discretization of 3D fokker planck

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The constraints

$$-\frac{\partial m}{\partial t} + \mathcal{L}^{*} m - \mu(t,a,x) = 0 \tag{1}$$

$$-\frac{\partial \tilde{m}}{\partial t} + \mathcal{L}^{0*} \tilde{m} + m(t,\bar{a},x) - \tilde{\mu}(t,a,x) = 0 \tag{2}$$

$$-\frac{\partial \hat{m}}{\partial t} + \mathcal{L}^{0*} \hat{m} - \hat{\mu}(t,a,x) = 0 \tag{3}$$

$t \in [0,\bar{t}]$, $a \in [0,\bar{a}]$, $X \in [0,\bar{x}]$

$$\mathcal{L}^{0*} m = -\frac{\partial}{\partial x}(mk(\theta - x)) + \frac{\delta^2}{2}\frac{\partial^2}{\partial x^2}(\tilde{m}x) \tag{4}$$

$$\mathcal{L}^{*} m = -\frac{\partial}{\partial x}(mk(\theta - x)) + \frac{\delta^2}{2}\frac{\partial^2}{\partial x^2}(\tilde{m}x) - \frac{\partial m}{\partial a} \tag{5}$$

$$\begin{gathered} i = 0 \cdots N_t - 2,\ t = (i+1)\Delta t \\ z = 0 \cdots N_a - 1 \\ j = 0 \cdots N_X - 1 \end{gathered}$$

$$\begin{aligned} &\frac{m[i][z][j] - m[i-1][z][j]}{\Delta t} \\ &+ \frac{m[i][z][j+1]k(\theta - x[j+1]) + m[i][z][j-1]k(\theta - x[j-1]) - 2m[i][z][j]k(\theta - x[j])}{2\Delta x} \\ &- \frac{\delta^2}{2}\frac{m[i][z][j+1]x[j+1] - 2m[i][z][j]x[j] + m[i][z][j-1]x[j-1]}{\Delta x^2} \qquad (6) \\ &+ \frac{m[i][z][j] - m[i][z-1][j]}{\Delta a} + \mu = 0 \qquad (7) \end{aligned}$$

$$\begin{aligned} &\frac{\hat{m}[i][j] - \hat{m}[i-1][j]}{\Delta t} \\ &+ \frac{\hat{m}[i][j+1]k(\theta - x[j+1]) + \hat{m}[i][j-1]k(\theta - x[j-1]) - 2\hat{m}[i][j]k(\theta - x[j])}{2\Delta x} \\ &- \frac{\delta^2}{2}\frac{\hat{m}[i][j+1]x[j+1] - 2\hat{m}[i][j]x[j] + \hat{m}[i][j-1]x[j-1]}{\Delta x^2} + \hat{\mu} = 0 \end{aligned} \tag{8}$$

$$
\begin{aligned}
&\frac{\tilde{m}[i][j]-\tilde{m}[i-1][j]}{\Delta t} \\
&+\frac{\tilde{m}[i][j+1]k(\theta-x[j+1])+\tilde{m}[i][j-1]k(\theta-x[j-1])-2\tilde{m}[i][j]k(\theta-x[j])}{2\Delta x} \\
&-\frac{\delta^2}{2}\frac{\tilde{m}[i][j+1]x[j+1]-2\tilde{m}[i][j]x[j]+\tilde{m}[i][j-1]x[j-1]}{\Delta x^2}+\tilde{\mu}-m[i][Na-1][j]=0
\end{aligned} \tag{9}
$$

$$
V[j]=1+\frac{\Delta t\delta^2 x[j]}{\Delta x}+\frac{\Delta t}{\Delta a} \tag{10}
$$

$$
V_0[j]=1+\frac{\Delta t\delta^2 x[j]}{\Delta x}+\frac{\Delta t}{\Delta a} \tag{11}
$$

$$
V_1[j]=-\frac{\Delta t\delta^2 x[j]}{2\Delta x^2}+\frac{\Delta t k(\theta-x[j])}{2\Delta x} \tag{12}
$$

$$
V_2[j]=-\frac{\Delta t\delta^2 x[j]}{2\Delta x^2}-\frac{\Delta t k(\theta-x[j])}{2\Delta x} \tag{13}
$$

$$
V[j]m[i][z][j]+V_1[j+1]m[i][z][j+1]+V_2m[i][z][j-1]-m[i-1][z][j]-\frac{\Delta t}{\Delta a}m[i][z-1][j]+\Delta t\mu=0 \tag{14}
$$

$$
V_0[j]\hat{m}[i][j]+V_1[j+1]\hat{m}[i][j+1]+V_2\hat{m}[i][j-1]-\hat{m}[i-1][j]+\Delta t\hat{\mu}=0 \tag{15}
$$

$$
V_0[j]\tilde{m}[i][j]+V_1[j+1]\tilde{m}[i][j+1]+V_2\tilde{m}[i][j-1]-\tilde{m}[i-1][j]+\Delta t\tilde{\mu}-\Delta t m[i][N_a-1][j]=0 \tag{16}
$$

