Review of "Three-Dimensional Adaptive Higher-Order Finite Element Simulation for Geo-Electromagnetics — A Marine CSEM Example"

1. Objective

This paper presents a three-dimensional vector finite element framework for frequency-domain geo-electromagnetic modeling, focusing on marine Controlled-Source Electromagnetics (CSEM).

The method combines:

- Nédélec edge elements,
- Higher-order polynomial approximation,
- Adaptive mesh refinement (AMR),
- Primary/secondary field decomposition.

2. Mathematical Model

Assuming time dependence $e^{-i\omega t}$, the secondary electric field satisfies:

$$\operatorname{curl}(\mu^{-1}\operatorname{curl}\mathbf{E}_s) - i\omega(\sigma - i\omega\varepsilon)\mathbf{E}_s = \operatorname{curl}\left([\mu_p^{-1} - \mu^{-1}]\operatorname{curl}\mathbf{E}_p\right) \tag{1}$$

$$- i\omega\left([\sigma_p - \sigma] - i\omega[\varepsilon_p - \varepsilon]\right)\mathbf{E}_p \tag{2}$$

Boundary condition:

$$\mathbf{n} \times \mathbf{E}_s = 0$$

3. Finite Element Discretization

The approximation reads:

$$\tilde{\mathbf{E}}_s = \sum_{j=1}^{n} e_j \boldsymbol{\phi}_j$$

The resulting linear system:

$$A\mathbf{e} = \mathbf{f}$$

with

$$A_{ij} = \int_{\Omega} (\operatorname{curl} \phi_i) \cdot \mu^{-1} (\operatorname{curl} \phi_j) \, d^3 r \tag{3}$$

$$- i\omega \int_{\Omega} \phi_i \cdot (\sigma - i\omega\varepsilon) \phi_j \, d^3 r \tag{4}$$

4. Error Estimation

Magnetic field approximation:

$$\tilde{\mathbf{H}}_s = (i\omega\mu)^{-1} \operatorname{curl} \tilde{\mathbf{E}}_s$$

Error indicator:

$$\eta_K = \int_K (\hat{\mathbf{H}}_s - \tilde{\mathbf{H}}_s) \cdot \mu (\hat{\mathbf{H}}_s - \tilde{\mathbf{H}}_s) \, d^3 r$$

5. Numerical Results

- Higher-order elements significantly improve convergence.
- $p = 2$ gives optimal accuracy-cost balance.
- Bathymetry induces strong 3-D effects.
- FEM outperforms finite volume near sources.

6. Conclusion

The paper demonstrates that adaptive higher-order Nédélec finite elements are highly effective for 3-D marine CSEM simulations, particularly in geometrically complex environments.