UNIVERSITY OF MICHIGAN-DEARBORN

Chapter 6 : Counting Methods

Introduction

What is Counting Methods?

- In many discrete problems, we are confronted with the problem of counting.
- In the chapter 4, we saw that in order to estimate the run time of an algorithm, we needed to count the number of times certain steps or loops were executed. *Counting* also plays a crucial role in probability theory.
- In this chapter, we develop several tools for counting. These techniques can be used to derive *the binomial theorem.*

UNIVERSITY OF MICHIGAN-DEARBORN

Basic Principles

Counting Methods

- **<u>Definition of the Multiplication Principle:</u>** If an activity can be constructed in t successive steps and step 1 can be done in n1 ways, steps 2 can then be done in n2 ways, …, and step t can then be done in nt ways, then the number of different possible activities is **n1 × n2 × …. × nt.**

- **<u>Example:</u>** The menu for Kay's Quick lunch is shown in this table.

Appetizers	
Nachos	**2.15**
Salad	**1.90**
Main courses	
Hamburger	**3.25**
Cheeseburger	**3.65**
Fish filet	**3.15**
beverages	
Tea	**0.70**
Milk	**0.85**
Cola	**0.75**
Root Beer	**0.75**

As you can see, it features two appetizers, three main courses, and four beverages.

How many different dinners consist of one main course and one beverage?

Counting Methods

- If we list all possible dinners consisting of one main course and one beverage, we obtain :

Hamburger/tea, Hamburger/Milk, Hamburger/ Cola, Hamburger/Root Beer,

Cheeseburger/Tea, Cheeseburger/Milk, Cheeseburger/Cola, Cheeseburger/Root Beer,

Fish Filet/ Tea, Fish Filet/ Milk, Fish Filet/ Cola, Fish Filet/ Root beer

- There are three main courses and four Beverages, we define

The total number of dinners = $4 \times 3 = 12$

- In this example, we found that the total number of dinners was equal to the product of numbers of each of the courses.

This example illustrate the **Multiplication Principle**.

We may summarize the Multiplication Principle by saying that we multiply together the numbers of ways of doing each step when an activity is constructed in successive steps.

Counting Methods

DEARBORN

- **Theorem:** According to the Multiplication Principle that a set with n elements has 2^n subsets.
- ***Proof:*** Use the Multiplication Principle to show that a set {X1, …, Xn} containing elements has 2^n subsets.

A subset can be constructed in n successive steps: pick or do not pick X1; pick or do not pick X2, …, pick or do not pick Xn. Each step can be done in two ways. Thus the number of possible subsets is

$$\underbrace{2 \times 2 \times \ldots \times 2}_{\text{n factors}} = \mathbf{2^n}$$

- **.Example :** In a digital picture, we wish to encode the amount of light at each point as

an eight-bit string. *How many values are possible at one point?*

An *eight-bit encoding* can be constructed in eight successive steps: Select the first bit; select the second bit; ….; select the eight bit. Since there are two ways to select each bit, by the Multiplication Principle the total number of eight-bit encodings is

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = \mathbf{2^8 = 256}$$

Counting Methods

DEARBORN

- **Definition of the Addition Principle:** Suppose that X1, …, Xt are sets and that the ith set Xi has ni elements. If {X1, …, Xt} is a pairwise disjoint family (if i ≠ j, Xi Π Xj = ϕ), , the number of possible elements that can be selected from X1 or X2 or … or Xt is

n1 + n2 + … + nt

(Equivalently, the union X1 U X2 U … U Xt contains n1 + n2 + … + nt elements)

- **Example 1:** In how many ways can we select two books from different subjects among five distinct computer science books, three distinct mathematics books, and two distinct art books?

Using the Multiplication principle, we find that we can select two books:

1) One from the computer science and one from mathematics, in 5 x 3 = 15 ways
2) One from the computer science and one from art, in 5 x 2 = 10 ways
3) One from mathematics and one from art, in 3 x 2 = 6 ways

Since these sets of selections are pairwise disjoint, we may use the Addition Principle to conclude that there are are

15 + 10 + 6 = 31

ways of selecting two books from different subjects

among the computer science, mathematics and arts books.

