discretization of 3D fokker planck

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The constraints

$$-\frac{\partial m}{\partial t} + \mathcal{L}^* m - \mu(t, a, x) = 0 \tag{1}$$

$$-\frac{\partial \tilde{m}}{\partial t} + \mathcal{L}^{0*} \tilde{m} + m(t, \bar{a}, x) - \tilde{\mu}(t, a, x) = 0 \tag{2}$$

$$-\frac{\partial \hat{m}}{\partial t} + \mathcal{L}^{0*} \hat{m} - \hat{\mu}(t, a, x) = 0 \tag{3}$$

$t \in [0, \bar{t}]$, $a \in [0, \bar{a}]$, $X \in [0, \bar{x}]$

$$\mathcal{L}^{0*} m = -\frac{\partial}{\partial x}(mk(\theta - x)) + \frac{\delta^2}{2}\frac{\partial^2}{\partial x^2}(\tilde{m}x(1 - x)) \tag{4}$$

$$\mathcal{L}^* m = -\frac{\partial}{\partial x}(mk(\theta - x)) + \frac{\delta^2}{2}\frac{\partial^2}{\partial x^2}(\tilde{m}x(1 - x)) - \frac{\partial m}{\partial a} \tag{5}$$

$$\begin{gathered} i = 0 \cdots N_t - 2,\ t = (i+1)\Delta t \\ z = 0 \cdots N_a - 1 \\ j = 0 \cdots N_X - 1 \end{gathered}$$

$$\begin{aligned} &\frac{m[i][z][j] - m[i-1][z][j]}{\Delta t} \\ &+ \frac{m[i][z][j+1]k(\theta - x[j+1]) - m[i][z][j-1]k(\theta - x[j-1])}{2\Delta x} \\ &- \frac{\delta^2}{2}\frac{m[i][z][j+1]x[j+1](1 - x[j+1]) - 2m[i][z][j]x[j](1 - x[j]) + m[i][z][j-1]x[j-1](1 - x[j-1])}{\Delta x^2} \quad (6) \\ &+ \frac{m[i][z][j] - m[i][z-1][j]}{\Delta a} + \mu[i][z][j] = 0 \quad (7) \end{aligned}$$

$$\begin{aligned} &\frac{\hat{m}[i][j] - \hat{m}[i-1][j]}{\Delta t} \\ &+ \frac{\hat{m}[i][j+1]k(\theta - x[j+1]) - \hat{m}[i][j-1]k(\theta - x[j-1])}{2\Delta x} \\ &- \frac{\delta^2}{2}\frac{\hat{m}[i][j+1]x[j+1](1 - x[j+1]) - 2\hat{m}[i][j]x[j](1 - x[j]) + \hat{m}[i][j-1]x[j-1](1 - x[j-1])}{\Delta x^2} + \hat{\mu}[i][j] = 0 \end{aligned} \tag{8}$$

$$\begin{aligned}&\frac{\tilde{m}[i][j]-\tilde{m}[i-1][j]}{\Delta t}\\&+\frac{\tilde{m}[i][j+1]k(\theta-x[j+1])-\tilde{m}[i][j-1]k(\theta-x[j-1])}{2\Delta x}\\&-\frac{\delta^2}{2}\frac{\tilde{m}[i][j+1]x[j+1]-2\tilde{m}[i][j]x[j]+\tilde{m}[i][j-1]x[j-1]}{\Delta x^2}+\tilde{\mu}[i][j]-m[i][Na-1][j]=0\end{aligned} \tag{9}$$

$$V[j]=1+\frac{\Delta t\delta^2 x[j](1-x[j])}{\Delta x}+\frac{\Delta t}{\Delta a} \tag{10}$$

$$V_0[j]=1+\frac{\Delta t\delta^2 x[j](1-x[j])}{\Delta x} \tag{11}$$

$$V_1[j]=-\frac{\Delta t\delta^2 x[j](1-x[j])}{2\Delta x^2}+\frac{\Delta tk(\theta-x[j])}{2\Delta x} \tag{12}$$

$$V_2[j]=-\frac{\Delta t\delta^2 x[j](1-x[j])}{2\Delta x^2}-\frac{\Delta tk(\theta-x[j])}{2\Delta x} \tag{13}$$

$$V[j]m[i][z][j]+V_1[j+1]m[i][z][j+1]+V_2[j-1]m[i][z][j-1]-m[i-1][z][j]-\frac{\Delta t}{\Delta a}m[i][z-1][j]+\Delta t\mu[i][z][j]=0 \tag{14}$$

$$V_0[j]\hat{m}[i][j]+V_1[j+1]\hat{m}[i][j+1]+V_2[j-1]\hat{m}[i][j-1]-\hat{m}[i-1][j]+\Delta t\hat{\mu}[i][j]=0 \tag{15}$$

$$V_0[j]\tilde{m}[i][j]+V_1[j+1]\tilde{m}[i][j+1]+V_2[j-1]\tilde{m}[i][j-1]-\tilde{m}[i-1][j]+\Delta t\tilde{\mu}[i][j]-\Delta tm[i][N_a-1][j]=0 \tag{16}$$

