Birman

Chapter 5

. Let $H = L_2(\mathbf{R})$, $D(A_0) =$
ator A_0 is symmetric. *The*
and only those) absolutely
ew of Lemma 1 we have
count, we get

$\forall u, v \in D(A^*)$.

in Section 1, Subsection 1,
operator (A_0 *is essentially*
to the same conclusion.
5) are trivial.
a restriction of A to the set
ons clearly satisfy (25) and
A we have considered in
ctrum is $\mathbf{R}$. Since the core
e have in our case $\hat{\sigma}(A) =$

$\in \mathbf{R}$, pick $h \in L_2$ and set

t compact.

P45
Thm 2.8.2 · $P_1 = P_{H_1}$, $P_2 = P_{H_2}$, H_1, H_2 SHTF AE.
(a) $P_1 + P_2$ is proj (b) $H_1 \perp H_2$ (c) $P_1 P_2 = 0$
& (a), (b), (c) $\Rightarrow$ $P_1 + P_2 = P_{H_1 \oplus H_2}$.

CHAPTER 5

Spectral Measure. Integration

P48 Cor 2.8.5. $P_1 P_2 = P_1 \iff P_1 \le P_2$ (or. $H_1 \subseteq H_2$)

The basic results of the theory of self-adjoint operators (as well as of normal operators) are presented in Chapter 6. These results consist in the so-called spectral resolution of an operator, i.e. in a representation of an operator as an integral with respect to a spectral measure whose values are commuting projections. Such a resolution completely characterizes the spectral properties of an operator, permits us to describe its unitary invariants and to construct a simple model for operators under consideration (see Chapter 7). Spectral resolution also permits us to construct a functional calculus for an operator. All these questions are of great importance both in the general theory and in applications.

In this preparatory chapter we investigate projection-valued (spectral) measures (§§1, 2 and 5) and integrals with respect to such measures (§§3 and 4). The material presented in the chapter considerably simplifies further study of the spectral theory.

1. Basic Notions

The theory of spectral measure rests on the theory of scalar measure (see §1.3) on the one hand and on the properties of orthogonal projections (see §2.8) on the other.

1. Let $(Y, \mathscr{A})$ be a measurable space, H be a Hilbert space and $\mathscr{P} = \mathscr{P}(H)$ be the set of orthogonal projections on H. Suppose $E: \mathscr{A} \to \mathscr{P}$ is a mapping satisfying the following conditions.

1°. *Countable additivity*: if $\{\delta_n\}$ is a finite or countable set of disjoint sets $\delta_n \in \mathscr{A}$ and $\delta = \bigcup_n \delta_n$ then $E(\delta) = s\text{-}\Sigma_n E(\delta_n)$. $E(\emptyset) = 0$?

2°. *Completeness*: $E(Y) = I$.

Then E is called *a spectral measure* on H and $(Y, \mathscr{A}, H, E)$ is referred to as *a spectral measure space*.

Let us derive some simple properties of spectral measures which follow from finite additivity. Below all subsets $\delta \subset Y$ under consideration are assumed to be measurable (i.e. $\delta \in \mathscr{A}$).

THEOREM 1. *Let E be a spectral measure on H. Then*

$$E(\delta_1)E(\delta_2) = E(\delta_2)E(\delta_1) = E(\delta_1 \cap \delta_2). \tag{1}$$

$\delta_1 \cap \delta_2 = \phi \Rightarrow E(\delta_1) + E(\delta_2) = E(\delta_1 \cup \delta_2)$ is a projection.

$\Rightarrow E(\delta_1)E(\delta_2) = 0.$

now if $\delta_1 = \delta_2 = \phi$ then $E(\phi) = E(\phi) + E(\phi)$; $E(\phi) + E(\phi)$ is proj ⇒ $E(\phi)E(\phi) = 0$ i.e. $P^2 = 0$ but $P = P^2$ ⇒ $P = 0$

In particular, by

$$\text{if}\quad \delta_1 \cap \delta_2 = \varnothing \quad \text{then} \quad E(\delta_1)E(\delta_2) = 0;$$

$$\text{if}\quad \delta_1 \subset \delta_2 \quad \text{then} \quad E(\delta_1) \leqslant E(\delta_2). \tag{3}$$

$E(\delta_1)E(\delta_2) = E(\delta_1) \leq E(\delta_1)$

Proof. Property (2) follows from 1° in view of Theorem 2.8.2. If $\delta_0 = \delta_1 \cap \delta_2 \neq \varnothing$, put $\delta^i = \delta_i \backslash \delta_0$, $i = 1, 2$. Multiplying the equalities $E(\delta_i) = E(\delta_0) + E(\delta^i)$ (the order is not essential) and taking into account (2), we obtain (1). Property (3) follows from (1) by Corollary 2.8.5. □

$E(\delta_1)E(\delta_2) = E(\delta_0)E(\delta_0) + E(\delta^1)E(\delta_0) + E(\delta^1)E(\delta^2) + E(\delta_0)E(\delta^2)$
$= E(\delta_0) + 0 + 0 + 0$. (✓).

Properties (1)–(3) mean that the spectral measure is *commutative*, *orthogonal* and *monotone*.

Besides $E(\delta)$, it is convenient to consider the corresponding subspaces $H(\delta) = E(\delta)H$. Properties (2) and (3) mean that

$$\text{if}\quad \delta_1 \cap \delta_2 = \varnothing \quad \text{then} \quad H(\delta_1) \perp H(\delta_2); \tag{4}$$

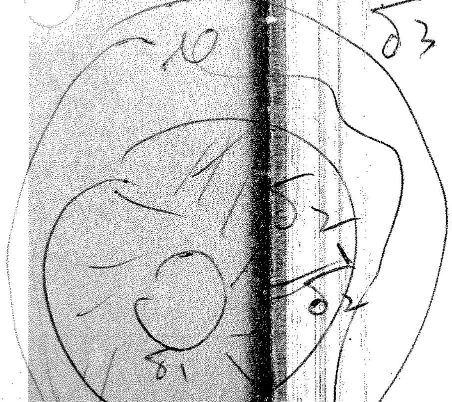
$$\text{if}\quad \delta_1 \subset \delta_2 \quad \text{then} \quad H(\delta_1) \subset H(\delta_2). \tag{5}$$

$\delta_1 \subseteq \delta_2$, $\delta \triangleq \delta_2 \backslash \delta_1$, $E(\delta_2) = E(\delta_1) + E(\delta)$

$E(\delta_2)E(\delta_1) = E(\delta_2 \backslash \delta_1) = E(\delta_1) \Rightarrow E(\delta_1) \leq E(\delta_2)$.

Condition 2° in the definition of spectral measure is not so essential: if it is not satisfied then $H(\delta) \subset H(Y)$ $(\forall \delta \in \mathscr{A})$ by (5) and so E can be considered as a spectral measure on $H(Y)$. $= E(Y)H$.

2. Let us proceed to the investigation of the *countable* additivity of spectral measures. Let $\{\delta_n\}$, $n = 1, 2, \ldots$ be a sequence of disjoint measurable sets. It follows from the finite additivity (see (4)) that the $H(\delta_n)$ are pairwise orthogonal. By Corollary 2.8.7 the series $\Sigma_n E(\delta_n)$ s-converges to the projection onto $\Sigma_n \oplus H(\delta_n)$. The condition of countable additivity implies that the latter projection must coincide with $E(\cup_n \delta_n)$. Let us mention some simple consequences of the countable additivity. good.



THEOREM 2. *Let* $\{\delta_n\}$, $n = 1, 2, \ldots$ *be measurable subsets of* Y *and* E *be a spectral measure.*

(1) *If* $\{\delta_n\}$ *increases then*

$$s\text{-}\lim E(\delta_n) = E(\cup_n \delta_n). \tag{6}$$

$\delta_2' = \delta_2 \backslash \delta_1$, $\delta_3' = \delta_3 \backslash \delta_2'$

(2) *If* $\{\delta_n\}$ *decreases then*

$$s\text{-}\lim E(\delta_n) = E(\cap_n \delta_n).$$

In particular,

$$\text{if}\quad \delta_1 \supset \delta_2 \supset \ldots \quad \text{and} \quad \cap_n \delta_n = \varnothing \quad \text{then} \quad s\text{-}\lim E(\delta_n) = 0. \tag{7}$$

Proof: (1) Put $\delta = \cup_n \delta_n$, $\delta_1' = \delta_1$, $\delta_n' = \delta_n \backslash \delta_{n-1}$ $(n > 1)$. Then δ_n''s are pairwise disjoint, $\cup_n \delta_n' = \delta$, and by 1° $E(\delta) = s\text{-}\Sigma_n E(\delta_n') = s\text{-}\lim E(\delta_n)$. (✓)

(2) The proof reduces to (1) by passage to the increasing sequence $\{Y \backslash \delta_n\}$. □

3. Every spectral measure generates a family of finite scalar measures defined on $\mathscr{A}$.

$Y \backslash \delta_n \triangleq \eta_n$. then $s\text{-}\lim E(\eta_n) = E(\cup \eta_n)$

$(\cap \delta_n) = \cap \eta_n$ $= s\text{-}\lim (I - E(\delta_n)) = I - E(\cap \delta_n)$

done.

(2)

(3)

n 2.8.2. If $\delta_0 = \delta_1 \cap \delta_2 \neq$
$(\delta_i) = E(\delta_0) + E(\delta^i)$ (the
obtain (1). Property (3)
□

commutative, orthogonal

onding subspaces $H(\delta) =$

(4)
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(6)

$E(\delta_n) = 0.$ (7)

1). Then δ_n''s are pairwise
$E(\delta_n)$.
g sequence $\{ Y\setminus\delta_n \}$. □

ar measures defined on $\mathscr{A}$.

Namely, for $f \in H$ put $\mu_f(\cup\delta_n) = (E(\cup\delta_n)f, f)$
$= \sum_n (E(\delta_n)f, f)$ b/c $E(\cup\delta_n)f = \lim_{M} \sum_{n=1}^{M} E(\delta_n)f$

$$\mu_f(\delta) = (E(\delta)f, f) = \| E(\delta) f \|^2. \tag{8}$$

$\mu_f = \mu_{f,f}$

The measure μ_f is countably additive in view of 1°. It follows from 2° that

$$\mu_f(Y) = \|f\|^2 \quad (\forall f \in H). \tag{9}$$

Besides μ_f, we consider also the complex measures

$$\mu_{f,g}(\delta) = (E(\delta)f, g) \quad (\forall f, g \in H). \tag{10}$$

Operations on such measures can be reduced to those on $\mu_f = \mu_{f,f}$ in view of (2.4.9) which in this case turns into

$$4\mu_{f,g}(\delta) = \mu_{f+g}(\delta) + i\mu_{f+ig}(\delta) - \mu_{f-g}(\delta) - i\mu_{f-ig}(\delta). \tag{11}$$ P34.

Note also that

$(E(\delta)g, f)$

$$= \mu_{g,f}(\delta) = \overline{\mu_{f,g}(\delta)}, = \overline{(E(\delta)f, g)} = \overline{(f, E(\delta)g)} \tag{12}$$

$$| \mu_{f,g}(\delta) |^2 \leqslant \mu_f(\delta)\mu_g(\delta). \tag{13}$$ i.e. $|(E(\delta)f, g)|^2 \le \|E(\delta)f\|^2 \|E(\delta)g\|^2$

The last inequality follows from $(Pf, g) = (P^2 f, g) = (Pf, Pg)$

$$|(E(\delta)f, g)| = |(E(\delta)f, E(\delta)g)| \leqslant \|E(\delta)f\| \cdot \|E(\delta)g\|.$$

If $\delta_n \in \mathscr{A}$, $n = 1, 2, \ldots$, are pairwise disjoint and $\delta = \cup_n \delta_n$ then by (13)

$$\sum_n | \mu_{f,g}(\delta_n) | \leqslant \sum_n \sqrt{\mu_f(\delta_n)}\ \sqrt{\mu_g(\delta_n)} \leqslant \Big[\sum_n \mu_f(\delta_n)\Big]^{1/2} \Big[\sum_n \mu_g(\delta_n)\Big]^{1/2}$$
$$= \sqrt{\mu_f(\delta)}\ \sqrt{\mu_g(\delta)}.$$

This means that the variation $|\mu_{f,g}|$ of $\mu_{f,g}$ satisfies

$$|\mu_{f,g}|(\delta) \leqslant \sqrt{\mu_f(\delta)}\ \sqrt{\mu_g(\delta)} \quad (\forall f, g \in H, \forall \delta \in \mathscr{A}). \tag{14}$$

In particular for $\delta = Y$ we have in view of (9)

$$|\mu_{f,g}|(Y) \leqslant \|f\| \cdot \|g\|. \tag{15}$$

As in the scalar case the sets $\delta \in \mathscr{A}$ with $E(\delta) = 0$ are called *sets of zero E-measure*. The notions 'E-almost everywhere' (E-a.e.), 'E-bounded', 'E-sup' have the usual meanings. In particular, if φ is a real measurable (with respect to $\mathscr{A}$) function on Y then

$\varphi(y) \le a$ except for some $\delta \in \mathscr{A}$ s.t. $E(\delta) = 0$ zero projection.

$$E\text{-sup}\ \varphi = \inf \{ a \in \mathbf{R} : \varphi(y) \leqslant a\ E\text{-a.e.} \}.$$

Clearly $E(\delta) = 0$ iff $\mu_f(\delta) = 0$, $\forall f \in H$. There exist vectors $g \in H$ (vectors of maximal type) such that $E(\delta) = 0$ iff $\mu_g(\delta) = 0$ (see Theorem 7.3.4 below). P171.

