discretization of 3D fokker planck

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Link between 2D and 3D: For renewable agents, $m(t,a,x)$ and $\mu(t,a,x)$ are 3d with time, age, state. $\hat{m}(t,x)$, $\hat{\mu}(t,x)$, $\tilde{m}(t,x)$, $\tilde{\mu}(t,x)$
For conventional agents, all variables are 2d with time and state.

$$-\frac{\partial m}{\partial t} + \mathcal{L}^* m - \mu(t,a,x) = 0 \tag{1}$$

Let

$$m^* = \int_{\mathcal{A}} m \; da$$

$$\mu^* = \int_{\mathcal{A}} \mu \; da$$

$$-\frac{\partial m^*}{\partial t} + \mathcal{L}^{0*} m - \int_{\mathcal{A}} \frac{\partial m}{\partial a} \; da - \mu^* = 0 \tag{2}$$

where

$$\mathcal{L}^* m = \mathcal{L}^{0*} m - \frac{\partial m}{\partial a}$$

$$\int_{\mathcal{A}} \frac{\partial m}{\partial a} \; da = m(t, a_{\max}, x) - m(t,0,x) = m(t, a_{\max}, x) - \hat{\mu}(t,x)$$

$$-\frac{\partial m^*}{\partial t} + \mathcal{L}^{0*} m - m(t, a_{\max}, x) + \hat{\mu}(t,x) - \mu^* = 0$$

Let

$$\bar{m} = m^* + \tilde{m}$$

As

$$-\frac{\partial \tilde{m}}{\partial t} + \mathcal{L}^{0*} \tilde{m} + m(t, \bar{a}, x) - \tilde{\mu}(t,a,x) = 0$$

So

$$-\frac{\partial \bar{m}}{\partial t} + \mathcal{L}^{0*} \bar{m} + \hat{\mu}(t,x) - \bar{\mu}(t,x) = 0$$

The constraints

$$-\frac{\partial m}{\partial t} + \mathcal{L}^* m - \mu(t,a,x) = 0 \tag{3}$$

$$-\frac{\partial \tilde{m}}{\partial t} + \mathcal{L}^{0*} \tilde{m} + m(t,\bar{a},x) - \tilde{\mu}(t,a,x) = 0 \tag{4}$$

$$-\frac{\partial \hat{m}}{\partial t} + \mathcal{L}^{0*} \hat{m} - \hat{\mu}(t,a,x) = 0 \tag{5}$$

$t \in [0,\bar{t}]$, $a \in [0,\bar{a}]$, $X \in [0,\bar{x}]$

$$\mathcal{L}^{0*} m = -\frac{\partial}{\partial x}(mk(\theta - x)) + \frac{\delta^2}{2}\frac{\partial^2}{\partial x^2}(\tilde{m}x(1-x)) \tag{6}$$

$$\mathcal{L}^{*} m = -\frac{\partial}{\partial x}(mk(\theta - x)) + \frac{\delta^2}{2}\frac{\partial^2}{\partial x^2}(\tilde{m}x(1-x)) - \frac{\partial m}{\partial a} \tag{7}$$

$$i = 0 \cdots N_t - 2,\ t = (i+1)\Delta t$$
$$z = 0 \cdots N_a - 1$$
$$j = 0 \cdots N_X - 1$$

$$\frac{m[i][z][j] - m[i-1][z][j]}{\Delta t}$$
$$+\frac{m[i][z][j+1]k(\theta - x[j+1]) - m[i][z][j-1]k(\theta - x[j-1])}{2\Delta x}$$
$$-\frac{\delta^2}{2}\frac{m[i][z][j+1]x[j+1](1-x[j+1]) - 2m[i][z][j]x[j](1-x[j]) + m[i][z][j-1]x[j-1](1-x[j-1])}{\Delta x^2} \tag{8}$$
$$+\frac{m[i][z][j] - m[i][z-1][j]}{\Delta a} + \mu[i][z][j] = 0 \tag{9}$$