When $i=0$, $t=\Delta t$

$$V[j]m[0][z][j]+V_1[j+1]m[0][z][j+1]+V_2[j-1]m[0][z][j-1]-\frac{\Delta t}{\Delta a}m[0][z-1][j]+\Delta t\mu[0][z][j]=m_0[j] \tag{17}$$

$$V_0[j]\hat{m}[0][j]+V_1[j+1]\hat{m}[0][j+1]+V_2[j-1]\hat{m}[0][j-1]+\Delta t\hat{\mu}[0][j]=\hat{m}_0[j] \tag{18}$$

$$V_0[j]\tilde{m}[0][j]+V_1[j+1]\tilde{m}[0][j+1]+V_2[j-1]\tilde{m}[0][j-1]+\Delta t\tilde{\mu}[0][j]-\Delta tm[0][N_a-1][j]=0 \tag{19}$$

capacity:

$$c(t)=\int_{\mathcal{O}}\int_{\mathcal{A}}m\ dadx \tag{20}$$

conventional offer:

$$\int_{\mathcal{O}}\int_{\mathcal{A}}Fm\ dadx+\int_{\mathcal{O}}\int_{\mathcal{A}}F\tilde{m}\ dx \tag{21}$$

for given i (t)

$$\sum_{j=0}^{N_X-1}F[j]\left(\left(\sum_{z=1}^{N_a-1}m[i][z][j]\Delta a\Delta x\right)+\tilde{m}[i][j]\Delta x\right) \tag{22}$$

Renewable offer:

$$\int_{\mathcal{O}}\int_{\mathcal{A}}Xm\ dadx+\int_{\mathcal{O}}\int_{\mathcal{A}}X\tilde{m}\ dx \tag{23}$$

for given i (t)

$$\sum_{j=0}^{N_X-1} X[j]\left(\left(\sum_{z=1}^{N_a-1} m[i][z][j]\Delta a\Delta x\right)+\tilde{m}[i][j]\Delta x\right) \tag{24}$$

objective function

$$\int_t\int_{\mathcal{O}}\int_{\mathcal{A}} e^{-\rho t}G\ m\ dtdxda+\int_t\int_{\mathcal{O}}\int_{\mathcal{A}} e^{-(\rho+\gamma)t}f\ \mu\ dtdxda-\int_{\mathcal{O}}\int_{\mathcal{A}} e^{-(\rho+\gamma)t}s\ \hat{\mu}\ dxda \tag{25}$$

$$\begin{aligned}\sum_{i=0}^{N_t-2}\sum_{j=0}^{N_a-1}\sum_{z=1}^{N_X-1} e^{-\rho t}G\ m[i][z][j]\ +\sum_{i=0}^{N_t-2}\sum_{j=0}^{N_a-1} e^{-\rho t}G\ \tilde{m}[i][z][j]\Delta t\Delta x\\ +\sum_{i=0}^{N_t-2}\sum_{j=0}^{N_a-1}\sum_{z=1}^{N_X-1} e^{-(\rho+\gamma)t}f\ \mu[i][z][j]\ \Delta t\Delta x\Delta a+\sum_{i=0}^{N_t-2}\sum_{j=0}^{N_a-1} e^{-(\rho+\gamma)t}f\ \tilde{\mu}[i][z][j]\ \Delta t\Delta x\\ -\sum_{i=0}^{N_t-2}\sum_{j=0}^{N_a-1} e^{-(\rho+\gamma)t}s\ \hat{\mu}[i][j]\ \Delta t\Delta x\end{aligned} \tag{26}$$

When $z=0$

$$m[i][0][j]=\hat{\mu}[i][j]$$

When $z=1$

$$V[j]m[i][1][j]+V_1[j+1]m[i][1][j+1]+V_2[j-1]m[i][1][j-1]-m[i-1][1][j]+\Delta t\mu[i][j]=\frac{\Delta t}{\Delta a}\hat{\mu}[i][j] \tag{27}$$

When $z=2,\cdots,N_a-1$

$$V[j]m[i][z][j]+V_1[j+1]m[i][z][j+1]+V_2[j-1]m[i][z][j-1]-m[i-1][z][j]-\frac{\Delta t}{\Delta a}m[i][z-1][j]+\Delta t\mu[i][j]=0 \tag{28}$$

$$V_0[j]\hat{m}[i][j]+V_1[j+1]\hat{m}[i][j+1]+V_2[j-1]\hat{m}[i][j-1]-\hat{m}[i-1][j]+\Delta t\hat{\mu}[i][j]=0 \tag{29}$$

$$V_0[j]\tilde{m}[i][j]+V_1[j+1]\tilde{m}[i][j+1]+V_2[j-1]\tilde{m}[i][j-1]-\tilde{m}[i-1][j]+\Delta t\tilde{\mu}[i][j]-\Delta t m[i][N_a-1][j]=0 \tag{30}$$

