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Assignment 1

1 The Fourth Lotka-Volterra Diagram

Recall the Lotka-Volterra model is:

$$\dot{N}_1 = r_1 N_1 \left(\frac{K_1 - N_1 - \alpha_{12} N_2}{K_1}\right)$$
$$\dot{N}_2 = r_2 N_2 \left(\frac{K_2 - N_2 - \alpha_{21} N_1}{K_2}\right)$$

where we should notice that, α_{12} is the impact on organism 1 by organism 2, α_{21} is the impact on organism 2 by organism 1. $\dot{N}_i$ is the change of population i over time, and r_i is the growth rate of the population i, and K_i is the carrying capacity for the population i.

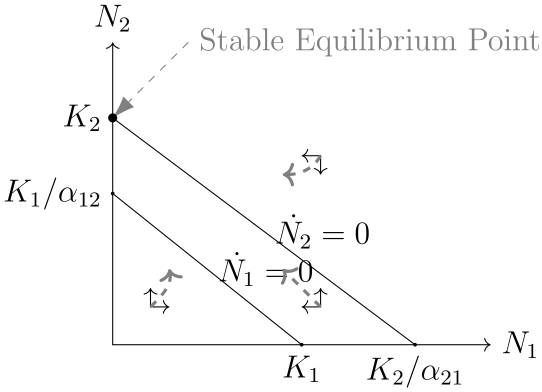

Figure 1: Lotka-Volterra competition model, phase space diagram

In Figure 1, where $K_1 > K_1/\alpha_{12}$ and $K_2/\alpha_{21} > K_1$, there's no unstable equilibrium, and there's one stable equilibrium, where N_1 dies out and only N_2 survives.

2 The Environment Preserve Team

First, we need to make some assumptions:

1. In those similar sites the team observed, the dynamical mechanics is similar, which means we can use the same model to explain all those sites.

2. In those sites, we assume that the capacity per area for each organism are the same. We define K_1 to be the capacity per area for organism 1 and K_2 for organism 2. Because two organisms share the same sites, the area is a same constant, so when we compare K_1 and K_2, we can safely ignore the area.

3. In those sites, the initial N_1 and N_2 can be any possible value, so we can rule out the possibility that the local extinction of organism 1 is caused by some special initial value.

So based on the evidence that organism 1 always extincts, we know that there's only one stable equilibrium point, and at that point, $N_1 = 0$. Among 4 possible cases in L-V model, Figure 1 in Question 1 shows the only possible case fit the dynamical mechanism of organism 1 and 2.

The colleague's suggestion is regularly reducing N_2 by a factor of ρ, by doing so, the equations become:

$$\dot{N_1} = r_1 N_1 (\frac{K_1 - N_1 - \alpha_{12} N_2}{K_1})$$
$$\dot{N_2} \approx r_2 N_2 (\frac{K_2 - N_2 - \alpha_{21} N_1}{K_2}) - N_2 \rho / h$$

where ρ is a constant means the number reduced each time, h is a constant means the time between any two reducing operations, the approximation means we use continuous reduction to approximate the discrete reduction.

The $-N_2\rho/h$ term will move isocline $\dot{N} = 0$ down to a new parallel position, the intersection of isocline $\dot{N} = 0$ on N_2-axis will become $K_2(1 - \frac{\rho}{r_2 h})$, and the intersection on N_1-axis will become $\frac{K_2}{\alpha_{21}}(1 - \frac{\rho}{r_2 h})$. Shows in Figure 2.

