University of Michigan-Dearborn

Chapter 8 : Graph Theory

Introduction

Graph Theory

Definition: A graph G consists of two sets V and E, where:

- V is a nonempty set of vertices
- E is a set of edges.

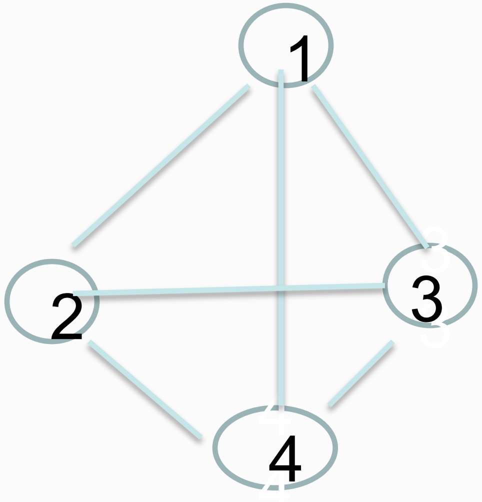
We denote a graph G

G = (V, E)

Graph Theory

- **<u>Example:</u> undirected Graph G1**

G_1

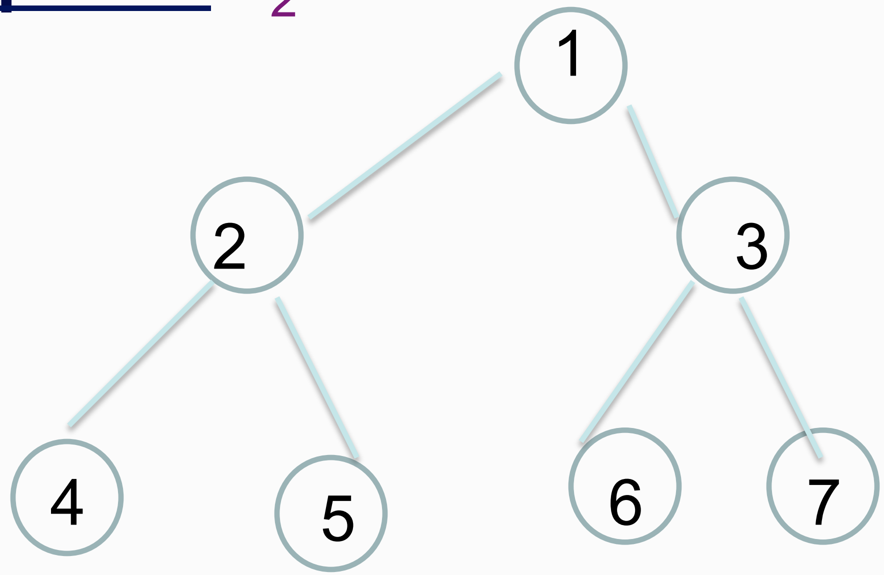


$G_1 = (V_1, E_1)$,

where $V_1 = \{1,2,3,4\}$

$$E_1 = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$$

- **Example 2:** G_2



$G_2 = (V_2, E_2)$

Where $V_2 = \{1,2,3,4,5,6,7\}$

$\mathbf{E_2 = \{(1,2), (1,3),(2,4),(2,5),(3,6),(3,7)\}}$

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- **Example 3:** Directed Graph G_3

