Adjoint-State Fréchet Derivatives for Full-Wave Frequency-Domain Maxwell Equations

1 Forward Problem

We consider the frequency-domain Maxwell system under the time convention $e^{-i\omega t}$.

Maxwell Equations

$$\nabla \times \boldsymbol{E} = i\omega \boldsymbol{B}, \tag{1}$$
$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_s + \sigma \boldsymbol{E} - i\omega\varepsilon \boldsymbol{E}, \tag{2}$$
$$\boldsymbol{B} = \mu \boldsymbol{H}. \tag{3}$$

Eliminating $\boldsymbol{H}$ and $\boldsymbol{B}$ yields the second-order equation for $\boldsymbol{E}$:

$$\nabla \times (\mu^{-1} \nabla \times \boldsymbol{E}) - i\omega\sigma \boldsymbol{E} - \omega^2 \varepsilon \boldsymbol{E} = \boldsymbol{f}, \tag{4}$$

where $\boldsymbol{f}$ denotes the source term (electric or magnetic dipole).
Define the forward operator:

$$\mathcal{A}(m)\boldsymbol{E} = \boldsymbol{f}, \tag{5}$$

with model parameters

$$m = (\sigma, \varepsilon, \mu).$$

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2 Misfit Functional

Assume data are electric field measurements:

$$J(m) = \frac{1}{2} \sum_r \left\| \boldsymbol{P}_r \boldsymbol{E} - \boldsymbol{d}_r \right\|^2. \tag{6}$$

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3 Fréchet Derivatives of the Forward Operator

Let

$$\sigma \to \sigma + \delta\sigma, \quad \varepsilon \to \varepsilon + \delta\varepsilon, \quad \mu \to \mu + \delta\mu.$$

Correspondingly,

$$\boldsymbol{E} \to \boldsymbol{E} + \delta\boldsymbol{E}.$$

Linearizing:

$$\mathcal{A}(m + \delta m)(\boldsymbol{E} + \delta\boldsymbol{E}) = \boldsymbol{f}. \tag{7}$$

Subtracting the original equation and neglecting higher-order terms:

$$\mathcal{A}(m)\delta\boldsymbol{E} + D_m\mathcal{A}[\delta m]\boldsymbol{E} = 0. \tag{8}$$

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3.1 Derivative w.r.t. Conductivity σ

Only the term $-i\omega\sigma\boldsymbol{E}$ depends on σ.

$$\delta(-i\omega\sigma\boldsymbol{E}) = -i\omega\delta\sigma\boldsymbol{E}.$$

Therefore,

$$D_\sigma\mathcal{A}[\delta\sigma]\boldsymbol{E} = i\omega\delta\sigma\boldsymbol{E}. \tag{9}$$

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3.2 Derivative w.r.t. Permittivity ε

$$\delta(-\omega^2\varepsilon\boldsymbol{E}) = -\omega^2\delta\varepsilon\boldsymbol{E}.$$

Thus,

$$D_\varepsilon\mathcal{A}[\delta\varepsilon]\boldsymbol{E} = \omega^2\delta\varepsilon\boldsymbol{E}. \tag{10}$$

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3.3 Derivative w.r.t. Permeability μ

We consider

$$\nabla \times (\mu^{-1}\nabla \times \boldsymbol{E}).$$

Using

$$\delta(\mu^{-1}) = -\mu^{-2}\delta\mu,$$

we obtain

$$\delta\left(\nabla \times (\mu^{-1}\nabla \times \boldsymbol{E})\right) = \nabla \times \left(-\mu^{-2}\delta\mu\nabla \times \boldsymbol{E}\right).$$

Hence,

$$D_\mu \mathcal{A}[\delta\mu]\boldsymbol{E} = -\nabla \times \left(\mu^{-2}\delta\mu \nabla \times \boldsymbol{E}\right). \tag{11}$$

