# Full Newton Hessian for Frequency-Domain Maxwell Inversion

## 1 Frequency-Domain Maxwell Equations

Assume time convention $e^{-i\omega t}$.

$$\nabla \times \boldsymbol{E} = i\omega \boldsymbol{B}, \tag{1}$$
$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_s + \sigma \boldsymbol{E} - i\omega\varepsilon \boldsymbol{E}, \tag{2}$$
$$\boldsymbol{B} = \mu \boldsymbol{H}. \tag{3}$$

Eliminate $\boldsymbol{H}$ and $\boldsymbol{B}$:

$$\boldsymbol{H} = \mu^{-1}\boldsymbol{B} = \mu^{-1}(1/i\omega)\nabla \times \boldsymbol{E}.$$

Substitute into Ampère's law:

$$\nabla \times (\mu^{-1}\nabla \times \boldsymbol{E}) - i\omega\sigma \boldsymbol{E} - \omega^2\varepsilon \boldsymbol{E} = \boldsymbol{f}.$$

Define forward operator:

$$\mathcal{A}(m)\boldsymbol{E} = \boldsymbol{f}, \quad m = (\sigma, \varepsilon, \mu).$$

## 2 Misfit Functional

$$J(m) = \frac{1}{2}\|\boldsymbol{P}\boldsymbol{E}(m) - \boldsymbol{d}\|^2.$$

## 3 First-Order Linearization

Let

$$m \to m + \delta m, \quad \boldsymbol{E} \to \boldsymbol{E} + \delta \boldsymbol{E}.$$

Linearize forward equation:

$$\mathcal{A}(m + \delta m)(\boldsymbol{E} + \delta \boldsymbol{E}) = \boldsymbol{f}.$$

Expand:

$$\mathcal{A}(m)\boldsymbol{E} + \mathcal{A}(m)\delta \boldsymbol{E} + D_m\mathcal{A}[\delta m]\boldsymbol{E} = \boldsymbol{f}.$$

Subtract original equation:

$$\mathcal{A}(m)\delta\boldsymbol{E} = -D_m\mathcal{A}[\delta m]\boldsymbol{E}.$$

Thus,

$$\delta\boldsymbol{E} = -\mathcal{A}^{-1}D_m\mathcal{A}[\delta m]\boldsymbol{E}.$$

# 4 Fréchet Derivatives

## 4.1 Derivative w.r.t. $\sigma$

$$D_\sigma\mathcal{A}[\delta\sigma]\boldsymbol{E} = i\omega\delta\sigma\boldsymbol{E}.$$

## 4.2 Derivative w.r.t. $\varepsilon$

$$D_\varepsilon\mathcal{A}[\delta\varepsilon]\boldsymbol{E} = \omega^2\delta\varepsilon\boldsymbol{E}.$$

## 4.3 Derivative w.r.t. $\mu$

Use

$$\delta(\mu^{-1}) = -\mu^{-2}\delta\mu.$$

Then

$$D_\mu\mathcal{A}[\delta\mu]\boldsymbol{E} = -\nabla\times(\mu^{-2}\delta\mu\nabla\times\boldsymbol{E}).$$

# 5 Adjoint Equation

Define Lagrangian:

$$\mathcal{L} = J(m) + \Re\langle\boldsymbol{\Lambda}, \mathcal{A}\boldsymbol{E} - \boldsymbol{f}\rangle.$$

Variation w.r.t. $\boldsymbol{E}$:

$$\delta\mathcal{L} = \langle\boldsymbol{P}^*(\boldsymbol{P}\boldsymbol{E} - \boldsymbol{d}), \delta\boldsymbol{E}\rangle + \langle\boldsymbol{\Lambda}, \mathcal{A}\delta\boldsymbol{E}\rangle.$$

Thus adjoint equation:

$$\mathcal{A}^\dagger\boldsymbol{\Lambda} = \boldsymbol{P}^*(\boldsymbol{P}\boldsymbol{E} - \boldsymbol{d}).$$

Explicit form:

$$\nabla\times(\mu^{-1}\nabla\times\boldsymbol{\Lambda}) + i\omega\sigma\boldsymbol{\Lambda} - \omega^2\varepsilon\boldsymbol{\Lambda} = \boldsymbol{q}.$$

# 6 Gradient

$$\delta J = -\Re\langle\boldsymbol{\Lambda}, D_m\mathcal{A}[\delta m]\boldsymbol{E}\rangle.$$

### 6.1 Conductivity

$$\nabla_\sigma J = -\Re(i\omega \boldsymbol{\Lambda} \cdot \boldsymbol{E}).$$

### 6.2 Permittivity

$$\nabla_\varepsilon J = -\Re(\omega^2 \boldsymbol{\Lambda} \cdot \boldsymbol{E}).$$

### 6.3 Permeability

Using integration by parts:

$$\nabla_\mu J = \Re\left(\mu^{-2}(\nabla \times \boldsymbol{E}) \cdot (\nabla \times \boldsymbol{\Lambda})\right).$$

## 7 Gauss–Newton Hessian

Jacobian:

$$\mathcal{J} = -\boldsymbol{P}\mathcal{A}^{-1} D_m \mathcal{A}.$$

Gauss–Newton approximation:

$$H_{GN} = \mathcal{J}^* \mathcal{J}.$$

## 8 Full Newton Hessian

Now include second-order field derivatives.

Second variation:

$$\delta^2 \boldsymbol{E} = -\mathcal{A}^{-1}\left(D_m^2 \mathcal{A}[\delta m_1, \delta m_2]\boldsymbol{E} + D_m \mathcal{A}[\delta m_1]\delta \boldsymbol{E}_2 + D_m \mathcal{A}[\delta m_2]\delta \boldsymbol{E}_1\right).$$

Full Hessian:

$$H_{Newton} = H_{GN} + H_{2nd}.$$

Second-order term:

$$H_{2nd} = \left\langle \boldsymbol{\Lambda}, D_m^2 \mathcal{A}[\cdot,\cdot]\boldsymbol{E} \right\rangle.$$

## 9 Explicit Second Derivatives

### 9.1 $\sigma$

Since linear:

$$D_\sigma^2 \mathcal{A} = 0.$$

### 9.2 $\varepsilon$

$$D_\varepsilon^2 \mathcal{A} = 0.$$

### 9.3 $\mu$

$$\delta^2(\mu^{-1}) = 2\mu^{-3}(\delta\mu_1)(\delta\mu_2).$$

Thus

$$D_\mu^2\mathcal{A} = 2\nabla \times (\mu^{-3}\delta\mu_1\delta\mu_2\nabla \times \boldsymbol{E}).$$

## 10 Final Newton Hessian Expression

$$H_{Newton} = \mathcal{J}^*\mathcal{J} + \Re\left\langle \boldsymbol{\Lambda}, D_m^2\mathcal{A}\boldsymbol{E}\right\rangle.$$

Only $\mu$ contributes to second-order term.