$\mu_f(\delta) = (E(\delta)f, f) = \|E(\delta)f\|^2$.

$E(\delta) = 0 \iff \|E(\delta)f\|^2 = 0$ for all $f \in H$

$\iff \mu_f(\delta) = 0, \forall f \in H$.

i.e. $E(\delta) = 0 \iff \|E(\delta)g\|^2 = 0$ here δ is fixed

all g & δ.

Rmk: $\delta = Y$, then $E(Y) = I$, such maximal g $\nexists$.

orthogonal.

Commuting.

2. Extension of a Spectral Measure. Product Measures

Thm 1. $\begin{cases} E(\delta_1)E(\delta_2) = E(\delta_2)E(\delta_1) = E(\delta_1 \cap \delta_2) \\ E(\phi) = 0 \\ \delta_1 \subseteq \delta_2 \Rightarrow E(\delta_1) \le E(\delta_2) \end{cases}$

The construction of a spectral measure usually begins with a countably additive projection-valued function defined on an algebra $\mathscr{A}^0$ (which is not necessarily a σ-algebra). Therefore the problem naturally arises to extend such a function to a measure defined at least on the minimal σ-algebra containing $\mathscr{A}^0$. Besides, it is not always easy to verify the countable additivity. The questions related to this matter are considered in this section.

1. Let $\mathscr{A}^0$ be an algebra of subsets $\delta \subset Y$ and E^0: $\mathscr{A}^0 \to \mathscr{P}(H)$ be a (finitely) additive function, i.e. such that for any disjoint $\delta_k \in \mathscr{A}^0$, $k = 1, \ldots, n$

$$E^0(\delta) = \sum_k E^0(\delta_k), \quad \delta = \bigcup_k \delta_k. \tag{1}$$

Suppose moreover $E^0(Y) = I$. Theorem 1 is valid for E^0 (the proof requires only the property of finite additivity).

Thm 2.8.8. $P, P_k, k \geq 1$ proj. and $P = w\text{-}\lim P_k$, then $P = s\text{-}\lim P_k$

Given $f \in H$ consider the scalar-valued function $\mu_f^0(\delta)$ on $\mathscr{A}^0$ defined by $\mu_f^0(\delta) = (E^0(\delta)f, f)$, $\delta \in \mathscr{A}^0$. The function E^0 may be countably additive, i.e. for any countable collection of disjoint sets $\delta_k \in \mathscr{A}^0$ (1) holds whenever $\delta = \bigcup_k \delta_k \in \mathscr{A}^0$.

Let us present several assertions simplifying the verification of countable additivity.

LEMMA 1. *Let E^0 be an additive projection-valued function on an algebra $\mathscr{A}^0$ of subsets of Y. If for each $f \in H$ the function μ_f^0 is countably additive on $\mathscr{A}^0$ then so is E^0.*

Proof. Let $\delta_k \in \mathscr{A}^0$, $k = 1, 2, \ldots$, be disjoint and $\delta = \bigcup_k \delta_k \in \mathscr{A}^0$. By the hypotheses for every $f \in H$

2.8.9.

$4\phi(x,y) = \phi(x+y) - \phi(x-y) + i\phi(x+iy) - i\phi(x-iy)$

$$(E^0(\delta)f, f) = \mu_f^0(\delta) = \sum_k \mu_f^0(\delta_k) = \sum_k (E^0(\delta_k)f, f).$$

Using (2.4.9) we can obtain the same for the sesqui-linear forms

$$(E^0(\delta)f, g) = \sum_k (E^0(\delta_k)f, g) \quad (\forall f, g \in H),$$

P46 Cool.

i.e. $E^0(\delta) = w\text{-}\Sigma_k E^0(\delta_k)$. By Theorem 2.8.8 the series is strongly convergent. □

$\Rightarrow E^0(\delta) = s\text{-}\Sigma_k E^0(\delta_k) \Rightarrow$ completely additive.

LEMMA 2. *Let E^0 be an additive projection-valued function on an algebra $\mathscr{A}^0$, upper semi-continuous at $\varnothing$ i.e. s-$\lim E^0(\partial_k) = 0$ for any decreasing sequence $\partial_n \in \mathscr{A}^0$ with $\bigcap_n \partial_n = \varnothing$. Then E^0 is countably additive on $\mathscr{A}^0$.*

Yeah

Proof. Let δ_k, δ be the same as in the proof of Lemma 1. Put $\partial_n = \bigcup_{k>n} \delta_k = \delta \setminus \bigcup_{k \leq n} \delta_k$. Then $\partial_n \in \mathscr{A}^0$ and $\bigcap_n \partial_n = \varnothing$. Therefore $E^0(\partial_n) = E^0(\delta) - \Sigma_{k \leq n} E^0(\delta_k) \xrightarrow{s} 0$, which is equivalent to (1). □

$x \in \bigcap_n \partial_n \Leftrightarrow x \in \partial_n, \forall n \geq 1 \Leftrightarrow x \in \bigcup_{k>n} \delta_k, \forall n \geq 1$ or $x \in \delta$ and $x \notin \bigcup_{k \leq n} \partial_n$ ↓ it's impossible

$\begin{cases} \partial_0 = \bigcup_{k>0} \delta_k = \bigcup_{k=1}^{\infty} \delta_k \\ \partial_1 = \bigcup_{k=2}^{\infty} \delta_k \\ \vdots \\ \partial_n \downarrow \phi \text{ if } \end{cases}$ $\forall x \in \delta = \bigcup \delta_k$, $x \notin \partial_n$ for some n $\Rightarrow \bigcap \partial_n = \phi \Rightarrow E^0(\partial_n) \xrightarrow{s} 0$

$\forall f \in H$

$\Rightarrow E^1(W')\, E^0(\delta_i'')\, f$

$= E^1(W') \left[\sum_i E^0(\delta_i'')\, f \right]$

$= \sum_i \boxed{E^1(W')\, E^0(\delta_i'')\, f}$

$= \sum_i \left[\sum_k E^0(\delta_k')\, E^0(\delta_i'')\, f \right]$

$= \sum_k \left(\sum_i E^0(\delta_k')\, E^0(\delta_i'')\, f \right)$ orthogonal sym.

change the order $=$

$\Downarrow$

$= \sum_{k,l} \left(E^0(\delta_k')\, E^0(\delta_l'')\, f \right)$

$E^0(\delta_k')\, E^0(\delta_l'')\, f \perp E^0(\delta_{k'}')\, E^0(\delta_{l'}'')\, f$ if $\begin{cases} k = k' \\ l = l' \end{cases}$ are not s.t.

they are equal b/c.

$\exists$

① $\sum_{k,l} a_{kl}$

② $\sum_k \sum_l a_{kl} = \sum_l \sum_k a_{kl}$

① $\neq$ ②

$\Rightarrow$

P46. $\{P_\alpha\}_\alpha$ fam. of proj. (commuting pairwise)

$H_\alpha = P_\alpha H$, $P_0 = P_{H_0}$, $H_0 = \bigcap_\alpha H_\alpha$.

Then P_0 is called the greatest lower bound of $\{P_\alpha\}$.

$$P_0 \triangleq \inf_\alpha P_\alpha$$

but this $\not\Rightarrow$ $(P_0 x, x) = \inf_\alpha (P_\alpha x, x) \quad \forall x \in H$. ⋯ (3)

for (3) to be valid, we have

Th 2.8.9 P47.

$\{P_\alpha\}$ be proj. commuting pairwisely and $\{P_\alpha\}$ closed under multiplication.

here $\hat{E}(\sigma) = \inf\{E(\omega) : \omega \supseteq \sigma, \omega \in A'\}$ indexed by ω

it's a family of proj. "commuting pairwisely" &

$E'(\omega_1) E'(\omega_2) = E'(\omega_1 \cap \omega_2)$ is still in this family b/c

$\omega_1 \supseteq \sigma, \omega_2 \supseteq \sigma \Rightarrow \omega_1 \cap \omega_2 \supseteq \sigma$

$\omega_1 \in A', \omega_2 \in A' \Rightarrow \omega_1 \cap \omega_2 \in A'$.

So. closed under multiplication

$$\Rightarrow (\hat{E}(\sigma) f, f) = \inf_{\substack{\omega \supseteq \sigma \\ \omega \in A'}} \{(E'(\omega) f, f)\}, \text{ for any } f \in H. \quad (1)$$

and $\hat{E}$ is still a projection b/c

$$(*) \quad \hat{E}(\sigma) = P_{H_0}, \quad H_0 = \bigcap_{\substack{\omega \supseteq \sigma \\ \omega \in A'}} E'(\omega) H$$

and RHS of (1) $= \inf_{\substack{\omega \supseteq \sigma \\ \omega \in A'}} \{\mu_f'(\omega)\} \triangleq (\mu_f')^{\wedge}(\sigma)$

So. $(**)$ $(\hat{E}(\sigma) f, f) = (\hat{\mu}_f)(\sigma)$. choose $\sigma \subseteq \gamma$ s.t.

$\hat{E}(\sigma) + \hat{E}(\gamma \setminus \sigma) = E'(\gamma) = I$

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H) be a (finitely) additive
$\ldots, n$

(1)

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$\delta = \bigcup_k \delta_k \in \mathscr{A}^0$. By the

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n $\mathscr{A}^0$.
f Lemma 1. Put $\partial_n =$
$\varnothing$. Therefore $E^0(\partial_n) =$
□

2. Let E^0 be a countably additive projection-valued function on an algebra $\mathscr{A}^0$ of subsets of Y and let $E^0(Y) = I$. Let us show that E^0 can be extended to a certain σ-algebra (to a spectral measure) using for this purpose the standard process of extension of a scalar measure (we recommend the reader to consult systematically §1.3, Sub-§§10–16 when reading the chapter). $\mu_f(\delta) = \|E(\delta)f\|^2$ if $\delta \in A^0$

For $f \in H$ consider the measure μ_f equal to the standard extension of the countably additive function μ_f^0, $\hat{\mu}_f$ being the corresponding outer measure defined on all subsets of Y.

First, we extend E^0 to the set $\mathscr{A}^1$ of all countable unions of the sets in $\mathscr{A}^0$. Namely, for $\omega \in \mathscr{A}^1$ put

Note: From (*) we know we can actually def the wk-convergence, then it has to be the strong convergence.

1 $$E^1(\omega) = s\text{-}\sum_k E^0(\delta_k), \tag{2}$$

where $\omega = \bigcup_k \delta_k$, $\delta_k \in \mathscr{A}^0$. Then

2 $$(E^1(\omega)f, f) = \sum_k \mu_f^0(\delta_k) = \mu_f(\omega), \quad \forall f \in H, \quad \cdots (*)$$

and consequently definition (2) does not depend on the choice of δ_k (since this is true for scalar measures). If $\omega', \omega'' \in \mathscr{A}^1$ and $\omega' = \bigcup_k \delta_k'$, $\omega'' = \bigcup_k \delta_k''$ are disjoint decompositions then $\omega' \cap \omega'' = \bigcup_{k,l} (\delta_k' \cap \delta_l'')$ also is a disjoint decomposition. Then

3 $$E^1(\omega' \cap \omega'') = s\text{-}\sum_{k,l} E^0(\delta_k' \cap \delta_l'') = s\text{-}\sum_{k,l} E^0(\delta_k')\, E^0(\delta_l'') = E^1(\omega')\, E^1(\omega''). \quad \text{LHS pg.}$$

This equality extends to the case of a finite number of sets in $\mathscr{A}^1$.

For an arbitrary $\delta \subset Y$ define its outer measure $\hat{E}(\delta)$ by

$$\hat{E}(\delta) = \inf\{E^1(\omega) : \omega \supset \delta, \omega \in \mathscr{A}^1\}, \tag{3}$$

P46

(the infimum of a family of projections is defined in §2.8, Sub-§4). The set of projections on the right-hand side of (3) is closed under multiplication. Therefore in view of Theorem 2.8.9 P44

$$(\hat{E}(\delta)f, f) = \hat{\mu}_f(\delta) \quad (\forall \delta \subset Y, \forall f \in H). \tag{4}$$

$= \inf(E(\omega)f, f) = \inf \mu_f(\omega) = \hat{\mu}_f(\delta)$. cool.

