Signals and Systems

Homework 7 — Due : Apr. 12 2024

Problem 1 (20 pts)**.** Consider three continuous-time periodic signals whose Fourier series representations are as follows:

$$x_1(t) = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k e^{jk\frac{2\pi}{50}t}, \quad x_2(t) = \sum_{k=-100}^{100} \cos(k\pi)\, e^{jk\frac{2\pi}{50}t}, \quad \text{and} \quad x_3(t) = \sum_{k=-100}^{100} j\sin\left(\frac{k\pi}{2}\right) e^{jk\frac{2\pi}{50}t}.$$

Use Fourier series properties to help answer the following questions:

(a) Which of the three signals is/are real valued?

(b) Which of the three signals is/are even?

Problem 2 (30 pts)**.** Let $x(t)$ and $y(t)$ both be continuous-time periodic signals having period T_0 and with Fourier series representations given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \qquad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}. \tag{1}$$

Suppose that $y(t)$ in the eq. (1) equals $x^*(t)$. Express the b_k in the equation in terms of a_k, and prove Parseval's relation for periodic signals – that is,

$$\frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

Problem 3 (20 pts)**.** Suppose that we are given the following information about a signal $x(t)$:

- $x(t)$ is a real signal.
- $x(t)$ is periodic with period 6 and has Fourier coefficeients a_k.
- $a_k = 0$ for $k = 0$ and $k > 2$.
- $x(t) = -x(t-3)$.
- $\frac{1}{6}\int_{-3}^{3} |x(t)|^2 dt = \frac{1}{2}$.
- a_1 is a positive real number.

Show that $x(t) = A\cos(Bt + C)$, and determine the values of the constants A, B, and C.

Problem 4 (30 pts)**.** For any

$$x(t) \xleftrightarrow{FS} a_k,$$

we can decompose $x(t)$ and a_k into even parts plus odd parts

$$x(t) = x_e(t) + x_o(t), \qquad a_k = a_{k,e} + a_{k,o}.$$

Prove that

$$x_e(t) \xleftrightarrow{FS} a_{k,e}, \qquad x_o(t) \xleftrightarrow{FS} a_{k,o}$$

Problem 1 (20 pts)**.** Consider three continuous-time periodic signals whose Fourier series representations are as follows:

$$x_1(t) = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k e^{jk\frac{2\pi}{50}t}, \quad x_2(t) = \sum_{k=-100}^{100} \cos(k\pi)\, e^{jk\frac{2\pi}{50}t}, \quad \text{and} \quad x_3(t) = \sum_{k=-100}^{100} j\sin\left(\frac{k\pi}{2}\right) e^{jk\frac{2\pi}{50}t}.$$

Use Fourier series properties to help answer the following questions:

(a) Which of the three signals is/are real valued?

(b) Which of the three signals is/are even?

(a) $\underline{x_1 \text{ is not real.}}$

for $k=0$, $\cos(0)\cdot e^0 = 1$

for $k \neq 0$, $\cos(-k\pi)\cdot e^{-jk\frac{2\pi}{50}t} = \cos(k\pi)\cdot\left[\cos\left(k\frac{2\pi}{50}t\right) - \cancel{j\sin\left(k\frac{2\pi}{50}t\right)}\right]$

$\cos(k\pi)\cdot e^{jk\frac{2\pi}{50}t} = \cos(k\pi)\cdot\left[\cos\left(k\frac{2\pi}{50}t\right) + \cancel{j\sin\left(k\frac{2\pi}{50}t\right)}\right]$

$\Rightarrow \underline{x_2 \text{ is real.}}$

for $k=0$, $j\sin(0)\, e^0 = 0$

for $k\neq 0$, $j\sin\left(\frac{-k\pi}{2}\right) e^{-jk\frac{2\pi}{50}t} = -j\sin\left(\frac{k\pi}{2}\right)\cdot\left[\cancel{\cos\left(k\frac{2\pi}{50}t\right)} - j\sin\left(k\frac{2\pi}{50}t\right)\right]$

$j\sin\left(\frac{k\pi}{2}\right) e^{jk\frac{2\pi}{50}t} = j\sin\left(\frac{k\pi}{2}\right)\cdot\left[\cancel{\cos\left(k\frac{2\pi}{50}t\right)} + j\sin\left(k\frac{2\pi}{50}t\right)\right]$

$\Rightarrow \underline{x_3 \text{ is real.}}$

(b) $x_1(t) = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k e^{jk\frac{2\pi}{50}t} = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k \cdot \left[\cos\left(k\frac{2\pi}{50}t\right) + \sin\left(k\frac{2\pi}{50}t\right)\right]$

$= \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k \cdot \left[\cos\left(k\frac{2\pi}{50}(-t)\right) - \sin\left(k\frac{2\pi}{50}(-t)\right)\right] \neq x_1(-t)$

x_1 is not even.