$$\frac{\hat{m}[i][j] - \hat{m}[i-1][j]}{\Delta t}$$
$$+\frac{\hat{m}[i][j+1]k(\theta - x[j+1]) - \hat{m}[i][j-1]k(\theta - x[j-1])}{2\Delta x}$$
$$-\frac{\delta^2}{2}\frac{\hat{m}[i][j+1]x[j+1](1-x[j+1]) - 2\hat{m}[i][j]x[j](1-x[j]) + \hat{m}[i][j-1]x[j-1](1-x[j-1])}{\Delta x^2} + \hat{\mu}[i][j] = 0 \tag{10}$$

$$\frac{\tilde{m}[i][j] - \tilde{m}[i-1][j]}{\Delta t}$$
$$+\frac{\tilde{m}[i][j+1]k(\theta - x[j+1]) - \tilde{m}[i][j-1]k(\theta - x[j-1])}{2\Delta x}$$
$$-\frac{\delta^2}{2}\frac{\tilde{m}[i][j+1]x[j+1] - 2\tilde{m}[i][j]x[j] + \tilde{m}[i][j-1]x[j-1]}{\Delta x^2} + \tilde{\mu}[i][j] - m[i][Na-1][j] = 0 \tag{11}$$

$$V[j] = 1 + \frac{\Delta t \delta^2 x[j](1-x[j])}{\Delta x} + \frac{\Delta t}{\Delta a} \tag{12}$$

$$V_0[j] = 1 + \frac{\Delta t \delta^2 x[j](1-x[j])}{\Delta x} \tag{13}$$

$$V_1[j] = -\frac{\Delta t\delta^2 x[j](1-x[j])}{2\Delta x^2} + \frac{\Delta t k(\theta - x[j])}{2\Delta x} \tag{14}$$

$$V_2[j] = -\frac{\Delta t\delta^2 x[j](1-x[j])}{2\Delta x^2} - \frac{\Delta t k(\theta - x[j])}{2\Delta x} \tag{15}$$

$$V[j]m[i][z][j]+V_1[j+1]m[i][z][j+1]+V_2[j-1]m[i][z][j-1]-m[i-1][z][j]-\frac{\Delta t}{\Delta a}m[i][z-1][j]+\Delta t\mu[i][z][j]=0 \tag{16}$$

$$V_0[j]\hat{m}[i][j] + V_1[j+1]\hat{m}[i][j+1] + V_2[j-1]\hat{m}[i][j-1] - \hat{m}[i-1][j] + \Delta t\hat{\mu}[i][j] = 0 \tag{17}$$

$$V_0[j]\tilde{m}[i][j] + V_1[j+1]\tilde{m}[i][j+1] + V_2[j-1]\tilde{m}[i][j-1] - \tilde{m}[i-1][j] + \Delta t\tilde{\mu}[i][j] - \Delta t m[i][N_a-1][j] = 0 \tag{18}$$

When $i = 0$, $t = \Delta t$

$$V[j]m[0][z][j]+V_1[j+1]m[0][z][j+1]+V_2[j-1]m[0][z][j-1]-\frac{\Delta t}{\Delta a}m[0][z-1][j]+\Delta t\mu[0][z][j]=m_0[j] \tag{19}$$

$$V_0[j]\hat{m}[0][j] + V_1[j+1]\hat{m}[0][j+1] + V_2[j-1]\hat{m}[0][j-1] + \Delta t\hat{\mu}[0][j] = \hat{m}_0[j] \tag{20}$$

$$V_0[j]\tilde{m}[0][j] + V_1[j+1]\tilde{m}[0][j+1] + V_2[j-1]\tilde{m}[0][j-1] + \Delta t\tilde{\mu}[0][j] - \Delta t m[0][N_a-1][j] = 0 \tag{21}$$

capacity:

$$c(t) = \int_{\mathcal{O}}\int_{\mathcal{A}} m \; dadx + \int_{\mathcal{O}} \tilde{m} \; dx \tag{22}$$

conventional offer: no age variable
for given i (t)

$$\sum_{j=0}^{N_X-1} F[j]\left(\sum_{z=1}^{N_a-1} m[i][z][j]\Delta a\Delta x\right) \tag{23}$$