Counting Methods

- **Example 2:** A six person committee composed of Alice, Ben, Connie, Dolph, Egbert, Francisco is to select a chairperson, secretary, and treasurer.

1) How many ways can this be done ?

We use the Multiplication Principle. The officers can be selected in three successive step: select the chairperson; select the secretary; select the treasurer. The chairperson can be selected in six ways. Once the chairperson has been selected, the secretary can be selected in five ways. After the selection of the chairperson and the secretary, the treasurer can be selected in four ways. Therefore, the total number of possibilities is **6 x 5 x 4 = 120**.

2) How many ways can this be done if Alice or Ben must be the chair?

If Alice is the chairperson, we have *5 x 4 = 20* ways to select the remaining officers. Similarly, if Ben is the chairperson, there are 20 ways to select the remaining officers. Since these cases are disjoint, by the Addition Principle, there are

20 + 20 = 40 possibilities.

3) In how many ways can this be done if Egbert must hold one of the offices ?

If Edge is the chairperson, we have *5 x 4 = 20* ways to select the remaining officers. Similarly, if Edge is secretary, there are 20 possibilities, and if Edge is treasurer, there are 20 possibilities. Since these three are pairwise disjoint, by the Addition Principle, there are **20 + 20 + 20 = 60** possibilities.

4) In how many ways can this be done if both Dolph and Francisco must hold office ?

If the activity of assigning Dolph, Francisco, and one other person to offices to be made up of three successive steps; Assign Dolph; assign Francisco; fill the remaining office. There are three ways to assign Dolph. One Dolph has been assigned, there are two ways to assign Francisco. One Dolph and Francisco have been assigned, there are four ways to fill the remaining office. By the Multiplication Principle, there are **3 x 2 x 4 = 24** possibilities.

Counting Methods

DEARBORN

- **Definition:** The *Inclusion-Exclusion Principle* generalizes the Addition Principle by giving a formula to compute the number of elements in a union without requiring the sets to be pairwise disjoint.

- **Theorem :** If X and Y are finite sets, then

$$|X \cup Y| = |X| + |Y| - |X \cap Y|$$

- **Proof:** Suppose that we want to count the number of eight-bit strings that start 10 or end 011 or both. Let X denote the set of eight-bit strings that start 10 and Y. We cannot use the Addition Principle and add | X | and | Y | to compute |X U Y| because the Addition Principle requires X and Y to be disjoint. Here X and Y are not disjoint; for example 10111011 $\in X \cap Y$.

Counting Methods

DEARBORN

- **Example :** A six person committee composed of Alice, Ben, Connie, Dolph, Egbert, Francisco is to select a chairperson, secretary, and treasurer.

How many selections are there in which either Alice or Dolph or both are officers?

Let X denote the set of selections in which Alice is an officer and let Y denote the set of selections in which Dolph is an officer. We must compute $|X \cup Y|$. Since X and Y are not disjoint (both Alice and Dolph could be officers), we cannot use the Addition Principle.

Instead, we use *the Inclusion- exclusion Principle*.

The number of selections in which Alice is an officer is $3 \times 5 \times 4 = 60$, that is $|X| = 60$. Similarly, the number of selections in which Dolph is an officer is $3 \times 5 \times 4 = 60$, that is $|Y| = 60$.

Now $X \cap Y$ is the set of selections in which both Alice and Dolph are officers. Thus the number of selections in which both Alice and Dolph are officers is $3 \times 2 \times 4 = 24$, that is, $|X \cap Y| = 24$.

The Inclusion-Exclusion Principle tells us that

$$|X \cup Y| = |X| + |Y| - |X \cap Y| = 60 + 60 - 24 = 96$$

Thus there are 96 selections in which either Alice or Dolph or both are officers.

Exercises

DEARBORN

Exercise 1: Use the Multiplication Principle:

The menu for Kay's Quick lunch is shown in this table.

Appetizers		
	Nachos	2.15
	Salad	1.90
Main courses		
	Hamburger	3.25
	Cheeseburger	3.65
	Fish filet	3.15
beverages		
	Tea	0.70
	Milk	0.85
	Cola	0.75
	Root Beer	0.75

How many dinners at Kay's Quick Lunch consist of one appetizer and one beverage?

