Noah Arthurs   Lecture Notes #1
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Counting

Based on a handout by Mehran Sahami

Although you may have thought you had a pretty good grasp on the notion of counting at the age of three, it turns out that you had to wait until now to learn how to really count. Aren't you glad you took this class now?! But seriously, below we present some properties related to counting which you may find helpful in the future.

Counting is important in the world of computer science for a few reasons. In order to understand probability on a fundamental level, it is useful to first understand counting. Moreover, while computers are fast, some problems require so much work that they would take an unreasonable amount of time to complete. We can use counting theory to reason about complexity.

1   Sum Rule

Sum Rule of Counting: If the outcome of an experiment can either be one of m outcomes or one of n outcomes, where none of the outcomes in the set of m outcomes is the same as the any of the outcomes in the set of n outcomes, then there are $m + n$ possible outcomes of the experiment.

Rewritten using set notation, the Sum Rule states that if the outcomes of an experiment can either be drawn from set A or set B, where $|A| = m$ and $|B| = n$, and $A \cap B = \emptyset$, then the number of outcomes of the experiment is $|A| + |B| = m + n$.

1.1   Example 1

Problem: You are running an on-line social networking application which has its distributed servers housed in two different data centers, one in San Francisco and the other in Boston. The San Francisco data center has 100 servers in it and the Boston data center has 50 servers in it. If a server request is sent to the application, how large is the set of servers it may get routed to?

Solution: Since the request can be sent to either of the two data centers and none of the machines in either data center are the same, the Sum Rule of Counting applies. Using this rule, we know that the request could potentially be routed to any of the 150 (= 100 + 50) servers.

2 Product Rule

Product Rule of Counting: If an experiment has two parts, where the first part can result in one of m outcomes and the second part can result in one of n outcomes regardless of the outcome of the first part, then the total number of outcomes for the experiment is mn.

Rewritten using set notation, the Product Rule states that if an experiment with two parts has an outcome from set A in the first part, where $|A| = m$, and an outcome from set B in the second part (regardless of the outcome of the first part), where $|B| = n$, then the total number of outcomes of the experiment is $|A||B| = mn$.

As we saw in class, this generalizes to an arbitrary number of parts:

If an experiment has r parts such that part i has n_i outcomes for all $i = 1, \ldots, n$, then the total number of outcomes for the experiment is $\prod_{i=1}^{r} n_i = n_1 \times n_2 \times \cdots \times n_r$.

2.1 Example 2

Problem: Two 6-sided dice, with faces numbered 1 through 6, are rolled. How many possible outcomes of the roll are there?

Solution: Note that we are not concerned with the total value of the two dice, but rather the set of all explicit outcomes of the rolls. Since the first die[1] can come up with 6 possible values and the second die similarly can have 6 possible values (regardless of what appeared on the first die), the total number of potential outcomes is 36 (= 6 * 6). These possible outcomes are explicitly listed below as a series of pairs, denoting the values rolled on the pair of dice:

$$
\begin{array}{cccccc}
(1, 1) & (1, 2) & (1, 3) & (1, 4) & (1, 5) & (1, 6) \\
(2, 1) & (2, 2) & (2, 3) & (2, 4) & (2, 5) & (2, 6) \\
(3, 1) & (3, 2) & (3, 3) & (3, 4) & (3, 5) & (3, 6) \\
(4, 1) & (4, 2) & (4, 3) & (4, 4) & (4, 5) & (4, 6) \\
(5, 1) & (5, 2) & (5, 3) & (5, 4) & (5, 5) & (5, 6) \\
(6, 1) & (6, 2) & (6, 3) & (6, 4) & (6, 5) & (6, 6)
\end{array}
$$

2.2 Example 3

Problem: Consider a hash table with 100 buckets. Two arbitrary strings are independently hashed and added to the table. How many possible ways are there for the strings to be stored in the table?

Solution: Each string can be hashed to one of 100 buckets. Since the results of hashing the first string do not impact the hash of the second, there are 100 * 100 = 10,000 ways that the two strings may be stored in the hash table.

