FEM Discretization of 3D Maxwell Equations with Dipole Sources

1. Frequency-domain Maxwell Equation (with Dipoles)

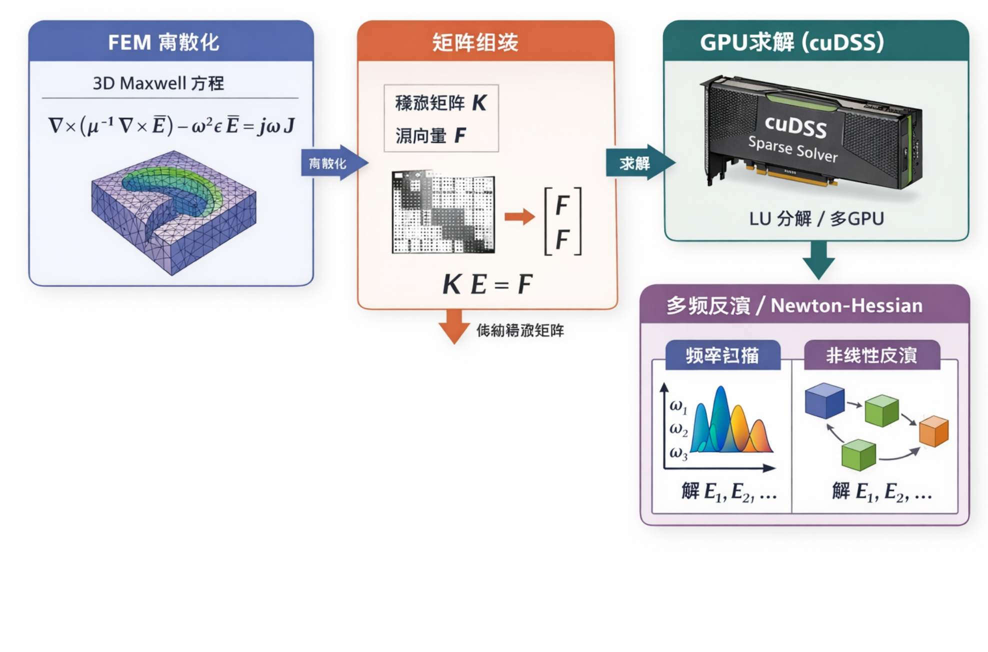
$$\nabla \times (\boldsymbol{\mu}^{-1} \nabla \times \boldsymbol{E}) - \omega^2 \boldsymbol{\epsilon} \boldsymbol{E} = j\omega \boldsymbol{J}_p + \nabla \times (\boldsymbol{m} \delta(\boldsymbol{r} - \boldsymbol{r}_0)) \tag{1}$$

Electric dipole source:

$$\boldsymbol{J}_p = -j\omega \boldsymbol{p} \delta(\boldsymbol{r} - \boldsymbol{r}_0) \tag{2}$$

Magnetic dipole source:

