Iteration Complexity of First-order Methods for Linear Constrained Composite Nonconvex Problems

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Paper

- Wei Liu, Qihang Lin, Yangyang Xu, *First-order Methods for Affinely Constrained Composite Non-convex Non-smooth Problems: Lower Complexity Bound and Near-optimal Methods.* Arxiv

- Wei Liu, Qihang Lin, Yangyang Xu, *Lower Complexity Bounds of First-order Methods for Affinely Constrained Composite Non-convex Problems.* Submitted to Mathematics of Operations Research (second round)

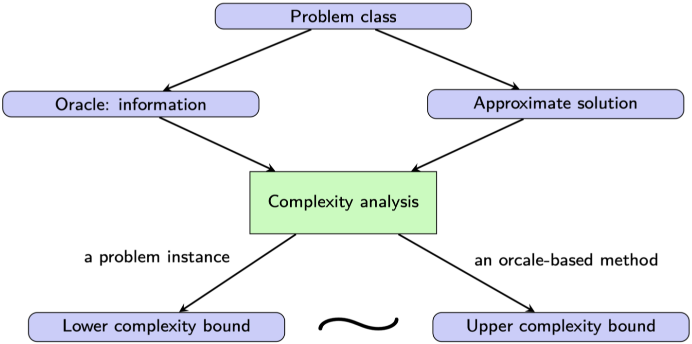
Problem class

$$\min_{\mathbf{x}\in\mathbb{R}^d} F_0(\mathbf{x}) := f_0(\mathbf{x}) + g(\mathbf{x}), \quad \text{s.t. } \mathbf{A}\mathbf{x} + \mathbf{b} = \mathbf{0} \tag{P}$$

- ∇f_0 is L-Lipschitz continuous
- $\inf_{\mathbf{x}} F_0(\mathbf{x}) > -\infty$
- $g(\mathbf{x}) = \bar{g}(\bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{b}})$, where $\bar{\mathbf{A}} \in \mathbb{R}^{\bar{n}\times d}$, $\bar{\mathbf{b}} \in \mathbb{R}^{\bar{n}}$, and $\bar{g} : \mathbb{R}^{\bar{n}} \to \mathbb{R} \cup \{+\infty\}$ is a proper closed convex function but potentially non-smooth
- relint(dom$(\bar{g})$) $\neq \emptyset$

Why lower complexity bounds

- provide understanding of the fundamental limit of a class of methods and the difficulty of a class of problems
- tell if existing methods could be improved
- guide to design "optimal" methods
- Opposite: Worst-case complexity/Upper complexity bounds


Problem class
Oracle: information
Approximate solution
Complexity analysis
a problem instance
an orcale-based method
Lower complexity bound
∼
Upper complexity bound

Two parts

Lower Complexity Bounds

Upper Complexity Bounds

A few existing lower bounds and motivating questions

Lower/upper bounds for unconstrained convex problems

Consider problem

$$f^* = \min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$$

where f is convex and L-smooth, i.e., $\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\| \le L\|\mathbf{x} - \mathbf{y}\|, \forall\, \mathbf{x}, \mathbf{y}$

- lower bound [Nesterov'04]: given $\varepsilon > 0$, there is a convex L-smooth function f such that $\Omega\left(\frac{\sqrt{L}\|\mathbf{x}^0 - \mathbf{x}^*\|}{\sqrt{\varepsilon}}\right)$ gradients needed to produce an ε-optimal solution $\bar{\mathbf{x}}$, i.e., $f(\bar{\mathbf{x}}) - f^* \le \varepsilon$, if

$$\mathbf{x}^k \in \mathbf{x}^0 + \mathsf{span}\{\nabla f(\mathbf{x}^0), \nabla f(\mathbf{x}^1), \ldots, \nabla f(\mathbf{x}^{k-1})\}$$

- upper bound: $O\left(\frac{\sqrt{L}\|\mathbf{x}^0 - \mathbf{x}^*\|}{\sqrt{\varepsilon}}\right)$ gradients enough

- [Nesterov'13]: the same upper bound holds for $\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}) + g(\mathbf{x})$ with oracle ∇f and $\mathbf{prox}_{\eta g}$

Observation: a nonsmooth proximable term g does not make a harder problem

Lower/upper bounds for unconstrained nonconvex problems

Consider problem

$$f^* = \min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})$$

where f is L-smooth but may be nonconvex, and f^* is finite

- lower bound [Carmon et. al'20]: given $\varepsilon > 0$, there is an L-smooth function f such that $\Omega(\frac{L(f(\mathbf{x}^0)-f^*)}{\varepsilon^2})$ gradients needed to produce an ε-stationary point $\bar{\mathbf{x}}$, i.e., $\|\nabla f(\bar{\mathbf{x}})\| \le \varepsilon$
- upper bound: $O(\frac{L(f(\mathbf{x}^0)-f^*)}{\varepsilon^2})$

