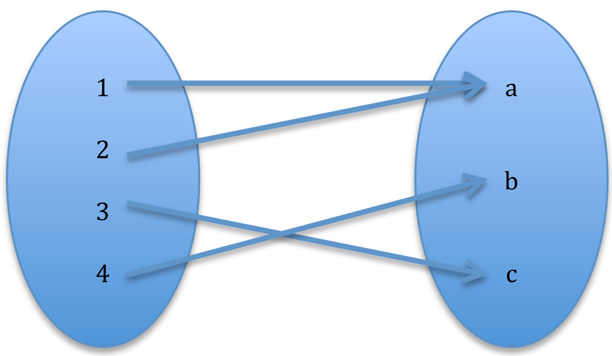
Chapter 3: Functions, Sequences and Relations
Correction of Exercises

I. Functions:

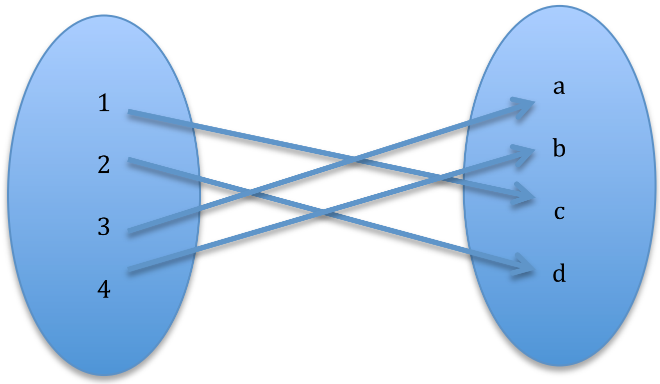
Exercise 1: Find the element of each set, draw a graph and determine if the function is one-to-one, onto or both. If it is one-to-one and onto, give the description of the inverse function as a set of ordered pairs, draw a graph and identify the element of each set.

- S = [(1,a),(2,a),(3,c),(4,b)]
  - The set S is a function from X to Y; the domain X={1, 2, 3, 4} and the range Y={a, b, c}. It is neither one-to-one, because two elements (1, 2) in X have the same element (a) in Y, nor onto, because one element (a) in Y, has two elements (1, 2) in X.
  - The inverse function: $\boldsymbol{S^{-1}}$ **= [(a,1),(a,2),(c,3),(b,4)]**
  - The graph of the element of each set:

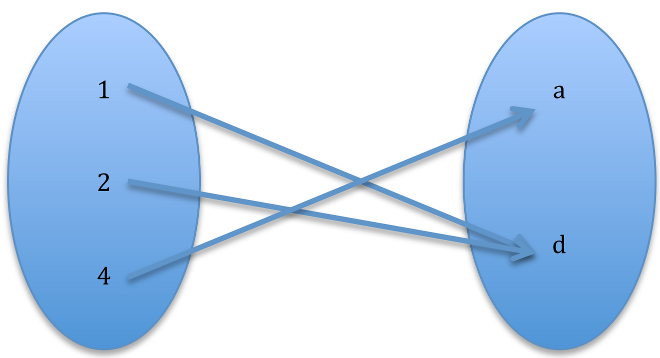


- K = [(1,c),(2,d),(3,a),(4,b)]
  - The set K is a function from X to Y; the domain X={1, 2, 3, 4} and the range Y={a, b, c, d}. It is one-to-one, because each element in X has at least one element in Y or onto, because each element in Y has at least one element in X.
  - The inverse function: $\boldsymbol{K^{-1}}$ **= [(c,1)(d,2),(a,3),(b,4)]**

- The graph of the element of each set:



- V = [(1,d),(2,d),(4,a)]
  - The set V is a function from X to Y; the domain X={1, 2, 4} and the range Y={a, d}. It is neither one-to-one, because two elements (1, 2) in X have the same element (d) in Y, nor onto, because one element (d) in Y, has two elements (1, 2) in X.
  - The inverse function: **V^{-1} = [(d,1),(d,2),(a,4)]**
  - The graph of the element of each set:



<u>Exercise 2:</u> Determine whether each function is one-to-one, onto, or both. The domain and codomain of each function is the set of all integers.

