proj_1_DynamicsTest

April 29, 2024

```
[ ]: from proj_1_module import *
     import numpy as np
     import matplotlib.pyplot as plt
     import scienceplots
     plt.style.use(['science', 'notebook', 'grid'])
```

1 Population model

$$\dot{P}(t) = (B - D)P(t), \qquad P(0) = P_0 \tag{1}$$

The analytic solution is given by

$$P(t) = P_0\, e^{\,(B-D)t} \tag{2}$$

```
[ ]: def test_dynamics_solve(birth_rate, death_rate, P0 = 100,stepSize = 0.001,␣
      ↪total_time = 100):
         def population_model(t, P, B = birth_rate, D = death_rate):
             return (B-D)*P
         B = birth_rate
         D = death_rate
         t0 = 0
         num_points = int(total_time / stepSize)

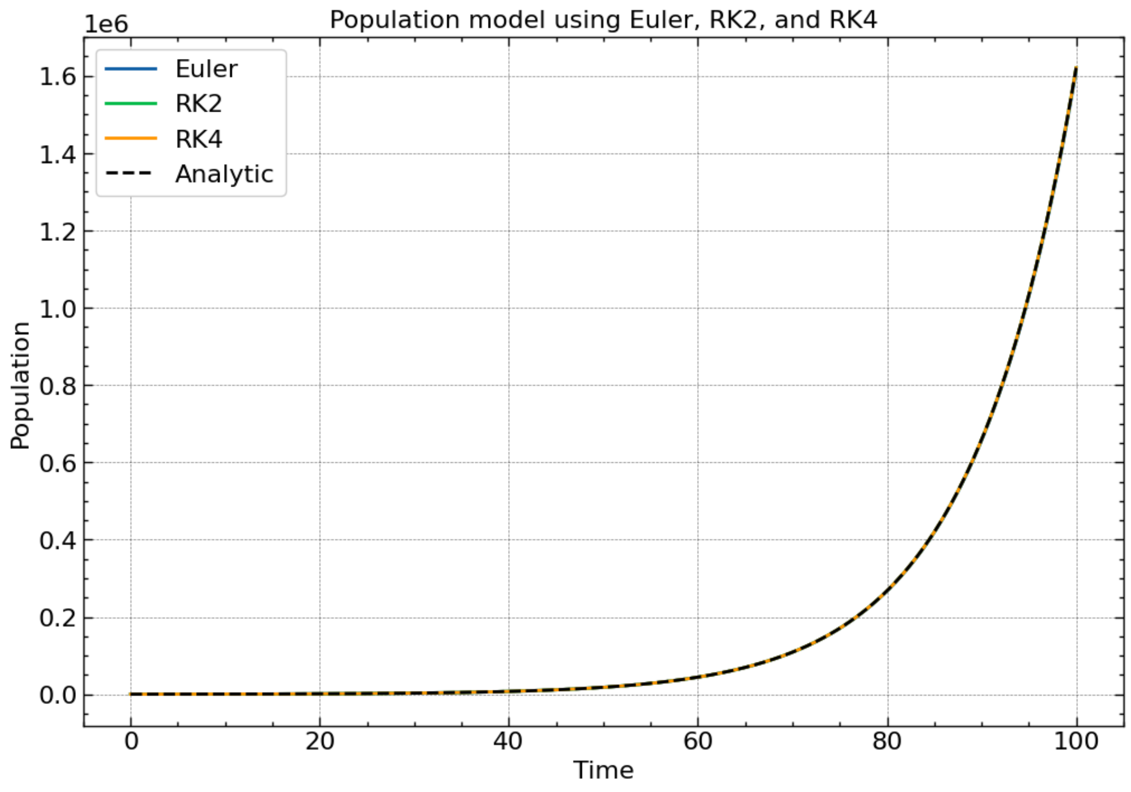


         T_Euler, S_Euler = dynamics_solve(population_model, D = 1, t_0 = t0,␣
      ↪s_0=P0, h = stepSize, N = num_points, method = 'Euler')
         T_RK2, S_RK2 = dynamics_solve(population_model, D = 1, t_0 = t0, s_0=P0, h␣
      ↪= stepSize, N = num_points, method = 'RK2')
         T_RK4, S_RK4 = dynamics_solve(population_model, D = 1, t_0 = t0, s_0=P0, h␣
      ↪= stepSize, N = num_points, method = 'RK4')
         S_ana = P0*np.exp( (B-D) * T_Euler)

         plt.figure(figsize=(12,8))
         plt.plot(T_Euler, S_Euler, label = 'Euler')
         plt.plot(T_RK2, S_RK2, label = 'RK2')
```

```
    plt.plot(T_RK4, S_RK4, label = 'RK4')
    plt.plot(T_Euler, S_ana, 'k--',label = 'Analytic')
    plt.legend()
    plt.xlabel('Time')
    plt.ylabel('Population')
    plt.title('Population model using Euler, RK2, and RK4')
    plt.show()
```

```
[ ]: test_dynamics_solve(birth_rate=0.1, death_rate=0.01, P0=200 ,stepSize=0.01,␣
      ↪total_time=100)
     print(1/np.abs(0.1-0.01))
```



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11.111111111111109
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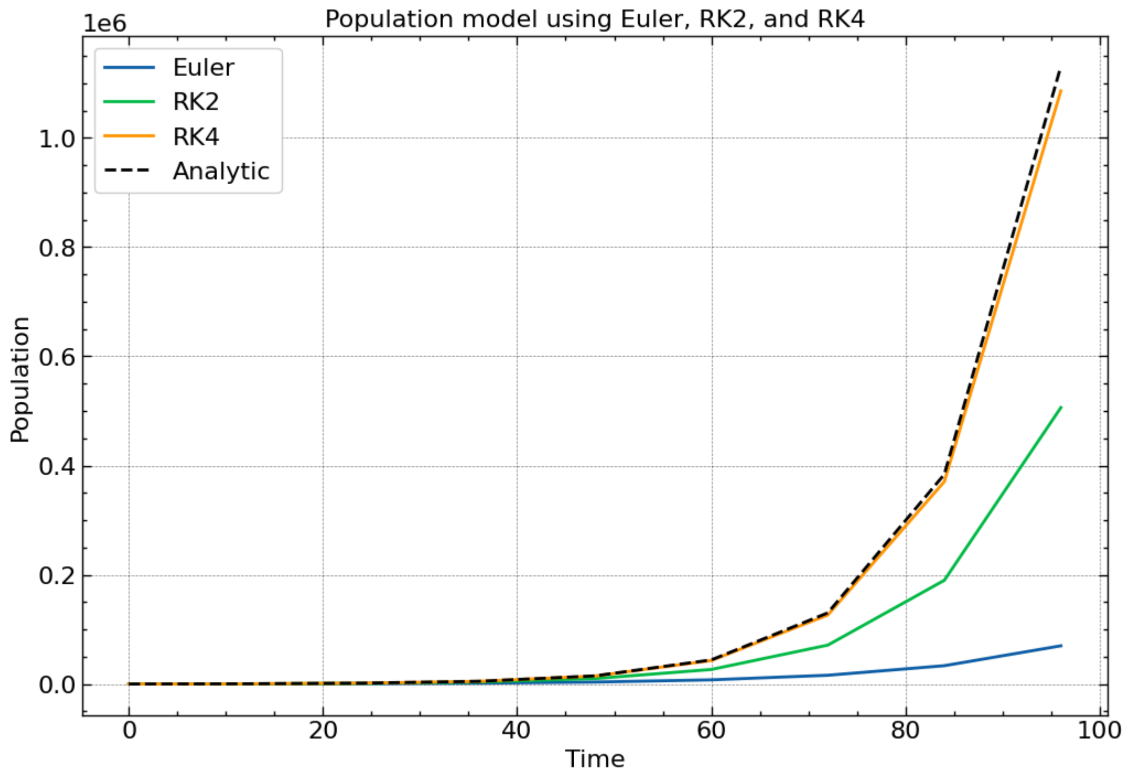
```
[ ]: test_dynamics_solve(birth_rate=0.1, death_rate=0.01, P0=200 ,stepSize=12,␣
      ↪total_time=100)
     print(1/np.abs(0.1-0.01))
```