When $i=0$, $t=\Delta t$

$$
V[j]m[0][z][j]+V_1[j+1]m[0][z][j+1]+V_2m[0][z][j-1]-\frac{\Delta t}{\Delta a}m[0][z-1][j]+\Delta t\mu=m_0[j] \tag{17}
$$

$$
V_0[j]\hat{m}[0][j]+V_1[j+1]\hat{m}[0][j+1]+V_2\hat{m}[0][j-1]+\Delta t\hat{\mu}=\hat{m}_0[j] \tag{18}
$$

$$
V_0[j]\tilde{m}[0][j]+V_1[j+1]\tilde{m}[0][j+1]+V_2\tilde{m}[0][j-1]+\Delta t\tilde{\mu}-\Delta t m[0][N_a-1][j]=0 \tag{19}
$$

When particle exit at $\bar{a}$

$$
-\frac{\partial m}{\partial t}+\mathcal{L}^*m-\mu(t,a,x)+m(t,\bar{a},x)=0 \tag{20}
$$

$$
\begin{aligned}
&\frac{m[i][z][j]-m[i-1][z][j]}{\Delta t} \\
&+\frac{m[i][z][j+1]k(\theta-x[j+1])+m[i][z][j-1]k(\theta-x[j-1])-2m[i][z][j]k(\theta-x[j])}{2\Delta x} \\
&-\frac{\delta^2}{2}\frac{m[i][z][j+1]x[j+1]-2m[i][z][j]x[j]+m[i][z][j-1]x[j-1]}{\Delta x^2}
\end{aligned} \tag{21}
$$

$$
+\frac{m[i][z][j]-m[i][z-1][j]}{\Delta a}+\mu-m[i][Na-1][j]=0 \tag{22}
$$

$$V[j]m[i][z][j]+V_1[j+1]m[i][z][j+1]+V_2m[i][z][j-1]-m[i-1][z][j]-\frac{\Delta t}{\Delta a}m[i][z-1][j]+\Delta t\mu-\Delta tm[i][Na-1][j]=0 \tag{23}$$

$$V_0[j]\hat{m}[i][j]+V_1[j+1]\hat{m}[i][j+1]+V_2\hat{m}[i][j-1]-\hat{m}[i-1][j]+\Delta t\hat{\mu}-\Delta tm[i][Na-1][j]=0 \tag{24}$$

capacity:

$$c(t)=\int_{\mathcal{O}}\int_{\mathcal{A}} m\ dadx \tag{25}$$

offer:

$$\int_{\mathcal{O}}\int_{\mathcal{A}} Fm\ dadx+\int_{\mathcal{O}}\int_{\mathcal{A}} F\tilde{m}\ dx \tag{26}$$

for given i (t)

$$\sum_{j=0}^{N_X-1} F[j]\left(\left(\sum_{z=0}^{N_a-1} m[i][z][j]\Delta a\Delta x\right)+\tilde{m}[i][j]\Delta x\right) \tag{27}$$

objective function

$$\int_t\int_{\mathcal{O}}\int_{\mathcal{A}} e^{-\rho t}G\ m\ dtdxda+\int_t\int_{\mathcal{O}}\int_{\mathcal{A}} e^{-(\rho+\gamma)t}f\ \mu\ dtdxda-\int_{\mathcal{O}}\int_{\mathcal{A}} e^{-(\rho+\gamma)t}s\ \hat{\mu}\ dxda \tag{28}$$

$$\sum_{i=0}^{N_t-2}\sum_{j=0}^{N_a-1}\sum_{z=0}^{N_X-1} e^{-\rho t}G\ m[i][z][j]\ \Delta t\Delta x\Delta a+\sum_{i=0}^{N_t-2}\sum_{j=0}^{N_a-1}\sum_{z=0}^{N_X-1} e^{-(\rho+\gamma)t}f\ \mu[i][z][j]\ \Delta t\Delta x\Delta a-\sum_{i=0}^{N_t-2}\sum_{j=0}^{N_a-1} e^{-(\rho+\gamma)t}s\ \hat{\mu}[i][j]\ \Delta t\Delta x \tag{29}$$