

Signals and Systems

Homework 10 — Due : May 17 2024

Problem 1 (20 pts). Determine the Fourier transform of the following signals:

(a) $x_1[n] = \sin\left(\frac{\pi}{4}n\right) + \cos\left(\frac{2\pi}{3}n\right)$

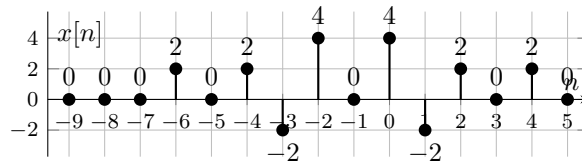
(b) $x_3[n] = 2^n \sin\left(\frac{\pi}{4}n\right) u[-n]$

Problem 2 (20 pts). Determine the inverse Fourier transform of the following signals:

(a) $X_1(e^{j\omega}) = \cos^2(3\omega) + \sin^2(\omega)$

(b) $X_2(e^{j\omega}) = A(\omega)e^{jB(\omega)}$, where $A(\omega) = \begin{cases} 0, & 0 \leq |\omega| < \pi/3 \\ 1, & \pi/3 \leq |\omega| < \pi \end{cases}$, and $B(\omega) = -\frac{2}{3}\omega$

Problem 3 (30 pts). Let $X(e^{j\omega})$ be the Fourier transform of the signal $x[n]$.



Perform the following calculations *without* explicitly evaluating $X(e^{j\omega})$.

(a) Find $X(e^{j0})$.

(b) Find $\Re\{X(e^{j\omega})\}$.

(c) Find $X(e^{j\pi})$.

(d) Evaluate $\int_{-\pi}^{\pi} X(e^{j\omega}) d\omega$.

(e) Evaluate $\int_{-\pi}^{\pi} \left| \frac{d}{d\omega} X(e^{j\omega}) \right|^2 d\omega$.

(f) Sketch $y[n]$ such that $\Im\{y[n]\} = \Re\{X(e^{j\omega})\}$.

Problem 4 (20 pts). Let $X(e^{j\omega})$ be the Fourier transform of a real signal $x[n]$. Show that $x[n]$ can be written as

$$x[n] = \int_0^{\pi} [B(\omega) \sin(\omega n) + C(\omega) \cos(\omega n)] d\omega$$

by finding expressions for $B(\omega)$ and $C(\omega)$ in terms of $X(e^{j\omega})$.

Problem 5 (10 pts). Consider a signal $y[n]$ with Fourier transform $Y(e^{j\omega})$. Suppose that $y[n] = x_{(4)}[n]$, where the signal $x[n]$ has a Fourier transform $X(e^{j\omega})$. Determine a real number α such that $0 < \alpha < 2\pi$ and $Y(e^{j\omega}) = X(e^{j(\omega-\alpha)})$.

Problem 1 (20 pts). Determine the Fourier transform of the following signals:

(a) $x_1[n] = \sin\left(\frac{\pi}{4}n\right) + \cos\left(\frac{2\pi}{3}n\right)$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \delta(\omega - \omega_0) e^{j\omega n} d\omega = e^{j\omega_0 n}$$

$$\sin\left(\frac{\pi}{4}n\right) = \frac{1}{2j} e^{j\frac{\pi}{4}n} - \frac{1}{2j} e^{-j\frac{\pi}{4}n}$$

$$\mathcal{F}\left\{\sin\left(\frac{\pi}{4}n\right)\right\} = \frac{\pi}{j} \delta\left(\omega - \frac{\pi}{4}\right) - \frac{\pi}{j} \delta\left(\omega + \frac{\pi}{4}\right)$$

$$\cos\left(\frac{2\pi}{3}n\right) = \frac{1}{2} e^{j\frac{2\pi}{3}n} + \frac{1}{2} e^{-j\frac{2\pi}{3}n}$$

$$\mathcal{F}\left\{\cos\left(\frac{2\pi}{3}n\right)\right\} = \pi \cdot \delta\left(\omega - \frac{2\pi}{3}\right) + \pi \delta\left(\omega + \frac{2\pi}{3}\right)$$

$$\mathcal{F}\{x_1[n]\} = \frac{\pi}{j} \delta\left(\omega - \frac{\pi}{4}\right) - \frac{\pi}{j} \delta\left(\omega + \frac{\pi}{4}\right) + \pi \delta\left(\omega - \frac{2\pi}{3}\right) + \pi \delta\left(\omega + \frac{2\pi}{3}\right)$$