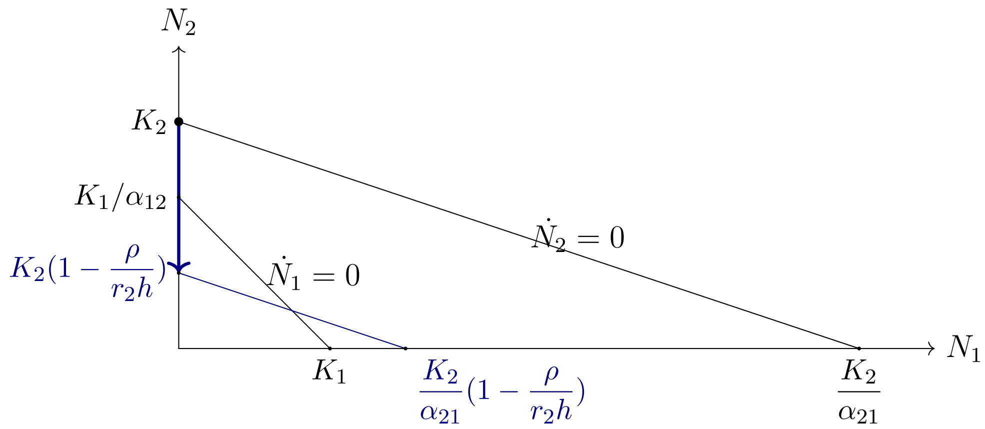

Figure 2: The isocline $\dot{N_2} = 0$ moves down because of reduction. Blue line is the result by regularly reducing N_2, and green line is the result by regularly reducing N_1.

Here we have three cases, (1) $a_{12}a_{21} < 1$, (2) $\alpha_{21}\alpha_{12} = 1$, (3) $\alpha_{21}\alpha_{12} > 1$.

1. * Case (1), when $\alpha_{21}\alpha_{12} < 1$, the isocline $\dot{N_2} = 0$ is steeper then $\dot{N_1} = 0$. So in this case, there's a range when moving isocline $\dot{N_2} = 0$, two isoclines will intersect in a way that will produce a stable equilibrium point, which means the organism 1 and 2 will be stable around that point. Figure 2 shows this case, and we know the range is $K_1 < \frac{K_2}{\alpha_{21}}(1 - \frac{\rho}{r_2 h})$ and $K_1/\alpha_{12} > K_2(1 - \frac{\rho}{r_2 h})$. Solve those inequalities, we have the range for ρ is ($r_2 h(1 - \frac{K_1}{K_2 \alpha_{12}})$, $r_2 h(1 - \frac{K_1 \alpha_{21}}{K_2})$).

2. Case (2), real world cannot have exactly same slope.

3. Case (3), when $\alpha_{21}\alpha_{12} > 1$, there's no stable equilibrium in between, so either N_1 goes to zero, or N_2 goes to zero.

The result of computer simulation confirms with the analysis on Case (1). We see the stable equilibrium appears and disappears.

