Signals and Systems

Homework 12 — Due : May 31 2024

Problem 1 (20 pts)**.** Given that

$$e^{-at}u(t) \xleftrightarrow{\mathfrak{L}} \frac{1}{s+a}, \quad \mathbf{Re}\{s\} > \mathbf{Re}\{-a\},$$

determine the function of time, $x(t)$, for the following Laplace transforms:

(a) $\frac{s+2}{s^2+7s+12}$, $\quad \mathbf{Re}\{s\} < -4$

(b) $\frac{s+1}{s^2+5s+6}$, $\quad -3 < \mathbf{Re}\{s\} < -2$

Problem 2 (20 pts)**.** We are given the following five facts about a real signal $x(t)$ with Laplace transform $X(s)$:

1. $X(s)$ has exactly two poles.
2. $X(s)$ has no zeros in the finite s-plane.
3. $X(s)$ has a pole at $s = 2 + j$.
4. $X(0) = 5$.
5. $e^{-3t}x(t)$ is not absolutely integrable.

Determine $X(s)$ and specify its region of convergence.

Problem 3 (20 pts)**.** A causal LTI system with impulse response $h(t)$ has the following properties:

1. When the input to the system is $x(t) = e^{2t}$ for all t, the output is $y(t) = (1/6)e^{2t}$ for all t.
2. The impulse response $h(t)$ satisfies the differential equation

$$\frac{d}{dt}h(t) + 2h(t) = e^{-4t}u(t) + bu(t),$$

   where b is an unknown constant.

Determine the system function $H(s)$ of the system, consistent with the information above. (There should be no unknown constants in your answer.)

Problem 4 (10 pts)**.** Consider the system characterized by the differential equation

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t),$$

with initial conditions $y(-1) = 1$ and $\frac{d}{dt}y(t)\Big|_{t=-1} = -1$. What is the output of the system with input $x(t) = 3u(t+1)$?

Problem 5 (30 pts)**.** Consider the following sub-problems:

(a) Find the inverse Laplace transform of $\frac{s}{(s+1)(s-1)}$, ROC: $\mathbf{Re}\{s\} > 1$. Let the answer be $x_1(t)$.

(b) Find the inverse Laplace transform of $\frac{s-1}{(s+1)s}$, ROC: $\cancel{-1 < \mathbf{Re}\{s\} < 1}$ $-1 < \mathbf{Re}\{s\} < 0$. Let the answer be $x_2(t)$.

(c) Find $x_1(t) * x_2(t)$.

Problem 1 (20 pts). Given that

$$e^{-at}u(t) \xleftrightarrow{\mathfrak{L}} \frac{1}{s+a}, \quad \mathbf{Re}\{s\} > \mathbf{Re}\{-a\},$$

determine the function of time, $x(t)$, for the following Laplace transforms:

(a) $\frac{s+2}{s^2+7s+12}$, $\quad \mathbf{Re}\{s\} < -4$

$$\frac{s+2}{s^2+7s+12} = \frac{s+2}{(s+3)(s+4)} = \frac{-1}{s+3} + \frac{2}{s+4}, \quad Re\{s\} < -4$$

$$x(t) = e^{-3t}u(-t) - 2e^{-4t}u(-t)$$

(b) $\frac{s+1}{s^2+5s+6}$, $\quad -3 < \mathbf{Re}\{s\} < -2$

$$\frac{s+1}{s^2+5s+6} = \frac{s+1}{(s+2)(s+3)} = \frac{-1}{s+2} + \frac{2}{s+3}, \quad -3 < Re\{s\} < -2$$

$$x(t) = e^{-2t}u(-t) + 2e^{-3t}u(t)$$

Problem 2 (20 pts). We are given the following five facts about a real signal $x(t)$ with Laplace transform $X(s)$:

1. $X(s)$ has exactly two poles.
2. $X(s)$ has no zeros in the finite s-plane.
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4. $X(0) = 5$.
5. $e^{-3t}x(t)$ is not absolutely integrable.