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```

The nature time scale of this problem is the time constant $\frac{1}{|B-D|}$. This represents the time at which there will be a $\frac{1}{e}$ percent change in the system. If our step size is smaller than this value, the simulation will work well since we are not using approximation to neglect big changes in the functional value. As we have demonstrated in the example above, as the time step get greater than this value, the numeric solver approach is bad for Euler and Rk2.

2 Simple Harmonic

$$H = \frac{1}{2}kq^2 + \frac{p^2}{2m}$$
$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$$
$$\dot{p} = -\frac{\partial H}{\partial q} = -kq$$

[ ]:
```
import math

def DynamicTest(spring=1, mass=1, Q0=1, P0=0, stepSize=0.1, total_time=10):
    k = spring
    m = mass
```

```
def dH_dq(q, p):
    return k * q

def dH_dp(q, p):
    return p / m

def Hamiltonian(q, p):
    return 0.5 * k * q**2 + 0.5 * (p**2) / m

def calculate_energy(q, p, m, k):
    A = np.sqrt((p**2) / (k * m) + q**2)
    return 0.5*k*A**2

def calculate_A_and_phi(x0, p0, m, k):
    omega = math.sqrt(k / m)
    A = math.sqrt((p0**2) / (k * m) + x0**2)
    phi = math.atan2(-p0 / (m * omega * A), x0 / A)
    return A, phi

# Initial conditions
q_0 = Q0
p_0 = P0
h = stepSize
num_points = int(total_time / stepSize)
methods = ["Euler", "RK2", "RK4", "SE", "SV"]

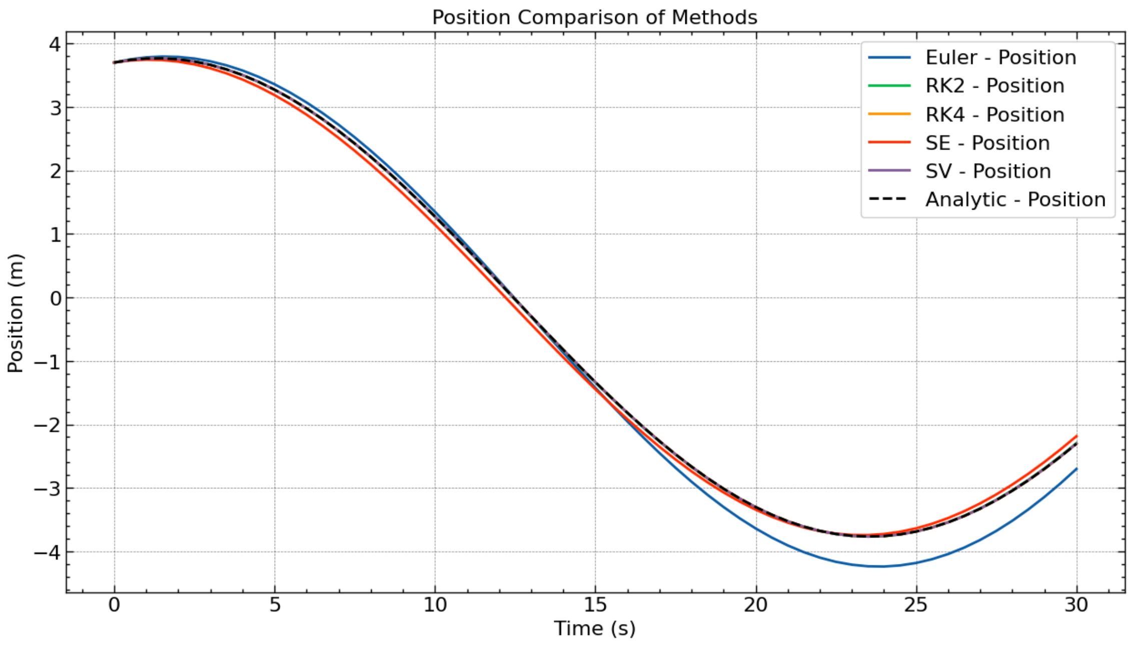
T_analytic = 0

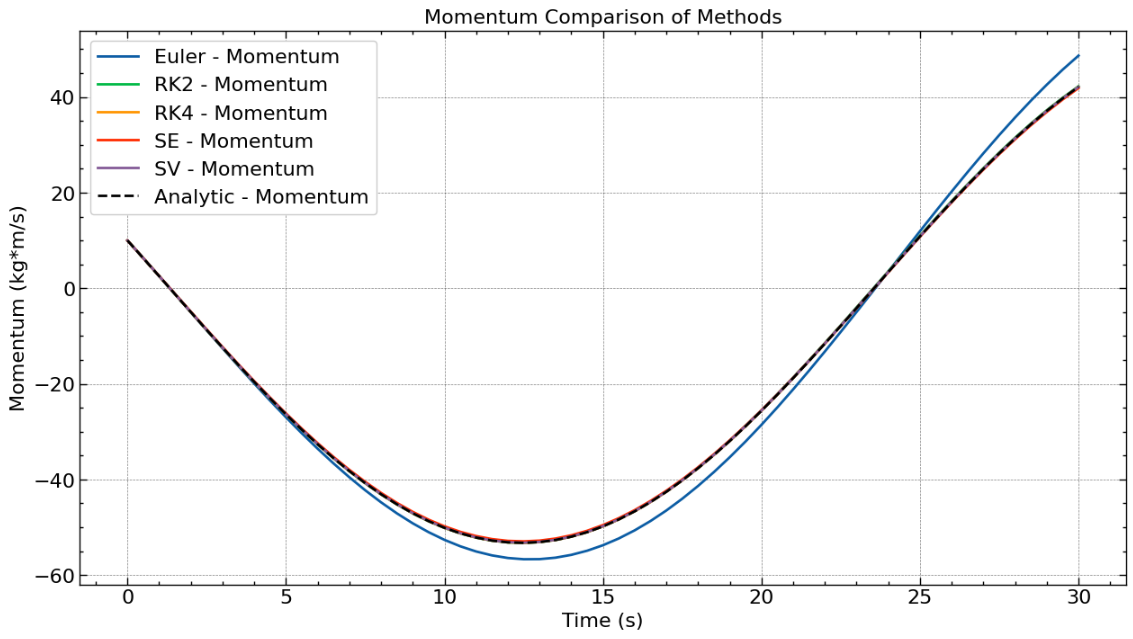
plt.figure(figsize=(15, 8))
for method in methods:
    T, Q, P = hamiltonian_solve(dH_dq, dH_dp, q_0=q_0, p_0=p_0, h=h,␣
↪N=num_points, method=method)
    T_analytic = T
    plt.plot(T, Q, label=f"{method} - Position")
A, phi = calculate_A_and_phi(Q0, P0, m, k)
omega = np.sqrt(k/m)
Q_analytic = A*np.cos(omega*T_analytic + phi)
P_analytic = -m*A*omega*np.sin(omega*T_analytic+phi)
plt.plot(T_analytic, Q_analytic,'k--' ,label = 'Analytic - Position')
plt.title('Position Comparison of Methods')
plt.xlabel('Time (s)')
plt.ylabel('Position (m)')
plt.legend()
plt.show()

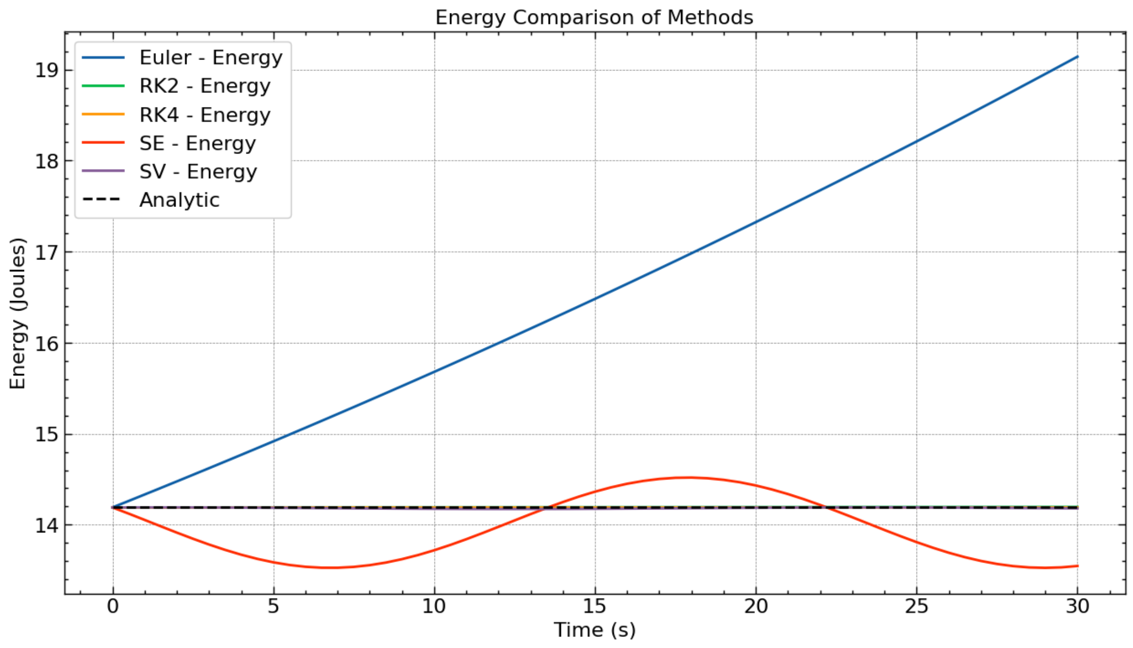
plt.figure(figsize=(15, 8))
for method in methods:
        T, Q, P = hamiltonian_solve(dH_dq, dH_dp, q_0=q_0, p_0=p_0, h=h,␣
↪N=num_points, method=method)
        plt.plot(T, P, label=f"{method} - Momentum")
    plt.plot(T_analytic, P_analytic, 'k--' ,label = 'Analytic - Momentum')
    plt.title('Momentum Comparison of Methods')
    plt.xlabel('Time (s)')
    plt.ylabel('Momentum (kg*m/s)')
    plt.legend()
    plt.show()

    plt.figure(figsize=(15, 8))
    for method in methods:
        T, Q, P = hamiltonian_solve(dH_dq, dH_dp, q_0=q_0, p_0=p_0, h=h,␣
↪N=num_points, method=method)
        E = Hamiltonian(Q, P)
        plt.plot(T, E, label=f"{method} - Energy")
    plt.plot(T, [calculate_energy(Q0, P0, m , k)] * (T.size), 'k--',label =␣
↪'Analytic')
    plt.title('Energy Comparison of Methods')
    plt.xlabel('Time (s)')
    plt.ylabel('Energy (Joules)')
    plt.legend()
    plt.show()
```

