Predefined Time Prescribed Performance Backstepping Control for Robotic Manipulators with Input Saturation

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摘要

本文研究了具有未知动力学、有界扰动以及执行器饱和条件下的机器人机械臂轨迹跟踪控制问题，提出了一种立于任何初始状态条件的预定义时间预设性能约束的自适应反步控制方法。针对传统预设性能控制中存在的初值奇异性和误差变换非光滑问题，针对每个通道独立设计了结合多项式性能函数与误差缩放函数的预定义时间误差变换结构，确保系统状态在预定义时间内严格且平滑地满足全局预设性能约束。为处理反步控制及执行器饱和输入引起的计算复杂度爆炸问题，设计了一种预定义时间饱和补偿器，以同时消除命令滤波与输入饱和的不利影响。此外，引入一阶滑模扰动观测器实时估计并补偿系统中未知的有界扰动，利用径向基函数神经网络对系统未建模动态进行辨识和补偿，从而提高了控制系统的鲁棒性和跟踪精度。同时，结合预定义时间理论和自适应动态障碍李雅普诺夫函数，严格证明了闭环系统在预定义时间内的全局稳定性和误差收敛性。数值和实验结果结果验证了所提出方法的有效性，与现有方法相比，本文方法在响应速度、稳态精度及鲁棒性能方面表现出明显优势。

Keywords: Prescribed performance control, predefined-time stability, adaptive dynamic barrier Lyapunov function, backstepping control, input saturation, robotic manipulators' trajectory tracking.

1 Global prescribed–performance function

为同时实现“*初值全局可行、预定义时间内平滑收紧*”与“*收敛后对饱和/扰动的自适应友好*”，构造一类全局预设性能函数（G-PPF），系统性解决传统 PPC/BLF-PPF 的三类痛点：(i) 初值越界诱发进入阶段奇异，常需投影/重置/非对称缩放等权宜处理；(ii) 在 T_p 处边界多为仅连续非 C^1，易在控制律中产生尖峰项并放大噪声；(iii) 收敛后采用固定且过窄管径，面对执行器饱和与外扰时保守与抖振并存。所提 G-PPF 以核函数 $\phi(b) = -\ln b$ 与光滑窗 $s(b)$ 协同保证 $\rho(0^+) = +\infty$ 与 $\dot{\rho}(T_p^-) = 0$，实现“*预定义时间内平滑收紧而初值全局合法*”。进一步，引入前段缩放 $\sigma(t)$（调速收紧）与后段“*呼吸式*”缩放 $\Sigma(t)$（稳态放宽）两级解耦调节，并按饱和残差与扰动强度低通自适应调整约束，避免长期保守。与此同时，给出 $\dot{\rho}(t)$ 的显式解析式，使 BLF 导数中的边界变化项可被严格抵消，不遗留常数残差，从而将闭环 Lyapunov 不等式自然规范为 PTS-ISS 形态。该构造与反步、饱和补偿、RBFNN 等模块无缝兼容，可在严格保证预定义时间稳定的同时提升稳态精度与执行器友好性。

1.1 Definition

令 $T_p > 0$ 为预定义收敛时间、$0 < p < 1$，定义归一化时间

$$b(t)=\big(t/T_p\big)^p\in(0,1],\qquad \dot b=\tfrac{p}{T_p}\big(t/T_p\big)^{p-1}>0,$$

取 C^1 光滑窗

$$s(b)=1-3b^2+2b^3,\quad s(0)=1,\ s(1)=0,\ s'(1)=0, \tag{1}$$

以及全局核函数

$$\phi(b)=-\ln b,\qquad \phi(b)\to+\infty\ (b\to 0^+),\quad \phi(1)=0. \tag{2}$$

对每个通道 $i=1,\dots,n$，定义全局性能函数

$$\rho_i(t)=\begin{cases} a+\sigma_i(t),\phi!\left(b(t)\right),s!\left(b(t)\right), & 0<t<T_p,\\ a\big[1+g(t),\Sigma_i(t)\big], & t\ge T_p,\end{cases} \tag{3}$$