Exercise 2: Use the Multiplication Principle: A man has eight shirts, four pairs of pants, and five pairs of shoes. How many different outfits are possible?

Exercise 3: **Use the Multiplication Principle:** Two dice are rolled, one blue and one red. How many outcomes are possible?

Exercises

Exercise 4: Use the addition Principle : Three departmental committees have 6, 12 and 9 members with no overlapping membership. In how many ways can these committees send one member to meet with the president?

Exercise 5: Two dices are rolled, one blue and one red.

① How many outcomes give the sum of 4?

② How many outcomes have the blue die showing 2?

③ How many outcomes have neither die showing 2?

Exercise 6: In group of 191 students, 10 are taking French, business, and music; 36 are taking French and Business; 20 are taking French and music; 18 are taking business and music; 65 are taking French; 76 are taking business; and 63 are taking music.

Use the inclusion-Exclusion Principle for three finite sets to determine how many students are not taking any of the three courses.

UNIVERSITY OF MICHIGAN-DEARBORN

Permutations and Combinations

Counting Methods

- **Definition:** A *permutation* is a selection of objects without regard to order.

A *permutation* of n distinct elements X1, …, Xn is an ordering of the n elements X1, …, Xn.

- **Example:** There are six permutations of three elements. If the elements are denoted A, B, C, the

six permutations are ABC, ACB, BAC, BCA, CAB, CBA

- **Theorem :** There are n! permutations of n elements.

By the Multiplication Principle, there are $n\,(n-1)\,(n-2)\ldots 2 \times 1 = n!$ Permutations of n elements.

- **Proof of the theorem for n = 4:** We found that there are 24 ways to order four candidates on a ballot; thus there are 24 permutations of four objects. The method that we used to count the number of distinct ballots containing four names may be used to derive a formula for the number of permutations of n elements. The proof of the following theorem for n = 4 :

$$B \longrightarrow A \longrightarrow D \longrightarrow C$$

Select 1^{st} element Select 2^{nd} element Select 3^{rd} element Select 4^{th} element

A permutation of ABCD is constructed by successively selecting the first element, then the second element, then the third element, and, finally, the fourth element.

Counting Methods

- **Example 1:** There are

  $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3{,}628{,}800$ permutations of 10 elements.

- **Example 2:** How many permutations of the letters ABCDEF contain the substring DEF?

To guarantee the presence of the pattern DEF in the substring, these three letters must be kept together in this order. The remaining letters : A, B, C can be placed arbitrarily.

DEF	A	B	C

By to the previous theorem, there are 4! Permutations of 4 objects. Thus the number of permutations of the letters ABCDEF that contain the substring DEF is

$4! = 24.$

Counting Methods

- **Definition :** A *r-permutation* of n distinct elements X1, …, Xn is an ordering of an r-element subset of {X1, …, Xn}. The number of r-permutations of a set of n distinct elements is denoted ***P(n, r)***.

- **Theorem :** $P(n, r) = n (n - 1) (n - 2) .. (n – r + 1) = n!/(n-r)!$, where $r <= n$.

  $P(n, n) = n!$

- **Example:** In how many ways can we select a chairperson, vice-chairperson, secretary, and treasurer from a group of 10 persons?

We need to count the number of orderings of four persons selected from a group of 10, since an ordering picks a chairperson as first pick, a vice-chairperson as a second pick, a secretary as a third pick, and a treasurer as a fourth pick.

By this theorem, the solution is

$P(10, 4) = 10 \times 9 \times 8 \times 7 = 5040$ **or** $P(10, 4) = 10!/(10 – 4)! = 10! / 6! = 5040$

Counting Methods

DEARBORN

- **Definition:** A combination is a selection of objects without regard to order.

Given a set X = {X1, …, Xn} containing n (distinct) elements,

a) An r-combination of X is an unordered selection of r-elements of X.

b) The number of r-combinations of a set of n distinct elements is denoted **C(n, r)** or $\binom{n}{r}$

- **Theorem :** C(n, r) = P(n, r)/r! = $\frac{n!}{r!(n-r)!}$, where r <= n.

- **Example 1:** In how many ways can we select a committee of three from a group of 10 distinct persons?

