Signals and Systems

Homework 13 — Due : Jun. 07 2024

Problem 1 (30 pts, 6 pts each)**.** The initial-value theorem states that, for a signal $x(t)$ with Laplace transform $X(s)$ and for which $x(t) = 0$ for $t < 0$, the initial value of $x(t)$ [i.e.,$x(0^+)$] can be obtained from $X(s)$ through the relation $x(0^+) = \lim_{s\to\infty} sX(s)$. First, we note that, since $x(t) = 0$ for $t < 0$, $x(t) = x(t)u(t)$. Next, expanding $x(t)$ as a Taylor series at $t = 0^+$, we obtain

$$x(t) = \left[x(0^+) + x^{(1)}(0^+)t + \cdots + x^{(n)}(0^+)\frac{t^n}{n!} + \cdots\right]u(t),$$

where $x^{(n)}(0^+)$ denotes the n^{th} derivative of $x(t)$ evaluated at $t = 0^+$.

(a) Determine The Laplace transform of an arbitrary term $x^{(n)}(0^+)(t^n/n!)u(t)$ in the Taylor series. (You may find it helpful to review Example 9.14.)

(b) From your result in part (a) and the expansion in the Taylor series, show that $X(s)$ can be expressed as

$$X(s) = \sum_{n=0}^{\infty} x^{(n)}(0^+)\frac{1}{s^{n+1}}.$$

(c) Demonstrate that $x(0^+) = \lim_{s\to\infty} sX(s)$ follows from the result of part (b).

(d) By first determining $x(t)$, verify the initial-value theorem for $X(s) = \frac{s+1}{(s+2)(s+3)}$.

(e) A more general form of the initial-value theorem states that if $x^{(n)}(0^+) = 0$ for $n < N$, then $x^{(N)}(0^+) = \lim_{s\to\infty} s^{N+1}X(s)$. Demonstrate that this more general statement also follows from the result in part (b).

Problem 2 (36 pts, 9 pts each)**.** Let

$$x_1(t) = e^{-2t}u(t) \quad \text{and} \quad x_2(t) = e^{-3(t+1)}u(t+1).$$

(a) Determine the unilateral Laplace transform $\mathcal{X}_1(s)$ and the bilateral Laplace transform $X_1(s)$ for the signal $x_1(t)$.

(b) Determine the unilateral Laplace transform $\mathcal{X}_2(s)$ and the bilateral Laplace transform $X_2(s)$ for the signal $x_2(t)$.

(c) Take the inverse bilateral Laplace transform of the product $X_1(s)X_2(s)$ to determine the signal $g(t) = x_1(t)*x_2(t)$.

(d) Show that the inverse unilateral Laplace transform of the product $\mathcal{X}_1(x)\mathcal{X}_2(x)$ is not the same as $g(t)$ for $t > 0^-$.

Problem 3 (12 pts, 6 pts each)**.** Determine the z-transform for each of the following sequences and write down the ROCs. Indicate whether the Fourier transform of the sequence exists.

(a) $x[n] = (\frac{1}{2})^{n+1}u[n]$

(b) $x[n] = (-\frac{1}{3})^n u[-n-2]$

Problem 4 (10 pts, 5 pts each)**.** Let $x[n] = (-1)^n u[n] + \alpha^n u[-n-n_0]$. Determine the constraints on the complex number α and the integer n_0, given that the ROC of $X(z)$ is $1 < |z| < 2$.

Problem 5 (12 pts)**.** Suppose that the algebraic expression for the z-transform of $x[n]$ is

$$X(z) = \frac{1 - \frac{1}{4}z^{-2}}{\left(1 + \frac{1}{4}z^{-2}\right)\left(1 + \frac{5}{4}z^{-1} + \frac{3}{8}z^{-2}\right)}.$$

How many different regions of convergence could correspond to $X(z)$?

Problem 1 (30 pts, 6 pts each). The initial-value theorem states that, for a signal $x(t)$ with Laplace transform $X(s)$ and for which $x(t) = 0$ for $t < 0$, the initial value of $x(t)$ [i.e.,$x(0^+)$] can be obtained from $X(s)$ through the relation $x(0^+) = \lim_{s\to\infty} sX(s)$. First, we note that, since $x(t) = 0$ for $t < 0$, $x(t) = x(t)u(t)$. Next, expanding $x(t)$ as a Taylor series at $t = 0^+$, we obtain

$$x(t) = \left[x(0^+) + x^{(1)}(0^+)t + \cdots + x^{(n)}(0^+)\frac{t^n}{n!} + \cdots\right] u(t),$$

where $x^{(n)}(0^+)$ denotes the n^{th} derivative of $x(t)$ evaluated at $t = 0^+$.

