Signals and Systems

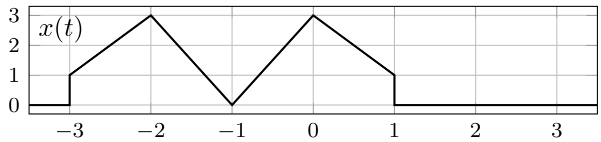
Homework 9 — Due : May 10, 2024

Problem 1 (20 pts)**.** Given the relationships

$$y(t) = x(t) * h(\cancel{t}-t) \quad \text{and} \quad g(t) = 2x(-3t) * h(3t).$$

Given that $x(t)$ has Fourier transform $X(j\omega)$ and $h(t)$ has Fourier transform $H(j\omega)$, show that $g(t)$ has the form $g(t) = Ay(Bt)$. Determine the values of A and B.

Problem 2 (30 pts)**.** Let $X(j\omega)$ denote the Fourier transform of th signal $x(t)$ depicted in the figure.



(a) $X(j\omega)$ can be expressed as $A(j\omega)e^{j\Theta(j\omega)}$, where $A(j\omega)$ and $\Theta(j\omega)$ are both real-values. Find $\Theta(j\omega)$.

(b) Find $X(j0)$.

(c) Find $\int_{-\infty}^{\infty} X(j\omega)d\omega$.

(d) Evaluate $\int_{-\infty}^{\infty} X(j\omega)\frac{2\sin(\omega)}{\omega}e^{-j\omega}d\omega$.

(e) Evaluate $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$.

(f) Sketch the inverse Fourier transform of $\mathbf{Re}\{X(j\omega)\}$.

Note: You should perform all these calculations without explicitly evaluating $X(j\omega)$.

Problem 3 (20 pts)**.** Consider an LTI system whose response to the input $x(t) = [e^{-t} + e^{-3t}]u(t)$ is $y(t) = [2e^{-t} - 2e^{-4t}]u(t)$.

(a) Find the frequency response and the impulse response of the system.

(b) Find the differential equation relating the input and the output of this system.

Problem 4 (20 pts)**.** The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the equation

$$\frac{d}{dt}y(t) + 10y(t) = x(t) + \int_{-\infty}^{\infty} z(\tau)x(t-\tau)d\tau.$$

where $z(t) = e^{-2t}u(t) + \delta(t)$. Find the frequency response and the impulse response of the system.

Problem 5 (10 pts)**.** Find the Fourier transform of

$$x(t) = \begin{cases} 1, \text{if } t > 0 \\ 0, \text{if } t = 0 \\ -1, \text{if } t < 0 \end{cases}$$

Problem 1 (20 pts). Given the relationships

$$y(t) = x(t) * h(\cancel{t}-t) \quad \text{and} \quad g(t) = 2x(-3t) * h(3t).$$

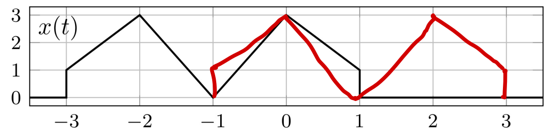
Given that $x(t)$ has Fourier transform $X(j\omega)$ and $h(t)$ has Fourier transform $H(j\omega)$, show that $g(t)$ has the form $g(t) = Ay(Bt)$. Determine the values of A and B.

$$Y(jw) = X(jw)\, H(-jw)$$

$$G(jw) = 2 \cdot \frac{1}{3} X\left(-\frac{jw}{3}\right) \cdot \frac{1}{3} H\left(\frac{jw}{3}\right) = \frac{2}{3} \cdot \frac{1}{3} Y\left(-\frac{jw}{3}\right)$$

$$\Rightarrow g(t) = \frac{2}{3} \cdot y(-3t) \;.\quad A = \frac{2}{3} \;.\; B = -3$$

Problem 2 (30 pts). Let $X(j\omega)$ denote the Fourier transform of th signal $x(t)$ depicted in the figure.



(a) $X(j\omega)$ can be expressed as $A(j\omega)e^{j\Theta(j\omega)}$, where $A(j\omega)$ and $\Theta(j\omega)$ are both real-values. Find $\Theta(j\omega)$.

Let $a(t) = x(t-1) \Rightarrow x(t) = a(t+1)$

$$X(jw) = F\{x(t)\} = F\{a(t+1)\} = A(jw) \cdot e^{jw}$$

$$\theta(jw) = w \quad \#$$

(b) Find $X(j0)$.

$$X(j0) = \int_{-\infty}^{\infty} x(t)\, e^{-j0t} dt = \int_{-\infty}^{\infty} x(t)\, dt = 7$$

(c) Find $\int_{-\infty}^{\infty} X(j\omega) d\omega$.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(jw)\, e^{jwt}\, dw$$

$$\int_{-\infty}^{\infty} X(jw)\, dw = 2\pi\, x(0) = 6\pi$$

(d) Evaluate $\int_{-\infty}^{\infty} X(j\omega)\frac{2\sin(\omega)}{\omega}e^{-j\omega}d\omega$.

