Chapter 2

Combinatorial Methods

Outline of Chapter 2

- 2.1 Introduction
- 2.2 Counting principle
- 2.3 Permutations
- 2.4 Combinations
- 2.5 Stirling formula

Section 2.2

Counting Principles

Theorem 2.1 Counting Principles

- **Theorem 2.1** *If the set E contains n elements and the set F contains m elements, there are nm ways in which we can choose, first, an element of E and then an element of F.*

- **Proof.**
- Let $E = \{a_1, a_2, \ldots, a_n\}$ and $F = \{b_1, b_2, \ldots, b_m\}$.
- The following rectangular array, which consists of nm elements, contains all possible ways that we can choose, first, an element of E and then an element of F.

$$
\begin{array}{l}
(a_1, b_1), (a_1, b_2), \ldots, (a_1, b_m) \\
(a_2, b_1), (a_2, b_2), \ldots, (a_2, b_m) \\
\vdots \\
(a_n, b_1), (a_n, b_2), \ldots, (a_n, b_m)
\end{array}
$$

Theorem 2.2 Generalized Counting Principle

- Let $E_1, E_2, \ldots, E_k$ be sets with $n_1, n_2, \ldots, n_k$ elements, respectively.
- Then there are

$$n_1 \times n_2 \times n_3 \times \cdots \times n_k$$

ways in which we can, first, choose an element of E_1, then an element of E_2, then an element of E_3, ... , and finally an element of E_k.

Example 2.1 and Remark 2.1

- How many outcomes are there if we throw 5 dice?
- **Solution.** Let $E_i, 1 \leq i \leq 5$, be the set of all possible outcomes of the i–th die.
- Then, $E_i = \{1, 2, 3, 4, 5, 6\}$.
- The number of outcomes equals the number of ways we can choose one element each from sets E_1, E_2, E_3, E_4, and finally E_5.
- From Theorem 2.2 there are 6^5.

- We assume that all outcomes (sample points) in the sample space are *equally likely*.
- In section 3.5, we introduce the concept of *independence*.
- Assuming that all tosses are indepdent to one another, we can show that all outcomes are equally likely.

Example 2.8 - Standard birthday problem

- What is the probability that at least two students in a class of size n have the same birthdays?
- Compute the probability for $n = 23, 30, 50, 60$.
- What is your intuition of this probability?

Solution

- Let $P(n)$ be the probability that no students have the same birthdays.

$$P(n) = \frac{365 \times 364 \times 363 \times \cdots \times [365 - (n-1)]}{365^n}$$

- The desired probability is $1 - P(n)$.
- For $n = 23, 30, 50, 60$, the answers are 0.507, 0.706, 0.970 and 0.995.

Number of Subsets of a Set

- The set of all subsets of A is called the **power set** of A.
- **Theorem 2.3** A set with n elements has 2^n subsets