Consider now those $\delta \subset Y$ for which

$$\hat{E}(\delta) + \hat{E}(Y \setminus \delta) = I. = E^0(Y) \tag{5}$$

The set of such δ's is denoted by $\mathscr{A}(E^0)$. The σ-algebra $\mathscr{A}(\mu_f^0)$ (see §1.3, Sub-§11) is denoted by $\mathscr{A}(f)$.

THEOREM 3. *The set $\mathscr{A}(E^0)$ is a σ-algebra containing $\mathscr{A}^0$ and admits the following description*

$$\mathscr{A}(E^0) = \bigcap_{f \in H} \mathscr{A}(f). \tag{6}$$

(4) ⇒ $\delta \in A(\mu_f^0) \iff$
$(\hat{E}(\delta)f, f) + (\hat{E}(Y \setminus \delta)f, f) = (f, f)$

Conversely.

$\sigma \in \mathcal{A}(H)$ $\forall f \Rightarrow$

$\hat{\mu}_f(\sigma) + \hat{\mu}_f(X \setminus \sigma) = \hat{\mu}_f(X)$ i.e.

by (4) $(\hat{E}(\sigma) f, f) + (\hat{E}(X \setminus \sigma) f, f) = (f, f)$ $\forall f$

$\Updownarrow$?

$\hat{E}(\sigma) + \hat{E}(X \setminus \sigma) = \hat{E}(X) = I$ $\forall f \in H$

$(P_1 f, f) + (P_2 f, f) = (f, f)$ ... (1)

$\Updownarrow$?

$P_1 + P_2 = I$ ... (2)

$\Big\{$ $((P_1+P_2) f, f) = (f, f)$ $\forall f$, then, $A \in B(H)$

(1) $\Rightarrow$ $A \triangleq P_1 + P_2$, $\cancel{(Af, f) = (f, f) \; \forall f}$

(1) $\Rightarrow$ $(P_1 f, g) + (P_2 f, g) = (f, g)$

$(P_1 f, f) - (P_1 g, g)$

b/c $\underline{4(P_1 f, g)} = (P_1 (f+g), f+g) - (P_1(f-g), f-g) + i(P_1(f+ig), f+ig) - i(P_1(f-ig), f-ig)$, $\forall f, g$

$\Rightarrow ((P_1+P_2) f, g) = (f, g)$

$\Rightarrow P_1 + P_2 = I$ (done).

How to apply Lemma 1 to E:

here from (4) we know

$$(\dot{E}(\delta)f, f) = \mu_f(\delta), \quad \forall \delta \in \mathcal{A}(E^0) = \bigcap_{f \in H} \mathcal{A}(f).$$

$\forall f$ μ_f is σ-add on ~~$\mathcal{A}(f)$~~ on $\mathcal{A}(E^0)$

then ~~$E(\delta)$ is~~ $\Rightarrow$ E is asuell.

here we use $\mathcal{A}(E^0)$ as our algebra in Lemma 1.

$(\hat{E}(\delta)f,f) = \mu_f(\delta), \ \forall \delta \in \mathcal{A}(E^0)$ σ-additive.

The restriction of $\hat{E}$ to $\mathcal{A}(E^0)$ is a spectral measure E extending E^0. $= \hat\mu_f(\delta) + \hat\mu_f(Y\setminus\delta)$

Proof. It follows from (4) and (5) that for $\delta \in \mathcal{A}(E^0)$, $f \in H$ yes

$$\hat{\mu}_f(\delta) + \hat{\mu}_f(Y \setminus \delta) = \mu_f(Y) = \|f\|^2 = (\hat{E}(\delta)f,f) + (\hat{E}(Y\setminus\delta)f,f) \quad (7)$$

LHS page.

which implies (see §1.3, Sub-§12) that $\delta \in \mathcal{A}(f)$ and so $\mathcal{A}(E^0) \subset \mathcal{A}(f)$. Conversely, if $\delta \in \mathcal{A}(f)$ for every $f \in H$ then (4) and (7) imply (5). Thus (6) is established. It follows from (6) that $\mathcal{A}(E^0)$ is a σ-algebra. Applying Lemma 1 to E we see that E is countably additive. Thus, E is a spectral measure. □

3. The spectral measure E constructed in Theorem 3 is called *the standard extension* of E^0. Besides E, we consider its restriction to the least σ-algebra $\mathcal{A}^0_{\min}$ containing $\mathcal{A}^0$. The passage to this restriction only diminishes the supply of sets of zero E-measure. The measure E on $\mathcal{A}^0_{\min}$ is uniquely determined by E^0. The following theorem clarifies the precise meaning of the above.

THEOREM 4. (1) *For every $\delta \in \mathcal{A}(E^0)$ there exists $\partial \in \mathcal{A}^0_{\min}$ such that $\delta \subset \partial$ and $E(\partial \setminus \delta) = 0$.*

(2) *If $\tilde{E}$ is a spectral measure extending E^0 to a σ-algebra $\tilde{\mathcal{A}}$ then $\tilde{E} = E$ on $\mathcal{A}^0_{\min}$.*

Proof. (1) Let $\{e_n\}$ be a dense subset of H. In view of Subsection 15 of Section 1.3 there exists a sequence $\{\partial_n\}$ in $\mathcal{A}^0_{\min}$ such that $\delta \subset \partial_n$ and $\mu_{e_n}(\partial_n \setminus \delta) = 0$. Put $\partial = \bigcap_n \partial_n$. Then $\partial \in \mathcal{A}^0_{\min}$, $\delta \subset \partial$ and $\mu_{e_n}(\partial \setminus \delta) = 0$. (2) The result follows directly from the same property of scalar measures. □

It can be shown that the standard extension E satisfies the following property of *N-completeness* (cf. §1.3, Sub-§7): if $\delta \in \mathcal{A}(E^0)$, $E(\delta) = 0$ and $\partial \subset \delta$, then $\partial \in \mathcal{A}(E^0)$. The restriction of E to $\mathcal{A}^0_{\min}$ does not necessarily satisfy this property. The standard extension is the minimal extension among all N-complete extensions, i.e. if in the hypotheses of Theorem 4 E is an N-complete measure then $\tilde{\mathcal{A}} \supset \mathcal{A}(E^0)$ and $\tilde{E} = E$ on $\mathcal{A}(E^0)$.

In the sequel we shall mainly be interested in those questions where the sets of zero E-measure can be neglected. The first part of Theorem 4 means that in this case it is sufficient to consider the measure E on $\mathcal{A}^0_{\min}$. good.

4. If Y is endowed with a metric (or topology) consistent with the structure of a measurable space, the properties of a spectral measure can be investigated in more detail.

Let Y be a complete separable metric space, $\mathcal{A} = \mathcal{A}^B(Y)$ be the σ-algebra of all Borel subsets of Y, and let $(Y, \mathcal{A}, H, E)$ be a spectral measure space. In this case E is called a *Borel* spectral measure. For every $f \in H$ the scalar measure μ_f is a finite Borel measure on Y. Therefore (see §1.3, Sub-§22) "tight."

$$\mu_f(\delta) = \sup\{\mu_f(\delta') : \delta' \subset \delta;\ \delta' \text{ is compact in } Y\}, \quad \forall \delta \in \mathcal{A}^B(Y). \quad (8)$$

Every additive function on $\mathcal{A}^B(Y)$ satisfying (8) is automatically countably additive

P19. Thm 6. $\mathcal{Y}_1, \mathcal{Y}_2$ polish-sp. $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2$ with prod. topo. (metric)

$(\mathcal{Y}_1, E_1, H)$, $(\mathcal{Y}_2, E_2, H)$ meas. spec. space (borel)

and $E_1 \cup E_2$. i.e. $E_1(\delta) \cup E_2(\partial)$, $\forall \delta \in \mathcal{A}^B(\mathcal{Y}_1)$, $\partial \in \mathcal{A}^B(\mathcal{Y}_2)$

Then $\exists !$ Borel meas E on $\mathcal{Y}$ s.t.

$$E(\delta \times \mathcal{Y}_2) = E_1(\delta), \ \forall \delta \in \mathcal{A}^B(\mathcal{Y}_1) \quad \cdots (11)$$

$$E(\mathcal{Y}_1 \times \partial) = E_2(\partial), \ \forall \partial \in \mathcal{A}^B(\mathcal{Y}_2) \quad \cdots (12)$$

proof: $\mathcal{A}_0^0$ be the set of $\triangle \subseteq \mathcal{Y}$ with $\triangle = \delta \times \partial$, $\delta \in \mathcal{A}^B(\mathcal{Y}_1)$, $\partial \in \mathcal{A}^B(\mathcal{Y}_2)$ (13)

and define $E^0(\triangle) = E^0(\delta \times \partial) = E_1(\delta) E_2(\partial)$ ... (14) $\Rightarrow E^0(\triangle)$ is proj.

and $E^0(\mathcal{Y}) = E^0(\mathcal{Y}_1 \times \mathcal{Y}_2) = I$.

Easy to show E^0 is additive (finite) on $\mathcal{A}_0^0$.

Denote by $\mathcal{A}^0$ the fin. collection union of sets in $\mathcal{A}_0^0$.

$\Rightarrow \mathcal{A}^0$ is an algebra ($\mathcal{A}_0^0$ is not). Notice $\mathcal{A}^0_{min} = \mathcal{A}^B(\mathcal{Y})$

We extend E^0 to $\mathcal{A}^B(\mathcal{Y})$.

- E^0 is add. on $\mathcal{A}_0^0 \Rightarrow E^0$ extends to $\mathcal{A}^0$ by additivity.
- Recall Thm 5:

$\mathcal{Y}$ is polish-sp.

$\mathcal{A}^0 \subseteq \mathcal{A}^B(\mathcal{Y})$ is an algebra,

$E^0 : \mathcal{A}^0 \to P(H)$ is add.

If $\forall \triangle \in \mathcal{A}^0$, $f \in H$, we have

$\mu_f^0(\triangle) = (E(\triangle) f, f) = \sup \{ \mu_f^0(w) : \bar{w} \subseteq \triangle, w \in \mathcal{A}^0; \bar{w} \text{ compact in } \mathcal{Y} \} \cdots (9)$

$\Downarrow$

E^0 is σ-additive on $\mathcal{A}^0$.

Pf: we only need μ_f^0 σ-add. $\forall f$.

From P7.

Now: we verify the conditions in Thm 5. We only need to establish (9) for all sets in (13).

Let $\mu_f^1(\delta) = (E_1(\delta) f, f)$, $\mu_f^2(\partial) = (E_2(\partial) f, f)$, From (8)

(8): μ_f is a fin. borel measure if E is. &

$\mu_f(\delta) = \sup \{ \mu_f(\delta') : \delta' \subseteq \delta, \delta' \text{ is compact in } \mathcal{Y} \}, \forall \delta \in \mathcal{A}^B(\mathcal{Y})$

Prg-130.

there $\exists$ cpt sets $\delta' \subseteq \delta$, $\partial' \subseteq \partial$ s.t.

$$\mu_f^1(\delta \setminus \delta') < \varepsilon, \quad \mu_f^2(\partial \setminus \partial') < \varepsilon \quad \cdots (15)$$

The set $\Delta' \triangleq \delta' \times \partial'$ is cpt in Y, and $\Delta' \in A^0$, $\Delta' \subseteq \Delta$

so with (15)

$$\Rightarrow \Delta \setminus \Delta' = [\delta \times (\partial \setminus \partial')] \cup [(\delta \setminus \delta') \times \partial']$$

$$E^0(\Delta \setminus \Delta') = E_1(\delta') E_2(\partial \setminus \partial') + E_1(\delta \setminus \delta') E_2(\partial')$$

$$\mu_f^0(\Delta \setminus \Delta') = (E_1(\delta) E_2(\partial \setminus \partial') f, f) + (E_1(\delta \setminus \delta') E_2(\partial') f, f)$$

~~$= (E_2(\partial \setminus \partial') E_1(\delta) f, f) + \cdots$~~

we need if $P_1 P_2 = P_0 = P_2 P_1$, then

$$(P_1 P_2 f, f) \leq (P_2 f, f)$$

$$P_1 P_2 = P_{H_0} = P_{H_1 \cap H_2} \leq P_{H_2} = P_2 \qquad \text{Yes.}$$

$$\leq (E_2(\partial \setminus \partial') f, f) + (E_1(\delta \setminus \delta') f, f)$$

$$= \mu_f^2(\partial \setminus \partial') + \mu_f^1(\delta \setminus \delta') < 2\varepsilon$$

$\Rightarrow$ (9) is (✓) $\Rightarrow$ E^0 is σ-add. on A^0

By Thm 3. Extend E^0 from A^0 to $A(E^0)$ and $A(E^0)$ is an σ-alg.

and $A(E^0) = \bigcap_{f \in H} A(f)$, $A(f) \triangleq A(\mu_f^0) \supseteq A^B(Y)$

$\Rightarrow A(E^0) \supseteq A^B(Y)$ good. A^0_{min}.

