

# Signals and Systems

Homework 7 — Due : Apr. 12 2024

**Problem 1** (20 pts). Consider three continuous-time periodic signals whose Fourier series representations are as follows:

$$x_1(t) = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k e^{jk\frac{2\pi}{50}t}, \quad x_2(t) = \sum_{k=-100}^{100} \cos(k\pi) e^{jk\frac{2\pi}{50}t}, \quad \text{and} \quad x_3(t) = \sum_{k=-100}^{100} j \sin\left(\frac{k\pi}{2}\right) e^{jk\frac{2\pi}{50}t}.$$

Use Fourier series properties to help answer the following questions:

- (a) Which of the three signals is/are real valued?
- (b) Which of the three signals is/are even?

**Problem 2** (30 pts). Let  $x(t)$  and  $y(t)$  both be continuous-time periodic signals having period  $T_0$  and with Fourier series representations given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}. \quad (1)$$

Suppose that  $y(t)$  in the eq. (1) equals  $x^*(t)$ . Express the  $b_k$  in the equation in terms of  $a_k$ , and prove Parseval's relation for periodic signals – that is,

$$\frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

**Problem 3** (20 pts). Suppose that we are given the following information about a signal  $x(t)$ :

- $x(t)$  is a real signal.
- $x(t)$  is periodic with period 6 and has Fourier coefficients  $a_k$ .
- $a_k = 0$  for  $k = 0$  and  $k > 2$ .
- $x(t) = -x(t - 3)$ .
- $\frac{1}{6} \int_{-3}^3 |x(t)|^2 dt = \frac{1}{2}$ .
- $a_1$  is a positive real number.

Show that  $x(t) = A \cos(Bt + C)$ , and determine the values of the constants  $A$ ,  $B$ , and  $C$ .

**Problem 4** (30 pts). For any

$$x(t) \xleftrightarrow{FS} a_k,$$

we can decompose  $x(t)$  and  $a_k$  into even parts plus odd parts

$$x(t) = x_e(t) + x_o(t), \quad a_k = a_{k,e} + a_{k,o}.$$

Prove that

$$x_e(t) \xleftrightarrow{FS} a_{k,e}, \quad x_o(t) \xleftrightarrow{FS} a_{k,o}$$

**Problem 1** (20 pts). Consider three continuous-time periodic signals whose Fourier series representations are as follows:

$$x_1(t) = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k e^{jk\frac{2\pi}{50}t}, \quad x_2(t) = \sum_{k=-100}^{100} \cos(k\pi) e^{jk\frac{2\pi}{50}t}, \quad \text{and} \quad x_3(t) = \sum_{k=-100}^{100} j \sin\left(\frac{k\pi}{2}\right) e^{jk\frac{2\pi}{50}t}.$$

Use Fourier series properties to help answer the following questions:

(a) Which of the three signals is/are real valued?

(b) Which of the three signals is/are even?

(a) let  $a_k = \left(\frac{1}{2}\right)^k$ ,  $a_k \neq a_{-k}^* \Rightarrow$   $x_1$  is not real.

for  $k=0$ ,  $\cos(0) \cdot e^0 = 1$

for  $k \neq 0$ ,  $\cos(-k\pi) \cdot e^{jk\frac{2\pi}{50}t} = \cos(k\pi) \cdot [\cos(k\frac{2\pi}{50}t) - j \sin(k\frac{2\pi}{50}t)]$

$\cos(k\pi) \cdot e^{jk\frac{2\pi}{50}t} = \cos(k\pi) \cdot [\cos(k\frac{2\pi}{50}t) + j \sin(k\frac{2\pi}{50}t)]$

$\Rightarrow$   $x_2$  is real.

for  $k=0$ ,  $j \sin(0) e^0 = 0$

for  $k \neq 0$ ,  $j \sin\left(\frac{-k\pi}{2}\right) e^{jk\frac{2\pi}{50}t} = -j \sin\left(\frac{k\pi}{2}\right) \cdot [\cos(k\frac{2\pi}{50}t) - j \sin(k\frac{2\pi}{50}t)]$

$j \sin\left(\frac{k\pi}{2}\right) e^{jk\frac{2\pi}{50}t} = j \sin\left(\frac{k\pi}{2}\right) \cdot [\cos(k\frac{2\pi}{50}t) + j \sin(k\frac{2\pi}{50}t)]$

