Math 6a - Problem Set 5

1. Alice, Bob and Charlie play a game with Bob and Charlie on one team and Alice on the other. There are 100 boxes, labelled 1 to 100, and 100 notes, again labelled 1 to 100. The game proceeds as follows:

   – With Bob watching, Alice places one note in each box as she pleases.

   – Bob is then allowed to pick two boxes and switch their notes. He may only do this once.

   – Alice sees Bob's move and then picks a number N between 1 and 100.

   – Bob now leaves and Charlie enters without speaking. Alice tells Charlie the number N.

   – Finally, Charlie may open 50 boxes to try and find the box with number N.

   If Charlie picks 50 boxes at random, they win with a 50% chance. Show that there is a strategy by which they can always win, no matter how Alice plays.

2. Let G be a group of order m whose elements are permutations of $\{1, 2, \ldots, n\}$ with the operation being composition. If not all permutations in G are even, show that exactly half the permutations are even and half are odd.

3. Suppose that the vertices of a regular pentagon are labelled clockwise by $1, 2, 3, 4, 5$. You are now allowed to move the labels by rotation or reflection. For example, starting with the initial configuration, reflecting through the line of symmetry that passes through the vertex 1 changes the labels to $1, 5, 4, 3, 2$, again in clockwise order. Show that no sequence of rotations and reflections can produce the sequence $2, 1, 3, 4, 5$ in clockwise order.

4. A group G is generated by a subset S if every element of G can be written as a product of elements from S, where we allow elements to be repeated.

   (a) Show that $(1\ 2)$ and $(1\ 2\ 3\ \cdots\ n)$ generate S_n.

   (b) Show that any set T of $\binom{n-1}{2} + 1$ distinct transpositions from S_n with $n \geq 3$ generates S_n.

5. Suppose that we want to colour the vertices of a cube with 4 colours and two colourings are considered the same if there is a rotation (but not a reflection) taking one to the other. How many distinct colourings are there?