- Then (11) & (12) follow from (14). (✓)

Uniqueness: $(\delta \times Y_2) \cap (Y_1 \times \partial) = \delta \times \partial \Rightarrow E^0(\delta \times \partial)$

$$= E^0((\delta \times Y_2) \cap (Y_1 \times \partial))$$

$$= E^0(\delta \times Y_2) E^0(Y_1 \times \partial)$$

$$= E^1(\delta) E^2(\partial) \quad \Big\} \Rightarrow$$

on A_0^0, E^0 is unique

$\Rightarrow$ on A^0, E^0 is unique

$\Rightarrow A^0_{min}$, E^0 is unique.

$\Rightarrow$ on "$A^B(Y)$. E^0 is unique Borel spec. meas. E.

 proof of that.

(see §1.3, Sub-§23). Let us establish the same for spectral measures, which simplifies the verification of countable additivity.

THEOREM 5. *Let Y be a complete separable metric space, $\mathscr{A}^0 \subset \mathscr{A}^B(Y)$ be an algebra, and $E^0\colon \mathscr{A}^0 \to \mathscr{P}(H)$ be an additive function. Suppose that for any $\Delta \in \mathscr{A}^0$, $f \in H$*

$$\mu_f^0(\Delta) = \sup\{\mu_f^0(\omega)\colon \bar{\omega} \subset \Delta,\ \omega \in \mathscr{A}^0,\ \bar{\omega} \text{ is compact in } Y\}, \tag{9}$$

Then E^0 is countably additive. clear.

Proof. In accordance with Lemma 1 it is sufficient to show that the measures μ_f^0 are countably additive; this follows from (9) in view of Subsection 23 of Section 1.3. □

The *support* supp E of a Borel spectral measure is defined as the least closed set in Y whose complement is of zero E-measure (cf. §1.3, Sub-§20). In the same way as for scalar measures it can be shown that the support always exists and a point $y \in Y$ belongs to supp E if and only if each its neighbourhood has non-zero E-measure. Incidentally, these facts follow directly from their scalar analogues (see §7.3, Sub-§3).

5. Consider now the product of commuting Borel spectral measures (for arbitrary spectral measure spaces $(Y_i, \mathscr{A}_i, H, E_i)$, $i = 1, 2$, with $E_1 \cup E_2$ the function E^0 defined in (14) can be non-countably additive, see Section 5).

Let Y_1, Y_2 be complete separable metric spaces, E_1, E_2 be Borel spectral measures on Y_1, Y_2 acting on the same Hilbert space H. Suppose that E_1, E_2 are commuting:

$$E_1(\delta) \cup E_2(\partial), \quad \forall \delta \in \mathscr{A}^B(Y_1), \forall \partial \in \mathscr{A}^B(Y_2). \tag{10}$$

Let $Y = Y_1 \times Y_2$ be endowed with a metric generating the product topology in Y (see §1.2, Sub-§8).

THEOREM 6. *Under the above assumptions there exists a unique Borel measure E on Y such that*

$$E(\delta \times Y_2) = E_1(\delta), \quad \forall \delta \in \mathscr{A}^B(Y_1), \tag{11}$$

$$E(Y \times \partial) = E_2(\partial), \quad \forall \partial \in \mathscr{A}^B(Y_2) \tag{12}$$

(E is called the product of E_1, E_2).

Proof. Let $\mathscr{A}_0^0$ be the set of $\Delta \subset Y$ with

$$\Delta = \delta \times \partial, \quad \delta \in \mathscr{A}^B(Y_1), \partial \in \mathscr{A}^B(Y_2). \tag{13}$$

For $\Delta \in \mathscr{A}_0^0$ put

$$E^0(\Delta) = E^0(\delta \times \partial) = E_1(\delta) E_2(\partial). \tag{14}$$

P45 Thm 2.8.4

$H_1, H_2 \leq H$. $H_0 = H_1 \cap H_2$ $P_1 = P_{H_1}$, $P_2 = P_{H_2}$, $P_0 = P_{H_0}$. Then TFAE: (a) $P_1 P_2$ is proj. (b) $P_1 \cup P_2$. (c) $H^1 := H_1 \ominus H_0$, $H^2 := H_2 \ominus H_0$ are orthogonal.

Then (a) $\Rightarrow$ $P_1 P_2 = P_0$.

In view of Theorem 2.8.4 it follows from (10) that $E^0(\Delta)$ is a projection. Clearly, $E^0(Y) = E_1(Y_1)E_2(Y_2) = I$. It is not difficult to show that E^0 is additive on $\mathscr{A}_0^0$ (verification is left to the reader; it is completely analogous to that of scalar case). Denote by $\mathscr{A}^0$ the collection of all finite unions of sets of $\mathscr{A}_0^0$. Clearly $\mathscr{A}^0$ is an algebra ($\mathscr{A}_0^0$ is not an algebra). Notice that the minimal σ-algebra containing $\mathscr{A}^0$ is $\mathscr{A}^B(Y)$. Our aim is to extend E^0 to $\mathscr{A}^B(Y)$. The function E_0, being additive on $\mathscr{A}_0^0$, extends to $\mathscr{A}^0$ by additivity. Let us verify that the function E^0 on $\mathscr{A}^0$ satisfies the hypothesis of Theorem 5.

Let $f \in H$. It is sufficient to establish (9) for the sets of the form (13). Let $\mu_f^{(i)}(\cdot) = (E_i(\cdot)f, f)$, $i = 1, 2$. In view of (8) for every $\varepsilon > 0$ there exist compact sets $\delta' \subset \delta$, $\partial' \subset \partial$ such that

$$\mu_f^{(1)}(\delta \setminus \delta') < \varepsilon, \qquad \mu_f^{(2)}(\partial \setminus \partial') < \varepsilon. \tag{15}$$

The set $\Delta' = \delta' \times \partial'$ is compact in Y, $\Delta' \in \mathscr{A}^0$, $\Delta' \subset \Delta$. This together with (15) yields

$$\Delta \setminus \Delta' = [\delta \times (\partial \setminus \partial')] \cup [(\delta \setminus \delta') \times \partial'],$$

$$E^0(\Delta \setminus \Delta') = \underset{\text{proj}}{E_1(\delta)} E_2(\partial \setminus \partial') + E_1(\delta \setminus \delta')E_2(\partial'),$$

$$\mu_f^0(\Delta \setminus \Delta') \leq \mu_f^{(2)}(\partial \setminus \partial') + \mu_f^{(1)}(\delta \setminus \delta') < 2\varepsilon.$$

Thus (9) is proved and so E^0 is countably additive on $\mathscr{A}^0$. By Theorem 3 there exists an extension of E^0 to a spectral measure E on the σ-algebra $\mathscr{A}(E^0) \supset \mathscr{A}^B(Y) = \mathscr{A}_{\min}^0$. Equalities (11) and (12) follow from (14).

Let us show that the spectral measure E is unique. Since $(\delta \times Y_2) \cap (Y_1 \times \partial) = \delta \times \partial$, equalities (11) and (12) in view of (1.1) uniquely determine E on $\mathscr{A}_0^0$ and so on $\mathscr{A}^0$. It follows from Theorem 4 that E is uniquely determined on $\mathscr{A}_{\min}^0 = \mathscr{A}^B(Y)$. □

Theorem 6 can obviously be generalized to the case of an arbitrary finite commuting number of spectral measures.

3. Integral with Respect to a Spectral Measure. Bounded Functions

δ of σ-measure
$\delta \in \mathscr{A}$
$E(\delta) = 0$

1. We proceed to the theory of integration of scalar functions with respect to a spectral measure. To this end we start with *simple* (i.e. taking a finite number of values) functions, then we consider the case of arbitrary bounded functions, and, finally, in Section 4 the case of unbounded functions.

Let $(Y, \mathscr{A}, H, E)$ be a spectral measure space. Denote by $L_\infty(Y, E)$ the set of E-bounded E-measurable functions on Y, the functions coinciding almost everywhere being identified as usual. The space $L_\infty(Y, E)$, endowed with the natural addition, multiplication, complex conjugation and the norm

$$\|\varphi\|_\infty = E\text{-sup}|\varphi(y)| \tag{1}$$

is an involutive commutative Banach algebra with unit $\mathbf{1}$ ($\mathbf{1}(y) = 1$ E–a.e. on Y).

Denote by $\Pi = \Pi(Y, E)$ the set of all E-measurable simple functions on Y.

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CHAPTER 5

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simple functions on Y.

Recall that a function φ is called simple if there exists a partition $\delta_1, \ldots, \delta_n$ of Y into disjoint measurable subsets such that φ is constant on each δ_k, $k = 1, 2, \ldots, n$. If χ_δ is the characteristic function of a set δ then

$$\varphi = \sum_{k \le n} c_k \chi_{\delta_k}, \tag{2}$$

where $\varphi \mid \delta_k \equiv c_k$. The set $\Pi(Y, E)$ is a dense subalgebra of $L_\infty(Y, E)$.

The integral of $\varphi \in \Pi(Y, E)$ *with respect to* E *is the operator* J_φ defined by

$$J_\varphi = \int \varphi \, \mathrm{d}E := \sum_{k \le n} c_k E(\delta_k), \tag{3}$$

where δ_k, c_k are taken from (2) (integration in this chapter is always taken over Y). This definition does not depend on the choice of a representation (2). This follows easily from the finite additivity of the spectral measure.

The basic properties of the integral defined on Π are direct consequences of (3):

$$J_{\alpha\varphi + \beta\psi} = \alpha J_\varphi + \beta J_\psi, \tag{4}$$

$$J_{\varphi\psi} = J_\varphi J_\psi = J_\psi J_\varphi, \tag{5}$$

$$(J_\varphi)^* = J_{\bar\varphi}, \tag{6}$$

$J_\varphi J_{\bar\varphi} = J_\varphi (J_\varphi)^* = J_{\bar\varphi} J_\varphi = (J_\varphi)^* J_\varphi$.

$$J_1 = I, \tag{7}$$

$\Rightarrow J_\varphi$ is normal, $\forall \varphi \in \Pi$

$$(J_\varphi f, g) = \int \varphi \, \mathrm{d}\mu_{f,g}, \tag{8}$$

$= \left(\sum_{k \le n} c_k E(\delta_k) f, g\right) = \sum_{k \le n} c_k (E(\delta_k) f, g) = \sum_{k \le n} c_k \mu_{f,g}(\delta_k) = \int \varphi \, d\mu_{f,g}$.

$$(J_\varphi f, f) = \int \varphi \, \mathrm{d}\mu_f, \tag{9}$$

$$\| J_\varphi f \|^2 = \int | \varphi |^2 \, \mathrm{d}\mu_f, \tag{10}$$

$\le |\varphi|^2_{\sup} \|f\|^2$

$$\| J_\varphi \| = E\text{-sup} \mid \varphi \mid. \tag{11}$$

Let us prove (5), (10) and (11) (the other properties are obvious).

In view of (4) it is sufficient to verify (5) for $\varphi = \chi_{\delta_1}$, $\psi = \chi_{\delta_2}$. Then $\varphi\psi = \chi_{\delta_1 \cap \delta_2}$ and

$$J_{\varphi\psi} = E(\delta_1 \cap \delta_2) = E(\delta_1)E(\delta_2) = J_\varphi J_\psi.$$

Equality (10) follows from (6), (5) and (9):

$$\| J_\varphi f \|^2 = (J_\varphi f, J_\varphi f) = (J_{\bar\varphi} J_\varphi f, f) = (J_{|\varphi|^2} f, f) = \int | \varphi |^2 \, \mathrm{d}\mu_f.$$

Finally, (10) implies that the left-hand side of (11) is not greater than the right-hand side. To ascertain that they are in fact equal, suppose that $E\text{-sup} \, |\varphi| = |c_1|$. Then $E(\delta_1) \neq 0$ and for $f \in H(\delta_1)$ we have $\| J_\varphi f \| = | c_1 | \cdot \| E(\delta_1) f \| = | c_1 | \cdot \| f \|$. This implies (11).

$\|J_\varphi f\| = \cancel{\sum (c_k E(\delta_k), f)}$

$\| \sum c_k E(\delta_k) f \|$

$= \|c_1 E(\delta_1) f\|$

$= |c_1| \|f\| \Rightarrow$ LHS RHS

$J_{\varphi_1\varphi_2} = J_{\varphi_1} J_{\varphi_2}$ in $B(H)$

2. The definition of the integral extends to $L_\infty(Y, E)$ by passage to the limit. Formula (3) defines the mapping

$$\mathbf{J} = \mathbf{J}(E) : \Pi(Y, E) \to \mathbf{B}(H); \qquad \mathbf{J}\varphi = J_\varphi.$$

(11): $\|J_\varphi\| = E\text{-}\sup |\varphi|$.