$x_2(t) = \sum_{k=-100}^{100} \cos(k\pi) e^{jk\frac{2\pi}{50}t} = \sum_{k=-100}^{100} \cos(k\pi) \cdot \left[\cos\left(k\frac{2\pi}{50}t\right) + j\sin\left(k\frac{2\pi}{50}t\right)\right]$

$= 1 + 2\sum_{k=1}^{100} \cos(k\pi) \cdot \cos\left(k \cdot \frac{2\pi}{50}t\right) = 1 + 2\sum_{k=1}^{100} \cos(k\pi) \cdot \cos\left(k \cdot \frac{2\pi}{50}(-t)\right)$

$= x_2(-t)$

$\Rightarrow$ x_2 is even.

$x_3(t) = \sum_{k=-100}^{100} j\sin\left(\frac{k\pi}{2}\right) e^{jk\frac{2\pi}{50}t} = \sum_{k=-100}^{100} j\sin\left(\frac{k\pi}{2}\right) \cdot \left[\cos\left(k\frac{2\pi}{50}t\right) + j\sin\left(k\frac{2\pi}{50}t\right)\right]$

$= -2\sum_{k=1}^{100} \sin\left(\frac{k\pi}{2}\right) \sin\left(k\frac{2\pi}{50}t\right) = 2\sum_{k=1}^{100} \sin\left(\frac{k\pi}{2}\right) \sin\left(k\frac{2\pi}{50}(-t)\right)$

$= -x_3(-t)$

$\Rightarrow$ x_3 is not even.

Problem 2 (30 pts)**.** Let $x(t)$ and $y(t)$ both be continuous-time periodic signals having period T_0 and with Fourier series representations given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \qquad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}. \tag{1}$$

Suppose that $y(t)$ in the eq. (1) equals $x^*(t)$. Express the b_k in the equation in terms of a_k, and prove Parseval's relation for periodic signals – that is,

$$\frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

Note that $(AB)^* = A^* B^*$, $(A+B)^* = A^* + B^*$

$$y(t) = x^*(t) = \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right)^* = \sum_{k=-\infty}^{\infty} (a_k e^{jk\omega_0 t})^* = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$\Rightarrow b_k = a_{-k}^*$ #

Let $c_k = \frac{1}{T_0}\int_0^{T_0} x(t)y(t)e^{-jk\omega_0 t}$, then $c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}$

$$c_0 = \sum_{n=-\infty}^{\infty} a_n b_{-n} = \sum_{n=-\infty}^{\infty} a_n a_n^* = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$= \frac{1}{T_0}\int_0^{T_0} x(t)y(t)\,dt = \frac{1}{T_0}\int_0^{T_0} x(t)x^*(t)\,dt = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt$$ #

$$(\underbrace{\cos A - j\sin A}_{A})(\underbrace{\cos B - j\sin B}_{B})$$

$= \cos A\cos B - \sin A\sin B - j(\sin A\cos B + \cos A\sin B)$

$= \cos(A+B) - j\sin(A+B)$

$$(\underbrace{\cos A + j\sin A}_{A^*})(\underbrace{\cos B + j\sin B}_{B^*})$$

$= \cos A\cos B - \sin A\sin B + j(\sin A\cos B + \cos A\sin B)$

$= \cos(A+B) + j\sin(A+B) = [\cos(A+B) - j\sin(A+B)]^*$

$A^*B^* = (AB)^*$

Problem 3 (20 pts). Suppose that we are given the following information about a signal $x(t)$:

- $x(t)$ is a real signal.
- $x(t)$ is periodic with period 6 and has Fourier coefficeients a_k.
- $a_k = 0$ for $k = 0$ and $k > 2$.
- $x(t) = -x(t-3)$.
- $\frac{1}{6}\int_{-3}^{3} |x(t)|^2 dt = \frac{1}{2}$.
- a_1 is a positive real number.

Show that $x(t) = A\cos(Bt + C)$, and determine the values of the constants A, B, and C.

