Predefined Time Prescribed Performance Backstepping Control for Robotic Manipulators with Input Saturation

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摘要

For robotic manipulators trajectory tracking with unknown dynamics, bounded disturbances, and input saturation, this work proposes an adaptive backstepping control method that combines predefined time convergence with global predefined performance constraints. A channel-wise predefined-time error transformation merges a polynomial performance function with error scaling. It removes initial-state singularities and nonsmooth mappings in conventional PPC and enforces strict, smooth, global bounds within a user-set time, even from unknown initial states. We also design a unified saturation compensator that cancels residuals from command filtering and actuator saturation, curbs the backstepping complexity explosion, and reduces online computation. A first-order sliding-mode disturbance observer and an RBFNN estimator run online to handle bounded disturbances and unmodeled dynamics. With predefined-time stability theory and an adaptive dynamic barrier Lyapunov function, we prove closed-loop boundedness and on-time convergence. Simulations and experiments show faster response, better steady-state accuracy, and stronger robustness than existing methods.

Keywords: Prescribed performance control, predefined-time stability, adaptive dynamic barrier Lyapunov function, backstepping control, input saturation, robotic manipulators' trajectory tracking.

1 Introduction

2 Problem and preliminary

2.1 System description

This work considers an uncertain nonlinear n-DOF robot manipulators system. Its dynamics are described as [1, 2]:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + \Delta(q,\dot{q},t) = \tau + d(t), \tag{1}$$

where $q, \dot{q}, \ddot{q} \in \mathbb{R}^n$ are the joint state vectors, $M(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$ represents the centrifugal effects, $G(q) \in \mathbb{R}^n$ represents the gravitational force, $\tau \in \mathbb{R}^n$ is the control joint input, $\Delta(q,\dot{q},t) \in \mathbb{R}^n$ is the uncertainty and unmodeled dynamics, $d(t) \in \mathbb{R}^n$ is an external disturbance, n is the number of DOF of the manipulators system.

For controller design, define the state $x = [x_1, x_2] = [q, \dot{q}] \in \mathbb{R}^{2n}$. The system eq. (1) can be written in the standard state space form:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = f + g\mathrm{sat}(u) + h, \end{cases} \tag{2}$$

where $f(x) = -M^{-1}(q)(C(q,\dot{q})x_2 + G(q))$, $M^{-1}(q) = M^{-1}(q)$, $h(x,t) = -M^{-1}(q)\Delta(q,\dot{q},t)$, and $x_1 = [x_{1,1}, x_{1,2}, ..., x_{1,n}]^\top$, $x_2 = [x_{2,1}, x_{2,2}, ..., x_{2,n}]^\top$, $\mathrm{sat}(u)$ is the saturation of control input $u = \tau$. Define the saturation error as

$$\Delta u_i = u_i - \mathrm{sat}(u_i) = \begin{cases} u_i - u_{\max}, & u_i \geq u_{\max}, \\ 0, & u_{\min} < u_i < u_{\max}, \quad i = 1, 2, ..., n, \\ u_i - u_{\min}, & u_i \leq u_{\min}, \end{cases} \tag{3}$$

where $u_{\max}, u_{\min}$ are the upper and lower saturation bounds of control input, respectively.

The controller design and stability analysis are conducted under the following physically justified assumptions.

Assumption 1 *[3] $M(q)$ is symmetric positive definite for all q. There exist constants $m_1, m_2 > 0$ such that $m_1 I \leq M(q) \leq m_2 I$, $\forall q \in \mathbb{R}^n$. Moreover, the standard structural property holds: $\dot{M}(q) - 2C(q,\dot{q})$ is skew-symmetric.*

Assumption 2 *The unmodeled dynamics are bounded: there exists $\bar{\Delta} > 0$ such that $\|\Delta(q,\dot{q},t)\| \leq \bar{\Delta}$, $\forall t \geq 0$. The disturbance and its derivative are bounded: there exist $\bar{d}, \bar{\dot{d}} > 0$ such that $\|d(t)\| \leq \bar{d}$ and $\|\dot{d}(t)\| \leq \bar{\dot{d}}$, $\forall t \geq 0$. The desired trajectory $q_d(t)$ is twice continuously differentiable, and there exist $\bar{q}_d, \bar{\dot{q}}_d, \bar{\ddot{q}}_d > 0$ such that $\|q_d(t)\| \leq \bar{q}_d$, $\|\dot{q}_d(t)\| \leq \bar{\dot{q}}_d$, $\|\ddot{q}_d(t)\| \leq \bar{\ddot{q}}_d$, $\forall t \geq 0$.*

2.2 Fuzzy logic system

Fuzzy logic systems (FLS) are universal approximators on compact sets and are widely employed to estimate unknown nonlinear mappings. Let the regressor be $\chi \in \mathbb{R}^m$. For the i-th channel, an FLS with N Takagi–Sugeno rules (singleton fuzzifier, product inference, and center-average defuzzifier) is constructed as:

$$\textit{Rule } \ell\textit{: If } \chi_1 \textit{ is } A_{1\ell} \textit{ and } \cdots \textit{ and } \chi_m \textit{ is } A_{m\ell}\textit{, then } y_i \textit{ is } \theta_{\ell,i}, \quad \ell = 1, \ldots, N,$$

where $\theta_{\ell,i} \in \mathbb{R}$ are consequent parameters to be learned. Denote the normalized basis vector by $\psi_i(\chi) = \big[\psi_1(\chi), \ldots, \psi_N(\chi)\big]^\top \in \mathbb{R}^N$. Then the FLS output used to approximate the unknown $f_i(\chi)$ is modeled as

$$h_i(\chi) = \theta_i^{*\top}\, \psi_i(\chi) + \varepsilon_i, \tag{4}$$

where $\theta_i^* = [\theta_{1,i}^*, \ldots, \theta_{N,i}^*]^\top \in \mathbb{R}^N$ is the ideal parameter vector within the chosen rule base, and ε_i is the bounded approximation error satisfying $|\varepsilon_i| \leq \bar{\varepsilon}$ for all χ in a compact set $\Omega_\chi \subset \mathbb{R}^m$, with a known constant $\bar{\varepsilon} > 0$.

The normalized basis functions $\psi_{\ell,i}(\chi)$ are computed from the firing strengths $\varpi_\ell(\chi)$:

$$\psi_{i,\ell}(\chi) = \frac{\varpi_\ell(\chi)}{\sum_{k=1}^N \varpi_k(\chi)}, \quad \varpi_\ell(\chi) = \prod_{\flat=1}^m \mu_{\flat,\ell}(\chi_j), \quad \ell = 1, \ldots, N, \tag{5}$$

where $\mu_{\flat,\ell}(\cdot)$ are the membership functions of the antecedent fuzzy sets $A_{\flat,\ell}$. In this paper we choose Gaussian membership functions

$$\mu_{\flat,\ell}(\chi_j) = \exp\Big(-\frac{(\chi_j - \mathcal{C}_{\flat,\ell})^2}{\mathcal{D}_{\flat,\ell}^2}\Big), \quad \mathcal{C}_{\flat,\ell} \in \mathbb{R},\ \mathcal{D}_{\flat,\ell} > 0,$$

with $\mathcal{C}_{\flat,\ell}$ and $\mathcal{D}_{\flat,\ell}$ denoting the centers and widths, respectively.

