# Chapter 2

## Combinatorial Methods

# Outline of Chapter 2

- 2.1 Introduction
- 2.2 Counting principle
- 2.3 Permutations
- 2.4 Combinations

# Section 2.2

## Counting Principles

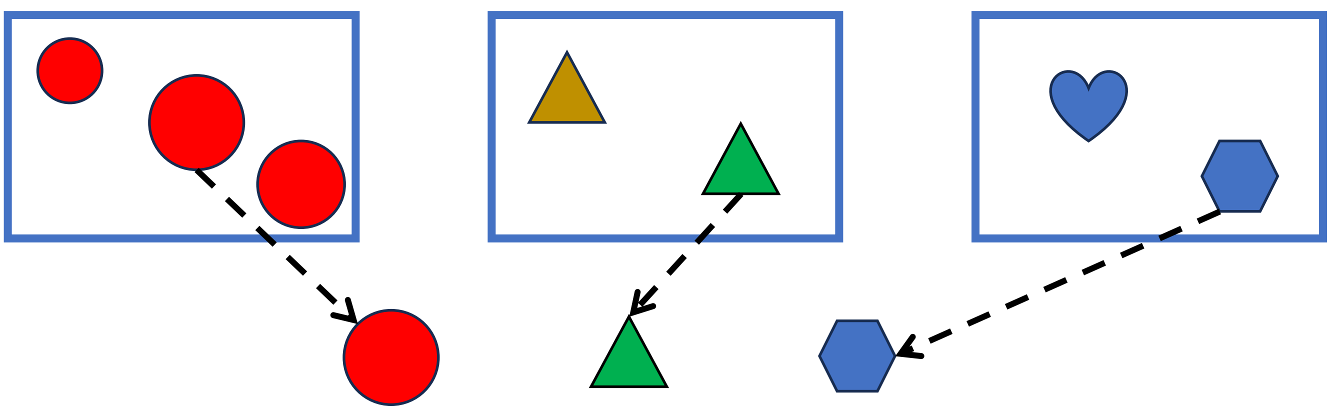
# Theorem 2.2 Generalized Counting Principle

- Let $E_1, E_2, \ldots, E_k$ be sets with $n_1, n_2, \ldots, n_k$ elements, respectively.
- Then there are

$$n_1 \times n_2 \times n_3 \times \cdots \times n_k$$

ways in which we can, first, choose an element of $E_1$, then an element of $E_2$, then an element of $E_3$, ... , and finally an element of $E_k$.

distinct



# Section 2.3

## Permutations

# Definition of Permutations

Distinguishable objects

- **Definition.** An *ordered* arrangement of $r$ objects from a set $A$ containing $n$ objects $(0 < r \leq n)$ is called an $r$-element permutation of $A$, or a permutation of the elements of $A$ taken $r$ at a time. The number of $r$ -element permutations of a set containing $n$ objects is denoted by $_nP_r$.

$$_nP_r = n(n-1)(n-2)\cdots(n-r+1)$$

$$_nP_n = n(n-1)(n-2)\cdots(n-n+1) = n!$$

$$_nP_r = \frac{n!}{(n-r)!}$$

# Distinguishable and Indistinguishable

- The formula for the number of the permutations is valid only if all objects are distinguishable.
- For example, the number of permutations of the eight letters in STANFORD is 8!.
- However, the number of permutations of the letters in BERKELEY is less than 8!.

- **Theorem 2.4** The number of distinguishable permutations of $n$ objects of $k$ different types, where $n_1$ are alike, $n_2$ are alike, . . . , $n_k$ are alike and $n = n_1 + n_2 + \cdots + n_k$, is

$$\boxed{\frac{n!}{n_1! \times n_2! \times \cdots \times n_k!}}$$

# Section 2.4

## Combinations

# Definition of Combinations

- In many combinatorial problems, unlike permutations, the order in which objects are arranged is immaterial.

Distinguishable objects

- **Definition.** An **unordered** arrangement of $r$ objects from a set $A$ containing $n$ objects $(r \leq n)$ is called an $r$-element combination of $A$, or a combination of the elements of $A$ taken $r$ at a time.

$$x \times r! = {}_nP_r$$

- The number of $r$-element combinations of $n$ objects is given by

$${}_nC_r = \frac{n!}{(n-r)!\,r!} \qquad {}_nC_r \times r! = {}_nP_r$$

# Theorem 2.5 (Binomial Expansion)

- **Theorem 2.5** *For any integer* $n \geq 0$,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

- **Proof.** From

$$(x+y)^n = (x+y)(x+y)\cdots(x+y) \qquad (*)$$

we obtain only terms of the form $x^{n-i}y^i$, $0 \leq i \leq n$.

- Therefore, all we have to do is to find out how many times the term $x^{n-i}y^i$ appears, $0 \leq i \leq n$.
- $x^{n-i}y^i$ emerges because $n-i$ pairs of parentheses in (*) contribute $x$ and $i$ pairs contribute $y$.