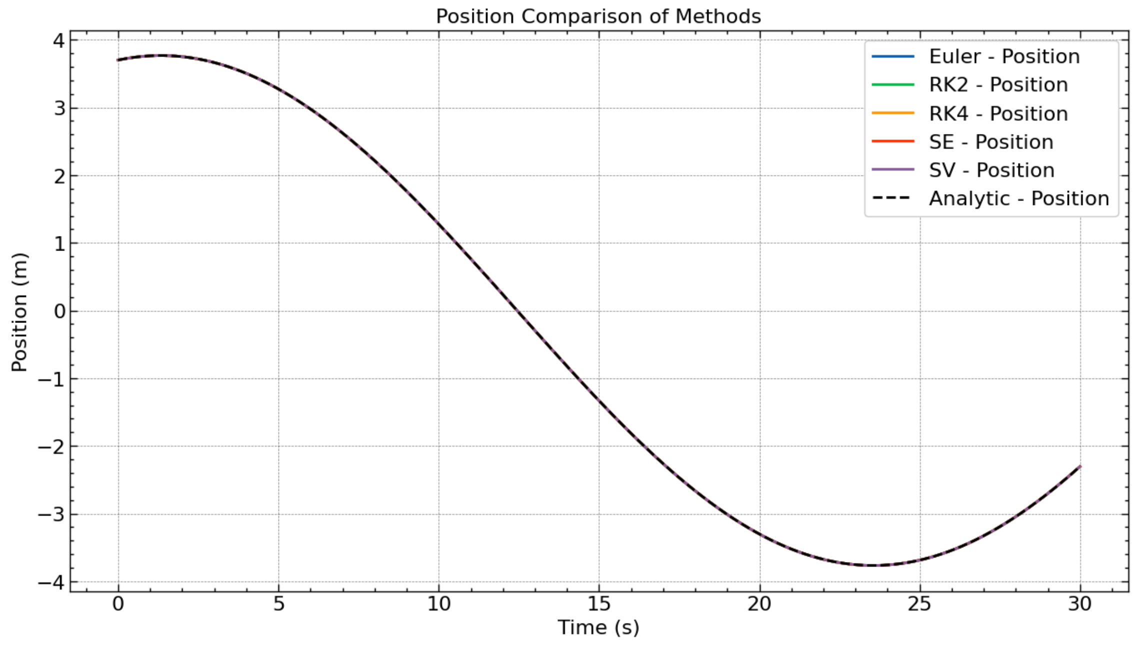
```
[ ]: DynamicTest(spring=2, mass=100, Q0=3.7, P0=10, stepSize=0.5, total_time=30)
```

