NONNORMAL

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1 STEIN'S EQUATION

Suppose the support of w is $[a, b]$, define

$$q(t) = \frac{g(t)}{v(t)}, \tag{1.1}$$ {{s6}}

$$Q(x) = \int_k^x q(t)dt \tag{1.2}$$ {{s3}}

and

$$p(x) = \frac{1}{c_k v(x)} \exp(-Q(x)), \tag{1.3}$$ {{s4}}

where c_k is a normalizing constant that makes the above density be a probability density.

Similar to Shao and Zhang [2016], we use the following stein's equation

$$v(w)f_z'(w) - g(w)f_z(w) = I_{\{w \leqslant z\}} - F(z), \tag{1.4}$$ {{s1}}

where $F(z)$ is the d.f. of χ_k^2. The solution of this equation is

$$\begin{aligned} f_z(w) &= \frac{1}{v(w)p(w)} \int_0^w (I_{\{t \leqslant z\}} - F(z))p(t)dt \\ &= \begin{cases} \frac{1}{v(w)p(w)} F(w)(1-F(z)) & \text{if } 0 \leqslant w \leqslant z; \\ \frac{1}{v(w)p(w)} F(z)(1-F(w)) & \text{if } z < w. \end{cases} \end{aligned} \tag{1.5}$$

From Shao and Zhang [2016], we can easily verify that $g(w)$ and $v(w)$ satisfy the condition (B1-B4) in this paper, so we know that there exist some constant $C_{1,k}$ and $C_{2,k}$ such that $|f_z|_\infty \leqslant C_{1,k}$ and $|f_z g|_\infty \leqslant C_{2,k}$.

2 PROOF OF THE RESULTS

2.1 Properties of the solution

Now, we will prove some properties of $f_z'(w) = \frac{I_{\{x \leqslant z\}} - F(z) + g(x)f_z(x)}{v(x)}$

Since $f_z g \leqslant C_{2,k}$, it is easy to see that for $x > 0$

$$|f_z'(x)| \leqslant \frac{1 + C_{2,k}}{2x} \tag{2.1}$$ {{eq:3_7_1}}

To have the monotonicity of $f_z'(x)$, we need following conditions

- M_1

$$\lim_{x\to a+}\frac{q(x)p(x)}{q'(x)+q^2(x)}-\frac{v'(x)p(x)}{v(x)(q'(x)+q^2(x))}=\lim_{x\to a+}\frac{v(x)p(x)(g(x)-v'(x))}{v(x)g'(x)-g(x)v'(x)+g^2(x)}\geqslant 0 \tag{2.2}$$

{{eq:5_16_1}}

$$\lim_{x\to b-}\frac{q(x)p(x)}{q'(x)+q^2(x)}-\frac{v'(x)p(x)}{v(x)(q'(x)+q^2(x))}=\lim_{x\to b-}\frac{v(x)p(x)(g(x)-v'(x))}{v(x)g'(x)-g(x)v'(x)+g^2(x)}\leqslant 0 \tag{2.3}$$

{{eq:5_16_6}}

- M_2

$$q'(x)+q^2(x)=\frac{v(x)g'(x)-g(x)v'(x)+g^2(x)}{v^2(x)}>0 \tag{2.4}$$

{{eq:5_16_2}}

- M_3

$$(-g(x)+v'(x))g''(x)+2(g'(x))^2-g'(x)v''(x)\geqslant 0 \tag{2.5}$$

{{eq:5_16}}

For monotonicity of f_z' we have the following lemmas.

{lem4}

Lemma 2.1. *If f_z is the solution of (1.4), under condition M_1-M_3 we have $f_z'(x)$ is increasing in the interval $[a,z]$ and interval $(z,b]$ respectively.*

Proof. For $x\leqslant z$,

$$\frac{1-F(z)+g(x)f_z(x)}{v(x)}=(1-F(z))\left(\frac{1}{v(x)}+q(x)\frac{F(x)}{v(x)p(x)}\right)=(1-F(z))h_1(x), \tag{2.6}$$

{{p2}}

where $h_1(x)=\frac{1}{v(x)}+q(x)\frac{F(x)}{v(x)p(x)}$.

$$h_1'(x)=\frac{1}{v(x)}\left(q(x)+(q'(x)+q^2(x))\frac{F(x)}{p(x)}-\frac{v'(x)}{v(x)}\right). \tag{2.7}$$

{{p3}}

By condition M_1

$$\lim_{x\to a}\frac{q(x)p(x)}{q'(x)+q^2(x)}+F(x)-\frac{p(x)}{x(q'(x)+q^2(x))}=0 \tag{2.8}$$

