UNIVERSITY OF MICHIGAN-DEARBORN

Chapter 3 : Functions, sequences and relations

Functions, sequences and relations

All of mathematics, as well as subjects that rely on mathematics, such as computer science and engineering, make use of functions, sequences and relations.

- A **function** assigns to each member of a set X exactly one member of a set Y. For example, functions are used to analyze the time needed to execute algorithms.
- A **sequence** is a special kind of functions. For example, a sequence takes order into account.
- **Relations** generalize the notion of functions. It's a set of ordered pairs. For example, relations are used to help user to access to information in a database.

Functions

Functions

- Let X and Y be sets. A *function f* from X to Y is a subset of the Cartesian product X×Y for $x \in X$ and $y \in Y$ with $(x, y) \in f$.
- *We denote a function f* from X to Y as $f : X \longrightarrow Y$
- The set X is called *the domain of f* and the set of Y is called *the codomain of f*. the set

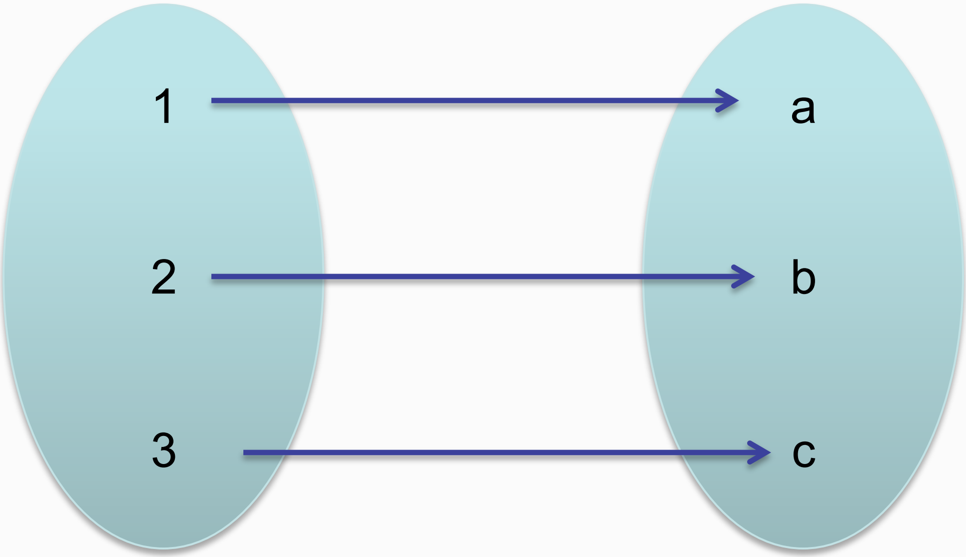
$$\{ y \mid (x, y) \in f \}$$

Which is a subset of the codomain Y and we called *the range of f.*

Functions

- **Example 1:** Let consider X = {1, 2, 3} and Y = {a, b, c}.

  f = {(1,a),(2,b),(3,c)}

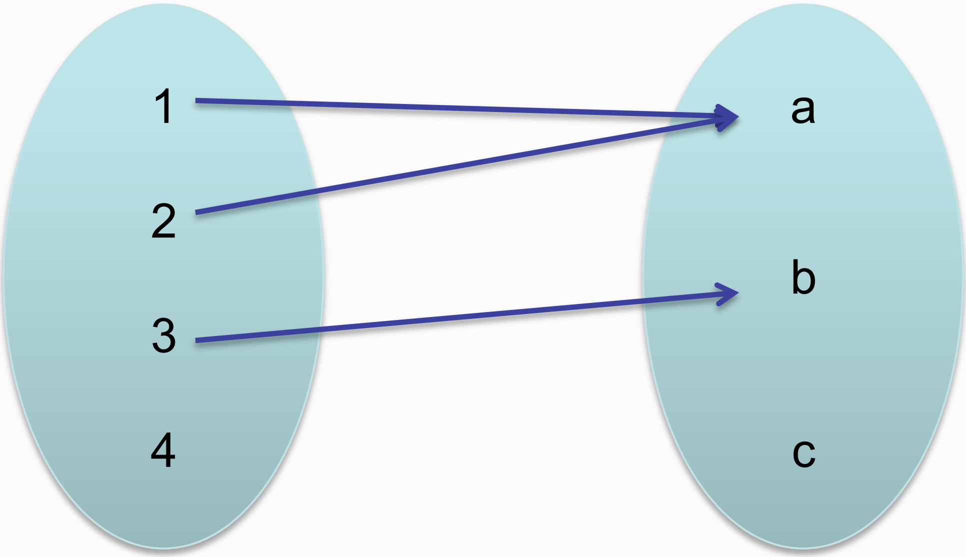


➢ ***f is a function from X = {1, 2, 3} to Y = {a, b, c} because all the element in X are assigned to Y.***

Functions

- **Example 2:** Let consider X = {1, 2, 3, 4} and Y = {a, b, c}.

$$f = \{(1,a),(2,a),(3,b)\}$$



➢ ***f is not a function from X = {1, 2, 3, 4} to Y = {a, b, c} because the element 4 in X is not assigned to Y.***

Functions

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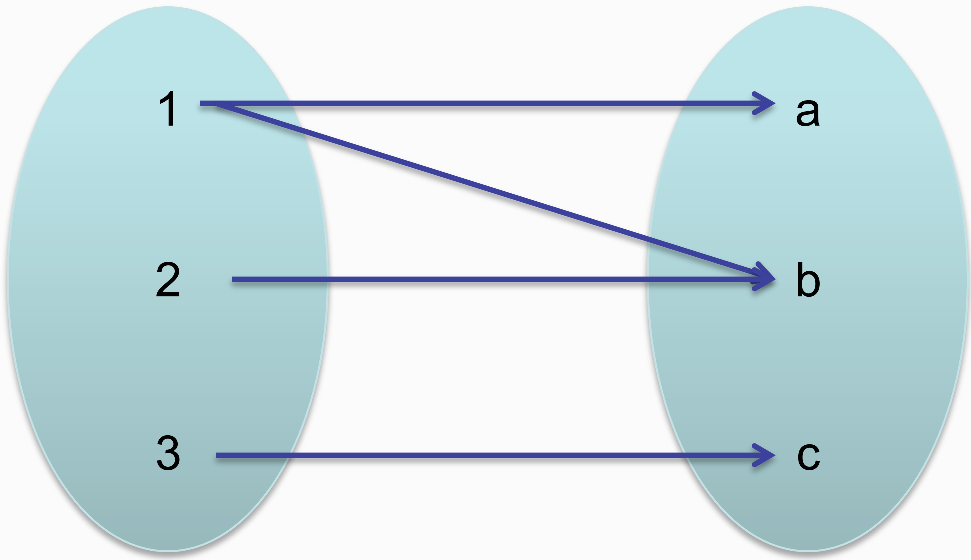
- **Example 2:** Let consider X = {1, 2, 3} and Y = {a, b, c}.

$$f = \{(1,a),(2,b),(3,c),(1, b)\}$$



- ***f is not a function from X = {1, 2, 3} to Y = {a, b, c} because the element 1 in X is not assigned a <u>unique</u> element in Y (1 is assigned to tow different values in Y which are a and b).***

Functions

- **Definition:** For each element x in the domain X, there is exactly one element y in the codomain Y with (x, y) $\in$ *f*.

