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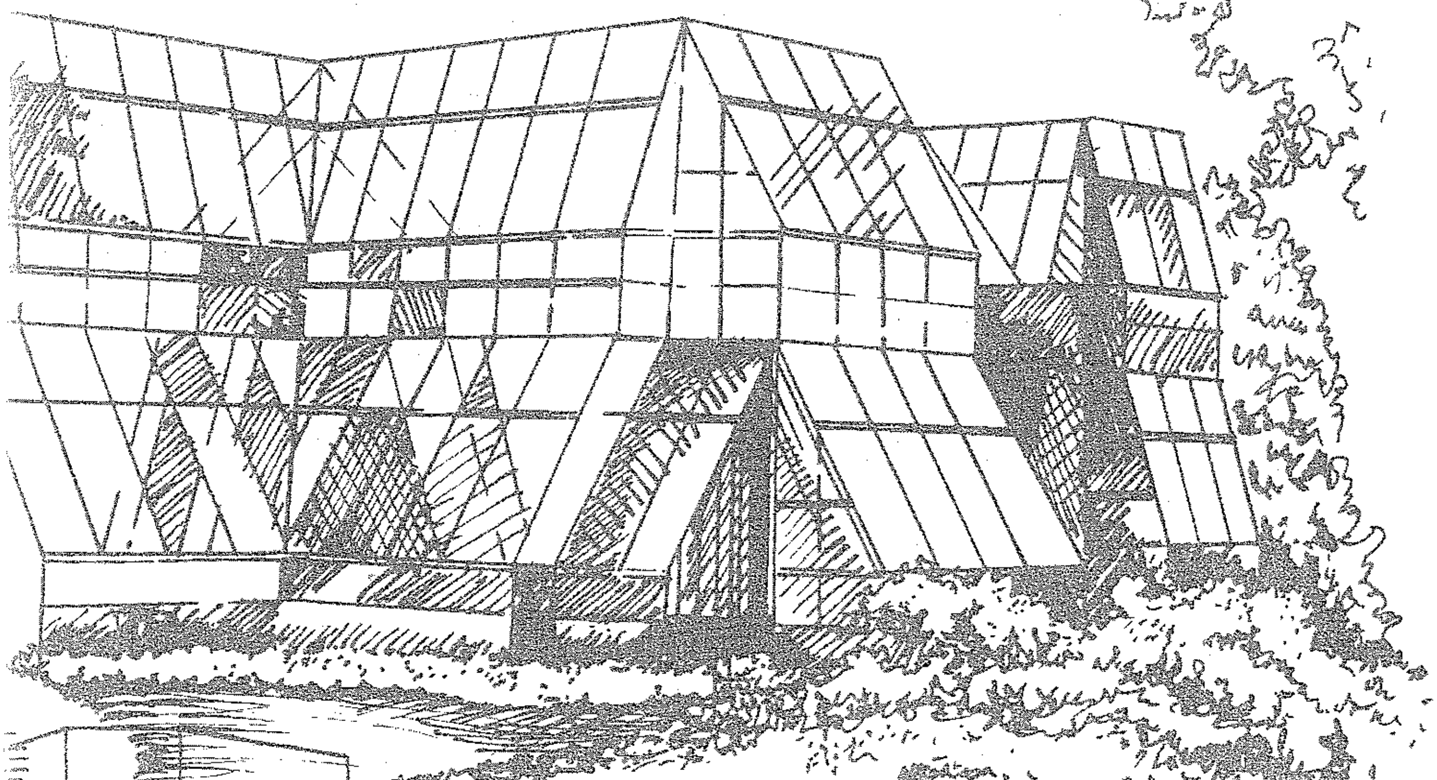

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Staggered grid for Maxwell's equations in arbitrary 3-D inhomogeneous anisotropic media

Sofia Davuducheva
Mathematical Department
Central Geophysical Expedition, Narodnogo Opolchenia 40-3, 123298 Moscow, Russia

Vladimir Druskin
Electromagnetics Department
Schlumbeger-Doll Reseach, Ridgefield, CT 06877-4108

Abstract

A conservative finite-difference approximation for Maxwell's equations with arbitrary 3-D variable tensor electrical conductivity, magnetic and dielectric permeabilities is developed. An example of the time-domain solution using spectral Lanczos decomposition is considered.