Since a committee is an unordered group of people, the answer is

C(10, 3) = P(10, 3) / 3! = (10 x 9 x 8) / 3! = 120

Counting Methods

- **Example 2:** In how many ways can we select a committee of two women and three men from a group of five distinct women and six distinct men?

Since a committee is an ordered group of people, we find the two women can be selected in

$$C(5, 2) = P(5, 2) / 2! = 10 \text{ ways}$$

Since a committee is an ordered group of people, we find the three men can be selected in

$$C(6, 3) = P(6, 3) / 3! = 20 \text{ ways}$$

The committee can be constructed in two successive steps: Select the women; select the men. By the Multiplication Principle, the total number of committees is

10 x 20 = 200.

Example 3: How many eight-bit strings contain exactly four 1's?

An eight-bit string containing four 1's is uniquely determined once we tell which bits are 1. Thus can be done in

$$C(8, 4) = P(8, 4)/4! = 70 \text{ ways.}$$

Exercises

Exercise 1: How many permutations are there of a. b, c, d?

Exercise 2: In how many ways can we select a chairperson, vice-chairperson, and recorder from a group of 11 persons?

Exercise 3: Determine how many strings can be formed by ordering the letters ABCDE :

① Contains the substring ACE

② Contains either the substring AE or the substring EA or both.

Exercise 4: In how many ways can five distinct Martians and eight distinct Jovians wait in line if no two Martians stand together?

Exercises

DEARBORN

Exercise 4: Suppose that a pizza parlor features four specialty pizzas and pizzas with three or fewer unique toppings (no choosing anchovies twice!) chosen from 17 toppings. How many different pizzas are there?

Exercise 5: Refer to a club consisting of six distinct men and seven distinct women.

In how ways can we select a committee of five persons?

Exercise 6: Find the number of unordered five card poked hands, select from an ordinary 52-card deck, having the properties indicated.

① Containing four aces

② Find the number of unordered 13 card bridge hands selected from an ordinary 52 card deck.

UNIVERSITY OF MICHIGAN-DEARBORN

Introduction to discrete Probability

Counting Methods

- *Probability theory* is the branch of mathematics concerned with :
  - *Probability*
  - *The analysis of random phenomena*

- ***Discrete probability theory*** deals with events that occur in countable sample spaces.
- *Examples of Discrete probability :*
  - Throwing dice
  - Experiments with decks of cards
  - Random walk
  - Tossing coins.

Counting Methods

- **Definition:** *Discrete probability theory* is the probability of an event to occur that was defined as number of cases favorable for the event, over the number of total outcomes possible in a sample space.

The probability P(E) of an event E from the finite sample space S is

$$P(E) = |E| / |S|$$

- **Example :** Two fair dice are rolled. What is the probability that the sum of the numbers on the dice is 10?

Since the first die can show any one of six numbers and the second die can show any one of six numbers, by the multiplication Principle there are :

6 x 6 = 36 possible sums: *the size of the sample space*.

There are three possible ways to obtain the sum of 10 : (4, 6), (5, 5), (6, 4). We can say that *the size of the event* is 3.

Therefore, the probability is **3/36 = 1/12.**

UNIVERSITY OF MICHIGAN-DEARBORN

Discrete Probability Theory

Counting Methods

- **Definition:** A *probability function P* assigns to each outcome x in a sample space S a number P(x) so that

$$0 \le P(x) \le 1, \qquad \text{for all } x \in S$$

And

$$\sum_{x \notin S} P(x) = 1$$

- **Example:** Suppose that a die is loaded so the number 2 through 6 are equally likely to appear, but that 1 is three times as likely as any other number to appear. To model this situation, we should have

$$P(2) = P(3) = P(4) = P(5) = P(6)$$

And $P(1) = 3\,P(2)$

Since $1 = P(1) + P(2) + P(3) + P(4) + P(5) + P(6) =$

$= 3\,P(2) + P(2) + P(2) + P(2) + P(2) + P(2) = 8\,P(2)$

We must have $P(2) = 1/8$. Therefore, $P(2) = P(3) = P(4) = P(5) = P(6) = 1/8$

And $P(1) = 3\,P(2) = 3/8$

Counting Methods

DEARBORN

- **Definition:** The *probability of an event E* is defined as the sum of the probabilities of the outcomes in E.

