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CS 109

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Calculation Reference

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1   Useful identities related to summations

Since it may have been a while since some folks have worked with summations, I just wanted to provide a reference on them that you may find useful in your future work. Here are some useful identities and rules related to working with summations. In the rules below, f and g are arbitrary real-valued functions.

Pulling a constant out of a summation:

$$\sum_{n=s}^{t} C \cdot f(n) = C \cdot \sum_{n=s}^{t} f(n), \text{ where } C \text{ is a constant.}$$

Eliminating the summation by summing over the elements:

$$\sum_{i=1}^{n} x = nx$$

$$\sum_{i=m}^{n} x = (n - m + 1)x$$

$$\sum_{i=s}^{n} f(C) = (n - s + 1)f(C), \text{ where } C \text{ is a constant.}$$

Combining related summations:

$$\sum_{n=s}^{j} f(n) + \sum_{n=j+1}^{t} f(n) = \sum_{n=s}^{t} f(n)$$

$$\sum_{n=s}^{t} f(n) + \sum_{n=s}^{t} g(n) = \sum_{n=s}^{t} [f(n) + g(n)]$$

Changing the bounds on the summation:

$$\sum_{n=s}^{t} f(n) = \sum_{n=s+p}^{t+p} f(n - p)$$

"Reversing" the order of the summation:

$$\sum_{n=a}^{b} f(n) = \sum_{n=b}^{a} f(n)$$

Arithmetic series:

$$\sum_{i=0}^{n} i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \text{(with a moment of silence for C. F. Gauss.)}$$

$$\sum_{i=m}^{n} i = \frac{(n-m+1)(n+m)}{2}$$

Arithmetic series involving higher order polynomials:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} = \left[\sum_{i=1}^{n}\right]^2$$

Geometric series:

$$\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$$

$$\sum_{i=m}^{n} x^i = \frac{x^{n+1}-x^m}{x-1}$$

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \text{ if } |x| < 1$$

More exotic geometric series:

$$\sum_{i=0}^{n} i2^i = 2 + 2^{n+1}(n-1)$$

$$\sum_{i=0}^{n} \frac{i}{2^i} = \frac{2^{n+1}-n-2}{2^n}$$

Taylor expansion of exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Binomial coefficient:

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Much more information on binomial coefficients is available in the Ross textbook.

2   Growth rates of summations

Besides solving a summation explicitly, it is also worthwhile to know some general growth rates on sums, so you can (tightly) bound a sum if you are trying to prove something in the big-Oh/Theta world. If you're not familiar with big-Theta (Θ) notation, you can think of it like big-Oh notation, but it actually provides a "tight" bound. Namely, big-Theta means that the function grows no more quickly and no more slowly than the function specified, up to constant factors, so it's actually more informative than big-Oh.

Here are some useful bounds:

$$\sum_{i=1}^{n} i^c = \Theta(n^{c+1}), \text{ for } c \geq 0.$$

$$\sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n)$$

$$\sum_{i=1}^{n} c^i = \Theta(c^n), \text{ for } c \geq 2.$$

3   A few identities related to products

Recall that the mathematical symbol Π represents a product of terms (analogous to Σ representing a sum of terms). Below, we give some useful identities related to products.

Definition of factorial:

$$\prod_{i=1}^{n} i = n!$$

Note that $0! = 1$ by definition.

Stirling's approximation for $n!$ is given below. This approximation is useful when computing $n!$ for large values of n (particularly when $n > 30$).

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n, \text{ or equivalently } n! \approx \sqrt{2\pi} n^{\left(n+\frac{1}{2}\right)} e^{-n}$$

Eliminating the product by multiplying over the elements:

$$\prod_{i=1}^{n} C = C^n, \text{ where } C \text{ is a constant.}$$

Combining products:

$$\prod_{i=1}^{n} f(i) \prod_{i=1}^{n} g(i) = \prod_{i=1}^{n} f(i) \cdot g(i)$$

Turning products into summations (by taking logarithms, assuming $f(i) > 0$ for all i):

$$\log\left(\prod_{i=1}^{n} f(i)\right) = \sum_{i=1}^{n} \log f(i)$$

4   Suggestions for computing permutations and combinations

For your problem set solutions it is fine for your answers to include factorials, exponentials, or combinations; you don't need to calculate those all out to get a single numeric answer. However, if you'd like to work with those in Python, R, or Microsoft Excel, here are a few functions you may find useful.

In Python:

`math.factorial(n)`	computes $n!$
`scipy.special.binom(n, m)`	computes $\binom{n}{m}$ (as a `float`)
`math.exp(n)`	computes e^n
`n ** m`	computes n^m

Names to the left of the dots (`.`) are modules that need to be imported before being used: `import math, scipy.special`.

In R:

`factorial(n)`	computes $n!$
`choose(n, m)`	computes $\binom{n}{m}$
`exp(n)`	computes e^n
`n^m`	computes n^m

In Microsoft Excel:

`FACT(n)`	computes $n!$
`COMBIN(n, m)`	$\binom{n}{m}$
`EXP(n)`	computes e^n
`POWER(n, m)`	computes computes n^m

To use functions in Excel, you need to set a cell to equal a function value. For example, to compute $3! \cdot \binom{5}{2}$, you would put the following in a cell:

```
= FACT(3) * COMBIN(5, 2)
```

Note the equals sign (=) at the beginning of the expression.

5 A little review of calculus

Since it may have been a while since you did calculus, here are a few rules that you might find useful. Note that you do not need to be able to solve integrals for this class. You only need to be able to set them up.

Product Rule for derivatives:

$$d(u \cdot v) = du \cdot v + u \cdot dv$$

Derivative of exponential function:

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}$$

Integral of exponential function:

$$\int e^u du = e^u$$

Derivative of natural logarithm:

$$\frac{d}{dx} \ln(x) = \frac{1}{x}$$

Integral of 1/x:

$$\int \frac{1}{x} dx = \ln(x)$$

Integration by parts (everyone's favorite!):

Choose a suitable u and dv to decompose the integral of interest:

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

Here's the underlying rule that integration by parts is derived from:

$$\int d(u \cdot v) = u \cdot v = \int du \cdot v + \int u \cdot dv$$

6 Bibliography

Additional information on sums and products can generally be found in a good calculus or discrete mathematics book. The discussion of summations above is based on Wikipedia (`http://en.wikipedia.org/wiki/Summation`).