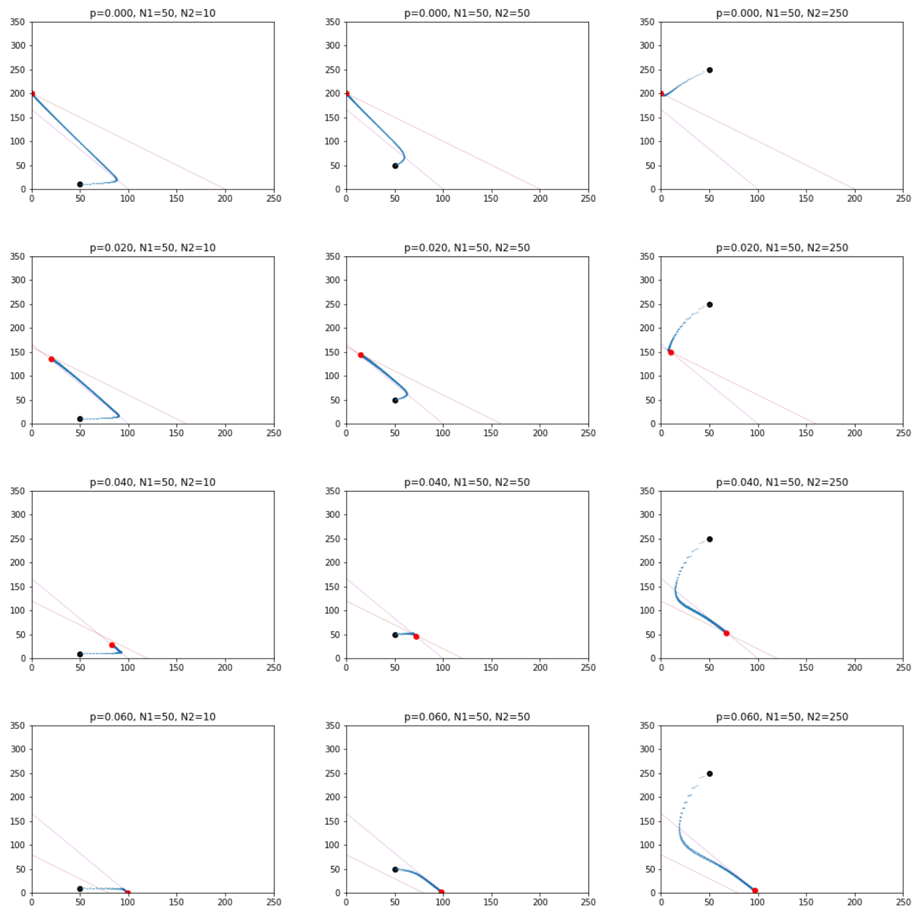

Figure 3: Simulation with different ρ's and initial conditions. Blue lines are the trajectories. When reducing only N_1 and $\alpha_{21}\alpha_{12} < 1$, the system will produce an equilibrium point while ρ is in the right range. After the isocline $\dot{N}_2 = 0$ completely pass the isocline $\dot{N}_1 = 0$, the stable equilibrium will be at the N2-axis.

Different with we only reduce N_1, in the case of we both reduce N_1 and N_2 by a factor of ρ, both isoclines move downwards. The intersection of isocline $\dot{N}_1 = 0$ on N_2-axis will become $K_1/\alpha_{12}(1 - \frac{\rho}{r_1 h})$. And the intersection on N_1-axis will become $K_1(1 - \frac{\rho}{r_1 h})$. So if the system meet the criterion of $\alpha_{12}\alpha_{21} < 1$, and p can satisfy that

$$K_1/\alpha_{12}(1 - \frac{\rho}{r_1 h}) > K_2(1 - \frac{\rho}{r_2 h})$$

$$K_2/\alpha_{21}(1 - \frac{\rho}{r_2 h}) > K_1(1 - \frac{\rho}{r_1 h})$$

there system will have a stable equilibrium.

Solve these inequalities, we have the interval:

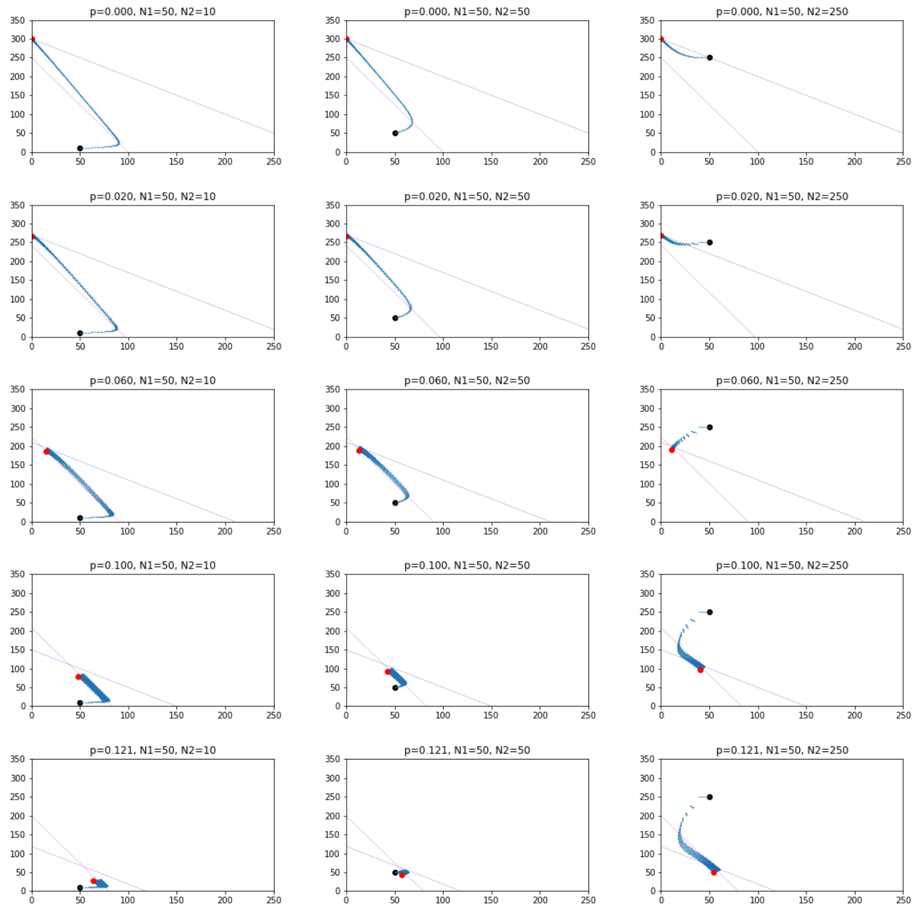

Figure 4: Simulation of reduce both organisms by a factor of ρ at a constant time interval h.

$$\rho \in (\frac{r_1 r_2(K_2\alpha_{12} - K_1)}{r_1 K_2 \alpha_{12} - r_2 K_1}, \frac{r_1 r_2(K_1\alpha_{21} - K_2)}{r_2 K_1 \alpha_{21} - r_1 K_2})$$

In this interval, along with the key criterion of $\alpha_{12}\alpha_{21} < 1$, we can adjust the system to equilibrium state.

We also implement the simulation, and the result is in Figure 4.

Another interesting strategy to achieve this goal, is just before N_1 dies out, suddenly reduce N_1 and N_2 by a large proportion, making N_1 and N_2 almost zero, and then the trajectory will start over again.

3 Implementation of Euler's and Heun's method.

Showed in Figure 5. For more details, please see Python code.

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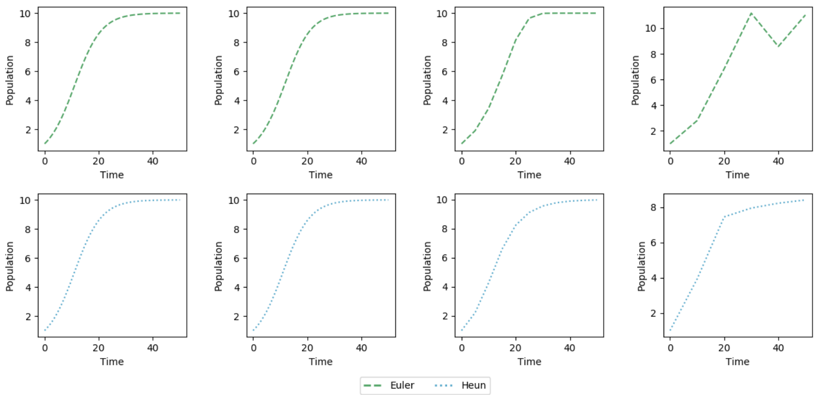

Figure 5: Implementation of Euler's and Heun's method.

4 SIS model simulation.

Showed in Figure 6. For more details, please see Python code.

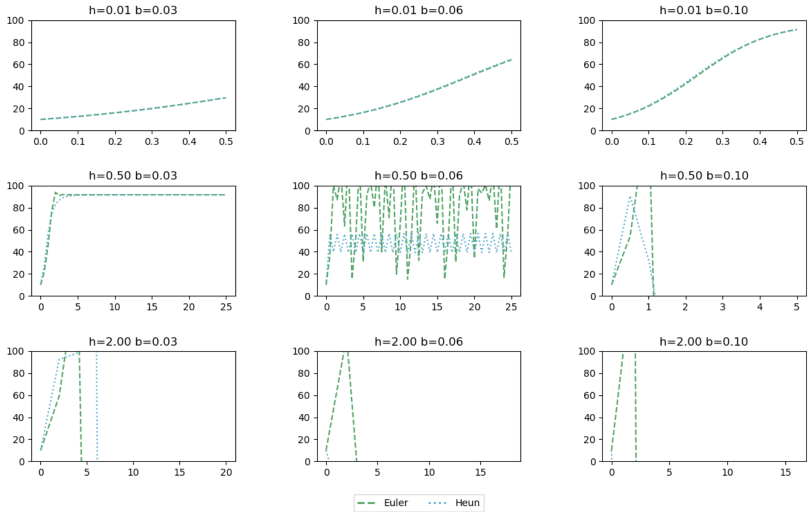

Figure 6: SIS model simulation.