Let E be an event. The probability of E, P(E) is

$$\mathbf{P(E)} = \sum_{x \in E} P(x)$$

- **Example:** Given the assumption of the previous example, the probability of an odd number is P(1) + P(3) + P(5) = 3/8 + 1/8 + 1/8 = 5/8

- **Theorem:** Let E be an event. The probability of $\overline{E}$, the complement of E, satisfies

$$P(E) + P(\overline{E}) = 1$$

- **Example:** The probability of obtaining no defective microprocessors is 0.9037. By the previous theorem, the probability of obtaining at least one defective microprocessor is

1 – 0.9037 = 0.0963

Counting Methods

- **Theorem:** *The probability of the union of two events.*

Let E1 and E2 be events, then

$$P(E1 \cup E2) = P(E1) + P(E2) - P(E1 \cap E2)$$

- **Example:** Two fair dice are rolled. What is the probability of getting doubles (two dice showing the same number) or a sum of 6?

We let E1 denote the event "get doubles" and E2 denote the event "get a sum of 6". Since doubles can be obtained in six ways, $P(E1) = 6/36 = 1/6$

Since the sum six can be obtained in five ways [(1, 5), (2, 4), (3, 3), (5, 1), (4, 2)].

$$P(E2) = 5/36$$

The event $E1 \cap E2$ is "get doubles and get a sum of 6". Since this last event can occur only one way (by getting a pair of 3s),

$$P(E1 \cap E2) = 1/36$$

By the previous theorem, the probability of getting doubles or a sum of 6 is

$$P(E1 \cup E2) = P(E1) + P(E2) - P(E1 \cap E2) = 1/6 + 5/36 - 1/36 = 5/18$$

Counting Methods

- **Theorem:** Two events E1 and E2 are *mutually exclusive* if **$E1 \cap E2 = \phi$**, then

$$P(E1 \cup E2) = P(E1) + P(E2)$$

- **Example:** Two fair dice are rolled. What is the probability of getting doubles (two dice showing the same number) or a sum of 5?

We let E1 denote the event "get doubles" and E2 denote the event "get a sum of 5".

Notice that E1 and E2 are mutually exclusive : You cannot get doubles and the sum of 5 simultaneously. Since doubles can be obtained in six ways, $P(E1) = 6/36 = 1/6$

Since the sum of 5 can be obtained in four ways [(1, 4), (2, 3), (3, 2), (4, 1)].

$$P(E2) = 4/36 = 1/9$$

By the previous theorem, the probability of getting doubles or a sum of 6 is

$$P(E1 \cup E2) = P(E1) + P(E2) = 1/6 + 1/9 = 5/18$$

Counting Methods

- **Definition:** Let E and F be events, and assume P(F) > 0. The *conditional probability of E given F* is

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

- **Example:** Weather records show that the probability of high barometric pressure is 0.80, and the probability of rain and high barometric pressure is 0.10.

Using the previous definition, the probability of rain given high barometric pressure is

$$P(R \mid H) = \frac{P(R \cap H)}{P(H)} = \frac{0.10}{0.80} = 0.125$$

Where R denotes the event “rain”, and H denotes the event “high barometric pressure”.

Counting Methods

DEARBORN

- **Definition:** If the probability of event E does not depend on event F in the sense that P(E|F)=P(E), we say that E and F are *independent events*. By the previous definition

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

Thus if E and F are *independent events*, $P(E) = P(E \mid F) = \frac{P(E \cap F)}{P(F)}$

Or

$$P(E \cap F) = P(E)P(F)$$

- **Example 1:** Intuitively, if we flip a fair coin twice, the outcome of the second toss does not depend on the outcome of the first toss (coins have no memory). For example, if H is the event "head on first toss", and T is the event "tail on second toss", we expect that events H and T are independent. Using the previous definition to verify that H and T are indeed independent.

The event H ∩ T is the event "head on first toss and tail on second toss". Thus P(H ∩ T)=1/4.

Since P(H) = ½ = P(T), we have

$$P(H \cap T) = \frac{1}{4} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = P(H)P(T)$$

Therefore, H and T are independent.