- f(x) = n +1

The function f is both one-to-one and onto. To prove that f is one-to-one, suppose that f(n) = f(m). Then n + 1 = m + 1.

Thus, n = m. Therefore, f is one-to-one.

To prove that f is onto, let m be an integer, m -1 = n. Then f(m – 1) = (m – 1) + 1 = m. Therefore, f is onto.

- $f(x) = | n |$

The function f is neither one-to-one nor onto. Since f(-1) = | -1 | = 1= f(1), f is not one-to-one. Since $f(n) \geq 0$ for all $n \in \mathbf{Z}$, $f(n) \neq -1$ for all $n \in \mathbf{Z}$. Therefore, f is not onto.

- $f(x) = n^2$

The function f is neither one-to-one nor onto. Since $f(-1) = (-1)^2 = f(1)$, f is not one-to-one. Since $f(n) \geq 0$ for all $n \in \mathbf{Z}$, $f(n) \neq -1$ for all $n \in \mathbf{Z}$. Therefore, f is not onto.

Exercise 3: Let each function is one-to-one on the specified domain X. If Y = range of f, we obtain a bijection from X to Y. Find each inverse function

- $f(x) = 4x + 2$ x = set of real numbers

Let y = 4x + 2 we find x = $(x - 2)/4$ $f^{-1} = (y - 2)/4$

- $f(x) = 3^x$ x = set of real numbers

Let y = 3x we find x = $\log y / \log 3$ $f^{-1} = \log y / \log 3$

- $f(x) = 3 + 1/x$ x = set of nonzero real numbers

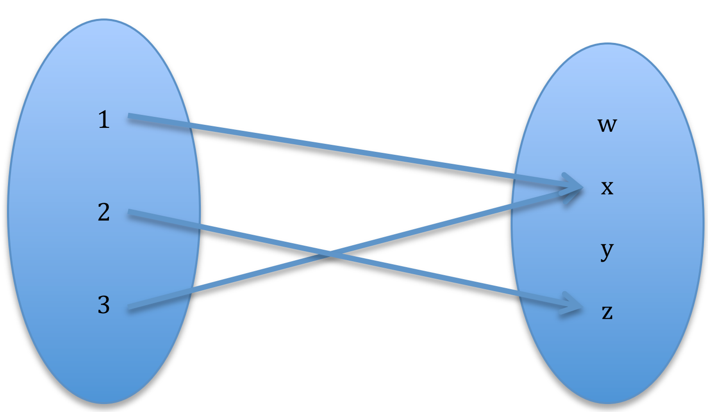
Let y = 3 + 1/x we find x = $1/(y - 3)$ $f^{-1} = 1/(y - 3)$

Exercise 4: Consider the function g = {(1, b), (2, c), (3, a)} from X = {1, 2, 3} to Y = {a, b, c, d}, and f = {(a, x), (b, x), (c, z), (d, w)}, a function from Y to Z = {w, x, y, z}.

1. Determine f o g as a set of ordered pairs.

f o g = {(3, x), (1, x), (2, z)}

2. Draw the arrow diagram of f o g.

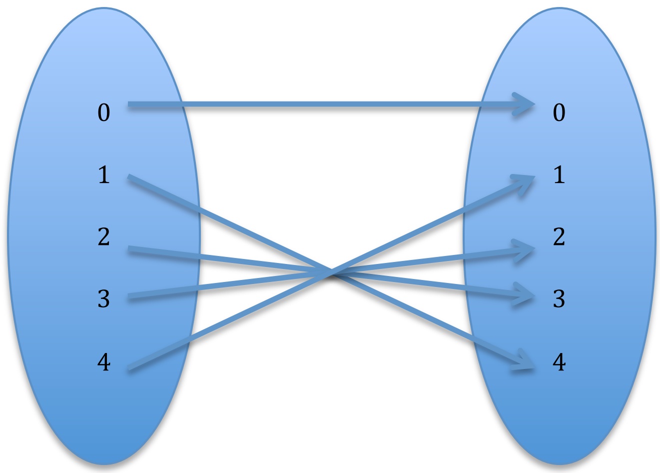


Exercise 5: Let f be the function from X = {0, 1, 2, 3, 4} to X defined by f(x) = 4x mod 5

1. Determine f as a set of ordered pairs.

   f(x) = {(0, 0), (1, 4), (2, 3), (3, 2), (4, 1)}

2. Draw the arrow diagram of f.



3. Determine if f is one-to-one or onto.

It is one-to-one, because each element in X has at least one element in X or onto, because each element in X has at least one element in X.