G_3

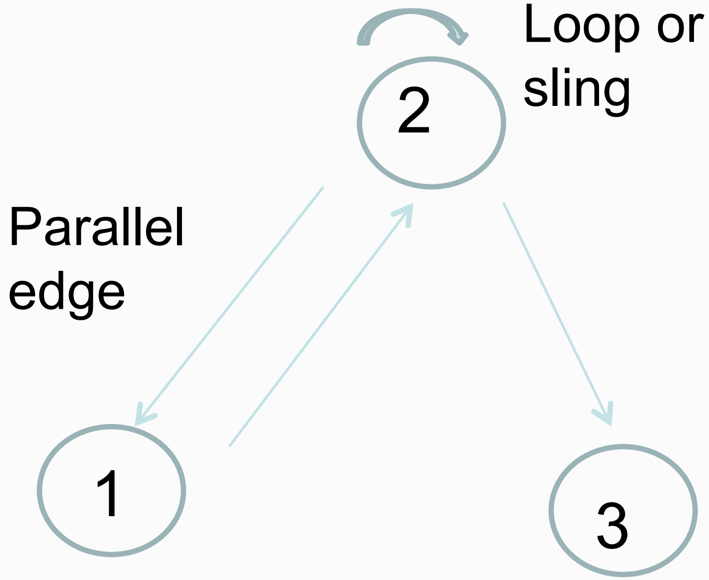
$G_3 = (V_3 , E_3)$

Where $V_3 = \{1,2,3\}$

$E_3 = \{(1,2),(2,1),(2,3)\}$

Graph Theory

- **Example 4:** G_4

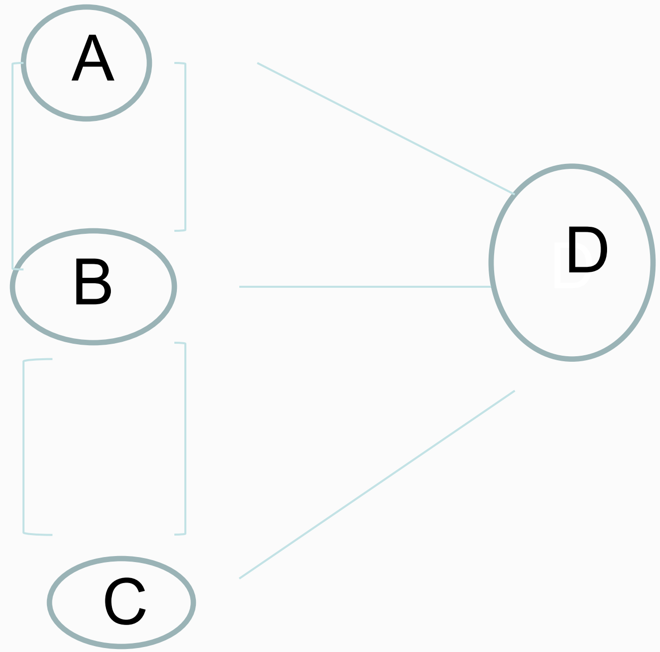


$G_4 = (V_4, E_4)$
Where $V_4 = \{1,2,3\}$

$E_4 = \{(1,2), (2,1),(2,2),(2,3)\}$

Graph Theory

- **Definition:** The degree of a vertex v, **deg (v)**, is the number of edges incident on v.



Deg (A) = 3

Deg (B) = 5

Deg (C) =3

Deg (D) = 3

- **Definition:**

A graph in which the vertices can be partitioned into disjoint sets V_1 and V_2 with every edge incident on one vertex in V_1 and one vertex in V_2 is called a

bipartite graph

Graph Theory

- **Example :** This graph is bipartite since if we let

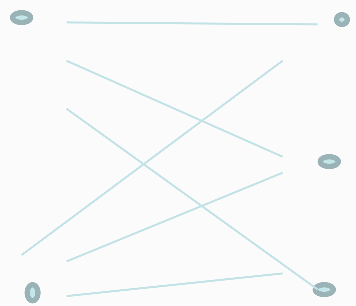
  V1 = {v1, v2, v3} and V2 = {v4, v5}

Each edge is incident
on one vertex in V1 and one vertex in V2

Graph Theory

- **Definition:**
  The graph $K_{m, n}$, called *the complete bipartite graph of m and n vertices*, has disjoint sets V1 of m vertices and V2 of n vertices. Every vertex in V1 is joined to every vertex in V2 by an edge.

- **Example:**



$K_{2, 3}$ is the complete bipartite graph on two and three vertices.

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Paths and Cycles

Graph Theory

- **Definition:**

Let V0 and Vn be vertices in a graph. A *path* from V0 to Vn of length n is an alternating sequence of n+1 vertices and n edges beginning with vertex V0 and ending with vertex Vn.