{{5_16_4}}

By some calculation and by condition M_3, we have

$$\begin{aligned}&\left(\frac{q(x)p(x)}{q'(x)+q^2(x)}+F(x)-\frac{v'(x)p(x)}{v(x)(q'(x)+q^2(x))}\right)'\\ =&\frac{p(x)}{v^2(x)(q'(x)+q^2(x))^2}\Big((-g(x)+v'(x))g''(x)+2(g'(x))^2-g'(x)v''(x)\Big)\geqslant 0\end{aligned} \tag{2.9}$$

{{p6}}

By condition M_2, $p(x)\geqslant 0$ and $v(x)\geqslant 0$, we have $h_1'(x)\geqslant 0$ for $x\geqslant a$, and $f_z'(x)=(1-F(z))h_1(x)$ is increasing for $x\leqslant z$.

Similarly, for $x>z$,

$$\frac{-F(z)+g(x)f_z(x)}{v(x)}=F(z)\left(-\frac{1}{v(x)}+q(x)\frac{1-F(x)}{v(x)p(x)}\right)=F(z)h_2(x). \tag{2.10}$$

{{p2_2}}

$$h_2'(x) = \frac{1}{v(x)}\left(-q(x) + (q'(x) + q^2(x))\frac{1-F(x)}{p(x)} + \frac{1}{x}\right). \tag{2.11}$$
{{p3_2}}

By condition M_1, we have

$$\lim_{x\to b} \frac{-q(x)p(x)}{q'(x)+q^2(x)} + 1 - F(x) + \frac{p(x)}{x(q'(x)+q^2(x))} = 0. \tag{2.12}$$
{{p5_2}}

By some calculation and by M_3, we have

$$\begin{aligned}&\left(\frac{-q(x)p(x)}{q'(x)+q^2(x)} + 1 - F(x) + \frac{p(x)}{x(q'(x)+q^2(x))}\right)'\\ =& -\frac{p(x)}{v^2(x)(q'(x)+q^2(x))^2}\Big((-g(x)+v'(x))g''(x) + 2(g'(x))^2 - g'(x)v''(x)\Big) \leqslant 0.\end{aligned} \tag{2.13}$$
{{p6_2}}

So $h_2'(x) \geqslant 0$ for $x > 0$, and $\frac{I_{\{x\leqslant z\}} - F(z) + g(x)f_z(x)}{v(x)}$ is increasing for $x > z$. $\square$

Combining the monotonicity of $\frac{I_{\{x\leqslant z\}} - F(z) + g(x)f_z(x)}{v(x)}$ and continuity of $f_z(x)g(x)$, we can easily have

$$f_z'(x) - \frac{I_{\{x\leqslant z\}}}{2z} = \frac{I_{\{x\leqslant z\}} - F(z) + g(x)f_z(x)}{v(x)} - \frac{I_{\{x\leqslant z\}}}{2z} \tag{2.14}$$
{{p7}}

is increasing for all $x \geqslant 0$.

Next, we are going to consider the property of $f'(x)$ near zero.

$$f_z'(x) = \frac{I_{\{x\leqslant z\}} - F(z) + g(x)f_z(x)}{v(x)}. \tag{2.15}$$
{{p8}}

It easy to verify that

$$\lim_{x\to 0+} I_{\{x\leqslant z\}} - F(z) + g(x)f_z(x) = 0 \tag{2.16}$$
{{p9}}

for $z > 0$. Now we can use L'Hô pital's rule to calculate $\lim\limits_{x\to 0+} f_z'(x)$,

$$\begin{aligned}&\lim_{x\to 0+} \frac{I_{\{x\leqslant z\}} - F(x) + g(x)f_z(x)}{v(x)}\\ =&(1-F(z))\lim_{x\to 0+}\frac{F(x) + \frac{p(x)v(x)}{g(x)}}{\frac{p(x)v^2(x)}{g(x)}}\\ =&(1-F(z))\lim_{x\to 0+}\frac{\left(F(x) + \frac{p(x)v(x)}{g(x)}\right)'}{\left(\frac{p(x)v^2(x)}{g(x)}\right)'}\\ =&(1-F(z))\lim_{x\to 0+}\frac{1}{k^2-2kx+x^2+2k}\\ =&(1-F(z))\frac{1}{k^2+2k}.\end{aligned} \tag{2.17}$$
{{p10}}

REFERENCES

Qi-Man Shao and Zhuo-Song Zhang. Identifying the limiting distribution by a general approach of stein's method. *Science China Mathematics*, 59(12):2379–2392, Dec 2016. ISSN 1869-1862. doi: 10.1007/s11425-016-0322-3. URL `https://doi.org/10.1007/s11425-016-0322-3`.