Predefined Time Prescribed Performance Backstepping Control for Robotic Manipulators with Input Saturation

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摘要

For robotic manipulators trajectory tracking with unknown dynamics, bounded disturbances, and input saturation, this work proposes an adaptive backstepping control method that combines predefined time convergence with global predefined performance constraints. A channel-wise predefined-time error transformation merges a polynomial performance function with error scaling. It removes initial-state singularities and nonsmooth mappings in conventional PPC and enforces strict, smooth, global bounds within a user-set time, even from unknown initial states. We also design a unified saturation compensator that cancels residuals from command filtering and actuator saturation, curbs the backstepping complexity explosion, and reduces online computation. A first-order sliding-mode disturbance observer and an RBFNN estimator run online to handle bounded disturbances and unmodeled dynamics. With predefined-time stability theory and an adaptive dynamic barrier Lyapunov function, we prove closed-loop boundedness and on-time convergence. Simulations and experiments show faster response, better steady-state accuracy, and stronger robustness than existing methods.

Keywords: Prescribed performance control, predefined-time stability, adaptive dynamic barrier Lyapunov function, backstepping control, input saturation, robotic manipulators' trajectory tracking.

1 Introduction

2 Problem and preliminary

2.1 System description

This work considers an uncertain nonlinear n-DOF robot manipulators system. Its dynamics are described as [? ?]:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + \Delta(q,\dot{q},t) = \tau + d(t), \tag{1}$$

where $q, \dot{q}, \ddot{q} \in \mathbb{R}^n$ are the joint state vectors, $M(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$ represents the centrifugal effects, $G(q) \in \mathbb{R}^n$ represents the gravitational force, $\tau \in \mathbb{R}^n$ is the control joint input, $\Delta(q,\dot{q},t) \in \mathbb{R}^n$ is the uncertainty and unmodeled dynamics, $d(t) \in \mathbb{R}^n$ is an external disturbance, n is the number of DOF of the manipulators system.

For controller design, define the state $x = [x_1, x_2] = [q, \dot{q}] \in \mathbb{R}^{2n}$. The system **??** can be written in the standard state space form:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = f(x) + g(q)\mathrm{sat}(u) + \omega(x,t), \end{cases} \tag{2}$$

where $f(x) = -M^{-1}(q)(C(q,\dot{q})x_2 + G(q))$, $g(q) = M^{-1}(q)$, $\omega(x,t) = -M^{-1}(q)(\Delta(q,\dot{q},t) - d(t))$, and $x_1 = [x_{1,1}, x_{1,2}, ..., x_{1,n}]^\top$, $x_2 = [x_{2,1}, x_{2,2}, ..., x_{2,n}]^\top$, $\mathrm{sat}(u)$ is the saturation of control input $u = \tau$. Define the saturation error as

$$\Delta u_i = u_i - \mathrm{sat}(u_i) = \begin{cases} u_i - u_{\max}, & u_i \geq u_{\max}, \\ 0, & u_{\min} < u_i < u_{\max}, \quad i = 1, 2, ..., n, \\ u_i - u_{\min}, & u_i \leq u_{\min}, \end{cases} \tag{3}$$

where $u_{\max}, u_{\min}$ are the upper and lower saturation bounds of control input, respectively.

The controller design and stability analysis are conducted under the following physically justified assumptions.

Assumption 1 *[?] $M(q)$ is symmetric positive definite for all q. There exist constants $m_1, m_2 > 0$ such that $m_1 I \leq M(q) \leq m_2 I$, $\forall q \in \mathbb{R}^n$. Moreover, the standard structural property holds: $\dot{M}(q) - 2C(q,\dot{q})$ is skew-symmetric.*

Assumption 2 *The unmodeled dynamics are bounded: there exists $\bar{\Delta} > 0$ such that $\|\Delta(q,\dot{q},t)\| \leq \bar{\Delta}$, $\forall t \geq 0$. The disturbance and its derivative are bounded: there exist $\bar{d}, \bar{\dot{d}} > 0$ such that $\|d(t)\| \leq \bar{d}$ and $\|\dot{d}(t)\| \leq \bar{\dot{d}}$, $\forall t \geq 0$. The desired trajectory $q_d(t)$ is twice continuously differentiable, and there exist $\bar{q}_d, \bar{\dot{q}}_d, \bar{\ddot{q}}_d > 0$ such that $\|q_d(t)\| \leq \bar{q}_d$, $\|\dot{q}_d(t)\| \leq \bar{\dot{q}}_d$, $\|\ddot{q}_d(t)\| \leq \bar{\ddot{q}}_d$, $\forall t \geq 0$.*

2.2 Preliminary

Lemma 1 *[?] Consider the system $\dot{x}(t) = f(x(t)), x(0) = x_0$, where $x \in \mathbb{R}^n$ is the system's state, and $x(0) = x_0$ is the initial state. $f(\cdot)$ is continuous and satisfies $f(0) = 0$. et $V : \mathbb{R}^n \to \mathbb{R}_+$ be continuously differentiable, positive definite, and radially unbounded, with $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. For given design parameters $0 < \eta < 1$, $T_p > 0$, and $0 < \varsigma < \infty$, suppose the closed-loop system satisfies*

$$\dot{V}(x) \leq -\frac{\pi}{\eta T_p}\left(V(x)^{1-\frac{\eta}{2}} + V(x)^{1+\frac{\eta}{2}}\right) + \varsigma, \tag{4}$$

Then the origin is predefined-time stable with convergence time bounded by T_p. Moreover,

$$\left\{\lim_{t\to T_p} x | V \leq \min\left\{ \left(\tfrac{2\,\eta\, T_p\, \varsigma}{\pi}\right)^{\frac{1}{1-\frac{\eta}{2}}}, \left(\tfrac{2\,\eta\, T_p\, \varsigma}{\pi}\right)^{\frac{1}{1+\frac{\eta}{2}}} \right\}\right\}, \tag{5}$$

In particular, if $\varsigma = 0$ then $V(T_p) = 0$.