(a) Determine The Laplace transform of an arbitrary term $x^{(n)}(0^+)(t^n/n!)u(t)$ in the Taylor series. (You may find it helpful to review Example 9.14.)

$$\mathcal{L}\left\{x^{(n)}(0^+)\frac{t^n}{n!}u(t)\right\} = \int_0^{\infty} x^{(n)}(0^+)\frac{t^n}{n!}e^{-st}\,dt = x^{(n)}(0^+)\int_0^{\infty}\frac{t^n}{n!}e^{-st}\,dt = x^{(n)}(0^+)\frac{1}{s^{n+1}}$$

(b) From your result in part (a) and the expansion in the Taylor series, show that $X(s)$ can be expressed as

$$X(s) = \sum_{n=0}^{\infty} x^{(n)}(0^+)\frac{1}{s^{n+1}}.$$

$$\mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt = \int_{-\infty}^{\infty}\left(\sum_{n=0}^{\infty} x^{(n)}(0^+)\frac{t^n}{n!}u(t)\right)e^{-st}\,dt$$

$$= \sum_{n=0}^{\infty}\left(\int_{-\infty}^{\infty} x^{(n)}(0^+)\frac{t^n}{n!}u(t)e^{-st}\,dt\right) = \sum_{n=0}^{\infty} x^{(n)}(0^+)\frac{1}{s^{n+1}} = X(s)$$

(c) Demonstrate that $x(0^+) = \lim_{s\to\infty} sX(s)$ follows from the result of part (b).

$$\lim_{s\to\infty} sX(s) = \lim_{s\to\infty}\sum_{n=0}^{\infty} x^{(n)}(0^+)\frac{1}{s^n} = \lim_{s\to\infty} x^{(0)}(0^+)\frac{1}{s^0} = x(0^+)$$

(d) By first determining $x(t)$, verify the initial-value theorem for $X(s) = \frac{s+1}{(s+2)(s+3)}$.

$$\frac{s+1}{(s+2)(s+3)} = \frac{-1}{s+2} + \frac{2}{s+3}, \quad x(t) = (-e^{-2t} + 2e^{-3t})u(t), \quad x(0^+) = 1$$

$$\lim_{s\to\infty}\frac{s(s+1)}{(s+2)(s+3)} = \lim_{s\to\infty}\frac{s^2+s}{s^2+5s+6} = 1 = x(0^+)$$

(e) A more general form of the initial-value theorem states that if $x^{(n)}(0^+) = 0$ for $n < N$, then $x^{(N)}(0^+) = \lim_{s\to\infty} s^{N+1}X(s)$. Demonstrate that this more general statement also follows from the result in part (b).

$$\sum_{n=N}^{\infty}\int_{-\infty}^{\infty} x^{(n)}(0^+)\frac{t^n}{n!}u(t)e^{-st}\,dt = \sum_{n=N}^{\infty} x^{(n)}(0^+)\frac{1}{s^{n+1}} = \mathcal{L}\{x(t)\} = X(s)$$

$$\lim_{s\to\infty} s^{N+1}X(s) = \lim_{s\to\infty}\sum_{n=N}^{\infty} x^{(n)}(0^+)s^{N-n} = x^{(N)}(0^+)$$

Problem 2 (36 pts, 9 pts each). Let

$$x_1(t) = e^{-2t}u(t) \quad \text{and} \quad x_2(t) = e^{-3(t+1)}u(t+1).$$

(a) Determine the unilateral Laplace transform $\mathcal{X}_1(s)$ and the bilateral Laplace transform $X_1(s)$ for the signal $x_1(t)$.

$$\mathcal{X}_1(s) = \mathcal{U}\{x_1(t)\} = \int_{0^-}^{\infty} e^{-2t}u(t)e^{-st}\,dt = \frac{1}{s+2}, \ \mathrm{Re}\{s\} > -2$$

$$X_1(s) = \mathcal{L}\{x_1(t)\} = \int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st}\,dt = \frac{1}{s+2}, \ \mathrm{Re}\{s\} > -2$$

(b) Determine the unilateral Laplace transform $\mathcal{X}_2(s)$ and the bilateral Laplace transform $X_2(s)$ for the signal $x_2(t)$.

$$\mathcal{X}_2(s) = \mathcal{U}\{x_2(t)\} = \int_{0^-}^{\infty} e^{-3(t+1)}u(t+1)\,e^{-st}\,dt = e^{-3}\int_{0^-}^{\infty} e^{-(s+3)t}\,dt$$

$$= -\frac{e^{-3}}{s+3}\left(e^{-(s+3)t}\right)_{0^-}^{\infty} = \frac{e^{-3}}{s+3}, \ \mathrm{Re}\{s\} > -3$$

$$X_2(s) = \mathcal{L}\{x_2(t)\} = \int_{-\infty}^{\infty} e^{-3(t+1)}u(t+1)e^{-st}\,dt = \int_{-1}^{\infty} e^{-3t-3}\cdot e^{-st}\,dt$$

$$= -\frac{e^{-3}}{s+3}\left(e^{-(s+3)t}\right)_{-1}^{\infty} = \frac{e^{-3}}{s+3}e^{s+3} = \frac{e^{s}}{s+3}, \ \mathrm{Re}\{s\} > -3$$