其中 $a>0$ 表示稳态精度下界。$\sigma_i(t)$ 调节 $t<T_p$ 阶段的收紧速率，$\Sigma_i(t)$ 则在 $t\ge T_p$ 阶段实现呼吸式放宽以顺应饱和与扰动：

$$\sigma_i(t)=\mathrm{Proj}*[\sigma*\min,,\sigma_{\max}]!\Big(\sigma_{i0}+k_u r_{u,i}+k_d r_{d,i}\Big), \tag{4}$$
$$\tau_u\dot r_{u,i}=-r_{u,i}+|\Delta u_i|,\qquad \tau_d\dot r_{d,i}=-r_{d,i}+|d_i|,$$

其中 $\sigma_{i0},k_u,k_d,\tau_u,\tau_d>0$，$\Delta u_i=u_i-\mathrm{sat}(u_i)$ 为饱和残差。$t\ge T_p$ 时，以 C^1 门控函数确保光滑拼接：

$$g(t)=1-\exp!\big(-\iota(t-T_p)^2\big),\quad \iota>0,\quad g(T_p)=0,\ g'(T_p)=0, \tag{5}$$

并设呼吸幅值

$$\Sigma_i(t)=\mathrm{Proj}*[,0,,\Sigma*\max,]!\Big(k_u r_{u,i}+k_d r_{d,i}+k_e r_{e,i}\Big), \tag{6}$$
$$\tau_e\dot r_{e,i}=-r_{e,i}+|e_i|,\qquad k_e,\tau_e>0.$$

1.2 Derivatives for exact compensation

由 $\phi'(b)=-1/b$、$s'(b)=-6b+6b^2$ 与 **??**–**??**，在 $0<t<T_p$ 有

$$\dot\rho_i=\dot\sigma_i,\phi(b),s(b)*\sigma_i\Big[\phi'(b),s(b)+\phi(b),s'(b)\Big]\dot b. \tag{7}$$

在 $t\ge T_p$，

$$\dot\rho_i=a\Big[g'(t),\Sigma_i(t)+g(t),\dot\Sigma_i(t)\Big],\qquad \dot\Sigma_i=\text{Proj-grad}\Big(k_u\dot r_{u,i}+k_d\dot r_{d,i}+k_e\dot r_{e,i}\Big). \tag{8}$$

在 BLF 框架中，诸如 $\dot V_{\mathrm{BLF}}\ni(\dot\rho_i\rho_i)\cdot(\cdot)$ 的边界变化项可用 **????** 精确抵消，避免残余常数项，便于将闭环不等式规范为 PTS-ISS 形式。

1.3 Properties

对 $\rho_i(t)$ 于 **??**–**??**，成立：

1. **全局初值合法性：** $\lim\limits_{t\to0^+}\rho_i(t)=+\infty$，故任意有限 $e_i(0)$ 均满足 $|e_i(0)|<\rho_i(0^+)$。
2. **预定义时间收紧与 C^1 拼接：** 由 $\phi(1)=0$、$s(1)=0$ 得 $\rho_i(T_p^-)=a$；由 $s'(1)=0$ 与 $g(T_p)=g'(T_p)=0$ 得 $\dot\rho_i(T_p^-)=\dot\rho_i(T_p^+)=0$，实现 C^1 级平滑衔接。
3. **单调与平滑（$0<t\le T_p$）：** 在 $\dot\sigma_i\le 0$（或 $\dot\sigma_i$ 由低通滤波器充分抑制）时，**??** 第二项严格为负，且第一项不为正，故 $\rho_i(t)$ 在 $(0,T_p]$ 上严格递减且 C^1。更一般地，只要 $|\dot\sigma_i|$ 足够小，单调性仍保持。
4. **$t\ge T_p$ 的自适应稳态友好性：** $\rho_i(t)=a,[1+g(t)\Sigma_i(t)]\ge a$，其中 g、Σ_i 有界且经低通滤波，保证边界正性与缓变性；Σ_i 随饱和残差与扰动强度自适应调整，可在不破坏约束的前提下降低保守性。

5. **实现友好的导数：** ??–?? 给出闭式导数，便于在 BLF 导数中做精确补偿，避免由未知边界变化引入的保守加项。

Remark 1 (Parameter guidelines and compatibility) 参数选取与实现建议：令 $\sigma_{\min} \le \sigma_{i0} \le \sigma_{\max}$，并取 $0 \le \Sigma_{\max} \le 0.5$ 将收敛后最大放宽幅度限制在 50

