Correction of the Quiz 04

Name:

Time: Complete and submit to the instructor

Evaluation:

- As described in the syllabus, the Quiz is 20% of the overall grade.

Exercise 1: Let the universe be the set U ={1, 2, 3, … , 10}.

Let A = {1, 4, 7, 9}, B = {1, 2, 3, 4, 5,6} and C = {2, 4, 7, 6, 8}.

List the elements of each set

- A ∩ (B U C)

{ 1, 4}

- $\bar{B}$ ∩ (C − A)

{8}

- (A ∩ B) – C

{1}

- $\overline{A \cap B}$ U C

{2, 3, 4, 5, 6, 7, 8, 9, 10}

- (A U B) – (C – B)

{1, 2, 3, 4, 5, 6, 9}

Exercise 2: Use the Mathematical induction to prove that the statement is verified.

1) Use the geometric sum to prove that

$$r^0 + r^1 + \ldots + r^n = \frac{r^{n+1}-1}{r-1}$$

For all $n \geq 0$ and $0 < r < 1$

1. Basic Step:

If $n = 0$, we have $\frac{r^{n+1}-1}{r-1} = \frac{r^{0+1}-1}{r-1} = 1$

For $n = 0$, the geometric sum is true.

2. Inductive step:

Let assume that the statement is true for $n = 0$. We must prove then show that the geometric sum is true for $n \geq 0$.

$$r^0 + r^1 + \ldots + r^{n+1} = \frac{r^{n+1}-1}{r-1} + r^{n+1} = \frac{r^{n+1}-1 + (r-1)\, r^{n+1}}{r-1} = \frac{r^{n+1}-1 + r \times r^{n+1} - r^{n+1}}{r-1}$$
$$= \frac{r^{n+2}-1}{r-1}$$

- ***Since the Basic step and the Inductive step have been verified, the Principle of Mathematical Induction tells us that the geometric sum is true for n ≥ 0.***

2) $1^2 + 2^2 + 3^2 + \ldots + n^2 = n\,(n+1)(2n+1)/6$ $\quad n \geq 1$

1. Basic Step:

If n = 1, we have $\mathbf{1\,(1+1)((2\times 1)+1)/6 = 1\times 2\times 3/6 = 1}$

For n = 1, the statement [$1^2 + 2^2 + 3^2 + \ldots + n^2 = n\,(n+1)(2n+1)/6$] is true for $n \geq 1$.

2. Inductive step:

Let assume that the statement is true for n = 1. We must prove that the inequality is true for $1^2 + 2^2 + 3^2 + \ldots + (n+1)^2 = (n+1)(n+2)(2n+3)/6$

$$\mathbf{1^2 + 2^2 + 3^2 + \ldots + (n+1)^2 = \frac{n\,(n+1)(2n+1)}{6} + (n+1)^2}$$

$$= \frac{\mathbf{n\,(n+1)(2n+1)}}{\mathbf{6}} + \frac{\mathbf{6\,(n+1)2}}{\mathbf{6}} = \frac{\mathbf{[n\,(n+1)(2n+1)] + 6\,(n+1)2}}{\mathbf{6}} = \frac{\mathbf{(n+1)\,[n(2n+1) + 6\,(n+1)]}}{\mathbf{6}}$$

$$= \frac{\mathbf{(n+1)\,[2n^2 + n + 6\,n + 6]}}{\mathbf{6}} = \frac{\mathbf{(n+1)\,[2n^2 + 7n + 6]}}{\mathbf{6}}$$

Theorem:

For $ax^2 + bx + c = 0$

Delta = $\Delta = b^2 - 4ac$

If $\sqrt{\Delta} > 0$: we have two solutions.

If $\sqrt{\Delta} < 0$: we don't have any solution.

$$x_1 = \frac{-b-\sqrt{\Delta}}{2a} \quad \textit{and} \quad x_2 = \frac{-b+\sqrt{\Delta}}{2a}$$

$$ax^2 + bx + c = a\,(x - x_1)(x - x_2)$$

In our exercise, we have a quadratic equation: $2n^2 + 7n + 6$

$$\Delta = b^2 - 4ac$$

$$\Delta = 7^2 - 4\times 2\times 6 = 49 - 48 = 1$$

$$\sqrt{\Delta} = \sqrt{1} = 1$$

$$x_1 = \frac{-7-\sqrt{1}}{2\times 2} = \frac{-8}{4} = -2$$

$$x_2 = \frac{-7+\sqrt{1}}{2\times 2} = \frac{-6}{4} = \frac{-3}{2}$$

$$2n^2 + 7n + 6 = 2\,(x - (-2))\,(x - (-\tfrac{3}{2})) = 2\,(x + 2)(x + \tfrac{3}{2})$$

$$1^2 + 2^2 + 3^2 + \ldots + (n+1)^2 = \frac{(n+1)\,[2n^2+7n+6]}{6} = \frac{(n+1)\,[2\,(x+2)\left(x+\frac{3}{2}\right)]}{6}$$

$$= \frac{(n+1)\,(x+2)(2x+3)}{6}$$

- ***Since the Basic step and the Inductive step have been verified, the Principle of Mathematical Induction tells us that the statement is true for all $n \geq 1$.***