Exercise 6: Let the function g = {(1, a), (2, c), (3, c)} be a function from X = {1, 2, 3} to Y = {a, b, c, d}. Let S = {1}, T = {1, 3}, U = {a} and V = {a, c}.

1. Determine g(S)

$$g(S) = \{a\}$$

2. Determine g(T)

$$g(T) = \{a, c\}$$

3. Determine $g^{-1}(U)$

$$g^{-1}(U) = \{1\}$$

4. Determine $g^{-1}(V)$

$$g^{-1}(V) = \{1, 2, 3\}$$

II. Sequences:

Exercise 1: Consider the sequence S defined by c, d, d, c, d, c

1. Find S1

   S1 = **c**

2. Find S4

   S1 = **c**

3. Determine S as a string

The string of S is ***cddcdc.***

Exercise 2: Consider the sequence T defined by Tn = 2n - 1

1. Find T1

   $T1 = 2 \times 1 - 1 = 1$

2. Find T100

   $T100 = 2 \times 100 - 1 = 199$

3. Find

   $\sum_{i=1}^{3} T_i = T_1 + T_2 + T_3 = 1 + 3 + 5 = 9$

4. Find

   $\prod_{i=3}^{6} T_i = T_3 \times T_4 \times T_5 \times T_6 = 5 \times 7 \times 9 \times 11 = 3465$

Exercise 3: Consider the sequence Q defined by Q1 = 8, Q2 = 12, Q3 = 12, Q4 = 28, Q5 = 33

1. Find

   $\sum_{i=2}^{4} Q_i = Q_2 + Q_3 + Q_4 = 12 + 12 + 28 = 52$

2. Find

   $\sum_{k=2}^{4} Q_k = Q_2 + Q_3 + Q_4 = 12 + 12 + 28 = 52$

3. Is Q increasing?

   Q is not increasing because $Q1 < Q2$ but $Q2 = Q3$.

4. Is Q decreasing?

   Q is not decreasing because $Q1 \ngtr Q2$.

5. Is Q nonincreasing?

   Q isn't nonincreasing because $Q2 \ngeq Q3$ and $Q4 \ngeq Q5$.

6. Is Q nondecreasing?

Q is nondecreasing because $Q1 \leq Q2$ and $Q2 = Q3$

Exercise 4: Consider the sequence A defined by $An = n^2 - 3n + 3$

1. Find

$\sum_{i=1}^{4} A_i = A_1 + A_2 + A_3 + A_4 = 1 + 1 + 3 + 7 = 12$

2. Find

$\sum_{j=3}^{5} A_j = A_3 + A_4 + A_5 = 3 + 7 + 13 = 23$

3. Find

$\prod_{i=1}^{2} A_i = A_1 \times A_2 = 1 \times 1 = 1$

4. Find

$\prod_{x=3}^{4} A_x = A_3 \times A_4 = 3 \times 7 = 21$

5. Is A increasing?

A is not increasing because $A1 < A2$ but $A2 = A1$.

6. Is A decreasing?

A is not decreasing because $A1 \ngtr A2$ and $A3 \ngtr A4$

7. Is A nonincreasing?

A isn't nonincreasing because $A2 \ngeq A3$ and $A4 \ngeq A5$

8. Is A nondecreasing?

A is nondecreasing because $A2 \leq A3$ and $A2 = A3$

Exercise 5: Consider the sequence Y and Z defined by

$Yn = 2^n - 1$ $\qquad$ $Zn = n\,(n-1)$

1. Find

$(\sum_{i=1}^{3} Y_i)\,(\sum_{i=1}^{3} Z_i) = (Y_1 + Y_2 + Y_3)\,(Z_1 + Z_2 + Z_3)$