Proof of Theorem 2.3

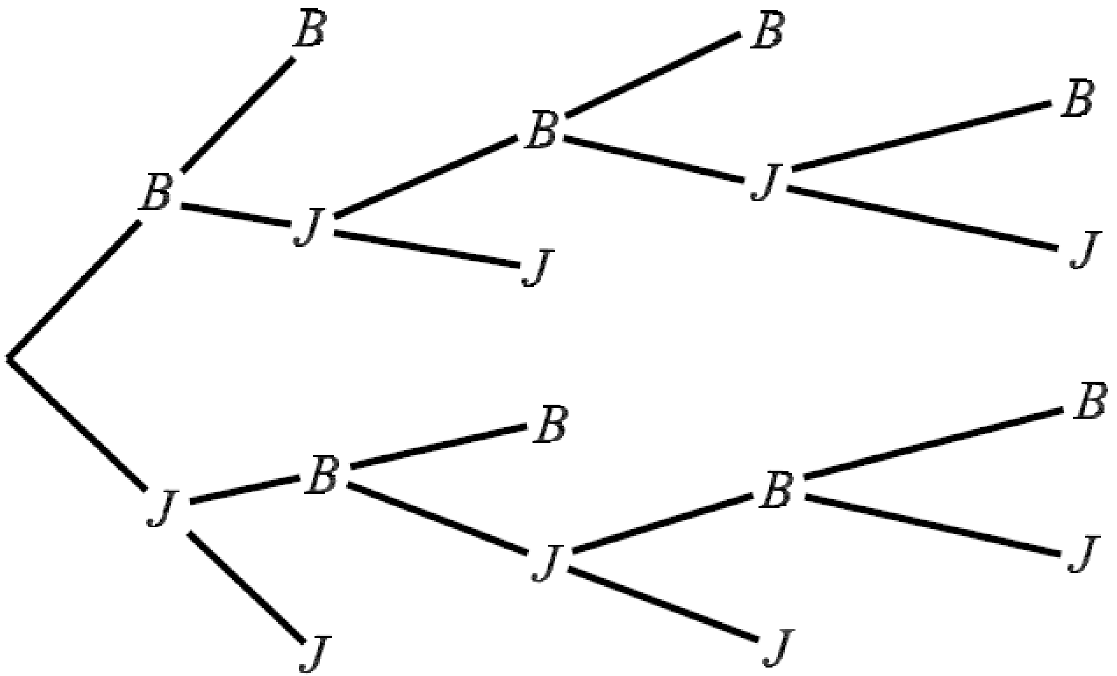
- Let $A = \{a_1, a_2, a_3, \ldots, a_n\}$ be a set with n elements.
- Let B be a subset of A.
- There is a one-to-one correspondence between B and A.
- We associate a sequence $b_1 b_2 b_3 \cdots b_n$, where

$$b_i = \begin{cases} 1 & \text{if } a_i \in B \\ 0 & \text{otherwise.} \end{cases}$$

- By the generalized counting principle, the number of sequences of 0's and 1's of length n is $2 \times 2 \times \cdots \times 2 = 2^n$.
- Thus the number of subsets of A is 2^n.

Tree Diagrams

- Tree diagrams are useful pictorial representations that break down a complex counting problem into smaller, more tractable ones.



Example 2.11

- Bill and John keep playing chess until one of them wins two games in a row or three games altogether.
- In what percent of all possible cases does the game end because Bill wins three games without winning two in a row?
- **Solution.**
- The tree diagram of Figure 2.1 illustrates all possible cases. The total number of possible cases is equal to the number of the endpoints of the branches, which is 10.
- The number of cases in which Bill wins three games without winning two in a row, as seen from the figure, is one.
- So the answer is 10%.
- Note that the probability of this event is not 0.10 because not all of the branches of the tree are equiprobable.

Section 2.3

Permutations

Definition of Permutations

Distinguishable objects

- **Definition.** An *ordered* arrangement of r objects from a set A containing n objects $(0 < r \leq n)$ is called an r-element permutation of A, or a permutation of the elements of A taken r at a time. The number of r -element permutations of a set containing n objects is denoted by ${}_nP_r$.

$$ {}_nP_r = n(n-1)(n-2)\cdots(n-r+1) $$

$$ {}_nP_n = n(n-1)(n-2)\cdots(n-n+1) = n! $$

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Example 2.15

- If five boys and five girls sit in a row in a random order, what is the probability that no two children of the same sex sit together?
- **Solution.**
- There are 10! ways for 10 persons to sit in a row.
- In order that no two of the same sex sit together, boys must occupy positions 1, 3, 5, 7, 9, and girls positions 2, 4, 6, 8, 10, or vice versa.
- In each case there are $5! \times 5!$ possibilities.
- Therefore, the desired probability is equal to

$$\frac{2 \times 5! \times 5!}{10!} \approx 0.008$$

Distinguishable and Indistinguishable

- The formula for the number of the permutations is valid only if all objects are distinguishable.
- For example, the number of permutations of the eight letters in STANFORD is 8!.
- However, the number of permutations of the letters in BERKELEY is less than 8!.