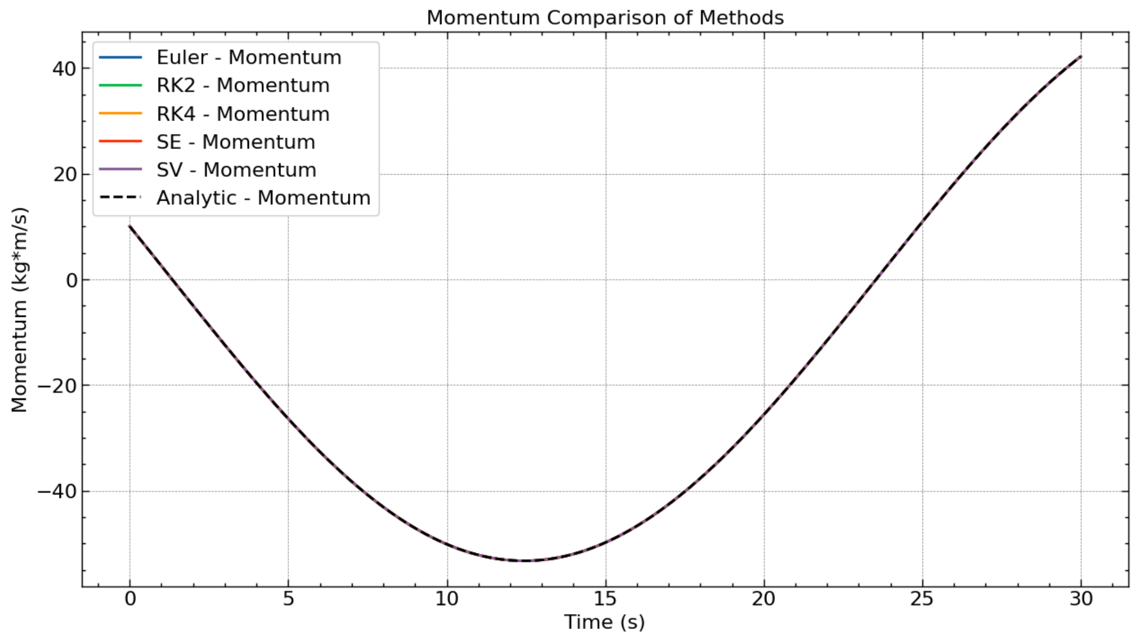


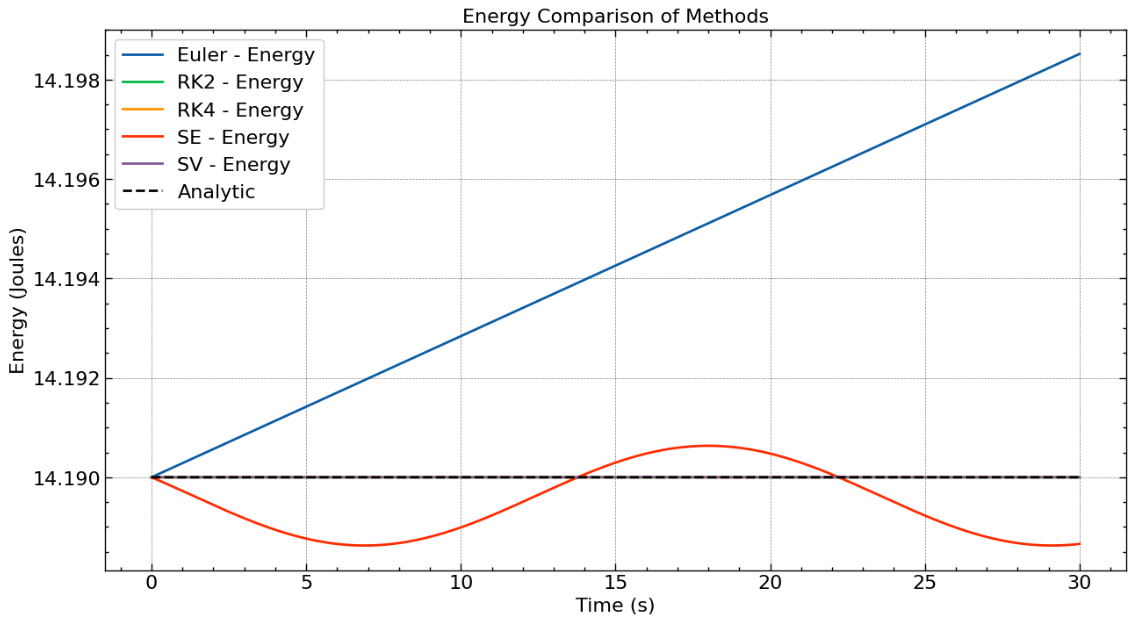




[ ]:
```
DynamicTest(spring=2, mass=100, Q0=3.7, P0=10, stepSize=0.001, total_time=30)
```

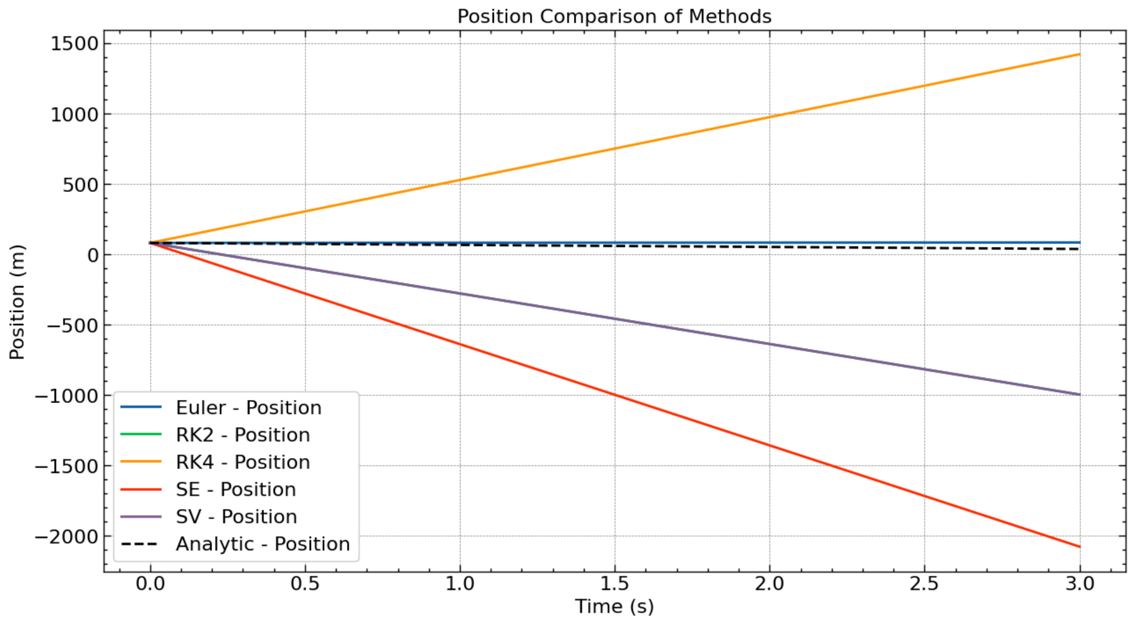

Position Comparison of Methods
Euler - Position
RK2 - Position
RK4 - Position
SE - Position
SV - Position
Analytic - Position
Position (m)
Time (s)
4
3
2
1
0
−1
−2
−3
−4
0
5
10
15
20
25
30

