} \geq \varphi(x)\cdot e^{x} \\ \varphi(y) \geq \varphi(x)\cdot e^{x-y} \geq \varphi(x)\cdot e^{-\varepsilon}, \quad y \in [x, x+\varepsilon] \end{cases}$$

$\cdots ②$

①,② $\Rightarrow$ $e^{-\varepsilon}(f_1 * \varphi)(x) = e^{-\varepsilon}\int_0^{\varepsilon} f_1(y)\,\varphi(x-y)\,dy \leq e^{-\varepsilon}\varphi(x)\cdot e^{\varepsilon} = \varphi(x)$

$x-\varepsilon \leq x-y \leq x$

$\leq$ (wait: $\geq$ below)

$e^{\varepsilon}(f_2 * \varphi)(x) = e^{\varepsilon}\int_{-\varepsilon}^{0} f_2(y)\,\varphi(x-y)\,dy \geq e^{\varepsilon}\cdot\varphi(x)\cdot e^{-\varepsilon} = \varphi(x)$

$x \leq x-y \leq x+\varepsilon$

P10.

Recall: for $\forall \varepsilon > 0$, $\exists$ f_1, f_2 s.t.

- $$e^{-\varepsilon}(f_1 * \varphi)(x) \le \varphi(x) \le e^{\varepsilon}(f_2 * \varphi)(x) \quad \cdots \quad (1)$$

- $$\lim_{x \to \infty} (\varphi * f)(x) = a \cdot \int_{-\infty}^{\infty} f(y)\,dy, \text{ for all } f \in L^1(\mathbb{R}).$$

let $x \to +\infty \Rightarrow$

$$\overline{\lim_{x \to +\infty}} \varphi(x) \le a \cdot e^{\varepsilon}$$

$$\underline{\lim_{x \to +\infty}} \varphi(x) \ge a \cdot e^{-\varepsilon}$$

ε is $\forall \Rightarrow \lim_{x \to \infty} \varphi(x) = a.$