Let $\hat{\theta}_i$ be the adaptive estimate of θ_i^* and define the parameter estimation error $\tilde{\theta}_i = \theta_i^* - \hat{\theta}_i$, $\hat{\theta}_i(0) = 0$. The ideal parameter vector θ_i^* is understood as the best fixed parameter within the chosen rule base that minimizes the worst-case approximation error over Ω_χ:

$$\theta_i^* = \arg\min_{\theta_i \in \mathbb{R}^N} \left\{ \sup_{\chi \in \Omega_\chi} \big|\, f_i(\chi) - \theta_i^\top \psi_i(\chi) \,\big| \right\}. \tag{6}$$

Assumption 3 *The ideal parameter vector θ_i^* is bounded, i.e., there exists $\bar{\theta} > 0$ such that $\|\theta_i^*\| \leq \bar{\theta}$.*

仿真参数设置： The parameters of FLS are set as $N = 25$, $\mathcal{C}_{\flat,\ell} \in \{-1, -0.5, 0, 0.5, 1\}$, $\mathcal{D}_{j,\ell} = 0.4$, $\bar{\theta} = 50$, $\varrho_i = 15$, $\kappa_i = 0.05$。

2.3 Preliminary

Lemma 1 *[4] (Predefined time stable theory). Consider the system $\dot{x}(t) = f(x(t)), x(0) = x_0$, where $x \in \mathbb{R}^n$ is the system's state, and $x(0) = x_0$ is the initial state. $f(\cdot)$ is continuous and satisfies $f(0) = 0$. et $V : \mathbb{R}^n \to \mathbb{R}_+$ be continuously differentiable, positive definite, and radially unbounded, with $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. For given design parameters $0 < \eta < 1$, $T_p > 0$, and $0 < \varsigma < \infty$, suppose the closed-loop system satisfies*

$$\dot{V}(x) \leq -\frac{\pi}{\eta T_p}\left(V(x)^{1-\frac{\eta}{2}} + V(x)^{1+\frac{\eta}{2}}\right) + \varsigma, \tag{7}$$

Then the origin is predefined-time stable with convergence time bounded by T_p. Moreover,

$$\left\{\lim_{t\to T_p} x | V \leq \min\left\{\left(\tfrac{2\,\eta\, T_p\, \varsigma}{\pi}\right)^{\frac{1}{1-\frac{\eta}{2}}}, \left(\tfrac{2\,\eta\, T_p\, \varsigma}{\pi}\right)^{\frac{1}{1+\frac{\eta}{2}}}\right\}\right\}, \tag{8}$$

In particular, if $\varsigma = 0$ then $V(T_p) = 0$.

Definition 1 (PTS–ISS) 考虑系统 $\dot{x} = f(x, u)$。若存在给定常数 $T > 0$、$\eta \in (0, 1)$ 以及类-$\mathcal{K}$ 函数 $\gamma(\cdot)$，使得对任意本质有界输入 $u \in L_\infty$，

$$\|x(T)\| \;\leq\; \gamma\big(\|u\|_\infty\big),$$

并且对所有 $t \geq T$ 状态保持在该球内；若 $u \equiv 0$ 则 $x(T) = 0$。则称系统在预定义时间 T 满足输入到状态稳定（Predefined-Time ISS, 简记 PTS–ISS）。

Theorem 1 (PTS–ISS 的 Lyapunov 判据) *设存在 C^1 Lyapunov 函数 $V : \mathbb{R}^n \to \mathbb{R}_+$ 与类-$\mathcal{K}_\infty$ 函数 α_1, α_2 使得*

$$\alpha_1(\|x\|) \;\leq\; V(x) \;\leq\; \alpha_2(\|x\|), \quad V(0) = 0,$$

并存在给定参数 $T > 0$, $\eta \in (0, 1)$ 以及类-$\mathcal{K}$ 函数 $\sigma(\cdot)$，使对所有 (x, u) 成立

$$\dot{V}(x, u) \;\leq\; -\frac{\pi}{\eta T}\Big(V^{1-\frac{\eta}{2}} + V^{1+\frac{\eta}{2}}\Big) \;+\; \sigma\big(\|u\|_\infty\big). \tag{9}$$

则对任意初值与本质有界输入 u，在 $t = T$ 时

$$V(T) \;\leq\; \min\left\{\left(\tfrac{2\eta T}{\pi}\,\sigma(\|u\|_\infty)\right)^{\frac{1}{1-\frac{\eta}{2}}},\;\; \left(\tfrac{2\eta T}{\pi}\,\sigma(\|u\|_\infty)\right)^{\frac{1}{1+\frac{\eta}{2}}}\right\}, \tag{10}$$

并且对所有 $t \geq T$ 有 $V(t) \leq V(T)$。若 $u \equiv 0$（从而 $\sigma(0) = 0$），则 $V(T) = 0$，即系统在预定义时间 T 到达原点。

证明 令 $r := \|u\|_\infty$，则 $\sigma(\|u\|_\infty) = \sigma(r)$ 为常数。由 eq. (9) 得

$$\dot{V} \;\leq\; -\frac{\pi}{\eta T}\Big(V^{1-\frac{\eta}{2}} + V^{1+\frac{\eta}{2}}\Big) \;+\; \sigma(r).$$

考虑比较系统

$$\dot{z} \;=\; -\frac{\pi}{\eta T}\Big(z^{1-\frac{\eta}{2}} + z^{1+\frac{\eta}{2}}\Big) \;+\; \sigma(r), \quad z(0) = V(0).$$

由比较引理得 $V(t) \leq z(t)$。对该标量系统应用预定义时间引理（即原文的 Lemma 1，将常数项 ς 替换为 $\sigma(r)$），即可得到 eq. (10)。进一步令

$$V^\star(r) := \min\left\{\left(\tfrac{2\eta T}{\pi}\,\sigma(r)\right)^{\frac{1}{1-\frac{\eta}{2}}},\;\; \left(\tfrac{2\eta T}{\pi}\,\sigma(r)\right)^{\frac{1}{1+\frac{\eta}{2}}}\right\},$$

则当 $V > V^\star(r)$ 时右端严格为负，故 V 下降；当 $V \leq V^\star(r)$ 时解不能越出该阈值。由于 $V(T) \leq V^\star(r)$，于是对所有 $t \geq T$ 有

$$V(t) \;\leq\; V^\star(\|u\|_\infty).$$

若 $u \equiv 0$（从而 $\sigma(0) = 0$），则 $V^\star(0) = 0$ 且 $V(T) = 0$。证毕。

□

Remark 1 (与原 Lemma 1（预定义时间稳定）的对比)

- **扰动建模层级：** Lemma 1 要求 $\dot V \le -\frac{\pi}{\eta T_p}(V^{1-\eta/2}+V^{1+\eta/2})+\varsigma$，其中 ς 为常数；本定理将该常数推广为输入范数的类-$\mathcal{K}$ 函数 $\sigma(\|u\|_\infty)$，从而可统一刻画外界扰动、建模不确定性、近似残差、饱和残差、滤波导数等所有残余影响（将其堆叠为“输入”并取 $\|\cdot\|_\infty$）。
- **结论形式：** 两者在 $t=T$ 都给出闭式“进入并保持”的半径；当 $u\equiv 0$（或 $\sigma(0)=0$）时，本定理退化为 Lemma 1 的“预定义时间到原点”结论；当 $u\not\equiv 0$ 而有界时，本定理给出预定义时间一致实用稳定（半径由 $\sigma(\|u\|_\infty)$ 决定），而 Lemma 1 只能处理常数半径。
- **可设定时间：** 两者右端均含“双幂核” $V^{1\mp\eta/2}$，系数 $\pi/(\eta T)$ 直接将到达/进入时间预设为 T；若工程上得到 $\dot V \le -c\big(V^{1-\eta/2}+V^{1+\eta/2}\big)+\sigma(\|u\|_\infty)$，取 $T=\pi/(\eta c)$ 可归约到本定理形式。

Remark 2 (在本文闭环中的实例化) 在本文闭环中可令

$$u \equiv [\,\Delta u,\ d,\ \dot d,\ \varepsilon,\ R\,],$$

分别表示饱和残差、外扰及其导数、RBF 近似残差、滤波导数等；可取

$$\sigma(\|u\|_\infty) \;=\; c_\Delta\|\Delta u\|_\infty^2 + c_d\|d\|_\infty^2 + c_{\dot d}\|\dot d\|_\infty^2 + c_\varepsilon\|\varepsilon\|_\infty^2 + c_R\|R\|_\infty^2 + \sigma(0),$$

其中 $\sigma(0)\ge 0$ 吸收 Young 不等式常数、泄漏/投影等小常数；通过增大 $\lambda,\kappa,\varpi,\iota$ 等参数可将 $\sigma(0)$ 压小，以减小最终半径。

Lemma 2 *[5] For $\mathcal{X},\mathcal{Y}\ge 0$, and $\mathcal{Z}_1,\mathcal{Z}_2>1$ with $1/\mathcal{Z}_1+1/\mathcal{Z}_2=1$, the following inequalities hold:*

$$\mathcal{X}\mathcal{Y} \le \frac{\mathcal{X}^{\mathcal{Z}_1}}{\mathcal{Z}_1}+\frac{\mathcal{Y}^{\mathcal{Z}_2}}{\mathcal{Z}_2}, \tag{11}$$