5 Global Precision of Heun's method.

Recall Taylor's expansion at time t is:

$$x(t+h) = x(t) + hf(x(t)) + \frac{h^2}{2}f'(x(t)) + \beta$$

where β is the h^3 and higher terms.

Recall Heun's method:

$$x_{e,i+1} = x_i + hf(x_i)$$

where $x_{e,i+1}$ is the approximation of Euler's method.

$$\begin{aligned} x_{i+1} &= x_i + \frac{h}{2}[f(x_i) + f(x_{e,i+1})] \\ &= x_i + \frac{h}{2}[f(x_i) + f(x_i + hf(x_i))] \\ &= x_i + \frac{h}{2}[f(x_i) + f(x_i) + hf'(x_i)] \\ &= x_i + \frac{h}{2}[2f(x_i) + hf'(x_i)] \\ &= x_i + hf(x_i) + \frac{h^2}{2}f'(x_i) \end{aligned}$$

When we use Heun's method to simulate continuous equation at $x_i = x(t)$, we use x_{i+1} to approximate $x(t+h)$, which means we ignore β terms. So the local error is h^3. And the global error is over T/h steps, where T is the total amount of time. Thus the global error is $h^3 T/h = h^2 T$.

6 Analyzing SIS, SIR, SIRS model

1. Why did we see cycles in the predator-prey model and not the SIS one?

   Recall SIS model:

$$\begin{aligned} \dot{S} &= -\beta SI + \gamma I \\ \dot{I} &= \beta SI - \gamma I \\ N &= S + I \end{aligned}$$

   By eliminating S, we get:

$$\dot{I} = \beta(N - I)I - \gamma I$$

   The system in fact is a 1-D system. So, no, no cycles in 1-D space.

2. Will there be cycles in SIR model?

   Recall SIR model:

$$\begin{aligned} \dot{S} &= -\beta S \\ \dot{I} &= \beta SI - \gamma I \\ \dot{R} &= \gamma I \\ N &= S + I + R \end{aligned}$$

   By staring at the equations, we know that S will monotonically decrease and R will monotonically increase, so these two variables cannot oscillate. So, no, at most one variable oscillating cannot make any cycle.

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Assignment 1

3. Will there be cycles in SIRS model?

   Recall SIRS model:

$$\begin{aligned}\dot{S} &= -\beta SI + \epsilon R\\ \dot{I} &= \beta SI - \gamma I\\ \dot{R} &= \gamma I - \epsilon R\\ N &= S + I + R\end{aligned}$$

   By eliminating R, replacing by $N - S - I$, we get:

$$\begin{aligned}\dot{S} &= -\beta SI + \epsilon(N - S - I)\\ \dot{I} &= \beta SI - \gamma I\end{aligned}$$

   So, in fact, this system is a 2-D system.

   Solving for $\dot{S} = 0$ and $\dot{I} = 0$, we get two isoclines:

$$\begin{aligned}S &= \frac{N - I}{1 + \frac{\beta}{\epsilon} I}\\ S &= \gamma/\beta\end{aligned}$$

   When $N > \gamma/\beta$, it is possible to form a cycle. The phase space looks like in Figure 7 .

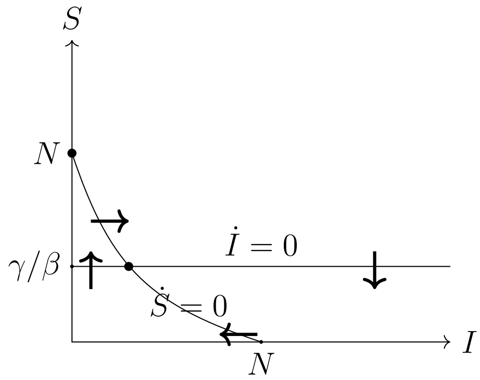

Figure 7: SIRS phase space with $N > \gamma/\beta$

   If $N <= \gamma/\beta$, there will be no space for trajectories to move around.

   Now, let's check two equilibrium points of the SIRS system. Because if the eigenvalue λ's of the Jacobian Matrix at an equilibrium point are both purely imaginary and conjugate to each other, just like the L-V predator-pray system, the system will have cycling trajectories.

The first one is a trivial one. $S^* = N$, $I^* = 0$, which means there's no infection at all. So it's not the case.