1 Introduction

In the last years, the staggered grid modeling approach [1] has been successfully applied to solving problems of calculating electromagnetic field in arbitrary 3-D isotropic media. However, developing this approach for practically important 3-D anisotropic models with arbitrary tensors of electrical conductivity, magnetic and dielectric permeability happens to be very complicated. Yee's algorithm is based on calculation of different electric field components at different space points, but the electrical conductivity tensor relates these components taken at the same point. To avoid this difficulty, we use another staggered grid, based on a general approach, suggested by V.Lebedev, [4]. This grid locates all components of electrical fields at the same points, however vectors of magnetic and electric fields are at different points and similar to Yee's approximation action of operators $\nabla \cdot \nabla \times$ or $\nabla \times \nabla$ gives complete cancellation on the grid and every discrete Maxwell's equation has all properties of its differential counterpart. It is important that conservation and Ohm' lows for the electrical current are valid locally on the grid. In isotropic media our grid approximation can be split in 4 uncoupled conventional staggered schemes. We have developed a code for the time domain diffusion problem.

2 Formulation of the problem

Consider 3-D anisotropic medium with arbitrary symmetric non-negatively definite 3×3 tensors of conductivity $S(x,y,z)$, magnetic $M(x,y,z)$ and dielectric $W(x,y,z)$ permeabilities. The electromagnetic field obeys the Maxwell system,

$$\nabla \times E = -M \frac{\partial H}{\partial t}, \tag{1}$$

$$\nabla \times H = S\,E + W \frac{\partial E}{\partial t} + J, \tag{2}$$

with the boundary conditions

$$E,\ H \to 0 \qquad \text{as} \quad (x,y,z) \to \infty, \tag{3}$$

where $E \equiv (E_x, E_y, E_z)$ and $H \equiv (H_x, H_y, H_z)$ are the electric and magnetic field vectors, respectively, while $J \equiv (J_x, J_y, J_z)$ is the external current.

3 Definition of staggered Lebedev's grid

Introduce 3-D grid,

$$Q = \{r_n : r_n = (x_i, y_j, z_k),\ n = (i, j, k),\ i = 0, \ldots, M_x,\ j = 0, \ldots, M_y,\ k = 0, \ldots, M_z\},$$
$$M = M_x\, M_y\, M_z, \quad \|n\| = i + j + k.$$

Let us denote subgrids as

$$P = \left\{r_n : \|n\| = 0, 2, 4, \ldots, 2\left[\tfrac{1}{2}M\right]\right\}, \quad R = \left\{r_n : \|n\| = 1, 3, 5, \ldots, 2\left[\tfrac{1}{2}(M-1)\right] + 1\right\},$$

and subgrid functions as $f^P = f(r_n)|_{r_n \in P}$, $f^R = f(r_n)|_{r_n \in R}$.

All functions with upper indices P or R will be considered as approximations on corresponding subgrids.

Define finite differences along the x axis,

$$\begin{aligned} f_x^P &= (f_{x\ i,j,k}^P), \quad f_{x\ i,j,k}^P = \frac{f^R{}_{i+1,j,k} - f^R{}_{i-1,j,k}}{x_{i+1} - x_{i-1}}, \\ f_x^R &= (f_{x\ i,j,k}^R), \quad f_{x\ i,j,k}^R = \frac{f^P{}_{i+1,j,k} - f^P{}_{i-1,j,k}}{x_{i+1} - x_{i-1}}, \end{aligned} \tag{4}$$

and similarly along the y and z axes.

To make these formulae sensible, in boundary nodes of the grid additional fictitious nodes x_{-1}, x_{M_x+1}, y_{-1}, y_{M_y+1}, z_{-1}, z_{M_z+1} are assumed to be introduced, where we put $f \equiv 0$. The finite differences defined in this way perform mapping from P to R and back.