Moreover, for any $\mathcal{Z}_1,\mathcal{Z}_2>0$ with $\mathcal{Z}_1+\mathcal{Z}_2=1$ and any $\mathcal{Z}_3>0$ (generalized weighted form),

$$|\mathcal{X}|^{\mathcal{Z}_1}|\mathcal{Y}|^{\mathcal{Z}_2} \le \mathcal{Z}_1\mathcal{Z}_3^{-\frac{\mathcal{Z}_2}{\mathcal{Z}_1}}|\mathcal{X}|+\mathcal{Z}_2\mathcal{Z}_3|\mathcal{Y}|. \tag{12}$$

Lemma 3 *[6] Let $\mathcal{X}_i\in\mathbb{R}$ and $\mathcal{Y}\in\mathbb{R}^+$. Then*

$$\sum_{i=1}^n|\mathcal{X}_i|^{\mathcal{Y}} \ge \left(\sum_{i=1}^n|\mathcal{X}_i|\right)^{\mathcal{Y}},\quad \mathcal{Y}\in(0,1), \tag{13}$$

$$\sum_{i=1}^n|\mathcal{X}_i|^{\mathcal{Y}} \ge n^{1-\mathcal{Y}}\left(\sum_{i=1}^n|\mathcal{X}_i|\right)^{\mathcal{Y}},\quad \mathcal{Y}\in(1,\infty). \tag{14}$$

3 Global prescribed–performance function

本小节的目标是构造一种全局性能函数（G-PPF），同时解决现有 PPC/BLF-PPF 常见的三类问题：其一，传统性能函数在初值越界时会出现进入阶段奇异，需依赖投影/重置/非对称缩放等权宜手段；其二，边界在 T_p 处常仅保证连续而非 C^1，易在控制律中诱发尖峰项并放大噪声；其三，收敛后仍以固定且过窄的管径面对执行器饱和与扰动，导致保守或抖振。为此，所提 G-PPF 通过 $\phi(b)=-\ln b$ 与光滑窗 $s(b)$ 保证 $\rho(0^+)=\infty$、$\dot\rho(T_p^-)=0$，在预定义时间内平滑收紧且初值全局合法；并以前段缩放 $\sigma(t)$ 与后段“呼吸式”缩放 $\Sigma(t)$ 解耦调节收敛路径与收敛后放宽，按饱和残差与扰动强度自适应调整约束，避免长期保守。与此同时，给出 $\dot\rho(t)$ 的解析表达，便于在 BLF 导数中实施精确补偿，不遗留常数残差，从而将总 Lyapunov 不等式规范为 PTS-ISS 形式。该构造与反步、饱和补偿、RBFNN 等模块无缝兼容，可在严格保证预定义时间稳定的同时提升稳态精度与执行器友好性。

Let $T_p>0$ be the predefined convergence time and $0<p<1$ a shaping exponent. Define the normalized time $b(t)=(t/T_p)^p\in(0,1]$, a C^1 window

$$s(b)=1-3b^2+2b^3,\quad s(0)=1,\ s(1)=0,\ s'(1)=0, \tag{15}$$

and the global kernel

$$\phi(b)=-\ln b, \tag{16}$$

which ensures $\phi(b) \to +\infty$ as $b \to 0^+$ and $\phi(1) = 0$. For each channel $i = 1, \ldots, n$, we define the global performance function

$$\rho_i(t) = \begin{cases} a + \sigma_i(t)\, \phi\left(b(t)\right)\, s\left(b(t)\right), & 0 < t < T_p, \\ a\Big[1 + g(t)\, \Sigma_i(t)\Big], & t \geq T_p, \end{cases} \tag{17}$$

where $a > 0$ is the steady-state accuracy.

The factor $\sigma_i(t)$ modulates the *pre-T_p* constriction rate, while $\Sigma_i(t)$ allows a *post-T_p* breathing of the tube in response to actuator saturation and disturbances:

$$\begin{aligned} &\sigma_i(t) = \operatorname{Proj}_{[\sigma_{\min},\, \sigma_{\max}]}\Big(\sigma_{i0} + k_u r_{u,i} + k_d r_{d,i}\Big), \\ &\tau_u \dot{r}_{u,i} = -r_{u,i} + \|\Delta u_i\|, \quad \tau_d \dot{r}_{d,i} = -r_{d,i} + \|d_i\|, \end{aligned} \tag{18}$$

with $\sigma_{i0}, k_u, k_d, \tau_u, \tau_d > 0$, and $\Delta u_i = u_i - \operatorname{sat}(u_i)$ the saturation residue. After T_p, a smooth gate ensures C^1 transition

$$g(t) = 1 - \exp\big(-\iota(t - T_p)^2\big), \quad \iota > 0, \quad g(T_p) = 0, \ g'(T_p) = 0, \tag{19}$$

and the post–convergence breathing amplitude is

$$\begin{aligned} &\Sigma_i(t) = \operatorname{Proj}_{[\,0,\, \Sigma_{\max}\,]}\Big(k_u r_{u,i} + k_d r_{d,i} + k_e r_{e,i}\Big), \\ &\tau_e \dot{r}_{e,i} = -r_{e,i} + |e_i|, \quad k_e, \tau_e > 0. \end{aligned} \tag{20}$$

Derivatives (for exact compensation).

For $0 < t < T_p$, using $\dot{b} = \frac{p}{T_p}(t/T_p)^{p-1}$, $\phi'(b) = -1/b$, and $s'(b) = -6b + 6b^2$,

$$\dot{\rho}_i = \dot{\sigma}_i\, \phi(b)\, s(b) + \sigma_i \Big[\phi'(b)\, s(b) + \phi(b)\, s'(b)\Big] \dot{b}. \tag{21}$$

For $t \geq T_p$,

$$\dot{\rho}_i = a\Big[g'(t)\, \Sigma_i(t) + g(t)\, \dot{\Sigma}_i(t)\Big], \quad \dot{\Sigma}_i = \operatorname{Proj-grad}\Big(k_u \dot{r}_{u,i} + k_d \dot{r}_{d,i} + k_e \dot{r}_{e,i}\Big). \tag{22}$$

In the control design, the terms $+\dot{\rho}_i/\rho_i$ are included to cancel the boundary variation terms in the BLF derivatives.

Properties 1 The function $\rho_i(t)$ in (17) has the following properties:

1. **Global initial admissibility:** $\lim\limits_{t \to 0^+} \rho_i(t) = +\infty$, so any finite $e_i(0)$ satisfies $|e_i(0)| < \rho_i(0^+)$.
2. **Predefined-time constriction:** $\rho_i(T_p^-) = a$ and, by (15), $\dot{\rho}_i(T_p^-) = 0$.
3. **Monotonicity and smoothness ($0 < t < T_p$):** $\rho_i(t)$ is strictly decreasing and C^1 on $(0, T_p]$.
4. **Post-T_p adaptation:** for $t \geq T_p$, $\rho_i(t) = a[1 + g(t)\Sigma_i(t)] \geq a$, with g and Σ_i bounded and low-pass filtered; thus ρ_i remains positive and slowly varying.
5. **Implementation-ready derivatives:** the explicit forms (21)–(22) allow exact compensation in the BLF-based Lyapunov analysis without leaving constant residues.

Remark 3 ***Parameter guidelines.***

Choose $\sigma_{\min} \leq \sigma_{i0} \leq \sigma_{\max}$ and $0 \leq \Sigma_{\max} \leq 0.5$ to cap the largest post–convergence relaxation at 50%; pick τ_u, τ_d, τ_e relatively large so that $\rho_i(t)$ varies slowly and does not excite the closed-loop; set $k_u, k_d \gg k_e$ to prioritize actuator/disturbance–driven breathing over transient error–driven relaxation.