Definition 1 考虑系统 $\dot{x} = f(x,u)$。若存在给定常数 $T > 0$、$\eta \in (0,1)$ 以及类-$\mathcal{K}$ 函数 $\gamma(\cdot)$，使得对任意本质有界输入 $u \in L_\infty$，

$$\|x(T)\| \ \leq \ \gamma\big(\|u\|_\infty\big),$$

并且对所有 $t \geq T$ 状态保持在该球内；若 $u = 0$ 则 $x(T) = 0$。则称系统在预定义时间 T 满足输入到状态稳定（Predefined-Time ISS, 简记 PTS–ISS）。

Theorem 1 (PTS–ISS 的 Lyapunov 判据) *设存在 C^1 Lyapunov 函数 $V : \mathbb{R}^n \to \mathbb{R}_+$ 与类-$\mathcal{K}_\infty$ 函数 $\mathcal{A}_1, \mathcal{A}_2$ 使得*

$$\mathcal{A}_1(\|x\|) \ \leq \ V(x) \ \leq \ \mathcal{A}_2(\|x\|), \quad V(0) = 0,$$

并存在给定参数 $T > 0$, $\eta \in (0,1)$ 以及类-$\mathcal{K}$ 函数 $\sigma(\cdot)$，使对所有 (x,u) 成立

$$\dot{V}(x,u) \ \leq \ -\frac{\pi}{\eta T}\Big(V^{1-\frac{\eta}{2}} + V^{1+\frac{\eta}{2}}\Big) \ + \ \sigma\big(\|u\|_\infty\big). \tag{6}$$

则对任意初值与本质有界输入 u，在 $t = T$ 时

$$V(T) \ \leq \ \min\left\{ \Big(\tfrac{2\eta T}{\pi}\,\sigma(\|u\|_\infty)\Big)^{\frac{1}{1-\frac{\eta}{2}}}, \ \Big(\tfrac{2\eta T}{\pi}\,\sigma(\|u\|_\infty)\Big)^{\frac{1}{1+\frac{\eta}{2}}} \right\}, \tag{7}$$

并且对所有 $t \geq T$ 有 $V(t) \leq V(T)$。若 $u = 0, \sigma(0) = 0$，则 $V(T) = 0$，即系统在预定义时间 T 到达原点。

证明 令 $r=\|u\|_\infty$，则 $\sigma(\|u\|_\infty)=\sigma(r)$ 为常数。由 **??** 得

$$\dot{V} \;\le\; -\frac{\pi}{\eta T}\Big(V^{\,1-\frac{\eta}{2}}+V^{\,1+\frac{\eta}{2}}\Big) \;+\; \sigma(r).$$

考虑比较系统

$$\dot{y} \;=\; -\frac{\pi}{\eta T}\Big(y^{\,1-\frac{\eta}{2}}+y^{\,1+\frac{\eta}{2}}\Big) \;+\; \sigma(r), \quad y(0)=V(0).$$

由比较引理得 $V(t)\le y(t)$。对该标量系统应用**??**，将常数项 ς 替换为 $\sigma(r)$, 即可得到 **??**。进一步令

$$V^\star(r)=\min\Big\{\Big(\tfrac{2\eta T}{\pi}\,\sigma(r)\Big)^{\frac{1}{1-\frac{\eta}{2}}},\ \ \Big(\tfrac{2\eta T}{\pi}\,\sigma(r)\Big)^{\frac{1}{1+\frac{\eta}{2}}}\Big\},$$

则当 $V>V^\star(r)$ 时右端严格为负，故 V 下降；当 $V\le V^\star(r)$ 时解不能越出该阈值。由于 $V(T)\le V^\star(r)$，于是对所有 $t\ge T$ 有

$$V(t) \;\le\; V^\star(\|u\|_\infty).$$

若 $u=0$, 则 $V^\star(0)=0$ 且 $V(T)=0$。证毕。

□

Remark 1 与原**??**对比, **??** 要求 $\dot{V}\le -\frac{\pi}{\eta T_p}(V^{1-\eta/2}+V^{1+\eta/2})+\varsigma$，其中 ς 为常数. 本定理将该常数推广为输入范数的类-$\mathcal{K}$ 函数 $\sigma(\|u\|_\infty)$，从而可统一刻画外界扰动、建模不确定性、饱和残差、滤波导数等所有残余影响,并将其整合为 $\|\cdot\|_\infty$。两者在 $t=T$ 都给出闭式进入并保持的半径；当 $u=0$或 $\sigma(0)=0$时，本定理退化为 Lemma 1 的预定义时间到原点结论；当 $u\neq 0$ 而有界时，本定理给出预定义时间一致实用稳定，收敛半径由 $\sigma(\|u\|_\infty)$ 决定，而 Lemma 1 只能处理常数半径。

Lemma 2 *[?] For $\mathcal{X},\mathcal{Y}\ge 0$, and $\mathcal{Z}_1,\mathcal{Z}_2>1$ with $1/\mathcal{Z}_1+1/\mathcal{Z}_2=1$, the following inequalities hold:*

$$\mathcal{X}\mathcal{Y}\le \frac{\mathcal{X}^{\mathcal{Z}_1}}{\mathcal{Z}_1}+\frac{\mathcal{Y}^{\mathcal{Z}_2}}{\mathcal{Z}_2}, \tag{8}$$

Moreover, for any $\mathcal{Z}_1,\mathcal{Z}_2>0$ with $\mathcal{Z}_1+\mathcal{Z}_2=1$ and any $\mathcal{Z}_3>0$ (generalized weighted form),

$$|\mathcal{X}|^{\mathcal{Z}_1}|\mathcal{Y}|^{\mathcal{Z}_2}\le \mathcal{Z}_1\mathcal{Z}_3^{-\frac{\mathcal{Z}_2}{\mathcal{Z}_1}}|\mathcal{X}|+\mathcal{Z}_2\mathcal{Z}_3|\mathcal{Y}|. \tag{9}$$

Lemma 3 *[?] Let $\mathcal{X}_i\in\mathbb{R}$ and $\mathcal{Y}\in\mathbb{R}^+$. Then*

$$\sum_{i=1}^{n}|\mathcal{X}_i|^{\mathcal{Y}}\ge\left(\sum_{i=1}^{n}|\mathcal{X}_i|\right)^{\mathcal{Y}}, \quad \mathcal{Y}\in(0,1), \tag{10}$$

$$\sum_{i=1}^{n}|\mathcal{X}_i|^{\mathcal{Y}}\ge n^{1-\mathcal{Y}}\left(\sum_{i=1}^{n}|\mathcal{X}_i|\right)^{\mathcal{Y}}, \quad \mathcal{Y}\in(1,\infty). \tag{11}$$

3 Global prescribed–performance function

为在预定义时间内实现全局可行的误差约束，并兼顾稳态阶段对饱和与扰动的友好性，本文构造一类全局性能函数（G-PPF）.

Let $T_p>0$ be the predefined convergence time and $0<p<1$. Define the normalized time $b(t)=(t/T_p)^p\in(0,1]$, a C^1 window

$$s(b)=1-3b^2+2b^3, \quad s(0)=1,\ s(1)=0,\ s'(1)=0, \tag{12}$$

and the global kernel

$$\phi(b)=-\ln b, \tag{13}$$

which ensures $\phi(b)\to+\infty$ as $b\to 0^+$ and $\phi(1)=0$.