$= (1 + 3 + 7)\,(0 + 2 + 6) = 11 \times 8 = 88$

2. Find

$(\sum_{i=1}^{5} Y_i)\,(\sum_{i=1}^{4} Z_i) = (Y_1 + Y_2 + Y_3 + Y_4 + Y_5)\,(Z_1 + Z_2 + Z_3 + Z_4)$

$= (1 + 3 + 7 + 15 + 24)\,(0 + 2 + 6 + 12) = 50 + 20 = 70$

3. Find

   $\sum_{i=1}^{4} Y_i\, Z_i = Y_1 Z_1 + Y_2\, Z_2 + Y_3\, Z_3 + Y_4\, Z_4$ =1×0 + 3×2 + 7×6 + 15×12 = 229

4. Find

   $(\sum_{i=3}^{4} Y_i)\ (\prod_{i=2}^{4} Z_i) = (Y_3 + Y_4)\ (Z_2 \times Z_3 \times Z_4)$ = (7 +15) (2 × 6 × 12)= 3168

III. Relations:

Exercise 1: Write the relation as a set of ordered pairs

1.

8840	Hammer
9921	Pliers
451	Paint
2207	Carpet

[(8840, Hammer), (9921, Pliers), (451, Paint), (2207, Carpet).

2.

a	a
b	b

[(a, a), (b, b)]

Exercise 2: Write the relation as a table of ordered pairs

1. R = {(a, 1), (b, 2), (a, 1), (c, 1)}

a	1
b	2
a	1
c	1

2. R = {(Roger, music), (Pat, History), (Ben, Math), (Pat, Polysci)}

Roger	Music
Pat	History
Ben	Math
pat	Polysci

Exercise 3: Write the relation as a set of ordered pairs.

1. [(a, b), (a, c), (b, a), (b, d), (c, c), (c, d)]
2. [(b, c), (c, b), (d, d)]

Exercise 4: Consider the relation R on the set {1, 2, 3, 4, 5} defined by the rule $(x, y) \in R$ if 3 divides $x - y$

1. List the element of R

R = {(1, 1), (1, 4), (2, 2), (2, 5), (3, 3), (4, 1), (4, 4), (5, 2), (5, 5)}

2. List the element of R^{-1}

R^{-1} = {(1, 1), (4, 1), (2, 2), (5, 2), (3, 3), (1, 4), (4, 4), (2, 5), (5, 5)}

3. Is the element of R is reflexive, symmetric, antisymmetric, transitive and/or partial order?

The relation **R from X to Y is a partial order** because R is reflexive, symmetric, antisymmetric and transitive.

- *Reflexive:* (1, 1), (2, 2), (3, 3), (4, 4) and (5, 5) are each in R.
- symmetric: (1, 4), (4, 1), (2, 5) and (5, 2) are each in R.
- Antisymmetric: (1, 1), (2, 2), (3, 3), (4, 4) and (5, 5) are each in R.
- Transitive: (1, 1), (2, 2), (3, 3), (4, 4) and (5, 5) are each in R.

Exercise 5: Consider the relation R on the set {1, 2, 3, 4, 5} defined by the rule $(x, y) \in R$ if $x + y \leq 6$

1. List the element of R

R = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (5, 1)}

2. List the element of R^{-1}

R^{-1} = R = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (5, 1)}

3. Is the element of R is reflexive, symmetric, antisymmetric, transitive and/or partial order?

The relation **R from X to Y is not a partial order** because R is neither reflexive nor transitive and antisymmetric but R is symmetric.

- *Reflexive:* (4, 4) and (5, 5) are not each in R.
- symmetric: (1, 4), (4, 1), (2, 4), (4, 2), (1, 5) and (5, 1) are each in R.
- Antisymmetric: *:* (4, 4) and (5, 5) are not each in R.
- Transitive: (4, 4) and (5, 5) are not each in R.

Exercise 6: Let R1 and R2 be the relations on {1, 2, 3, 4} given by

R1 = {(1, 1), (1, 2), (3, 4), (4, 2)}

R2 = {(1, 1), (2, 1), (3, 1), (4, 4), (2, 2)}

1. List the element of R1 o R2

R1 o R2 = {(1, 1), (2, 1), (3, 1), (3, 2), (2, 2), (4, 2)}

2. List the element of R2 o R1

R2 o R1 = {(1, 1), (4, 1), (3, 4), (1, 2), (4, 2)}

IV. Equivalence Relation:

Exercise 1: Determine whether the relation is an equivalence relation on x, y $\in$ {1, 2, 3, 4, 5}, If the relation is an equivalence relation, list the equivalence classes.