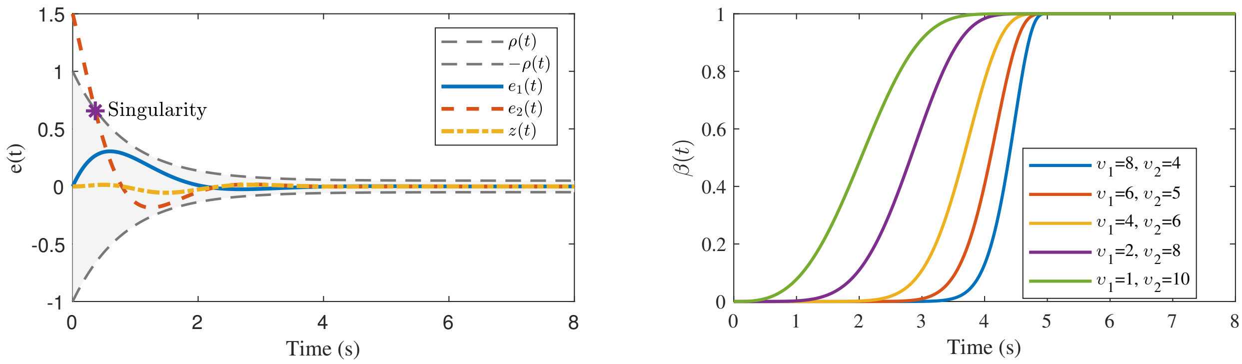

图 1 全局动态变化PPC.

4 Main results

4.1 Controller design

This work presents a controller design methodology that integrates the backstepping approach with BLFs to achieve both predefined-time convergence and prescribed performance constraints. The structure of the proposed control algorithm is shown in fig. 2.

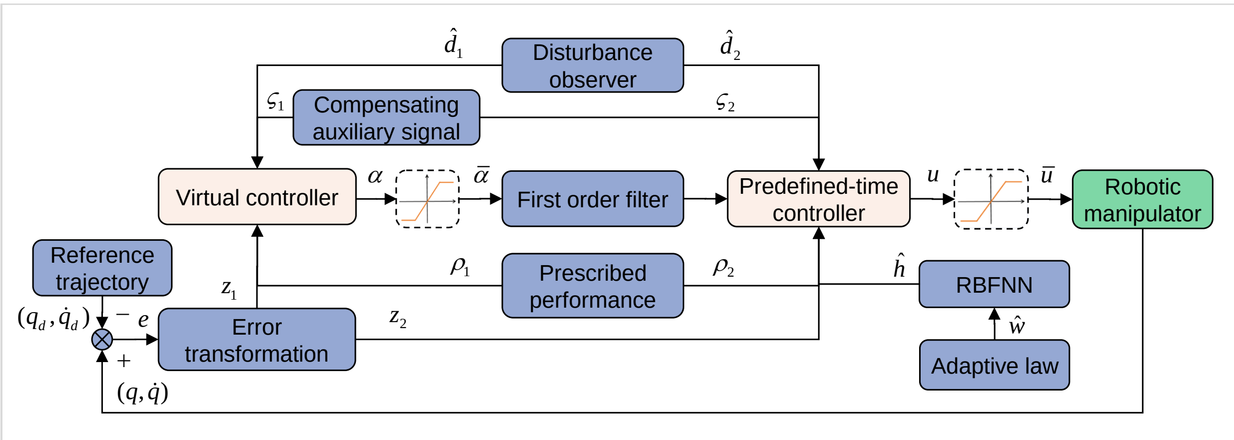

图 2 The structure of the proposed controller

In a trajectory tracking control system, to guarantee that tracking error meets the prescribed dynamic and steady-state performance requirements, define the tracking error as

$$\begin{cases} z_1 = x_1 - q_d, \\ z_2 = x_2 - \alpha^f - \zeta, \end{cases} \tag{23}$$

where α is the virtual control, α^f is a filtered version of α, and $\zeta \in \mathbb{R}^n$ are dynamic anti-saturation compensators of actual controller.

To smooth the signal and facilitate the design of the subsequent control law, the filter of the virtual controller α^f is introduced with the following dynamic equations:

$$\lambda \dot{\alpha}^f = -\left(\alpha^f - \alpha\right), \tag{24}$$

where $0 < \lambda < 1, \alpha^f(0) = \alpha(0)$. Define the filtering error is $\tilde{\alpha} = \alpha - \alpha^f$.

为将饱和残差在预设时间 $T_\zeta < cT_p$（$0 < c < 1$）内抽干，针对每个通道 i 引入单通道动态补偿信号 ζ_i，并选

$$\begin{cases} \dot{\zeta} = \delta - \mu_2 \zeta - M^{-1}(q)\Delta u, \\ \delta = -\frac{\pi}{2\gamma T_\zeta}\left[\mathrm{sig}(\zeta)^{1-\gamma} + \mathrm{sig}(\zeta)^{1+\gamma}\right], \end{cases} \tag{25}$$

其中 $0<\gamma<1$, $\mu_2>0$, $\mathrm{sig}(\zeta)^{1\pm\gamma}=|\zeta|^{1\pm\gamma}\mathrm{sig}(\zeta)$, 并取初值 $\zeta_i(0)=0$。

In the first step, a BLF is used as an energy function for the error dynamics. We choose:

$$V_1=\frac{1}{2}\sum_{i=1}^{n}\frac{\rho_{1,i}^2 z_{1,i}^2}{\rho_{1,i}^2-z_{1,i}^2}. \tag{26}$$

求V_1 的时间导数为

$$\dot V_1=\sum_{i=1}^{n}\left[\frac{\rho_{1,i}^4\, z_{1,i}}{\left(\rho_{1,i}^2-z_{1,i}^2\right)^2}\dot z_{1,i}-\frac{\rho_{1,i}\, z_{1,i}^4}{\left(\rho_{1,i}^2-z_{1,i}^2\right)^2}\dot\rho_{1,i}\right] \tag{27}$$

记 $P_{j,i}=\frac{\rho_{j,i}^4}{(\rho_{j,i}^2-z_{j,i}^2)^2}>0, Q_{j,i}=\frac{z_{j,i}^4}{(\rho_{j,i}^2-z_{j,i}^2)^2}>0, \Phi_{j,i}=\frac{z_{j,i}\left(\rho_{j,i}^2-z_{j,i}^2\right)}{\rho_{j,i}^4}=\frac{z_{j,i}(\rho_{j,i}^2-z_{j,i}^2)}{P_{j,i}}, j\in\{1,2\}$, $\dot z_{1,i}=z_{2,i}-\dot q_{d,i}+\alpha_i^f+\zeta_i$, 将它们代入 BLF 导数后，并根据公式（），得到

$$\dot V_1=\sum_{i=1}^{n}\Big[P_{1,i}z_{1,i}z_{2,i}+P_{1,i}z_{1,i}(\alpha_i+\zeta_i-\dot q_{d,i})-P_{1,i}z_{1,i}\tilde\alpha_i-Q_{1,i}\rho_{1,i}\dot\rho_{1,i}\Big]$$

Let $V_{j,i}=\frac{1}{2}\frac{\rho_{j,i}^2 z_{j,i}^2}{\rho_{j,i}^2-z_{j,i}^2}, \Psi(V_{j,i})=\frac{\pi}{\eta T_p}\Big((V_{j,i})^{1-\eta/2}+n^{-\frac{\eta}{2}}(V_{j,i})^{1+\eta/2}\Big)$, 我们设计

$$\mathcal{K}_{1,i}(z_{j,i},\rho_{j,i})=\frac{\rho_{j,i}^2}{2}\frac{\Psi(V_{j,i})}{V_{j,i}}=\frac{\pi\,\rho_{j,i}^2}{2\eta T_p}\Big((V_{j,i})^{-\eta/2}+n^{-\frac{\eta}{2}}(V_{j,i})^{\eta/2}\Big).$$

把它代回，可设计虚拟控制律为：

$$\alpha_i=\dot q_{d,i}-\zeta_i+\frac{z_{1,i}^3}{\rho_{1,i}^3}\,\dot\rho_{1,i}-\mathcal{K}_{1,i}(z_{1,i},\rho_{1,i})\Phi_{1,i}-k_{1,i}\rho_{1,i}^2\Phi_{1,i}, \tag{28}$$

where $k_1=\mathrm{diag}\{k_{1,1},k_{1,i},\cdots,k_{1,n}\}>0$, $k_{1,i}>0$,

则一步 Lyapunov 导数化简为

$$\dot V_1\le\sum_{i=1}^{n}\Big[P_{1,i}z_{1,i}z_{2,i}-\Psi(V_{1,i})-P_{1,i}z_{1,i}\tilde\alpha_i-k_{1,i}\frac{\rho_{1,i}^2 z_{1,i}^2}{\rho_{1,i}^2-z_{1,i}^2}\Big].$$

