

# discretization of 3D fokker planck

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Link between 2D and 3D: For renewable agents,  $m(t, a, x)$  and  $\mu(t, a, x)$  are 3d with time, age, state.  
 $\hat{m}(t, x)$ ,  $\hat{\mu}(t, x)$ ,  $\tilde{m}(t, x)$ ,  $\tilde{\mu}(t, x)$   
 For conventional agents, all variables are 2d with time and state.

$$-\frac{\partial m}{\partial t} + \mathcal{L}^* m - \mu(t, a, x) = 0 \quad (1)$$

Let

$$\begin{aligned} m^* &= \int_{\mathcal{A}} m \, da \\ \mu^* &= \int_{\mathcal{A}} \mu \, da \\ -\frac{\partial m^*}{\partial t} + \mathcal{L}^{0*} m - \int_{\mathcal{A}} \frac{\partial m}{\partial a} \, da - \mu^* &= 0 \end{aligned} \quad (2)$$

where

$$\begin{aligned} \mathcal{L}^* m &= \mathcal{L}^{0*} m - \frac{\partial m}{\partial a} \\ \int_{\mathcal{A}} \frac{\partial m}{\partial a} \, da &= m(t, a_{\max}, x) - m(t, 0, x) = m(t, a_{\max}, x) - \hat{\mu}(t, x) \\ -\frac{\partial m^*}{\partial t} + \mathcal{L}^{0*} m - m(t, a_{\max}, x) + \hat{\mu}(t, x) - \mu^* &= 0 \end{aligned}$$

Let

$$\bar{m} = m^* + \tilde{m}$$

As

$$-\frac{\partial \tilde{m}}{\partial t} + \mathcal{L}^{0*} \tilde{m} + m(t, \bar{a}, x) - \tilde{\mu}(t, a, x) = 0$$

So

$$-\frac{\partial \bar{m}}{\partial t} + \mathcal{L}^{0*} \bar{m} + \hat{\mu}(t, x) - \bar{\mu}(t, x) = 0$$

The constraints

$$-\frac{\partial m}{\partial t} + \mathcal{L}^* m - \mu(t, a, x) = 0 \quad (3)$$

$$-\frac{\partial \tilde{m}}{\partial t} + \mathcal{L}^{0*} \tilde{m} + m(t, \bar{a}, x) - \tilde{\mu}(t, a, x) = 0 \quad (4)$$

$$-\frac{\partial \hat{m}}{\partial t} + \mathcal{L}^{0*} \hat{m} - \hat{\mu}(t, a, x) = 0 \quad (5)$$

$$t \in [0, \bar{t}], a \in [0, \bar{a}], X \in [0, \bar{x}]$$

$$\mathcal{L}^{0*} m = -\frac{\partial}{\partial x}(mk(\theta - x)) + \frac{\delta^2}{2} \frac{\partial^2}{\partial x^2}(\tilde{m}x(1 - x)) \quad (6)$$

$$\mathcal{L}^* m = -\frac{\partial}{\partial x}(mk(\theta - x)) + \frac{\delta^2}{2} \frac{\partial^2}{\partial x^2}(\tilde{m}x(1 - x)) - \frac{\partial m}{\partial a} \quad (7)$$

$$i = 0 \cdots N_t - 2, t = (i + 1)\Delta t$$

$$z = 0 \cdots N_a - 1$$

$$j = 0 \cdots N_X - 1$$

$$\begin{aligned} & \frac{m[i][z][j] - m[i-1][z][j]}{\Delta t} \\ & + \frac{m[i][z][j+1]k(\theta - x[j+1]) - m[i][z][j-1]k(\theta - x[j-1])}{2\Delta x} \\ & - \frac{\delta^2}{2} \frac{m[i][z][j+1]x[j+1](1 - x[j+1]) - 2m[i][z][j]x[j](1 - x[j]) + m[i][z][j-1]x[j-1](1 - x[j-1])}{\Delta x^2} \end{aligned} \quad (8)$$