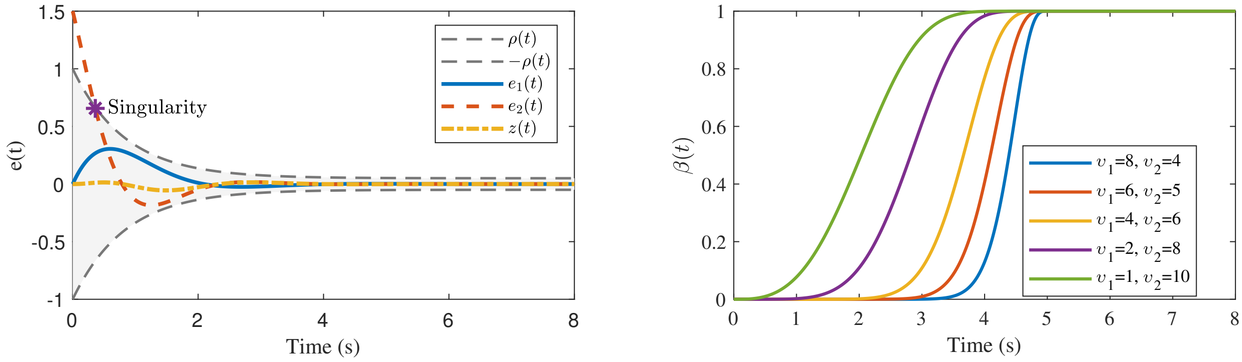

图 1 全局动态 G-PPF：预定义时间内平滑收紧与收敛后“呼吸式”放宽。

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与原稿相比的关键改进要点：（1）动机与三类痛点更聚焦，建立到设计要素的“一一对应”；（2）显式强调 C^1 拼接与 BLF 导数的“精确补偿”关系，服务于后续 PTS-ISS 证明链；（3）单调性给出可操作的充分条件（$\dot\sigma_i \le 0$ 或“足够小”）；（4）参数整定与实现细节（Proj-grad、p 的物理含义）前置，降低读者复现实践门槛；（5）行文删繁就简，保留必要数学细节与工程语义。

1.4 System description

求V_1 的时间导数为

$$\dot V_1 = \sum_{i=1}^{n}\left[\frac{\rho_{1,i}^4\, z_{1,i}}{\left(\rho_{1,i}^2 - z_{1,i}^2\right)^2}\dot z_{1,i} - \frac{\rho_{1,i}\, z_{1,i}^4}{\left(\rho_{1,i}^2 - z_{1,i}^2\right)^2}\dot\rho_{1,i}\right] \tag{9}$$

记 $P_{j,i} = \frac{\rho_{j,i}^4}{(\rho_{j,i}^2 - z_{j,i}^2)^2} > 0, Q_{j,i} = \frac{z_{j,i}^4}{(\rho_{j,i}^2 - z_{j,i}^2)^2} > 0, \Phi_{j,i} = \frac{z_{j,i}\left(\rho_{j,i}^2 - z_{j,i}^2\right)}{\rho_{j,i}^4} = \frac{z_{j,i}(\rho_{j,i}^2 - z_{j,i}^2)}{P_{j,i}}, j \in \{1,2\}$, $\dot z_{1,i} = z_{2,i} - \dot q_{d,i} + \alpha_i^f + \zeta_i$, 将它们代入 BLF 导数后，并根据公式（），得到

$$\dot V_1 = \sum_{i=1}^{n}\Big[P_{1,i} z_{1,i} z_{2,i} + P_{1,i} z_{1,i}(\alpha_i + \zeta_i - \dot q_{d,i}) - P_{1,i} z_{1,i}\tilde\alpha_i - Q_{1,i}\rho_{1,i}\dot\rho_{1,i}\Big]$$

Let $V_{j,i} = \frac{1}{2}\frac{\rho_{j,i}^2 z_{j,i}^2}{\rho_{j,i}^2 - z_{j,i}^2}, \Psi(V_{j,i}) = \frac{\pi}{\eta T_p}\Big((V_{j,i})^{1-\eta/2} + n^{-\frac{\eta}{2}}(V_{j,i})^{1+\eta/2}\Big)$, 我们设计

$$\mathcal{K}_{1,i}(z_{j,i}, \rho_{j,i}) = \frac{\rho_{j,i}^2}{2}\frac{\Psi(V_{j,i})}{V_{j,i}} = \frac{\pi\,\rho_{j,i}^2}{2\eta T_p}\Big((V_{j,i})^{-\eta/2} + n^{-\frac{\eta}{2}}(V_{j,i})^{\eta/2}\Big).$$