In the second step, a BLF is used as an energy function for the error dynamics. We choose:

$$V_2=\frac{1}{2}\sum_{i=1}^{n}\frac{\rho_{2,i}^2 z_{2,i}^2}{\rho_{2,i}^2-z_{2,i}^2}. \tag{29}$$

Using $\dot z_{2,i}=\dot x_{2,i}-\dot\alpha_i^f-\dot\zeta_i$, we obtain the time derivative of V_2 is as

$$\begin{aligned}\dot V_2&=\sum_{i=1}^{n}\Big[\,P_{2,i}z_{2,i}\,(\dot x_{2,i}-\dot\alpha_i^f-\dot\zeta_i)-Q_{2,i}\rho_{2,i}\,\dot\rho_{2,i}\Big]\\&=\sum_{i=1}^{n}\Big[\,P_{2,i}z_{2,i}\,\Big(f_i(x)+h_i(x,t)+g_i(u_i-\Delta u_i)+d'(t)-\dot\alpha_i^f-\delta_i\;+\;\mu_2\,\zeta_i\;+\;g_i\,\Delta u_i-\tfrac{z_2^3}{\rho_{2,i}^3}\dot\rho_{2,i}\Big)\Big]\end{aligned} \tag{30}$$

Let $Q_2=\mathrm{col}\{Q_{2,i}\}$, $P_1=\mathrm{col}\{P_{1,i}\}$, $\Phi_2=\mathrm{col}\{\Phi_{2,i}\}$. The control input torque is designed as

$$\begin{aligned}u=&C(q,\dot q)x_2+G(q)\\&+M(q)\Big[\dot\alpha^f+\delta-\mu_2\,\zeta-\frac{P_1}{P_2}z_1+\tfrac{z_2^3}{\rho_{2,i}^3}\dot\rho_{2,i}-\mathcal{K}_2(z_2,\rho_{2,i})\Phi_2-\hat h(\chi)-\hat\omega-k_s\,\mathrm{sgn}(z_2)-k_2\rho_{2,i}^2\Phi_{2,i}\Big]\end{aligned} \tag{31}$$

where $k_2=\mathrm{diag}\{k_{2,1},k_{2,i},\cdots,k_{2,n}\}>0$, $k_{2,i}>0$, $k_s>0$.

Let $\chi=\{q,\dot q\}\in\mathbb{R}^m$ be the regressor and $\psi(\chi)\in\mathbb{R}^N$ the normalized basis. Approximate the structured uncertainty by $\hat h_i(\chi)=\hat\theta_i^\top\psi_i(\chi)$, with $\theta_i\in\mathbb{R}^N$ adapted online by

$$\dot{\hat\theta}_i=-\kappa_i\,\hat\theta_i,\qquad \varrho_i,\kappa_i>0. \tag{32}$$

将结构不确定估计误差和外扰的剩余项定义为 $\omega(x,t)=h(x)-\hat h(\chi)+M^{-1}(q)d(t)$。

为避免直接微分速度，采用一阶跟踪微分器作为观测器

$$
\begin{cases}
\dot{\vartheta} = -\omega_d\,\vartheta + \omega_d\,x_2,\\
\dot{\hat{x}}_2 = \omega_d(x_2 - \vartheta),\\
\dot{\hat{\omega}} = -\Lambda_o\,\hat{\omega} + \Lambda_o\left(\dot{\hat{x}}_2 - f(x) - M^{-1}(q)\,\mathrm{sat}(u) - \hat{h}(\chi)\right),
\end{cases}
\tag{33}
$$

where $\Lambda_o = \mathrm{diag}\{\lambda_{o,i}\} > 0, \omega_d > 0, \vartheta(0) = x_2(0), \hat{\omega}(0) = 0$, 设观测误差为 $\tilde{\omega} = \omega - \hat{\omega}$. 其动力学为

$$
\dot{\tilde{\omega}} = -\Lambda_o\,\tilde{\omega} \;+\; \Delta_\omega, \qquad \Delta_\omega := \dot{\omega} - \Lambda_o\,\varepsilon_d,
$$

where $\Lambda_o = \mathrm{diag}\{\lambda_{o,i}\} > 0, w_d > 0, \vartheta(0) = x_2(0)$,$\varepsilon_d := \dot{\hat{x}}_2 - \dot{x}_2$ 为跟踪微分器误差。

根据公式可得到

$$
\dot{V}_2 \le + \sum_{i=1}^{n}\left(-k_{2,i}\frac{\rho_{2,i}^2 z_{2,i}^2}{\rho_{2,i}^2 - z_{2,i}^2} - \Psi(V_{2,i}) - P_{2,i}z_{2,i}\tilde{\omega}_i\right) - k_s\sum_{i=1}^{n}P_{2,i}\,|z_{2,i}|
\tag{34}
$$

5 Stability analysis

We analyze the stability of the closed-loop system under the proposed controller. A composite Lyapunov function eq. (36) is constructed by combining the BLF-based error energy, the RBFNN weight estimation error $\tilde{\theta}_i$, the virtual-control filtering error $\tilde{\alpha}$, and the disturbance-estimation errors $\tilde{d}_{1,i}, \tilde{d}_{2,i}$. The composite function is able to analyze the trajectory error dynamics, adaptive weight convergence and the preservation of performance constraints in a unified manner. Sufficient conditions for the global stability of the system and for all signals to be bounded are given by the derivation of its time derivative inequality. Ultimately, we show that in any initial state, the system trajectory will converge within a tight set at a set predefined-time T_p, thus achieving global predefined-time stability.

Theorem 2 *Under Assumptions 1 to 3 and the controller in* **????** *with observers* **????** *and weight update* **??**, *and under certain parameter conditions. At this point, for any initial condition, the closed-loop system is predefined-time stable. The scaled errors $z_{1,i}(t), z_{2,i}(t)$ enter a compact set no later than T_p, hence $|e_i(t)| < \rho_i(t)$ for all t, and $e_i(t)$ converges to a prescribed small neighborhood of the origin.*

证明 The composite Lyapunov function is constructed as follows

$$
V = \sum_{i=1}^{n} V_{2,i} + \tfrac{1}{2}\tilde{\alpha}^\top\tilde{\alpha} + \tfrac{1}{2\gamma_o}\tilde{w}^\top\tilde{w} + \sum_{i=1}^{n}\tfrac{1}{2\varrho_i}\|\tilde{\theta}_i\|^2
\tag{35}
$$

The time derivative of eq. (36) is given by

$$
\begin{aligned}
\dot{V} \le & \sum_{i=1}^{n}\left(-k_{1,i}\frac{\rho_{1,i}^2 z_{1,i}^2}{\rho_{1,i}^2 - z_{1,i}^2} - \Psi(V_{1,i}) - P_{1,i}z_{1,i}\tilde{\alpha}_i\right) + \sum_{i=1}^{n}\left(-k_{2,i}\frac{\rho_{2,i}^2 z_{2,i}^2}{\rho_{2,i}^2 - z_{2,i}^2} - \Psi(V_{2,i}) - P_{2,i}z_{2,i}\tilde{\omega}_i\right)\\
& - k_s\sum_{i=1}^{n}P_{2,i}\,|z_{2,i}| + \tilde{\alpha}^\top\dot{\tilde{\alpha}} + \tilde{\omega}^\top\dot{\tilde{\omega}} + \sum_{i=1}^{n}\tfrac{1}{\varrho_i}\tilde{\theta}_i^\top\dot{\tilde{\theta}}_i\\
\le & \sum_{i=1}^{n}\left(-k_{1,i}\frac{\rho_{1,i}^2 z_{1,i}^2}{\rho_{1,i}^2 - z_{1,i}^2} - \Psi(V_{1,i})\right) + \sum_{i=1}^{n}\left(-k_{2,i}\frac{\rho_{2,i}^2 z_{2,i}^2}{\rho_{2,i}^2 - z_{2,i}^2} - \Psi(V_{2,i})\right)\\
& - \sum_{i=1}^{n}P_{2,i}z_{2,i}\tilde{\omega}_i + \tilde{\omega}^\top\dot{\tilde{\omega}} - k_s\sum_{i=1}^{n}P_{2,i}\,|z_{2,i}| + \tilde{\alpha}^\top\dot{\tilde{\alpha}} - \sum_{i=1}^{n}P_{1,i}z_{1,i}\tilde{\alpha}_i + \sum_{i=1}^{n}\tfrac{1}{\varrho_i}\tilde{\theta}_i^\top\dot{\tilde{\theta}}_i
\end{aligned}
\tag{36}
$$