(b) $x_3[n] = 2^n \sin\left(\frac{\pi}{4}n\right) u[-n]$

$$x_3[n] = 2^n \cdot \frac{1}{2j} \cdot \left(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right) u[-n]$$

$$\frac{1}{2j} \sum_{n=-\infty}^0 2^n \cdot \left(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right) e^{-j\omega n}$$

$$= \frac{1}{2j} \sum_{n=-\infty}^0 \left([2 \cdot e^{j(\omega - \frac{\pi}{4})}]^n - [2 \cdot e^{-j(\omega + \frac{\pi}{4})}]^n \right)$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} \left(\frac{e^{j(\omega - \frac{\pi}{4})}}{2} \right)^n - \frac{1}{2j} \sum_{n=0}^{\infty} \left(\frac{e^{j(\omega + \frac{\pi}{4})}}{2} \right)^n$$

$$= \frac{1}{2j} \lim_{n \rightarrow \infty} \frac{1 - [2^{-1} e^{j(\omega - \frac{\pi}{4})}]^n}{1 - 2^{-1} e^{j(\omega - \frac{\pi}{4})}} - \frac{1}{2j} \sum_{n=0}^{\infty} \frac{1 - [2^{-1} e^{j(\omega + \frac{\pi}{4})}]^n}{1 - 2^{-1} e^{j(\omega + \frac{\pi}{4})}}$$

$$= \frac{1}{2j} \cdot \frac{2}{2 - e^{j(\omega - \frac{\pi}{4})}} - \frac{1}{2j} \cdot \frac{2}{2 - e^{j(\omega + \frac{\pi}{4})}}$$

$$= \frac{1}{j} \left(\frac{1}{2 - e^{j(\omega - \frac{\pi}{4})}} - \frac{1}{2 - e^{j(\omega + \frac{\pi}{4})}} \right)$$

Problem 2 (20 pts). Determine the inverse Fourier transform of the following signals:

$$(a) \quad X_1(e^{j\omega}) = \cos^2(3\omega) + \sin^2(\omega) \quad \sum_{n=-\infty}^{\infty} \delta[n-n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

$$\cos^2(3\omega) = \frac{1}{4} e^{j6\omega} + \frac{1}{4} e^{-j6\omega} + \frac{1}{2}$$

$$\mathcal{F}^{-1}\{\cos^2(3\omega)\} = \frac{1}{4} \delta[n+6] + \frac{1}{4} \delta[n-6] + \frac{1}{2} \delta[n]$$

$$\sin^2(\omega) = \left(-\frac{1}{4}\right) e^{j2\omega} + \left(-\frac{1}{4}\right) e^{-j2\omega} + \frac{1}{2}$$

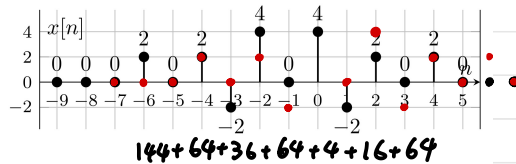
$$\mathcal{F}^{-1}\{\sin^2(\omega)\} = \left(-\frac{1}{4}\right) \delta[n+2] + \left(-\frac{1}{4}\right) \delta[n-2] + \frac{1}{2} \delta[n]$$