Counting Methods

- **<u>Example 2:</u>** Joe and Alicia take a final examination in discrete mathematics. The probability that Joe passes is 0.70, and the probability that Alicia passes is 0.95. Assuming that events "Joe passes" and "Alicia passes" are independent, find the probability that Joe or Alicia, or both, passes the final exam.

We let J denote the event "Joe passes the final exam" and A denote the events "Alicia passes the final exam". We are asked to compute $P(J \cup A)$.

According to the *theorem of the union of two events*, we says that

$$P(J \cup A) = P(J) + P(A) - P(J \cap A)$$

Since we are given P(J) and P(A), we need only compute $P(J \cap A)$. Because the events J and A are independent. According to the definition of *the independent of two events*, we says that

$$P(J \cup A) = P(J)\,P(A) = (0.70)\,(0.95) = 0.665$$

Therefore,

$$\mathbf{P(J \cup A) = P(J) + P(A) - P(J \cap A) = 0.70 + 0.95 - 0.665 = 0.985}$$

Counting Methods

- **Definition:** Bayes' Theorem is useful in computing the probability of a class given a set of features.

- **Bayes' Theorem:** Suppose that the possible classes are C1, …, Cn. Suppose further that each pair of classes is mutually exclusive and each item to be classified belongs to one of the classes. For a feature set F, we have

$$P(C_j \mid F) = \frac{P(F \mid C_j)P(C_j)}{\sum_{i=1}^{n} P(F \mid C_i)P(C_i)}$$

Counting Methods

- **Example:** At the telemarking firm sellPhone, Dale, Rusty, and Lee make calls. The following table shows the percentage of call each caller makes and the percentage of persons who are annoyed and hang up on each caller:

	caller		
	Dale	Rusty	Lee
Percent of calls	40	25	35
Percent of hang-ups	20	55	30

Let D denote the event "Dale made the call", let R denote the event "Rusty made the call", let L denote the event "Lee made the call" and H denote the event "the caller hung up". Find P(D), P(R), P(L), P(H|D), P(H|R), P(H|L), P(D|H), P(R|H), P(L|H) and P(H).

Since Dale made 40% of the calls, Rusty made 25% of the calls and Lee made 35% of the calls, we obtain similarly: $P(D) = 0.4$ $P(R) = 0.25$ $P(L) = 0.35$

Since Dale, Rusty and Lee made the call, The table shows similarly for Dale, Rusty and Lee that 20%, 55% and 30% of the persons hung up, therefore, :

$P(H|D) = 0.2$ $P(H|R) = 0.55$ $P(H|L) = 0.30$

Counting Methods

To compute P(D|H), we use *Bayes' Theorem*:

$$P(D|H)=\frac{P(H|D)P(D)}{P(H|D)P(D)+P(H|R)P(R)+P(H|L)P(L)}=\frac{(0.20)(0.40)}{(0.20)(0.40)+(0.55)(0.25)+(0.30)(0.35)}=0.248$$

$$P(R|H)=\frac{P(H|R)P(R)}{P(H|D)P(D)+P(H|R)P(R)+P(H|L)P(L)}=\frac{(0.55)(0.25)}{(0.20)(0.40)+(0.55)(0.25)+(0.30)(0.35)}=0.426$$

Again using the Bayes' Theorem or noting that $P(D|H)+P(R|H)+P(L|H)=1$

We obtain $P(L|H)=0.326$

Finally, the proof of *Bayes' Theorem* shows that

$$P(H)=P(H|D)P(D)+P(H|R)P(R)+P(H|L)P(L)=(0.20)(0.40)+(0.55)(0.25)+(0.30)(0.35)=0.3225$$

Exercises

Exercise 1: A die is loaded so that the number 2 through 6 are equally likely to appear, but 1 is three times as any other number to appear. One die is rolled.

What is the probability of getting a 5?

Exercise 2: A dice that are loaded so that the number 2, 4 and 6 are equally likely to appear. 1, 3 and 5 are also equally likely to appear, but 1 is three times as likely as 2 is to appear.

1. One dice is rolled. What is the probability of not getting a 5?
2. two dice is rolled. What is the probability of getting double?

Exercise 3: Suppose that a coin is flipped and a die is rolled. Let E1 denote the event "the coin shows a tail", let E2 denote the event "the die shows a 3", let E3 denote the event "the coin shows heads and the die shows an odd number".