Equality (11) means that $\mathbf{J}$ is an isometric (and *a fortiori* continuous) mapping from the normed algebra $\Pi(Y, E)$ (with norm (1)) to $\mathbf{B}(H)$. This mapping can be extended to an isometric mapping from $L_\infty(Y, E)$ to $\mathbf{B}(H)$. The extended mapping is denoted by $\mathbf{J}$ as well. The value J_φ of $\mathbf{J}$ at $\varphi \in L_\infty(Y, E)$ is called *the integral of φ with respect to E*. In other words for $\varphi \in L_\infty(Y, E)$ we have $J_\varphi = \int \varphi \, dE :=$ u-lim J_{φ_n}, *where $\{\varphi_n\}$ is an arbitrary sequence of simple functions converging to φ in $L_\infty(Y, E)$.*

$\varphi = \tilde{\varphi}$ a.e. $[E]$; $J_\varphi = J_{\tilde\varphi}$; $J_\varphi : J_\varphi$

Since in $\mathbf{B}(H)$ the linear operations, the multiplication, the norm, and the involution $T \mapsto T^*$ are continuous with respect to u-convergence, it follows that (4)–(6) and (11) hold for arbitrary functions in $L_\infty(Y, E)$. Adjoining (7) to the above properties, we come to the central result of the theory of integration with respect to a spectral measure.

Yes

THEOREM 1. *The mapping $\mathbf{J}: \varphi \mapsto J_\varphi$ is an isometric isomorphism of the Banach algebra $L_\infty(Y, E)$ with unit $\mathbf{1}$ and involution $\varphi \mapsto \bar{\varphi}$ onto a commutative subalgebra of $\mathbf{B}(H)$ with unit I and involution $T \mapsto T^*$.*

Normal op.

$J_\varphi J_{\bar\varphi} = J_{\varphi\bar\varphi} = J_{\bar\varphi\varphi} = J_{\bar\varphi} J_\varphi$

Now we present some useful consequences of Theorem 1. (1) Every operator J_φ is normal. (2) J_φ is self-adjoint iff φ is real E-a.e. on Y. (3) J_φ is unitary iff $|\varphi(y)| = 1$ E-a.e. on Y. (4) Every subspace $H(\delta)$, $\delta \in \mathscr{A}$, reduces all operators J_φ (this follows from (5) with $\psi = \chi_\delta$ because of Theorem 3.6.1).

Theorem 1 does not cover properties (8)–(10) which however hold for arbitrary $\varphi \in L_\infty(Y, E)$ as well. Indeed, if $\varphi \in \Pi$ and $\varphi_n \to \varphi$ in $L_\infty(Y, E)$ then

$$(J_\varphi f, g) = \lim (J_{\varphi_n} f, g) = \lim \int \varphi_n \, d\mu_{f,g} = \int \varphi \, d\mu_{f,g}.$$

bdd. conv. thm for scalar measure integration

In particular, for $g = f$ we get (9). Equality (10), as in the case $\varphi \in \Pi$, can be deduced from (6), (5) and (9). Let us mention also the following result.

(5) $J_{\psi\varphi} = J_\psi J_\varphi = J_\varphi J_\psi$.
(3.6.1) $H(\sigma)$ red T iff $T \cup P_1$
Now $H(\delta)$ red J_φ
$H(\delta) = E(\delta)H$, let $\psi = \chi_\delta$, then $J_\psi = E(\delta)$, thus $J_\psi = P_1$ in (3.6.1)
$J_\psi J_\varphi = J_\varphi J_\psi$
$P_1 T = T P_1$
$T \cup P_1 \Rightarrow H(\delta)$ red all J_φ.

THEOREM 2. *If $\varphi_n \in L_\infty(Y, E)$ are uniformly bounded and $\varphi_n \to \varphi$ E-a.e. then* s-lim $J_{\varphi_n} = J_\varphi$.

here φ_n is not simple anymore, this is why

Proof. Given $f \in H$, it follows from (4) and (10)

$$\|J_\varphi f - J_{\varphi_n} f\|^2 = \|J_{\varphi - \varphi_n} f\|^2 = \int |\varphi - \varphi_n|^2 \, d\mu_f.$$

$J_\varphi \neq u\text{-}\lim J_{\varphi_n}$.

The right-hand side tends to zero as $n \to \infty$ in view of Lebesgue's Theorem. □

3. In conclusion we present one more useful assertion of a technical nature.

LEMMA 3. *Let $\varphi \in L_\infty(Y, E)$, $f \in H$, and $h = J_\varphi f$. Then*

$J_\varphi f = \lim_n J_{\varphi_n} f$

$$\mu_h(\delta) = \int_\delta |\varphi|^2 \, d\mu_f \quad (\forall \delta \in \mathscr{A}) \tag{12}$$

$\varphi_n \to \varphi$ in L_∞ then $\varphi_n \chi_\delta \to \varphi \chi_\delta$

$(E(\delta)h, h) = (E(\delta)J_\varphi f, J_\varphi f) = (E(\delta)J_\varphi f, E(\delta)J_\varphi f) = (J_{\chi_\delta\varphi} f, J_{\chi_\delta\varphi} f) = \text{UP}.$

$\Gamma \uparrow \quad \delta_n = \{\ |\varphi(v)| \le n\}$.

$E(\delta_n) \xrightarrow{\text{s}} I$.

$f_n = E(\delta_n) f \to f$ in H

$\int |\varphi|^2 \, d\mu_{f_n}$

here $|\varphi| \geq 0$ '' measurable — consider $\int_{\delta_n} |\varphi|^2 \, d\mu_f$

$\underline{\underline{h = \int |\varphi|^2 f}}$ $\quad \mu_h(\delta)$

(1) $\cancel{\int |\varphi|^2 d\mu_{f_n}} =$

$f_n = E(\delta_n) f = \int \chi_{\delta_n} f$ "$= h$"

(2) $\mu_{f_n}(\delta) = \mu_h(\delta) = \int_\delta |\chi_{\delta_n}|^2 \, d\mu_f$

(3) $\int |\varphi|^2 \, d\mu_{f_n} = \int |\varphi|^2 \chi_{\delta_n}^2 \, d\mu_f$.

Rmk: $h = \int \varphi f$, $\varphi \in L^\infty$ then

$d\mu_h = |\varphi|^2 \, d\mu_f$

abs. cont.

by passage to the limit.

ntinuous) mapping from
). This mapping can be
. The extended mapping
is called *the integral of* φ
have $J_\varphi = \int \varphi \, dE :=$
ctions converging to φ *in*

ion, the norm, and the
vergence, it follows that
). Adjoining (7) to the
eory of integration with

morphism of the Banach
commutative subalgebra

1. (1) Every operator J_φ
Y. (3) J_φ is unitary iff
, re...es all operators J_φ
6.1,
wever hold for arbitrary
$_\infty(Y, E)$ then

the case $\varphi \in \Pi$, can be
ollowing result.

' *and* $\varphi_n \to \varphi$ *E-a.e. then*

sgue's Theorem. □

a technical nature.

n

P11 $d\tau = \psi \, d\mu$ or $\tau \ll \mu$.

$d\mu_h = |\varphi|^2 d\mu_f$. b/c
$\mu_h(\delta) = \int_\delta |\varphi|^2 d\mu_f$

and for any E-measurable $\Phi \geq 0$

$$\int \Phi \, d\mu_h = \int \Phi \, |\varphi|^2 \, d\mu_f. \tag{13}$$

Proof. Equality (13) follows directly from (12) (see §1.5, Sub-§15). Let us verify (12):

$$\mu_h(\delta) = \| E(\delta) J_\varphi f \|^2 = \| J_{\varphi\chi_\delta} f \|^2 = \int_\delta |\varphi|^2 \, d\mu_f \quad ✓. \qquad \square$$

4. Integral with Respect to a Spectral Measure. Unbounded Functions

$\|J_\varphi f\|^2 = \int |\varphi|^2 d\mu_f \;\; \forall \varphi \in L_\infty(Y, E)$

1. We proceed to the integrals of unbounded functions starting from (3.10).

Denote by $S(Y, E)$ the space of all E-a.e. finite E-measurable functions on Y. Then $S(Y, E)$ is an algebra with unit $\mathbf{1}$ and involution $\varphi \to \bar{\varphi}$ but it is not normed. For every $\varphi \in S(Y, E)$ define the set D_φ as follows

$$D_\varphi = \{ f \in H : \int |\varphi|^2 \, d\mu_f < \infty \}. \tag{1}$$

Lem 3.3. $\varphi \in L_\infty(Y, E), f \in H$. $h \triangleq J_\varphi f$. Then $\mu_h(\delta) = \int_\delta |\varphi|^2 d\mu_f$ & $\Phi \geq 0$ or $\Phi \in S^+(Y, E)$ $\int \Phi \, d\mu_h = \int \Phi |\varphi|^2 d\mu_f$.

LEMMA 1. *The set* D_φ *is linear and dense in H.*

Proof. It follows from (1.8) that (1.8) $f \in H$, $\mu_f(\delta) \triangleq (E(\delta)f, f) = \|E(\delta)f\|^2$

$$\mu_{f+g}(\delta) = \| E(\delta)(f+g) \|^2 \leq 2 \| E(\delta) f \|^2 + 2 \| E(\delta) g \|^2 = 2\mu_f(\delta) + 2\mu_g(\delta).$$

This implies that D_φ is linear. Put

$$\delta_n = \{ y \in Y : |\varphi(y)| \leq n \}.$$

s-lim $E(\delta_n) = E(\bigcup \delta_n)$. check $E(\bigcup \delta_n) = E(Y) = I$ b/c $E(Y \setminus (\bigcup \delta_n)) = 0$ $E(Y) = I = E(Y \setminus (\bigcup \delta_n)) + E(\bigcup \delta_n) = 0 + E(\bigcup \delta_n)$

The set $Y \setminus \bigcup_n \delta_n$ has zero E-measure and so $E(\delta_n) \xrightarrow{s} I$ by Theorem 1.2. For any $f \in H$ we have $f_n = E(\delta_n) f \to f$. To show that D_φ is dense it is sufficient to check that $f_n \in D_\varphi$. But in view of Lemma 3.3 $f_n = J_{\chi_{\delta_n}} f$

$$\int |\varphi|^2 \, d\mu_{f_n} = \int |\varphi|^2 \chi_{\delta_n}^2 \, d\mu_f \leq n^2 \int d\mu_f = n^2 \| f \|^2. \;\; \cdots (*) \qquad \square$$

$\Rightarrow f_n \in D_\varphi \Rightarrow \bar{D}_\varphi = H$

$\int d\mu_f = \|f\|^2$ b/c $\mu_f(\delta) = \|E(\delta)f\|^2$ now $\int d\mu_f = \mu_f(Y) = \|E(Y)f\|^2 = \|f\|^2 \cdots (*)$

A sequence of functions $\{\varphi_n\}$ is called *M-convergent* to $\varphi \in S(Y, E)$ if

$$\varphi_n \in L_\infty(Y, E) \quad (\forall n); \qquad \varphi_n(y) \to \varphi(y) \quad (E\text{-a.e.});$$

$$\exists C > 0 : |\varphi_n(y)|^2 \leq C(|\varphi(y)|^2 + 1) \quad (\forall n, E\text{-a.e.}).$$

In particular, the sequence of *truncations* $\varphi_{[n]}$ *M*-converges to $\varphi \in S(Y, E)$, where $\varphi_{[n]} := \chi_{\delta_n} \varphi$ and δ_n is defined by (2).

Suppose $\{\varphi_n\}$ *M*-converges to $\varphi \in S(Y, E)$. Then for $f \in D_\varphi$ (3.4) and (3.10) yield

L_∞ (3.4) $J_{\alpha\varphi + \beta\psi} = \alpha J_\varphi + \beta J_\psi$ & (12) 3.10 $\|J_\varphi f\|^2 = \int |\varphi|^2 d\mu_f$.

$$\| J_{\varphi_m} f - J_{\varphi_n} f \|^2 = \int |\varphi_n - \varphi_m|^2 \, d\mu_f \to 0 \quad (m, n \to \infty),$$

$\Downarrow$ J_{φ_n} has a strong limit ~~not $J_{\varphi_n} \notin B(H)$~~

$f \in D_\varphi$ means $\int |\varphi|^2 d\mu_f < \infty$

$h \triangleq$ s-lim J_{φ_n}, $J_{\varphi_n} \in B(H)$.