Renewable offer:

$$\int_{\mathcal{O}}\int_{\mathcal{A}} Xm \; dadx + \int_{\mathcal{O}} X\tilde{m} \; dx \tag{24}$$

for given i (t)

$$\sum_{j=0}^{N_X-1} X[j]\left(\left(\sum_{z=1}^{N_a-1} m[i][z][j]\Delta a\Delta x\right) + \tilde{m}[i][j]\Delta x\right) \tag{25}$$

objective function

$$\begin{aligned}\int_t\int_{\mathcal{O}}\int_{\mathcal{A}} e^{-\rho t}G \; m \; dtdxda + \int_t\int_{\mathcal{O}} e^{-\rho t}G \; \tilde{m} \; dtdx + \int_t\int_{\mathcal{O}}\int_{\mathcal{A}} e^{-(\rho+\gamma)t}f \; \mu \; dtdxda \\ + \int_t\int_{\mathcal{O}} e^{-(\rho+\gamma)t}f \; \tilde{\mu} \; dtdx - \int_{\mathcal{O}}\int_{\mathcal{A}} e^{-(\rho+\gamma)t}s \; \hat{\mu} \; dxda\end{aligned} \tag{26}$$

$$\sum_{i=0}^{N_t-2}\sum_{j=0}^{N_a-1}\sum_{z=1}^{N_X-1} e^{-\rho t}G\ m[i][z][j]\ +\sum_{i=0}^{N_t-2}\sum_{j=0}^{N_a-1} e^{-\rho t}G\ \tilde{m}[i][z][j]\Delta t\Delta x$$
$$+\sum_{i=0}^{N_t-2}\sum_{j=0}^{N_a-1}\sum_{z=1}^{N_X-1} e^{-(\rho+\gamma)t}f\ \mu[i][z][j]\ \Delta t\Delta x\Delta a+\sum_{i=0}^{N_t-2}\sum_{j=0}^{N_a-1} e^{-(\rho+\gamma)t}f\ \tilde{\mu}[i][z][j]\ \Delta t\Delta x$$
$$-\sum_{i=0}^{N_t-2}\sum_{j=0}^{N_a-1} e^{-(\rho+\gamma)t}s\ \hat{\mu}[i][j]\ \Delta t\Delta x \tag{27}$$

When $z=0$

$$m[i][0][j]=\hat{\mu}[i][j]$$

When $z=1$

$$V[j]m[i][1][j]+V_1[j+1]m[i][1][j+1]+V_2[j-1]m[i][1][j-1]-m[i-1][1][j]+\Delta t\mu[i][j]=\frac{\Delta t}{\Delta a}\hat{\mu}[i][j] \tag{28}$$

When $z=2,\cdots,N_a-1$

$$V[j]m[i][z][j]+V_1[j+1]m[i][z][j+1]+V_2[j-1]m[i][z][j-1]-m[i-1][z][j]-\frac{\Delta t}{\Delta a}m[i][z-1][j]+\Delta t\mu[i][j]=0 \tag{29}$$

$$V_0[j]\hat{m}[i][j]+V_1[j+1]\hat{m}[i][j+1]+V_2[j-1]\hat{m}[i][j-1]-\hat{m}[i-1][j]+\Delta t\hat{\mu}[i][j]=0 \tag{30}$$

$$V_0[j]\tilde{m}[i][j]+V_1[j+1]\tilde{m}[i][j+1]+V_2[j-1]\tilde{m}[i][j-1]-\tilde{m}[i-1][j]+\Delta t\tilde{\mu}[i][j]-\Delta t m[i][N_a-1][j]=0 \tag{31}$$

Summing up with z from 1 to N_a-1. Take

$$\bar{m}[i][j]=\left(\sum_{z=1}^{N_a-1} m[i][j]\Delta a\right)+\tilde{m}[i][j]$$

$$\bar{\mu}[i][j]=\left(\sum_{z=1}^{N_a-1} \mu[i][j]\Delta a\right)+\tilde{\mu}[i][j]$$

$$V_0[j]\bar{m}[i][j]+V_1[j+1]\bar{m}[i][j+1]+V_2[j-1]\bar{m}[i][j-1]-\bar{m}[i-1][j]+\Delta t m\bar{u}[i][j]-\Delta t\hat{\mu}[i][j]=0 \tag{32}$$