(V0, e1, V1, e2, V2, …, Vn-1, en, Vn)

In which edge ei is incident on vertices Vi-1 and Vi for i = 1, …, n.

Graph Theory

- **Example:**

In this graph (1, e1, 2, e2, 3, e3, 4, e4, 2)

Is a path of length 4 from vertex 1 to vertex 2.

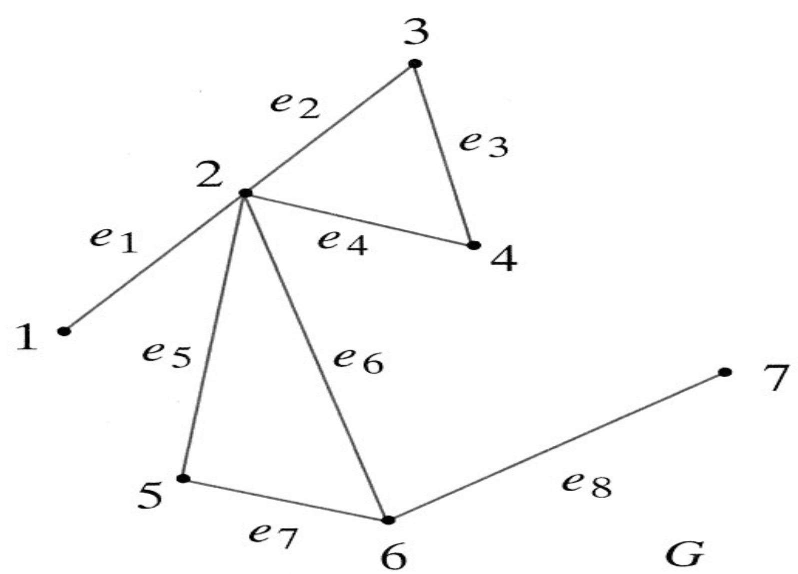
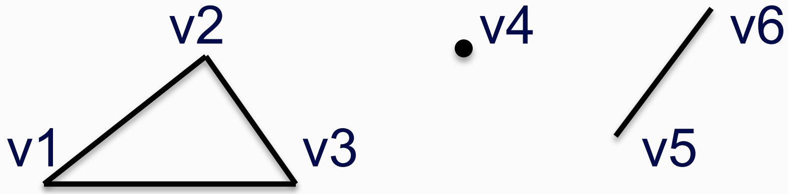


Figure 8.2.1 A connected graph with paths $(1, e_1, 2, e_2, 3, e_3, 4, e_4, 2)$ of length 4 and (6) of length 0.

Graph Theory

- A simple cycle is a cycle of the form (v0, v1, ,,vn) where v0 = vn and v0,v1, ,,vn-1 are distinct.
- A *graph G is connected* if given any vertices v and w, there is a path from v to w.
- **Example** : the following *graph is NOT connected*:

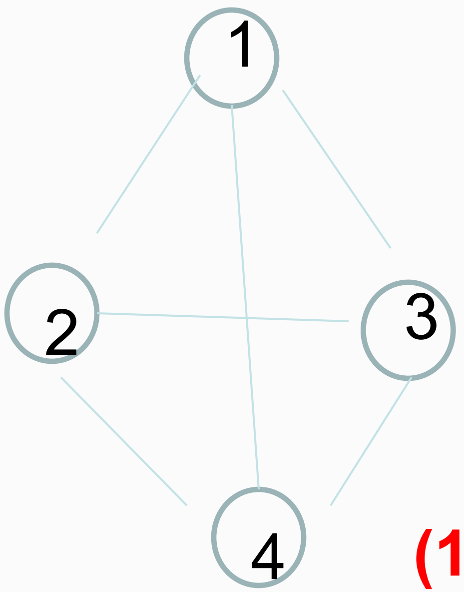


There is no path from v2 to v5

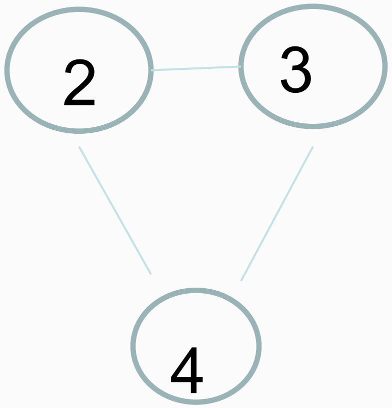
Graph Theory

- **Definition of the Subgraphs:** Let G = (V, E) be a graph. A graph G' = (V', E') is a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$.