- **Theorem 2.4** The number of distinguishable permutations of n objects of k different types, where n_1 are alike, n_2 are alike, . . . , n_k are alike and $n = n_1 + n_2 + \cdots + n_k$, is

$$\boxed{\frac{n!}{n_1! \times n_2! \times \cdots \times n_k!}}$$

Making Indistinguishable Objects Distinguishable

- Distinguish the 3 *E*s in *BERKELEY* by marking them E_1, E_2, and E_3

$$\begin{array}{ll} BYE_1RE_2LE_3K & BYE_2RE_3LE_1K \\ BYE_1RE_3LE_2K & BYE_3RE_1LE_2K \\ BYE_2RE_1LE_3K & BYE_3RE_2LE_1K \end{array}$$

= *BERKELEY*

8!/3!

Example 2.17

- In how many ways can we paint 11 offices so that four of them will be painted green, three yellow, two white, and the remaining two pink?
- **Solution.**
- Let ``ggypgwpygwy" represent the sitution in which the first two offices are painted green, the third one yellow, and so on, with similar representation for other cases.
- Then the answer is equal to the number of distinguishable permutations of ``ggypgwpygwy" which by Theorem 2.4 is $11!/(4! \times 3! \times 2! \times 2!)$

Section 2.4

Combinations

Definition of Combinations

- In many combinatorial problems, unlike permutations, the order in which objects are arranged is immaterial.

Distinguishable objects

- **Definition.** An **unordered** arrangement of r objects from a set A containing n objects $(r \leq n)$ is called an r-element combination of A, or a combination of the elements of A taken r at a time.

ABC BCA
ACB CAB
BAC CBA

$$x \times r! = {}_nP_r$$

- The number of r-element combinations of n objects is given by

$$\binom{n}{r} = {}_nC_r = \frac{n!}{(n-r)!\,r!} \qquad {}_nC_r \times r! = {}_nP_r$$

the same combination

Notations and Some Useful Identities

Notation: By the symbol $\binom{n}{r}$ (read: n choose r) we mean the number of all r-element combinations of n objects. Therefore, for $r \leq n$,

$n! = n \cdot (n-1) \cdots 2 \cdot 1$

$0! \overset{def}{=} 1$

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}.$$

Observe that $\binom{n}{0} = \binom{n}{n} = 1$ and $\binom{n}{1} = \binom{n}{n-1} = n$. Also, for any $0 \leq r \leq n$,

$$\binom{n}{r} = \binom{n}{n-r}$$

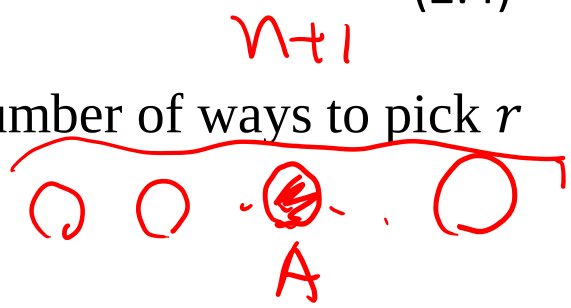
A Useful Identity

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1} \qquad (2.4)$$

- Eq. (2.4) can be proved algebraically or verified combinatorially. Let us prove (2.4) by a combinatorial argument.
- Consider a set of $n+1$ objects, $\{a_1, a_2, \ldots, a_n, a_{n+1}\}$. There are $\binom{n+1}{r}$ r-element combinations of this set.
- Now we separate these r-element combinations into two disjoint classes:
  1. one class consisting of all r-element combinations of $\{a_1, a_2, \ldots, a_n\}$ and
  2. another consisting of all $(r-1)$-element combinations of $\{a_1, a_2, \ldots, a_n\}$ attached to a_{n+1}.
- The latter class contains $\binom{n}{r-1}$ elements and the former contains $\binom{n}{r}$ elements, showing that (2.4) is valid.

An Alternative Explanation of (2.4)

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1} \tag{2.4}$$



- The left hand side of the above equation is the number of ways to pick r out of $n + 1$ objects.
- Tag one of the $n + 1$ objects. Call it A.
- There are two possibilities.
- In the first case, A **is** selected as one of the r objects.
  - In this case, we further select $r - 1$ objects from the rest n objects.
  - The number of selections is the second term in (2.4).
- In the second case, A is **not** selected as one of the r objects.
  - In this case, we select r objects from the rest n objects.
  - The number of selections is the first term in (2.4).