For each channel $i=1,\dots,n$, we define the global performance function

$$\rho_i(t)=\begin{cases} a+\sigma_i(t)\,\phi\left(b(t)\right)\,s\left(b(t)\right), & 0<t<T_p,\\ a\Big[1+g(t)\,\Sigma_i(t)\Big], & t\ge T_p,\end{cases} \tag{14}$$

where $a > 0$ is the steady-state accuracy. The factor $\sigma_i(t)$ modulates the constriction rate when $0 < t < T_p$, while $\Sigma_i(t)$ allows adjustment in response to actuator saturation and disturbances when $t \geq T_p$:

$$\sigma_i(t) = \mathrm{Proj}_{[\sigma_{\min},\,\sigma_{\max}]}\Big(\sigma_0 + k_u r_{u,i} + k_d r_{d,i}\Big), \tag{15}$$

$$\tau_u \dot{r}_{u,i} = -r_{u,i} + \|\Delta u_i\|, \tag{16}$$

$$\tau_d \dot{r}_{d,i} = -r_{d,i} + \|d_i\|, \tag{17}$$

with $\sigma_0, k_u, k_d, \tau_u, \tau_d > 0$.

When $t \geq T_p$, a smooth gate ensures C^1 transition

$$g(t) = 1 - \exp\big(-\iota(t - T_p)^2\big)\,, \tag{18}$$

where $\iota > 0, g(T_p) = 0, g'(T_p) = 0$. The post-convergence adjustable amplitude is

$$\begin{aligned} \Sigma_i(t) &= \mathrm{Proj}_{[\,0,\,\Sigma_{\max}\,]}\Big(k_u r_{u,i} + k_d r_{d,i} + k_e r_{e,i}\Big), \\ \tau_e \dot{r}_{e,i} &= -r_{e,i} + |e_i|, \quad k_e, \tau_e > 0. \end{aligned} \tag{19}$$

Since $\phi'(b) = -1/b$, $s'(b) = -6b + 6b^2$, $\dot{b} = \frac{p}{T_p}(t/T_p)^{p-1}$, and **??**. For $0 < t < T_p$, there is

$$\dot{\rho}_i = \dot{\sigma}_i\,\phi(b)\,s(b) + \sigma_i\Big[\phi'(b)\,s(b) + \phi(b)\,s'(b)\Big]\dot{b}. \tag{20}$$

For $t \geq T_p$,

$$\begin{aligned} \dot{\rho}_i &= a\Big[g'(t)\,\Sigma_i(t) + g(t)\,\dot{\Sigma}_i(t)\Big], \\ \dot{\Sigma}_i &= \text{Proj-grad}\Big(k_u \dot{r}_{u,i} + k_d \dot{r}_{d,i} + k_e \dot{r}_{e,i}\Big). \end{aligned} \tag{21}$$

In the control design, the terms $+\dot{\rho}_i/\rho_i$ are included to cancel the boundary variation terms in the BLF derivatives.

Properties 1 The function $\rho_i(t)$ in (**??**) has the following properties: (i) $\lim_{t\to 0^+} \rho_i(t) = +\infty$, so any finite $e_i(0)$ satisfies $|e_i(0)| < \rho_i(0^+)$. (ii) 由 $\phi(1) = 0$、$s(1) = 0$ 得 $\rho_i(T_p^-) = a$；由 $s'(1) = 0$ 与 $g(T_p) = g'(T_p) = 0$ 得 $\dot{\rho}_i(T_p^-) = \dot{\rho}_i(T_p^+) = 0$，实现 C^1 级平滑衔接. (iii) When $0 < t < T_p$, $\rho_i(t)$ is strictly decreasing and C^1 on $(0, T_p]$. (iv) For $t \geq T_p$, $\rho_i(t) = a[1 + g(t)\Sigma_i(t)] \geq a$, with g and Σ_i bounded and low-pass filtered, thus ρ_i remains positive and slowly varying.

Remark 2 我们构造的一类全局预设性能函数，系统性解决传统 PPC/BLF-PPF 的三类痛点：(i) 初值越界诱发进入阶段奇异，常需投影/重置/非对称缩放等权宜处理；(ii) 在 T_p 处边界多为仅连续非 C^1，易在控制律中产生尖峰项并放大噪声；(iii) 收敛后采用固定且过窄管径，面对执行器饱和与外扰时保守与抖振并存。所提 G-PPF 实现预定义时间内平滑收紧而初值全局合法。而且进一步，引入前段$\sigma(t)$调速收紧，与后段 $\Sigma(t)$稳态放宽两级解耦调节，并按饱和残差与扰动强度低通自适应调整约束，避免长期保守。与此同时，给出 $\dot{\rho}(t)$ 的显式解析式，使 BLF 导数中的边界变化项可被严格抵消，不遗留常数残差，从而将闭环 Lyapunov 不等式自然规范为 PTS-ISS 形态。该构造可在严格保证预定义时间稳定的同时提升稳态精度与执行器友好性。

Remark 3 ***Parameter guidelines.***

Choose $\sigma_{\min} \leq \sigma_0 \leq \sigma_{\max}$ and $0 \leq \Sigma_{\max} \leq 0.5$ to cap the largest post–convergence relaxation at 50%; pick τ_u, τ_d, τ_e relatively large so that $\rho_i(t)$ varies slowly and does not excite the closed-loop; set $k_u, k_d \gg k_e$ to prioritize actuator/disturbance–driven breathing over transient error–driven relaxation.

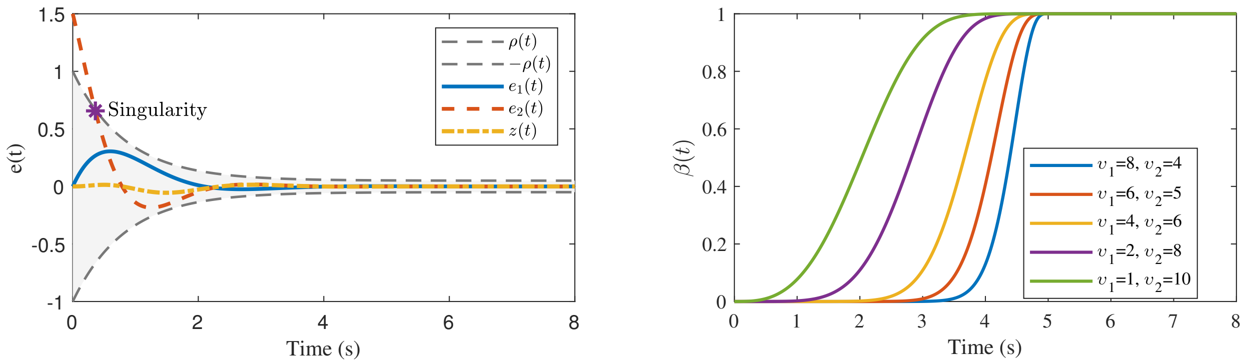

图 1 全局动态变化PPC.