$$f(x) = y$$

- **Example 1:** Let consider X = {1, 2, 3} and Y = {a, b, c} and *f* be the function defined *f*(x) = y

$$f(1) = a$$

$$f(2) = b$$

$$f(3) = c$$

- **Example 2:** Let consider X = {1, -2, 0} and Y = {a, b, c} and *f* be the function defined *f*(x) = x^2

$$f(1) = 1$$

$$f(-2) = 4$$

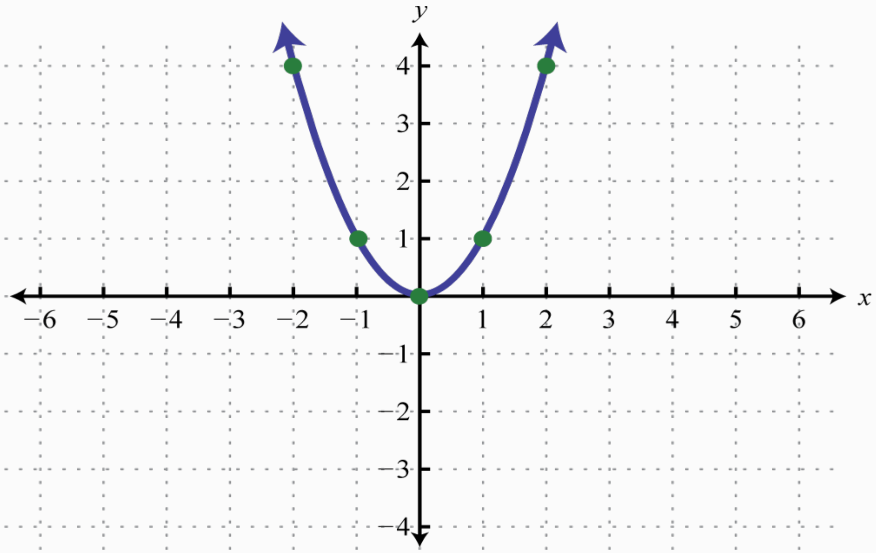
$$f(0) = 0$$

Functions

- Another way to visualize a function is to draw a graph.
- The **graph of a function** *f* whose domain and codomain are subsets of the real numbers is obtained by plotting points in the plane that correspond to the elements in f.
- The domain is contained in the horizontal axis (x) and the codomain is contained in the vertical axis (y).
- **Example:** The graph of the function *f*(x) = x^2

$$f(x) = x^2$$

x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4



Functions

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- **Definition:** If x is an integer and y is a positive integer, we define x mod y to be the remainder when x is divisible by y.
- **Example 1:** We have

  6 mod 2 = 0 , 5 mod 1 = 0 , 8 mod 12 = 8 , 199673 mod 2 = 1

- **Example 2:** We must store or retrieve the number n, we might take as first choice for a location n, n mod 11. We have h(n) = n mod 11

  15 mod 11 = 4 , 558 mod 11 = 8 , 32 mod 11 = 10 , 132 mod 11 = 0 ,

  102 mod 11 = 3 , 5 mod 11 = 5 , 257 mod 11 = 6

0	1	2	3	4	5	6	7	8	9	10
132			102	15	5	257		558		32

➢ ***We call this approach a hash function that takes a data item to be stored or retrieved and computes the first choice for n location for the item.***

Functions

- **Definition:** The *floor* of x, denoted $\lfloor x \rfloor$, is the greatest integer less than or equal to x. The *ceiling* of x, denoted $\lceil x \rceil$, is the least integer greater than or equal to x.
- **Example 1:**

$\lfloor 8.3 \rfloor = 8$, $\lceil 9.1 \rceil = 10$, $\lfloor -8.7 \rfloor = -9$, $\lceil -11.3 \rceil = -11$, $\lceil 6 \rceil = 6$, $\lceil -8 \rceil = -8$

- **Example 2:** Let consider a function $f(u) = 80 + 17\lceil u - 1 \rceil$, $0 \le u \le 13$.

If u = 3.7, $f(u) = 80 + 17\lceil 3.7 - 1 \rceil = 80 + 17 \times \lceil 2.7 \rceil = 80 + 17 \times 3 = 131$

If u = 2, $f(u) = 80 + 17\lceil 2 - 1 \rceil = 80 + 17 \times \lceil 1 \rceil = 80 + 17 \times 1 = 97$

If u =0, $f(u) = 80 + 17\lceil 0 - 1 \rceil = 80 + 17 \times \lceil -1 \rceil = 80 + 17 \times (-1) = 63$

Functions

- **Definition:** A function f from X to Y is said to be *one-to-one* (*or injective*) if for each $y \in Y$, there is at most one $x \in X$ with $\boldsymbol{f(x) = y}$.
- **Example 1:** The function $f = \{(1,b),(3,a),(2,c)\}$

From X = {1, 2, 3} and Y = {a, b, c} is one-to-one.

- **Example 2:** The function $f = \{(1,a),(2,b),(3,a)\}$

Is not one-to-one since $f(1) = a = f(3)$

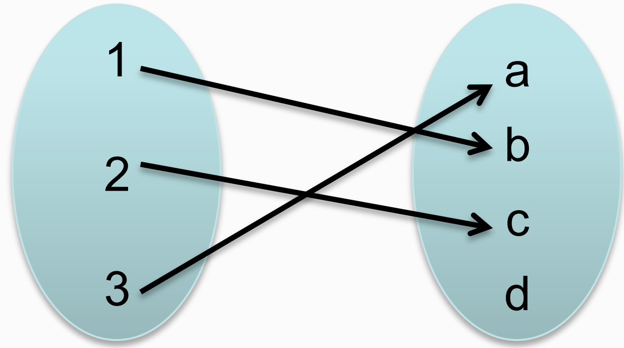
- **Example 3:** If X is the set of persons who have social security numbers and we assign each person $x \in X$ his or her social security number SS(x). We obtain a one-to-one function since distinct persons are always assigned distinct social security numbers. It is because this correspondence is one-to-one that the government uses social security numbers as identifiers.

Functions

- **Example 4:** Let consider this function $f = \{(1,b),(3,a),(2,c)\}$

Where $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$

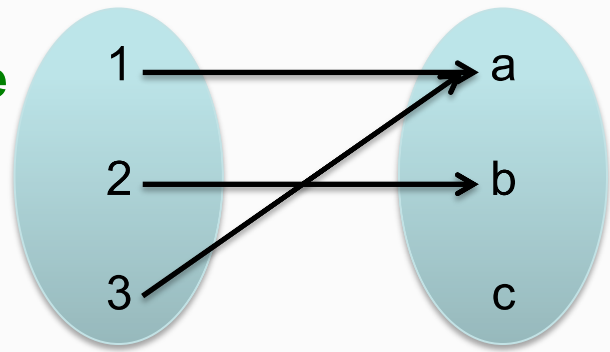
This function is one-to-one because each element in X has at most one element in Y.



- **Example 5:** Let consider this function $f = \{(1,a),(2,b),(3,a)\}$

Where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$

This function is not one-to-one because two elements in X has the same element in Y.



Functions

- **Definition:** For a function *f* from X to Y is one-to-one and equivalent to: For all x1, x2 $\in$ X, if *f*(x1) = *f*(x2), then x1 = x2.

$$(f(x1) = f(x2)) \longrightarrow (x1 = x2)$$

- **Example :** We have to prove that the function *f*(n) = 2n + 1 from the set of positive integers to the set of positive integers is one-to-one.

We must show that for all positive integers n1 and n2, if *f*(n1)=*f*(n2), then n1=n2.

So, suppose that *f*(n1)=*f*(n2). Using the definition of *f*, we can say that

$$2n1 + 1 = 2n2 + 1$$

$$2n1 = 2n2$$

$$n1 = n2$$

Therefore, ***f* is one-to-one**.

