Chapter 2: Mathematical Induction
Correction of Exercises

- ***Exercise 1:*** Using the induction, verify that each equation is true for every positive integer $n \geq 1$.

a) $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

1. Basic Step:

If n = 1, we have $\mathbf{1^2 = 1}$.

For n = 1, the statement $[1 + 3 + 5 + \ldots + (2n - 1) = n^2]$ is true for $n \geq 1$.

2. Inductive step:

Let assume that the statement is true for n = 1. We must prove that the inequality is true for $[1 + 3 + 5 + \ldots + (2n - 1) + (2n+1) = (n+1)^2]$.

$$\begin{aligned} \mathbf{1 + 3 + 5 + \ldots + (2n - 1) + (2n+1)} &= \mathbf{n^2 + (2n + 1)} \\ &= \mathbf{n^2 + 2n + 1} \\ &= \mathbf{(n + 1)^2} \end{aligned}$$

- ***Since the Basic step and the Inductive step have been verified, the Principle of Mathematical Induction tells us that the statement is true for all $n \geq 1$.***

b) $1^2 + 2^2 + 3^2 + \ldots + n^2 = n\,(n + 1)(2n + 1)/6$

1. Basic Step:

If n = 1, we have $\mathbf{1\,(1 + 1)((2 \times 1) + 1)/6 = 1 \times 2 \times 3/6 = 1}$

For n = 1, the statement $[1^2 + 2^2 + 3^2 + \ldots + n^2 = n\,(n + 1)(2n + 1)/6]$ is true for $n \geq 1$.

2. Inductive step:

Let assume that the statement is true for n = 1. We must prove that the inequality is true for $1^2 + 2^2 + 3^2 + \ldots + (n+1)^2 = (n + 1)(n + 2)(2n + 3)/6$

$$\mathbf{1^2 + 2^2 + 3^2 + \ldots + (n+1)^2 = \frac{n\,(n+1)(2n+1)}{6} + (n+1)^2}$$

$$\mathbf{= \frac{n\,(n+1)(2n+1)}{6} + \frac{6\,(n+1)2}{6} = \frac{[n\,(n+1)(2n+1)] + 6\,(n+1)2}{6} = \frac{(n+1)\,[n(2n+1) + 6\,(n+1)]}{6}}$$

$$\mathbf{= \frac{(n+1)\,[2n^2 + n + 6\,n + 6]}{6} = \frac{(n+1)\,[2n^2 + 7n + 6]}{6}}$$

Theorem:

$$For\ ax^2 + bx + c = 0$$

$$Delta = \Delta = b^2 - 4ac$$

If $\sqrt{\Delta} > 0$ *: we have two solutions.*

If $\sqrt{\Delta} < 0$ *: we don't have any solution.*

$$x_1 = \frac{-b-\sqrt{\Delta}}{2a} \quad and \quad x_2 = \frac{-b+\sqrt{\Delta}}{2a}$$

$$ax^2 + bx + c = a\,(x - x_1)(x - x_2)$$

In our exercise, we have a quadratic equation: $2n^2 + 7n + 6$

$$\Delta = b^2 - 4ac$$

$$\Delta = 7^2 - 4\times 2\times 6 = 49 - 48 = 1$$

$$\sqrt{\Delta} = \sqrt{1} = 1$$

$$x_1 = \frac{-7-\sqrt{1}}{2\times 2} = \frac{-8}{4} = -2$$

$$x_2 = \frac{-7+\sqrt{1}}{2\times 2} = \frac{-6}{4} = \frac{-3}{2}$$

$$2n^2 + 7n + 6 = 2\,(x - (-2))\,(x - (-\tfrac{3}{2})) = 2\,(x+2)(x + \tfrac{3}{2})$$

$$1^2 + 2^2 + 3^2 + \ldots + (n+1)^2 = \frac{(n+1)\,[2n^2+7n+6]}{6} = \frac{(n+1)\,[2\,(x+2)\left(x+\frac{3}{2}\right)]}{6}$$

$$= \frac{(n+1)\,(x+2)(2x+3)}{6}$$

- ***Since the Basic step and the Inductive step have been verified, the Principle of Mathematical Induction tells us that the statement is true for all n ≥ 1.***

◆ ***Exercise 2:*** Using the induction, verify the inequality.

$$\frac{1}{2n} \leq \frac{1 \times 3 \times 5 \times \ldots \times (2n-1)}{2 \times 4 \times 6 \times \ldots \times (2n)} \qquad n = 1, 2, \ldots$$

1. Basic Step:

If n = 1, we have $\frac{1}{2 \times 1}$ - $\frac{1}{2}$

For n = 1, the statement is true.

