

Signals and Systems

Homework 5 — Due : Mar. 29 2024

Problem 1 (5 pts). Consider an LTI system whose input $x(t)$ and output $y(t)$ are related by the differential equation

$$\frac{d}{dt}y(t) + 4y(t) = x(t),$$

the system also satisfies the condition of initial rest. Given $x(t) = e^{(-1+3j)t}u(t)$, what is $y(t)$?

Problem 2 (25 pts). Consider a system whose input $x(t)$ and output $y(t)$ satisfy the first-order differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t).$$

- (a) Given the auxiliary condition $y(1) = 1$, use a counterexample to show that the system is not linear.
- (b) Given the auxiliary condition $y(1) = 1$, use a counterexample to show that the system is not time invariant.
- (c) Given the auxiliary condition $y(1) = 1$, show that the system is incrementally linear.
- (d) Given the auxiliary condition $y(1) = 0$, show that the system is linear but not time invariant.
- (e) Given the auxiliary condition $y(0) + y(4) = 0$, show that the system is linear but not time invariant.

Problem 3 (30 pts). Given the equation

$$x(t) * u_1(t) = \int_{-\infty}^{+\infty} x(t - \tau)u_1(\tau)d\tau = x'(t) \quad (\text{eq. 1})$$

for any signal $x(t)$, and from it we derived the relationship

$$\int_{-\infty}^{+\infty} g(\tau)u_1(\tau)d\tau = -g'(0). \quad (\text{eq. 2})$$

- (a) Show that (eq. 2) is an equivalent characterization of $u_1(t)$ by showing that (eq. 2) implies (eq. 1).
[Hint: Fix t , and define the signal $g(\tau) = x(t - \tau)$.]
- (b) Let $f(t)$ be a given signal. Show that

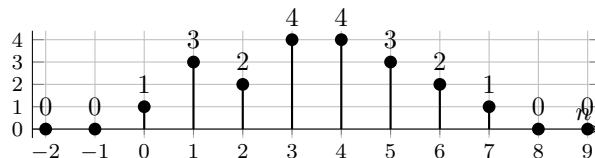
$$f(t)u_1(t) = f(0)u_1(t) - f'(0)\delta(t)$$

by showing that both functions have the same operational definitions.

Problem 4 (25 pts). Consider the LTI system initially at rest and described by the difference equation

$$y[n] + 2y[n - 1] = x[n] + 2x[n - 2].$$

Plot the response of this system to the input depicted in the figure by solving the difference equation recursively.



Problem 5 (15 pts). Evaluate the following integrals:

- (a) $\int_{-\infty}^{\infty} u_0(t) \cos(t)dt$
- (b) $\int_0^5 \sin(2\pi t)\delta(t + 3)dt$
- (c) $\int_{-5}^5 u_1(1 - \tau) \cos(2\pi\tau)d\tau$

Problem 1 (5 pts). Consider an LTI system whose input $x(t)$ and output $y(t)$ are related by the differential equation

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the system also satisfies the condition of initial rest. Given $x(t) = e^{(-1+3j)t}u(t)$, what is $y(t)$?

$$y_h' + 4y_h = 0, \quad I(t) = e^{\int 4 dt} = e^{4t}, \quad e^{4t} \cdot y_h = \int 0 dt = \text{const}, \quad y_h = \text{const} \cdot e^{-4t}$$

$$\text{Let } y_p = A e^{(-1+3j)t}, \quad (3+3j) A e^{(-1+3j)t} = e^{(-1+3j)t} \text{ for } t > 0, \quad A = \frac{1-j}{6}$$

$$y = [\text{const} \cdot e^{-4t} + \frac{1-j}{6} e^{(-1+3j)t}] u(t), \quad \text{initial rest } y(0) = 0$$

$$\text{const} \cdot e^{-4} + \frac{1-j}{6} \cdot e^{-1+3j} = e^{-1+3j}, \quad \text{const} \cdot e^{-4} = \frac{5+j}{6} \cdot e^{-1+3j}, \quad \text{const} = \frac{5+j}{6} \cdot e^{3+3j}$$

$$y = \left[\frac{5+j}{6} \cdot e^{3+3j} \cdot e^{-4t} + \frac{1-j}{6} e^{(-1+3j)t} \right] u(t)$$

Problem 2 (25 pts). Consider a system whose input $x(t)$ and output $y(t)$ satisfy the first-order differential equation

$$y_h = \text{const.} \cdot e^{-2t} \quad \frac{dy(t)}{dt} + 2y(t) = x(t).$$

- (a) Given the auxiliary condition $y(1) = 1$, use a counterexample to show that the system is not linear.
- (b) Given the auxiliary condition $y(1) = 1$, use a counterexample to show that the system is not time invariant.
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- (e) Given the auxiliary condition $y(0) + y(4) = 0$, show that the system is linear but not time invariant.