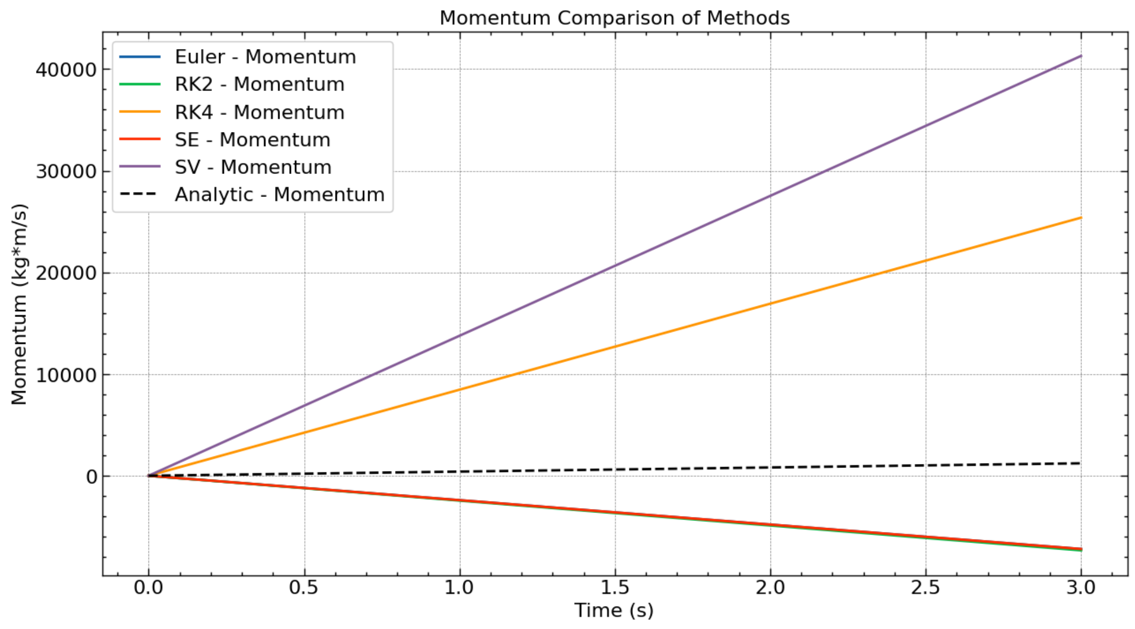

Momentum Comparison of Methods
Euler - Momentum
RK2 - Momentum
RK4 - Momentum
SE - Momentum
SV - Momentum
Analytic - Momentum
Momentum (kg*m/s)
Time (s)
40
20
0
−20
−40
0
5
10
15
20
25
30

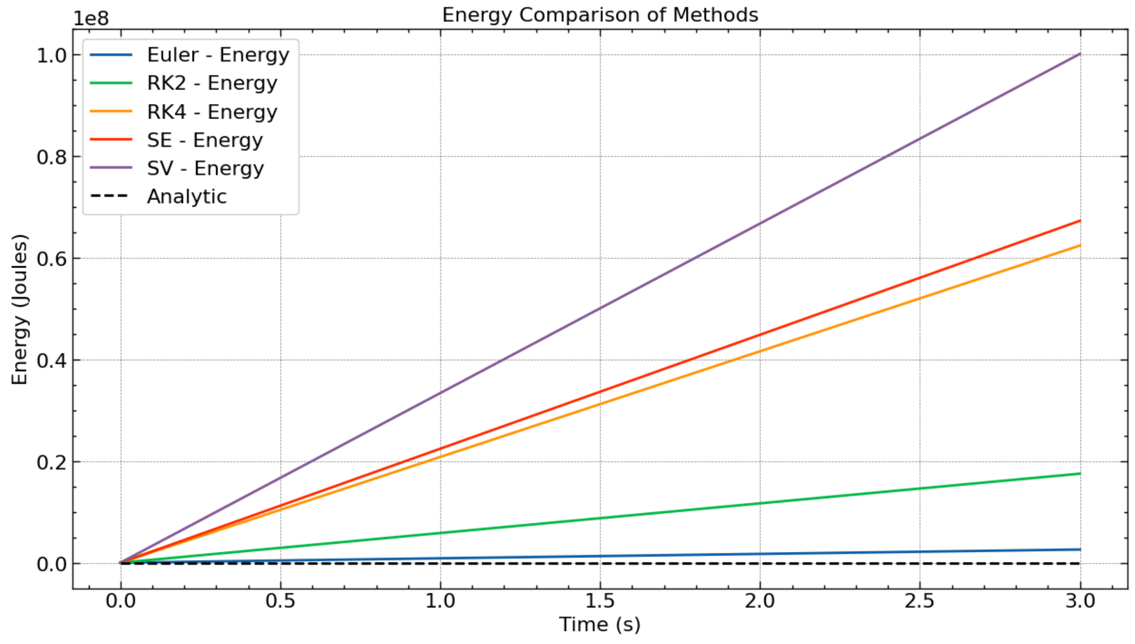


As we can see, as we make the step size very small, all methods match the exact solution very well.

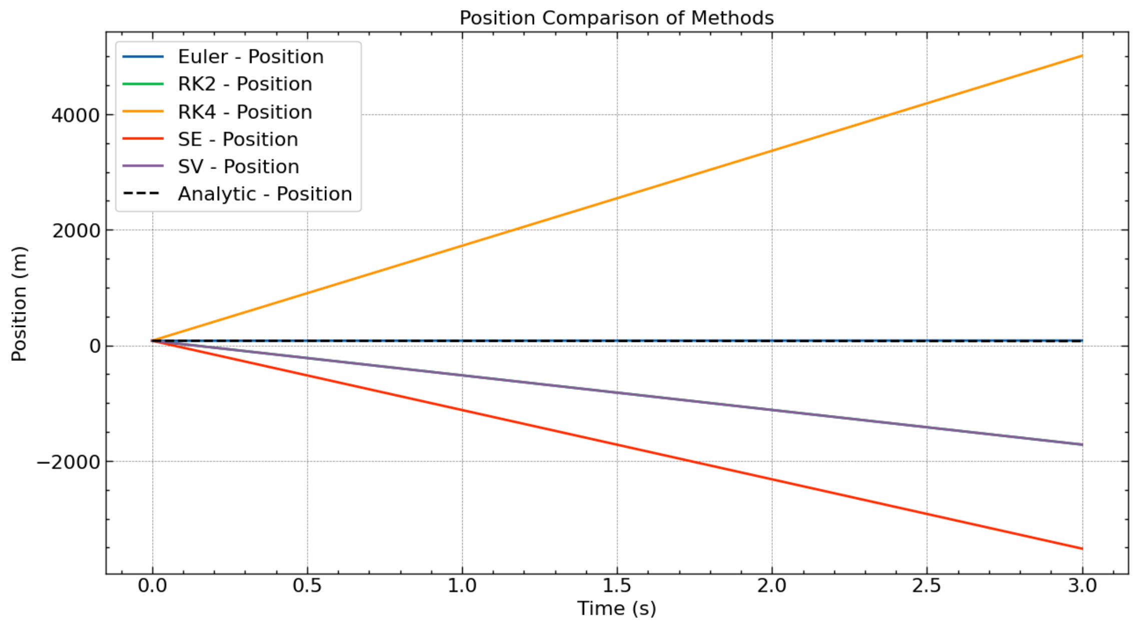
```
[ ]: DynamicTest(spring=30, mass=10, Q0=80, P0=10, stepSize=3, total_time=5)
```