Functions

- **Definition:** A function f from X to Y is not one-to-one and equivalent to: For all (x1, x2)$\in$ X, if f(x1) = f(x2), then x1 ≠ x2.

$$(f(x1) = f(x2)) \longrightarrow (x1 \neq x2)$$

- **Example :** We have to prove that the function $f(n) = 2^n - n^2$ from the set of positive integers to the set of integers is not one-to-one.

We must find positive integers n1 and n2, and n1 ≠ n2, such that

$$f(n1) = f(n2)$$

$$\text{If } n1 = 2,\ f(2) = 2^2 - 2^2 = 0$$

$$\text{If } n2 = 4,\ f(4) = 2^4 - 4^2 = 0$$

We find that, f(2) = f(4) but n1 ≠ n2

Therefore, ***f* is not one-to-one**.

Functions

- **Definition:** If *f* is a function from X to Y and the range of *f* is Y, *f* is said to be **onto** Y (or an *onto function or a surjective function*).
- **Example 1:** The function $f = \{(1,a),(2,c),(3,b)\}$

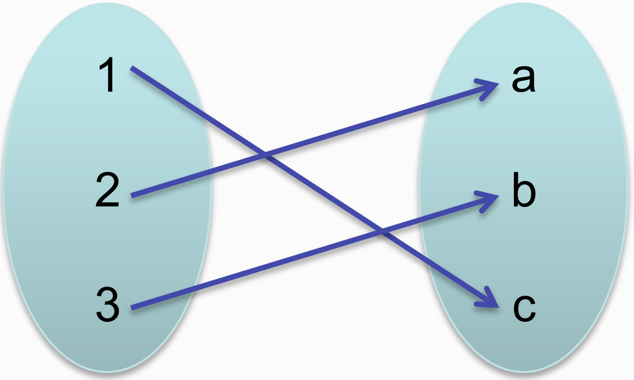
From X = {1, 2, 3} and Y = {a, b, c}. This function is one-to-one and onto Y.

- **Example 2:** Let consider this function $f = \{(1,c),(2,a),(3,b)\}$

Where X = {1, 2, 3} and Y = {a, b, c}

This function is one-to-one and onto.

- **f is one-to-one because each element in X has at least one element in Y.**
- **f is onto because each element in Y has at least one element from X pointing to it.**



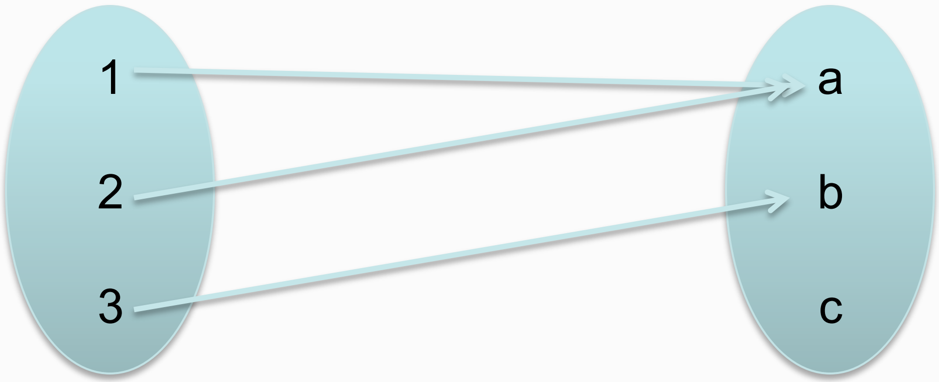
Functions

- **Example 3:** Let consider this function $f = \{(1,a),(2,a),(3,b)\}$

Where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$

This function is neither one-to-one nor onto.

- **f is not one-to-one because two elements (1 and 2) in X have the same element in Y (a).**
- **f is not onto because one element in Y (a) has two elements from X (1 and 2) pointing to them.**



Functions

- **Definition:** A function f from X to Y is onto and equivalent to: For all $y \in Y$, there exists $x \in X$ such that $\boldsymbol{f(x) = y}$
- **Example :** We have to prove that the function $f(u) = {}^{1}\!/\!{}_{u^2}$ from the set X of non-zero real numbers to the set Y of the positive real numbers is onto Y.

We must show that for every $y \in Y$, there exists $u \in X$ such that $f(u) = y$.

$$f(u) = y$$

$${}^{1}\!/\!{}_{u^2} = y$$

$$u = \pm 1/\sqrt{y}$$

Notice that $1/\sqrt{y}$ is defined because y is a positive real number. If we take u to be the positive square root : $\boldsymbol{u = 1/\sqrt{y}}$

Thus, for every $y \in Y$, there exists u, namely $u = 1/\sqrt{y}$ such that

$$f(u) = f(1/\sqrt{y}) = 1/(1/\sqrt{y})^2 = y$$

Therefore, ***f* is onto Y**.

Functions

- **Definition:** A function f from X to Y is not onto and equivalent to: For all y $\in$ Y, there exists x $\in$ X such that $f(x) \neq y$
- **Example 1:** We have to prove that the function $f(n) = 2n - 1$

from the set X of positive integers to the set Y of positive integers is *not onto* Y.

We must find an element m $\in$ Y such that for all n $\in$ X, $f(n) \neq m$. Since $f(n)$ is an odd integer for all n, we may choose for y any positive, even integer, for example, $y = 2$. Then y $\in$ Y and

$$f(n) \neq y \qquad \text{For all } n \in X$$

Therefore, f is not onto y.

Functions

- **Definition:** A function *f* from X to Y that is both one-to one and onto is called a **bijection**.
- **Example 1:** The function $f = \{(1,a),(2,c),(3,b)\}$

From X = {1, 2, 3} and Y = {a, b, c}. This function is both one-to-one and onto from X to Y and f is called a bijection.

- **Example 2:** If *f* is a bijection from a finite set X to a finite set Y, then |X| = |Y|, that is, the sets have the same cardinality and are the same size. For example:

$$f = \{(1,a),(2,b),(3,c),(4,d)\}$$

This function is bijection from X = {1, 2, 3, 4} and Y = {a, b, c, d}. Both sets have four elements. In effects, *f* counts the elements in Y :

$f(1) = a$ is the first element in Y

$f(2) = b$ is the second element in Y

$f(3) = c$ is the third element in Y

$f(4) = d$ is the fourth element in Y

Functions

- **Definition:** A function *f* from X to Y is called inverse where *f is* one-to one and onto, equivalent to: $(x,y) \in f$ *, and* denoted f^{-1} .
- **Example 1:** Let consider the function

$$f = \{(1,a),(2,c),(3,b)\}$$ Where X = {1, 2, 3} and Y = {a, b, c}

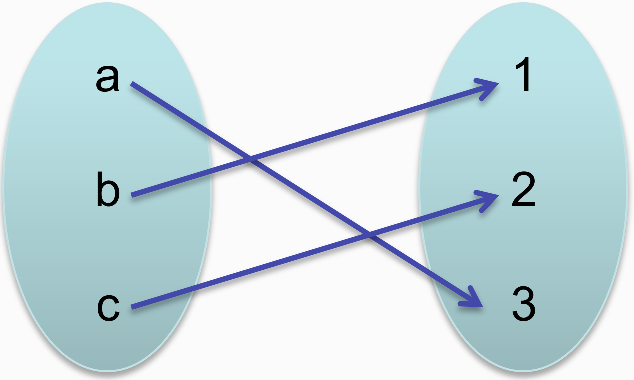
$$f^{-1} = \{(a,1),(c,2),(b,3)\}$$

- **Example 2:** Let consider this function $f = \{(1,b),(2,c),(3,a)\}$, X = {1, 2, 3} and Y = {a, b, c}

So, $f^{-1} = \{(b,1),(c,2),(a,3)\}$

f^{-1} is the inverse of the function f.