4 Grid Maxwell system and its scalar stationary limits

Using Lebedev's grid and considering the above-defined matrices, and magnetic and electric fields on the corresponding subgrids, we can easily rewrite the grid matrix Ohm law, and also similar laws for M and W, as

$$J^R = S^R E^R, \quad B^P = M^P H^P, \quad D^R = W^R E^R.$$

Then we obtain a grid approximation of Maxwell system (1)–(2),

$$\begin{aligned} \tilde{\nabla} \times E^R &= -M^P \frac{\partial H^P}{\partial t}, \\ \tilde{\nabla} \times H^P &= S^R\, E^R + W^R\, \frac{\partial E^R}{\partial t} + J^R, \end{aligned} \tag{5}$$

where $\tilde{\nabla}\times$ is the grid **curl** naturally defined using (4). From (5), one can easily obtain the stationary-limit anisotropic equations for the DC electric and magnetic potentials:

$$\begin{aligned} E^R &= -\tilde{\nabla} u^P, \\ \tilde{\nabla}\left(S^R \tilde{\nabla} u^P\right) &= -\tilde{\nabla} J^R \end{aligned} \tag{6}$$

(DC potential equation),

$$H^P = \widetilde{\nabla} v^R,$$
$$\widetilde{\nabla}\left(M^P \widetilde{\nabla} v^R\right) = 0 \tag{7}$$

(magnetostatic equation).

The following system for E can be derived from (5):

$$-\widetilde{\nabla}\times\left[\left(M^P\right)^{-1}\widetilde{\nabla}\times E^R\right] = S^R\frac{\partial E^R}{\partial t} + W^R\frac{\partial^2 E^R}{\partial t^2} + \frac{\partial J^R}{\partial t}. \tag{8}$$

In the case when $J = -\varphi_0(x,y,z)\chi(t)$, where φ_0 is a finite function and χ is Heavyside's function (switch-on time mode), we can put in (8) $\partial J^R/\partial t = 0$ for $t > 0$, adding the initial condition for E:

$$E^R\big|_{t=0} = \varphi_0^R. \tag{9}$$

It can be easily shown that $E^R(t)$ approaches the solution of equation (6) as $t \to +\infty$ (see in [1]). Equation (3) can be replaced by the condition

$$E^R \times \nu = 0, \quad i = 0, M_x; \quad j = 0, M_y; \quad k = 0, M_z,$$

where ν is the internal normal (to the boundary) vector. Such a substitution does not spoil accuracy, if the region of the grid Q is large enough.

Provided that the function $E(x,y,z)$ is smooth and can be expanded in a Taylor series, one can show that the difference operator on the left hand side of equation (8) is of the second order of approximation (if the grid steps are constant).

5 Connection with the isotropic case

Let us consider the case when the matrices S^R, M^R, W^R are diagonal. Break the grid Q into four clusters 000, 101, 011, 110. Let the cluster 000 contain elements of the type $E_x^R(2j+1, 2j, 2k)$ or, for brevity, $E_x^R(100)$, and also $E_y^R(010)$, $E_z^R(001)$, $H_x^P(011)$, $H_y^P(101)$, $H_z^P(110)$; the cluster 101 contain elements $E_x^R(001)$, $E_y^R(111)$, $E_z^R(100)$, $H_x^R(110)$, $H_y^R(000)$, $H_z^R(011)$, and so on. It can be easily shown that the finite-difference operator $\widetilde{\nabla}$ defined by formulae (4) performs mapping only within one cluster. Coupling between clusters in equation (8) is performed by non-diagonal elements of matrices S^R, M^P, W^R only. So, for diagonal matrices equations (8) can be divided into four independent sets within each cluster. Each vector equation (8) at each cluster coincides with a standard Yee system (see in [1]), while potential equations (6) and (7) are nothing but the standard seven points' 3D difference schemes.