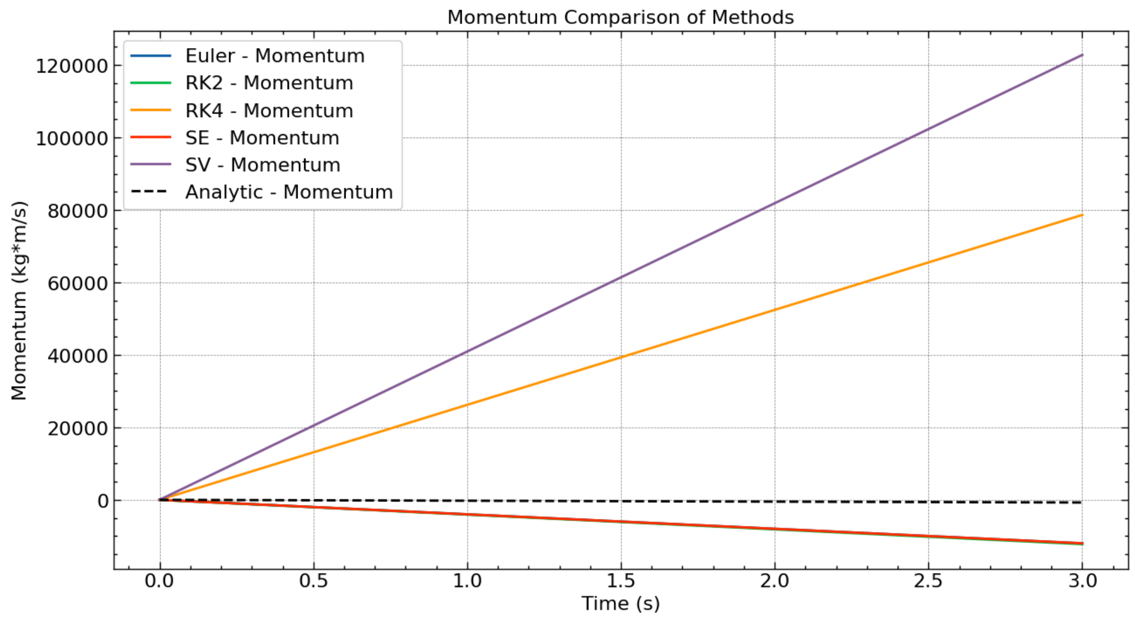


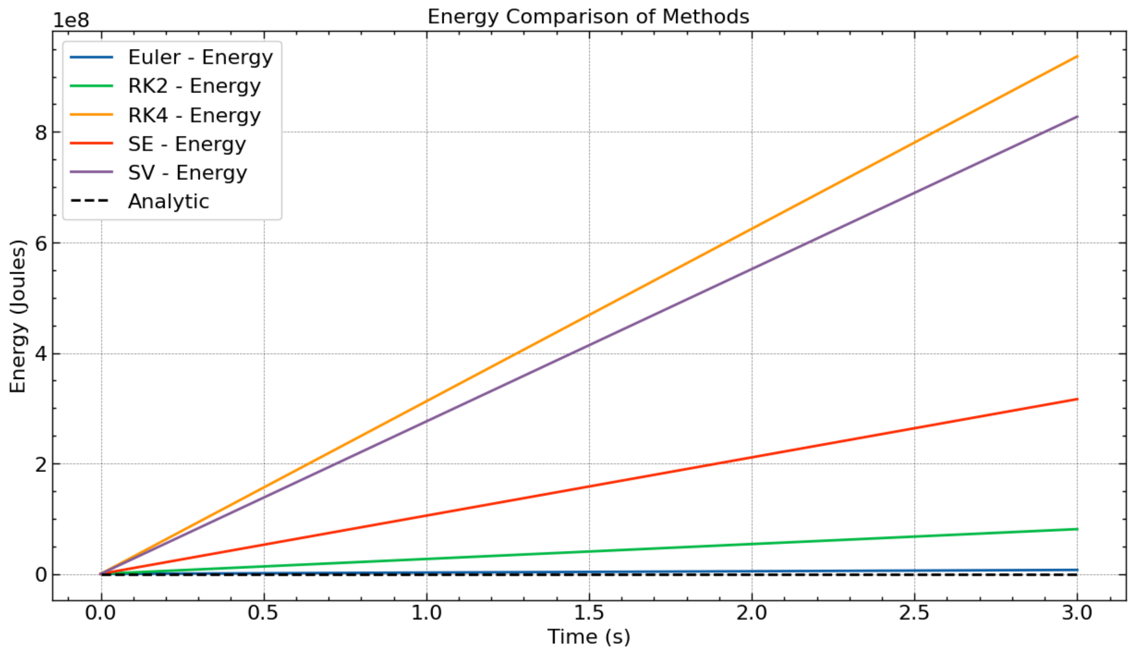




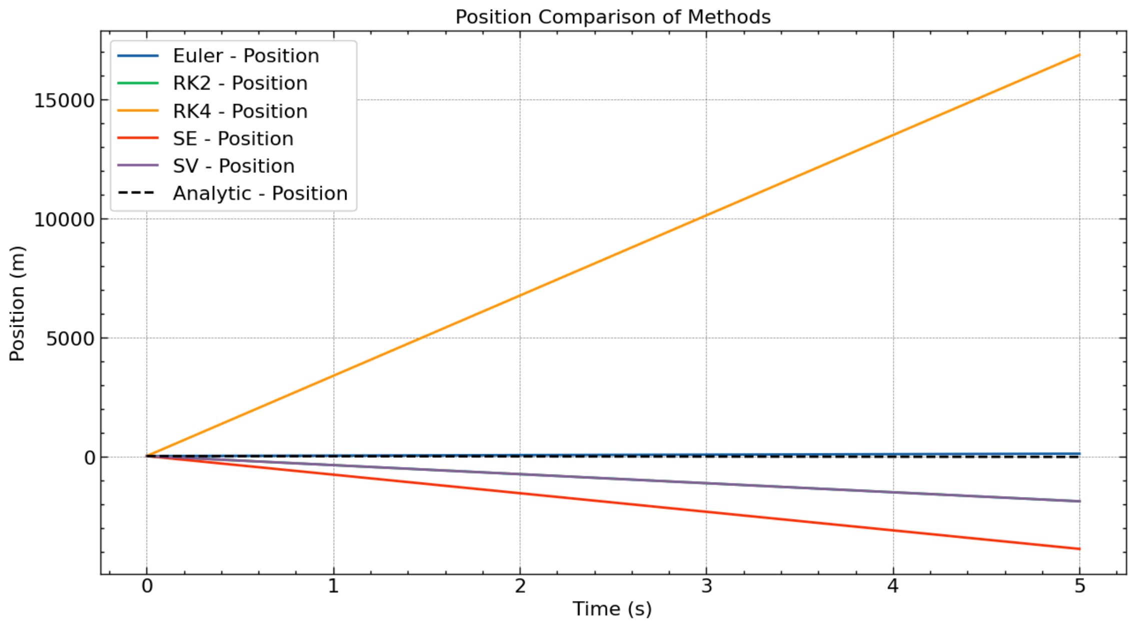
```
[ ]: DynamicTest(spring=50, mass=10, Q0=80, P0=10, stepSize=3, total_time=5)
```