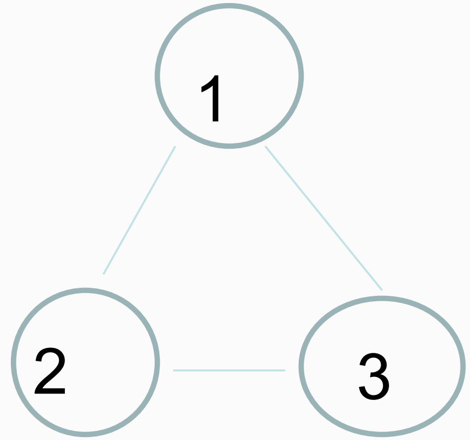
- **Example:** **G_1**



(1)



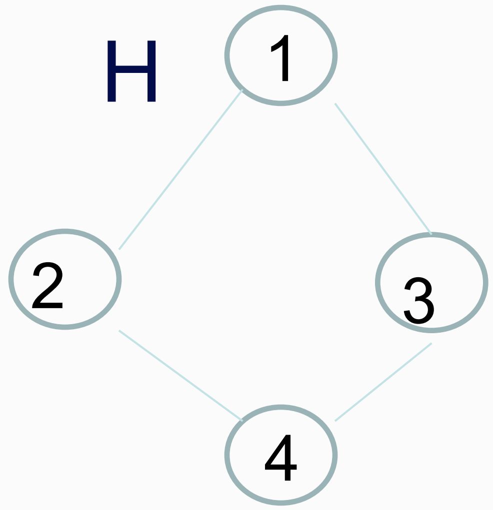
(2)



(1)and (2) are subgraphs of G_1

Graph Theory

- **Definition:** A *connected component H* of an undirected graph G is a maximal connected subgraph.



H is a connected component of G1

Graph Theory

- **Definition:**

❖ A path from v to w of length n is an edge sequence from v to w of length n in which the edges are distinct.

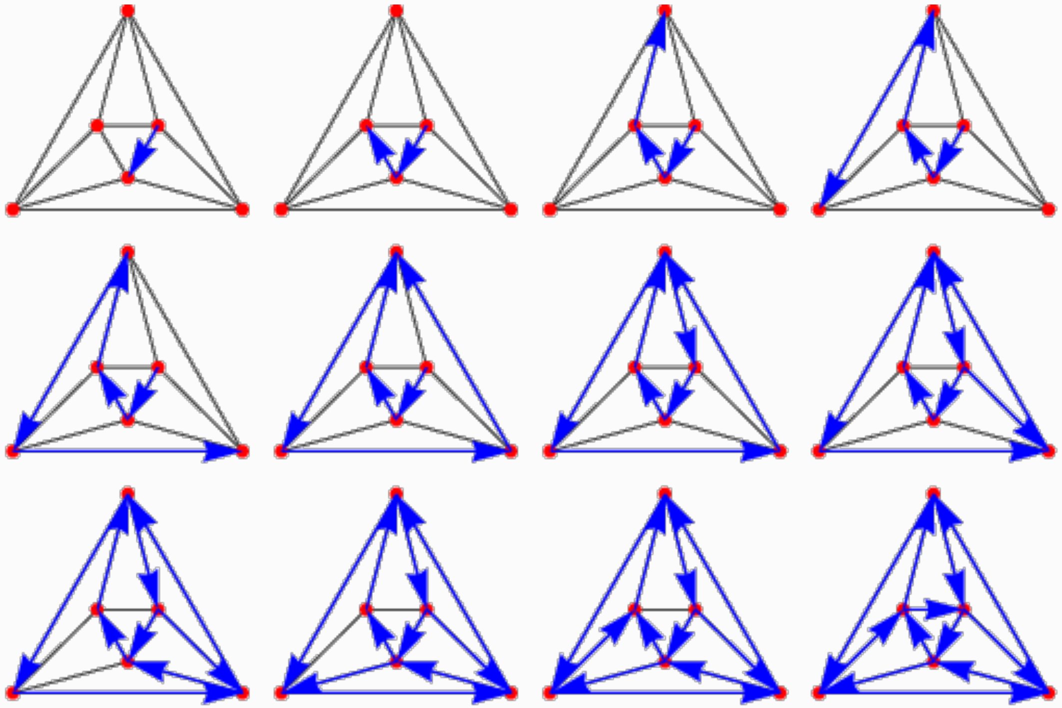
❖ A simple path from v to w of length n is a path of the form v=v0, v1,,,,vn=w, where vi!= vj.