6 Approximation of grid coefficients

Let us consider a homogeneous isotropic medium containing rectangular anisotropic blocks B_l, $l = 1, \ldots, L$, without intersections, inclined in arbitrary direction, with constant tensors S_l, M_l, W_l. For simplicity, let us take the case when these tensors are diagonal in the coordinate system $O'x'y'z'$ connected with the axes of the block B_l. Then the problem of calculating the average conductivity, and the magnetic and dielectric permeabilities of an elementary cell with the center at the point i, j, k in the coordinate system $O'x'y'z'$, denoted by $\tilde{S}'_{ijk}, \tilde{M}'_{ijk}$ and $\tilde{W}'_{ijk}$, respectively[1], can be reduced to a conventional isotropic averaging. Namely, every diagonal component of these tensors

[1] Here and below we omit the upper indices P, R.

can be calculated as the average value of the corresponding (component over an elementary cell. It is important that computing, for example, $(\bar{S}'_{ijk})_{x'x'}$, we take the arithmetic mean over y' and z', and the harmonic mean over x', in accordance to the rules of addition of resistances in electric circuits. Other components of this tensor can be computed analogously. The essential condition is that all blocks intersecting the cell i, j, k are inclined in the same direction. In such a way, all these tensors $\bar{S}'_{ijk}, \bar{M}'_{ijk}, \bar{W}'_{ijk}$ remain diagonal. As it is shown below, the realization of the algorithm requires to calculate 3×3 matrix functions M^{-1} and $S^{-1/2}, W^{-1/2}$ that can be easily done by direct inversion and SVD respectively. Returning to the initial coordinate system $Oxyz$ by means of a transformation G (which is a combination of planar rotations), we find

$$\bar{S}_{ijk}^{-1/2} = G\ \left(\bar{S}'_{ijk}\right)^{-1/2}\ G^*,$$

and also analogous formulae for $M^{-1}, W^{-1/2}$.

The presented way of calculating the tensors of conductivity and magnetic and electrical permeabilities can be also applied to some more general cases: (i) a homogeneous isotropic medium containing arbitrary homogeneous blocks with arbitrary tensors of anisotropy; (ii) a homogeneous anisotropic medium containing arbitrary homogeneous isotropic blocks; (iii) a homogeneous anisotropic medium containing arbitrary homogeneous anisotropic blocks, so that the main axes of anisotropy tensors are collinear to the ones of the analogous tensors of the medium. In all these cases it is also possible, within each elementary cell, to perform the transformation to the coordinate system where the anisotropy tensors are diagonal.

7 A way to solve the problem

First, consider the case $W \equiv 0$ (quasistationary approximation). By analogy with [1], let us rewrite equation (8) in a form which is more convenient for calculation. Let us substitute

$$E^R_{ijk} = \frac{1}{\sqrt{\rho^R_{ijk}}}\ \left(\left(S^R\right)^{-1/2}\ E^{*R}\right)_{ijk},$$

where the weight ρ^R_{ijk} is defined as

$$\begin{gathered}\rho^R_{ijk} = (x_{i+1} - x_{i-1})(y_{j+1} - y_{j-1})(z_{k+1} - z_{k-1}),\\ i = 0, \ldots, M_x, \quad j = 0, \ldots, M_y, \quad k = 0, \ldots, M_z.\end{gathered}$$

Then equations (8), (9) can be rewritten as

$$\begin{aligned} AE^{*R} + \frac{dE^{*R}}{dt} &= 0, \quad t > 0,\\ E^{*R}\big|_{t=0} &= \varphi_0^{*R}, \end{aligned} \tag{10}$$

where the operator A is defined by

$$A = \frac{1}{\sqrt{\rho^R}}\ \left(S^R\right)^{-1/2}\ \tilde{\nabla} \times \left\{\left(M^P\right)^{-1}\ \tilde{\nabla} \times \left[\frac{1}{\sqrt{\rho^R}}\ \left(S^R\right)^{-1/2}\right]\right\}.$$

By analogy with [1], and using the symmetry and non-negative definiteness of the tensors M^P, S^R, we can easily show that the operator A is symmetric and non-negatively definite in R^n with the usual scalar product. From here, it follows that the difference-differential scheme is stable.

The solution of equations (10) can be found as

$$E^{*R}(t) = \exp(-tA)\ \varphi_0^{*R}.$$

This vector is calculated by the Spectral Lanczos Decomposition Method (SLDM) (see in [2]). This is a modification of the Lanczos method intended for computation of matrix functions.