(c) Take the inverse bilateral Laplace transform of the product $X_1(s)X_2(s)$ to determine the signal $g(t) = x_1(t) * x_2(t)$.

$$X_1(s)X_2(s) = e^{s}\frac{1}{(s+2)(s+3)} = e^{s}\left(\frac{1}{s+2} - \frac{1}{s+3}\right)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+3}\right\} = \left(e^{-2t} - e^{-3t}\right)u(t) \Rightarrow g(t) = \left(e^{-2(t+1)} - e^{-3(t+1)}\right)u(t+1)$$

(d) Show that the inverse unilateral Laplace transform of the product $\mathcal{X}_1(x)\mathcal{X}_2(x)$ is not the same as $g(t)$ for $t > 0^-$.

$$\mathcal{X}_1(s)\mathcal{X}_2(s) = \frac{e^{-3}}{(s+2)(s+3)} = e^{-3}\left(\frac{1}{s+2} - \frac{1}{s+3}\right), \ \mathrm{Re}\{s\} > -2$$

$$\mathcal{U}^{-1}\{\mathcal{X}_1(s)\mathcal{X}_2(s)\} = e^{-3}\left(e^{-2t} - e^{-3t}\right)u(t) \neq g(t)$$

Problem 3 (12 pts, 6 pts each). Determine the z-transform for each of the following sequences and write down the ROCs. Indicate whether the Fourier transform of the sequence exists.

(a) $x[n] = (\frac{1}{2})^{n+1}u[n]$

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\right)^{n+1}u[n]\,z^{-n} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2z}\right)^{n}$$

The ROC is $|z| > \frac{1}{2}$. The Fourier transform exists.

(b) $x[n] = (-\frac{1}{3})^{n}u[-n-2]$

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty}\left(-\frac{1}{3}\right)^{n}u[-n-2]\,z^{-n} = \sum_{n=-\infty}^{-2}\left(\frac{1}{-3z}\right)^{n} = \sum_{n=2}^{\infty}(-3z)^{n}$$

The ROC is $0 \le |z| < \frac{1}{3}$. The Fourier transform doesn't exist.

Problem 4 (10 pts, 5 pts each). Let $x[n] = (-1)^n u[n] + \alpha^n u[-n-n_0]$. Determine the constraints on the complex number α and the integer n_0, given that the ROC of $X(z)$ is $1 < |z| < 2$.

$$\mathcal{Z}\{x[n]\} = \sum_{n=0}^{\infty}(-1)^n z^{-n} + \sum_{n=-\infty}^{-n_0} a^n z^{-n} = \sum_{i=0}^{\infty}(-z)^{-n} + \sum_{n=n_0}^{\infty}\left(\frac{z}{a}\right)^{n}$$

$\because$ ROC is $1 < |z| < 2$ $\therefore |a| = 2$, $n_0 > -\infty$

Problem 5 (12 pts). Suppose that the algebraic expression for the z-transform of $x[n]$ is

$$X(z) = \frac{1 - \frac{1}{4}z^{-2}}{\left(1 + \frac{1}{4}z^{-2}\right)\left(1 + \frac{5}{4}z^{-1} + \frac{3}{8}z^{-2}\right)}.$$

How many different regions of convergence could correspond to $X(z)$?

$$X(z) = \frac{1-\frac{1}{4}z^{-2}}{(1+\frac{1}{4}z^{-2})(1+\frac{5}{4}z^{-1}+\frac{3}{8}z^{-2})} = \frac{\cancel{(1+\frac{1}{2}z^{-1})}(1-\frac{1}{2}z^{-1})}{\frac{1}{8}(1+\frac{1}{4}z^{-2})(3z^{-1}+4)\cancel{(z^{-1}+2)}\,2}$$

$$= \frac{1-\frac{1}{2}z^{-1}}{(1+\frac{1}{4}z^{-2})(1+\frac{3}{4}z^{-1})} = \frac{1-\frac{1}{2}z^{-1}}{(\frac{1}{2}z^{-1}+i)(\frac{1}{2}z^{-1}-i)(1+\frac{3}{4}z^{-1})}$$

pole : $z = \frac{1}{-2i}$, $z = \frac{1}{2i}$, $z = -\frac{3}{4}$

$\left|\frac{1}{-2i}\right| = \left|\frac{1}{2i}\right| \neq \left|-\frac{3}{4}\right| \Rightarrow 3$ different regions