UNIVERSITY OF MICHIGAN-DEARBORN

Chapter 2 : Mathematical Induction

Mathematical Induction

DEARBORN

- Mathematical induction can be used in more profound way.
- Let S_n denote the sum of the first n positive integers :

$$S_n = 1 + 2 + \dots + n$$

Suppose that: $S_n = n(n+1)/2$

A sequence of statement can be made : S1 = 1(1+1)/2 = 1

S2 = 2(2+1)/2 = 3

S3 = 3(3+1)/2 = 6

- ***We must show that for all n, if equation n is true : Sn = n(n+1)/2 then, equation n+1 is also true. Sn+1= (n+1)(n+2)/2***

- **Definition :** The Principle of Mathematical Induction consists of two steps :
  - ① ***Basic step :*** Prove that S(1) is true.
  - ② ***Inductive step :*** Assuming that S (n) is true for n $\geq$ 1, prove that S(n+1) is true

Then, S(n) is true for every positive integer n.

Mathematical Induction

DEARBORN

1. Basis Step: $S(1): 1 = 1(2)/2 = 1$ is true

2. Inductive Step: If $S(n) = n(n+1)/2$ is true.

$$S(n+1) = n(n+1)/2 + (n+1)$$

$$= \{n(n+1) + 2(n+1)\}/2$$

$$=(n^2 + n) + 2n +2) /2$$

$$=(n^2 + 3n +2)/2$$

$$= \{(n+1)(n+2)\}/2$$

So, $S(n)$ is true for every positive integer n.

Mathematical Induction

- ***Definition :*** n factorial is defined as follows:

$$n! = \begin{cases} 1 & \text{if } n=0 \ (0! = 1) \\ n(n-1)(n-2)\ldots.2\times 1 & \text{if } n \geq 1 \end{cases}$$

- *Example 1:*

$0! = 1! = 1$

$3! = 3\times(3-1)\times(3-2)=6$

$6! = 6\times(6-1)\times(6-2)\times(6-3)\times(6-4)\times(6-5)=720$

Mathematical Induction

DEARBORN

- *Example 2:* Use the induction to show that

$$n! \geq 2^{n-1} \quad \text{for all } n \geq 1$$

1. Basic step : The statement is true if n = 1.

This is easily accomplished, since $1! = 1 \geq 1 = 2^{(1-1)} = 2^0$

2. Inductive step : If $n! \geq 2^{n-1}$ is true. We must then prove that the inequality is true for n+1: $(n+1)! \geq 2^n$

$$\begin{aligned}(n+1)! &= (n+1)(n!)\\ &\geq (n+1)\,2^{n-1}\\ &\geq 2\times 2^{n-1} && \text{since } n+1 \geq 2\\ &= 2^n\end{aligned}$$

So, the statement is true for all $n \geq 1$.

Mathematical Induction

- We must use the induction to show that if $r \neq 1$,

$$a + ar^1 + ar^2 + \ldots + ar^n = \frac{a(r^{n+1} - 1)}{r - 1} \quad \text{for all } n \geq 0$$

We called the sum on the left the **geometric sum** which $a \neq 0$ and $r \neq 0$.

1. Basic step : (n=0)

We have to prove that the geometric sum is true for n=0.

For n=0, the geometric sum becomes: $a = \frac{a(r^1 - 1)}{r - 1}$ which is true.

2. Inductive step : Let assume that the geometric sum is true for n.

$$a + ar^1 + ar^2 + \ldots + ar^{n+1} = \frac{a(r^{n+1} - 1)}{r - 1} + a(r^{n+1}) = \frac{a(r^{n+1} - 1)}{r - 1} + \frac{ar^{n+1}(r - 1)}{r - 1}$$

$$= \frac{a(r^{n+2} - 1)}{r - 1}$$

- ***Since the Basic step and the Inductive step have been verified, the Principle of Mathematical Inductive tell us that the Geometric Sum is true for all $n \geq 0$.***

Mathematical Induction

DEARBORN

- *Example 1:* As an example of the use of the geometric sum, if we take a= 1 and r = 2, we obtain the formula

$$1 + (1\times2)^1 + (1\times2)^2 + (1\times2)^3 + \ldots + (1\times2)^n = \frac{2^{n+1}-1}{2-1} = 2^{n+1} - 1$$

- *Example 2:* Consider $S(n) = 1 + 3 + \ldots + (2n-1)$ for n = 1, 2, 3, 4.

n	1 + 3+ … + (2n-1)
1	1
2	3
3	5
4	7

.

Mathematical Induction

DEARBORN

- *Example 3:* We will use induction to show that $5^n - 1$ is divisible by 4 for all $n \geq 1$.

1. Basic step: (n=1)

If $n = 1$, $5^n - 1 = 5^1 - 1 = 4$, which is divisible by 4.

2. Inductive step:

We assume that $5^n - 1$ is divisible by 4. We must then show that

$5^{n+1} - 1$ is divisible by 4.

$$5^{n+1} - 1 = (5 \times 5^n) - 1 = (4 \times 5^n) + (1 \times 5^n) - 1 = 4 \times 5^n + 5^n - 1$$

By the inductive assumption, $5^n - 1$ is divisible by 4 and, since 4×5^n is divisible by 4, the sum $(5^n - 1) + 4 \times 5^n = 5^{n+1} - 1$ is divisible by 4.

- ***Since the Basic step and the Inductive step have been verified, the Principle of Mathematical Induction tells us that $5^n - 1$ is divisible by 4 for all $n \geq 1$.***

EXERCICES

DEARBORN

- ***Exercise 1:*** Using the induction, verify that each equation is true for every positive integer n ≥ 1

a) $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

b) $1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

- ***Exercise 2:*** Using the induction, verify the inequality. n ≥ 1

$$1/2n \leq 1*3*5*\ldots*(2n-1) / 2*4*6*\ldots*(2n)$$

n=1, 2, …

- ***Exercise 3:*** Use the geometric sum to prove that

$$r^0 + r^1 + \ldots + r^n = (r^{n+1} - 1)/r - 1$$

For all n ≥ 0 and $0 \leq r \leq 1$

- ***Exercise 4:***

a) Prove that $7^n - 1$ is divisible by 6, for all n ≥ 1.

b) Prove that $11^n - 6$ is divisible by 5, for all n $\geq$ 1