$\Rightarrow$   $x_3$  is real.

$$\begin{aligned}
 (b) \quad x_1(t) &= \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k e^{jk \frac{2\pi}{50} t} = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k \cdot \left[ \cos\left(k \frac{2\pi}{50} t\right) + j \sin\left(k \frac{2\pi}{50} t\right) \right] \\
 &= \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k \cdot \left[ \cos\left(k \frac{2\pi}{50} (-t)\right) - \sin\left(k \frac{2\pi}{50} (-t)\right) \right] \neq x_1(-t)
 \end{aligned}$$

$x_1$  is not even.

$$\begin{aligned}
 x_2(t) &= \sum_{k=-100}^{100} \cos(k\pi) e^{jk \frac{2\pi}{50} t} = \sum_{k=-100}^{100} \cos(k\pi) \cdot \left[ \cos\left(k \frac{2\pi}{50} t\right) + j \sin\left(k \frac{2\pi}{50} t\right) \right] \\
 &= 1 + 2 \sum_{k=1}^{100} \cos(k\pi) \cdot \cos\left(k \frac{2\pi}{50} t\right) = 1 + 2 \sum_{k=1}^{100} \cos(k\pi) \cdot \cos\left(k \frac{2\pi}{50} (-t)\right) \\
 &= x_2(-t)
 \end{aligned}$$

$\Rightarrow x_2$  is even.

$$\begin{aligned}
 x_3(t) &= \sum_{k=-100}^{100} j \sin\left(\frac{k\pi}{2}\right) e^{jk \frac{2\pi}{50} t} = \sum_{k=-100}^{100} j \sin\left(\frac{k\pi}{2}\right) \cdot \left[ \cos\left(k \frac{2\pi}{50} t\right) + j \sin\left(k \frac{2\pi}{50} t\right) \right] \\
 &= -2 \sum_{k=1}^{100} \sin\left(\frac{k\pi}{2}\right) \sin\left(k \frac{2\pi}{50} t\right) = 2 \sum_{k=1}^{100} \sin\left(\frac{k\pi}{2}\right) \sin\left(k \frac{2\pi}{50} (-t)\right) \\
 &= -x_3(-t)
 \end{aligned}$$

$\Rightarrow x_3$  is not even.

**Problem 2** (30 pts). Let  $x(t)$  and  $y(t)$  both be continuous-time periodic signals having period  $T_0$  and with Fourier series representations given by

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Suppose that  $y(t)$  in the eq. (1) equals  $x^*(t)$ . Express the  $b_k$  in the equation in terms of  $a_k$ , and prove Parseval's relation for periodic signals – that is,

$$\frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

Note that  $(AB)^* = A^* B^*$ ,  $(A+B)^* = A^* + B^*$

$$\begin{aligned} y(t) = x^*(t) &= \left( \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right)^* = \sum_{k=-\infty}^{\infty} (a_k e^{jk\omega_0 t})^* = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} \end{aligned}$$

$$\Rightarrow b_k = a_{-k}^*$$

$$\text{Let } c_k = \frac{1}{T_0} \int_0^{T_0} x(t) y(t) e^{-jk\omega_0 t} dt, \text{ then } c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}$$

$$c_0 = \sum_{n=-\infty}^{\infty} a_n b_{-n} = \sum_{n=-\infty}^{\infty} a_n a_n^* = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$= \frac{1}{T_0} \int_0^{T_0} x(t) y(t) dt = \frac{1}{T_0} \int_0^{T_0} x(t) x^*(t) dt = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt$$

$$\overset{A}{(\cos A - j \sin A)} \overset{B}{(\cos B - j \sin B)}$$

$$= \cos A \cos B - \sin A \sin B - j(\sin A \cos B + \cos A \sin B)$$

$$= \cos(A+B) - j \sin(A+B)$$

$$\overset{A^*}{(\cos A + j \sin A)} \overset{B^*}{(\cos B + j \sin B)}$$

$$= \cos A \cos B - \sin A \sin B + j(\sin A \cos B + \cos A \sin B)$$

$$= \cos(A+B) + j \sin(A+B) = [\cos(A+B) - j \sin(A+B)]^* \\ \underline{A^* B^* = (AB)^*}$$

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- $a_1$  is a positive real number.

