Linear Algebra

Vector Space

- **linear combination**: A ***linear combination*** of a list $v_1, \dots, v_m$ of vectors in V is a vector of the form
  $$a_1 v_1 + \cdots + a_m v_m,$$
  where $a_1, \dots, a_m \in \mathbf{F}$.
- **span**: The set of all linear combinations of a list of vectors $v_1, \dots, v_m$ in V is called the ***span*** of $v_1, \dots, v_m$, denoted $\operatorname{span}(v_1, \dots, v_m)$. In other words,
  $$\operatorname{span}(v_1, \dots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \dots, a_m \in \mathbf{F}\}.$$
  The span of the empty list $(\,)$ is defined to be $\{0\}$.
- **spans**: If $\operatorname{span}(v_1, \dots, v_m)$ equals V, we say that $v_1, \dots, v_m$ ***spans*** V.
- **finite-dimensional vector space**: A vector space is called ***finite-dimensional*** if some list of vectors in it spans the space.
  - 子主题
- **$\mathcal{P}_m(\mathbf{F})$**: For m a nonnegative integer, $\mathcal{P}_m(\mathbf{F})$ denotes the set of all polynomials with coefficients in $\mathbf{F}$ and degree at most m.
- **infinite-dimensional vector space**:
  - A vector space is called ***infinite-dimensional*** if it is not finite-dimensional.
  - $\mathcal{P}(\mathbf{F})$ is infinite-dimensional.
- **linearly independent**: A list $v_1, \dots, v_m$ of vectors in V is called ***linearly independent*** if the only choice of $a_1, \dots, a_m \in \mathbf{F}$ that makes $a_1 v_1 + \cdots + a_m v_m$ equal 0 is $a_1 = \cdots = a_m = 0$.
- **basis**: A ***basis*** of V is a list of vectors in V that is linearly independent and spans V.
- **dimension**:
  - The ***dimension*** of a finite-dimensional vector space is the length of any basis of the vector space.
  - $\dim \mathbf{F}^n = n$ because the standard basis of $\mathbf{F}^n$ has length n.
    $\dim \mathcal{P}_m(\mathbf{F}) = m + 1$ because the basis $1, z, \dots, z^m$ of $\mathcal{P}_m(\mathbf{F})$ has length $m + 1$.

Linear Maps

- **linear map**: A ***linear map*** from V to W is a function $T : V \to W$ with the following properties:
  additivity
  $T(u + v) = Tu + Tv$ for all $u, v \in V$;
  homogeneity
  $T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbf{F}$ and all $v \in V$.
- **$\mathcal{L}(V, W)$**: The set of all linear maps from V to W is denoted $\mathcal{L}(V, W)$.
- **null space**: For $T \in \mathcal{L}(V, W)$, the ***null space*** of T, denoted $\operatorname{null} T$, is the subset of V consisting of those vectors that T maps to 0:
  $$\operatorname{null} T = \{v \in V : Tv = 0\}.$$
- **range**: For T a function from V to W, the ***range*** of T is the subset of W consisting of those vectors that are of the form Tv for some $v \in V$:
  $$\operatorname{range} T = \{Tv : v \in V\}.$$
- **row rank, column rank**:
  - The ***row rank*** of A is the dimension of the span of the rows of A in $\mathbf{F}^{1,n}$.
  - The ***column rank*** of A is the dimension of the span of the columns of A in $\mathbf{F}^{m,1}$.
- **rank**: The ***rank*** of a matrix $A \in \mathbf{F}^{m,n}$ is the column rank of A.

Eigenvalues, Eigenvectors, and Invariant Subspaces

- **eigenvalue**: Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbf{F}$ is called an ***eigenvalue*** of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.
- **eigenvector**: Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$ is an eigenvalue of T. A vector $v \in V$ is called an ***eigenvector*** of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.
- **eigenspace**: Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. The ***eigenspace*** of T corresponding to λ, denoted $E(\lambda, T)$, is defined by
  $$E(\lambda, T) = \operatorname{null}(T - \lambda I).$$
  In other words, $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along with the 0 vector.