Equation (32) is same as 2D model.
Boundary cases: When $z\neq 0$, $j=0$, $i\neq 0$

$$V[0]m[i][z][0]+V_1[1]m[i][z][1]-m[i-1][z][0]-\frac{\Delta t}{\Delta a}m[i][z-1][0]+\Delta t\mu[i][z][0]=0 \tag{33}$$

$$V_0[0]\hat{m}[i][0]+V_1[1]\hat{m}[i][1]-\hat{m}[i-1][0]+\Delta t\hat{\mu}[i][0]=0 \tag{34}$$

$$V_0[0]\tilde{m}[i][0]+V_1[1]\tilde{m}[i][1]-\tilde{m}[i-1][0]+\Delta t\tilde{\mu}[i][0]-\Delta t m[i][N_a-1][0]=0 \tag{35}$$

When $z\neq 0$, $j=N_X-1$, $i\neq 0$

$$V[N_X-1]m[i][z][N_X-1]+V_2[N_X-2]m[i][z][N_X-2]-m[i-1][z][N_X-1]-\frac{\Delta t}{\Delta a}m[i][z-1][N_X-1]+\Delta t\mu[i][z][N_X-1]=0 \tag{36}$$

$$V_0[N_X-1]\hat{m}[i][N_X-1]+V_2[N_X-2]\hat{m}[i][N_X-2]-\hat{m}[i-1][N_X-1]+\Delta t\hat{\mu}[i][N_X-1]=0 \tag{37}$$

$$V_0[N_X-1]\tilde{m}[i][N_X-1]+V_2[N_X-2]\tilde{m}[i][N_X-2]-\tilde{m}[i-1][N_X-1]+\Delta t\tilde{\mu}[i][N_X-1]-\Delta t m[i][N_a-1][N_X-1]=0 \tag{38}$$

When $z \neq 0$, $i = 0$, $j \neq 0$, $j \neq N_X - 1$

$$V[j]m[0][z][j]+V_1[j+1]m[0][z][j+1]+V_2[j-1]m[0][z][j-1]-\frac{\Delta t}{\Delta a}m[0][z-1][j]+\Delta t\mu[0][j]=m_0[i][j] \tag{39}$$

$$V_0[j]\hat{m}[0][j]+V_1[j]\hat{m}[0][j+1]+V_2[j-1]\hat{m}[0][j-1]+\Delta t\hat{\mu}[0][j]=0 \tag{40}$$

$$V_0[j]\tilde{m}[0][j]+V_1[j+1]\tilde{m}[0][j+1]+V_2[j-1]\tilde{m}[0][j-1]+\Delta t\tilde{\mu}[0][j]-\Delta t m[0][N_a-1][j]=0 \tag{41}$$

When $z \neq 0$, $i = 0$, $j = 0$

$$V[0]m[0][z][0]+V_1[1]m[0][z][1]-\frac{\Delta t}{\Delta a}m[0][z-1][0]+\Delta t\mu[0][z][0]=0 \tag{42}$$

$$V_0[0]\hat{m}[i][0]+V_1[1]\hat{m}[0][1]+\Delta t\hat{\mu}[0][0]=0 \tag{43}$$

$$V_0[0]\tilde{m}[0][0]+V_1[1]\tilde{m}[0][1]+\Delta t\tilde{\mu}[0][0]-\Delta t m[0][N_a-1][0]=0 \tag{44}$$

When $z \neq 0$, $i = 0$, $j = N_X - 1$

$$V[N_X-1]m[0][z][N_X-1]+V_2[N_X-2]m[0][z][N_X-2]-\frac{\Delta t}{\Delta a}m[0][z-1][N_X-1]+\Delta t\mu[0][z][N_X-1]=0 \tag{45}$$

$$V_0[N_X-1]\hat{m}[0][N_X-1]+V_2[N_X-2]\hat{m}[0][N_X-2]+\Delta t\hat{\mu}[0][N_X-1]=0 \tag{46}$$

$$V_0[N_X-1]\tilde{m}[0][N_X-1]+V_2[N_X-2]\tilde{m}[0][N_X-2]+\Delta t\tilde{\mu}[0][N_X-1]-\Delta t m[0][N_a-1][N_X-1]=0 \tag{47}$$