Example 2.23

- From an ordinary deck of 52 cards, seven cards are drawn at random and without replacement. What is the probability that at least one of the cards is a king?
- **Solution.** In $\binom{52}{7}$ ways seven cards can be selected from an ordinary deck of 52 cards.
- In $\binom{48}{7}$ of these, none of the cards selected is a king.
- Therefore, the desired probability is

$$P(\text{at least one king}) = 1 - P(\text{no kings}) = 1 - \frac{\binom{48}{7}}{\binom{52}{7}} = 0.4496$$

A Common Mistake in solving Example 2.23

- A common mistake is to calculate this and similar probabilities as follows:
- To make sure that there is at least one king among the seven cards drawn, we will first choose a king;
  - there are $\binom{4}{1}$ possibilities
- Then we choose the remaining six cards from the remaining 51 cards;
  - there are $\binom{51}{6}$ possibilities
- Thus the answer is

$$\frac{\binom{4}{1}\binom{51}{6}}{\binom{52}{7}} = 0.5385$$

Why is the Analysis on the Last Page Wrong?

- This solution is wrong because it counts some of the possible outcomes several times.
- For example, the hand K_H, 5_C, 6_D, 7_H, K_D, J_C, and 9_S is counted twice:
  - once when K_H is selected as the first card from the kings and 5_C, 6_D, 7_H, K_D, J_C, and 9_S from the remaining 51, and
  - once when K_D is selected as the first card from the kings and K_H, 5_C, 6_D, 7_H, J_C, and 9_S from the remaining 51 cards.

Example 2.25

- Show that the number of different ways n indistinguishable objects can be placed into k distinguishable cells is

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$

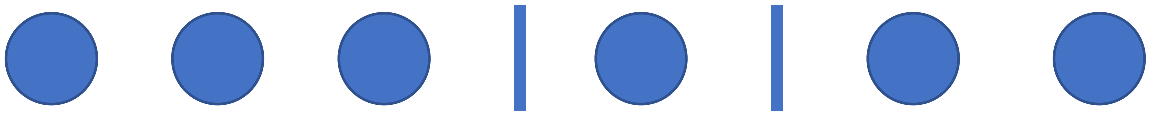
Example: $n = 4, k = 3$

- This problem is equivalent to the number of non-negative solutions of the following equation

$$x_1 + x_2 + \cdots + x_k = n$$

Number of **Non-negative** Solutions

- Consider n indistinguishable objects and $k-1$ indistinguishable plates.
- The number of objects arranged between the $(i-1)$-th and the i-th plates is the value of x_i.



$\boxed{n = 6, k = 3}$

- The number of possible arrangement is

$$\frac{(n+k-1)!}{n!\,(k-1)!}$$

Number of **Positive** Solutions

- Consider again the equation

$$x_1 + x_2 + \cdots + x_k = n$$

- How many **distinct positive integer** solutions does it have?
- This problem can be changed into the number of non-negative integer solutions by introducing new variables.