❖ A cycle is a path from v to v.

Graph Theory

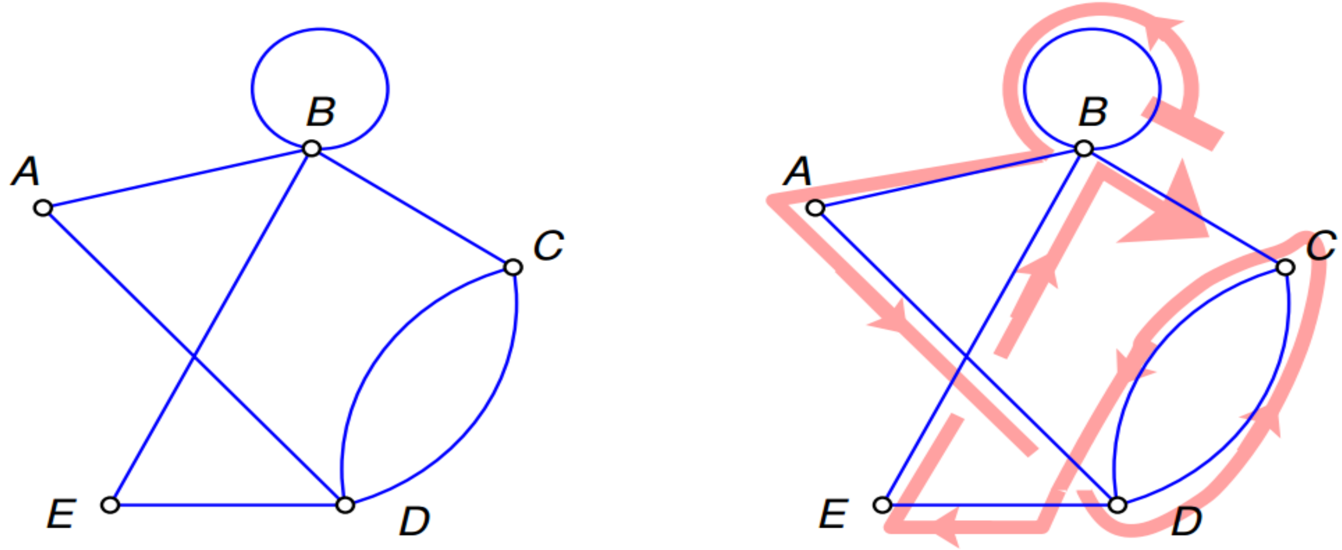
- *An Euler Cycle*

If a graph G is connected and every vertex has an even degree, *G has an Euler cycle*.