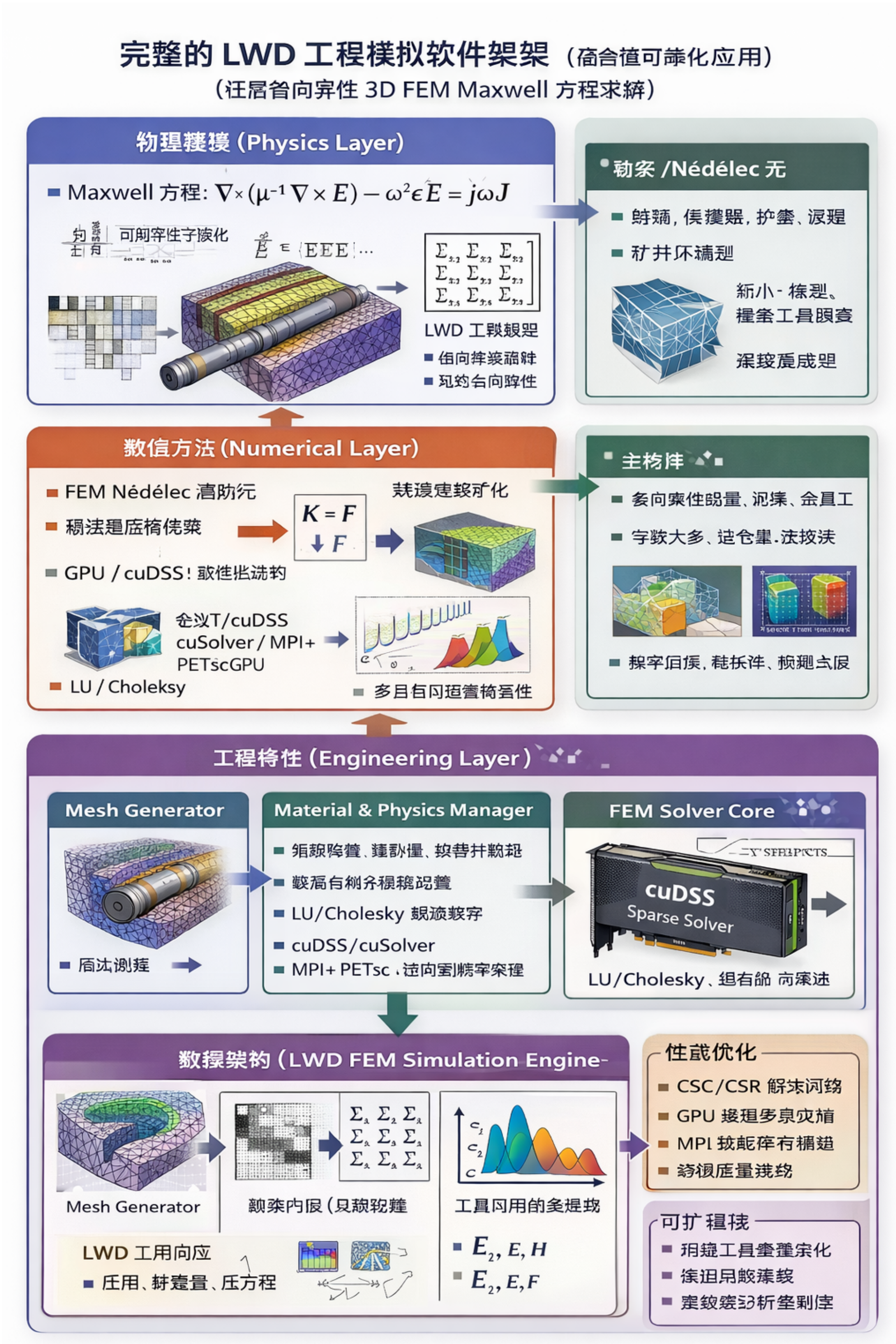
$$\boldsymbol{J}_m = \nabla \times (\boldsymbol{m} \delta(\boldsymbol{r} - \boldsymbol{r}_0)) \tag{3}$$



2. Weak Form (FEM)

Choose test functions $\boldsymbol{v} \in H(\mathrm{curl}, \Omega)$:

$$\int_\Omega (\nabla \times \boldsymbol{v}) \cdot \boldsymbol{\mu}^{-1} (\nabla \times \boldsymbol{E}), d\Omega - \omega^2 \int_\Omega \boldsymbol{v} \cdot \boldsymbol{\epsilon} \boldsymbol{E}, d\Omega = \int_\Omega \boldsymbol{v} \cdot (j\omega \boldsymbol{J}_p + \boldsymbol{J}_m), d\Omega \tag{4}$$



