Math 6a - Problem Set 3

1. A Rubik's cube is a $3 \times 3 \times 3$ cube of unit cubes, so that in its original position all the cubes on the surface of a particular face have a colour associated to that face. In place of the central cube is a mechanism which allows you to rotate each face of the cube mixing up the colours. The Rubik's cube can be viewed as a puzzle where the goal is to go from any position to the original position. Describe an algorithm which works out how to go from a particular position to the original position in the smallest number of moves.

2. Suppose that G is a connected $2k$-regular graph for some $k \geq 1$.

   (a) Show that if G has an even number of edges, then there is a k-regular subgraph H of G with the same vertex set.

   (b) What happens if G has an odd number of edges?

3. Consider the directed multigraph with loops whose vertex set consists of all 2^n possible 0/1-sequences of length n, where there is an edge from u to v if the last $n-1$ entries in u are the same as the first $n-1$ entries in v. Show that this graph admits an Eulerian cycle and use this to show that there is a 0/1-sequence of length 2^{n+1} containing every possible 0/1-sequence of length $n+1$ as a subsequence (provided we are allowed to spool around when we reach the end of the sequence). Give a concrete example with this property when $n = 4$.

4. Write $\chi(G)$ for the smallest k such that there is a k-colouring of the graph G.

   (a) Show that the number of edges $e(G)$ in a graph G is at least $\binom{\chi(G)}{2}$.

   (b) Show that if G has no isolated vertices and is not a complete graph, then $e(G) > \binom{\chi(G)}{2}$.

5. The greedy algorithm for colouring a graph G starts with an ordering of the vertices of G, say $v_1, v_2, \ldots, v_n$, and an ordered list of colours, say $1, 2, \ldots,$ and colours $v_1, v_2, \ldots, v_n$ one at a time in order, colouring v_i with the smallest colour which has not already appeared on any neighbour v_j of v_i with $j < i$.

   (a) Show that there is an ordering of the vertices of any graph G such that the greedy algorithm colours the vertices with $\chi(G)$ colours.

   (b) Show that there is a bipartite graph G and an ordering of its vertices such that the greedy algorithm uses 2020 colours.