$(B(H), SOT)$ is complete

No. $\|J_{\varphi_n}\|$ is not bdd. b/c $f \in D_\varphi$ not $\forall f \in H$

$J_\varphi = \int \varphi \, dE$

$J_\varphi f = \int \varphi \, dE_f$, when $\varphi_n \xrightarrow{M} \varphi$.

$= \lim_{n \to \infty} J_{\varphi_n} f$, maybe $\overset{?}{=} \varphi 1_{\delta_n}$.

$D_{\varphi+\psi} \qquad D_\varphi$

$\parallel \qquad\qquad \parallel$

$f \in D J_{\varphi+\psi} = D J_\varphi$, where $\psi \in L^\infty(\Sigma, E)$

$g \in D J_\varphi$

$\Updownarrow \qquad\qquad \Updownarrow$

$\int |\varphi+\psi|^2 \, d\mu_f < \infty \qquad \int |\varphi|^2 \, d\mu_f < \infty$

now $= \int |\varphi|^2 + |\psi|^2 + 2\mathrm{Re}\langle \varphi, \psi \rangle \, d\mu_f$

then $\int |\varphi|^2 \, d\mu_f < \infty$

always have $\int |\psi|^2 \, d\mu_f < \infty$ $\Big\} \Rightarrow \int |\langle \varphi, \psi \rangle| \, d\mu_f < \infty$

on the other hand. $\int |\varphi+\psi|^2 \, d\mu_f < \infty$

$\Rightarrow \quad \varphi = (\varphi+\psi) - \psi \in D J_\varphi$ as well

so we ~~fin~~ showed $D J_{\varphi+\psi} = D J_\varphi$.

here $\varphi \in S(Y, E)$, $f \in D_\varphi$

and hence the limit $h = \lim J_{\varphi_n} f$ exists. It does not depend on $\{\varphi_n\}$, since for any $\{\varphi'_n\}$ M-convergent to φ we have

def $J_\varphi : D_\varphi \to H$.

$$\| J_{\varphi'_n} f - J_{\varphi_n} f \|^2 = \int | \varphi'_n - \varphi_n |^2 \, d\mu_f \to 0.$$

Put

$$J_\varphi f := \lim J_{\varphi_n} f \quad (\forall f \in D_\varphi). \tag{3}$$

The mapping $f \mapsto J_\varphi f$ is obviously linear. *The operator J_φ defined by* (3) (*where $\{\varphi_n\}$ is an arbitrary sequence M-convergent to φ*) *with domain D_φ is called the integral of $\varphi \in S(Y, E)$ with respect to E.* We also use the following notation $J_\varphi = \int \varphi \, dE$, $J_\varphi f = \int \varphi \, dEf$. $J_\varphi f = \int (\varphi \, dE) f = \int \varphi \, dE f$.

Passing to the limit in (3), we get

$$\| J_\varphi f \|^2 = \lim \| J_{\varphi_n} f \|^2 = \lim \int | \varphi_n |^2 \, d\mu_f = \int | \varphi |^2 \, d\mu_f.$$

Therefore formula (~~13~~ 10) remains valid for unbounded φ:

$$\| J_\varphi f \|^2 = \int | \varphi |^2 \, d\mu_f \quad (\forall f \in D_\varphi). \tag{4}$$

(8), (9), (10) true for φ unbdd and $f \in D_\varphi$, $g \in H$

The same can be said about (3.8) and (3.9):

(1.14) $|\mu_{f,g}|(\sigma) \le \sqrt{\mu_f(\sigma)} \cdot \sqrt{\mu_g(\sigma)}$

(1.15) $|\mu_{f,g}|(Y) \le \|f\| \cdot \|g\|$

$f \in D_\varphi$ means $\int |\varphi|^2 d\mu_f < \infty$

$$(J_\varphi f, g) = \int \varphi \, d\mu_{f,g} \quad (\forall f \in D_\varphi, \forall g \in H), \tag{5}$$

$$(J_\varphi f, f) = \int \varphi \, d\mu_f \quad (\forall f \in D_\varphi). \tag{6}$$

P13 (1.5.13): $\varphi_1, \varphi_2 \in L_2(Y, \mu_1), L_2(Y, \mu_2)$ resp. Then $\varphi_1 \cdot \varphi_2 \in L_1(Y, |\tau|)$, where τ s.t. $|\tau(\sigma)|^2 \le \mu_1(\sigma)\mu_2(\sigma)$.

i.e. $\varphi \in L_2(Y, \mu_f)$, $1 \in L_2(Y, \mu_g)$ b/c $\int 1 \, d\mu_g = \|g\|^2 < \infty$

It is sufficient to establish (5). Let $\{\varphi_n\}$ M-converge to φ. Then in accordance with (1.14) and (1.5.13) $(J_{\varphi_n} f, g) \to (J_\varphi f, g)$ too.

$$\left| \int \varphi_n \, d\mu_{f,g} - \int \varphi \, d\mu_{f,g} \right|^2 \le \int | \varphi_n - \varphi |^2 \, d\mu_f \int d\mu_g \to 0.$$

$\Rightarrow \left|\int \varphi_1 \varphi_2 \, d\tau\right|^2 \le \int |\varphi_1|^2 d\mu_1 \int |\varphi_2|^2 d\mu_2$.

Now, passing to the limit in (3.8), we obtain (5). (3.8): $(J_\varphi f, g) = \int \varphi \, d\mu_{f,g}$

Let $\varphi \in S(Y, E)$, $\psi \in L_\infty(Y, E)$. If $\{\varphi_n\}$ M-converges to φ, then $\{\varphi_n + \psi\}$ obviously M-converges to $\varphi + \psi$. Passage to the limit in $J_{\varphi_n} f + J_\psi f = J_{\varphi_n + \psi} f$ shows that under the above assumptions

$\psi_n = \psi$

$$J_\varphi f + J_\psi f = J_{\varphi + \psi} f \quad (\forall f \in D_\varphi). \tag{7}$$

$\psi \in L_\infty \Rightarrow D_\psi = H$.

Below this fact will be considerably extended (see Theorem 3).

The following lemma indicates conditions on $\{\varphi_n\}$ ensuring the convergence of $J_{\varphi_n} f$ to $J_\varphi f$. These conditions are considerably weaker than the M-convergence. In particular, they depend on $f \in D_\varphi$. Moreover, equalities (3.12) and (3.13) hold for unbounded functions.

$|\mu_{f,g}|(\sigma) \le \sqrt{\mu_f(\sigma)} \cdot \sqrt{\mu_g(\sigma)}$

now $\Rightarrow \varphi \cdot 1 \in L_1(Y, \mu_{f,g})$

i.e. $\left|\int |\varphi| \, d\mu_{f,g}\right| \le \int |\varphi|^2 d\mu_f \cdot \int d\mu_g = \|J_\varphi f\|^2 \cdot \|g\|^2 \Rightarrow$ RHS of (5) makes sense. Then

$\Rightarrow$ (**) makes sense

$\varphi_n \in B(Y, \mu) \Rightarrow \varphi_n \in L_2(Y, \mu_f) \Rightarrow$ (**)

Lemma 2. $f \in D_\varphi$ i.e. $\int |\varphi|^2 d\mu_f < \infty$. $h = J_\varphi f$ = "$\lim J_{\varphi_n} f$" $\varphi_n \xrightarrow{m} \varphi$ & $\varphi_n \in L_\infty(Y, E)$

Then. (a) $\left\{ \begin{array}{l} \text{if } \varphi_n \in S(Y,E),\ f \in D_{\varphi_n}\ (\forall n) \text{ and} \\ \varphi_n \to \varphi \text{ in } L^2(Y, \mu_f) \end{array} \right\} \overset{\text{then}}{\Longrightarrow} J_{\varphi_n} f \to h$ in H.

(b) Lemma 3.3 holds for φ, f.

(a). Let $\tilde{\varphi}_n \xrightarrow{M} \varphi$. Then From (*): $\|J_\varphi f\|^2 = \int |\varphi|^2 d\mu_f$, if $f \in D_\varphi$

& (**): $J_\varphi + J_\psi = J_{\varphi+\psi}$ or $J_\varphi^f + J_\psi f = J_{\varphi+\psi} f$, $\forall f \in D_\varphi$ & $\psi \in L_\infty(Y,E)$

$$\Rightarrow \|\underset{\substack{\uparrow \\ S(Y,E)}}{J_{\varphi_n}} f - \underset{\substack{\uparrow \\ L_\infty(Y,E)}}{J_{\tilde{\varphi}_n}} f\|^2 = \|J_{\varphi_n - \tilde{\varphi}_n} f\|^2 = \int |\varphi_n - \tilde{\varphi}_n|^2 d\mu_f \to 0.$$

now $J_{\tilde{\varphi}_n} f \to J_\varphi f \Rightarrow J_{\varphi_n} f \to J_\varphi f$ as well.

Summary of (a)

$$\left\{ \begin{array}{l} f \in D_\varphi,\ f \in D_{\varphi_n},\ \forall n \geq 1 \\ \varphi_n \in S(Y,E) \text{ s.t. } \varphi_n \to \varphi \text{ in } L^2(Y,E) \end{array} \right. \overset{\text{then}}{\Longrightarrow} \underset{\overset{\shortparallel}{h_n}}{J_{\varphi_n} f} \to J_\varphi f \triangleq h$$

$h_n \to h$.

(b) we need show:

If $\left\{ \begin{array}{l} \varphi \in S(Y,E), \\ f \in D_\varphi \\ h \triangleq J_\varphi f \end{array} \right. \overset{\text{then}}{\Longrightarrow} \left\{ \begin{array}{l} \mu_h(\delta) = \int_\delta |\varphi|^2 d\mu_f \quad \text{and for any} \\ \phi \in S_+(Y,E),\ \int \phi\, d\mu_h = \int \phi |\varphi|^2 d\mu_f. \end{array} \right.$

obv. we only need to show $\mu_h(\delta) = \int_\delta |\varphi|^2 d\mu_f$. ... (3.12)

pf: now $\mu_h(\delta) = \|E(\delta) h\|^2 \underset{\substack{E(\delta) \text{ is} \\ \text{cont.}}}{=} \lim \|E(\delta) J_{\tilde{\varphi}_n} f\|^2 = \lim \|J_{\chi_\delta} J_{\tilde{\varphi}_n} f\|^2$

$= \lim \|J_{\chi_\delta \tilde{\varphi}_n} f\|^2$

next "$J_{\chi_\delta} z = E(\delta) z$"

$= \lim \int_\delta |\tilde{\varphi}_n|^2 d\mu_f$

$= \int_\delta |\varphi|^2 d\mu$. (✓)

d on $\{\varphi_n\}$, since for any

(3)

$_\varphi$ *defined by* (3) (*where domain* D_φ *is called the* the following notation

(4)

(5)

(6)

Then in accordance with

).

s to φ, then $\{\varphi_n + \psi\}$ $J_{\varphi_n} f + J_\psi f = J_{\varphi_n + \psi_n} f$

(7)

m 3).

ring the convergence of the M-convergence. In .12) and (3.13) hold for

$J_{\varphi_n} f \to J f \Leftarrow f \in D_{\varphi_n}, f \in D_\varphi$ ① and $\varphi_n, \varphi \in S(Y, E)$ ... ② & $\varphi_n \to \varphi$ in $L_2(Y, \mu_f)$

LEMMA 2. *Let* $\varphi \in S(Y, E)$, $f \in D_\varphi$ *and* $h = J_\varphi f$. *Then*

(*a*) *if* $\varphi_n \in S(Y, E)$, $f \in D_{\varphi_n}$ $(\forall n)$ *and* $\varphi_n \to \varphi$ *in* $L_2(Y, \mu_f)$, *then* $J_{\varphi_n} f \to h$;

(*b*) *Lemma 3.3 holds for* φ, f.

Proof. (a) Let $\{\tilde\varphi_n\}$ M-converge to φ. Then in view of (7) and (4)

$$\| J_{\varphi_n} f - J_{\tilde\varphi_n} f \|^2 = \| J_{\varphi_n - \tilde\varphi_n} f \|^2 = \int |\varphi_n - \tilde\varphi_n|^2 \, d\mu_f \to 0.$$

Since $J_{\tilde\varphi_n} f \to J_\varphi f$, we have $J_{\varphi_n} f \to J_\varphi f$.

(b) It is sufficient to prove (3.12) which follows from

very important

$$\mu_h(\delta) = \| E(\delta) h \|^2 = \lim \| E(\delta) J_{\tilde\varphi_n} f \|^2 = \lim \int_\delta |\tilde\varphi_n|^2 \, d\mu_f = \int_\delta |\varphi|^2 \, d\mu_f. \qquad \square$$

2. Now we are going to ascertain properties (3.4)–(3.6) for $\varphi \in S(Y, E)$. First of all, notice that $D_\varphi \subset D_\psi$ whenever $\varphi, \psi \in S(Y, E)$ and

$$|\psi(y)| \leqslant C(1 + |\varphi(y)|) \quad (E\text{-a.e.}) \qquad (8)$$

? how ~~to show this~~?