In the case when $W(x,y,z) \neq 0$, but $S(x,y,z) \equiv 0$, by means of a simple substitution, one can also obtain an equation analogous to (10):

$$Au + \frac{d^2u}{dt^2} = 0,$$

$$u|_{t=0} = \varphi_0; \quad \left.\frac{du}{dt}\right|_{t=0} = \varphi_1.$$

The solution of this equation can be found as

$$u(t) = \cos\left(t\sqrt{A}\right)\ \varphi_0 + A^{-0.5}\sin\left(t\sqrt{A}\right)\ \varphi_1.$$

Afterwards, SLDM can be employed to calculate the matrix functions.

It is also not difficult to find the solution of the corresponding equation for the electric field in the frequency domain (see in [1]).

Basing on the presented algorithm, a FORTRAN77-program, MAXANIS, for calculating non-stationary electromagnetic field in 3D anisotropic media has been written. In the program the case $W \equiv 0$ has been realized (quasistationary approximation). The number of spatial nodes for E^R and, therefore, the program's CPU and memory requirements[2] are four times higher than the ones for the isotropic case examined in [3]. So, it requires 45 Mb RAM, the maximal size of the grid being $80 \times 80 \times 80$ nodes.

Using the program MAXANIS, we have calculated the electric field of the elementary dipole P in the switch-off time mode for a model containing an inclined anisotropic stratum depicted in fig. 1. The dipole axis is perpendicular to the stratum, and the angle between this axis and Ox is equal to α. At this dipole axis we calculate the collinear component $E_\parallel$ of the electric field. The distance between the dipole and the center of the stratum is OB = 7.07 km, while the width of the stratum is AC = 1 km. The results of these calculations are presented in fig. 2. Both curves show $E_\parallel$ versus the distance to the dipole, r. The upper curve corresponds to the time $t = 0$ (DC-case), while for the lower one $t = 100\ s$ (after switching off the current). The dots correspond to the case $\alpha = 0$ (both the dipole and the stratum are not inclined with respect to the grid), but the crosses show the results at $\alpha = \pi/4$. It is easy to see that the results are almost independent of the incline angle of the model with respect to the grid. Thus, the MAXANIS program makes it possible to deal, with a sufficient accuracy, with anisotropic models containing blocks inclined in arbitrary directions.

Acknowledgments

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References

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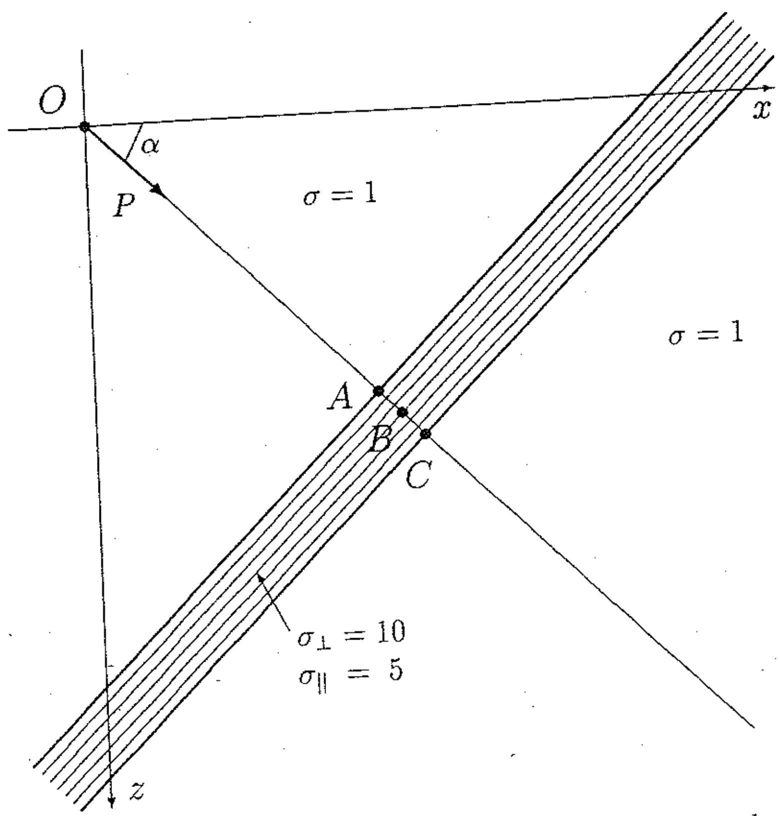