Exercise 3:

1) Define a sequence S as

$$\mathbf{S_n = 2^n + 4 \times 3^n} \qquad n \geq 0$$

1) Find S0 : $S0 = 2^0 + 4 \times 3^0 = 1 + 4 \times 1 = 5$

2) Find S1 : $S1 = 2^1 + 4 \times 3^1 = 2 + 4 \times 3 = 14$

3) Find a formula of Si : $Si = 2^i + 4 \times 3^i$

4) Find a formula for Sn-1 : $Sn\text{-}1 = 2^{n-1} + 4 \times 3^{n-1}$

5) Find a formula for Sn-2 : $Sn\text{-}2 = 2^{n-2} + 4 \times 3^{n-2}$

6) Prove that {Sn} satisfies: Sn = 5 Sn-1 – 6 Sn-2 for all $n \geq 2$

$$5\, Sn\text{-}1 - 6\, Sn\text{-}2 = 5 \times (2^{n-1} + 4 \times 3^{n-1}) - 6 \times (2^{n-2} + 4 \times 3^{n-2})$$

$$= (5\times2 - 6) \times 2^{n-2} + (5\times4\times3 - 6\times4) \times 3^{n-2}$$

$$= 4 \times 2^{n-2} + 36 \times 3^{n-2}$$

$$= 2^2 \times 2^{n-2} + (4 \times 3^2) \times 3^{n-2}$$

$$= 2^n + 4 \times 3^n = \mathbf{Sn}$$

2) Consider the sequence A defined by $A_n = n^2 - 3n + 3$

1. Find

$\sum_{i=1}^{4} A_i = A_1 + A_2 + A_3 + A_4 = 1 + 1 + 3 + 7 = 12$

2. Find

$\sum_{j=3}^{5} A_j = A_3 + A_4 + A_5 = 3 + 7 + 13 = 23$

3. Find

$\prod_{i=1}^{2} A_i = A_1 \times A_2 = 1 \times 1 = 1$

4. Find

$\prod_{x=3}^{4} A_x = A_3 \times A_4 = 3 \times 7 = 21$

5. Is A increasing?

A is not increasing because A1 < A2 but A2 = A1.

6. Is A decreasing?

A is not decreasing because A1 ≯ A2 and A3 ≯ A4

7. Is A nonincreasing?

A isn't nonincreasing because A2 ≱ A3 and A4 ≱ A5

8. Is A nondecreasing?

A is nondecreasing because A2 ≤ A3 and A2 = A3

Exercise 4:

a) Use the mathematical induction to show that

$$f_n^{\,2} = f_{n-1} f_{n+1} + (-1)^{n+1} \qquad \text{for all } n \geq 2$$

a) Basic step: (n = 2)

$$f_2^2 = 1 = 1 \times 2 - 1 = f_1\ f_3 + (-1)^3$$

b) Inductive step: Assume that the statement is true.

$$f_n f_{n+2} + (-1)^{n+2} = f_n \,(f_{n+1} + f_n) + (-1)^{n+2}$$

$$= f_n\ f_{n+1} + f_n^{\,2} + (-1)^{n+2}$$

$$= f_n\ f_{n+1} + (f_{n-1} f_{n+1} + (-1)^{n+1}) + (-1)^{n+2}$$

$$= f_{n+1}\ (f_n + f_{n-1}) = f_{n+1}^{\;2}$$

We can conclude that the statement is true.

b) Let consider the formula

$$s_1 = 2, \qquad s_n = s_{n-1} + 2n \qquad \text{for all } n \geq 2$$

Write the recursive algorithm that computes: $s_n = 2 + 4 + 6 + \ldots + 2n$.