4 Main results

4.1 Controller design

This work presents a controller design methodology that integrates the backstepping approach with BLFs to achieve both predefined-time convergence and prescribed performance constraints. The structure of the proposed control algorithm is shown in **??**.

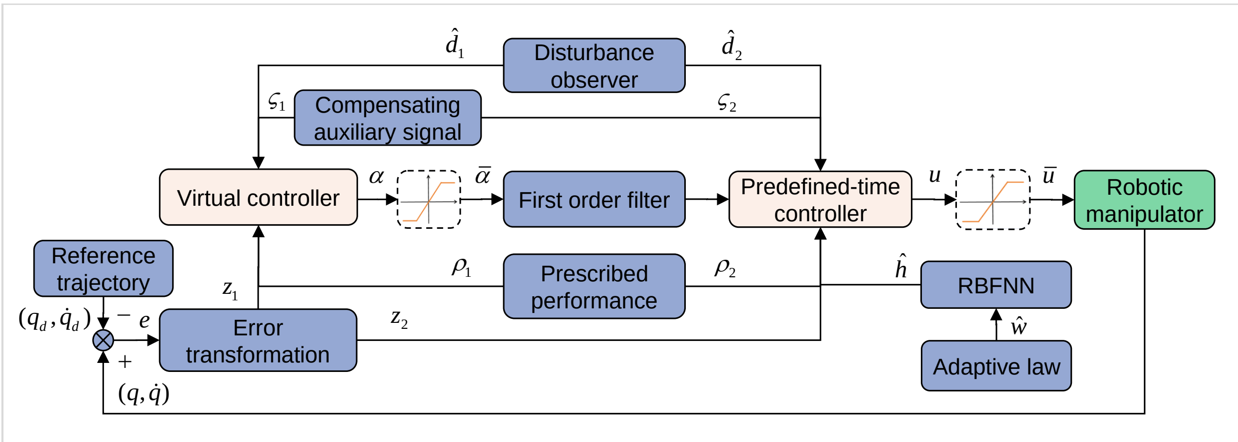

图 2 The structure of the proposed controller

In a trajectory tracking control system, to guarantee that tracking error meets the prescribed dynamic and steady-state performance requirements, define the tracking error as

$$
\begin{cases} z_1 = x_1 - q_d, \\ z_2 = x_2 - \alpha^f - \zeta, \end{cases} \tag{22}
$$

where α is the virtual control, α^f is a filtered version of α, and $\zeta \in \mathbb{R}^n$ are dynamic anti-saturation compensators of actual controller.

To smooth the signal and facilitate the design of the subsequent control law, the filter of the virtual controller α^f is introduced with the following dynamic equations:

$$
\beta\dot{\alpha}^f = -\left(\alpha^f - \alpha\right), \tag{23}
$$

where $0 < \beta < 1, \alpha^f(0) = \alpha(0)$. Define the filtering error is $\tilde{\alpha} = \alpha - \alpha^f$.

为将饱和残差在预设时间 $T_\zeta < cT_p$（$0 < c < 1$）内抽干，针对每个通道 i 引入单通道动态补偿信号 ζ_i，并选

$$
\begin{cases} \dot{\zeta} \;=\; \delta \;-\; \mu_2\,\zeta \;-\; M^{-1}(q)\Delta u, \\ \delta \;=\; -\frac{\pi}{2\gamma T_\zeta}\left[\mathrm{sig}(\zeta)^{1-\gamma} + \mathrm{sig}(\zeta)^{1+\gamma}\right], \end{cases} \tag{24}
$$

其中 $0<\gamma<1,\ \mu_2>0$, $\mathrm{sig}(\zeta)^{1\pm\gamma}=|\zeta|^{1\pm\gamma}\mathrm{sig}(\zeta)$, 并取初值 $\zeta_i(0)=0$。

In the first step, a BLF is used as an energy function for the error dynamics. We choose:

$$V_1=\frac{1}{2}\sum_{i=1}^{n}\frac{\rho_{1,i}^2 z_{1,i}^2}{\rho_{1,i}^2-z_{1,i}^2}. \tag{25}$$

求V_1 的时间导数为

$$\dot V_1=\sum_{i=1}^{n}\left[\frac{\rho_{1,i}^4\,z_{1,i}}{\left(\rho_{1,i}^2-z_{1,i}^2\right)^2}\dot z_{1,i}-\frac{\rho_{1,i}\,z_{1,i}^4}{\left(\rho_{1,i}^2-z_{1,i}^2\right)^2}\dot\rho_{1,i}\right] \tag{26}$$

记 $P_{j,i}=\frac{\rho_{j,i}^4}{(\rho_{j,i}^2-z_{j,i}^2)^2}>0, Q_{j,i}=\frac{z_{j,i}^4}{(\rho_{j,i}^2-z_{j,i}^2)^2}>0, \Phi_{j,i}=\frac{z_{j,i}\left(\rho_{j,i}^2-z_{j,i}^2\right)}{\rho_{j,i}^4}, \rho_{j,i}^2-z_{j,i}^2\le\rho_{j,i}^2,$, 将$\dot z_{1,i}=z_{2,i}-\dot q_{d,i}+\alpha_i^f+\zeta_i$,代入 BLF 导数后，并根据公式（），得到

$$\dot V_1=\sum_{i=1}^{n}\Big[P_{1,i}z_{1,i}z_{2,i}+P_{1,i}z_{1,i}(\alpha_i+\zeta_i-\dot q_{d,i})-P_{1,i}z_{1,i}\tilde\alpha_i-Q_{1,i}\rho_{1,i}\dot\rho_{1,i}\Big]$$

Let $V_{j,i}=\frac{1}{2}\frac{\rho_{j,i}^2 z_{j,i}^2}{\rho_{j,i}^2-z_{j,i}^2}, \Psi(V_{j,i})=\frac{\pi}{\eta T_p}\Big((V_{j,i})^{1-\eta/2}+n^{-\frac{\eta}{2}}(V_{j,i})^{1+\eta/2}\Big)$, 我们设计

$$\mathcal{K}_{1,i}(z_{j,i},\rho_{j,i})=\frac{\rho_{j,i}^2}{2}\frac{\Psi(V_{j,i})}{V_{j,i}}.$$

把它代回，可设计虚拟控制律为：

$$\alpha_i=\dot q_{d,i}-\zeta_i+\frac{z_{1,i}^3}{\rho_{1,i}^3}\,\dot\rho_{1,i}-\mathcal{K}_{1,i}(z_{1,i},\rho_{1,i})\Phi_{1,i}-k_{1,i}\rho_{1,i}^2\Phi_{1,i}, \tag{27}$$

where $k_1=\mathrm{diag}\{k_{1,1},k_{1,i},\cdots,k_{1,n}\}>0$, $k_{1,i}>0$,

则一步 Lyapunov 导数化简为

$$\dot V_1\le\sum_{i=1}^{n}\Big[P_{1,i}z_{1,i}z_{2,i}-\Psi(V_{1,i})-P_{1,i}z_{1,i}\tilde\alpha_i-k_{1,i}\frac{\rho_{1,i}^2 z_{1,i}^2}{\rho_{1,i}^2-z_{1,i}^2}\Big].$$