The second one is

$$S^* = \frac{\gamma}{\beta}$$
$$I^* = \frac{\epsilon}{\epsilon+\gamma}(N - \frac{\gamma}{\beta})$$

The Jacobian Matrix of the system is:

$$J = \begin{bmatrix} \frac{\partial \dot{S}}{\partial S} & \frac{\partial \dot{S}}{\partial I} \\ \frac{\partial \dot{I}}{\partial S} & \frac{\partial \dot{I}}{\partial I} \end{bmatrix}$$

$$= \begin{bmatrix} -\epsilon - \beta I & -\epsilon - \beta S \\ \beta I & \beta S - \gamma \end{bmatrix}$$

So the Jacobian Matrix at the second equilibrium point is:

$$J^* = \begin{bmatrix} -\epsilon - \beta(\frac{\epsilon}{\epsilon+\gamma}(N - \frac{\gamma}{\beta})) & -\epsilon - \beta(\frac{\gamma}{\beta}) \\ \beta(\frac{\epsilon}{\epsilon+\gamma}(N - \frac{\gamma}{\beta})) & \beta(\frac{\gamma}{\beta}) - \gamma \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\epsilon(\epsilon+\beta N)}{\epsilon+\gamma} & -(\epsilon+\gamma) \\ \frac{\epsilon(\beta N - \gamma)}{\epsilon+\gamma} & 0 \end{bmatrix}$$

The eigenvalues of this matrix are the solutions of this equation:

$$\mathbf{det}(J^* - \lambda I) = 0$$
$$(-\frac{\epsilon(\epsilon+\beta N)}{\epsilon+\gamma} - \lambda)(-\lambda) + (\epsilon+\gamma)(\frac{\epsilon(\beta N - \gamma)}{\epsilon+\gamma}) = 0$$
$$\lambda^2 - (-\frac{\epsilon(\beta N + \epsilon)}{\epsilon+\gamma})\lambda + \epsilon(\beta N - \gamma) = 0$$

in the form of $\lambda^2 - \tau\lambda + \Delta = 0$:

$$\tau = -\frac{\epsilon(\epsilon+\beta N)}{\epsilon+\gamma}$$
$$\Delta = \epsilon(\beta N - \gamma)$$

In order to get pure imaginary solutions, we need $\tau = 0$, but we have $\tau < 0$ since those parameters are all positive real number (if $\epsilon = 0$ then it will be a SIR model). And also we have $\Delta > 0$ (since $N > \dfrac{\gamma}{\beta}$), so according to the classification of equilibrium, it is a stable equilibrium point.

And an interesting guess is, whenever cycles exist, the cycles will perpendicular to the isoclines while pass the isoclines. And in this model, we can see in Figure 9 the direction of the trajectory will always bend towards the equilibrium point while passing isocline $\dot{S} = 0$, so the trajectory will shrink every round, which means the trajectory will form a spiral, not any cycles.

So, our answer to this question is, there are no cycles in a SIRS model, and by simulation, we can clearly see in Figure 8 the spiral around the stable equilibrium as the system reaches a point at which there is no net change in the populations in the model.