◆ The inverse is obtained by reversing all of the element of Y to X.



Functions

- **Definition:** Let g be a function from X to Y and l *f* be a function from Y to Z. The *composition* of *f* with g, denoted ***f* o g**, is the function

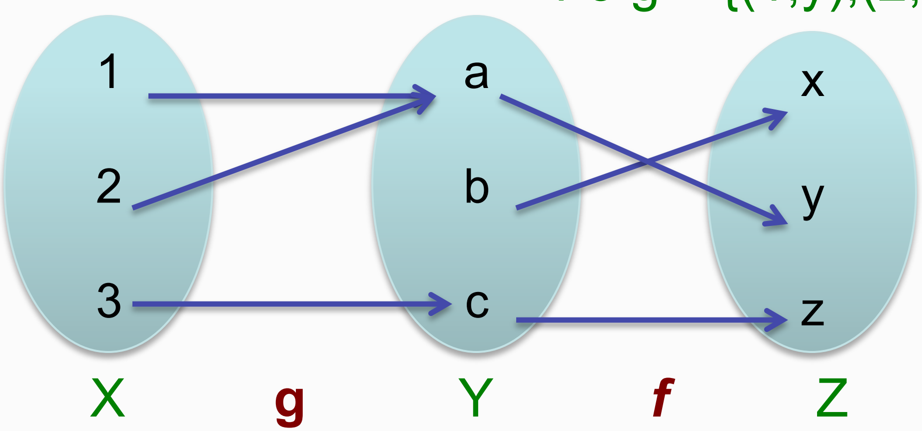
$$(f \circ g)(x) = f(\, g(x)\,)$$

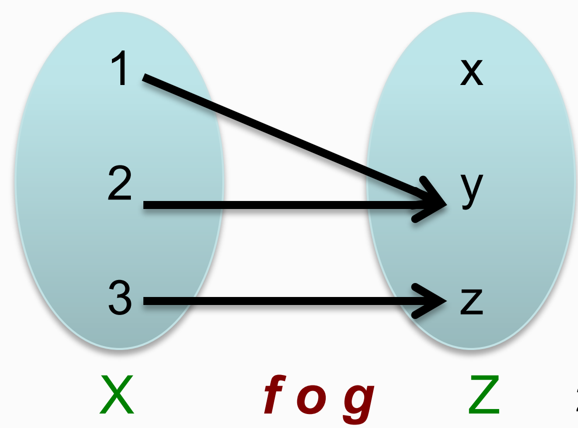
From X to Z.

- **Example :** Let consider the function *g* = {(1,a),(2,a),(3,c)}

From X = {1, 2, 3} and Y = {a, b, c}, and *f* = {(a,y),(b,x),(c,z)} a function from Y to Z = {x, y, z}, the composition function from X to Z is the function

f o g = {(1,y),(2,y),(3,z)}





Exercises

Exercise 1: Find the element of each set, draw a graph and determine if the the function is one-to-one, onto or both. If it is one-to-one and onto, give the description of the inverse function as a set of ordered pairs, draw a graph and identify the element of each set.

- S = [(1,a),(2,a),(3,c),(4,b)]
- K = [(1,c),(2,d),(3,a),(4,b)]
- V = [(1,d),(2,d),(4,a)]

Exercise 2: Determine whether each function is one-to-one, onto, or both. The domain and codomain of each function is the set of all integers.

- $f(x) = n + 1$
- $f(x) = | n |$
- $f(x) = n^2$

Exercise 3: Let each function is one-to-one on the specified domain X. If Y = range of f, we obtain a bijection from X to Y. Find each inverse function

❖ $f(x) = 4x + 2$	x = set of real numbers
❖ $f(x) = 3^x$	x = set of real numbers
❖ $f(x) = 3 + 1/x$	x = set of nonzero real numbers

Exercises

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Exercise 4: Consider the function g = {(1, b), (2, c), (3, a)} from X = {1, 2, 3} to Y = {a, b, c, d}, and

f = {(a, x), (b, x), (c, z), (d, w)},

a function from Y to Z = {w, x, y, z}.

1. Determine f o g as a set of ordered pairs. D
2. Draw the arrow diagram of f o g.

Exercise 5: Let f be the function from X = {0, 1, 2, 3, 4} to X defined by f(x) = 4x mod 5

1. Determine f as a set of ordered pairs.
2. Draw the arrow diagram of f.
3. Determine if f is one-to-one or onto.

Exercise 6: Let the function g = {(1, a), (2, c), (3, c)} be a function from X = {1, 2, 3} to Y = {a, b, c, d}. Let S = {1}, T = {1, 3}, U = {a} and V = {a, c}.

1. Determine g(S)
2. Determine g(T)
3. Determine $g^{-1}(U)$
4. Determine $g^{-1}(V)$

Sequences and strings

Sequences and strings

- **Definition:** A *sequence* is a special type of function in which the domain

consists of a set of consecutive integers.

Let **Sn** denoted the entire sequence:

S1, S2, S3, S4, S5, …

We use the notation Sn to denote the single element of the sequence S at *index* n.

- **Example:** Consider the sequence S

2, 4, 6, …, 2n, …

S1 : The first element of the sequence is 2

S2 : The second element of the sequence is 4

S3 : The third element of the sequence is 6

Sn : The nth element of the sequence is 2n

S1 = 2, S2 = 4, S3 = 6, …, Sn = 2n

Sequences and strings

- **Definition:**

① If the domain of the sequence is infinite, we say that the *sequence is infinite*.

② If the domain of the sequence is finite, we say that the *sequence is finite*.

- **Example 1:** Consider the sequence D

3, 5, 7, …, 3n, …

D1 = 3, D2 = 5, D3 = 7, Dn = 3n

The sequence D is infinite, we can write $\{Sn\}_{n=k}^{\infty}$

- **Example 2:** Consider the sequence T

-1, 0, 1, 2, 3

T1 = -1, T2 = 0, T3 = 1, T4 = 2, T5 = 3

The sequence D is finite, we can write $\{T_n\}_{n=-1}^{3}$

Sequences and strings

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- **Example 3:** If X is the sequence defined by

$$Xn = \frac{1}{2^n} \qquad -1 \le n \le 4$$

The elements of X are

$$n = -1 \longrightarrow X_{-1} = 2$$
$$n = 0 \longrightarrow X_0 = 1$$
$$n = 1 \longrightarrow X_1 = 1/2$$
$$n = 2 \longrightarrow X_2 = 1/4$$
$$n = 3 \longrightarrow X_3 = 1/8$$
$$n = 4 \longrightarrow X_4 = 1/16$$