(a) $y = y_h + y_p = \text{const.} \cdot e^{-2t} + y_p$

Let $x_1(t) = 1$, $y_1 = \frac{1}{2} e^2 e^{-2t} + \frac{1}{2}$

Let $x_2(t) = 0$, $y_2 = e^2 \cdot e^{-2t}$

Let $x_3(t) = x_1(t) + x_2(t) = x_1(t)$ implies that $y_3(t) = y_1(t) \neq y_1(t) + y_2(t)$

\Rightarrow not linear

(b) Let $x_1(t) = e^{2t}$, $y_1 = \left(e^2 - \frac{e^4}{4}\right) e^{-2t} + \frac{1}{4} e^{2t}$

Let $x_2(t) = x_1(t-s)$, $y_2 = \left(e^2 - \frac{e^4}{4}\right) e^{-2(t-s)} + \frac{1}{4} e^{2(t-s)} \neq y_1(t-s)$

\Rightarrow not time invariant

(c) Let $\frac{d}{dt} y_1(t) + 2y_1(t) = x_1(t)$, $y_1(1) = 0$

Let $\frac{d}{dt} y_2(t) + 2y_2(t) = x_2(t)$, $y_2(1) = 0$

Let α, β be constants

$$\frac{d}{dt} (\alpha y_1(t) + \beta y_2(t)) + 2 (\alpha y_1(t) + \beta y_2(t)) = \alpha x_1(t) + \beta x_2(t)$$

$$x_3(t) = \alpha x_1(t) + \beta x_2(t) \longrightarrow y_3(t) = y_1(t) + y_2(t) \text{ and } y_3(1) = y_1(1) + y_2(1) = 0$$

\Rightarrow linear for the auxiliary condition: $y(1) = 0$

\Rightarrow for $y(1) = 1$, 可视为 $y(1) = 0$ 的系统加上一个固定的偏移量

\Rightarrow incrementally linear

(d) From (c) we can see that for the condition $y(1) = 0$, then the system is linear.

$$\text{Let } x_1(t) = e^{2t}, y_1(t) = -\frac{e^4}{4} e^{-2t} + \frac{1}{4} e^{2t}$$

$$\text{Let } x_2(t) = x_1(t-s) = \frac{e^{2t}}{e^{2s}}, y_2(t) = -\frac{e^4}{4} e^{-2(t-s)} + \frac{1}{4} e^{2(t-s)} \neq y_1(t-s)$$

\Rightarrow not time invariant

(e) Let $\frac{d}{dt} y_1(t) + 2y_1(t) = x_1(t), y_1(0) + y_1(4) = 0$

$$\frac{d}{dt} y_2(t) + 2y_2(t) = x_2(t), y_2(0) + y_2(4) = 0$$

Let α, β be constants

$$\frac{d}{dt} (\alpha y_1(t) + \beta y_2(t)) + 2(\alpha y_1(t) + \beta y_2(t)) = \alpha x_1(t) + \beta x_2(t)$$

$$y_3(0) + y_3(4) = y_1(0) + y_1(4) + y_2(0) + y_2(4)$$

$$\Rightarrow x_3(t) = \alpha x_1(t) + \beta x_2(t) \longrightarrow y_3(t) = \alpha y_1(t) + \beta y_2(t)$$

\Rightarrow linear

$$\text{Let } x_1(t) = e^{2t}, y_1(t) = \frac{1}{4} e^{2t} + \text{const} \cdot e^{-2t}$$

$$y(0) = \frac{1}{4} + \text{const}, y(4) = \frac{e^8}{4} + \text{const} \cdot e^{-8}$$

$$\frac{1+e^8}{4} + (1+e^{-8}) \text{const} = 0, \text{const} = -\frac{1+e^8}{4(1+e^8)}$$

$$y_1(t) = \frac{-1-e^8}{4(1+e^8)} e^{-2t} + \frac{1}{4} e^{2t}$$

$$\text{Let } x_2(t) = x_1(t-s), y_2(t) = \text{const} \cdot e^{-2t} + \frac{1}{4} e^{2t-2s}$$

$$\text{const} + \frac{1}{4e^{2s}} + \frac{\text{const}}{e^8} + \frac{e^8}{4e^{2s}} = 0$$

$$\text{const} (1+e^{-8}) = \frac{-1-e^8}{4e^{2s}}, \text{const} = \frac{-1-e^8}{4(1+e^8)} \cdot e^{-2s}$$

$$y_2(t) = \frac{-1-e^8}{4(1+e^8)} e^{-2(t+s)} + \frac{1}{4} e^{2(t-s)} \neq y_1(t-s)$$

\Rightarrow not time invariant

Problem 3 (30 pts). Given the equation

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for any signal $x(t)$, and from it we derived the relationship

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[Hint: Fix t , and define the signal $g(\tau) = x(t-\tau)$.]

(b) Let $f(t)$ be a given signal. Show that

$$f(t) u_1(t) = f(0) u_1(t) - f'(0) \delta(t)$$

by showing that both functions have the same operational definitions.