In the second step, a BLF is used as an energy function for the error dynamics. We choose:

$$V_2=\frac{1}{2}\sum_{i=1}^{n}\frac{\rho_{2,i}^2 z_{2,i}^2}{\rho_{2,i}^2-z_{2,i}^2}. \tag{28}$$

Using $\dot z_{2,i}=\dot x_{2,i}-\dot\alpha_i^f-\dot\zeta_i$, we obtain the time derivative of V_2 is as

$$\begin{aligned}\dot V_2&=\sum_{i=1}^{n}\Big[P_{2,i}z_{2,i}\,(\dot x_{2,i}-\dot\alpha_i^f-\dot\zeta_i)-Q_{2,i}\rho_{2,i}\,\dot\rho_{2,i}\Big]\\&=\sum_{i=1}^{n}\Big[P_{2,i}z_{2,i}\,\Big(f_i(x)+h_i(x,t)+g_i(u_i-\Delta u_i)+d'(t)-\dot\alpha_i^f-\delta_i\;+\;\mu_2\,\zeta_i\;+\;g_i\,\Delta u_i-\tfrac{z_2^3}{\rho_{2,i}^3}\dot\rho_{2,i}\Big)\Big]\end{aligned} \tag{29}$$

Let $Q_2=\mathrm{col}\{Q_{2,i}\}$, $P_1=\mathrm{col}\{P_{1,i}\}$, $\Phi_2=\mathrm{col}\{\Phi_{2,i}\}$. The control input torque is designed as

$$\begin{aligned}u=&C(q,\dot q)x_2+G(q)\\&+M(q)\left[\dot\alpha^f+\delta-\mu_2\,\zeta-\frac{P_1}{P_2}z_1+\tfrac{z_2^3}{\rho_2^3}\dot\rho_2-\mathcal{K}_2(z_2,\rho_2)\Phi_2-\hat h(\chi)-\hat\omega-k_s\,\mathrm{sgn}(z_2)-k_2\rho_2^2\Phi_2\right]\end{aligned} \tag{30}$$

where $k_2=\mathrm{diag}\{k_{2,1},k_{2,i},\cdots,k_{2,n}\}>0$, $k_{2,i}>0$, $k_s>0$.

为结构不确定和额外扰动,避免直接微分速度，采用一阶跟踪微分器作为观测器

$$\begin{cases}\dot\vartheta=-\omega_d\,\vartheta+\omega_d\,x_2,\\ \dot{\hat x}_2=\omega_d(x_2-\vartheta),\\ \dot{\hat\omega}=-\Lambda\,\hat\omega+\Lambda\Big(\hat x_2-f(x)-g(q)\,\mathrm{sat}(u)\Big),\end{cases} \tag{31}$$

where $\Lambda = \mathrm{diag}\{\lambda_1, ..., \lambda_n\}, \lambda_i > 0, \omega_d > 0, \vartheta \in \mathbb{R}^n, \vartheta(0) = x_2(0), \hat{\omega}(0) = 0$, 设观测误差为 $\tilde{\omega} = \omega - \hat{\omega}$. 其动力学为

$$\dot{\tilde{\omega}} = -\Lambda\,\tilde{\omega} \;+\; \Delta_\omega, \qquad \Delta_\omega = \dot{\omega} - \Lambda\,\varepsilon_d, \tag{32}$$

$\varepsilon_d = \hat{\dot{x}}_2 - \dot{x}_2$ 为跟踪微分器误差,一致有界并可收敛.

根据公式可得到

$$\dot{V}_2 \le + \sum_{i=1}^{n} \left(-k_{2,i} \frac{\rho_{2,i}^2 z_{2,i}^2}{\rho_{2,i}^2 - z_{2,i}^2} - \Psi(V_{2,i}) - P_{2,i} z_{2,i} \tilde{\omega}_i \right) - k_s \sum_{i=1}^{n} P_{2,i} \left| z_{2,i} \right| \tag{33}$$

5 Stability analysis

We analyze the stability of the closed-loop system under the proposed controller. A composite Lyapunov function **??** is constructed by combining the BLF-based error energy, the virtual-control filtering error $\tilde{\alpha}$, and the Observer error $\tilde{\omega}$. Sufficient conditions for the global stability of the system and for all signals to be bounded are given by the derivation of its time derivative inequality. Finally, we prove that, for any initial state, the trajectories enter a compact set no later than the predefined time T_p. This establishes global predefined-time stability.

Theorem 2 *Under* **??????** *and the controller in* **????** *with observers* **????** *and weight update* **??**, *and under certain parameter conditions. At this point, for any initial condition, the closed-loop system is predefined-time stable. The scaled errors $z_{1,i}(t), z_{2,i}(t)$ enter a compact set no later than T_p, hence $|e_i(t)| < \rho_i(t)$ for all t, and $e_i(t)$ converges to a prescribed small neighborhood of the origin.*

证明 The composite Lyapunov function is constructed as follows

$$\mathcal{L} = V_1 + V_2 + \tfrac{1}{2}\|\tilde{\alpha}\|^2 + \tfrac{1}{2}\|\tilde{\omega}\|^2 \tag{34}$$

The time derivative of **??** is given by

$$\begin{aligned} \dot{\mathcal{L}} \le & \sum_{i=1}^{n} \left(-k_{1,i} \frac{\rho_{1,i}^2 z_{1,i}^2}{\rho_{1,i}^2 - z_{1,i}^2} - \Psi(V_{1,i}) - P_{1,i} z_{1,i} \tilde{\alpha}_i \right) + \sum_{i=1}^{n} \left(-k_{2,i} \frac{\rho_{2,i}^2 z_{2,i}^2}{\rho_{2,i}^2 - z_{2,i}^2} - \Psi(V_{2,i}) - P_{2,i} z_{2,i} \tilde{\omega}_i \right) \\ & - k_s \sum_{i=1}^{n} P_{2,i} \left| z_{2,i} \right| + \tilde{\alpha}^\top \dot{\tilde{\alpha}} + \tilde{\omega}^\top \dot{\tilde{\omega}} \\ \le & \sum_{i=1}^{n} \left(-\Psi(V_{1,i}) - \Psi(V_{2,i}) \right) + \sum_{i=1}^{n} \left(-2k_{1,i} V_{1,i} - k_{2,i} V_{2,i} \right) - \sum_{i=1}^{n} P_{2,i} z_{2,i} \tilde{\omega}_i + \tilde{\omega}^\top \dot{\tilde{\omega}} - k_s \sum_{i=1}^{n} P_{2,i} \left| z_{2,i} \right| \\ & + \tilde{\alpha}^\top \dot{\tilde{\alpha}} - \sum_{i=1}^{n} P_{1,i} z_{1,i} \tilde{\alpha}_i \end{aligned} \tag{35}$$