Sequences and strings

- **Example 4:** Define a sequence S as

$$S_n = 2^n + 4 \times 3^n \qquad n \geq 0$$

1) Find S0 : $S_0 = 2^0 + 4 \times 3^0 = 1 + 4 \times 1 = 5$
2) Find S1 : $S_1 = 2^1 + 4 \times 3^1 = 2 + 4 \times 3 = 14$
3) Find a formula of Si : $S_i = 2^i + 4 \times 3^i$
4) Find a formula for Sn-1 : $S_{n-1} = 2^{n-1} + 4 \times 3^{n-1}$
5) Find a formula for Sn-2 : $S_{n-2} = 2^{n-2} + 4 \times 3^{n-2}$
6) Prove that {Sn} satisfies: $S_n = 5 S_{n-1} – 6 S_{n-2}$ for all $n \geq 2$

$$5 S_{n-1} – 6 S_{n-2} = 5 \times (2^{n-1} + 4 \times 3^{n-1}) – 6 \times (2^{n-2} + 4 \times 3^{n-2})$$

$$= (5\times2 – 6) \times 2^{n-2} + (5\times4\times3 – 6\times4) \times 3^{n-2}$$

$$= 4 \times 2^{n-2} + 36 \times 3^{n-2}$$

$$= 2^2 \times 2^{n-2} + (4 \times 3^2) \times 3^{n-2}$$

$$= 2^n + 4 \times 3^n = S_n$$

Sequences and strings

- **Definition:**
  - A sequence S is **increasing** if $S_n < S_{n+1}$ for all n for which n and n+1 are in the domain of the sequence.
  - A sequence S is **decreasing** if $S_n > S_{n+1}$ for all n for which n and n+1 are in the domain of the sequence.
  - A sequence S is **nondecreasing** if $S_n \leq S_{n+1}$ for all n for which n and n+1 are in the domain of the sequence.
  - A sequence S is **nonincreasing** if $S_n \geq S_{n+1}$ for all n for which n and n+1 are in the domain of the sequence.

Sequences and strings

- **Example 1:** The sequence

$$2, 5, 13, 104, 300$$

is increasing and nondecreasing

- **Example 2:** The sequence

$$A_i = \frac{1}{i} \qquad i \geq 1$$

is decreasing and not nonincreasing

- **Example 3:** The sequence

$$100, 90, 90, 74, 74, 74, 30$$

is nonincreasing, but it is not decreasing.

- **Example 4:** The sequence

$$100$$

Is increasing, decreasing, nonincreasing and nondecreasing since there is no value of i for which both i and i+1 are indexes.

Sequences and strings

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- **Definition:** Let {Sn} be a sequence defined for n = m, m+1, …, and let n1, n2, … be an increasing sequence whose values are in the set {m, m+1, …}. We call the sequence {Sn} a *subsequence* of Sn.
- **Example 1:** The sequence b, c

Is a subsequence of the sequence Tn $1 \le n \le 5$

T1 = a, T2 = a, T3 = b, T4 = c, T5 = q

- *The **subsequence** is obtained from the sequence Tn by choosing the third and the fourth terms.*

- **Example 2:** The sequence c, b

Is not a subsequence of the sequence Tn $1 \le n \le 5$

T1 = a, T2 = a, T3 = c, T4 = b, T5 = q

- *Its **not a subsequence** of the sequence Tn since the order of terms in the sequence Tn is not maintained.*

Sequences and strings

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- **Definition:** If {ai} is a sequence, we define

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_n \quad \text{or} \quad \prod_{i=m}^{n} a_i = a_m \times a_{m+1} \times \ldots . \times a_n$$

The formalism $\sum_{i=m}^{n} a_i$ is called the *sum (or sigma) notation* and $\prod_{i=m}^{n} a_i$ is called the *product notation*. We called i the *index*, m the *lower limit* and n the *upper limit*.

- **Example 1:** Let a be the sequence defined by a_n= 2n , n ≥1. Then

$$\sum_{i=1}^{3} a_i = a_1 + a_2 + a_3 = 2 + 4 + 6 = 12$$

$$\prod_{i=1}^{3} a_i = a_1 \times a_2 \times a_3 = 2 \times 4 \times 6 = 48$$

Sequences and strings

- **<u>Example 2:</u>** The geometric sum $a + ar^1 + ar^2 + \ldots + ar^n$

Can be written compactly using the sum notation as $\sum_{i=0}^{n} ar^i$

- **<u>Example 3:</u>** Let consider the sum $\sum_{i=0}^{n} ir^{n-i}$

Where i = j – 1

$ir^{n-i} = (j - 1)\, r^{n-(j-1)} = (j - 1)\,)\, r^{n-j+1}$

Since j = i + 1, when i = 0, j = 1. Thus the lower limit for j is 1. Similarly, when i = n, j = n+1, and the upper limit for j is n+1. Therefore,

$$\sum_{i=0}^{n} ir^{n-i} = \sum_{j=1}^{n+1} (j-1) r^{n-j+1}$$

Sequences and strings

- **Definition 1:** A *string* is a finite sequence of characters. In

programming languages, strings can be used to denote text. For example, in Java: `“Let’s read Rolling Stone”`

Denotes the string consisting of the sequence of characters : let’s read Rolling Stone.

- **Definition 2:** A string over X, where X is finite set, is a finite sequence of elements from X.
- **Example 1:** Let X = {a, b, c}. If we let

$$T1 = b, T2 = a, T3 = a, T4 = c$$

We obtain a string over X. The string is written baac.

Sequences and strings

- **Definition :**
  - ❖Since a string is a sequence, order is taken into account. For example, the string *baac* is different from the string *acab*.
  - ❖Repetition in a string can be specified by superscripts. For example, the string bbaaac may be written b^2a^3c.
  - ❖The string with no element is called *null string* and is denoted **λ**.

Exercises

Exercise 1: Consider the sequence S defined by c, d, d, c, d, c

1. Find S1
2. Find S4
3. Determine S as a string

Exercise 2: Consider the sequence T defined by Tn = 2n - 1

1. Find T1
2. Find T100
3. Find $\sum_{i=1}^{3} T_i$
4. Find $\prod_{i=3}^{6} T_i$

Exercise 3: Consider the sequence Q defined by Q1 = 8, Q2 = 12, Q3 = 12, Q4 = 28, Q5 = 33

1. Find $\sum_{i=2}^{4} Q_i$
2. Find $\sum_{k=2}^{4} Q_k$
3. Is Q increasing?
4. Is Q decreasing?
5. Is Q nonincreasing?
6. Is Q nondecreasing?

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Exercises

Exercise 4: Consider the sequence A defined by An = n^2 - 3n +3

1. Find $\sum_{i=1}^{4} A_i$
2. Find $\sum_{J=3}^{5} A_J$
3. Find $\prod_{i=1}^{2} A_i$
4. Find $\prod_{x=3}^{4} A_x$
5. Is A increasing?
6. Is A decreasing?
7. Is A nonincreasing?
8. Is A nondecreasing?

Exercise 5: Consider the sequence Y and Z defined by

Yn = 2^n – 1 Zn = n(n-1)

1. Find $\left(\sum_{i=1}^{3} Y_i\right)\left(\sum_{i=1}^{3} Z_i\right)$
2. Find $\left(\sum_{i=1}^{5} Y_i\right)\left(\sum_{i=1}^{4} Z_i\right)$
3. Find $\sum_{i=1}^{4} Y_i Z_i$
4. Find $\left(\sum_{i=3}^{4} Y_i\right)\left(\prod_{i=2}^{4} Z_i\right)$

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Relations

Relations

- **Definition 1:** We define a relation to be a set of ordered pairs.
- **Definition 2:** A *binary* relation R from a set X *to a set* Y is a subset of the Cartesian product X × Y.
  - If $(x, y) \in R$, We write **x R y** and say that x *is related to* y.
  - If X = Y, we call R a *binary relation* on X.
- **Example 1:** In this table, We consider the first element of the ordered pair to be related to the second element of the ordered pair.

Relation of students to courses

Student	Bill	Mary	Bill	Beth	Beth	Dave
Course	CompSci	Math	Art	History	CompSci	Math

If we let X = {Bill, Mary, Beth, Dave}

And Y= {CompSci, Math, Art, History}

The relation R can be written **R = {(Bill, CompSci), (Mary, Math), (Bill, Art), (Beth, History), (Beth, CompSci), (Dave, Math)}.**

Since (Beth, History) $\in$ R, we may write **Beth R History**.