把它代回，可设计虚拟控制律为：

$$\alpha_i = \dot q_{d,i} - \zeta_i + \frac{z_{1,i}^3}{\rho_{1,i}^3}\dot\rho_{1,i} - \mathcal{K}_{1,i}(z_{1,i}, \rho_{1,i})\Phi_{1,i} - k_{1,i}\rho_{1,i}^2\Phi_{1,i}, \tag{10}$$

where $k_1 = \mathrm{diag}\{k_{1,1}, k_{1,i}, \cdots, k_{1,n}\} > 0$, $k_{1,i} > 0$,

则一步 Lyapunov 导数化简为

$$\dot V_1 \le \sum_{i=1}^{n}\left[P_{1,i} z_{1,i} z_{2,i} - \Psi(V_{1,i}) - P_{1,i} z_{1,i}\tilde\alpha_i - k_{1,i}\frac{\rho_{1,i}^2 z_{1,i}^2}{\rho_{1,i}^2 - z_{1,i}^2}\right].$$

In the second step, a BLF is used as an energy function for the error dynamics. We choose:

$$V_2 = \frac{1}{2}\sum_{i=1}^{n} \frac{\rho_{2,i}^2 z_{2,i}^2}{\rho_{2,i}^2 - z_{2,i}^2}. \tag{11}$$

Using $\dot{z}_{2,i} = \dot{x}_{2,i} - \dot{\alpha}_i^f - \dot{\zeta}_i$, we obtain the time derivative of V_2 is as

$$\begin{aligned}\dot{V}_2 &= \sum_{i=1}^{n} \Big[P_{2,i} z_{2,i} \left(\dot{x}_{2,i} - \dot{\alpha}_i^f - \dot{\zeta}_i\right) - Q_{2,i}\rho_{2,i}\,\dot{\rho}_{2,i} \Big] \\ &= \sum_{i=1}^{n} \Big[P_{2,i} z_{2,i} \,\Big(f_i(x) + h_i(x,t) + g_i(u_i - \Delta u_i) + d'(t) - \dot{\alpha}_i^f - \delta_i \;+\; \mu_2\,\zeta_i \;+\; g_i\,\Delta u_i - \tfrac{z_2^3}{\rho_{2,i}^3}\dot{\rho}_{2,i}\Big)\Big]\end{aligned} \tag{12}$$

Let $Q_2 = \mathrm{col}\{Q_{2,i}\}$, $P_1 = \mathrm{col}\{P_{1,i}\}$, $\Phi_2 = \mathrm{col}\{\Phi_{2,i}\}$. The control input torque is designed as

$$\begin{aligned} u =& C(q,\dot{q})x_2 + G(q) \\ &+ M(q)\left[\dot{\alpha}^f + \delta - \mu_2\,\zeta - \frac{P_1}{P_2} z_1 + \tfrac{z_2^3}{\rho_{2,i}^3}\dot{\rho}_{2,i} - \mathcal{K}_2(z_2,\rho_{2,i})\Phi_2 - \hat{h}(\chi) - \hat{\omega} - k_s\,\mathrm{sgn}(z_2) - k_2\rho_{2,i}^2\Phi_{2,i}\right]\end{aligned} \tag{13}$$

where $k_2 = \mathrm{diag}\{k_{2,1}, k_{2,i}, \cdots, k_{2,n}\} > 0$, $k_{2,i} > 0$, $k_s > 0$.