$$X(s) = \frac{k}{(s-2-j)(s-2+j)}, \quad X(0) = \frac{k}{(-2-j)(-2+j)} = \frac{k}{5} = 5, \quad k = 25, \quad R\{s\} < -2$$

$$X(s) = \frac{5}{(s-2-j)(s-2+j)} = \frac{\frac{5}{2j}}{s-2-j} - \frac{\frac{5}{2j}}{s-2+j} = \frac{1}{2j}\left(\frac{1}{s-(2+j)} - \frac{1}{s-(2-j)}\right)$$

$$x(t) = \frac{5}{2j}\left(-e^{(2+j)t} + e^{(2-j)t}\right)u(-t) = \frac{5}{2j}e^{2t}\left(e^{-jt} - e^{jt}\right)u(-t) = -5\sin(t)\cdot u(-t)$$

Problem 3 (20 pts). A causal LTI system with impulse response $h(t)$ has the following properties:

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2. The impulse response $h(t)$ satisfies the differential equation

$$\frac{d}{dt}h(t) + 2h(t) = e^{-4t}u(t) + bu(t),$$

   where b is an unknown constant.

Determine the system function $H(s)$ of the system, consistent with the information above. (There should be no unknown constants in your answer.)

$$s\,H(s) + 2\,H(s) = \frac{1}{s+4} + \frac{b}{s}$$

$$H(s) = \frac{1}{(s+2)(s+4)} + \frac{b}{s(s+2)} = \frac{\frac{1}{2}}{s+2} - \frac{\frac{1}{2}}{s+4} + \frac{\frac{b}{2}}{s} - \frac{\frac{b}{2}}{s+2}$$

$$e^{st} \longrightarrow H(s)\,e^{st} \Rightarrow e^{2t} \longrightarrow H(2)\,e^{2t}$$

$$H(2) = \frac{1}{24} + \frac{b}{8} = \frac{1}{6}, \quad b = 1$$

$$\Rightarrow H(s) = \frac{2s+4}{s(s+2)(s+4)} = \frac{2}{s(s+4)}$$

Problem 4 (10 pts). Consider the system characterized by the differential equation

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t),$$

with initial conditions $y(-1) = 1$ and $\frac{d}{dt}y(t)\big|_{t=-1} = -1$. What is the output of the system with input $x(t) = 3u(t+1)$?

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s\left(s\,Y(s) - y(-1^-)\right) - y'(-1^-)$$

$$= s^2\,Y(s) - s\,y(-1^-) - y'(-1^-) = s^2\,Y(s) - s + 1$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = s\,Y(s) - y(-1^-) = s\,Y(s) - 1, \quad \mathcal{L}\{3\,u(t+1)\} = \frac{3e^s}{s}$$

$$s^2\,Y(s) - s + 1 + 3s\,Y(s) - 3 + 2\,Y(s) = \frac{3e^s}{s}$$

$$Y(s) = \frac{3e^s + s^2 + 2s}{s(s^2+3s+2)} = \frac{3e^s}{s(s+1)(s+2)} + \frac{s+2}{(s+1)(s+2)}$$

$$= 3e^s\left(\frac{-1}{s+1} + \frac{\frac{1}{2}}{s+2} + \frac{\frac{1}{2}}{s}\right) + \frac{1}{s+1}$$

$$y(t) = \left(-3e^{-(t+1)} + \frac{3}{2}e^{-2(t+1)} + \frac{3}{2}\right)u(t+1) + e^{-t}u(t)$$

Problem 5 (30 pts). Consider the following sub-problems:

(a) Find the inverse Laplace transform of $\frac{s}{(s+1)(s-1)}$, ROC: $\mathbf{Re}\{s\} > 1$. Let the answer be $x_1(t)$.

$$\frac{s}{(s+1)(s-1)} = \frac{1}{2}\cdot\frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{s-1}, \quad x_1(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{t}u(t)$$

(b) Find the inverse Laplace transform of $\frac{s-1}{(s+1)s}$, ROC: $\cancel{-1 < \mathbf{Re}\{s\} < 1}$ $-1 < \mathbf{Re}\{s\} < 0$. Let the answer be $x_2(t)$.

$$\frac{s-1}{(s+1)s} = \frac{2}{s+1} - \frac{1}{s}, \quad x_2(t) = 2e^{-t}u(t) - u(t)$$

(c) Find $x_1(t) * x_2(t)$.

$$\frac{s}{(s+1)(s-1)} \cdot \frac{s-1}{(s+1)s} = \frac{1}{(s+1)^2}, \quad x_1(t) * x_2(t) = te^{-t}u(t)$$