UNIVERSITY OF MICHIGAN-DEARBORN

Chapter 4 : Algorithms

Recursive Algorithms

Recursive Algorithms

A *recursive algorithm* is an algorithm that contains a recursive function. Recursive is a powerful, elegant and natural way to solve a large class of problems.

Recursive Algorithms

- **Definition :** The *Fibonacci sequence* $\{ f_n \}$ is defined by the equations

$$f_0 = 1$$

$$f_1 = 1$$

$$f_2 = 1$$

$$f_n = f_{n-1} + f_{n-2} \qquad \text{for all } n \geq 3$$

The Fibonacci sequence begins

1, 1, 2, 3, 5, 8, 13, …

In mathematical terms, the sequence F_n of Fibonacci numbers is defined by the recursive relation $\mathbf{f_n = f_{n-1} + f_{n-2}}$

By definition, the first two numbers in the Fibonacci sequence are either 1 and 1, or 0 and 1, depending on the chosen starting point of the sequence, and each subsequent number is the sum of the previous two numbers.

Recursive Algorithms

- **Exercise 1:** Use the mathematical induction to show that

$$\sum_{k=1}^{n} f_k = f_{n+2} - 1$$

for all $n \geq 1$

Recursive Algorithms

$$\sum_{k=1}^{n} f_k = f_{n+2} - 1$$

for all n ≥ 1

1. Basic step (n=1):

We must show that

$$\sum_{k=1}^{1} f_k = f_3 - 1$$

Since $\sum_{k=1}^{1} f_k = f_1 = 1$ and $f_3 - 1 = 2 - 1 = 1$, the equation is verified.

2. Inductive step : We assume the statement is true and we must prove case n+1

$$\sum_{k=1}^{n+1} f_k = f_{n+3} - 1$$

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Now
$$\sum_{k=1}^{n+1} f_k = \sum_{k=1}^{n} f_k + f_{n+1}$$

$$= (f_{n+2} - 1) + f_{n+1}$$ by the induction assumption

$$= f_{n+2} + f_{n+1} - 1$$

$$= f_{n+3} - 1$$

The last equality is true because of the definition of the Fibonacci numbers:

$$f_n = f_{n-1} - f_{n-2}$$ for all n ≥ 3

Since the basic step and the inductive step have been verified, the given equation is true for all n ≥ 1.

Exercises

Exercise 2: Use the mathematical induction to show that

$$f_n^2 = f_{n-1} f_{n+1} + (-1)^{n+1} \quad \text{for all } n \geq 2$$

Exercises

$$f_n^{\,2} = f_{n-1}\, f_{n+1} + (-1)^{n+1} \qquad \text{for all } n \geq 2$$

a) Basic step: (n = 2)

$$f_2^2 = 1 = 1 \times 2 - 1 = f_1 \; f_3 + (-1)^3$$

b) Inductive step: Assume that the statement is true.

$$f_n\, f_{n+2} + (-1)^{n+2} = f_n\, (\, f_{n+1} + f_n\,) + (-1)^{n+2}$$

$$= f_n\; f_{n+1} + f_n^{\,2} + (-1)^{n+2}$$

$$= f_n\; f_{n+1} + (f_{n-1}\, f_{n+1} + (-1)^{n+1}\,) + (-1)^{n+2}$$

$$= f_{n+1}\, (\, f_n + f_{n-1}\,) = f_{n+1}^{\;2}$$

We can conclude that the statement is true.

Exercises

Exercise 3: Use the mathematical induction to show that

$$\sum_{k=1}^{n} f_k^2 = f_n f_{n+1}$$

for all n ≥ 1

Exercises

a) Basic step: (n = 1)

$$f_1^2 = 1^2 = 1 = 1 \times 1 = f_1 \times f_2$$

b) inductive step:

$$\sum_{k=1}^{n+1} f_k^2 = \sum_{k=1}^{n} f_k^2 + f_{n+1}^2 = f_n f_{n+1} + f_{n+1}^2 = f_{n+1} (f_n + f_{n+1}) = f_{n+1} f_{n+2}$$

We can conclude that the statement is true.

Exercises

Exercise 4: Let consider that

$$1 + 2 + \ \ldots + n = An^2 + Bn + C$$

For all n, and for some constant A, B and C.

1) Assuming that this is true, plug in n = 1, 2, 3 to obtain three equations in the three unknowns A, B and C.
2) Solve for A, B and C with the three equations obtained in the previous question.
3) Prove using the mathematical induction that the statement is true.

Exercises

Exercise 4 :

_Let consider that

$1 + 2 + \ldots + n = An^2 + Bn + C$

For all n, and for some constant A, B and C.

1) Assuming that this is true, plug in n = 1, 2, 3 to obtain three equations in the three unknowns A, B and C.

When n = 1, we obtain

$1 = A + B + C$

When n = 2, we obtain

$3 = 4A + 2B + C$

When n = 3, we obtain

$6 = 9A + 3B + C$

Exercises

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2) Solve for A, B and C with the three equations obtained in the previous question.

Solving this system for A, B, C. We obtain

A = B = ½, C = 0

We obtain this formula $1 + 2 + \ldots + n = n^2 2 + n 2 + 0 = n\,(n+1) 2$

3) Prove using the mathematical induction that the statement is true.

We must show that for all n, if equation n is true : Sn = n(n+1)/2 then, equation n+1 is also true. Sn+1= (n+1)(n+2)/2

Basis Step: S(1) : 1 =1(2)/2 = 1 is true

Inductive Step: If S(n) = n(n+1)/2 is true.

S(n+1) = n(n+1)/2 + (n+1)

= {n(n+1) + 2(n+1)}/2

= (n^2 + n) + 2n +2) /2

= (n^2 + 3n +2)/2

= {(n+1) (n+2)}/2

So, S(n) is true for every positive integer n.