2. Inductive step:

Let assume that the statement is true for n = 1. We must prove that the inequality is true for $\frac{1}{2(n+1)} \leq \frac{1 \times 3 \times 5 \times \ldots \times (2n-1)(2n+1)}{2 \times 4 \times 6 \times \ldots \times (2n)(2n+2)}$

$$\frac{\mathbf{1 \times 3 \times 5 \times \ldots \times (2n-1)(2n+1)}}{\mathbf{2 \times 4 \times 6 \times \ldots \times (2n)(2n+2)}} \geq \frac{\mathbf{1}}{\mathbf{2n}} \times \frac{\mathbf{(2n+1)}}{\mathbf{(2n+2)}}$$
$$= \frac{\mathbf{2n+1}}{\mathbf{2n}} \times \frac{\mathbf{1}}{\mathbf{2(n+1)}} \geq \frac{\mathbf{1}}{\mathbf{2(n+1)}}$$

➢ ***Since the Basic step and the Inductive step have been verified, the Principle of Mathematical Induction tells us that the statement is true.***

◆ ***Exercise 3:*** Use the **geometric sum** to prove that

$$r^0 + r^1 + \ldots + r^n = \frac{r^{n+1}-1}{r-1}$$

For all n ≥ 0 and 0 < r < 1

1. Basic Step:

If n = 0, we have $\frac{r^{n+1}-1}{r-1} = \frac{r^{0+1}-1}{r-1} = 1$

For n = 0, the geometric sum is true.

2. Inductive step:

Let assume that the statement is true for n = 0. We must prove then show that the geometric sum is true for n ≥ 0.

$$r^0 + r^1 + \ldots + r^{n+1} = \frac{r^{n+1}-1}{r-1} + r^{n+1} = \frac{r^{n+1}-1 + (r-1)\, r^{n+1}}{r-1} = \frac{r^{n+1}-1 + r \times r^{n+1} - r^{n+1}}{r-1}$$
$$= \frac{r^{n+2}-1}{r-1}$$

➢ ***Since the Basic step and the Inductive step have been verified, the Principle of Mathematical Induction tells us that the geometric sum is true for n ≥ 0.***

- ***Exercise 4:***

a) Prove that 7^n - 1 is divisible by 6, for all $n \geq 1$.

1. Basic Step:
If $n = 1$, we have $7^n - 1 = 7^1 - 1 = 6$, which is divisible by 6.

2. Inductive step:
We assume that 7^n - 1 is divisible by 6. We must then show that 7^{n+1} -1 is divisible by 6.

$$7^{n+1} - 1 = 7 \times 7^n - 1 = (6 + 1) \times 7^n - 1 = 6 \times 7^n + 1 \times (7^n - 1)$$

If $n = 6$, $7^{6+1} - 1 = 823543 - 1 = 823542$ $\Longrightarrow$ $823542/6 = 137257$

By the inductive assumption, 7^n -1 is divisible by 6 and, since 6×7^n is divisible by 6, the sum $6 \times 7^n + 1 \times (7^n - 1) = 7^{n+1} - 1$ is divisible by 6.

- ***Since the Basic step and the Inductive step have been verified, the Principle of Mathematical Induction tells us that 7^n -1 is divisible by 6 for all $n \geq 1$.***

b) Prove that 11^n - 6 is divisible by 5, for all $n \geq 1$

1. Basic Step:
If $n = 1$, we have $11^n - 6 = 11^1 - 6 = 5$, which is divisible by 5.

2. Inductive step:
We assume that 11^n - 6 is divisible by 5. We must then show that 11^{n+1} -6 is divisible by 5.

$$11^{n+1} - 6 = 11 \times 11^n - 6 = (5+6) \times 11^n - 6 = 5 \times 11^n + 6 \times (11^n - 1)$$

If $n = 5$, $11^{5+1} - 6 = 1771561 - 6 = 1771555$ $\Longrightarrow$ $1771555/5 = 354311$

By the inductive assumption, 11^n - 6 is divisible by 5 and, since 5×11^n is divisible by 5, the sum $5 \times 11^n + 6 \times (11^n - 1) = 11^{n+1} - 6$ is divisible by 5.

- ***Since the Basic step and the Inductive step have been verified, the Principle of Mathematical Induction tells us that*** 11^n - 6 ***is divisible by 5 for all $n \geq 1$.***