Relations

- **<u>Example 2:</u>** Let X = {2, 3, 4} and Y = {3, 4, 5, 6, 7}

If we define a relation R from X and Y by

$(x, y) \in R$ **if X divides Y**

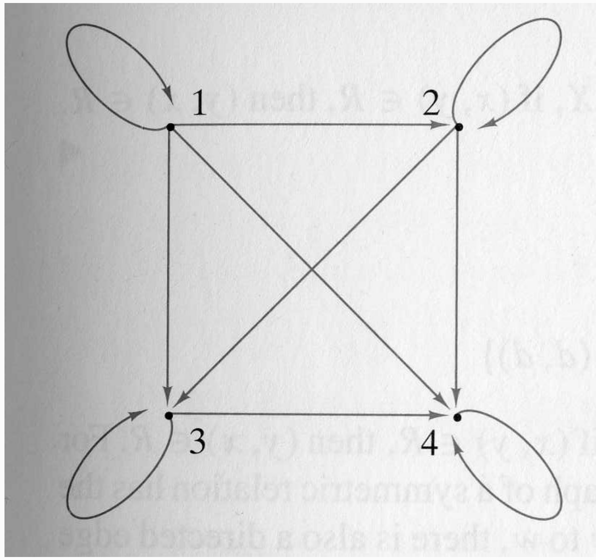
We obtain R = {(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)}

If we write R as a table, we obtain

X	Y
2	4
2	6
3	3
3	6
4	4

Relations

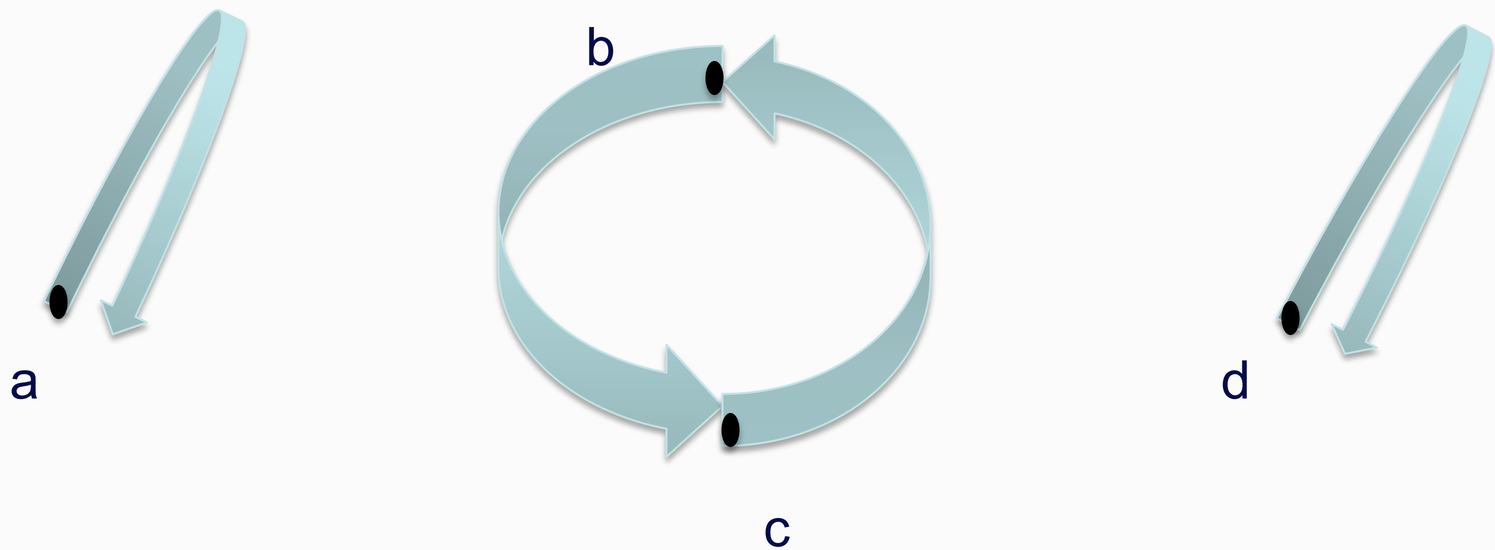
- **Example 3:** Let R be the relation on X ={1, 2, 3, 4} defined by (x, y) $\in$ R if $x \le y$, x, y $\in$X.



Then, **R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}**

Relations

- **Example 4:** Let R be the relation on X ={1, 2, 3, 4} given by this digraph



R = {(a, a), (b, c), (c, b), (d, d)}

Relations

- **Definition :** We define a relation R on a set X is ***reflexive*** if

$(x, x) \in R$ for every $x \in X$.

- **Example 1:** The relation $R = \{(1, 1), (2, 2), (3, 3)\}$ defined by $(x, y) \in R$ is $x \leq y$, $x, y \in X$, is reflexive because for each element $x \in X$, $(x, x) \in R$; specially, (1, 1), (2, 2) and (3, 3) are each in R.

- **Example 2:** The relation $R = \{(a, a), (b, c), (c, b), (d,d)\}$

On $X =\{a, b, c, d\}$ is not reflexive. For example, $b \in R$ but $(b, b) \notin R$ and $(c, c) \notin R$.

Relations

- **Definition :** We define a relation R on a set X is ***symmetric*** if

for all $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.

- **Example 1 :** The relation

$$R = \{(a, a), (b, c), (c, b), (d, d), (a, b), (b, a), (c, a), (a, c), (d, a), (a, d), (b, d), (d, b), (c, d), (d, c), (b, b), (c, c)\}$$

On $X = \{a, b, c, d\}$ is symmetric because for all x, y, if $(x, y) \in R$, then $(y, x) \in R$. for example, (b, c) is in R and (c, b) is also in R.

- **Example 2 :** The relation $R = \{(1,1), (2,1), (2,2)\}$ defined by $(x, y) \in R$, if $x \leq y$, $x, y \in X$, is not symmetric. For example, $(2, 1) \in R$ but $(1, 2) \notin R$.

Relations

- **Definition :** We define a relation R on a set X is ***antisymmetric*** if for all x, y$\in$ X, if (x, y)$\in$ R, and (y, x) $\in$R then x = y.

- **Example :** The relation R on R = {(1,1), (2,1), (1, 2), (2,2)} defined by (x,y)$\in$R, If x $\leq$ y, x, y $\in$X, is antisymmetric Because for all x, y, if (x, y)$\in$R and (y, x)$\in$R, then x = y.

Relations

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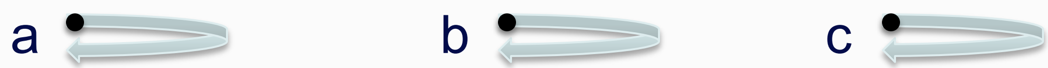
- <u>**Definition :**</u> We define a relation R has ***no members*** of the form (x, y),

$x \neq y$, we say that the equivalent characterization of "antisymmetric" for all $x, y \in X$, if $x \neq y$, $(x, y) \notin R$, and $(y, x) \notin R$.

- <u>**Example 1:**</u> The relation R = {(a, a), (b, b), (c, c)} on X ={a, b, c}



  - ➢ The relation R is reflexive and antisymmetric. This example shows that "antisymmetric" is not the same as "not symmetric"

- <u>**Example 2:**</u> The relation

$$R = \{(b, a), (a, b)\}$$

on X = {a, b, c, d} is symmetric and not antisymmetric because both (b, a) and (a, b) are in R. There are two directed edge between b and a.