Input : n

Output : 2 + 4 + 6 + … + 2n.

```
sum(s, n) {
      if (n ==1)
          return 2
      return sum(n – 1) + 2
}
```

Exercise 5:

1. Find the greatest common divisor of each pair of integers:

   a) 0, 17

A divisor of zero would be any integer n such that another unique integer m can be found with nm = 0. For example, n can be 1 or 17.
The positive divisors of 17 are 1, 17.

gcd(0, 17) = 17

   b) $3^2 \times 7^3 \times 11$, $3^2 \times 7^3 \times 11$

A divisor of $3^2 \times 7^3 \times 11$ would be any integer n such that another unique integer m can be found with nm = $3^2 \times 7^3 \times 11$.

$$\mathbf{gcd(3^2 \times 7^3 \times 11, 3^2 \times 7^3 \times 11) = 3^2 \times 7^3 \times 11}$$

2. Find the least common multiple of each pair of integers:

   a) 5, 25

The prime factorization of 5 = 1 × 5.

The prime factorization of 25 = 5 × 5.

lcm(5, 25) = 1 ×5 × 5 = 25

   b) 20, 40

The prime factorization of 20 = $\mathbf{2^2 \times 5}$

The prime factorization of 40 = $2 \times \mathbf{2^2 \times 5}$

lcm(20, 40) = $\mathbf{2^2 \times 2 \times 5 = 40}$

3. Express each binary (base 2) number in decimal (base 10).

   a) $1001 = 1 \times \mathbf{2^3} + 0 \times \mathbf{2^2} + 0 \times \mathbf{2^1} + 1 \times \mathbf{2^0} = 1 + 0 + 0 + 8 =$ **9**

   b) $100000 = 1 \times \mathbf{2^5} + 0 \times \mathbf{2^4} + 0 \times \mathbf{2^3} + 0 \times \mathbf{2^2} + 0 \times \mathbf{2^1} + 0 \times \mathbf{2^0}$

   $= 0 + 0 + 0 + 0 + 0 + 32 =$ **32**

4. Express each decimal (base 10) number in binary (base 2).

a) 43

2) 43	quotient = 21	remainder = 1	1's bit
2) 21	quotient = 10	remainder = 1	2's bit
2) 10	quotient = 5	remainder = 0	4's bit
2) 5	quotient = 2	remainder = 1	8's bit
2) 2	quotient = 1	remainder = 0	16's bit
2) 1	quotient = 0	remainder = 1	32's bit
0			

Binary number = 101011

b) 254

2) 254	quotient = 127	remainder = 0	1's bit
2) 127	quotient = 63	remainder = 1	2's bit
2) 63	quotient = 31	remainder = 1	4's bit
2) 31	quotient = 15	remainder = 1	8's bit
2) 15	quotient = 7	remainder = 1	16's bit
2) 7	quotient = 3	remainder = 1	32's bit
2) 3	quotient = 1	remainder = 1	64's bit
2) 1	quotient = 0	remainder = 1	128's bit
0			

Binary number = 11111110

5. Express each hexadecimal (base 16) number in decimal (base 10)

    a) 3A

$3A = 3 * \mathbf{16}^{1} + 10 * \mathbf{16}^{0} = 58$

    b) A03

$A03 = 10 * \mathbf{16}^{2} + 0 * \mathbf{16}^{1} + 3 * \mathbf{16}^{0} = 10 * 256 + 3 = 2560 + 3 = 256$

6. Add the binary numbers with base 2.

    a) 1001 + 1111

The binary number added = 11000

7. Add the Hexadecimal numbers with base 16.

$F0BA_{16}$ and $E9AD_{16}$

	1	1		
	F	0	B	A16
+	E	9	A	D16
1	D	A	6	716

A + D = 10 + 13 = 23 = 16 + 7 = 1716

1 + B + A = 1 + 11 + 10 = 22 = 16 + 6 = 1616

1 + 0 + 9 = 10 = A

F + E = 15 + 14 = 29 = 16 + 13 = 1D16

Hexadecimal addition = 1DA6716

8. Use the Euclidean algorithm to find the greatest common divisor of each pair of integers.

   a) 60, 90

90 mod 60 = 30

60 mod 30 = 0

so gcd(60, 90) = 30

   b) 30, 105

- Since 105 mod 30 = 15, we can write gcd(105, 30) = gcd(30, 15)
- Since 30 mod 15 = 0, we can write gcd(30, 15) = gcd(15, 0)

By inspection, gcd(15, 0) = 15. Therefore,

$$gcd(105, 30) = gcd(30, 15) = gcd(15, 0) = 15$$

FORMULA

THE SETS

Let U be a universal set and let A, B, and C be subsets of U. The following properties hold.