Discretize $\frac{\partial m}{\partial a}$ using central difference

$$\frac{m[i][z][j]-m[i-1][z][j]}{\Delta t}$$
$$+\frac{m[i][z][j+1]k(\theta-x[j+1])-m[i][z][j-1]k(\theta-x[j-1])}{2\Delta x}$$
$$-\frac{\delta^2}{2}\frac{m[i][z][j+1]x[j+1](1-x[j+1])-2m[i][z][j]x[j](1-x[j])+m[i][z][j-1]x[j-1](1-x[j-1])}{\Delta x^2} \quad (48)$$
$$+\frac{m[i][z+1][j]-m[i][z-1][j]}{2\Delta a}+\mu[i][z][j]=0 \quad (49)$$

$$\frac{\hat{m}[i][j]-\hat{m}[i-1][j]}{\Delta t}$$
$$+\frac{\hat{m}[i][j+1]k(\theta-x[j+1])-\hat{m}[i][j-1]k(\theta-x[j-1])}{2\Delta x}$$
$$-\frac{\delta^2}{2}\frac{\hat{m}[i][j+1]x[j+1](1-x[j+1])-2\hat{m}[i][j]x[j](1-x[j])+\hat{m}[i][j-1]x[j-1](1-x[j-1])}{\Delta x^2}+\hat{\mu}[i][j]=0 \quad (50)$$

$$\frac{\tilde{m}[i][j]-\tilde{m}[i-1][j]}{\Delta t}$$
$$+\frac{\tilde{m}[i][j+1]k(\theta-x[j+1])-\tilde{m}[i][j-1]k(\theta-x[j-1])}{2\Delta x}$$
$$-\frac{\delta^2}{2}\frac{\tilde{m}[i][j+1]x[j+1]-2\tilde{m}[i][j]x[j]+\tilde{m}[i][j-1]x[j-1]}{\Delta x^2}+\tilde{\mu}[i][j]-m[i][Na-1][j]=0 \quad (51)$$

$$V_0[j]=1+\frac{\Delta t\delta^2 x[j](1-x[j])}{\Delta x} \quad (52)$$

$$V_1[j]=-\frac{\Delta t\delta^2 x[j](1-x[j])}{2\Delta x^2}+\frac{\Delta t k(\theta-x[j])}{2\Delta x} \quad (53)$$

$$V_2[j]=-\frac{\Delta t\delta^2 x[j](1-x[j])}{2\Delta x^2}-\frac{\Delta t k(\theta-x[j])}{2\Delta x} \quad (54)$$

$$V_0[j]m[i][z][j]+V_1[j+1]m[i][z][j+1]+V_2[j-1]m[i][z][j-1]-m[i-1][z][j]$$
$$+\frac{\Delta t}{2\Delta a}m[i][z+1][j]-\frac{\Delta t}{2\Delta a}m[i][z-1][j]+\Delta t\mu[i][z][j]=0 \quad (55)$$

$$V_0[j]\hat{m}[i][j]+V_1[j+1]\hat{m}[i][j+1]+V_2[j-1]\hat{m}[i][j-1]-\hat{m}[i-1][j]+\Delta t\hat{\mu}[i][j]=0 \quad (56)$$

$$V_0[j]\tilde{m}[i][j]+V_1[j+1]\tilde{m}[i][j+1]+V_2[j-1]\tilde{m}[i][j-1]-\tilde{m}[i-1][j]+\Delta t\tilde{\mu}[i][j]-\Delta t m[i][N_a-1][j]=0 \quad (57)$$

When $z=N_a-1$

$$V_0[j]m[i][z][j]+V_1[j+1]m[i][z][j+1]+V_2[j-1]m[i][z][j-1]-m[i-1][z][j]$$
$$-\frac{\Delta t}{2\Delta a}m[i][z-1][j]+\Delta t\mu[i][z][j]=0 \quad (58)$$

The sum of Equation (55) for z from 1 to N_a-2, and sum with Equation (57) and (58) is not same as 2d cases. So it is not good to use central difference for $\frac{\partial m}{\partial a}$.

