# Math 6a - Problem Set 2

1. The number 1288119601 is composite. Find a Miller–Rabin witness for this fact.

2. (a) Show that $n! \leq n^n$ for all natural numbers $n$.

   (b) Show that for any natural number $k$,

   $$\left(1+\frac{1}{k}\right)^k < \left(1+\frac{1}{k+1}\right)^{k+1}.$$

   (c) Letting $e = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k$ (you may assume that this exists), show that

   $$n! > \frac{n^n}{e^n}$$

   for all natural numbers $n$.

3. (a) A lottery consists of 47 balls from which 6 are chosen at random. You win the grand prize if and only if you can pick these six numbers in advance. What is your chance of winning?

   (b) What are your chances if you also have to stipulate the order in which the 6 balls come out?

   (c) You hear through the grapevine that the lottery has been fixed and there are always three numbers taken from $1, 2, \ldots, 24$ and three numbers from $25, 26, \ldots, 47$. Now what are your chances?

   (d) Suppose now that the lottery has $n$ numbers, where $n$ is even, and 6 are again chosen. How much does knowing that there are always three numbers from $1, 2, \ldots, n/2$ and three from $n/2+1, \ldots, n$ improve your chances?

4. A graph has 6 vertices and 12 edges. Each vertex has degree either 2 or 5. How many vertices have degree 2 and how many have degree 5?

5. (a) Show that every graph with at least two vertices contains two vertices with the same degree.

   (b) Show that in a connected graph any two paths of maximum length intersect.

   (c) Show that if a tree is not a path, it must have at least three leaves.