Let $\chi = \{q, \dot{q}\} \in \mathbb{R}^m$ be the regressor and $\psi(\chi) \in \mathbb{R}^N$ the normalized basis. Approximate the structured uncertainty by $\hat{h}_i(\chi) = \hat{\theta}_i^\top \psi_i(\chi)$, with $\theta_i \in \mathbb{R}^N$ adapted online by

$$\dot{\hat{\theta}}_i = \varrho_i\, P_{2,i} z_{2,i} \psi(\chi) - \kappa_i\,\hat{\theta}_i, \qquad \varrho_i, \kappa_i > 0. \tag{14}$$

将结构不确定估计误差和外扰的剩余项定义为 $\omega(x,t) = h(x) - \hat{h}(\chi) + M^{-1}(q)d(t)$。

为避免直接微分速度，采用一阶跟踪微分器作为观测器

$$\begin{cases} \dot{\vartheta} = -\omega_d\,\vartheta + \omega_d\,x_2, \\ \hat{\dot{x}}_2 = \omega_d(x_2 - \vartheta), \\ \dot{\hat{\omega}} = -\Lambda_o\,\hat{\omega} + \Lambda_o\left(\hat{\dot{x}}_2 - f(x) - M^{-1}(q)\,\mathrm{sat}(u) - \hat{h}(\chi)\right), \end{cases} \tag{15}$$

where $\Lambda_o = \mathrm{diag}\{\lambda_{o,i}\} > 0, \omega_d > 0, \vartheta(0) = x_2(0), \hat{\omega}(0) = 0$, 设观测误差为 $\tilde{\omega} = \omega - \hat{\omega}$. 其动力学为

$$\dot{\tilde{\omega}} = -\Lambda_o\,\tilde{\omega} \;+\; \Delta_\omega, \qquad \Delta_\omega := \dot{\omega} - \Lambda_o\,\varepsilon_d,$$

where $\Lambda_o = \mathrm{diag}\{\lambda_{o,i}\} > 0, w_d > 0, \vartheta(0) = x_2(0)$,$\varepsilon_d := \hat{\dot{x}}_2 - \dot{x}_2$ 为跟踪微分器误差。

根据公式可得到

$$\dot{V}_2 \le +\sum_{i=1}^{n}\left(-k_{2,i}\frac{\rho_{2,i}^2 z_{2,i}^2}{\rho_{2,i}^2 - z_{2,i}^2} - \Psi(V_{2,i}) - P_{2,i}z_{2,i}\tilde{\omega}_i\right) - k_s\sum_{i=1}^{n} P_{2,i}\,|z_{2,i}| \tag{16}$$

$$\begin{aligned}\dot{V} \le& \sum_{i=1}^{n}\left(-k_{1,i}\frac{\rho_{1,i}^2 z_{1,i}^2}{\rho_{1,i}^2 - z_{1,i}^2} - \Psi(V_{1,i}) - P_{1,i}z_{1,i}\tilde{\alpha}_i\right) + \sum_{i=1}^{n}\left(-k_{2,i}\frac{\rho_{2,i}^2 z_{2,i}^2}{\rho_{2,i}^2 - z_{2,i}^2} - \Psi(V_{2,i}) - P_{2,i}z_{2,i}\tilde{\omega}_i\right) \\ &- k_s\sum_{i=1}^{n} P_{2,i}\,|z_{2,i}| + \tilde{\alpha}^\top\dot{\tilde{\alpha}} + \tilde{\omega}^\top\dot{\tilde{\omega}} + \sum_{i=1}^{n}\tfrac{1}{\varrho_i}\tilde{\theta}_i^\top\dot{\tilde{\theta}}_i \\ \le& \sum_{i=1}^{n}\left(-k_{1,i}\frac{\rho_{1,i}^2 z_{1,i}^2}{\rho_{1,i}^2 - z_{1,i}^2} - \Psi(V_{1,i})\right) + \sum_{i=1}^{n}\left(-k_{2,i}\frac{\rho_{2,i}^2 z_{2,i}^2}{\rho_{2,i}^2 - z_{2,i}^2} - \Psi(V_{2,i})\right) \\ &- \sum_{i=1}^{n} P_{2,i}z_{2,i}\tilde{\omega}_i + \tilde{\omega}^\top\dot{\tilde{\omega}} - k_s\sum_{i=1}^{n} P_{2,i}\,|z_{2,i}| + \tilde{\alpha}^\top\dot{\tilde{\alpha}} - \sum_{i=1}^{n} P_{1,i}z_{1,i}\tilde{\alpha}_i + \sum_{i=1}^{n}\tfrac{1}{\varrho_i}\tilde{\theta}_i^\top\dot{\tilde{\theta}}_i\end{aligned} \tag{17}$$

$$\rho_{2,i}^2 - z_{2,i}^2 \le \rho_{2,i}^2$$

Let

$$\mathcal{L} = -\sum_{i=1}^{n} P_{2,i} z_{2,i} \tilde{\omega}_i \; + \; \tilde{\omega}^\top \dot{\tilde{\omega}} \; - \; k_s \sum_{i=1}^{n} P_{2,i} |z_{2,i}|$$