(The converse is also true: if $D_\varphi \subset D_\psi$ then (8) holds with some constant C.)

THEOREM 3. *Let* $\varphi, \psi \in S(Y, E)$, $\alpha, \beta \in \mathbf{C}$. *Then*

$$D(\alpha J_\varphi + \beta J_\psi) = D_\varphi \cap D_\psi, \qquad (9)$$

$$\alpha J_\varphi + \beta J_\psi \subset J_{\alpha\varphi + \beta\psi}. \qquad (10)$$

closure

$\alpha\varphi_n + \beta\psi_n \in L_\infty(Y, E) \Rightarrow f \in D_{\alpha\varphi_n + \beta\psi_n} = H \Rightarrow \cdots (*)$

Proof. Equality (9) holds by definition. If $f \in D_\varphi \cap D_\psi$ then in view of (1) $\alpha\varphi + \beta\psi \in L_2(Y, \mu_f)$. Let $\{\varphi_n\}$, $\{\psi_n\}$ M-converge to φ, ψ respectively. Then $\alpha\varphi_n + \beta\psi_n \to \alpha\varphi + \beta\psi$ in $L_2(Y, \mu_f)$. Now (10) follows from Lemma 2(a) and from (3.4):

$$J_{\alpha\varphi + \beta\psi} f = \lim J_{\alpha\varphi_n + \beta\psi_n} f \overset{(*)}{=} \lim (\alpha J_{\varphi_n} f + \beta J_{\psi_n} f) = \alpha J_\varphi f + \beta J_\psi f. \qquad \square$$

THEOREM 4. *Let* $\varphi, \psi \in S(Y, E)$. *Then*

$$D(J_\varphi J_\psi) = D_{\varphi\psi} \cap D_\psi, \qquad (11)$$

$$J_\varphi J_\psi \subset J_{\varphi\psi}. \qquad (12)$$

Proof. Let $f \in D(J_\varphi J_\psi)$. This means $f \in D_\psi$ and $h = J_\psi f \in D_\varphi$. The latter, in turn, means $f \in D_{\varphi\psi}$. Indeed, in view of Lemma 2(b) $\int |\varphi\psi|^2 \, d\mu_f = \int |\varphi|^2 \, d\mu_h$. This proves (11).

To establish (12) suppose first that φ is bounded and $\{\psi_n\}$ M-converges to ψ. Then $\{\varphi\psi_n\}$ M-converges to $\varphi\psi$ and so for $f \in D_\psi$ we have

$$J_\varphi J_\psi f = \lim J_\varphi (J_{\psi_n} f) = \lim J_{\varphi\psi_n} f = J_{\varphi\psi} f.$$

Let $\varphi \in S(Y, E)$ and $\{\varphi_n\}$ M-converge to φ. Notice that $f \in D_\varphi \cap D_{\varphi\psi}$ implies the convergence of $\varphi_n\psi$ to $\varphi\psi$ in $L_2(Y, \mu_f)$. In view of Lemma 2(a) this yields ⇓

D_ψ

φ is bdd ⇓ $D(J_\varphi J_\psi) = D_{\varphi\psi} \cap D_\psi = D_\psi$

$\int |\psi|^2 |\varphi|^2 d\mu_f < \infty \Leftrightarrow \int |\varphi\psi|^2 d\mu_f < \infty$ ⇒ from $|\varphi_n|^2 \le C(|\varphi|^2 + 1)$

$|\varphi_n\psi|^2 \le |\varphi|^2|\psi|^2 + C|\psi|^2$ and LCT ⇒ $\in L^2$.

where $\|E(\sigma)f\|^2 =$

$$\int_\sigma |\bar\varphi[n]|^2 d\mu_g$$

$$\overline{\varphi[n]} = \bar\varphi \, \chi_{\sigma_n} \quad \text{where} \quad \sigma_n \triangleq \{y \mid |\bar\varphi(y)| \le n\}$$

and $\overline{\varphi[n]} \in L_\infty(Y, E)$

and so $f_n = T_{\overline{\varphi[n]}}\, g$ is well-def. recall $\begin{cases} g \in D(T_\varphi^*) \\ (T_\varphi f, g) = (f, h) \end{cases}$

$$\|E(\sigma) T_{\overline{\varphi[n]}}\, g\|^2$$

$f_n \in D_\varphi$?

b/c $\int |\varphi|^2 d\mu_{f_n}$ b/c

$$\|E(\sigma) f_n\|^2 = \mu_{f_n}(\sigma) = (E(\sigma) f_n, f_n)$$

$$\|E(\sigma) f\|^2 = \mu_f(\sigma) = (E(\sigma) f, f)$$

$\bar{\varphi}_{[n]} = \bar{\varphi}\chi_{\delta_n}$, $\delta_n \triangleq \{y \in Y \mid |\varphi(y)| \leq n\}$

$\Rightarrow \bar{\varphi}_{[n]} \in L_\infty(Y, E)$ and $\bar{\varphi}_{[n]} \xrightarrow[\text{in } L^2]{\mu} \bar{\varphi}$ ~~a~~ $d\mu_f$ $\cdots$ ①

show $f_n \in D_\varphi$,

we have $\begin{cases} f_n = J_{\bar{\varphi}_{[n]}} g & \text{okay.} \\ g \in D(J_\varphi^*). \end{cases}$ $f_n \in D_\varphi$ iff $\int |\varphi|^2 d\mu_{f_n} < \infty$.

we have $\mu_{f_n}(\delta) = \|E(\delta) f_n\|^2 = \|E(\delta) J_{\bar{\varphi}_{[n]}} g\|^2$

$= \|J_{\chi_\delta \bar{\varphi}_{[n]}} g\|^2 = \int_\delta |\bar{\varphi}_{[n]}|^2 d\mu_g$.

$\boxed{\begin{array}{l} \text{Lem 3. } \varphi \in L_\infty,\ f \in H,\ h = J_\varphi f, \text{ then} \\ \begin{cases} \mu_h(\delta) = \int_\delta |\varphi|^2 d\mu_f & \delta \in A \quad \& \\ \int \phi\, d\mu_h = \int \phi |\varphi|^2 d\mu_f, & \phi \geq 0 \end{cases} \end{array}}$

~~$\int |\varphi|^2 d\mu_{f_n} =$~~

now $\begin{cases} \bar{\varphi}_{[n]} \in L_\infty \\ g \in H \\ f_n = J_{\bar{\varphi}_{[n]}} g \end{cases} \Rightarrow$ $\mu_{f_n}(\delta) = \int_\delta |\bar{\varphi}_{[n]}|^2 d\mu_g$ & (yes).

$\int \phi\, d\mu_{f_n} = \int \phi |\bar{\varphi}_{[n]}|^2 d\mu_g$ $\cdots \Rightarrow$

$\Rightarrow \int |\varphi|^2 d\mu_{f_n} = \int |\varphi|^2 |\bar{\varphi}_{[n]}|^2 d\mu_g$

$= \int \underbrace{|\varphi_{[n]}|^2 |\bar{\varphi}_{[n]}|^2}_{\in L_\infty} d\mu_g < \infty$.

$\Rightarrow f_n \in D_\varphi$.

$$J_\varphi J_\psi f = \lim J_{\varphi_n}(J_\psi f) = \lim J_{\varphi_n \psi} f = J_{\varphi\psi} f. \qquad \square$$

REMARK. If under the hypothesis of Theorem 4

$$|\varphi| + |\psi| \leqslant C(1 + |\varphi\psi|). \tag{13}$$

then it follows from (8) and (11) that $D(J_\varphi J_\psi) = D_{\varphi\psi} = D(J_\psi J_\varphi)$ and

$$J_\varphi J_\psi = J_\psi J_\varphi = J_{\varphi\psi}. \tag{14}$$

Similarly, if under the hypothesis of Theorem 3 $|\varphi| + |\psi| \leqslant C(1 + |\alpha\varphi + \beta\psi|)$, then the inclusion in (10) turns into equality.

Property (3.6) remains true for unbounded φ:

THEOREM 5. *Let* $\varphi \in S(Y, E)$. *Then* $D(J_\varphi^*) = D_\varphi$ *and*

$$J_\varphi^* = J_{\bar\varphi}. \tag{15}$$

Proof. Let $f, g \in D_\varphi$ $(= D_{\bar\varphi})$. It follows from (5) and (1.12) that

$$(f, J_{\bar\varphi} g) = \overline{(J_{\bar\varphi} g, f)} = \int \overline{\bar\varphi \, d\mu_{g,f}} = \int \varphi \, d\mu_{f,g} = (J_\varphi f, g),$$

i. e. $J_{\bar\varphi} \subset J_\varphi^*$. Conversely, let $g \in D(J_\varphi^*)$ and $h = J_\varphi^* g$. Then

$$(J_\varphi f, g) = (f, h) \quad (\forall f \in D_\varphi). \tag{16}$$

Put, in particular, $f_n = J_{\bar\varphi_{[n]}} g$. Then $f_n \in D_\varphi$. Taking into account the equality $\varphi\bar\varphi_{[n]} = |\varphi_{[n]}|^2$, from (12) and (16) we get

$$(f_n, h) = (J_\varphi J_{\bar\varphi_{[n]}} g, g) = (J_{|\varphi_{[n]}|^2} g, g) = \int |\varphi_{[n]}|^2 d\mu_g = \| f_n \|^2.$$

Consequently, $\| f_n \| \leqslant \| h \|$ or, the same,

$$\int |\varphi_{[n]}|^2 d\mu_g \leqslant \| h \|^2.$$

This implies $\int |\varphi|^2 d\mu_g \leqslant \| h \|^2$ which means $g \in D_\varphi$. $\square$

THEOREM 6. *For every* $\varphi \in S(Y, E)$ *the operator* J_φ *is closed and normal.*

Proof. Theorem 3.3.2 implies that J_φ is closed (since $J_\varphi = (J_{\bar\varphi})^*$) while (13) and (14) imply (in view of $2|\varphi| \leqslant |\varphi|^2 + 1$) that J_φ is normal. $\square$

As in the case of bounded φ the operator J_φ is self-adjoint iff $\overline{\varphi(y)} = \varphi(y)$ E-a.e.

The following theorem contains further information about (10) and (12).

THEOREM 7. *Let* $\varphi, \psi \in S(Y, E)$. *Then*

$$\overline{\alpha J_\varphi + \beta J_\psi} = J_{\alpha\varphi + \beta\psi}. \tag{17}$$

$$\overline{J_\varphi J_\psi} = \overline{J_\psi J_\varphi} = J_{\varphi\psi}. \tag{18}$$

$E(\delta) \Rightarrow \mu_{Ef}$ bdd by $|\pi|^2 \mu_f$

by thm: $J_{\chi_{\delta_n}} = E(\delta_n)$

□

(13)

) $(J_\psi J_\varphi)$ and

(14)

$| \leqslant C(1 + |\alpha\varphi + \beta\psi|)$,

(15)

12) that

),

n

(16)

ccount the equality $\varphi\overline{\varphi}_{[n]}$

$= \|J_n\|^2$.

□

closed and normal.

$= (J_{\overline{\varphi}})^*)$ while (13) and

□

djoint iff $\overline{\varphi(y)} = \varphi(y)$

out (10) and (12).

(17)

(18)

Proof. Since the proof of (17) is analogous to that of (18), we ascertain (18) only. Theorem 6 together with (12) means that $J_{\varphi\psi}$ is a closed extension of $J_\psi J_\varphi$. We have to prove that $D(J_\psi J_\varphi)$ is dense in $D(J_{\varphi\psi}) = D_{\varphi\psi}$ in the graph norm. Let $f \in D_{\varphi\psi}$ and $f_n = E(\delta_n)f$ where δ_n is defined in (2). We have seen in the proof of Lemma 1 that $f_n \in D_\varphi$ and $f_n \to f$. Now $J_\varphi f_n = J_\varphi J_{\chi_{\delta_n}} f = J_{\varphi_{[n]}} f$. Note that $\psi\varphi_{[n]} \to \psi\varphi$ in $L_2(Y, \mu_f)$. Thus, in view of Lemma 2(a)

$D_{J(\psi)} = D_\psi$? we need $f_n \in D_{\psi\varphi}$.

$$J_\psi J_\varphi f_n = J_\psi J_{\varphi_{[n]}} f = J_{\psi\varphi_{[n]}} f \to J_{\varphi\psi} f.$$

Thus we have proved that $f_n \in D(J_\psi J_\varphi)$, $f_n \to f$ and $J_\psi J_\varphi f_n \to J_{\varphi\psi} f$. □

LHS. page (yellow)

3. Let us proceed to the commutation properties of J_φ.

meaning: it means $TE(\delta) = E(\delta)T$ for $\forall \delta \in \mathcal{A}$.