在 BLF 约束内会存在 $|z_{1,i}(t)| < \rho_{1,i}(t)$ 不变，故存在常数 $v \in (0,1)$ 使$|z_{1,i}(t)|/\rho_{1,i}(t) \le 1 - v, \forall t \in (0, +\infty)$. 令 $c_v = \frac{1}{(2v - v^2)^2} > 0$, 可得到 $|P_{1,i} z_{1,i}| \le c_v |z_{1,i}| \le c_v \sqrt{2V_{1,i}}$. Then, we get

$$|P_{1,i} z_{1,i} \tilde{\alpha}_i| \le c_v \sqrt{2V_{1,i}}\, |\tilde{\alpha}_i| \le \varepsilon V_{1,i} + \frac{c_v^2}{2\varepsilon} \tilde{\alpha}_i^2,$$

where $\varepsilon > 0$ is a positive constant.

由 $\beta \dot{\alpha}^f = -(\alpha^f - \alpha)$ 得 $\dot{\tilde{\alpha}} = \dot{\alpha} - \frac{1}{\beta}\tilde{\alpha}$，从而

$$\tilde{\alpha}^\top \dot{\tilde{\alpha}} \le -\frac{1}{2\beta} \|\tilde{\alpha}\|^2 + \beta\,\mathcal{E}(t),$$

其中 $\mathcal{E}(t)$ is a continuous and bounded function, 并且由有界信号构成。

将其与 (**??**) 相加，可得

$$\begin{aligned}\mathcal{L}_\alpha =& -\sum_{i=1}^{n} P_{1,i} z_{1,i}\tilde{\alpha}_i - k_{1,i}\sum_{i=1}^{n}\frac{\rho_{1,i}^2 z_{1,i}^2}{\rho_{1,i}^2 - z_{1,i}^2} + \tilde{\alpha}^\top\dot{\tilde{\alpha}} \\ \le& \,\varepsilon\sum_{i=1}^{n} V_{1,i} + \frac{c_v^2}{2\varepsilon}\sum_{i=1}^{n}\tilde{\alpha}_i^2 - k_{1,i}\sum_{i=1}^{n}\frac{\rho_{1,i}^2 z_{1,i}^2}{\rho_{1,i}^2 - z_{1,i}^2} - \frac{1}{2\beta}\|\tilde{\alpha}\|^2 + \beta\,\mathcal{E}(t) \\ \le& \,(\varepsilon - 2k_{1,i})\sum_{i=1}^{n} V_{1,i} - \Big(\frac{1}{2\beta} - \frac{c_v^2}{2\varepsilon}\Big)\|\alpha\|^2 + \beta\,\mathcal{E}(t)\end{aligned} \tag{36}$$

结合**??**, and let

$$\begin{aligned}\mathcal{L}_\omega \;=&\; -\sum_{i=1}^{n} P_{2,i} z_{2,i}\tilde{\omega}_i \;+\; \tilde{\omega}^\top\dot{\tilde{\omega}} \;-\; k_s\sum_{i=1}^{n} P_{2,i}|z_{2,i}| \\ =&\; -\tilde{\omega}^\top\Lambda\tilde{\omega} \;-\sum_{i=1}^{n} P_{2,i} z_{2,i}\tilde{\omega}_i \;-\; k_s\sum_{i=1}^{n} P_{2,i}|z_{2,i}| \;+\; \tilde{\omega}^\top\Delta_\omega.\end{aligned} \tag{37}$$

取任意 $\epsilon_1, \epsilon_2 \in (0,1)$。用矩阵带权（$\Lambda$）的 Young 不等式, 可得

$$\big|\tilde{\omega}^\top(P_2 \odot z_2)\big| = \big|(\Lambda^{1/2}\tilde{\omega})^\top\Lambda^{-1/2}(P_2 \odot z_2)\big| \le \tfrac{\epsilon_1}{2}\tilde{\omega}^\top\Lambda\tilde{\omega} + \tfrac{1}{2\epsilon_1}\big\|\Lambda^{-1/2}(P_2 \odot z_2)\big\|^2,$$

$$\tilde{\omega}^\top\Delta_\omega \le \tfrac{\epsilon_2}{2}\tilde{\omega}^\top\Lambda\tilde{\omega} + \tfrac{1}{2\epsilon_2}\big\|\Lambda^{-1/2}\Delta_\omega\big\|^2.$$

代回$\mathcal{L}_\omega$ 可得

$$\mathcal{L}_\omega \le -\Big(1 - \tfrac{\epsilon_1+\epsilon_2}{2}\Big)\tilde{\omega}^\top\Lambda\tilde{\omega} + \sum_{i=1}^{n}\Big[\tfrac{1}{2\epsilon_1\,\lambda_i}(P_{2,i}|z_{2,i}|)^2 - k_s P_{2,i}|z_{2,i}|\Big] + \tfrac{1}{2\epsilon_2}\big\|\Lambda^{-1/2}\Delta_\omega\big\|^2.$$

采用用配方法处理，有

$$\sum_{i=1}^{n}\Big[\tfrac{1}{2\epsilon_1\,\lambda_i}(P_{2,i}|z_{2,i}|)^2 - k_s P_{2,i}|z_{2,i}|\Big] \le -\,\tfrac{\epsilon_1}{2}\,k_s^2\,\mathrm{tr}(\Lambda) \;\le\; 0.$$