$$F^{-1}\left\{\frac{2\sin\omega}{\omega}\right\} = \begin{cases} 1, & |t|<1 \\ 0, & |t|>1 \end{cases} = h(t)$$

$$x(-1) * h(-1) = \int_{-\infty}^{\infty} x(\tau)\, h(-1-\tau)\, d\tau = \int_{-2}^{0} x(\tau)d\tau = 3$$

$$\int_{-\infty}^{\infty} X(j\omega)\cdot\frac{2\sin(\omega)}{\omega}e^{-j\omega}d\omega = 2\pi\cdot 3 = 6\pi$$

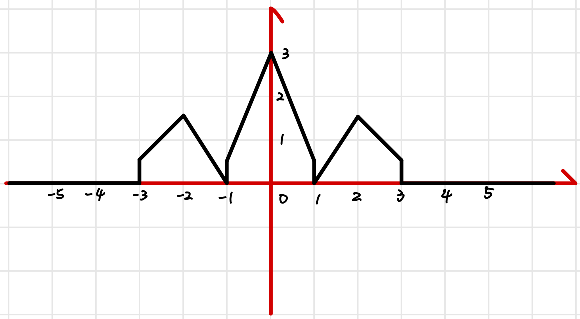
(e) Evaluate $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$.

$$2\pi\int_{-\infty}^{\infty}|x(t)|^2 dt = 4\pi\left(\int_0^1 9t^2 dt + \int_0^1 (4t^2-12t+9)dt\right)$$

$$= 4\pi\left(3+\frac{13}{3}\right) = \frac{88}{3}\pi$$

(f) Sketch the inverse Fourier transform of $\mathbf{Re}\{X(j\omega)\}$.

$$F^{-1}\{Re\{X(j\omega)\}\} = \frac{1}{2}[x(t)+x(-t)]$$



Problem 3 (20 pts). Consider an LTI system whose response to the input $x(t) = [e^{-t} + e^{-3t}]u(t)$ is $y(t) = [2e^{-t} - 2e^{-4t}]u(t)$.

(a) Find the frequency response and the impulse response of the system.

$$F\{x(t)\} = \int_0^{\infty} e^{-t} \cdot e^{-j\omega t}\, dt + \int_0^{\infty} e^{-3t} \cdot e^{-j\omega t}\, dt$$

$$= \int_0^{\infty} e^{-(j\omega+1)t}\, dt + \int_0^{\infty} e^{-(j\omega+3)t}\, dt$$

$$= \frac{1}{j\omega+1} + \frac{1}{j\omega+3} = \frac{2j\omega+4}{(j\omega+1)(j\omega+3)}$$

$$F\{y(t)\} = \frac{2}{j\omega+1} - \frac{2}{j\omega+4} = \frac{6}{(j\omega+1)(j\omega+4)}$$

Frequency Response

$$H(j\omega) = \frac{6\cdot(j\omega+3)}{(j\omega+4)\cdot 2\cdot(j\omega+2)} = \frac{3}{2}\cdot\frac{2j\omega+6}{(j\omega+4)\cdot(j\omega+2)} = \underline{\frac{3}{2}\left(\frac{1}{j\omega+4} + \frac{1}{j\omega+2}\right)}$$

$$h(t) = F^{-1}\{H(j\omega)\} = \underline{\frac{3}{2}\left(e^{-4t} + e^{-2t}\right)u(t)}$$ Impulse Response

(b) Find the differential equation relating the input and the output of this system.

$$H(j\omega) = \frac{3j\omega+9}{(j\omega)^2 + 6j\omega + 8} = \frac{Y(j\omega)}{X(j\omega)}$$

$$\Rightarrow 3\cdot\frac{d}{dt}x(t) + 9\,x(t) = \frac{d^2}{dt^2}y(t) + 6\cdot\frac{d}{dt}y(t) + 8\,y(t)$$

Problem 4 (20 pts). The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the equation

$$\frac{d}{dt}y(t) + 10y(t) = x(t) + \int_{-\infty}^{\infty} z(\tau)x(t-\tau)d\tau.$$

where $z(t) = e^{-2t}u(t) + \delta(t)$. Find the frequency response and the impulse response of the system.

$$\frac{d}{dt} h(t) + 10\, h(t) = \delta(t) + \int_{-\infty}^{\infty} [e^{-2\tau} u(\tau) + \delta(\tau)]\, \delta(t-\tau)\, d\tau$$

$$j\omega H(j\omega) + 10\, H(j\omega) = 1 + \frac{1}{j\omega+2} + 1$$

$$H(j\omega) = \frac{2j\omega+5}{(j\omega+2)(j\omega+10)} = \frac{1}{8}\cdot\frac{1}{j\omega+2} + \frac{15}{8}\cdot\frac{1}{j\omega+10}$$

$$h(t) = \frac{1}{8}e^{-2t}u(t) + \frac{15}{8}e^{-10t}u(t)$$

Problem 5 (10 pts). Find the Fourier transform of

$$x(t) = \begin{cases} 1, \text{if } t > 0 \\ 0, \text{if } t = 0 \\ -1, \text{if } t < 0 \end{cases}$$

$$F\{u(t)\} = \frac{1}{j\omega}, \quad F\{u(-t)\} = \frac{1}{-j\omega}$$

$$F\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt = \int_{-\infty}^{0} (-1) e^{-j\omega t} dt + \int_{0}^{\infty} 1 \cdot e^{-j\omega t} dt$$

$$= \frac{1}{j\omega} - \frac{1}{-j\omega} = \frac{2}{j\omega}$$