Relations

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- **Definition :** We define a relation R on a set X is ***transitive*** if for all $x, y, z \in X$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

- **Example 1:** The relation R on X = {(1, 2), (2, 3), (1, 3)} is defined by $(x, y) \in R$ if $x \leq y$, $x, y \in X$, is transitive because all x, y, z, if (x, y) and $(y, z) \in R$, then $(x, z) \in R$.
- **Example 2:** The relation

$$R = \{(a, a), (b, c), (c, b), (d, d)\}$$

on X = {1, 2, 3, 4} is not transitive. For example, (b, c) and (c, b) are in R, but (b, b) and (c, c) are not in R.

Relations

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- **Definition :** We define a relation R on a set X is ***partial order*** if R is reflexive, antisymmetric, and transitive.

- **Example :** The relation

R ={(1, 1),(2, 2), (3, 3), (1, 2), (2, 3), (1, 3), (2, 1), (3, 2), (3, 1))}

On X ={1, 2, 3, 4, 5} is reflexive, antisymmetric, and transitive.

◆ *Reflexive :* (1, 1), (2, 2) and (3, 3) are each in R.

◆ Antisymmetric : (1, 1), (2, 2), (3, 3), (4, 4) and (5, 5) are each in R.

◆ Transitive : (1, 2), (2, 3) and (1, 3)are each in R.

Relations

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- **Definition :** We define a relation R from X to Y. The ***inverse*** of R, denoted R^{-1}, is the relation from Y to X defined by

$$\mathbf{R^{-1} = \{(y, x) \mid (x, y) \in R\}}$$

- **Example :** If we define a relation R from X ={2, 3, 4} to Y = {3, 4, 5, 6, 7} by $(x, y) \in R$ if x divides y.

  We obtain R ={(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)}

  The inverse of this relation is

$$R^{-1}=\{(4, 2), (6, 2), (3, 3), (6, 3), (4, 4)\}$$

  We might describe this relation as "is divisible by".

Relations

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- **Definition :** We define a relation R1 from X to Y and R2 from Y to Z. The ***composition*** of R1 and R2, denoted R1 o R2, is the relation from X to Z defined by

$$R1 \circ R2 = \{(x, z) \mid (x, y) \in R1 \text{ and } (y, z) \in R2\}$$

- **Example :** The composition of the relations

$$R1 = \{(1, 2), (1, 6), (2, 4), (3, 4), (3, 6), (3, 8)\}$$

And $R2 = \{(2, u), (4, s), (4, t), (6, t), (8, u)\}$

Is $R1 \circ R2 = \{(1, u), (1, t), (2, s), (2, t), (3, t), (3, u), (3, s)\}$

For example, $(1, u) \in R1 \circ R2$ because $(1, 2) \in R1$ and $(2, u) \in R2$.

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Exercises

Exercise 1: Write the relation as a set of ordered pairs

1.

8840	Hammer
9921	Pliers
451	Paint
2207	Carpet

2.

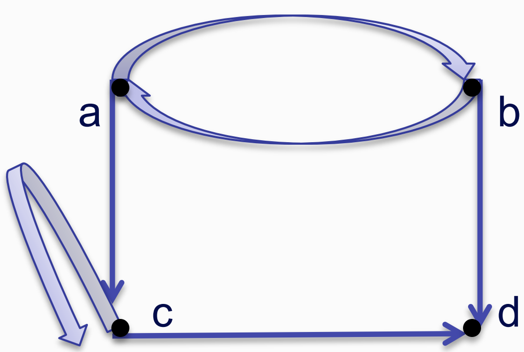
a	a
b	b

Exercise 2: Write the relation as a table of ordered pairs

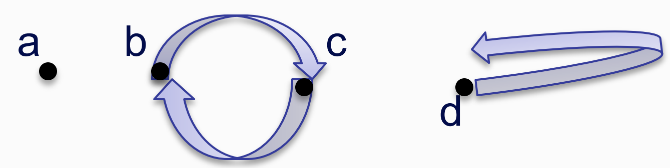
1. R = {(a, 1), (b, 2), (a, 1), (c, 1)}
2. R = {(roger, music), (pat, History), (Ben, Math), (Pat, Polysci)}

Exercise 3: Write the relation as a set of ordered pairs.

1.



2.



Exercises

Exercise 4: Consider the relation R on the set {1, 2, 3, 4, 5} defined by the rule $(x, y) \in R$ if 3 divides $x - y$

1. List the element of R
2. List the element of R^{-1}
3. Is the element of R is reflexive, symmetric, antisymmetric, transitive and/or partial order?

Exercise 5: Consider the relation R on the set {1, 2, 3, 4, 5} defined by the rule $(x, y) \in R$ if $x + y \leq 6$

1. List the element of R
2. List the element of R^{-1}
3. Is the element of R is reflexive, symmetric, antisymmetric, transitive and/or partial order?

Exercise 6: Let R1 and R2 be the relations on {1, 2, 3, 4} given by

$$R1 = \{(1, 1), (1, 2), (3, 4), (4, 2)\}$$

$$R2 = \{(1, 1), (2, 1), (3, 1), (4, 4), (2, 2)\}$$

1. List the element of R1 o R2
2. List the element of R o R1

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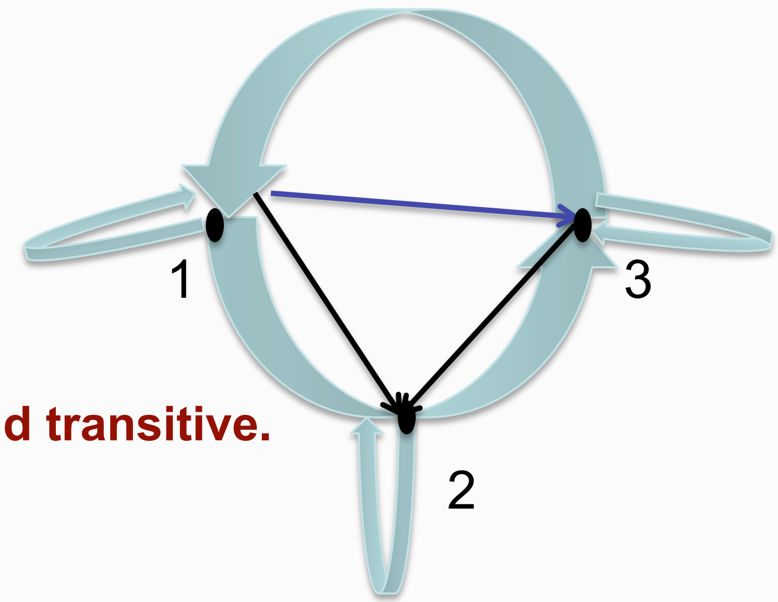
Equivalence Relations

Equivalence Relations

- **Definition :** We define a relation that is reflexive, symmetric, and transitive on a set X is called an ***equivalence relation*** on X.
- **Example 1:** Let consider the relation R is an equivalence relation on X = {1, 2, 3}

$$R = \{(1, 1), (1, 3), (3, 1), (2, 1), (1, 2), (2, 3), (3, 2), (3, 3)\}$$

The digraph of the relation R :



➢ **R is reflexive, symmetric and transitive.**

Equivalence Relations

- **Example 2:** Let consider the relation

R = {(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (5, 1), (5, 3), (5, 5), (4, 5), (5, 4), (3, 4), (4, 3), (2, 4), (4, 2)}

On X ={1, 2, 3, 4, 5}.