$$\mathcal{F}^{-1}\{X_1(e^{j\omega})\} = \delta[n] - \frac{1}{4} \delta[n+2] - \frac{1}{4} \delta[n-2] + \frac{1}{4} \delta[n+6] + \frac{1}{4} \delta[n-6]$$

$$(b) \quad X_2(e^{j\omega}) = A(\omega) e^{jB(\omega)}, \text{ where } A(\omega) = \begin{cases} 0, & 0 \leq |\omega| < \pi/3 \\ 1, & \pi/3 \leq |\omega| < \pi \end{cases}, \text{ and } B(\omega) = -\frac{2}{3}\omega$$

$$\begin{aligned} \mathcal{F}^{-1}\{X_2(e^{j\omega})\} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} A(\omega) e^{-j\frac{2}{3}\omega} \cdot e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \left(\int_{-\pi}^{-\frac{\pi}{3}} e^{-j\frac{2}{3}\omega} \cdot e^{j\omega n} d\omega + \int_{\frac{\pi}{3}}^{\pi} e^{-j\frac{2}{3}\omega} \cdot e^{j\omega n} d\omega \right) \\ &= \frac{1}{2\pi} \left(\left. \frac{e^{j(n-\frac{2}{3})\omega}}{j(n-\frac{2}{3})} \right|_{-\pi}^{-\frac{\pi}{3}} + \left. \frac{e^{j(n-\frac{2}{3})\omega}}{j(n-\frac{2}{3})} \right|_{\frac{\pi}{3}}^{\pi} \right) \\ &= \frac{1}{\pi(n-\frac{2}{3})} \cdot [\sin((n-\frac{2}{3})\pi) - \sin((n-\frac{2}{3})\frac{\pi}{3})] \\ &= \text{sinc}\left(n\pi - \frac{2\pi}{3}\right) - 3 \text{sinc}\left(\frac{\pi}{3}n - \frac{2\pi}{9}\right) \end{aligned}$$

Problem 3 (30 pts). Let $X(e^{j\omega})$ be the Fourier transform of the signal $x[n]$.



Perform the following calculations *without* explicitly evaluating $X(e^{j\omega})$.

(a) Find $X(e^{j0})$.

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n] = 12$$

(b) Find $\angle X(e^{j\omega})$. *let $\angle X(e^{j\omega}) = \angle \omega_x$*

$x[n+1]$ is real and even. $F\{x[n+1]\} = e^{j\omega} X(e^{j\omega})$ is also real and even.

$$\angle e^{j\omega} X(e^{j\omega}) = \angle(\omega + \omega_x) = \angle 0, \omega_x = -\omega \Rightarrow X(e^{j\omega}) = e^{-j\omega}$$

(c) Find $X(e^{j\pi})$.

$$\begin{aligned} X(e^{j\pi}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\pi n} = \sum_{n=-\infty}^{\infty} x[n] \{\cos(\pi n) - j \sin(\pi n)\} \\ &= (2 + 2 + 4 + 2) \times 2 = 20 \end{aligned}$$

(d) Evaluate $\int_{-\pi}^{\pi} X(e^{j\omega}) d\omega$.

$$\begin{aligned} \int_{-\pi}^{\pi} X(e^{j\omega}) d\omega &= 2\pi \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega} d\omega \\ &= 2\pi \cdot X[0] = 8\pi \end{aligned}$$

(e) Evaluate $\int_{-\pi}^{\pi} \left| \frac{d}{d\omega} X(e^{j\omega}) \right|^2 d\omega$.