Show that  $x(t) = A \cos(Bt + C)$ , and determine the values of the constants  $A$ ,  $B$ , and  $C$ .

$$x(t) \text{ is real} \rightarrow a_k = a_k^* \rightarrow \begin{cases} a_1 \text{ is real} \rightarrow a_{-1} = a_1 \\ a_2 = a_2^* \end{cases}$$

$$\text{Let } A_1 = |a_1| = |a_{-1}|, \quad A_2 = |a_2| = |a_2^*| = |a_{-2}|$$

$$T_0 = 6 \rightarrow \omega_0 = \frac{\pi}{3}$$

$$a_k = 0 \text{ for } k=0 \text{ and } k>2$$

$$\rightarrow x(t) = a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_{-2} e^{-j2\omega_0 t}$$

$$= a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + a_1 \cdot \cos(\omega_0 t) + a_2^* \cos(2\omega_0 t)$$

$$= A_1 \cos(\omega_0 t) + A_2 \cos(2\omega_0 t + \theta) + A_1 \cos(\omega_0 t) + A_2 \cos(2\omega_0 t - \theta)$$

$$= 2A_1 \cos\left(\frac{\pi}{3}t\right) + A_2 \cos\left(\frac{2\pi}{3}t + \theta\right) + A_2 \cos\left(\frac{2\pi}{3}t - \theta\right)$$

$$-x(t-3) = -2A_1 \cos\left(\frac{\pi}{3}t - \pi\right) - A_2 \cos\left(\frac{2\pi}{3}t - 2\pi + \theta\right) - A_2 \cos\left(\frac{2\pi}{3}t - 2\pi - \theta\right)$$

$$= A_1 \cos\left(\frac{\pi}{3}t\right) - A_2 \cos\left(\frac{2\pi}{3}t + \theta\right) - A_2 \cos\left(\frac{2\pi}{3}t - \theta\right)$$

$$x(t) = A_1 \cos\left(\frac{\pi}{3}t\right) + A_2 \cos\left(\frac{2\pi}{3}t + \theta\right) + A_2 \cos\left(\frac{2\pi}{3}t - \theta\right)$$

$$\rightarrow A_2 = 0 \rightarrow x(t) = 2A_1 \cos\left(\frac{\pi}{3}t\right)$$

$$\frac{1}{6} \int_{-3}^3 |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 = a_1^2 + a_{-1}^2 = \frac{1}{2} \rightarrow a_1 = \frac{1}{2}$$

$$\rightarrow x(t) = 2 \cdot \frac{1}{2} \cos\left(\frac{\pi}{3} t\right) = 1 \cdot \cos\left(\frac{\pi}{3} t + 0\right)$$

$$\rightarrow A=1, B=\frac{\pi}{3}, C=0$$

**Problem 4** (30 pts). For any

$$x(t) \xleftrightarrow{FS} a_k,$$

we can decompose  $x(t)$  and  $a_k$  into even parts plus odd parts

$$x(t) = x_e(t) + x_o(t),$$

$$a_k = a_{k,e} + a_{k,o}.$$

Prove that

$$x_e(t) \xleftrightarrow{FS} a_{k,e},$$

$$x_o(t) \xleftrightarrow{FS} a_{k,o}$$

$$\left. \begin{aligned} x_e(t) &= \frac{1}{2} [x(t) + x(-t)] \\ x_o(t) &= \frac{1}{2} [x(t) - x(-t)] \end{aligned} \right\} x(t) = x_e(t) + x_o(t)$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-j k \omega_0 t} dt$$

$$a_{k,e} = \frac{1}{T} \int_T x_e(t) e^{-j k \omega_0 t} dt = \frac{1}{2T} \left[ \int_T x(t) e^{-j k \omega_0 t} dt + \int_T x(-t) e^{-j k \omega_0 t} dt \right] = \frac{1}{2} [a_k(t) + a_k(-t)]$$

$$a_{k,o} = \frac{1}{T} \int_T x_o(t) e^{-j k \omega_0 t} dt = \frac{1}{2T} \left[ \int_T x(t) e^{-j k \omega_0 t} dt - \int_T x(-t) e^{-j k \omega_0 t} dt \right] = \frac{1}{2} [a_k(t) - a_k(-t)]$$

$$\Rightarrow a_k = a_{k,e} + a_{k,o}, \quad x_e \xleftrightarrow{FS} a_{k,e}, \quad x_o \xleftrightarrow{FS} a_{k,o}$$