这是一个有利的负常数，为了简洁可直接丢弃而保持上界。进一步用 $\tilde{\omega}^\top\Lambda\tilde{\omega} \ge \lambda_{\min}\|\tilde{\omega}\|^2$, 于是得到

$$\mathcal{L}_\omega \;\le\; -\Big(1 - \tfrac{\epsilon_1+\epsilon_2}{2}\Big)\lambda_{\min}\|\tilde{\omega}\|^2 \;+\; \tfrac{1}{2\epsilon_2}\big\|\Lambda^{-1/2}\Delta_\omega\big\|^2,$$

$$\begin{aligned}\dot{\mathcal{L}} \le& \sum_{i=1}^{n}\left(-k_{1,i}\frac{\rho_{1,i}^2 z_{1,i}^2}{\rho_{1,i}^2 - z_{1,i}^2} - \Psi(V_{1,i})\right) + \sum_{i=1}^{n}\left(-k_{2,i}\frac{\rho_{2,i}^2 z_{2,i}^2}{\rho_{2,i}^2 - z_{2,i}^2} - \Psi(V_{2,i})\right) \\ &+ \frac{\varepsilon}{2}V_{1,i} + \frac{c_v^2}{2\varepsilon}\tilde{\alpha}_i^2 - \frac{1}{2\lambda}\|\tilde{\alpha}\|^2 + \beta\,\mathcal{E}(t) - \Big(1 - \tfrac{\epsilon_1+\epsilon_2}{2}\Big)\lambda_{\min}\|\tilde{\omega}\|^2 \;+\; \tfrac{1}{2\epsilon_2}\big\|\Lambda^{-1/2}\Delta_\omega\big\|^2 \\ \le& \sum_{i=1}^{n}\left(-k_{1,i}\frac{\rho_{1,i}^2 z_{1,i}^2}{\rho_{1,i}^2 - z_{1,i}^2} - \Psi(V_{1,i})\right) + \sum_{i=1}^{n}\left(-k_{2,i}\frac{\rho_{2,i}^2 z_{2,i}^2}{\rho_{2,i}^2 - z_{2,i}^2} - \Psi(V_{2,i})\right) \\ &+ \frac{\varepsilon}{2}V_{1,i} + \frac{c_v^2}{2\varepsilon}\tilde{\alpha}_i^2 - \frac{1}{2\beta}\|\tilde{\alpha}\|^2 + \beta\,\mathcal{E}(t) - \Big(1 - \tfrac{\epsilon_1+\epsilon_2}{2}\Big)\lambda_{\min}\|\tilde{\omega}\|^2 \;+\; \tfrac{1}{2\epsilon_2}\big\|\Lambda^{-1/2}\Delta_\omega\big\|^2\end{aligned} \tag{38}$$

根据公式可得到

$$\dot{\mathcal{L}} \le -\,(2k_{1,i} - \varepsilon)V_{1,i} - 2k_{2,i}V_{2,i} - \Big(\frac{1}{2\beta} - \frac{c_v^2}{2\varepsilon}\Big)\tilde{\alpha}_i^2 - \Big(1 - \tfrac{\epsilon_1+\epsilon_2}{2}\Big)\lambda_{\min}\|\tilde{\omega}\|^2 + \frac{1}{2\epsilon_2}\big\|\Lambda^{-1/2}\Delta_\omega\big\|^2 + \beta\,\mathcal{E}(t) \tag{39}$$

After organizing **??**, the principal negative qualitative and constant bounded terms of Lyapunov's derivative can be obtained, which further leads to

$$\dot{\mathcal{L}} \le -\mathcal{M}\mathcal{L} + \mathcal{N}(t), \tag{40}$$

with

$$\mathcal{M} = \min\left\{\min_i\left(2k_{1,i} - \varepsilon\right), 2k_{2,i}, \min_i\left(\frac{1}{2\beta} - \frac{c_v^2}{2\varepsilon}\right), \min_i\left(1 - \tfrac{\epsilon_1+\epsilon_2}{2}\right)\right\} > 0, \tag{41}$$

$$\mathcal{N}(t) = \frac{1}{2\varepsilon}\big\|\Lambda^{-1/2}\Delta_\omega\big\|^2 + \beta\,\mathcal{E}(t). \tag{42}$$

From the above **????????**, as long as the controller parameters are chosen reasonably, $\mathcal{L}$ is 一致有界并可收敛, which implies all signals $z_1, z_2, \tilde{\alpha}, \tilde{\omega}$ are bounded within the compact set $\mathcal{Q}$, 从而存在常数$\phi > 0$ such that

$$(z_1, z_2, \tilde{\alpha}, \tilde{\omega}) \in \mathcal{Q} = \left\{\|(z_1, z_2, \tilde{\alpha}, \tilde{\omega})\| \le \sqrt{\phi}\right\}. \tag{43}$$

According to **??????** and **??**, it follows that

$$
\begin{aligned}
\dot{\mathcal{L}} \leq & -\sum_{i=1}^{n}\Psi(V_{1,i})-\sum_{i=1}^{n}\Psi(V_{2,i})-\frac{\pi}{\eta T_p}\sum_{i=1}^{n}\left[\left(\tfrac{1}{2}\tilde{\alpha}^2\right)^{1-\frac{\eta}{2}}+\left(\tfrac{1}{2}\tilde{\omega}^2\right)^{1-\frac{\eta}{2}}\right] \\
& -n^{\frac{\eta}{2}}\frac{\pi}{\eta T_p}\sum_{i=1}^{n}\left[\left(\tfrac{1}{2}\tilde{\alpha}^2\right)^{1+\frac{\eta}{2}}+\left(\tfrac{1}{2}\tilde{\omega}^2\right)^{1+\frac{\eta}{2}}\right]+\frac{2\pi}{\eta T_p}\sum_{i=1}^{n}\left[\left(\frac{\phi}{2}\right)^{1+\frac{\eta}{2}}+n^{\frac{\eta}{2}}\left(\frac{\phi}{2}\right)^{1+\frac{\eta}{2}}\right]+\mathcal{N}(t) \\
\leq & -\frac{\pi}{\eta T_p}\left[\sum_{i=1}^{n}V_{1,i}+\sum_{i=1}^{n}V_{2,i}+\left(\tfrac{1}{2}\tilde{\alpha}^2\right)+\left(\tfrac{1}{2}\tilde{\omega}^2\right)\right]^{1-\frac{\eta}{2}}-\frac{\pi}{\eta T_p}\left[\sum_{i=1}^{n}V_{1,i}+\sum_{i=1}^{n}V_{2,i}+\left(\tfrac{1}{2}\tilde{\alpha}^2\right)+\left(\tfrac{1}{2}\tilde{\omega}^2\right)\right]^{1+\frac{\eta}{2}} \\
& +\frac{2\pi}{\eta T_p}\sum_{i=1}^{n}\left[\left(\frac{\phi}{2}\right)^{1-\frac{\eta}{2}}+n^{\frac{\eta}{2}}\left(\frac{\phi}{2}\right)^{1+\frac{\eta}{2}}\right]+\mathcal{N}(t) \\
\leq & -\frac{\pi}{\eta T_p}\left(\mathcal{L}^{1-\frac{\eta}{2}}+\mathcal{L}^{1+\frac{\eta}{2}}\right)+\mathcal{R}(t)
\end{aligned} \tag{44}
$$

where

$$
\mathcal{R}(t)=\frac{2\pi}{\eta T_p}\sum_{i=1}^{n}\left[\left(\frac{\phi}{2}\right)^{1-\frac{\eta}{2}}+n^{\frac{\eta}{2}}\left(\frac{\phi}{2}\right)^{1+\frac{\eta}{2}}\right]+\mathcal{N}(t). \tag{45}
$$