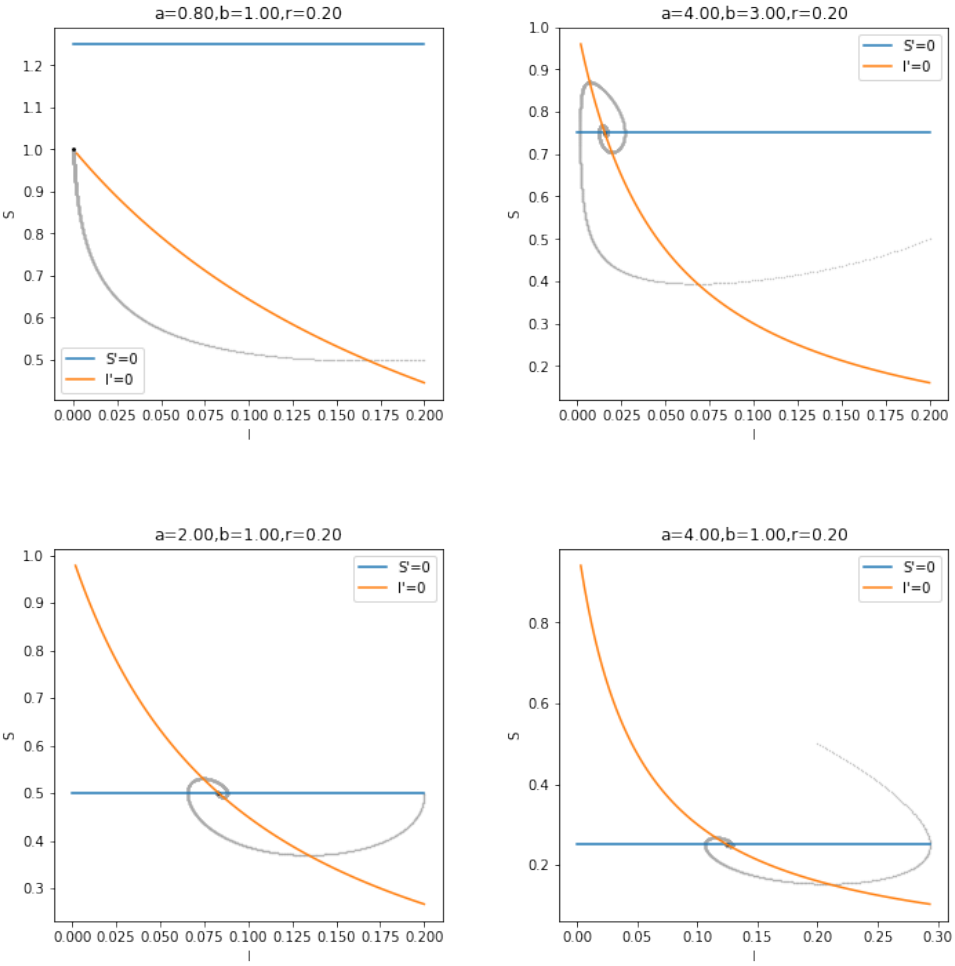

Figure 8: Simulation of SIRS model with different parameter sets.

4. Should we expect chaos in classic epidemic models?

No. Because in order for chaos to happen in continuous system, we need at least 3-dimension. The most complicated SIRS model only has two free variables, which is a 2-dimensional system.

7 Rumor Spreading Example

We propose a model to describe the spreading of a rumor in a closed population. For each individual in the population, we assume there are three states:

A: the individual knows that the rumor is false;

B: the individual is susceptible and may believe in the rumor;

C: the individual who thinks the rumor is true.

And our assumptions of contact are:

1. if A contacts C, and C becomes A with some probability α;
2. if B contacts C, B becomes C with some probability β that;
3. if A contacts B, nothing happens, they won't mention the rumor.

Then, our equations of the system are:

$$\begin{aligned}
\dot{A} &= \alpha AC \\
\dot{B} &= -\beta BC \\
\dot{C} &= \beta BC - \alpha AC \\
N &= A + B + C
\end{aligned}$$

The A, B, and C are the state variables, representing the population of each state respectively, and N is the total population of the closed system.

The system will always reach a stable equilibrium point, no matter how the parameters changes and the initial condition is, because the population C will always die out and all the derivatives become zero.

Figure 9 and Figure 10 shows different initial conditions with different parameters.

In the real-world, if α is high means the evidence against the rumor is strong, and if β is high means the rumor is convincing. So we can see, if the evidence against the rumor is strong, the rumor will die out quickly. If initially there are more people who know the rumor is false, it may stop the spreading of the rumor at early stage. And not every rumor can largely spread, sometimes (like in the right top corner) after the rumor has gone, there are still many people never heard of the rumor.