For 2d model, the relaxed version of this optimisation problem is the following

$$\sup_{(m,\mu)\in\mathcal{A}(m_0)} \int_0^T \int_{\mathbb{R}} f(t,x,\tilde{m}_t)m_t(dx)dt + \int_0^T \int_{\mathbb{R}} F(t,x,\tilde{\mu})\mu(dx,dt)$$
$$- \frac{\gamma}{2}(\int_0^T \int_{\mathbb{R}} f^2(t,x,\tilde{m}_t)m_t(dx) - \int_0^T \left(\int_{\mathbb{R}} f(t,x,\tilde{m}_t)m_t(dx)\right)^2 dt, \tag{59}$$

where $\mathcal{A}(m_0)$ denotes the set of all couples $(m,\mu) \in \mathcal{P}_{\leq 1}(\mathbb{R}) \times \mathcal{P}([0,T]\times\mathbb{R})$ such that

$$\int_0^T \int_{\mathbb{R}} \varphi(t,x)\mu(dt,dx) = \int_{\mathbb{R}} \varphi(0,x)m_0(dx) + \int_0^T \int_{\mathbb{R}} (\partial_t + \mathcal{L}_X)\varphi(t,x)m_t(dx)dt. \tag{60}$$

Discrete time by $t_i = i\Delta t = \frac{iT}{n_t}$
Assume $\mathcal{R} = (\underline{x}, \bar{x})$
Discretize the state space by $\underline{x} = x_0 < x_1 < \cdots < x_{n_s} = \bar{x}$
The objective function

$$\int_t \int_{\mathcal{O}} \left(e^{-\rho t}Gm - \frac{\gamma}{2}e^{-2\rho t}G^2 m\right)\, dtdx + \int_t \int_{\mathcal{O}} e^{-(\rho+\gamma)t} f\ \mu\ dtdx - \int_t \int_{\mathcal{O}} e^{-(\rho+\gamma)t} s\ \hat{\mu}\ dxda + \frac{\gamma}{2}\int_t \left(\int_{\mathcal{O}} e^{-\rho t}Gm\ dx\right)^2 dt \tag{61}$$

The last term

$$\left(\sum_{j=0}^{N_X-1} e^{-\rho t}Gm[i][z][j]\Delta x\right)^2 = \sum_{j_1=0}^{N_X-1}\sum_{j_2=0}^{N_X-1} (e^{-\rho t}Gm[i][j_1]\Delta x)(e^{-\rho t}Gm[i][j_2]\Delta x) \tag{62}$$

Full discretization:

$$\sum_{i=0}^{N_t-1}\sum_{j=0}^{N_t-1} \left(e^{-\rho t}Gm[i][j] - \frac{\gamma}{2}e^{-2\rho t}G^2 m[i][j]\right)\Delta t\Delta x + \sum_{i=0}^{N_t-1}\sum_{j=0}^{N_t-1} e^{-(\rho+\gamma)t} f\ \mu[i][j]\ \Delta t\Delta x$$
$$- \sum_{i=0}^{N_t-1}\sum_{j=0}^{N_t-1} e^{-(\rho+\gamma)t} s\ \hat{\mu}[i][j]\ \Delta t\Delta x + \frac{\gamma}{2}\sum_{i=0}^{N_t-1}\sum_{j_1=0}^{N_X-1}\sum_{j_2=0}^{N_X-1} (e^{-\rho t}Gm[i][j_1]\Delta x)(e^{-\rho t}Gm[i][j_2]\Delta x)\Delta t \tag{63}$$

For 3d model, the objective

$$\sup_{(m,\mu)\in\mathcal{A}(m_0)} \int_0^T \int_{\mathbb{R}}\int_{\mathbb{A}} f(t,x,\tilde{m}_t)m_t(dx)dtda + \int_0^T \int_{\mathbb{R}}\int_{\mathbb{A}} F(t,x,\tilde{\mu})\mu(dx,dt,da)$$
$$- \frac{\gamma}{2}(\int_0^T \int_{\mathcal{O}} f^2(t,x,\tilde{m}_t)m_t(dx) - \int_0^T \left(\int_{\mathbb{R}} f(t,x,\tilde{m}_t)m_t(dx)\right)^2 dt, \tag{64}$$