(a) THEOREM 8. *(a) An operator $T \in \mathbf{B}(H)$ commuting with a spectral measure E commutes with any J_φ, $\varphi \in S(Y, E)$. (b) The subspaces $H(\delta) = E(\delta)H$, $\delta \in \mathscr{A}$ reduce all the operators J_φ.*

$\varphi \in S(Y, E)$ $\Rightarrow$ (15). $\forall f \in D_\varphi$, $g \in H$ ↳ $(J_\varphi f, g) = \int \varphi \, d\mu_{f,g}$

Proof. (a) For every $f, g \in H$, $\delta \in \mathscr{A}$ we have

(1). Dfn of D_φ (✓)

$$\mu_{Tf,\,g}(\delta) = (E(\delta)\,Tf, g) = (E(\delta)f, T^*g) = \mu_{f,\,T^*g}(\delta), \tag{19}$$

$$\mu_{Tf}(\delta) = \|E(\delta)\,Tf\|^2 = \|TE(\delta)f\|^2 \leqslant \|T\|^2 \mu_f(\delta). \tag{20}$$

Let $\varphi \in S(Y, E)$ and $f \in D_\varphi$. It follows from (1) and (20) that $Tf \in D_\varphi$. According to (19) and (5)

$\int |\varphi|^2 d\mu_{(Tf)} < \infty$ b/c   Yes.

$J_\varphi Tf = TJ_\varphi f$ $\Leftarrow$

$$(J_\varphi Tf, g) = \int \varphi \, d\mu_{Tf,\,g} = \int \varphi \, d\mu_{f,\,T^*g} = (J_\varphi f, T^*g) = (TJ_\varphi f, g).$$

Thus $T \cup J_\varphi$. (by Dfn: $T \cup J_\varphi$ means $TJ_\varphi \subseteq J_\varphi T$ and so $Tf \in D_\varphi$, $\forall f \in D_\varphi$")

P_1. Thm 3.6.1 H_1 red T iff $T \cup P_1$. Then $T \cong E(\delta)$, we need $E(\delta)$ commutes $E(\eta)$ any $\eta \in \mathcal{A}$ yes!! $E(\delta)$ commutes J_φ, $\forall \varphi \in S(Y, E)$ then $E(\delta)H$ red. J_φ

(b) This follows from (a) with $T = E(\delta)$ because of Theorem 3.6.1. (✓) □

If two spectral measures are unitarily equivalent then so are the corresponding operators J_φ. More precisely,

* ~~Hontal~~ Means. of Hilbert

THEOREM 9. *Let E, E' be spectral measures on $(Y, \mathscr{A})$ acting on Hilbert spaces H, H'. Let V be a unitary mapping from H onto H' such that $VE(\delta) = E'(\delta)V$, $\forall \delta \in \mathscr{A}$. Then $VJ_\varphi = J'_\varphi V$, $\forall \varphi \in S(Y, E) = S(Y, E')$.*

The proof is completely analogous to that of Theorem 8(a). $\boxed{d\mu_f = d\mu_{Vf}}$

4. Now we are going to discuss a 'change of variables' in the integrals J_φ. More precisely, we deal with transformations of a spectral measure.

Let $(Y, \mathscr{A}, H, E)$ be a spectral measure space, Z be a set, and π be a mapping of Y onto Z. Denote by $\mathscr{A}'$ the σ-algebra of subsets $\partial \subset Z$ such that $\pi^{-1}(\delta) \in \mathscr{A}$. For $\partial \in \mathscr{A}'$ put $F(\partial) = E(\pi^{-1}(\partial))$. Then F is a spectral measure. Below we use the clear notation μ^E_f, J^F_φ. It follows from the definition of F that

$$\mu^F_{f,\,g}(\partial) = \mu^E_{f,\,g}(\pi^{-1}(\partial)) \quad (\partial \in \mathscr{A}', f, g \in H). \tag{21}$$

$Y \xrightarrow{\pi} Z$
$A' \ni \partial$ iff $\pi^{-1}(\partial) \in A$.
$\partial \in A' \Rightarrow$ define $F(\partial) = E(\pi^{-1}(\partial))$
Then F is a spectral measure

μ^E_f, φJ^F_φ

$\mu^F_{f,g}(\partial) \stackrel{?}{=} (F(\partial)f, g) = (E(\pi^{-1}(\partial))f, g) = \mu^E_{f,g}(\pi^{-1}(\partial))$

$\partial \in A'$ $f, g \in H$

we consider φ(x) as Borel measurable function.

and E defined on Borel-σ-alg.

THEOREM 10. *The inclusions* $\varphi \in S(Z, F)$ *and* $\varphi \circ \pi \in S(Y, E)$ *are equivalent. Besides*

$$\int_Z \varphi \, \mathrm{d}F = \int_Y (\varphi \circ \pi) \, \mathrm{d}E. \tag{22}$$

Proof. For every $\Delta \in \mathscr{A}^B(\mathbf{C})$ we have $\pi^{-1}[\varphi^{-1}(\Delta)] = (\varphi \circ \pi)^{-1}(\Delta)$. Hence φ is F-measurable iff $\varphi \circ \pi$ is E-measurable. Moreover, the two functions are almost everywhere finite simultaneously. The equality $\int_Z |\varphi|^2 \, \mathrm{d}\mu_f^F = \int_Y |\varphi \circ \pi|^2 \, \mathrm{d}\mu_f^E$ follows from (21) (with $g = f$) because of the formula in Section 1.5, Subsection 9. This implies $D(J_\varphi^F) = D(J_{\varphi \circ \pi}^E)$. The equality

$$\int_Z \varphi \, \mathrm{d}\mu_{f,g}^F = \int_Y (\varphi \circ \pi) \, \mathrm{d}\mu_{f,g}^E, \quad f \in D(J_\varphi^F), \, g \in H,$$

which also follows from (21), means that $(J_\varphi^E f, g) = (J_{\varphi \circ \pi}^E f, g)$. This proves (22). □

two Hilbert space. direct sum.

5. In conclusion notice that if $(Y, \mathscr{A}, H_k, E_k)$, $k = 1, 2$, are spectral measure spaces and $H = H_1 \oplus H_2$ then

$$E(\delta) := E_1(\delta) \oplus E_2(\delta), \quad \delta \in \mathscr{A}, \tag{23}$$

$\varphi \in S(Y, E)$ for example let $\varphi = \chi_\delta$, $\delta \in A$.

is a spectral measure on H. The integrals J_φ^E with respect to E admit the decomposition

$$J_\varphi^E = \int \varphi \, \mathrm{d}E = \int \varphi \, \mathrm{d}E_1 \oplus \int \varphi \, \mathrm{d}E_2 = J_\varphi^{E_1} \oplus J_\varphi^{E_2}. \tag{24}$$

then $\int \varphi \, dE_1$, $\int \varphi \, dE_2 = E_2(\delta)$ $= E_1(\delta)$ ↓ then $\int \varphi \, dE = E_1(\delta) \oplus E_2(\delta) = E(\delta)$

5. An Example of Commuting Spectral Measures whose Product is not Countably Additive

The example presented here has been constructed in [23].

not Borel measure

1. Here μ is Lebesgue measure and μ^* is outer Lebesgue measure on $\mathbf{R}$. Measurability means μ-measurability. To construct the example we need the following fact of set theory (see e.g. [22], §16). Halmos §16 P67-72.

There exists a set Y *in* $[0, 1]$ (obviously non-measurable) *such that*

$$\mu^*(Y) = \mu^*([0, 1] \setminus Y) = 1. \tag{1}$$

here Y is not polish.

It follows from (1) that for any measurable set $e \subset [0, 1]$

$\mu^*(\delta)$ below

$$\mu^*(e \cap Y) = \mu^*(e \setminus Y) = \mu(e). \tag{2}$$

Indeed, put $\delta = e \cap Y$, $e' = [0, 1] \setminus e$, $\delta' = e' \cap Y$. Then in view of (1)

$$1 = \mu^*(Y) \leqslant \mu^*(\delta) + \mu^*(\delta') \leqslant \mu(e) + \mu(e') = 1. \tag{3}$$

Therefore all inequalities in (3) should be equalities and, in particular, $\mu^*(\delta) = \mu(e)$. The proof for $e \setminus Y$ is the same.

$S(Y, E)$ *are equivalent.*

(22)

$\varphi \circ \pi)^{-1}(\Delta)$. Hence φ is
wo functions are almost
$_f^F = \int_Y |\varphi \circ \pi|^2 \, d\mu_f^E$ fol-
ction 1.5, Subsection 9.

$g)$. This proves (22). □

spectral measure spaces

(23)

E admit the decomposi-

(24)

vhose Product is not

ıeasure on **R**. Measura-
need the following fact

such that

(1)

1]

(2)

n view of (1)

(3)

in particular, $\mu^*(\delta) =$

It follows from (2) that $e \cap Y$, $e \backslash Y$ are non-measurable provided $\mu(e) > 0$.

2. Let Y satisfy (1) and $\delta = e \cap Y$ for a measurable set $e \subset [0, 1]$. Suppose $\tilde{e} \subset [0, 1]$ is another measurable set such that $\delta = \tilde{e} \cap Y$ and $e_0 := e \cap \tilde{e}$. Then $\delta = e_0 \cap Y$ and hence $(e \setminus e_0) \cap Y = \varnothing$. It follows from (2) that $\mu(e \setminus e_0) = 0$. Clearly $\mu(\tilde{e} \setminus e_0) = 0$ as well. Thus, if $e \cap Y = \tilde{e} \cap Y$ *for measurable* $e, \tilde{e}$ then

$$\mu(e \, \Delta \tilde{e}) = 0, \tag{4}$$

where $e \, \Delta e' := (e \backslash e') \cup (e' \backslash e)$.

Consider now the class $\mathscr{A}$ of the sets $\delta \subset Y$ which can be represented as $\delta = e \cap Y$. Clearly, it is a σ-algebra.

3. Let $H = L_2[0, 1]$ and $E(\delta)$, $\delta \in \mathscr{A}$ be the multiplication by the characteristic function χ_e of the corresponding set e, it follows from (4) that $E(\delta)$ is well defined. Clearly $E(\delta)$ is a projection and $(Y, \mathscr{A}, H, E)$ is a spectral measure space. Let us explain why E is countably additive. Let $\delta_k \in \mathscr{A}$, $\delta_k \cap \delta_l = \varnothing$ for $k \neq l$ and $\delta = \bigcup_k \delta_k$. Then in obvious notation $\delta = e \cap Y$, $e = \bigcup_k e_k$, and (2) implies that $\mu(e_k \cap e_l) = 0$ for $k \neq l$. Therefore $\chi_e(t) = \Sigma_k \chi_{e_k}(t)$ a.e., hence $E(\delta) = \Sigma_k E(\delta_k)$.

4. Equalities (1) hold for $Y_1 = Y$ iff they hold for $Y_2 = [0, 1] \backslash Y$. Let $\mathscr{A}_i$ and E_i, $i = 1, 2$, be the corresponding σ-algebra and spectral measure on $H = L_2[0, 1]$. Clearly, E_1 and E_2 are commuting.

As in (2.14) put

$$E^\circ(\delta \times \partial) = E_1(\delta) E_2(\partial), \quad \delta \in \mathscr{A}_1, \, \partial \in \mathscr{A}_2.$$

Let E be the extension of E° to the algebra $\mathscr{A}_0$ generated by the sets $\delta \times \partial$ (see §1.2, Sub-§2). Then E is additive. Let us show that it is not countably additive.

Given $n \in \mathbf{Z}_+$ consider

$$e_{n,k} = [(k-1)2^{-n}, k2^{-n}), \quad k < 2^n, \qquad e_{n,2^n} = [1 - 2^{-n}, 1].$$

Then $\bigcup_{k=1}^{2^n} e_{n,k} = [0, 1]$. Put $\delta_{n,k} = e_{n,k} \cap Y_1$, $\partial_{n,k} = e_{n,k} \cap Y_2$ and $w_n = \bigcup_{1 \leqslant k \leqslant 2^n} (\delta_{n,k} \times \partial_{n,k})$. The sets $w_n \subset Y_1 \times Y_2$ are decreasing and w_n lies within the $2^{-(n + 1/2)}$-neighbourhood of the diagonal of $[0, 1] \times [0, 1]$. This together with $Y_1 \cap Y_2 = \varnothing$ implies $\bigcap_n w_n = \varnothing$. Nevertheless $E(w_n) = \Sigma_{1 \leqslant k \leqslant 2^n} E_1(\delta_{n,k}) E_2(\partial_{n,k})$ is just multiplication by $\Sigma_{1 \leqslant k \leqslant 2^n} \chi_{e_{n,k}} = 1$. Therefore $E(w_n) = I$ and in view of Theorem 1.2 E is not countably additive.