1. List the element of the event E1 or E2.
2. Are E1 and E3 mutually exclusive?

Exercise 4: Six microprocessors are randomly selected from 100 microprocessors among which 10 are defective. Find the probability of obtaining no defective microprocessors.

Exercises

DEARBORN

Exercise 5: A family with four children. Assume that it is equally probable for a boy or a girl to be born.

1. What is the probability of all girls?
2. What is the probability of all girls given that there is at least one girl?
3. Are the events “there are children of both sexes” and “there is at most one boy independent”?

Exercise 5: A company that buys computers from three vendors and tracks the number of defective machines. The following table shows the results.

	vendor		
	Acme	Dotcom	Nuclear
Percent purchased	55	10	35
Percent defective	1	3	3

Let A denote the event “the computer was purchased from Acme”, let D denote the event “the computer was purchased from Dotcom”, let N denote the event “the computer was purchased from Nuclear, and let B denote the event “the computer was defective”.

1. Find P(A), P(D) and P(N).
2. Find P(B|A), P(B|D) and P(B|N).
3. Find P(A|B), P(D|B) and P(N|B).
4. Find P(B)

UNIVERSITY OF MICHIGAN-DEARBORN

Binomial Coefficients and Combinatorial Identities

Counting Methods

- **Definition:** The Binomial Theorem gives a formula for the coefficients in the expansion of

$$(a+b)^n = \underbrace{(a+b)(a+b)\ldots(a+b)}_{\text{n factors}}$$

- **Binomial Theorem:** If a and b are real number and n is a positive integer, then

$$(a+b)^n = \sum_{k=0}^{n} C(n,k)a^{n-k}b^k$$

- **Example:**

$$(a+b)^3 = (a+b)(a+b)(a+b)$$
$$= aaa + aab + aba + abb + baa + bab + bba + bbb$$
$$= a^3 + a^2b + a^2b + b^2a + a^2b + b^2a + b^2a + b^3$$
$$= a^3 + 3a^2b + 3b^2a + b^3$$

Counting Methods

- **Definition:** The number C(n, r) are known as *binomial coefficients* because they appear of the binomial a + b raised to a power.

- **Example 1:** Taking n = 3 in the previous theorem, we obtain

$$(a+b)^3 = C(3,0)a^3b^0 + C(3,1)a^2b^1 + C(3,2)a^1b^2 + C(3,3)a^0b^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

- **Example 2:** Expand $(3x-2y)^4$ using the *Binomial Theorem*.

If we take a = 3x, b = -2y and n =4 in the Binomial Theorem, we obtain

$$(3x-2y)^4 = (a+b)^4 = C(4,0)a^4b^0 + C(4,1)a^3b^1 + C(4,2)a^2b^2 + C(4,3)a^1b^3 + C(4,4)a^0b^4$$

$$= C(4,0)(3x)^4(-2y)^0 + C(4,1)(3x)^3(-2y)^1 + C(4,2)(2x)^2(-3y)^2 + C(4,3)(2x)^1(-3y)^3 + C(4,4)(2x)^0(-3y)^4$$

$$= 3^4x^4 + 4\times 3^3x^3(-2y) + 6\times 3^2x^2(-2y)^2 + 4\times 3x(-2y)^3 + (-2y)^4$$

$$= 81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4$$

Counting Methods

- **Example 3:** Find the coefficient of a^5b^4 in the expansion of $(a+b)^9$

The term involving a^5b^4 arises in the Binomial Theorem by taking n = 9 and k = 4:

$$C(n,k)a^{n-k}b^k = C(9,4)a^5b^4 = 126a^5b^4$$

Thus the coefficient of a^5b^4 is 126.

Counting Methods

- We can write the binomial coefficients in a triangle from known as ***Pascal's triangle***. The border consists of 1's and any interior value is the sum of the two numbers above it.
- *Example of Pascal's triangle*:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Exercises

Exercise 1: Expand $(x+y)^4$ using the Binomial Theorem.

Exercise 2: Prove

$$n(1+x)^{n-1} = \sum_{k=1}^{n} C(n,k)kx^{k-1}$$