  - The relation is reflexive because (1, 1), (2, 2), (3, 3), (4, 4) and (5, 5) ∈ R.
  - The relation is symmetric because whenever (x, y) is in R, (y, x) is also in R.
  - The relation is transitive because whenever (x, y) and (y, z) are in R, (x, z) is also in R.

➢ **Since R is reflexive, symmetric, and transitive, R is an equivalence relation on X = {1, 2, 3, 4, 5}.**

Equivalence Relations

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- **<u>Example 2:</u>** Let consider the relation R on X = {(1, 1), (2, 2), (1, 2)} defined by (x, y)$\in$ R if $x \leq y$, x, y $\in$ X, is not equivalence relation because R is not symmetric. For example, (1, 2) $\in$ R, but (2, 1) $\notin$ R. The relation R is reflexive and transitive.

- **<u>Example 3:</u>** Let consider the relation

$$R = \{(a, a), (b, c), (c, b), (d, d)\}$$

on X = {a, b, c, d} is not an equivalence relation because R is neither reflexive nor transitive. It’s not reflexive because, for example, (b, b) $\notin$ R. it is not transitive because, for example, (b, c) and (c, b) are in R, but (b, b) in not in R.

Equivalence Relations

- **Definition :** Let R an equivalence relation on a set X. For each a$\in$ R, let **[a] = {x $\in$ X | x R a}**

We define [a] is the set of all elements in X that are related to a. We called the sets [a] *the equivalence classes of X given by* the relation R.

- **Example 1:** Let consider the relation

R = {(1, 1), (1, 3), (3, 1), (2, 1), (1, 2), (2, 3), (3, 2), (3, 3)}

On X ={1, 2, 3} is an equivalence relation. The equivalence class [1] containing 1 consists of all x such that (x, 1) $\in$ R. Therefore,

[1] = [2] = [3] = {1, 2, 3}

Equivalence Relations

- **Example 2:** Let consider the relation

$$R = \{(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (5, 1), (5, 3), (5, 5)\}$$

On $X = \{1, 2, 3, 4, 5\}$ is an equivalence relation. The equivalence class [1] containing 1 consists of all x such that $(x, 1) \in R$ and the equivalence class [2] containing 2 consists of all x such that $(x, 2) \in R$
Therefore,

$$[1] = [3] = [5] = \{1, 3, 5\}$$

$$[2] = [4] = \{2, 4\}$$

Equivalence Relations

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- **<u>Example 3:</u>** Let consider the relation

$$R = \{(a, a), (b, b), (c, c)\}$$

On X ={a, b, c} is reflexive, symmetric and transitive. Thus R is an equivalence relation. The equivalence class [a] containing a consists of all x such that $(x, a) \in R$, the equivalence class [b] containing b consists of all x such that $(x, b) \in R$ and the equivalence class [c] containing c consists of all x such that $(x, c) \in R$.

The equivalence classes are

$$[a] = \{a\}$$

$$[b] = \{b\}$$

$$[c] = \{c\}$$

Exercises

Exercise 1: Determine whether the relation is an equivalence relation on x, y∈{1, 2, 3, 4, 5} . If the relation is an equivalence relation, list the equivalence classes.

1. [(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1)]
2. [(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 5), (5, 1), (3, 5), (5, 3), (1, 3), (3, 1)]
3. {(x, y)| 3 divides by x +y }

Exercise 2: Determine the members of the equivalence relation on {1, 2, 3, 4} defined by the given partition. Also find the equivalence classes [1], [2], [3] and [4].

1. {{1, 2}, {3, 4}}
2. {{1, 2, 3}, {4}}

Exercise 3: Let R be a reflexive relation on X satisfying: for all x, y, z ∈ X, is x R y and y R z, then z R x. Prove that R is an equivalence relation.

Matrices of Relations

Matrices of Relations

- **Definition :** A matrices of relations is a convenient way to represent a relation R from X to Y.
- **Example 1:** The matrix of the relation

$$R = \{(1, b), (1, d), (2, c), (3, c), (3, b), (4, a)\}$$

From X = {1, 2, 3, 4} to Y = {a, b, c, d} relative to the orderings 1, 2, 3, 4 and a, b, c, d is

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Matrices of Relations

- **<u>Example 2:</u>** The matrix of the relation

$$R = \{(1, b), (1, d), (2, c), (3, c), (3, b), (4, a)\}$$

relative to the orderings 2, 3, 4, 1 and d, b, a, c is

$$\begin{array}{c c} & \begin{array}{cccc} d & b & a & c \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \\ 1 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Obviously, the matrix of a relation from X to Y is dependent on the orderings of X and Y.

Matrices of Relations

- **<u>Example 3:</u>** The matrix of the relation R from {2, 3, 4} to {5, 6, 7, 8}, relative to the orderings 2, 3, 4 and 5, 6, 7, 8, defined by

x R y if x divides y

$$\begin{array}{c c} & \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Matrices of Relations

- **Example 4:** The matrix of the relation

$$R = \{(a, a), (b, b), (c, c), (d, d), (b, c), (c, b)\}$$

On {a, b, c , d}, relative to the ordering a, b, c, d is

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Matrices of Relations

- **Example 5:** Let R1 be the relation from X = {1, 2, 3} to Y = {a, b} defined by

$$R1 = \{(1, a), (2, b), (3, a), (3, b)\}$$

and let R2 be the relation from Y = {a, b} to Z = {x, y, z} defined by

$$R2 = \{(a, x), (a, y), (b, y), (b, z)\}$$

The matrix of R1 relative to the orderings 1, 2, 3 and a, b

$$A1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array}$$

Matrices of Relations

The matrix of R1 relative to the orderings a, b and x, y, z

$$A2 = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

The product of these matrices is

$$A1 \times A2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

Matrices of Relations

- The *ik*th entry in A1 × A2 is computed a

$$\begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ i & \begin{pmatrix} s & t \end{pmatrix} \end{array} \quad \begin{array}{c} k \\ \begin{pmatrix} u \\ v \end{pmatrix} \end{array} = s \times u + t \times v$$

Matrices of Relations

- **<u>Example 6:</u>** The matrix of the relation

$$R = \{(a, a), (b, b), (c, c), (d, d), (b, c), (c, b)\}$$

On {a, b, c , d}, relative to the ordering a, b, c, d is

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Its square is

$$A^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Matrices of Relations

- **Example 7:** The matrix of the relation

$$R = \{(a, a), (b, b), (c, c), (d, d), (a, c), (c, b)\}$$

On {a, b, c , d}, relative to the ordering a, b, c, d is

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Its square is

$$A^2 = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercises

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<u>Exercise 1:</u> Find the matrix of the relation R from X to Y relative to the orderings given

1. R = {(1, δ), (2, α), (2, Σ), (3, β), (3, Σ)} ordering of X ={1, 2, 3} and Y = {α, β, Σ, δ}.
2. R = {(1, 2), (2, 3), (3, 4), (4, 5)} ordering of X ={1, 2, 3, 4}

<u>Exercise 2:</u> Consider the matrix

$$\begin{array}{c c} & \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

1. Write the relation R, given by the matrix, as a set of ordered pairs.
2. Find the matrix of the inverse of the relation R, given by the matrix.

<u>Exercise 3:</u> Let the relations R1 = {(1, x), (1, y), (2, x), (3, x)}; R2 = {(x, b), (y, b), (y, a), (y, c)} ordering of X ={1, 2, 3} , Y = {x, y} and Z = {a, b, c}.

1. Find the matrix A1 of the relation R1
2. Find the matrix A2 of the relation R2
3. Find the matrix product A1 A2
4. Find the relation R2 o R1
5. Find the matrix of the relation R2 o R1