$x(t)$ is real. $\longrightarrow a_k = a_{-k}^* \rightarrow a_1$ is real $\longrightarrow a_{-1} = a_1$

$\rightarrow a_{-2} = a_2^*$

Let $A_1 = |a_1| = |a_{-1}|$, $A_2 = |a_2| = |a_2^*| = |a_{-2}|$

$T_0 = 6 \longrightarrow w_0 = \frac{\pi}{3}$

$a_k = 0$ for $k=0$ and $k>2$

$$\longrightarrow x(t) = a_1 e^{jw_0t} + a_2 e^{j2w_0t} + a_{-1}e^{-jw_0t} + a_{-2}e^{-j2w_0t}$$

$$= a_1 \cos(w_0t) + a_2 \cos(2w_0t) + a_1 \cdot \cos(w_0t) + a_2^* \cos(2w_0t)$$

$$= A_1 \cos(w_0t) + A_2 \cos(2w_0t + \theta) + A_1 \cos(w_0t) + A_2 \cos(2w_0t - \theta)$$

$$= 2A_1 \cos\left(\frac{\pi}{3}t\right) + A_2 \cos\left(\frac{2\pi}{3}t + \theta\right) + A_2 \cos\left(\frac{2\pi}{3}t - \theta\right)$$

$$-x(t-3) = -2A_1 \cos\left(\frac{\pi}{3}t - \pi\right) - A_2 \cos\left(\frac{2\pi}{3}t - 2\pi + \theta\right) - A_2 \cos\left(\frac{2\pi}{3}t - 2\pi - \theta\right)$$

$$= A_1 \cos\left(\frac{\pi}{3}t\right) - A_2 \cos\left(\frac{2\pi}{3}t + \theta\right) - A_2 \cos\left(\frac{2\pi}{3}t - \theta\right)$$

$$x(t) = A_1 \cos\left(\frac{\pi}{3}t\right) + A_2 \cos\left(\frac{2\pi}{3}t + \theta\right) + A_2 \cos\left(\frac{2\pi}{3} - \theta\right)$$

$$\longrightarrow A_2 = 0 \longrightarrow x(t) = 2a_1 \cos\left(\frac{\pi}{3}t\right)$$

$$\frac{1}{6}\int_{-3}^{3}|x(t)|^2 dt = \sum_{k=-\infty}^{\infty}|a_k|^2 = a_1^2 + a_1^2 = \frac{1}{2} \longrightarrow a_1 = \frac{1}{2}$$

$$\longrightarrow x(t) = 2\cdot\frac{1}{2}\cos\left(\frac{\pi}{3}t\right) = 1\cdot\cos\left(\frac{\pi}{3}t + 0\right)$$

$$\longrightarrow A = 1 \ , \ B = \frac{\pi}{3} \ , \ C = 0$$

Problem 4 (30 pts). For any

$$x(t) \overset{FS}{\longleftrightarrow} a_k,$$

we can decompose $x(t)$ and a_k into even parts plus odd parts

$$x(t) = x_e(t) + x_o(t), \qquad a_k = a_{k,e} + a_{k,o}.$$

Prove that

$$x_e(t) \overset{FS}{\longleftrightarrow} a_{k,e}, \qquad x_o(t) \overset{FS}{\longleftrightarrow} a_{k,o}$$

$$\left.\begin{aligned} x_e(t) &= \frac{1}{2}[x(t) + x(-t)] \\ x_o(t) &= \frac{1}{2}[x(t) - x(-t)] \end{aligned}\right\} x(t) = x_e(t) + x_o(t)$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

$$a_{k,e} = \frac{1}{T}\int_T x_e(t)e^{-jk\omega_0 t}dt = \frac{1}{2T}\left[\int_T x(t)e^{-jk\omega_0 t}dt + \int_T x(-t)e^{jk\omega_0 t}dt\right] = \frac{1}{2}[a_k(t) + a_k(-t)]$$

$$a_{k,o} = \frac{1}{T}\int_T x_o(t)e^{-jk\omega_0 t}dt = \frac{1}{2T}\left[\int_T x(t)e^{-jk\omega_0 t}dt - \int_T x(-t)e^{jk\omega_0 t}dt\right] = \frac{1}{2}[a_k(t) - a_k(-t)]$$

$$\Rightarrow a_k = a_{k,e} + a_{k,o} \ , \ x_e \overset{FS}{\longleftrightarrow} a_{k,e} \ , \ x_o \overset{FS}{\longleftrightarrow} a_{k,o}$$