1. Consider a forward SDE

$$dx_t = f(x_t, t)\,dt + g(x_t, t)dW_t,$$

show that the corresponding probability flow ODE is written as

$$dx_t = \left[f(x_t, t) - \frac{\partial}{\partial x} g^2(x_t, t) - \frac{g^2(x_t, t)}{2} \frac{\partial}{\partial x} \log p(x_t, t) \right] dt. \quad \text{———} \; \bigstar$$

By Fokker - Planck Equation : $\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}[f(x,t)\,p(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[g^2(x,t)\,p(x,t)]$

This eqn is a form of conti. eqn. $\frac{\partial p}{\partial t} = -\frac{\partial J_{SDE}}{\partial x}$, where J_{SDE} is the probability current of the stochastic process. By factoring the FPE, we can identity this current:

$$J_{SDE}(x,t) = f(x,t)\,p(x,t) - \frac{1}{2}\frac{\partial}{\partial x}[g^2(x,t)p(x,t)] \quad \text{——} \; \circledast$$

Let $\bigstar$ becomes to $dx_t = v(x,t)\,dt$ —— $\odot$, which $v(x,t) = f(x_t,t) - \frac{\partial}{\partial x} g^2(x_t,t) - \frac{g^2(x_t,t)}{2}\frac{\partial}{\partial x}\log p(x_t,t)$

The evolution of a density $p(x,t)$ under this deterministic flow is given by the standard Continuity Eqn.:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}[v(x,t)p(x,t)]$$

The probability current J_{ODE} for this flow is simply: $J_{ODE}(x,t) = v(x,t)\,p(x,t)$ —— $\otimes$

For the $\odot$ to be the "probability flow" corresponding to the $\circledast$, they must both describe the exact same probability density evolution $p(x,t)$. This means their $\frac{\partial p}{\partial t}$ terms must be identical, which implies their probability currents must be equal. $J_{SDE} = J_{ODE}$

Subst. $\circledast$ and $\otimes$:

$$f(x,t)\,p(x,t) - \frac{1}{2}\frac{\partial}{\partial x}[g^2(x,t)p(x,t)] = v(x,t)\,p(x,t)$$

$$\Rightarrow \quad vp = fp - \frac{1}{2}\frac{\partial}{\partial x}(g^2 p)$$

$$\Rightarrow \quad v = f - \frac{1}{2p}\frac{\partial}{\partial x}(g^2 p)$$

$$\Rightarrow \quad \frac{\partial}{\partial x}(g^2 p) = \left(\frac{\partial g^2}{\partial x}\right)p + g^2\left(\frac{\partial p}{\partial x}\right)$$

$$\Rightarrow \quad v = f - \frac{1}{2p}\left[\left(\frac{\partial g^2}{\partial x}\right)p + g^2\left(\frac{\partial p}{\partial x}\right)\right]$$

$$\Rightarrow \quad v = f - \frac{1}{2}\frac{\partial g^2}{\partial x} - \frac{g^2}{2}\left(\frac{1}{p}\frac{\partial p}{\partial x}\right)$$

$$\frac{\partial}{\partial x}\log p(x,t) = \frac{1}{p(x,t)}\frac{\partial p(x,t)}{\partial x}$$

, then $v = f - \frac{1}{2}\frac{\partial g^2}{\partial x} - \frac{g^2}{2}\frac{\partial}{\partial x}\log p$

$$v(x_t,t) = f(x_t,t) - \frac{1}{2}\frac{\partial}{\partial x}g^2(x_t,t) - \frac{g^2(x_t,t)}{2}\frac{\partial}{\partial x}\log p(x_t,t)$$

Hence, $dx_t = \left[f(x_t,t) - \frac{1}{2}\frac{\partial}{\partial x}g^2(x_t,t) - \frac{g^2(x_t,t)}{2}\frac{\partial}{\partial x}\log p(x_t,t)\right]dt$ #