完整的 LWD 工程模拟软件架架
3D FEM Maxwell 方程
(Physics Layer)
Maxwell 方程: ∇×(μ⁻¹∇×E) − ω²εE = jωJ
LWD 工
/Nédélec 元
数值方法 (Numerical Layer)
FEM Nédélec
K = F
GPU / cuDSS
T/cuDSS
cuSolver / MPI+
PETScGPU
LU / Choleksy
工程 (Engineering Layer)
Mesh Generator
Material & Physics Manager
LU/Cholesky
cuDSS/cuSolver
MPI+ PETSc
FEM Solver Core
cuDSS
Sparse Solver
LU/Cholesky
(LWD FEM Simulation Engine-
Mesh Generator
LWD 工用向应
性能优化
CSC/CSR
GPU
MPI
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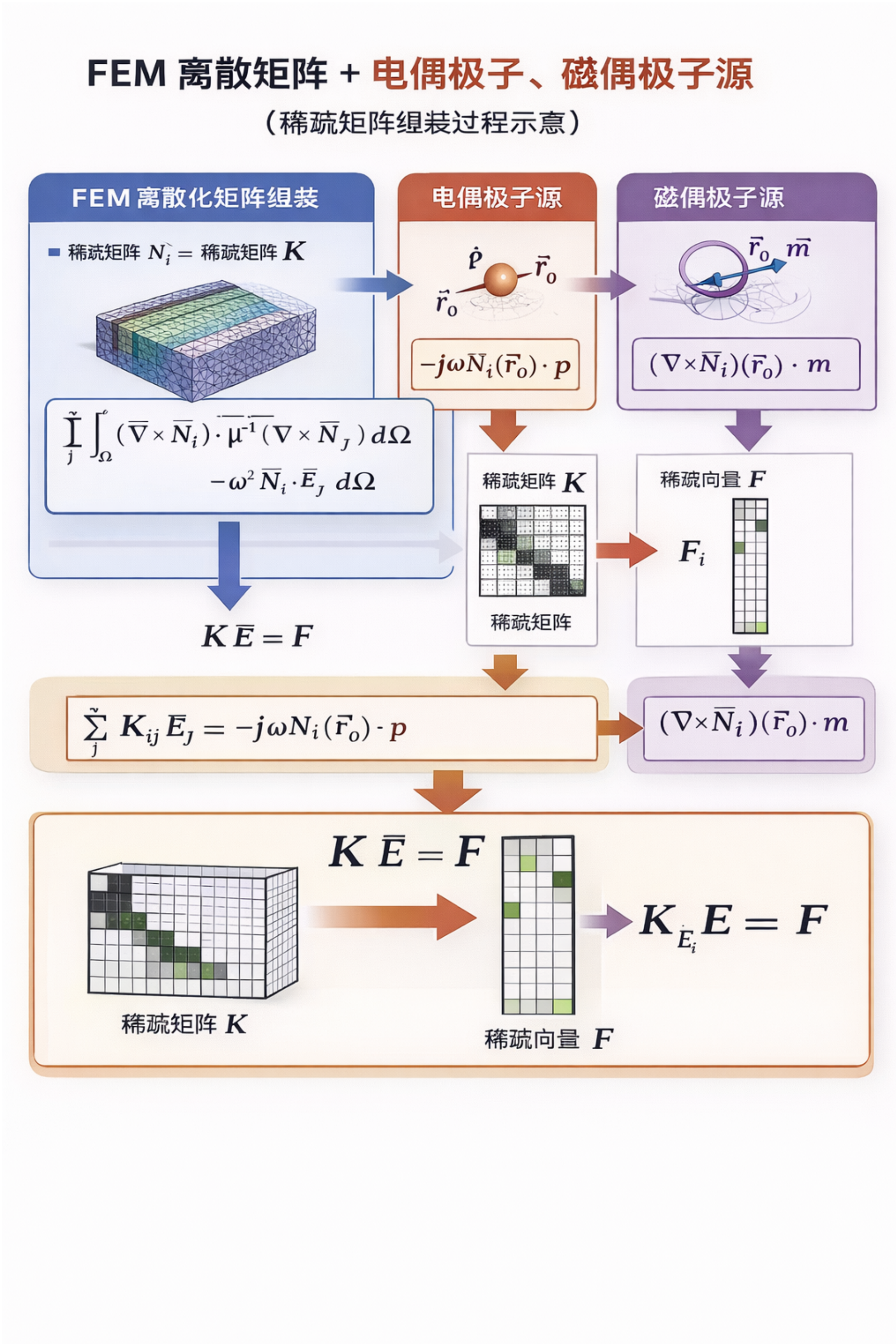
3. FEM Discretization

Expand the electric field with Nédélec basis functions:

$$\boldsymbol{E} \approx \sum_{j=1}^{N} E_j \boldsymbol{N}_j \tag{5}$$

The resulting sparse linear system:

$$\sum_{j=1}^{N} \left[\int_\Omega (\nabla \times \boldsymbol{N}_i) \cdot \boldsymbol{\mu}^{-1} (\nabla \times \boldsymbol{N} * j), d\Omega - \omega^2 \int *\Omega \boldsymbol{N}_i \cdot \boldsymbol{\epsilon} \boldsymbol{N}_j, d\Omega \right] E_j = -j\omega \boldsymbol{N}_i(\boldsymbol{r}_0) \cdot \boldsymbol{p} + (\nabla \times \boldsymbol{N}_i)(\boldsymbol{r}_0) \cdot \boldsymbol{m} \tag{6}$$



4. Sparse Matrix Form (Programming-Friendly)

$$\boldsymbol{K}\boldsymbol{E}=\boldsymbol{F},\quad \boldsymbol{K}*ij=\int *\Omega(\nabla\times\boldsymbol{N}_i)\cdot\boldsymbol{\mu}^{-1}(\nabla\times\boldsymbol{N}*j),d\Omega-\omega^2\int *\Omega\boldsymbol{N}_i\cdot\boldsymbol{\epsilon}\boldsymbol{N}_j,d\Omega\ \ \boldsymbol{F}_i\ =-j\omega\boldsymbol{N}_i(\boldsymbol{r}_0)\cdot\boldsymbol{p} \tag{7}$$

5. Implementation Notes

- Use tetrahedral or hexahedral Nédélec elements for 3D mesh.
- Material tensors $\boldsymbol{\mu}$ and $\boldsymbol{\epsilon}$ may vary per element for anisotropy.
- Apply PEC/PMC or PML boundary conditions by modifying the matrix $\boldsymbol{K}$ accordingly.
- Electric dipole contributions: $\boldsymbol{F}_i^p = -j\omega \boldsymbol{N}_i(\boldsymbol{r}_0) \cdot \boldsymbol{p}$
- Magnetic dipole contributions: $\boldsymbol{F}_i^m = (\nabla \times \boldsymbol{N}_i)(\boldsymbol{r}_0) \cdot \boldsymbol{m}$
- Solve $\boldsymbol{K}\boldsymbol{E} = \boldsymbol{F}$ using GPU or CPU sparse solvers like cuDSS, PETSc, or MUMPS.