- the same upper bound holds for $\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) + g(\mathbf{x})$ with oracle ∇f and $\mathbf{prox}_{\eta g}$ to produce $\bar{\mathbf{x}}$ such that $\mathrm{dist}(\mathbf{0}, \nabla f(\bar{\mathbf{x}}) + \partial g(\bar{\mathbf{x}})) \le \varepsilon$

Observation: a nonsmooth proximable term g does not make a harder problem

Lower/upper bounds for linear-constrained nonconvex problems

Consider problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}), \text{ s.t. } \mathbf{A}\mathbf{x}+\mathbf{b}=\mathbf{0}$$

where f is L-smooth but may be nonconvex

- lower bound [Sun-Hong'19]: given $\varepsilon>0$, there is an L-smooth function f such that $\Omega\left(\frac{L\kappa(\mathbf{A})(f(\mathbf{x}^0)-\inf_{\mathbf{x}} f(\mathbf{x}))}{\varepsilon^2}\right)$ oracle calls to $\nabla f(\cdot)$, $\mathbf{A}(\cdot)$, and $\mathbf{A}^\top(\cdot)$ to produce an ε-KKT point
- upper bound: $\tilde{O}\left(\frac{L\kappa(\mathbf{A})(f(\mathbf{x}^0)-\inf_{\mathbf{x}} f(\mathbf{x}))}{\varepsilon^2}\right)$

Questions: for the regularized problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} F(\mathbf{x}) := f(\mathbf{x})+g(\mathbf{x}), \text{ s.t. } \mathbf{A}\mathbf{x}+\mathbf{b}=\mathbf{0}$$

where g is closed convex and $\mathbf{prox}_{\eta g}$ can be accessed

Q1. Will g make the problem harder?

Q2. What is a "tight" lower complexity bound?

Upper bounds for linear-constrained regularized nonconvex problems

- [Zhang-Luo'22]: when g is the indicator function of $\{\mathbf{x} : \bar{\mathbf{A}}\mathbf{x} \leq \bar{\mathbf{b}}\}$

$$O\left(\frac{\hat{\kappa}^2 L^3 (F(\mathbf{x}^0) - \inf_{\mathbf{x}} F(\mathbf{x}))}{\varepsilon^2}\right)$$

  where $\hat{\kappa}$ the condition number of $[\mathbf{A}; \bar{\mathbf{A}}]$

- [Lin-Ma-X.'22; Kong-Melo-Monteiro'23]: when Slater's condition holds, i.e., $\exists\, \mathbf{x} \in \mathrm{relint}(\mathrm{dom}(g)) : \mathbf{A}\mathbf{x} + \mathbf{b} = \mathbf{0}$, then $\tilde{O}(\varepsilon^{-2.5})$ oracles are enough

Observation: higher than the lower bound for linear-constrained nonregularized nonconvex problems; thus we guess g may make a harder problem

This talk: (nearly) close the gap between lower and upper bounds for a class of linear-constrained regularized nonconvex problems

Problem class

$$\min_{\mathbf{x}\in\mathbb{R}^d} F_0(\mathbf{x}) := f_0(\mathbf{x}) + g(\mathbf{x}), \quad \text{s.t. } \mathbf{A}\mathbf{x} + \mathbf{b} = \mathbf{0} \tag{P}$$

- ∇f_0 is L-Lipschitz continuous
- $\inf_{\mathbf{x}} F_0(\mathbf{x}) > -\infty$
- $g(\mathbf{x}) = \bar{g}(\bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{b}})$, where $\bar{\mathbf{A}} \in \mathbb{R}^{\bar{n}\times d}$, $\bar{\mathbf{b}} \in \mathbb{R}^{\bar{n}}$, and $\bar{g} : \mathbb{R}^{\bar{n}} \to \mathbb{R} \cup \{+\infty\}$ is a proper closed convex function but potentially non-smooth
- relint(dom$(\bar{g})$) $\neq \emptyset$

Algorithm Class I by a strong oracle

- **a strong oracle:** $\left(\nabla f_0(\cdot), \mathbf{A}(\cdot), \mathbf{A}^\top(\cdot)\right)$ and $\mathbf{prox}_{\eta g}(\cdot)$ for any $\eta > 0$
- iterates $\left\{\mathbf{x}^{(t)}\right\}_{t=0}^{\infty}$ satisfies: $\mathbf{x}^{(t)} \in \mathbf{span}\left(\left\{\boldsymbol{\xi}^{(t)}, \boldsymbol{\zeta}^{(t)}\right\}\right) \,\forall\, t \geq 1$, where

$$\boldsymbol{\xi}^{(t)} \in \mathbf{span}\left(\left\{\mathbf{A}^\top \mathbf{b}\right\} \bigcup \cup_{s=0}^{t-1} \left\{\mathbf{x}^{(s)}, \nabla f_0(\mathbf{x}^{(s)}), \mathbf{A}^\top \mathbf{A}\mathbf{x}^{(s)}\right\}\right)$$

$$\boldsymbol{\zeta}^{(t)} \in \left\{\mathbf{prox}_{\eta g}(\boldsymbol{\xi}^{(t)}) \mid \eta > 0\right\}$$