Combined with eq. (4) and ??, we have $h_i - \hat{h}_i = \theta_i^{*\top}\psi_i - \hat{\theta}_i^\top\psi_i = \tilde{\theta}_i^\top\psi_i, \tilde{\theta}_i = \theta_i^* - \hat{\theta}_i, \tilde{\theta}_i^\top\hat{\theta}_i = \tilde{\theta}_i^\top\theta_i - \|\tilde{\theta}_i\|^2 \le \bar{\theta}_i\|\tilde{\theta}_i\| - \|\tilde{\theta}_i\|^2, \dot{\tilde{\theta}}_i = -\dot{\hat{\theta}}_i$, and combined with Young inequalities $\bar{\theta}_i\|\tilde{\theta}_i\| \le \frac{1}{2}\|\tilde{\theta}_i\|^2 + \frac{1}{2}\bar{\theta}_i^2$, this yields

$$\begin{aligned}
&\sum_{i=1}^n \frac{1}{\varrho_i}\tilde{\theta}_i^\top\dot{\tilde{\theta}}_i \\
&= -\sum_i \frac{1}{\varrho_i}\tilde{\theta}_i^\top\left(-\kappa_i\hat{\theta}_i\right) \\
&= -\sum_{i=1}^n \frac{\kappa_i}{\varrho_i}\tilde{\theta}_i^\top\hat{\theta}_i = \sum_{i=1}^n \frac{\kappa_i}{\varrho_i}\theta_i\|\tilde{\theta}_i\| - \sum_{i=1}^n \frac{\kappa_i}{\varrho_i}\|\tilde{\theta}_i\|^2 \le -\frac{\kappa_i}{2\varrho_i}\sum_{i=1}^n\|\tilde{\theta}_i\|^2 + \frac{\kappa_i}{2\varrho_i}\sum_{i=1}^n\bar{\theta}_i^2,
\end{aligned} \tag{37}$$

From eq. (24) and ??, the time derivative of the filtering error $\dot{\tilde{\alpha}}$ is given as

$$\sum_{i=1}^n \tilde{\alpha}^\top\dot{\tilde{\alpha}} = \sum_{i=1}^n \tilde{\alpha}\dot{\alpha} - \frac{1}{\lambda}\sum_{i=1}^n\tilde{\alpha}^2, \tag{38}$$

根据????,可知 $\frac{d}{dt}\bar{\alpha}$ 为分段连续函数，在整个定义域上仅在两端处存在有限阶不连续点，但整体仍属于 piecewise continuous 函数。虚拟控制律 α 是由状态反馈设计得到的连续可导函数，其导数 $\dot{\alpha}$ 可由系统状态和参考轨迹导数构成，因此在闭环系统稳定的前提下为有界函数。由此，令$R_i = \frac{d}{dt}\bar{\alpha}$, R_i is a bounded continuous function. Then by employing Young's inequality, we obtain:

$$\sum_{i=1}^n \tilde{\alpha}\dot{\tilde{\alpha}} \le -\frac{\lambda}{2}\sum_{i=1}^n\tilde{\alpha}^2 + \frac{1}{2\lambda}\sum_{i=1}^n R_i^2. \tag{39}$$

Let

$$\mathcal{L} = -\sum_{i=1}^n P_{2,i}z_{2,i}\tilde{\omega}_i \;+\; \tilde{\omega}^\top\dot{\tilde{\omega}} \;-\; k_s\sum_{i=1}^n P_{2,i}|z_{2,i}|$$

利用不等式 $ab \le k\,a + \frac{b^2}{4k}$（$(\sqrt{k\,a} - \frac{b}{2\sqrt{k}})^2 \ge 0$）并取 $a = P_{2,i}|z_{2,i}|$, $b = |\tilde{\omega}_i|$, $k = k_s$，对每个通道有

$$-P_{2,i}z_{2,i}\tilde{\omega}_i \;\le\; P_{2,i}|z_{2,i}|\,|\tilde{\omega}_i| \;\le\; k_sP_{2,i}|z_{2,i}| + \frac{1}{4k_s}\tilde{\omega}_i^2.$$

于是

$$-k_s\sum_{i=1}^n P_{2,i}|z_{2,i}| - \sum_{i=1}^n P_{2,i}z_{2,i}\tilde{\omega}_i \;\le\; \frac{1}{4k_s}\|\tilde{\omega}\|^2$$

由观测器误差$\dot{\tilde{\omega}} = -\Lambda_o\tilde{\omega} + \Delta_\omega, \Lambda_o = \mathrm{diag}\{\lambda_{o,i}\} > 0$, 并根据Young不等式得

$$\tilde{\omega}^\top\dot{\tilde{\omega}} = -\tilde{\omega}^\top\Lambda_o\tilde{\omega} + \tilde{\omega}^\top\Delta_\omega \;\le\; -\tilde{\omega}^\top\Lambda_o\tilde{\omega} + \frac{\varepsilon}{2}\tilde{\omega}^\top\Lambda_o\tilde{\omega} + \frac{1}{2\varepsilon}\Delta_\omega^\top\Lambda_o^{-1}\Delta_\omega \le -\left(1 - \tfrac{\varepsilon}{2}\right)\tilde{\omega}^\top\Lambda_o\tilde{\omega} + \frac{1}{2\varepsilon}\|\Lambda_o^{-1/2}\Delta_\omega\|^2,$$

把（A）与（B）相加：

$$\mathcal{L} \le -\left(1 - \tfrac{\varepsilon}{2}\right)\tilde{\omega}^\top\Lambda_o\tilde{\omega} + \frac{1}{4k_s}\|\tilde{\omega}\|^2 + \frac{1}{2\varepsilon}\|\Lambda_o^{-1/2}\Delta_\omega\|^2.$$

利用 $\tilde{\omega}^\top\Lambda_o\tilde{\omega} \ge \lambda_{o,\min}\|\tilde{\omega}\|^2$，得到

$$\mathcal{L} \le \; -\Big(\big(1 - \tfrac{\varepsilon}{2}\big)\lambda_{o,\min} - \tfrac{1}{4k_s}\Big)\|\tilde{\omega}\|^2 + \frac{1}{2\varepsilon}\|\Lambda_o^{-1/2}\Delta_\omega\|^2.$$

令 $\varepsilon = \frac{1}{2}$（简洁且常用），净负系数为 $\frac{3}{4}\lambda_{o,\min} - \frac{1}{4k_s}$。只要 $\boxed{k_s \;>\; \dfrac{1}{3\,\lambda_{o,\min}}}$

$\frac{1}{2\varepsilon}\|\Lambda_o^{-1/2}\Delta_\omega\|^2$ 作为输入项（由 $\Delta_\omega = \dot{\omega} - \Lambda_o\varepsilon_d$ 决定）并入 $\sigma(\|r(t)\|)$。

In ??, according to lemma 2, and using ????, $\dot{\tilde{d}}_{j,i} = \dot{d}_{j,i} + \hbar_{j,i}\,\mathrm{sgn}(s_{j,i}) + \varpi_{j,i}\hat{d}_{j,i} - \hbar_{j,i}\Gamma_{j,i}/\omega_j$. With $\|\dot{d}_{j,i}\| \le \bar{\dot{d}}$ and Young's inequality, we obtain

$$\frac{1}{\hbar_{j,i}}\tilde{d}_{j,i}\dot{d}_{j,i} \le \frac{1}{2\hbar_{j,i}}\tilde{d}_{j,i}^2 + \frac{1}{2\hbar_{j,i}}\dot{d}_{j,i}^2, \tag{40a}$$

$$\tilde{d}_{j,i}\,\mathrm{sgn}(s_{j,i}) \le \frac{1}{2}\tilde{d}_{j,i}^2 + \frac{1}{2}, \tag{40b}$$

$$\frac{\varpi_{j,i}}{\hbar_{j,i}}\tilde{d}_{j,i}\hat{d}_{j,i} = \frac{\varpi_{j,i}}{\hbar_{j,i}}\tilde{d}_{j,i}(d_{j,i} - \tilde{d}_{j,i}) \le -\frac{\varpi_{j,i}}{2\hbar_{j,i}}\tilde{d}_{j,i}^2 + \frac{\varpi_{j,i}}{2\hbar_{j,i}}d_{j,i}^2, \tag{40c}$$

$$\left|\frac{z_{j,i}d_{j,i}}{\omega_j(\rho_{j,i}^2 - z_{j,i}^2)}\right| \le \frac{\epsilon_j}{2\omega_j}\frac{z_{j,i}^2}{\rho_{j,i}^2 - z_{j,i}^2} + \frac{1}{2\epsilon_j\omega_j(\rho_{j,i}^2 - z_{j,i}^2)}d_{j,i}^2, \tag{40d}$$

where $\epsilon_j > 0$ is arbitrary.