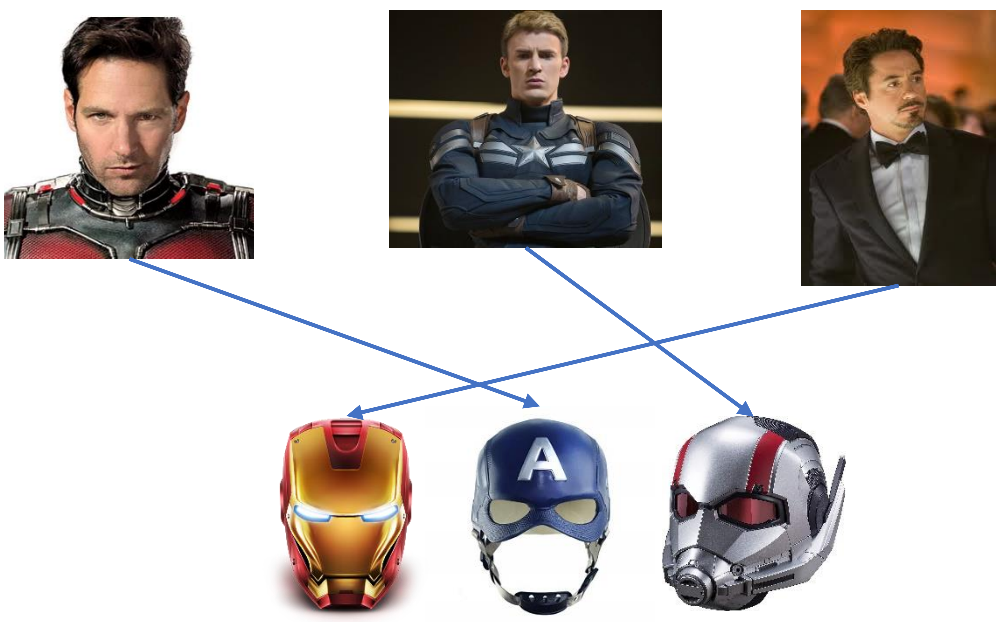
$$y_i = x_i - 1, \text{for } 1 \le i \le k$$
$$y_1 + y_2 + \cdots + y_k = n - k$$
$$\binom{(n-k)+k-1}{n-k} = \binom{n-1}{n-k} = \binom{n-1}{k-1}$$

Example 2.28

- An absentminded professor wrote n letters and sealed them in envelopes before writing the addresses on the envelopes.
- Then he wrote the n addresses on the envelopes at random.
- What is the probability that at least one letter was addressed correctly?

Equivalent version of Example 2.28

- n men throw their (distinct) helmets on the floor
- Every one picks up a helmet randomly
- What is the probability that at least one man picks up his helmet correctly?



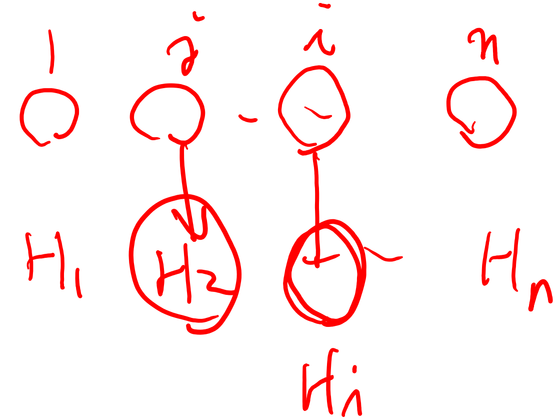
- As n approaches infinity, does this probability approaches
  - 0, or
  - 1, or
  - something in between?

Solution of Example 2.28

- Let E_i be the event that the ith man picks his hat correctly.
- Then, $E_1 \cup E_2 \cup \cdots \cup E_n$ is the event that at least one man picks his hat correctly
- We use the inclusion-exclusion formula to calculate $P(E_1 \cup E_2 \cup \cdots \cup E_n)$ (Chapter 1, page 20 of the textbook).

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P(E_i E_j) + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} P(E_i E_j E_k) - \cdots + (-1)^{n-1} P(E_1 E_2 \cdots E_n)$$

Solution Steps of Example 2.28



- First, we show that $P(E_i) = (n-1)!/n!$
- Then, we show that $P(E_i \cap E_j) = (n-2)!/n!$
- In general,

$$P(E_i \cap E_j \cap E_k) = (n-3)!/n!$$

- We need to count the number of terms in the inclusion-exclusion formula
  - There are n terms of the form $P(E_i)$.
  - There are ${}_nC_2$ terms of the form $P(E_i \cap E_j)$.
  - There are ${}_nC_3$ terms of the form $P(E_i \cap E_j \cap E_k)$.