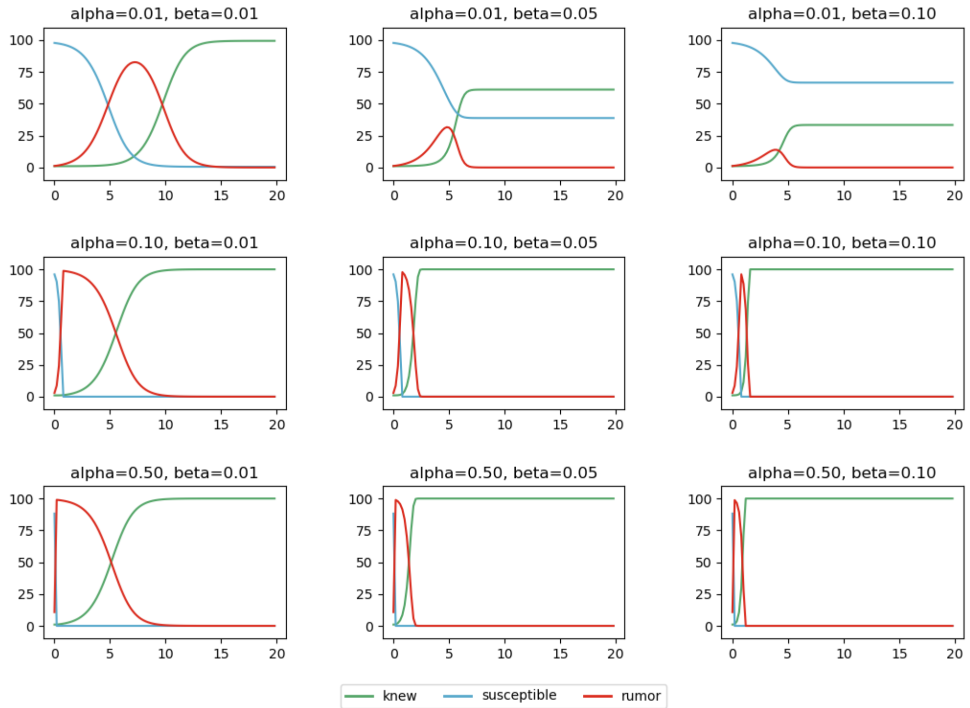

Figure 9: Simulation with initial condition [A,B,C] = [1,98,1]

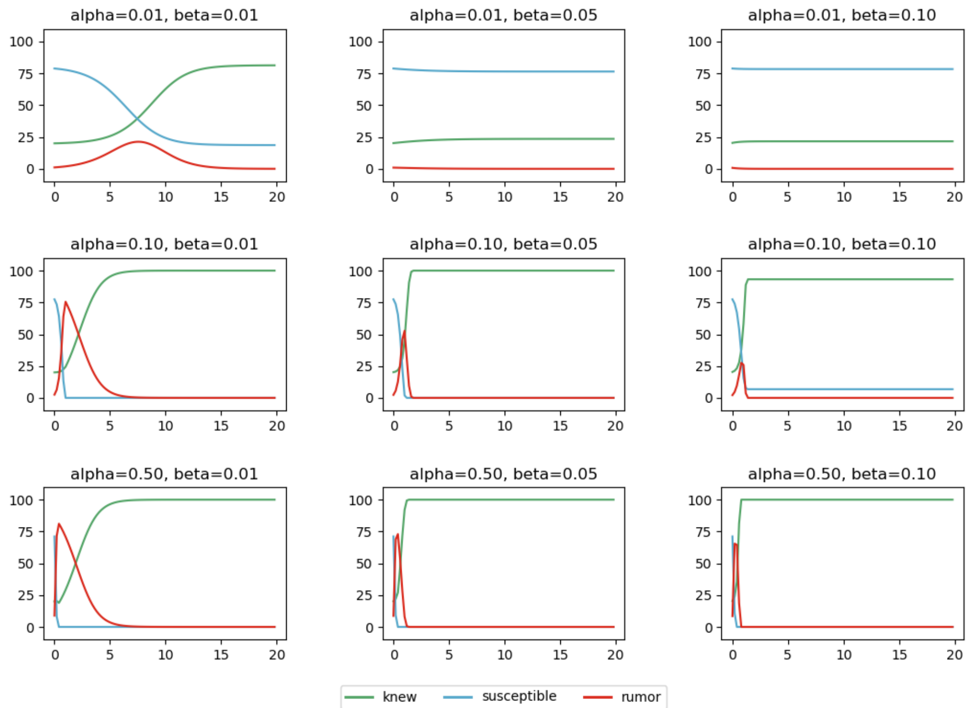

Figure 10: Simulation with initial condition [A,B,C] = [20,79,1]