Figure 1: A model containing an inclined anisotropic block

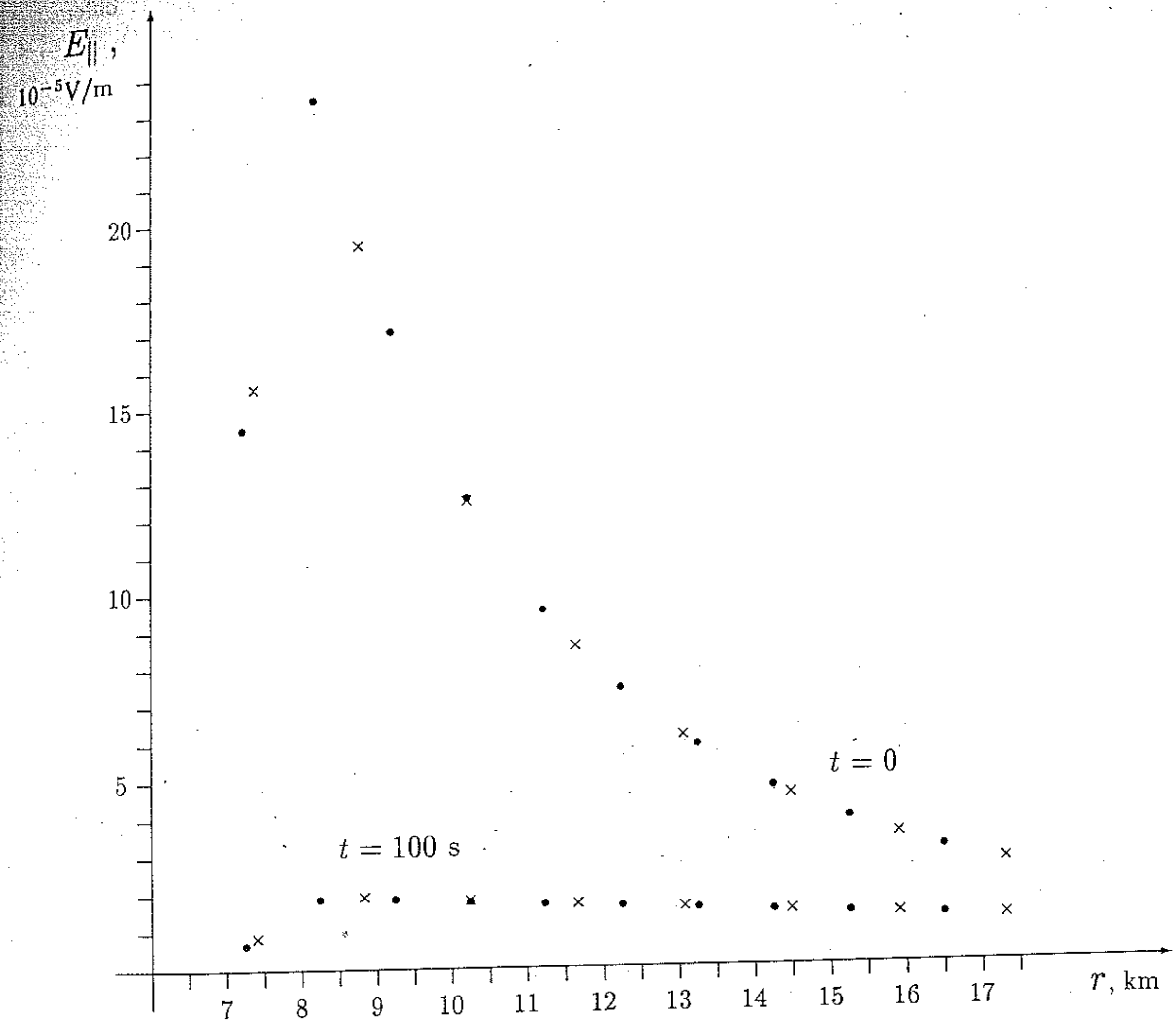

Figure 2: Electric field versus the distance to the dipole

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