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4 Adjoint-State Formulation

Define the Lagrangian:

$$\mathcal{L} = J(m) + \operatorname{Re}\left(\langle \boldsymbol{\Lambda}, \mathcal{A}\boldsymbol{E} - \boldsymbol{f} \rangle\right). \tag{12}$$

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4.1 Adjoint Equation

Taking variation w.r.t. $\boldsymbol{E}$:

$$\delta\mathcal{L} = \langle \boldsymbol{P}^*(\boldsymbol{P}\boldsymbol{E} - \boldsymbol{d}), \delta\boldsymbol{E} \rangle + \langle \boldsymbol{\Lambda}, \mathcal{A}\delta\boldsymbol{E} \rangle.$$

Using adjoint operator $\mathcal{A}^\dagger$:

$$\mathcal{A}^\dagger \boldsymbol{\Lambda} = \boldsymbol{P}^*(\boldsymbol{P}\boldsymbol{E} - \boldsymbol{d}).$$

For Maxwell operator:

$$\nabla \times (\mu^{-1}\nabla \times \boldsymbol{\Lambda}) + i\omega\sigma\boldsymbol{\Lambda} - \omega^2\varepsilon\boldsymbol{\Lambda} = \boldsymbol{q}. \tag{13}$$

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5 Gradient Derivation

$$\delta J = -\operatorname{Re}\langle \boldsymbol{\Lambda}, D_m\mathcal{A}[\delta m]\boldsymbol{E} \rangle.$$

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5.1 Gradient w.r.t. σ

$$\delta J = -\operatorname{Re}\int \boldsymbol{\Lambda} \cdot (i\omega\delta\sigma\boldsymbol{E})\, dV.$$

$$= -\operatorname{Re}\int i\omega\delta\sigma(\boldsymbol{\Lambda} \cdot \boldsymbol{E})\, dV.$$

Thus,

$$\boxed{\frac{\partial J}{\partial \sigma} = -\operatorname{Re}\left(i\omega\boldsymbol{\Lambda} \cdot \boldsymbol{E}\right)} \tag{14}$$

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5.2 Gradient w.r.t. ε

$$\delta J = -\operatorname{Re} \int \omega^2 \delta\varepsilon (\boldsymbol{\Lambda} \cdot \boldsymbol{E}) \, dV.$$

$$\boxed{\frac{\partial J}{\partial \varepsilon} = -\operatorname{Re}\left(\omega^2 \boldsymbol{\Lambda} \cdot \boldsymbol{E}\right)} \tag{15}$$

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5.3 Gradient w.r.t. μ

$$\delta J = -\operatorname{Re} \int \boldsymbol{\Lambda} \cdot \left(-\nabla \times (\mu^{-2} \delta\mu \nabla \times \boldsymbol{E})\right) \, dV.$$

Using integration by parts:

$$\int \boldsymbol{\Lambda} \cdot \nabla \times \boldsymbol{X} = \int (\nabla \times \boldsymbol{\Lambda}) \cdot \boldsymbol{X}.$$

Therefore,

$$\delta J = -\operatorname{Re} \int \mu^{-2} \delta\mu (\nabla \times \boldsymbol{E}) \cdot (\nabla \times \boldsymbol{\Lambda}) \, dV.$$

$$\boxed{\frac{\partial J}{\partial \mu} = \operatorname{Re}\left(\mu^{-2} (\nabla \times \boldsymbol{E}) \cdot (\nabla \times \boldsymbol{\Lambda})\right)} \tag{16}$$

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6 Final Explicit Gradient Expressions

$$\nabla_\sigma J = -\operatorname{Re}\left(i\omega \boldsymbol{\Lambda} \cdot \boldsymbol{E}\right), \tag{17}$$

$$\nabla_\varepsilon J = -\operatorname{Re}\left(\omega^2 \boldsymbol{\Lambda} \cdot \boldsymbol{E}\right), \tag{18}$$

$$\nabla_\mu J = \operatorname{Re}\left(\mu^{-2} (\nabla \times \boldsymbol{E}) \cdot (\nabla \times \boldsymbol{\Lambda})\right). \tag{19}$$