(a) Associative laws:

$$(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$$

(b) Commutative laws:

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

(c) Distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(d) Identity laws:

$$A \cup \varnothing = A, \quad A \cap U = A$$

(e) Complement laws:

$$A \cup \overline{A} = U, \quad A \cap \overline{A} = \varnothing$$

(f) Idempotent laws:

$$A \cup A = A, \quad A \cap A = A$$

(g) Bound laws:

$$A \cup U = U, \quad A \cap \varnothing = \varnothing$$

(h) Absorption laws:

$$A \cup (A \cap B) = A, \quad A \cap (A \cup B) = A$$

(i) Involution law:

$$\overline{\overline{A}} = A$$

(j) 0/1 laws:

$$\overline{\varnothing} = U, \quad \overline{U} = \varnothing$$

(k) De Morgan's laws for sets:

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}, \quad \overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

Proof The proofs are left as exercises (Exercises 44–54, Section 2.1) to be done after more discussion of logic and proof techniques.

$$\bar{A} = \mathrm{U} - \mathrm{A}$$

MATHEMATICAL INDUCTION

The Principle of Mathematical Induction consists of two steps:

- *Basic step :* Prove that S(1) is true.
- *Inductive step :* Assuming that S (n) is true for n ≥1, prove that S(n+1) is true

Then, S(n) is true for every positive integer n.

THE SEQUENCES

A *sequence* is a special type of function in which the domain consists of a set of consecutive integers.

Let **Sn** denoted the entire sequence:

S1, S2, S3, S4, S5, …

We use the notation Sn to denote the single element of the sequence S at *index* n.

- A sequence S is **increasing** if Sn < Sn+1 for all n for which n and n+1 are in the domain of the sequence.
- A sequence S is **decreasing** if Sn > Sn+1 for all n for which n and n+1 are in the domain of the sequence.
- A sequence S is **nondecreasing** if Sn ≤ Sn+1 for all n for which n and n+1 are in the domain of the sequence.
- A sequence S is **nonincreasing** if Sn ≥ Sn+1 for all n for which n and n+1 are in the domain of the sequence.
-

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$$

$$\prod_{i=m}^{n} a_i = a_m \times a_{m+1} \times \ldots \times a_n$$

ANALYSIS OF ALGORITHMS

Analysis an algorithm refers to the process of deriving estimates for the time and space needed to execute the algorithm.

- To derive a theta notation, you must derive both big oh and omega notation.
- An other way to derive big oh, omega and theta estimations is to use known results:

Expression	Name	Estimate
$a_k n^k + a_{k-1} n^{k-1} +$,,, $+ a_1 n + a_0$	Polynomial	**$\Theta(n^k)$**
$1 + 2 + \ldots + n$	Arithmetic Sum (Case k = 1 for Next entry)	**$\Theta(n^2)$**
$1^k + 2^k + \ldots + n^k$	Sum of Powers	**$\Theta(n^{k+1})$**
lg n!	Log n Factorial	**$\Theta(n \lg n)$**

Definition: Let f and g be functions with domain {1, 2, 3, …}

We write $f(n) = O(g(n))$

We define f(n) is of order at most g(n) or f(n) is big oh of g(n) if there exists a positive constant C1 such that

$$|f(n)| \leq C1\,|g(n)|$$

For all but finitely many positive integers n.

We write $f(n) = \Omega(g(n))$

We define f(n) is of order at least g(n) or f(n) is omega of g(n) if there exists a positive constant C2 such that

$$|f(n)| \geq C2\,|g(n)|$$

For all but finitely many positive integers n.

We write $f(n) = \Theta(g(n))$

We define f(n) is of order g(n) or f(n) is theta of g(n) if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

- ***By replacing O by Ω and "at most" by "at least", we obtain the definition of what it means for the best-case, worst-case, or average-case time of an algorithm to be of order at least g(n). If the best-case time required by an algorithm is O(g(n)) and Ω(g(n)), we say that the best-case time required by an algorithm is Θ(g(n)) .***

THE RECURSIVE ALGORITHMS

The *Fibonacci sequence* $\{ f_n \}$ is defined by the equations

$$f_0 = 0, f_1 = 1, f_2 = 1$$

$$\mathbf{f_n = f_{n-1} + f_{n-2}} \qquad \text{for all } n \geq 3$$

The Fibonacci sequence begins

0, 1, 1, 2, 3, 5, 8, 13, …

In mathematical terms, the sequence F_n of Fibonacci numbers is defined by the recursive relation $\quad f_n = f_{n-1} + f_{n-2}$

HEXADECIMAL TO DECIMAL CONVERSION CHART

Hexadecimal	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**A**	**B**	**C**	**D**	**E**	**F**
Decimal	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**

Introduction to number theory :

The ***Euclidean algorithm*** is an odd, famous, and efficient algorithm for finding the greatest common divisor of two integers,

The Euclidean algorithm is based on the fact that if $r = a \bmod b$, then

$$\mathbf{gcd(a, b) = gcd(b, r)}$$