

Signals and Systems

Homework 12 — Due : May 31 2024

Problem 1 (20 pts). Given that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad \text{Re}\{s\} > \text{Re}\{-a\},$$

determine the function of time, $x(t)$, for the following Laplace transforms:

(a) $\frac{s+2}{s^2+7s+12}, \quad \text{Re}\{s\} < -4$

(b) $\frac{s+1}{s^2+5s+6}, \quad -3 < \text{Re}\{s\} < -2$

Problem 2 (20 pts). We are given the following five facts about a real signal $x(t)$ with Laplace transform $X(s)$:

1. $X(s)$ has exactly two poles.
2. $X(s)$ has no zeros in the finite s -plane.
3. $X(s)$ has a pole at $s = 2 + j$.
4. $X(0) = 5$.
5. $e^{-3t}x(t)$ is not absolutely integrable.

Determine $X(s)$ and specify its region of convergence.

Problem 3 (20 pts). A causal LTI system with impulse response $h(t)$ has the following properties:

1. When the input to the system is $x(t) = e^{2t}$ for all t , the output is $y(t) = (1/6)e^{2t}$ for all t .
2. The impulse response $h(t)$ satisfies the differential equation

$$\frac{d}{dt}h(t) + 2h(t) = e^{-4t}u(t) + bu(t),$$

where b is an unknown constant.

Determine the system function $H(s)$ of the system, consistent with the information above. (There should be no unknown constants in your answer.)

Problem 4 (10 pts). Consider the system characterized by the differential equation

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t),$$

with initial conditions $y(-1) = 1$ and $\frac{d}{dt}y(t)|_{t=-1} = -1$. What is the output of the system with input $x(t) = 3u(t+1)$?

Problem 5 (30 pts). Consider the following sub-problems:

- (a) Find the inverse Laplace transform of $\frac{s}{(s+1)(s-1)}$, ROC: $\text{Re}\{s\} > 1$. Let the answer be $x_1(t)$.
- (b) Find the inverse Laplace transform of $\frac{s-1}{(s+1)s}$, ROC: $-1 < \text{Re}\{s\} < 0$. Let the answer be $x_2(t)$.
- (c) Find $x_1(t) * x_2(t)$.

Problem 1 (20 pts). Given that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad \operatorname{Re}\{s\} > \operatorname{Re}\{-a\},$$

determine the function of time, $x(t)$, for the following Laplace transforms:

(a) $\frac{s+2}{s^2+7s+12}, \quad \operatorname{Re}\{s\} < -4$

$$\frac{s+2}{s^2+7s+12} = \frac{s+2}{(s+3)(s+4)} = \frac{-1}{s+3} + \frac{2}{s+4}, \quad \operatorname{Re}\{s\} < -4$$

$$x(t) = e^{-3t}u(-t) - 2e^{-4t}u(-t)$$

(b) $\frac{s+1}{s^2+5s+6}, \quad -3 < \operatorname{Re}\{s\} < -2$

$$\frac{s+1}{s^2+5s+6} = \frac{s+1}{(s+2)(s+3)} = \frac{-1}{s+2} + \frac{2}{s+3}, \quad -3 < \operatorname{Re}\{s\} < -2$$

$$x(t) = e^{-2t}u(-t) + 2e^{-3t}u(t)$$

Problem 2 (20 pts). We are given the following five facts about a real signal $x(t)$ with Laplace transform $X(s)$:

1. $X(s)$ has exactly two poles.
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Determine $X(s)$ and specify its region of convergence.

$$X(s) = \frac{k}{(s-2-j)(s-2+j)}, \quad X(0) = \frac{k}{(-2-j)(-2+j)} = \frac{k}{5} = 5, \quad k = 25, \quad \operatorname{Re}\{s\} < 2$$

$$X(s) = \frac{25}{(s-2-j)(s-2+j)} = \frac{\frac{25}{2j}}{s-2-j} - \frac{\frac{25}{2j}}{s-2+j} = \frac{25}{2j} \left(\frac{1}{s-(2+j)} - \frac{1}{s-(2-j)} \right)$$

Problem 3 (20 pts). A causal LTI system with impulse response $h(t)$ has the following properties:

1. When the input to the system is $x(t) = e^{2t}$ for all t , the output is $y(t) = (1/6)e^{2t}$ for all t .
2. The impulse response $h(t)$ satisfies the differential equation

$$\frac{d}{dt}h(t) + 2h(t) = e^{-4t}u(t) + bu(t),$$

where b is an unknown constant.