利用不等式 $ab \le k\,a + \frac{b^2}{4k}$（$(\sqrt{k\,a} - \frac{b}{2\sqrt{k}})^2 \ge 0$）并取 $a = P_{2,i}|z_{2,i}|,\ b = |\tilde{\omega}_i|,\ k = k_s$，对每个通道有

$$-P_{2,i} z_{2,i} \tilde{\omega}_i \;\le\; P_{2,i}|z_{2,i}|\,|\tilde{\omega}_i| \;\le\; k_s P_{2,i}|z_{2,i}| + \frac{1}{4k_s}\tilde{\omega}_i^2.$$

于是

$$-k_s \sum_{i=1}^{n} P_{2,i}|z_{2,i}| - \sum_{i=1}^{n} P_{2,i} z_{2,i} \tilde{\omega}_i \;\le\; \frac{1}{4k_s}\,\|\tilde{\omega}\|^2$$

由观测器误差$\dot{\tilde{\omega}} = -\Lambda_o\,\tilde{\omega} + \Delta_\omega, \Lambda_o = \mathrm{diag}\{\lambda_{o,i}\} > 0$, 并根据Young不等式得

$$\tilde{\omega}^\top \dot{\tilde{\omega}} = -\tilde{\omega}^\top \Lambda_o \tilde{\omega} + \tilde{\omega}^\top \Delta_\omega \;\le\; -\tilde{\omega}^\top \Lambda_o \tilde{\omega} + \frac{\varepsilon}{2}\tilde{\omega}^\top \Lambda_o \tilde{\omega} + \frac{1}{2\varepsilon}\Delta_\omega^\top \Lambda_o^{-1} \Delta_\omega \le -\Big(1 - \tfrac{\varepsilon}{2}\Big)\tilde{\omega}^\top \Lambda_o \tilde{\omega} + \frac{1}{2\varepsilon}\big\|\Lambda_o^{-1/2}\Delta_\omega\big\|^2,$$

把（A）与（B）相加：

$$\mathcal{L} \le -\Big(1 - \tfrac{\varepsilon}{2}\Big)\tilde{\omega}^\top \Lambda_o \tilde{\omega} + \frac{1}{4k_s}\|\tilde{\omega}\|^2 + \frac{1}{2\varepsilon}\big\|\Lambda_o^{-1/2}\Delta_\omega\big\|^2.$$

利用 $\tilde{\omega}^\top \Lambda_o \tilde{\omega} \ge \lambda_{o,\min}\|\tilde{\omega}\|^2$，得到

$$\mathcal{L} \le \; -\Big(\big(1 - \tfrac{\varepsilon}{2}\big)\lambda_{o,\min} - \tfrac{1}{4k_s}\Big)\|\tilde{\omega}\|^2 + \frac{1}{2\varepsilon}\big\|\Lambda_o^{-1/2}\Delta_\omega\big\|^2.$$

令 $\varepsilon = \frac{1}{2}$（简洁且常用），净负系数为 $\frac{3}{4}\lambda_{o,\min} - \frac{1}{4k_s}$。只要 $\boxed{k_s \; > \; \dfrac{1}{3\,\lambda_{o,\min}}}$

$\frac{1}{2\varepsilon}\|\Lambda_o^{-1/2}\Delta_\omega\|^2$ 作为输入项（由 $\Delta_\omega = \dot{\omega} - \Lambda_o \varepsilon_d$ 决定）并入 $\sigma(\|r(t)\|)$。

Statements and Declarations

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Conflicts of Interest

The authors declare no conflict of interest.

Author contribution

Shuli Liu: Writing-Original Draft, Methodology, Software, Visualization, Data curation. Yi Liu:Resources, Investigation, Formal analysis, Conceptualization, Writing-Review & Editing. Jingang Liu: Project administration, Validation, Funding acquisition, Writing-Review & Editing. Yin Yang: Supervision, Funding acquisition, Writing-Review & Editing. All authors have reviewed and approved the final version of the manuscript.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.