Summing up with z from 1 to N_a-1. Take

$$\bar{m}[i][j]=\left(\sum_{z=1}^{N_a-1} m[i][j]\Delta a\right)+\tilde{m}[i][j]$$

$$\bar{\mu}[i][j]=\left(\sum_{z=1}^{N_a-1} \mu[i][j]\Delta a\right)+\tilde{\mu}[i][j]$$

$$V_0[j]\bar{m}[i][j]+V_1[j+1]\bar{m}[i][j+1]+V_2[j-1]\bar{m}[i][j-1]-\bar{m}[i-1][j]+\Delta t\bar{mu}[i][j]-\Delta t\hat{\mu}[i][j]=0 \tag{31}$$

Equation (31) is same as 2D model.
Boundary cases: When $z\neq 0$, $j=0$, $i\neq 0$

$$V[0]m[i][z][0]+V_1[1]m[i][z][1]-m[i-1][z][0]-\frac{\Delta t}{\Delta a}m[i][z-1][0]+\Delta t\mu[i][z][0]=0 \tag{32}$$

$$V_0[0]\hat{m}[i][0] + V_1[1]\hat{m}[i][1] - \hat{m}[i-1][0] + \Delta t\hat{\mu}[i][0] = 0 \tag{33}$$

$$V_0[0]\tilde{m}[i][0] + V_1[1]\tilde{m}[i][1] - \tilde{m}[i-1][0] + \Delta t\tilde{\mu}[i][0] - \Delta t m[i][N_a-1][0] = 0 \tag{34}$$

When $z \neq 0$, $j = N_X - 1$, $i \neq 0$

$$V[N_X-1]m[i][z][N_X-1] + V_2[N_X-2]m[i][z][N_X-2] - m[i-1][z][N_X-1] - \frac{\Delta t}{\Delta a}m[i][z-1][N_X-1] + \Delta t\mu[i][z][N_X-1] = 0 \tag{35}$$

$$V_0[N_X-1]\hat{m}[i][N_X-1] + V_2[N_X-2]\hat{m}[i][N_X-2] - \hat{m}[i-1][N_X-1] + \Delta t\hat{\mu}[i][N_X-1] = 0 \tag{36}$$

$$V_0[N_X-1]\tilde{m}[i][N_X-1] + V_2[N_X-2]\tilde{m}[i][N_X-2] - \tilde{m}[i-1][N_X-1] + \Delta t\tilde{\mu}[i][N_X-1] - \Delta t m[i][N_a-1][N_X-1] = 0 \tag{37}$$

When $z \neq 0$, $i = 0$, $j \neq 0$, $j \neq N_X - 1$

$$V[j]m[0][z][j] + V_1[j+1]m[0][z][j+1] + V_2[j-1]m[0][z][j-1] - \frac{\Delta t}{\Delta a}m[0][z-1][j] + \Delta t\mu[0][j] = m_0[i][j] \tag{38}$$

$$V_0[j]\hat{m}[0][j] + V_1[j]\hat{m}[0][j+1] + V_2[j-1]\hat{m}[0][j-1] + \Delta t\hat{\mu}[0][j] = 0 \tag{39}$$

$$V_0[j]\tilde{m}[0][j] + V_1[j+1]\tilde{m}[0][j+1] + V_2[j-1]\tilde{m}[0][j-1] + \Delta t\tilde{\mu}[0][j] - \Delta t m[0][N_a-1][j] = 0 \tag{40}$$

When $z \neq 0$, $i = 0$, $j = 0$

$$V[0]m[0][z][0] + V_1[1]m[0][z][1] - \frac{\Delta t}{\Delta a}m[0][z-1][0] + \Delta t\mu[0][z][0] = 0 \tag{41}$$

$$V_0[0]\hat{m}[i][0] + V_1[1]\hat{m}[0][1] + \Delta t\hat{\mu}[0][0] = 0 \tag{42}$$

$$V_0[0]\tilde{m}[0][0] + V_1[1]\tilde{m}[0][1] + \Delta t\tilde{\mu}[0][0] - \Delta t m[0][N_a-1][0] = 0 \tag{43}$$

When $z \neq 0$, $i = 0$, $j = N_X - 1$

$$V[N_X-1]m[0][z][N_X-1] + V_2[N_X-2]m[0][z][N_X-2] - \frac{\Delta t}{\Delta a}m[0][z-1][N_X-1] + \Delta t\mu[0][z][N_X-1] = 0 \tag{44}$$

$$V_0[N_X-1]\hat{m}[0][N_X-1] + V_2[N_X-2]\hat{m}[0][N_X-2] + \Delta t\hat{\mu}[0][N_X-1] = 0 \tag{45}$$

$$V_0[N_X-1]\tilde{m}[0][N_X-1] + V_2[N_X-2]\tilde{m}[0][N_X-2] + \Delta t\tilde{\mu}[0][N_X-1] - \Delta t m[0][N_a-1][N_X-1] = 0 \tag{46}$$