令

$$
\overline{\mathcal{R}}_t=\operatorname*{ess\,sup}_{0\leq s\leq t}\mathcal{R}(s),
$$

考虑比较系统

$$
\dot{y}=-\frac{\pi}{\eta T_p}\left(y^{1-\frac{\eta}{2}}+y^{1+\frac{\eta}{2}}\right)+\overline{\mathcal{R}}_t,\qquad y(0)=\mathcal{L}(0).
$$

由比较引理有 $\mathcal{L}(t)\leq y(t)$。对上述标量系统应用预定义时间稳定引理（你文中的 Lemma 1），得到在 $t=T_p$ 时

$$
y(T_p)\ \leq\ \gamma\big(\overline{\mathcal{R}}_{T_p}\big)=\min\left\{\left(\tfrac{2\eta T_p\,\overline{\mathcal{R}}_{T_p}}{\pi}\right)^{\frac{1}{1-\frac{\eta}{2}}},\ \left(\tfrac{2\eta T_p\,\overline{\mathcal{R}}_{T_p}}{\pi}\right)^{\frac{1}{1+\frac{\eta}{2}}}\right\}.
$$

于是

$$
\mathcal{L}(t)\ \leq\ o\big(\mathcal{L}(0),t\big)\ +\ \gamma\Big(\sup_{0\leq s\leq t}\mathcal{R}(s)\Big),
$$

where $o(t)=0\ \ (t\geq T_p)$. 当$\mathcal{R}=0$时, $\mathcal{L}(T_p)=0$；有界输入时，在 $t=T_p$ 以前吸入到半径 $\gamma(\|\mathcal{R}\|_{\infty,[0,T_p]})$ 的紧集内。等价地，设

$$
\Omega=\big\{\mathcal{S}:\ \mathcal{L}(T_p)\leq\gamma\big(\overline{\mathcal{R}}_{T_p}\big)\big\},\quad \mathcal{S}=[z_1,z_2,\tilde{\alpha},\tilde{\omega}]^\top,
$$

Remark 4 若 Δ_ω, $\mathcal{E}(t)\in L_\infty$，则 $\mathcal{R}(t)\in L_\infty$，因而 $\overline{\mathcal{R}}_{T_p}<\infty$，结论成立；当 $\mathcal{R}=0$ 时得到严格的预定义时间收敛。

The stability proof is completed.

□

Theorem 3 *在适当参数（$0<\gamma<1,\ \mu>0,\ T_\zeta>0$）下，补偿器 (**??**) 使所有通道的 $\zeta_i(t)$ 在 $t=T_\zeta$ 之前严格收敛到零或给定的极小邻域；若 $\Delta u_i=0$（饱和不被触发），则 ζ_i 在 T_ζ 内收敛到原点。该性质与主闭环的 BLF–PTS 设计可并行成立。*

证明 针对每个i通道，设计Lyapunov 函数如下

$$
\mathcal{L}_{\zeta,i}\ =\ \frac{1}{2}\zeta_i^2.
$$

对$\mathcal{L}_{\zeta,i}$求时间导数。并结合$\zeta_i\,\mathrm{sig}(\zeta_i)^\alpha=|\zeta_i|^{\alpha+1}$ 得

$$
\dot{\mathcal{L}}_{\zeta,i}=-\frac{\pi}{2\gamma T_\zeta}\Big((\mathcal{L}_{\zeta,i})^{1-\frac{\gamma}{2}}+(\mathcal{L}_{\zeta,i})^{1+\frac{\gamma}{2}}\Big)\ -\ \mu\zeta_i^2\ -\ g_i\zeta_i\Delta u_i.
$$

对交叉项用 Young 不等式：

$$
|g_i\zeta_i\Delta u_i|\ \leq\ \frac{\mu}{2}\zeta_i^2+\frac{g_i^2}{2\mu}(\Delta u_i)^2.
$$

合并上式并丢弃额外负项 $-\frac{\mu}{2}\sum\zeta_i^2\leq 0$

$$
\dot{\mathcal{L}}_{\zeta,i}\ \leq\ -\frac{\pi}{2\gamma T_\zeta}\Big((\mathcal{L}_{\zeta,i})^{1-\frac{\gamma}{2}}+(\mathcal{L}_{\zeta,i})^{1+\frac{\gamma}{2}}\Big)\ +\ \sum_{i=1}^{n}\frac{g_i^2}{2\mu}(\Delta u_i)^2.
$$

因为执行器幅值有限，Δu 本质有界：$\|\Delta u\|_\infty < \infty$。令 $\bar g \ge \max_i |g_i|$，则 $\frac{\bar g^2}{2\mu}\|\Delta u\|_\infty^2 \triangleq \varphi(\|\Delta u\|_\infty)$，其中 $\varphi(r)=\frac{\bar g^2}{2\mu}r^2 \in \mathcal{K}$。因此

$$\dot{\mathcal{L}}_{\zeta,i} \ \le\ -\frac{\pi}{\gamma T_\zeta}\Big(\mathcal{L}_{\zeta,i}^{1-\frac{\gamma}{2}} + \mathcal{L}_{\zeta,i}^{1+\frac{\gamma}{2}}\Big) \ +\ \varphi(\|\Delta u\|_\infty),$$

由引理可得在 $t = T_\zeta$：

$$\mathcal{L}_{\zeta,i}(T_\zeta) \ \le\ \min\Big\{\Big(\tfrac{2\gamma T_\zeta\,\varphi(\|\Delta u\|_\infty)}{\pi}\Big)^{\frac{1}{1-\frac{\gamma}{2}}},\ \ \Big(\tfrac{2\gamma T_\zeta\,\varphi(\|\Delta u\|_\infty)}{\pi}\Big)^{\frac{1}{1+\frac{\gamma}{2}}}\Big\},$$

并且对所有 $t \ge T_\zeta$ “进入并保持”。若饱和未触发（$\Delta u_i = 0, \varphi(0) = 0$），则 $\mathcal{L}_{\zeta,i}(T_\zeta) = 0$，从而在 T_ζ 内到达原点。

Remark 5 本证明严格地把饱和残差 Δu 视为输入，给出 PTS–ISS 形式的微分不等式；这样既覆盖饱和被触发的情形，又在无饱和时自然退化为 PTS 到达原点的结论。

□

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Statements and Declarations

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Conflicts of Interest

The authors declare no conflict of interest.

Author contribution

Shuli Liu: Writing-Original Draft, Methodology, Software, Visualization, Data curation. Yi Liu:Resources, Investigation, Formal analysis, Conceptualization, Writing-Review & Editing. Jingang Liu: Project administration, Validation, Funding acquisition, Writing-Review & Editing. Yin Yang: Supervision, Funding acquisition, Writing-Review & Editing. All authors have reviewed and approved the final version of the manuscript.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.