An Euler Path

Euler Paths and Euler Circuits

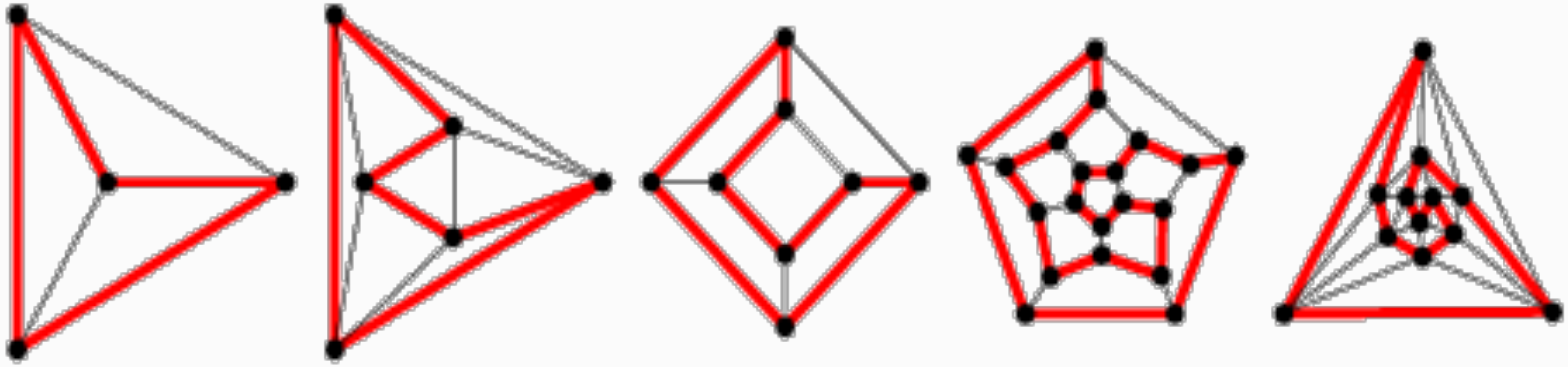


An Euler path: BBADCDEBC

Graph Theory

- ***The Hamiltonian Cycle:***

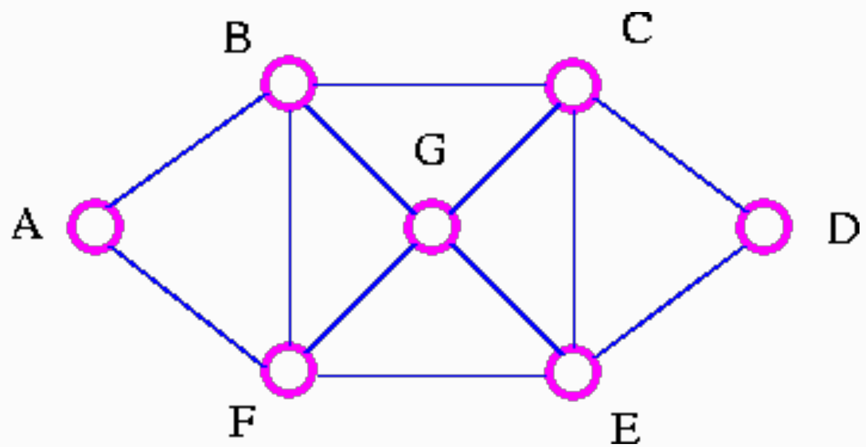
It is a route that begins and ends at the same vertex and which passes through each vertex of G exactly once.



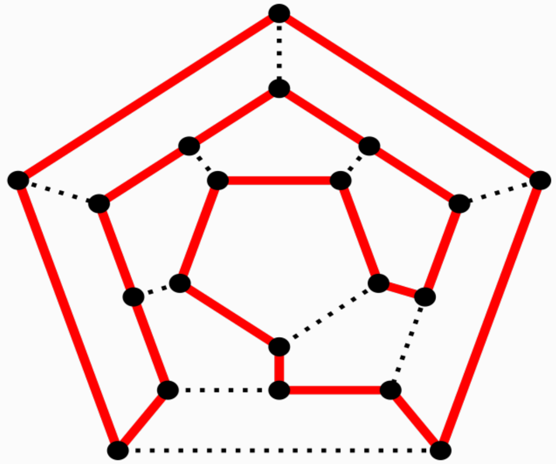
Graph Theory

- *An Euler Cycle vs a Hamiltonian Cycle*

An Euler cycle visits each edge once, whereas a Hamiltonian cycle visits each vertex once.