Determine the system function $H(s)$ of the system, consistent with the information above. (There should be no unknown constants in your answer.)

$$s H(s) + 2 H(s) = \frac{1}{s+4} + \frac{b}{s}$$

$$H(s) = \frac{1}{(s+2)(s+4)} + \frac{b}{s(s+2)} = \frac{\frac{1}{2}}{s+2} - \frac{\frac{1}{2}}{s+4} + \frac{\frac{b}{2}}{s} - \frac{\frac{b}{2}}{s+2}$$

$$e^{st} \rightarrow H(s) e^{st} \Rightarrow e^{2t} \rightarrow H(2) e^{2t}$$

$$H(2) = \frac{1}{2 \cdot 4} + \frac{b}{8} = \frac{1}{6}, \quad b = 1$$

$$\Rightarrow H(s) = \frac{2s+4}{s(s+2)(s+4)} = \frac{2}{s(s+4)}$$

Problem 4 (10 pts). Consider the system characterized by the differential equation

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with initial conditions $y(-1) = 1$ and $\left.\frac{d}{dt}y(t)\right|_{t=-1} = -1$. What is the output of the system with input $x(t) = 3u(t+1)$?

$$\text{Let } y_1(t) = y(t-1), \quad x_1(t) = x(t-1), \quad \frac{d^2}{dt^2}y_1(t) + 3\frac{d}{dt}y_1(t) + 2y_1(t) = x_1(t)$$

$$\frac{d^2}{dt^2}y(t-1) + 3\frac{d}{dt}y(t-1) + 2y(t-1) = x(t-1) \quad \cdot \quad \text{Let } x_1(t) = 3u(t), \quad x(t) = 3u(t+1)$$

$$\mathcal{L}\left\{\frac{d}{dt}y_1(t)\right\} = sY_1(s) - y_1(0) \quad \cdot \quad \mathcal{L}\left\{\frac{d^2}{dt^2}y_1(t)\right\} = s^2Y_1(s) - sy_1(0) - y_1'(0)$$

$$s^2Y_1(s) - s + 1 + 3sY_1(s) - 3 + 2Y_1(s) = \frac{3}{s}$$

$$Y_1(s) = \frac{3}{s(s+1)(s+2)} + \frac{s+2}{(s+1)(s+2)} = \frac{3}{2} \cdot \frac{1}{s} - 3 \cdot \frac{1}{s+1} + \frac{3}{2} \cdot \frac{1}{s+2} + \frac{1}{s+1}$$

$$y_1(t) = y(t-1) = \left(-\frac{3}{2} - 2e^{-t} + \frac{3}{2}e^{-2t}\right)u(t)$$

$$y(t) = \left(\frac{3}{2} - 2e^{-(t+1)} + \frac{3}{2}e^{-2(t+1)}\right)u(t+1)$$

Problem 5 (30 pts). Consider the following sub-problems:

- (a) Find the inverse Laplace transform of $\frac{s}{(s+1)(s-1)}$, ROC: $\text{Re}\{s\} > 1$. Let the answer be $x_1(t)$.

$$\frac{s}{(s+1)(s-1)} = \frac{1}{2} \cdot \frac{1}{s+1} + \frac{1}{2} \cdot \frac{1}{s-1} \quad , \quad x_1(t) = \frac{1}{2} e^{-t} u(t) + \frac{1}{2} e^t u(t)$$

- (b) Find the inverse Laplace transform of $\frac{s-1}{(s+1)s}$, ROC: $-1 < \text{Re}\{s\} < 0$. Let the answer be $x_2(t)$.

$$\frac{s-1}{(s+1)s} = \frac{2}{s+1} - \frac{1}{s} \quad , \quad x_2(t) = 2 e^{-t} u(t) + u(-t)$$

- (c) Find $x_1(t) * x_2(t)$.

$$\frac{s}{(s+1)(s-1)} \cdot \frac{s-1}{(s+1)s} = \frac{1}{(s+1)^2} \quad , \quad x_1(t) * x_2(t) = t e^{-t} u(t)$$