- Substitute these results into the inclusion-exclusion formula.

$$P\left(\bigcup_{i=1}^{n} E_i\right) = n\frac{(n-1)!}{n!} - \binom{n}{2}\frac{(n-2)!}{n!} + \cdots +$$

$$(-1)^{n-2}\binom{n}{n-1}\frac{(n-(n-1))!}{n!} + (-1)^{n-1}\binom{n}{n}\frac{1}{n!}$$

- The expression above is simplified as

$$P\left(\bigcup_{i=1}^{n} E_i\right) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{(-1)^{n-1}}{n!}$$

$$= 1 - \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

probability of at least one match

$$P(\text{no match}) = 1 - P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=0}^{n} (-1)^i / i!$$

Taylor Expansion of the Exponential Function

- Taylor expansion of exponential functions

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

- Probability of at least one match is

$x = -1$

$$P\left(\bigcup_{i=1}^{n} E_i\right) \approx 1 - \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 - \frac{1}{e} \approx 0.632.$$

- Probability of no match is

$$P(\text{no match}) = 1 - P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=0}^{\infty} \frac{x^i}{i!} \approx \frac{1}{e}.$$

Theorem 2.5 (Binomial Expansion)

- **Theorem 2.5** *For any integer* $n \geq 0$,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

- **Proof.** From

$$(x+y)^n = (x+y)(x+y)\cdots(x+y) \qquad (*)$$

we obtain only terms of the form $x^{n-i}y^i$, $0 \leq i \leq n$.

- Therefore, all we have to do is to find out how many times the term $x^{n-i}y^i$ appears, $0 \leq i \leq n$.
- $x^{n-i}y^i$ emerges because $n - i$ pairs of parentheses in (*) contribute x and i pairs contribute y.

Example 2.30

- Evaluate the sum $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

$$\sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 1^i = (1 + 1)^n = 2^n.$$

Example 2.31

- Evaluate the sum $\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n}$.
- **Solution.**

$$i\binom{n}{i} = i \cdot \frac{n!}{i!\,(n-i)!} = \frac{n \cdot (n-1)!}{(i-1)!\,(n-i)!} = n\binom{n-1}{i-1}$$

- So

$$\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} =$$
$$n\left[\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{n-1}\right] = n \cdot 2^{n-1}$$

by example 2.30

Theorem 2.6 Multinomial Expansion

- **Theorem 2.6** *In the expansion of*

$$(x_1 + x_2 + \cdots + x_k)^n,$$

the coefficient of the term

$$x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}, \quad n_1 + n_2 + \cdots + n_k = n$$

is

$$\frac{n!}{n_1! n_2! \cdots n_k!}.$$

Therefore,

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_1 + n_2 + \cdots + n_k = n} \frac{n!}{n_1! n_2! \cdots n_k!} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

Proof of Theorem 2.6

- Distribute n distinguishable balls into k distinguishable cells so that
  - n_i balls are placed in cell $i, i = 1, 2, \dots, k$
  - $n_1 + n_2 + \cdots + n_k = n$
- How many ways are there?

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

$$= \frac{n!}{(n-n_1)!\,n_1!} \times \frac{(n-n_1)!}{(n-n_1-n_2)!\,n_2!} \times \frac{(n-n_1-n_2)!}{(n-n_1-n_2-n_3)!\,n_3!} \times \cdots$$

$$\times \frac{(n-n_1-n_2-n_3-\cdots-n_{k-1})!}{(n-n_1-n_2-n_3-\cdots-n_{k-1}-n_k)!\,n_k!} = \frac{n!}{n_1!\,n_2!\,n_3!\,\cdots\,n_k!}$$

Section 2.5

Stirling’s Formula

Theorem 2.7 Stirling's Formula

- **Theorem 2.7** (Stirling's Formula)

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n}$$

where the sign $\sim$ means

$$\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n}\, n^n e^{-n}} = 1$$

Stirling's Formula is very accurate

n	$n!$	$\sqrt{2\pi n}\ n^n e^{-n}$	$R(n)$
1	1	0.922	1.084
2	2	1.919	1.042
5	120	118.019	1.017
8	40,320	39,902.396	1.010
10	3,628,800	3,598,695.618	1.008
12	479,001,600	475,687,486.474	1.007

Homework 2

- Section 2.2: B.30, B.36
- Section 2.3: A.29, B.34
- Section 2.4: A.47, B.51, B.66

- Due date: 5 pm, Wednesday, March 15, 2022