$$+ \frac{m[i][z][j] - m[i][z-1][j]}{\Delta a} + \mu[i][z][j] = 0 \quad (9)$$

$$\begin{aligned} & \frac{\hat{m}[i][j] - \hat{m}[i-1][j]}{\Delta t} \\ & + \frac{\hat{m}[i][j+1]k(\theta - x[j+1]) - \hat{m}[i][j-1]k(\theta - x[j-1])}{2\Delta x} \\ & - \frac{\delta^2}{2} \frac{\hat{m}[i][j+1]x[j+1](1 - x[j+1]) - 2\hat{m}[i][j]x[j](1 - x[j]) + \hat{m}[i][j-1]x[j-1](1 - x[j-1])}{\Delta x^2} + \hat{\mu}[i][j] = 0 \end{aligned} \quad (10)$$

$$\begin{aligned} & \frac{\tilde{m}[i][j] - \tilde{m}[i-1][j]}{\Delta t} \\ & + \frac{\tilde{m}[i][j+1]k(\theta - x[j+1]) - \tilde{m}[i][j-1]k(\theta - x[j-1])}{2\Delta x} \\ & - \frac{\delta^2}{2} \frac{\tilde{m}[i][j+1]x[j+1] - 2\tilde{m}[i][j]x[j] + \tilde{m}[i][j-1]x[j-1]}{\Delta x^2} + \tilde{\mu}[i][j] - m[i][Na-1][j] = 0 \end{aligned} \quad (11)$$

$$V[j] = 1 + \frac{\Delta t \delta^2 x[j](1 - x[j])}{\Delta x} + \frac{\Delta t}{\Delta a} \quad (12)$$

$$V_0[j] = 1 + \frac{\Delta t \delta^2 x[j](1 - x[j])}{\Delta x} \quad (13)$$

$$V_1[j] = -\frac{\Delta t \delta^2 x[j](1-x[j])}{2\Delta x^2} + \frac{\Delta t k(\theta - x[j])}{2\Delta x} \quad (14)$$

$$V_2[j] = -\frac{\Delta t \delta^2 x[j](1-x[j])}{2\Delta x^2} - \frac{\Delta t k(\theta - x[j])}{2\Delta x} \quad (15)$$

$$V[j]m[i][z][j] + V_1[j+1]m[i][z][j+1] + V_2[j-1]m[i][z][j-1] - m[i-1][z][j] - \frac{\Delta t}{\Delta a}m[i][z-1][j] + \Delta t \mu[i][z][j] = 0 \quad (16)$$

$$V_0[j]\hat{m}[i][j] + V_1[j+1]\hat{m}[i][j+1] + V_2[j-1]\hat{m}[i][j-1] - \hat{m}[i-1][j] + \Delta t \hat{\mu}[i][j] = 0 \quad (17)$$

$$V_0[j]\tilde{m}[i][j] + V_1[j+1]\tilde{m}[i][j+1] + V_2[j-1]\tilde{m}[i][j-1] - \tilde{m}[i-1][j] + \Delta t \tilde{\mu}[i][j] - \Delta t m[i][N_a-1][j] = 0 \quad (18)$$

When  $i = 0$ ,  $t = \Delta t$

$$V[j]m[0][z][j] + V_1[j+1]m[0][z][j+1] + V_2[j-1]m[0][z][j-1] - \frac{\Delta t}{\Delta a}m[0][z-1][j] + \Delta t \mu[0][z][j] = m_0[j] \quad (19)$$

$$V_0[j]\hat{m}[0][j] + V_1[j+1]\hat{m}[0][j+1] + V_2[j-1]\hat{m}[0][j-1] + \Delta t \hat{\mu}[0][j] = \hat{m}_0[j] \quad (20)$$

$$V_0[j]\tilde{m}[0][j] + V_1[j+1]\tilde{m}[0][j+1] + V_2[j-1]\tilde{m}[0][j-1] + \Delta t \tilde{\mu}[0][j] - \Delta t m[0][N_a-1][j] = 0 \quad (21)$$

capacity:

$$c(t) = \int_{\mathcal{O}} \int_{\mathcal{A}} m \, da \, dx \quad (22)$$

conventional offer:

$$\int_{\mathcal{O}} \int_{\mathcal{A}} F m \, da \, dx + \int_{\mathcal{O}} \int_{\mathcal{A}} F \tilde{m} \, dx \quad (23)$$

for given  $i(t)$

$$\sum_{j=0}^{N_X-1} F[j] \left( \left( \sum_{z=1}^{N_a-1} m[i][z][j] \Delta a \Delta x \right) + \tilde{m}[i][j] \Delta x \right) \quad (24)$$