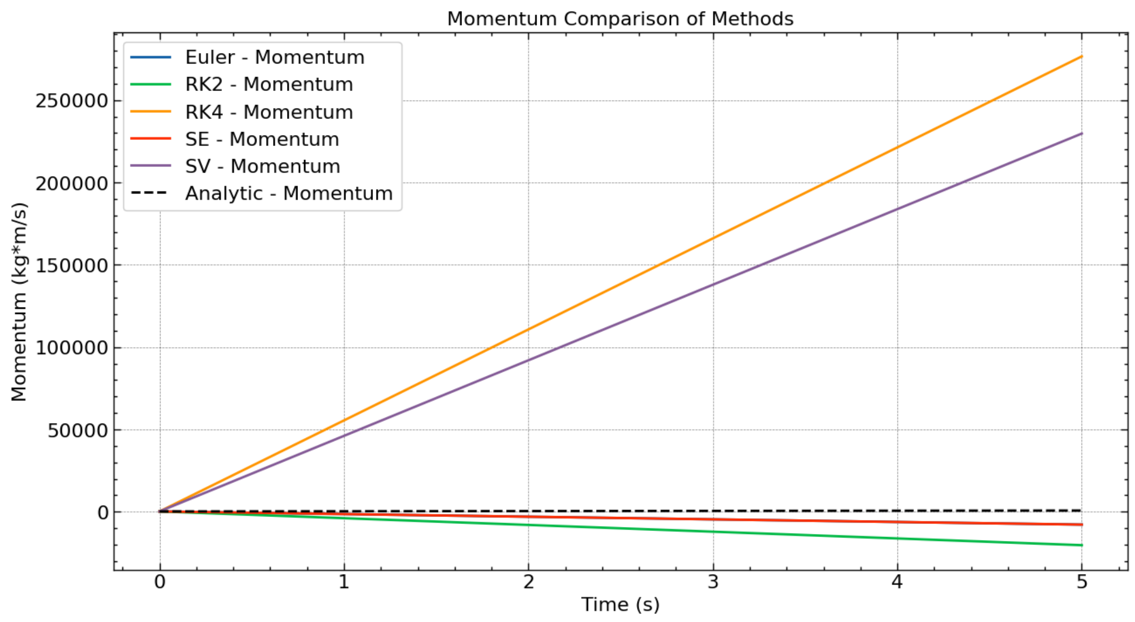

Position Comparison of Methods
Euler - Position
RK2 - Position
RK4 - Position
SE - Position
SV - Position
Analytic - Position
Position (m)
4000
2000
0
−2000
0.0
0.5
1.0
1.5
2.0
2.5
3.0
Time (s)

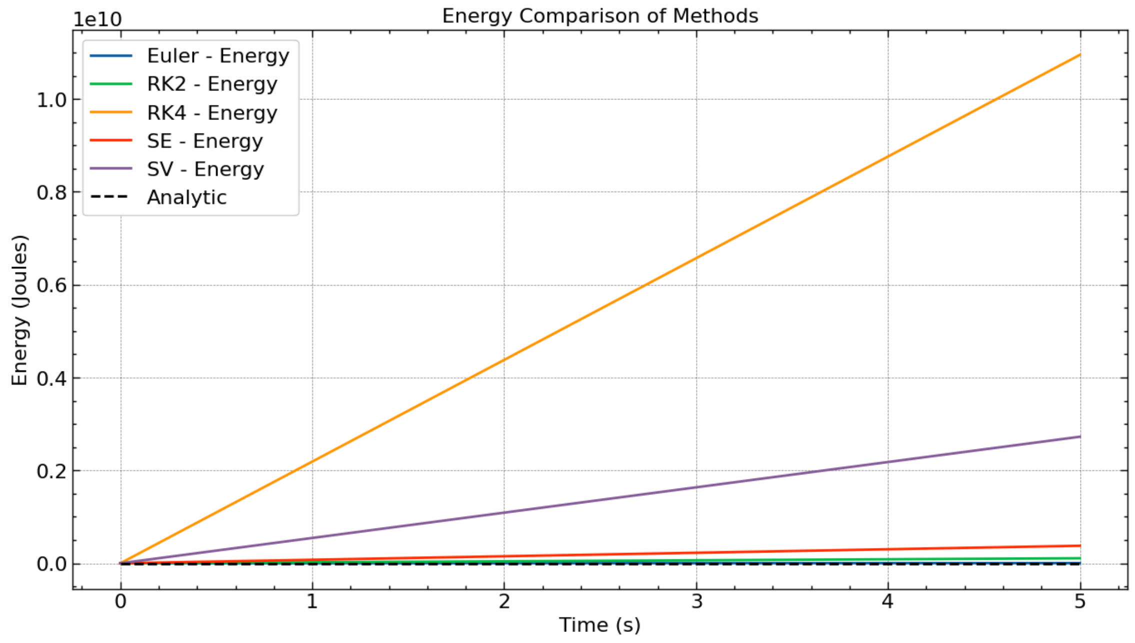

Momentum Comparison of Methods
Euler - Momentum
RK2 - Momentum
RK4 - Momentum
SE - Momentum
SV - Momentum
Analytic - Momentum
Momentum (kg*m/s)
120000
100000
80000
60000
40000
20000
0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
Time (s)



```
[ ]: DynamicTest(spring=50, mass=10, Q0=32, P0=200, stepSize=5, total_time=5)
```