$$\int_t \int_{\mathcal{O}}\int_{\mathcal{A}} \left(e^{-\rho t}Gm - \frac{\gamma}{2}e^{-2\rho t}G^2 m^2\right)\, dtdxda + \int_t \int_{\mathcal{O}} \left(e^{-\rho t}G\tilde{m} - \frac{\gamma}{2}e^{-2\rho t}G^2 \tilde{m}^2\right)\, dtdx \tag{65}$$
$$+ \int_t \int_{\mathcal{O}}\int_{\mathcal{A}} e^{-(\rho+\gamma)t} f\ \mu\ dtdxda + \int_t \int_{\mathcal{O}} e^{-(\rho+\gamma)t} f\ \tilde{\mu}\ dtdx - \int_t \int_{\mathcal{O}} e^{-(\rho+\gamma)t} s\ \hat{\mu}\ dxda$$
$$+ \frac{\gamma}{2}\int_t \left(\int_{\mathcal{O}}\int_{\mathcal{A}} e^{-\rho t}Gm\ dxda + \int_{\mathcal{O}} e^{-\rho t}G\tilde{m}dx\right)^2 dt \tag{66}$$

The linear terms

$$\sum_{i=0}^{N_t-1}\sum_{j=0}^{N_t-1}\sum_{z=1}^{N_a-1}\left(e^{-\rho t}Gm[i][z][j]-\frac{\gamma}{2}e^{-2\rho t}G^2m[i][z][j]\right)\Delta t\Delta x\Delta a$$
$$+\sum_{i=0}^{N_t-1}\sum_{j=0}^{N_t-1}\left(e^{-\rho t}G\tilde{m}[i][j]-\frac{\gamma}{2}e^{-2\rho t}G^2\tilde{m}[i][j]\right)\Delta t\Delta x$$
$$+\sum_{i=0}^{N_t-1}\sum_{j=0}^{N_t-1}\sum_{z=1}^{N_a-1}e^{-(\rho+\gamma)t}f\ \mu[i][z][j]\ \Delta t\Delta x\Delta a$$
$$+\sum_{i=0}^{N_t-1}\sum_{j=0}^{N_t-1}e^{-(\rho+\gamma)t}f\ \tilde{\mu}[i][j]\Delta t\Delta x-\sum_{i=0}^{N_t-1}\sum_{j=0}^{N_t-1}e^{-(\rho+\gamma)t}s\ \hat{\mu}[i][j]\ \Delta t\Delta x \tag{67}$$

The non-linear term

$$\begin{aligned}
&\frac{\gamma}{2}\sum_{i=0}^{N_t-1}\left(\sum_{j=0}^{N_X-1}\sum_{z=1}^{N_a-1}e^{-\rho t}Gm[i][z][j]\Delta x\Delta a\ +\sum_{k=0}^{N_X-1}e^{-\rho t}G\tilde{m}[i][j]\Delta x\right)^2\Delta t\\
&=\frac{\gamma}{2}\sum_{i=0}^{N_t-1}\sum_{j_1=0}^{N_X-1}\sum_{z_1=1}^{N_a-1}\sum_{j_2=0}^{N_X-1}\sum_{z_2=1}^{N_a-1}(e^{-\rho t}Gm[i][z_1][j_1]\Delta x\Delta a)(e^{-\rho t}Gm[i][z_2][j_2]\Delta x\Delta a)\Delta t\\
&+\frac{\gamma}{2}\sum_{i=0}^{N_t-1}\sum_{k_1=0}^{N_X-1}\sum_{k_2=0}^{N_X-1}(e^{-\rho t}G\tilde{m}[i][z][j]\Delta x)(e^{-\rho t}G\tilde{m}[i][k]\Delta x)\Delta t\\
&+\frac{\gamma}{2}\sum_{i=0}^{N_t-1}\sum_{j=0}^{N_X-1}\sum_{z=1}^{N_a-1}\sum_{k=0}^{N_X-1}(e^{-\rho t}Gm[i][z][j]\Delta x\Delta a)(e^{-\rho t}G\tilde{m}[i][k]\Delta x)\Delta t
\end{aligned} \tag{68}$$