Renewable offer:

$$\int_{\mathcal{O}} \int_{\mathcal{A}} X m \, da \, dx + \int_{\mathcal{O}} \int_{\mathcal{A}} X \tilde{m} \, dx \quad (25)$$

for given  $i(t)$

$$\sum_{j=0}^{N_X-1} X[j] \left( \left( \sum_{z=1}^{N_a-1} m[i][z][j] \Delta a \Delta x \right) + \tilde{m}[i][j] \Delta x \right) \quad (26)$$

objective function

$$\int_t \int_{\mathcal{O}} \int_{\mathcal{A}} e^{-\rho t} G m \, dt \, da \, dx + \int_t \int_{\mathcal{O}} \int_{\mathcal{A}} e^{-(\rho+\gamma)t} f \mu \, dt \, da \, dx - \int_{\mathcal{O}} \int_{\mathcal{A}} e^{-(\rho+\gamma)t} s \hat{\mu} \, da \, dx \quad (27)$$

$$\begin{aligned}
& \sum_{i=0}^{N_t-2} \sum_{j=0}^{N_a-1} \sum_{z=1}^{N_X-1} e^{-\rho t} G m[i][z][j] + \sum_{i=0}^{N_t-2} \sum_{j=0}^{N_a-1} e^{-\rho t} G \tilde{m}[i][z][j] \Delta t \Delta x \\
& + \sum_{i=0}^{N_t-2} \sum_{j=0}^{N_a-1} \sum_{z=1}^{N_X-1} e^{-(\rho+\gamma)t} f \mu[i][z][j] \Delta t \Delta x \Delta a + \sum_{i=0}^{N_t-2} \sum_{j=0}^{N_a-1} e^{-(\rho+\gamma)t} f \tilde{\mu}[i][z][j] \Delta t \Delta x \\
& - \sum_{i=0}^{N_t-2} \sum_{j=0}^{N_a-1} e^{-(\rho+\gamma)t} s \hat{\mu}[i][j] \Delta t \Delta x \quad (28)
\end{aligned}$$

When  $z = 0$

$$m[i][0][j] = \hat{\mu}[i][j]$$

When  $z = 1$

$$V[j]m[i][1][j] + V_1[j+1]m[i][1][j+1] + V_2[j-1]m[i][1][j-1] - m[i-1][1][j] + \Delta t \mu[i][j] = \frac{\Delta t}{\Delta a} \hat{\mu}[i][j] \quad (29)$$

When  $z = 2, \dots, N_a - 1$

$$V[j]m[i][z][j] + V_1[j+1]m[i][z][j+1] + V_2[j-1]m[i][z][j-1] - m[i-1][z][j] - \frac{\Delta t}{\Delta a} m[i][z-1][j] + \Delta t \mu[i][j] = 0 \quad (30)$$

$$V_0[j]\hat{m}[i][j] + V_1[j+1]\hat{m}[i][j+1] + V_2[j-1]\hat{m}[i][j-1] - \hat{m}[i-1][j] + \Delta t \hat{\mu}[i][j] = 0 \quad (31)$$

$$V_0[j]\tilde{m}[i][j] + V_1[j+1]\tilde{m}[i][j+1] + V_2[j-1]\tilde{m}[i][j-1] - \tilde{m}[i-1][j] + \Delta t \tilde{\mu}[i][j] - \Delta t m[i][N_a-1][j] = 0 \quad (32)$$

Summing up with  $z$  from 1 to  $N_a - 1$ . Take

$$\begin{aligned}
\bar{m}[i][j] &= \left( \sum_{z=1}^{N_a-1} m[i][z][j] \Delta a \right) + \tilde{m}[i][j] \\
\bar{\mu}[i][j] &= \left( \sum_{z=1}^{N_a-1} \mu[i][z][j] \Delta a \right) + \tilde{\mu}[i][j]
\end{aligned}$$

$$V_0[j]\bar{m}[i][j] + V_1[j+1]\bar{m}[i][j+1] + V_2[j-1]\bar{m}[i][j-1] - \bar{m}[i-1][j] + \Delta t \bar{m}[i][j] - \Delta t \bar{\mu}[i][j] = 0 \quad (33)$$

Equation (33) is same as 2D model.