1. [(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1)]

The equivalence relation: [1] = [3] = {1, 3}, [2] = {2}, [4] = {4}, [5] = {5}.

2. [(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 5), (5, 1), (3, 5), (5, 3), (1, 3), (3, 1)]

The equivalence relation: [1] = [3] = [5] = {1, 3, 5}, [2] = {2}, [4] = {4}.

3. {(x, y)| 3 divides by x +y }

{(1, 2), (2, 1), (1, 5), (5, 1), (4, 5), (5, 4)}

This relation is neither transitive nor reflexive because (1, 1), (2, 2), (4, 4) and (5, 5) don't belong to the relation. We can conclude that it is not an equivalence relation.

Exercise 2: Determine the members of the equivalence relation on {1, 2, 3, 4} defined by the given partition. Also find the equivalence classes [1], [2], [3] and [4].

1. {{1, 2}, {3, 4}}

[(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)]

[1] = [2] = {1, 2}, [3] = [4] = {3, 4}

2. {{1, 2, 3}, {4}}

[(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4)]

[1] = [2] = [3] = {1, 2, 3}, [4] = {4}

Exercise 3: Let R be a reflexive relation on X satisfying: for all x, y, z $\in$ X, is x R y and y R z, then z R x. Prove that R is an equivalence relation.

Since x R y is reflexive, we can define y R y is reflexive. If we take z = y, in the given condition, we have y R x. Therefore, R is symmetric.

We can say that x R y and y R z are reflexive, in this case we can conclude by x R z.

Since R is symmetric, x R z. Therefore, R is transitive. Since R is reflexive, symmetric and transitive, R is an equivalence relation.

V. Matrices of Relations:

Exercise 1: Find the matrix of the relation R from X to Y relative to the orderings given

1. R = {(1, δ), (2, α), (2, Σ), (3, β), (3, Σ)} ordering of X ={1, 2, 3} and Y = {α, β, Σ, δ}.

	α	β	Σ	δ
1	0	0	0	1
2	1	0	1	0
3	0	1	1	0

2. $R = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$ ordering of $X = \{1, 2, 3, 4\}$

	1	2	3	4	5
1	0	1	0	0	0
2	0	0	1	0	0
3	0	0	0	1	0
4	0	0	0	0	1

<u>Exercise 2:</u> Consider the matrix

	w	x	y	z
a	1	0	1	0
b	0	0	0	0
c	0	0	1	0
d	1	1	1	1

1. Write the relation R, given by the matrix, as a set of ordered pairs.

$R = \{(a, w), (a, y), (c, y), (d, w), (d, x), (d, y), (d, z)\}$.

2. Find the matrix of the inverse of the relation R, given by the matrix.

$R^{-1} = \{(w, a), (y, a), (y, c), (w, d), (x, d), (y, d), (z, d)\}$.

The domain = $\{w, x, y, z\}$ and the codomain = $\{a, b, c, d\}$

	a	b	c	d
w	1	0	0	1
x	0	0	0	1
y	1	0	1	1
z	0	0	0	1

<u>Exercise 3:</u> Let the relations $R1 = \{(1, x), (1, y), (2, x), (3, x)\}$; $R2 = \{(x, b), (y, b), (y, a), (y, c)\}$ ordering of $X = \{1, 2, 3\}$, $Y = \{x, y\}$ and $Z = \{a, b, c\}$.

1. Find the matrix A1 of the relation R1

	x	y
1	1	1
2	1	0
3	1	0

2. Find the matrix A2 of the relation R2

	a	b	c
x	0	1	0
y	1	1	1

3. Find the matrix product A1 A2

0	2	1
0	1	0
0	1	0

4. Find the relation R2 o R1

R2 o R1 = {(1, b), (2, b), (3, b), (1, a), (1, c)}

5. Find the matrix of the relation R2 o R1

	a	b	c
1	1	1	1
2	0	1	0
3	0	1	0