Math 6a - Problem Set 4

1. Show that any graph has a bipartite subgraph containing at least half of its edges.

2. A graph is planar if it may drawn on a page so that edges only intersect at their endpoints.

   (a) Write v for the number of vertices in G, e for the number of edges and f for the number of faces (including the outer, infinite face). Prove, by induction or otherwise, that if G is a connected planar graph, then
   $$v - e + f = 2.$$

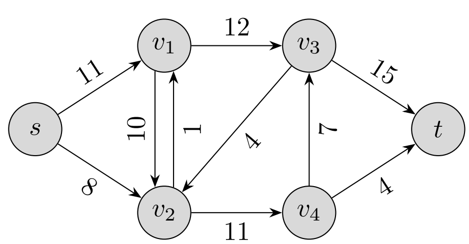
   (b) Use this to show that $K_{3,3}$, the complete bipartite graph with 3 vertices on either side, is not planar.

3. A travelling salesperson is required to follow a route that visits every city exactly once and returns to the starting point. How can she find the best route? In the language of graph theory, we have a complete graph on n vertices, a positive weight $w(xy)$ 'the distance' assigned to each pair of vertices xy and we are required to find a minimum weight 'Hamilton cycle', a cycle that uses all n vertices. We say that an algorithm is an r-approximation algorithm if it always finds some Hamilton cycle with weight that is at most r times as large as the minimum possible.

   Show that Prim's algorithm for finding minimum spanning trees[1] gives a 2-approximation for the travelling salesperson problem (TSP) under the additional assumption that the distances satisfy the triangle inequality $w(xz) \leq w(xy) + w(yz)$ for all x, y and z.

4. Show that if G is an r-regular bipartite graph with $2n$ vertices, then G contains a matching of size n.

5. Apply the Ford–Fulkerson algorithm to find a maximum flow in the following network:



[1] In this context, the algorithm works as follows: Start with any vertex v_1. Suppose that at step t we have a sequence of distinct vertices $s_t = (v_1, \ldots, v_t)$. Pick $1 \leq i \leq t$ and $u \notin \{v_1, \ldots, v_t\}$ that minimises $w(v_i u)$. Define s_{t+1} by inserting u into s_t after v_i. Output the Hamilton cycle defined by s_n.