Boundary cases: When  $z \neq 0, j = 0, i \neq 0$

$$V[0]m[i][z][0] + V_1[1]m[i][z][1] - m[i-1][z][0] - \frac{\Delta t}{\Delta a} m[i][z-1][0] + \Delta t \mu[i][z][0] = 0 \quad (34)$$

$$V_0[0]\hat{m}[i][0] + V_1[1]\hat{m}[i][1] - \hat{m}[i-1][0] + \Delta t \hat{\mu}[i][0] = 0 \quad (35)$$

$$V_0[0]\tilde{m}[i][0] + V_1[1]\tilde{m}[i][1] - \tilde{m}[i-1][0] + \Delta t \tilde{\mu}[i][0] - \Delta t m[i][N_a-1][0] = 0 \quad (36)$$

When  $z \neq 0, j = N_X - 1, i \neq 0$

$$V[N_X-1]m[i][z][N_X-1]+V_2[N_X-2]m[i][z][N_X-2]-m[i-1][z][N_X-1]-\frac{\Delta t}{\Delta a}m[i][z-1][N_X-1]+\Delta t\mu[i][z][N_X-1]=0 \quad (37)$$

$$V_0[N_X-1]\hat{m}[i][N_X-1]+V_2[N_X-2]\hat{m}[i][N_X-2]-\hat{m}[i-1][N_X-1]+\Delta t\hat{\mu}[i][N_X-1]=0 \quad (38)$$

$$V_0[N_X-1]\tilde{m}[i][N_X-1]+V_2[N_X-2]\tilde{m}[i][N_X-2]-\tilde{m}[i-1][N_X-1]+\Delta t\tilde{\mu}[i][N_X-1]-\Delta tm[i][N_a-1][N_X-1]=0 \quad (39)$$

When  $z \neq 0, i = 0, j \neq 0, j \neq N_X - 1$

$$V[j]m[0][z][j]+V_1[j+1]m[0][z][j+1]+V_2[j-1]m[0][z][j-1]-\frac{\Delta t}{\Delta a}m[0][z-1][j]+\Delta t\mu[0][j]=m_0[i][j] \quad (40)$$

$$V_0[j]\hat{m}[0][j]+V_1[j]\hat{m}[0][j+1]+V_2[j-1]\hat{m}[0][j-1]+\Delta t\hat{\mu}[0][j]=0 \quad (41)$$

$$V_0[j]\tilde{m}[0][j]+V_1[j+1]\tilde{m}[0][j+1]+V_2[j-1]\tilde{m}[0][j-1]+\Delta t\tilde{\mu}[0][j]-\Delta tm[0][N_a-1][j]=0 \quad (42)$$

When  $z \neq 0, i = 0, j = 0$

$$V[0]m[0][z][0]+V_1[1]m[0][z][1]-\frac{\Delta t}{\Delta a}m[0][z-1][0]+\Delta t\mu[0][z][0]=0 \quad (43)$$

$$V_0[0]\hat{m}[i][0]+V_1[1]\hat{m}[0][1]+\Delta t\hat{\mu}[0][0]=0 \quad (44)$$

$$V_0[0]\tilde{m}[0][0]+V_1[1]\tilde{m}[0][1]+\Delta t\tilde{\mu}[0][0]-\Delta tm[0][N_a-1][0]=0 \quad (45)$$

When  $z \neq 0, i = 0, j = N_X - 1$

$$V[N_X-1]m[0][z][N_X-1]+V_2[N_X-2]m[0][z][N_X-2]-\frac{\Delta t}{\Delta a}m[0][z-1][N_X-1]+\Delta t\mu[0][z][N_X-1]=0 \quad (46)$$

$$V_0[N_X-1]\hat{m}[0][N_X-1]+V_2[N_X-2]\hat{m}[0][N_X-2]+\Delta t\hat{\mu}[0][N_X-1]=0 \quad (47)$$

$$V_0[N_X-1]\tilde{m}[0][N_X-1]+V_2[N_X-2]\tilde{m}[0][N_X-2]+\Delta t\tilde{\mu}[0][N_X-1]-\Delta tm[0][N_a-1][N_X-1]=0 \quad (48)$$