An Euler cycle



A Hamiltonian Cycle

Representations of Graph

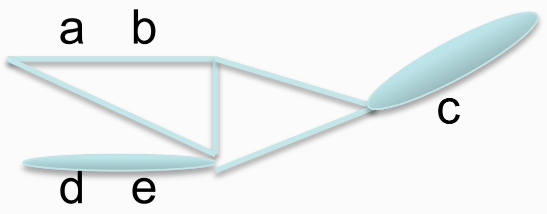
Graph Theory

- In the preceding sections we represented a graph by drawing it. Sometimes, as for example in using a computer to analyze a graph, we need a more formal representations.
- Our first method of representing a graph uses

the adjacency matrix.

Graph Theory

- **Example 1:** Consider the graph of the figure:



To obtain the adjacency matrix of this graph:

1. we first select an ordering of the vertices, say a, b, c, d, e.
2. Next, we label the rows and columns of a matrix with the ordered vertices.

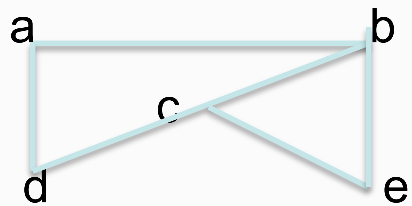
The entry in this matrix in row i, column j, i ≠ j, is the number of edges incident on i and j. If i = j, the entry is twice the number of loops incident on i.

The adjacency matrix for this graph is

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 2 & 0 \end{pmatrix} \end{array} = A$$

Graph Theory

- **Example 2:** Consider the graph of the figure:



The adjacency matrix of the simple graph of this figure :

$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array} = A$$

A, A^2, A^3 ,,,,,

$A^2 = A * A = ?$

Graph Theory

A, A^2, A^3 ,,,,,

$A^2 = A * A =$

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 & 1 \\ 2 & 1 & 3 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Consider the entry for row a, column c in A^2, obtained by multiplying pairwise the entries in row a by the entries in column c of the matrix A and summing:

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = 0 * 0 + 1 * 1 + 0 * 0 + 1 * 1 + 0 * 1 = 2$$

Graph Theory

- **<u>Example 3:</u>** After the previous example, we showed that if A is the matrix of the graph of the previous graph, then

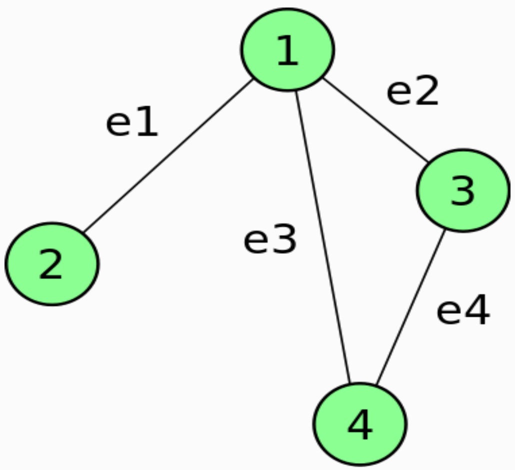
$$A^2 = \begin{pmatrix} 2 & 0 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 & 1 \\ 2 & 1 & 3 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

By multiplying,

$$A^4 = A^2 * A^2 = \begin{pmatrix} 2 & 0 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 & 1 \\ 2 & 1 & 3 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 & 1 \\ 2 & 1 & 3 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & 11 & 1 & 6 \\ 3 & 15 & 7 & 11 & 8 \\ 11 & 7 & 15 & 3 & 8 \\ 1 & 11 & 3 & 9 & 6 \\ 6 & 8 & 8 & 6 & 8 \end{pmatrix}$$

Graph Theory

- **Definition: An** *Incident Matrix* is a **matrix** that shows the relationship between two classes of objects. If the first class is X and the second is Y, the **matrix** has one row for each element of X and one column for each element of Y.



$$\begin{array}{c c} & \begin{array}{cccc} e1 & e2 & e3 & e4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

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Isomorphism of Graphs

Graph Theory

- **Definition:** *Isomorphism of Graphs*

Two **graphs** which contain the same number of **graph** vertices connected in the same way are said to be ***isomorphic***.