After organizing ??, the principal negative qualitative and constant bounded terms of Lyapunov's derivative can be obtained, which further leads to

$$\dot{V} \leq -rV + \sigma, \tag{41}$$

with

$$r = \min\left\{\min_i\left(\frac{2k_{1,i}-\epsilon_1}{\omega_1}\right), \min_i\left(\frac{2k_{2,i}-\epsilon_2}{\omega_2}\right), \min_i(\kappa_i), \lambda, \min_i\left(\varpi_{1,i}-1-\hbar_{1,i}\right), \min_i\left(\varpi_{2,i}-1-\hbar_{2,i}\right)\right\} > 0, \tag{42}$$

$$\sigma = \frac{\kappa_i}{2\varrho_i}\sum_{i=1}^{n}\bar{\theta}_i^2 + \frac{1}{2\lambda}\sum_{i=1}^{n}R_i^2 + \frac{n}{4\omega_1\iota_1} + \frac{n}{4\omega_2\iota_2} + H. \tag{43}$$

From the above ?? and eqs. (41) to (43), as long as the controller parameters are chosen reasonably, V is converge exponentially, which implies all signals $z_{1,i}, z_{2,i}, \tilde{\alpha}, \tilde{\theta}_i, \tilde{d}_{1,i}, \tilde{d}_{2,i}$ are bounded within the compact set $\mathcal{Q}$. Let

$$\mathcal{Q} = \left\{(z_{1,i}, z_{2,i}, \tilde{\alpha}, \tilde{\theta}_i, \tilde{d}_{1,i}, \tilde{d}_{2,i}) : \|(z, \tilde{\alpha}, \tilde{w}, \tilde{d})\| \leq \sqrt{\phi}\right\}. \tag{44}$$

According to eqs. (36) and (44) and ??, it follows that

$$\begin{aligned}
\dot{V} \leq &- \sum_{i=1}^{n}\Gamma_{1,i}z_{1,i}\left[\frac{\pi}{\eta T_p}\left(2^{-1} + n^{\frac{\eta}{2}}2^{-\frac{\eta}{2}-1}\left(\frac{z_{1,i}^2}{\rho_{1,i}^2 - z_{1,i}^2}\right)^{\frac{\eta}{2}}\right)\right] \\
&- \sum_{i=1}^{n}\Gamma_{2,i}z_{2,i}\left[\frac{\pi}{\eta T_p}\left(2^{-1} + n^{\frac{\eta}{2}}2^{-\frac{\eta}{2}-1}\left(\frac{z_{2,i}^2}{\rho_{2,i}^2 - z_{2,i}^2}\right)^{\frac{\eta}{2}}\right)\right] \\
&- \frac{\pi}{\eta T_p}\sum_{i=1}^{n}\left[\left(\frac{1}{2\,\varrho_i}\tilde{\theta}_i^2\right)^{\frac{\eta}{2}-1} + \left(\frac{1}{2}\tilde{\alpha}^2\right)^{\frac{\eta}{2}-1} + \left(\frac{1}{2\hbar_{1,i}}\tilde{d}_{1,i}^2\right)^{\frac{\eta}{2}-1} + \left(\frac{1}{2\hbar_{2,i}}\tilde{d}_{2,i}^2\right)^{\frac{\eta}{2}-1}\right] \\
&- n^{\frac{\eta}{2}}\frac{\pi}{\eta T_p}\sum_{i=1}^{n}\left[\left(\frac{1}{2\,\varrho_i}\tilde{\theta}_i^2\right)^{\frac{\eta}{2}+1} + \left(\frac{1}{2}\tilde{\alpha}^2\right)^{\frac{\eta}{2}+1} + \left(\frac{1}{2\hbar_{1,i}}\tilde{d}_{1,i}^2\right)^{\frac{\eta}{2}+1} + \left(\frac{1}{2\hbar_{2,i}}\tilde{d}_{2,i}^2\right)^{\frac{\eta}{2}+1}\right] \\
&+ \sum_{i=1}^{n}\frac{2\pi}{\eta T_p}\left[\left(\frac{\phi}{2}\right)^{1-\frac{\eta}{2}} + n^{\frac{\eta}{2}}\left(\frac{\phi}{2}\right)^{1+\frac{\eta}{2}}\right] + \sigma.
\end{aligned} \tag{45}$$

From lemma 2, letting $\mathcal{X} = 1/2\sum_{i=1}^{n}\ln\left(\rho_{j,i}^2/\rho_{j,i}^2 - e_{j,i}^2\right), \mathcal{Y} = 1, \mathcal{Z}_1 = 1-\eta/2, \mathcal{Z}_2 = \eta/2, \mathcal{Z}_3 = (\eta/2)^{(\eta/(2-\eta))}$, we have

$$\left(\frac{1}{2}\sum_{i=1}^{n}\ln\frac{\rho_{j,i}^2}{\rho_{j,i}^2 - e_{j,i}^2}\right)^{1-\frac{\eta}{2}} \leq \frac{1}{2}\sum_{i=1}^{n}\ln\frac{\rho_{j,i}^2}{\rho_{j,i}^2 - e_{j,i}^2} + \left(1 - \frac{\eta}{2}\right)\left(\frac{\eta}{2}\right)^{\frac{\eta}{2-\eta}}. \tag{46}$$

Quoting lemma 3, and combined with ?? and eq. (46), we get

$$\begin{aligned}
\dot{V} \leq &- \frac{\pi}{\eta T_p}\left(\frac{1}{2}\sum_{i=1}^{n}\ln\frac{\rho_{1,i}^2}{\rho_{1,i}^2 - z_{1,i}^2} + \frac{1}{2}\sum_{i=1}^{n}\ln\frac{\rho_{2i}^2}{\rho_{2i}^2 - z_{2,i}^2} + \sum_{i=1}^{n}\frac{1}{2\,\varrho_i}\tilde{\theta}_i^2 + \sum_{i=1}^{n}\frac{1}{2}\tilde{\alpha}^2 + \frac{1}{2\hbar_{1,i}}\tilde{d}_{1,i}^2 + \frac{1}{2\hbar_{2,i}}\tilde{d}_{2,i}^2\right)^{1-\frac{\eta}{2}} \\
&- \frac{\pi}{\eta T_p}\left(\frac{1}{2}\sum_{i=1}^{n}\ln\frac{\rho_{1,i}^2}{\rho_{1,i}^2 - z_{1,i}^2} + \frac{1}{2}\sum_{i=1}^{n}\ln\frac{\rho_{2i}^2}{\rho_{2i}^2 - z_{2,i}^2} + \sum_{i=1}^{n}\frac{1}{2\,\varrho_i}\tilde{\theta}_i^2 + \sum_{i=1}^{n}\frac{1}{2}\tilde{\alpha}^2 + \frac{1}{2\hbar_{1,i}}\tilde{d}_{1,i}^2 + \frac{1}{2\hbar_{2,i}}\tilde{d}_{2,i}^2\right)^{1+\frac{\eta}{2}} \\
&+ \sum_{i=1}^{n}\frac{2\pi}{\eta T_p}\left[\left(\frac{\phi}{2}\right)^{1-\frac{\eta}{2}} + n^{\frac{\eta}{2}}\left(\frac{\phi}{2}\right)^{1+\frac{\eta}{2}}\right] + (2-\eta)\left(\frac{\eta}{2}\right)^{\frac{\eta}{2-\eta}} + \sigma \\
\leq &- \frac{\pi}{\eta T_p}\left(V^{1-\frac{\eta}{2}} + V^{1+\frac{\eta}{2}}\right) + \varsigma,
\end{aligned} \tag{47}$$

where

$$\varsigma = \sum_{i=1}^{n}\frac{2\pi}{\eta T_p}\left((\frac{\phi}{2})^{1-\frac{\eta}{2}} + n^{\frac{\eta}{2}}(\frac{\phi}{2})^{1+\frac{\eta}{2}}\right) + (2-\eta)\left(\frac{\eta}{2}\right)^{\frac{\eta}{2-\eta}} + \sigma. \tag{48}$$