2.3 Example 4

Problem: California license plates prior to 1982 had only 6-place license plates, where the first three places were uppercase letters A-Z, and the last three places were numeric 0-9. How many such 6-place license plates were possible pre-1982?

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[1]"die" is the singular form of the word "dice" (which is the plural form).

Solution: We can treat each of the positions of the license plate as a separate part of an overall six-part experiment. That is, the first three parts of the experiment each have 26 outcomes, corresponding to the letters $A - Z$, and the last three parts of the experiment each have 10 outcomes, corresponding to the digits $0 - 9$. By the General Principle of Counting, we have $26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000$ possible license plates.

Interestingly enough the current population of California is 39.5 million residents as of 2017, so this would not nearly be enough license plates such that each person can own one vehicle. Fortunately, in 1982, California changed to 7-place license plates by prepending a numeric digit, resulting in $10 \times 26 \times 26 \times 26 \times 10 \times 10 \times 10 = 175,760,000$ possible 7-place license plates. This is enough for each resident in California to own approximately 4.5 vehicles.

2.4 *Example 5: Unique configurations of Go*

The number of atoms in the observable universe is about 10 to the 80th power (10^{80}). This measure is frequently used by computer scientists as a canonical really big number. There certainly are a lot of atoms in the universe. As a leading expert said,

"Space is big. Really big. You just won't believe how vastly, hugely, mind-bogglingly big it is. I mean, you may think it's a long way down the road to the chemist, but that's just peanuts to space."
- Douglas Adams

This number is often used to demonstrate tasks that computers will never be able to solve. Problems can quickly grow to such an absurd size through the product rule of counting. For example, let's say we wanted to write an AI algorithm to play the game of Go, and we need to store each possible board configuration. How many boards might we have to store?

A Go board has 19×19 points where a user can place a stone. Each of the points can be in one of three states: empty, occupied by black or occupied by white. By the product rule of counting, we can compute the number of unique board configurations. Each board point is a unique choice where you can decide to have one of the three options in the set Black, White, No Stone so there are $3^{(19 \times 19)} \approx 10^{172}$ possible board configurations. It turns out "only" about 10^{170} of those positions are legal. That is about the square of the number of atoms in the universe. In other-words: if there was another universe of atoms for every single atom, only then would there be as many atoms in the universe as there are unique configurations of a Go board. Not even the snazziest datastructure can hold that many configurations.

3 The Inclusion-Exclusion Principle

Inclusion-Exclusion Principle: If the outcome of an experiment can either be drawn from set A or set B, and sets A and B may potentially overlap (i.e., it is not guaranteed that $A \cap B = \emptyset$), then the number of outcomes of the experiment is $|A \cup B| = |A| + |B| - |A \cap B|$.

Note that the Inclusion-Exclusion Principle generalizes the Sum Rule of Counting for arbitrary sets A and B. In the case where $A \cap B = \emptyset$, the Inclusion-Exclusion Principle gives the same result as the Sum Rule of Counting since $|\emptyset| = 0$.

The Inclusion-Exclusion principle helps to make sure we aren't counting any element more than once. If you over-count, then you have to subtract off the number of elements that were double counted.

3.1 Example 6

Problem: An 8-bit string (one byte) is sent over a network. The valid set of strings recognized by the receiver must either start with 01 or end with 10. How many such strings are there?

Solution: The potential bit strings that match the receiver's criteria can either be the 64 strings that start with 01 (since that last 6 bits are left unspecified, allowing for $2^6 = 64$ possibilities) or the 64 strings that end with 10 (since the first 6 bits are unspecified). Of course, these two sets overlap, since strings that start with 01 and end with 10 are in both sets. There are $2^4 = 16$ such strings (since the middle 4 bits can be arbitrary). Casting this description into corresponding set notation, we have: $|A| = 64$, $|B| = 64$, and $|A \cap B| = 16$, so by the Inclusion-Exclusion Principle, there are $64 + 64 - 16 = 112$ strings that match the specified receiver's criteria.