Discretize  $\frac{\partial m}{\partial a}$  using central difference

$$\begin{aligned}
& \frac{m[i][z][j] - m[i-1][z][j]}{\Delta t} \\
& + \frac{m[i][z][j+1]k(\theta - x[j+1]) - m[i][z][j-1]k(\theta - x[j-1])}{2\Delta x} \\
& - \frac{\delta^2}{2} \frac{m[i][z][j+1]x[j+1](1-x[j+1]) - 2m[i][z][j]x[j](1-x[j]) + m[i][z][j-1]x[j-1](1-x[j-1])}{\Delta x^2} \quad (49) \\
& + \frac{m[i][z+1][j] - m[i][z-1][j]}{2\Delta a} + \mu[i][z][j] = 0 \quad (50)
\end{aligned}$$

$$\begin{aligned}
& \frac{\hat{m}[i][j] - \hat{m}[i-1][j]}{\Delta t} \\
& + \frac{\hat{m}[i][j+1]k(\theta - x[j+1]) - \hat{m}[i][j-1]k(\theta - x[j-1])}{2\Delta x} \\
& - \frac{\delta^2}{2} \frac{\hat{m}[i][j+1]x[j+1](1-x[j+1]) - 2\hat{m}[i][j]x[j](1-x[j]) + \hat{m}[i][j-1]x[j-1](1-x[j-1])}{\Delta x^2} + \hat{\mu}[i][j] = 0 \quad (51)
\end{aligned}$$

$$\begin{aligned}
& \frac{\tilde{m}[i][j] - \tilde{m}[i-1][j]}{\Delta t} \\
& + \frac{\tilde{m}[i][j+1]k(\theta - x[j+1]) - \tilde{m}[i][j-1]k(\theta - x[j-1])}{2\Delta x} \\
& - \frac{\delta^2}{2} \frac{\tilde{m}[i][j+1]x[j+1] - 2\tilde{m}[i][j]x[j] + \tilde{m}[i][j-1]x[j-1]}{\Delta x^2} + \tilde{\mu}[i][j] - m[i][Na-1][j] = 0 \quad (52)
\end{aligned}$$

$$V_0[j] = 1 + \frac{\Delta t \delta^2 x[j](1-x[j])}{\Delta x} \quad (53)$$

$$V_1[j] = -\frac{\Delta t \delta^2 x[j](1-x[j])}{2\Delta x^2} + \frac{\Delta t k(\theta - x[j])}{2\Delta x} \quad (54)$$

$$V_2[j] = -\frac{\Delta t \delta^2 x[j](1-x[j])}{2\Delta x^2} - \frac{\Delta t k(\theta - x[j])}{2\Delta x} \quad (55)$$

$$\begin{aligned}
& V_0[j]m[i][z][j] + V_1[j+1]m[i][z][j+1] + V_2[j-1]m[i][z][j-1] - m[i-1][z][j] \\
& + \frac{\Delta t}{2\Delta a}m[i][z+1][j] - \frac{\Delta t}{2\Delta a}m[i][z-1][j] + \Delta t\mu[i][z][j] = 0 \quad (56)
\end{aligned}$$

$$V_0[j]\hat{m}[i][j] + V_1[j+1]\hat{m}[i][j+1] + V_2[j-1]\hat{m}[i][j-1] - \hat{m}[i-1][j] + \Delta t\hat{\mu}[i][j] = 0 \quad (57)$$

$$V_0[j]\tilde{m}[i][j] + V_1[j+1]\tilde{m}[i][j+1] + V_2[j-1]\tilde{m}[i][j-1] - \tilde{m}[i-1][j] + \Delta t\tilde{\mu}[i][j] - \Delta t m[i][Na-1][j] = 0 \quad (58)$$

When  $z = Na - 1$

$$\begin{aligned}
& V_0[j]m[i][z][j] + V_1[j+1]m[i][z][j+1] + V_2[j-1]m[i][z][j-1] - m[i-1][z][j] \\
& - \frac{\Delta t}{2\Delta a}m[i][z-1][j] + \Delta t\mu[i][z][j] = 0 \quad (59)
\end{aligned}$$

The sum of Equation (56) for  $z$  from 1 to  $Na - 2$ , and sum with Equation (58) and (59) is not same as 2d cases. So it is not good to use central difference for  $\frac{\partial m}{\partial a}$ .