Formally, two **graphs** and with **graph** vertices are said to be **isomorphic** if there is a permutation of such that is in the set of **graph** edges.

Graph Theory

- **Example:** ***Isomorphism of graphs***

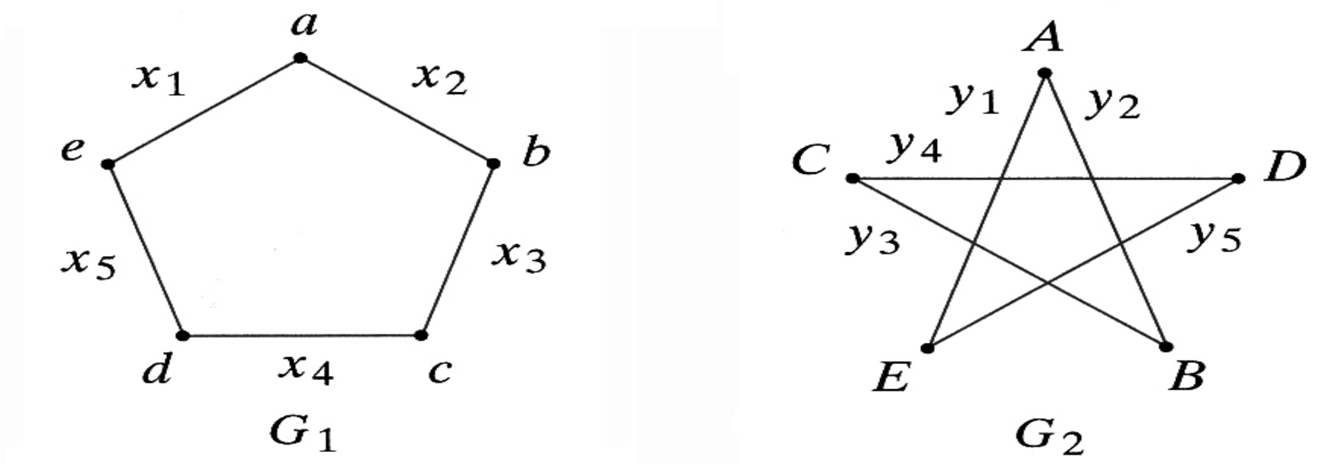


Figure 8.6.1
Isomorphic graphs.

An isomorphism for the graphs G1 and G2 of figure 8.6.1 is defined by

f(a) = A, f(b) = B, f(c) = C, f(d) = D, f(e) = E

g(xi) = yi, i = 1, …, 5

Graph Theory

- **Theorem:** Graphs *G_1 and G_2 are isomorphic* if and only if for some ordering of their vertices, *their adjacency matrices are equal.*
- **Example:**

The adjacency matrix of graph G1 in Figure 8.6.1 relative to the vertex ordering a, b, c, d, e

$$\begin{array}{c|c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Is equal to the adjacency matrix of graph G2 in Figure 8.6.1 relative to the vertex ordering A, B, C, D, E,

$$\begin{array}{c|c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Graphs G_1 and G_2 (fig. 8.6.1) are isomorphic because their adjacency matrices are identical.

Graph Theory

- **Example:**

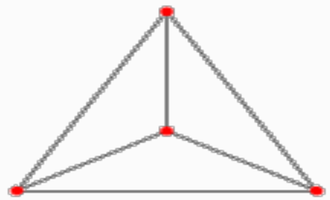
Graphs G_1 and G_2 are NOT isomorphic.

G_1 has 7 edges and G_2 has 6 edges.

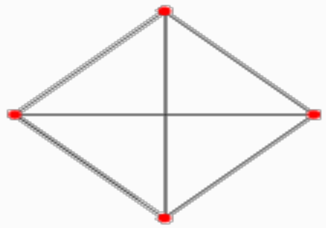
Graph Theory

- **Definition:** *Planar Graphs*

A graph is planar if it can be drawn in the plane without its edge crossing.



planar



non-planar