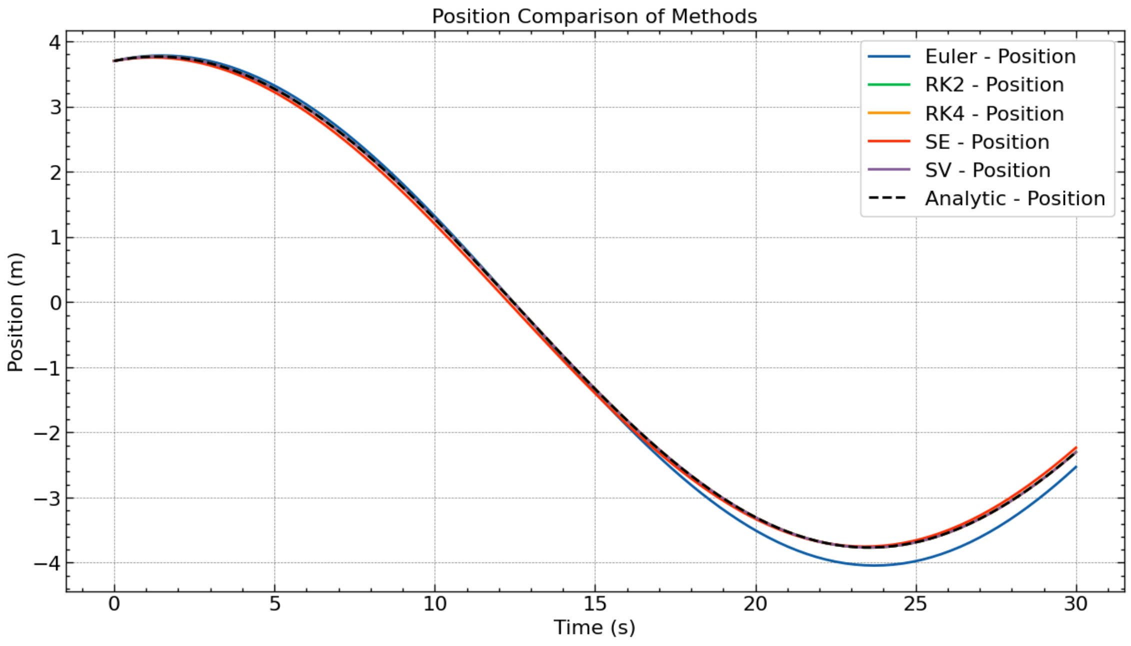






RK4 tends to over shoot position the most, and SV tends to over shoot momentum the most. SE and RK2 tends to underestimate momentum. Surprisingly, when the step size is not appriate, RK4 overestimate energy the most, whereas the Euler method conserve energy the best under this condition. The deviation in energy: RK4 > SV > SE > RK2 > Euler when stepsize is large.

```
[ ]: DynamicTest(spring=2, mass=100, Q0=3.7, P0=10, stepSize=0.3, total_time=30)
```



Position Comparison of Methods
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RK2 - Position
RK4 - Position
SE - Position
SV - Position
Analytic - Position
Position (m)
Time (s)
4
3
2
1
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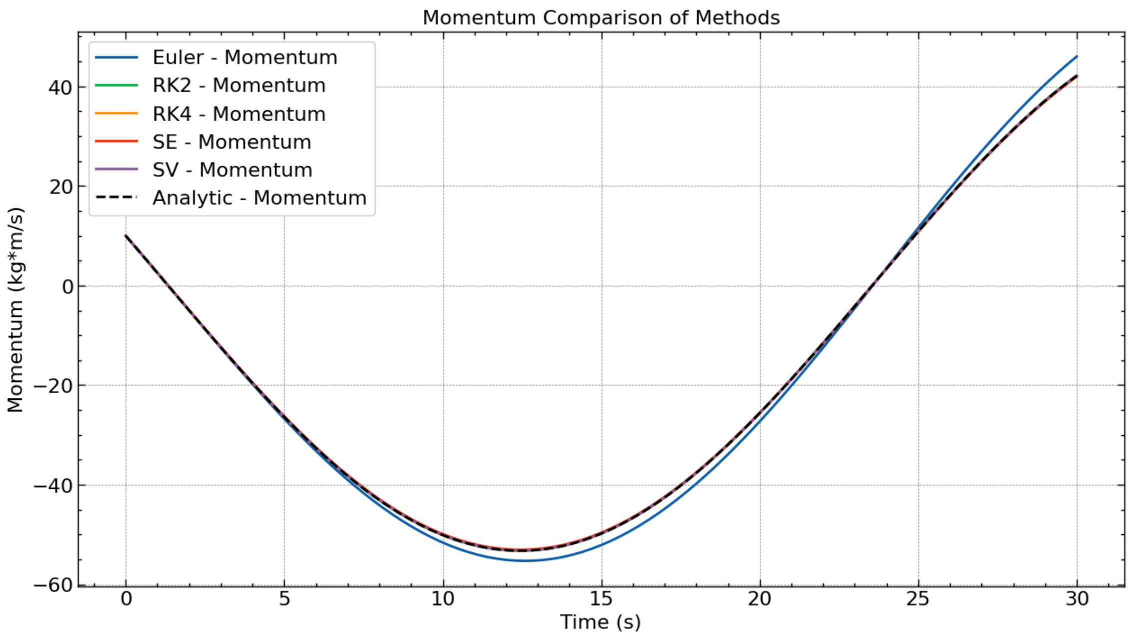
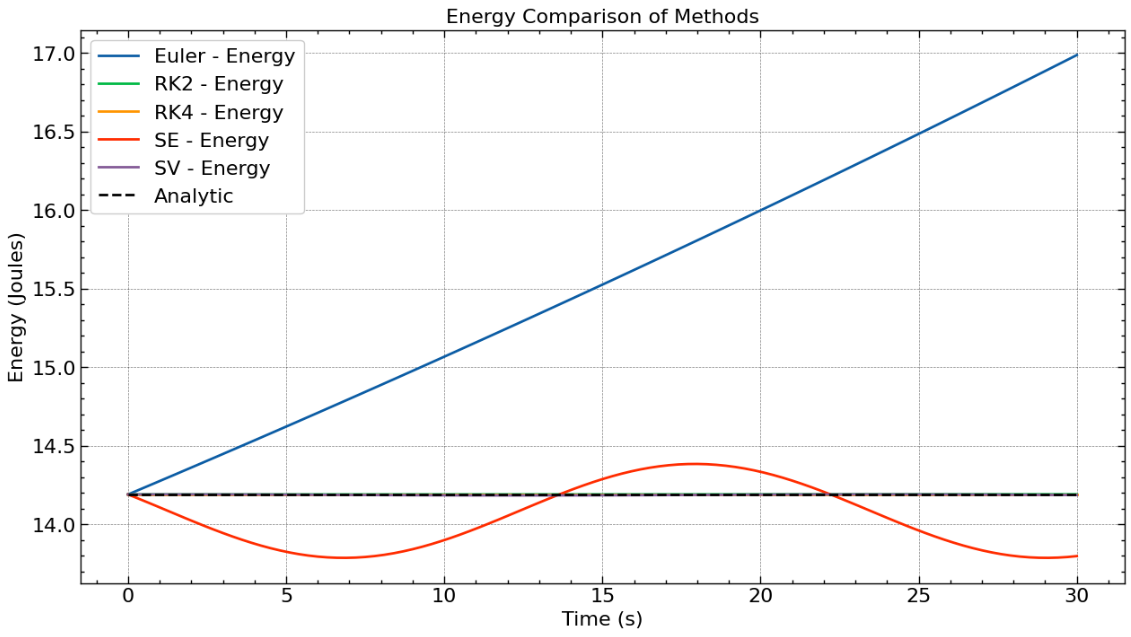
−1
−2
−3
−4
0
5
10
15
20
25
30



Momentum Comparison of Methods
Euler - Momentum
RK2 - Momentum
RK4 - Momentum
SE - Momentum
SV - Momentum
Analytic - Momentum
Momentum (kg*m/s)
Time (s)
40
20
0
−20
−40
−60
0
5
10
15
20
25
30



On the other hand, when step size is made small, RK4 and SV seems to be the most reliable at solving the equation, whereas SE and Euler has tiny fluctuation in energy. However, they are all actually very good when the step size is small.