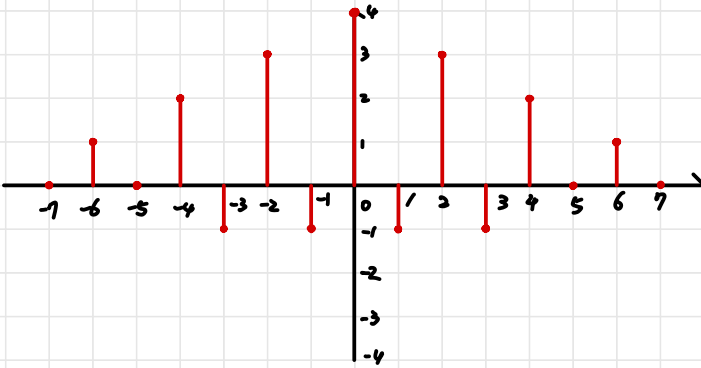
$$\frac{d}{d\omega} X(e^{j\omega}) = \frac{d}{d\omega} \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} (-jn) x[n] e^{-j\omega n}$$

$$F^{-1}\left\{\frac{d}{d\omega} X(e^{j\omega})\right\} = -jn x[n]$$

$$\int_{-\pi}^{\pi} \left| \frac{d}{d\omega} X(e^{j\omega}) \right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |-jn x[n]|^2 = 2\pi \sum_{n=-\infty}^{\infty} n^2 x^2[n] = 2\pi \cdot 392 = 784\pi$$

(f) Sketch $y[n]$ such that $\mathfrak{F}\{y[n]\} = \text{Re}\{X(e^{j\omega})\}$.

$$y[n] = \frac{1}{2} (x[n] + x[-n])$$



Problem 4 (20 pts). Let $X(e^{j\omega})$ be the Fourier transform of a real signal $x[n]$. Show that $x[n]$ can be written as

$$x[n] = \int_0^\pi [B(\omega) \sin(\omega n) + C(\omega) \cos(\omega n)] d\omega$$

by finding expressions for $B(\omega)$ and $C(\omega)$ in terms of $X(e^{j\omega})$.

$$X[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) [\cos(\omega n) + j \sin(\omega n)] d\omega$$

$$X^*(e^{-j\omega}) = X(e^{j\omega})$$

$$= \int_{-\pi}^{\pi} \left[\frac{X(e^{j\omega})}{2\pi} \cos(\omega n) + \frac{j}{2\pi} X(e^{j\omega}) \sin(\omega n) \right] d\omega$$

$$= \int_0^\pi \left[\frac{X(e^{j\omega})}{2\pi} \cos(\omega n) + \frac{j X(e^{j\omega})}{2\pi} \sin(\omega n) \right] d\omega + \int_0^\pi \left[\frac{X(e^{-j\omega})}{2\pi} \cos(-\omega n) + \frac{j X(e^{-j\omega})}{2\pi} \sin(-\omega n) \right] d\omega$$

$$= \int_0^\pi \frac{1}{2\pi} [X(e^{j\omega}) + X(e^{-j\omega})] \cos(\omega n) d\omega + \int_0^\pi \frac{j}{2\pi} [X(e^{j\omega}) - X(e^{-j\omega})] \sin(\omega n) d\omega$$

$$= \int_0^\pi \frac{1}{\pi} \text{Re}\{X(e^{j\omega})\} \cos(\omega n) d\omega + \int_0^\pi \frac{-j}{\pi} \text{Im}\{X(e^{j\omega})\} \sin(\omega n) d\omega$$

$$B(\omega) = \frac{-j}{\pi} \text{Im}\{X(e^{j\omega})\}$$

$$C(\omega) = \frac{1}{\pi} \text{Re}\{X(e^{j\omega})\}$$

Problem 5 (10 pts). Consider a signal $y[n]$ with Fourier transform $Y(e^{j\omega})$. Suppose that $y[n] = x_{(4)}[n]$, where the signal $x[n]$ has a Fourier transform $X(e^{j\omega})$. Determine a real number α such that $0 < \alpha < 2\pi$ and $Y(e^{j\omega}) = Y(e^{j(\omega-\alpha)})$.

α is a period of $Y(e^{j\omega})$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{(4)}[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n] e^{j\omega 4n} = X(e^{j4\omega})$$

$$\frac{2\pi}{4} = \frac{\pi}{2} \quad , \quad \alpha = \frac{\pi}{2}$$