(a) Let $g(\tau) = x(t-\tau)$, $-g'(0) = \int_{-\infty}^{\infty} g(\tau) u_1(\tau) d\tau = \int_{-\infty}^{\infty} f(t-\tau) u_1(\tau) d\tau = x(t) * u_1(t) = x'(t)$

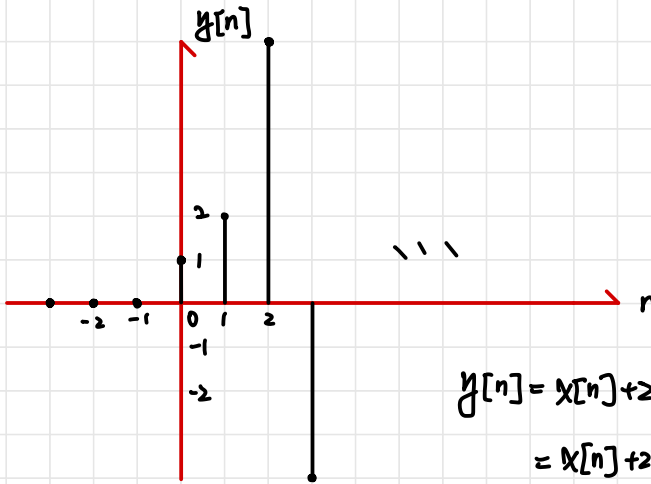
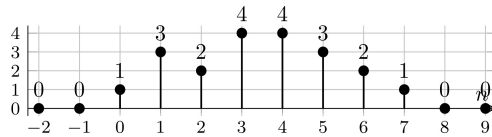
(b) $x(t) * f'(t) u_1(t)$ let $g(t-\tau) = f(\tau)$
 $= \int_{-\infty}^{\infty} x(t-\tau) f(\tau) u_1(\tau) d\tau$ let $p(t-\tau) = x(t-\tau) g(t-\tau)$
 $= \int_{-\infty}^{\infty} x(t-\tau) g(t-\tau) u_1(\tau) d\tau$ from (a), we can have $-g'(t) = f'(0)$
 $= \int_{-\infty}^{\infty} p(t-\tau) u_1(\tau) d\tau$
 $= p(t) * u_1(t) = \frac{d}{dt} p(t)$
 $= \frac{d}{dt} (x(t) g(t))$
 $= g(t) x'(t) + g'(t) x(t)$
 $= \underline{f(0) x'(t) - f'(0) x(t)}$ Δ

$$\begin{aligned} & x(t) * (f(0) u_1(t) - f'(0) \delta(t)) \\ &= f(0) \int_{-\infty}^{\infty} x(t-\tau) u_1(\tau) d\tau - f'(0) \int_{-\infty}^{\infty} x(t-\tau) \delta(\tau) d\tau \\ &= \underline{f(0) x'(t) - f'(0) x(t)} \end{aligned}$$

Problem 4 (25 pts). Consider the LTI system initially at rest and described by the difference equation

$$y[n] + 2y[n-1] = x[n] + 2x[n-2].$$

Plot the response of this system to the input depicted in the figure by solving the difference equation recursively.



$$y[n] = x[n] + 2x[n-2] - 2y[n-1] \quad -8 \rightarrow +16 \leftarrow$$

$$= x[n] + 2x[n-2] - 2x[n-1] - 4x[n-3] + 4x[n-2] + 8x[n-4]$$

$$= x[n] - 2x[n-1] + 6x[n-2] - 12x[n-3] \dots$$

$$= x[n] - 2x[n-1] + \sum_{k=2}^{\infty} (3 \cdot 2^{k-1} (-1)^k x[n-k])$$

$$y[n] = 0 \text{ for } n \leq -1$$

$$y[0] = 1, y[1] = 1$$

$$y[0] = 1$$

$$y[1] = 3 - 2 = 1$$

$$y[2] = 4 - 2 = 2$$

$$y[3] = 10 - 4 = 6$$

$$y[4] = 8 - 12 = -4$$

$$y[5] = 11 + 8 = 19$$

$$y[6] = 10 - 38 = -28$$

$$y[7] = 7 + 56 = 63$$

$$y[8] = 4 - 126 = -122$$

$$y[9] = 2 + 244 = 246$$

\vdots

Problem 5 (15 pts). Evaluate the following integrals:

(a) $\int_{-\infty}^{\infty} u_0(t) \cos(t) dt$

(b) $\int_0^5 \sin(2\pi t) \delta(t+3) dt$

(c) $\int_{-5}^5 u_1(1-\tau) \cos(2\pi\tau) d\tau$

(a) $\int_{-\infty}^{\infty} u_0(t) \cos(t) dt = 1$

(b) $\int_0^5 \sin(2\pi t) \delta(t+3) dt = 0$

(c) $\int_{-5}^5 u_1(1-\tau) \cos(2\pi\tau) d\tau$

$$= -\delta(1-\tau) \cos(2\pi\tau) \Big|_{\tau=-5}^5 - 2\pi \int_{-5}^5 \delta(1-\tau) \sin(2\pi\tau) d\tau$$

$$= 0 - 2\pi \sin(2\pi) \int_{-5}^5 \delta(1-\tau) d\tau = 0$$