In summary, eq. (48) is strictly controlled by the sum of the negative definite term and the bounded term ς. From lemma 1 then the closed loop system is globally stable at a predefined-time. Remember $A = \left[z_1, z_2, \tilde{\theta}, \tilde{\alpha}, \tilde{d}_1, \tilde{d}_2\right]^{\mathsf{T}} \in \mathbb{R}^{6n}$. And take the Lyapunov function $V(A) = \|A\|_2^2$, combined with the previous derivation and eq. (47), it can be seen that, for any initial condition, the closed-loop system globally converges to a compact set Ω within a predefined-time T_p. After that, $V(A)$ does not increase any further, which ensures that the system remains stable within this set. The Ω is as follows

$$\Omega = \left\{A : \lim_{t\to T_p} V(A) \leq \min\left\{\left(\tfrac{2\eta T_p\varsigma}{\pi}\right)^{\frac{1}{1-\frac{\eta}{2}}}, \left(\tfrac{2\eta T_p\varsigma}{\pi}\right)^{\frac{1}{1+\frac{\eta}{2}}}\right\}\right\}. \tag{49}$$

The stability proof is completed.

$\square$

Theorem 3 *在适当参数（$0<\gamma<1$, $\mu>0$, $T_\zeta>0$）下，补偿器 (25) 使所有通道的 $\zeta_i(t)$ 在 $t=T_\zeta$ 之前严格收敛到零或给定的极小邻域；若 $\Delta u_i\equiv 0$（饱和不被触发），则 ζ_i 在 T_ζ 内收敛到原点。该性质与主闭环的 BLF–PTS 设计可并行成立。*

证明 需要套用的 PTS–ISS 引理（简述）

若存在正定径向有界的 $V(x)$ 以及常数 $T>0$, $\eta\in(0,1)$ 和类-$\mathcal{K}$ 函数 $\sigma(\cdot)$，使闭环满足

$$\alpha_1(\|x\|)\le V(x)\le\alpha_2(\|x\|),\quad \dot V\ \le\ -\frac{\pi}{\eta T}\Big(V^{1-\frac{\eta}{2}}+V^{1+\frac{\eta}{2}}\Big)\ +\ \sigma(\|u\|_\infty),$$

则对任意本质有界输入 u，在 $t=T$ 时

$$V(T)\ \le\ \min\Big\{\Big(\tfrac{2\eta T\,\sigma(\|u\|_\infty)}{\pi}\Big)^{\frac{1}{1-\frac{\eta}{2}}},\ \ \Big(\tfrac{2\eta T\,\sigma(\|u\|_\infty)}{\pi}\Big)^{\frac{1}{1+\frac{\eta}{2}}}\Big\},$$

并进入且保持对应紧集；若 $u\equiv 0$，则 $V(T)=0$（预定义时间到达原点）。

设计Lyapunov 函数如下

$$V_{i,\zeta}\ =\ \frac{1}{2}\zeta_i^2.$$

对$V_{i,\zeta}$求时间导数。并结合$\zeta_i\,\mathrm{sig}(\zeta_i)^\alpha=|\zeta_i|^{\alpha+1}$ 得

$$\dot V_{i,\zeta}=-\frac{\pi}{2\gamma T_\zeta}\Big((V_{i,\zeta})^{1-\frac{\gamma}{2}}+(V_{i,\zeta})^{1+\frac{\gamma}{2}}\Big)\ -\ \mu\zeta_i^2\ -\ g_i\zeta_i\Delta u_i.$$

对交叉项用 Young 不等式：

$$|g_i\zeta_i\Delta u_i|\ \le\ \frac{\mu}{2}\zeta_i^2+\frac{g_i^2}{2\mu}(\Delta u_i)^2.$$

合并上式并丢弃额外负项 $-\frac{\mu}{2}\sum\zeta_i^2\le 0$

$$\boxed{\dot V_{i,\zeta}\ \le\ -\frac{\pi}{2\gamma T_\zeta}\Big((V_{i,\zeta})^{1-\frac{\gamma}{2}}+(V_{i,\zeta})^{1+\frac{\gamma}{2}}\Big)\ +\ \underbrace{\sum_{i=1}^{n}\frac{g_i^2}{2\mu}(\Delta u_i)^2}_{\triangleq\ \Lambda(t)}\ .}$$

(3) 将输入依赖项规整为 $\mathcal{K}$ 项. 因为执行器幅值有限，Δu 本质有界：$\|\Delta u\|_\infty<\infty$。令 $\bar g\ge\max_i|g_i|$，则

$$\Lambda_i(t)\ \le\ \frac{\bar g^2}{2\mu}\|\Delta u\|_\infty^2\ \triangleq\ \sigma_u(\|\Delta u\|_\infty),$$

其中 $\sigma_u(r)=\frac{\bar g^2}{2\mu}r^2\in\mathcal{K}$。因此

$$\dot V_{i,\zeta}\ \le\ -\frac{\pi}{\gamma T_\zeta}\Big(V_{i,\zeta}^{1-\frac{\gamma}{2}}+V_{i,\zeta}^{1+\frac{\gamma}{2}}\Big)\ +\ \sigma_u(\|\Delta u\|_\infty),$$

且 $V_{i,\zeta}$ 满足 $\alpha_1(\|\zeta\|)=\alpha_2(\|\zeta\|)=\frac{1}{2}\|\zeta\|^2$。

(4) 套用 PTS–ISS 引理并得出结论. 取 $T=T_\zeta$, $\eta=\gamma$, $\sigma(\cdot)=\sigma_u(\cdot)$，由引理可得在 $t=T_\zeta$：

$$V_{i,\zeta}(T_\zeta)\ \le\ \min\Big\{\Big(\tfrac{2\gamma T_\zeta\,\sigma_u(\|\Delta u\|_\infty)}{\pi}\Big)^{\frac{1}{1-\frac{\gamma}{2}}},\ \ \Big(\tfrac{2\gamma T_\zeta\,\sigma_u(\|\Delta u\|_\infty)}{\pi}\Big)^{\frac{1}{1+\frac{\gamma}{2}}}\Big\},$$

并且对所有 $t\ge T_\zeta$ “进入并保持”。若饱和未触发（$\Delta u_i\equiv 0\Rightarrow\sigma_u(0)=0$），则 $V_{i,\zeta}(T_\zeta)=0\Rightarrow\zeta_i(T_\zeta)=0$，从而在 T_ζ 内到达原点。至此，定理成立。

Remark 4 本证明严格地把饱和残差 Δu 视为“输入”，给出 PTS–ISS 形式的微分不等式；这样既覆盖“饱和被触发”的情形（给出与 $\|\Delta u\|_\infty$ 成比例的 ISS 半径），又在“无饱和”时自然退化为 PTS 到达原点的结论。

□

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Statements and Declarations

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Conflicts of Interest

The authors declare no conflict of interest.

Author contribution

Shuli Liu: Writing-Original Draft, Methodology, Software, Visualization, Data curation. Yi Liu:Resources, Investigation, Formal analysis, Conceptualization, Writing-Review & Editing. Jingang Liu: Project administration, Validation, Funding acquisition, Writing-Review & Editing. Yin Yang: Supervision, Funding acquisition, Writing-Review & Editing. All authors have reviewed and approved the final version of the manuscript.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.