  - **Examples:** quadratic penalty or augmented Lagrangian based first-order methods [Kong-Melo-Monteiro'19 '20]

Lower complexity bound

Let $\epsilon > 0$ and $L > 0$ be given. Then for any $\omega \in [0, \frac{150\pi\epsilon}{L})$, there exists an instance of problem (P) such that at least

$$\left\lceil \frac{\kappa([\bar{\mathbf{A}};\mathbf{A}])L\Big(F_0(\mathbf{x}^{(0)}) - \inf_{\mathbf{x}} F_0(\mathbf{x})\Big)}{36000\pi^2}\epsilon^{-2} \right\rceil$$

oracles are needed by any algorithm in the considered class to obtain a point that is ω-close to an ϵ-KKT point of the instance

- It can hold $\kappa([\bar{\mathbf{A}};\mathbf{A}]) \gg \kappa(\mathbf{A})$, $[\bar{\mathbf{A}};\mathbf{A}] := \begin{bmatrix} \bar{\mathbf{A}} \\ \mathbf{A} \end{bmatrix}$
- Hence, existence of g makes a harder problem even with $\mathbf{prox}_{\eta g}$
  This answers Q1

Algorithm Class II by a more practical oracle

- $g(\mathbf{x}) = \bar{g}(\bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{b}})$
- **a practical oracle:** $\left(\nabla f_0(\cdot), \mathbf{A}(\cdot), \mathbf{A}^\top(\cdot), \bar{\mathbf{A}}(\cdot), \bar{\mathbf{A}}^\top(\cdot)\right)$, and $\mathbf{prox}_{\eta\bar{g}}(\cdot)$
- iterate sequence $\left\{(\mathbf{x}^{(t)}, \mathbf{y}^{(t)})\right\}_{t=0}^{\infty}$ satisfies: for all $t \geq 1$

$$\mathbf{x}^{(t)} \in \mathbf{span}\Big(\left\{\mathbf{A}^\top\mathbf{b}, \bar{\mathbf{A}}^\top\bar{\mathbf{b}}\right\} \bigcup \cup_{s=0}^{t-1}\left\{\mathbf{x}^{(s)}, \nabla f_0(\mathbf{x}^{(s)}), \mathbf{A}^\top\mathbf{A}\mathbf{x}^{(s)}, \bar{\mathbf{A}}^\top\bar{\mathbf{A}}\mathbf{x}^{(s)}, \bar{\mathbf{A}}^\top\mathbf{y}^{(s)}\right\}\Big)$$

$$\mathbf{y}^{(t)} \in \mathbf{span}\left(\left\{\boldsymbol{\xi}^{(t)}, \boldsymbol{\zeta}^{(t)}\right\}\right), \text{ where}$$

$$\boldsymbol{\xi}^{(t)} \in \mathbf{span}\left(\left\{\bar{\mathbf{b}}\right\} \bigcup \cup_{s=0}^{t-1}\left\{\mathbf{y}^{(s)}, \bar{\mathbf{A}}\bar{\mathbf{A}}^\top\mathbf{y}^{(s)}, \bar{\mathbf{A}}\mathbf{x}^{(s)}\right\}\right)$$

$$\boldsymbol{\zeta}^{(t)} \in \left\{\mathbf{prox}_{\eta\bar{g}}(\boldsymbol{\xi}^{(t)}) \mid \eta > 0\right\}$$

  - **Examples:** quadratic penalty or augmented Lagrangian based first-order methods (such as linearized ADMM) for

$$\min_{\mathbf{x},\mathbf{y}} F(\mathbf{x},\mathbf{y}) := f_0(\mathbf{x}) + \bar{g}(\mathbf{y}), \quad \text{s.t. } \mathbf{A}\mathbf{x} + \mathbf{b} = \mathbf{0}, \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{b}} = \mathbf{y}$$

  - **Examples:** existing FOMs on solving minimax-structured optimization [Thekumparampil et al.'19, Lin et al.'20, Xu et al.'23]

Lower complexity bound under the practical oracle

Let $\epsilon > 0$ and $L > 0$ be given. Then for any $\omega \in [0, \frac{150\pi\epsilon}{L})$, there exists an instance of problem (P) such that at least

$$\left\lceil \frac{\kappa([\bar{\mathbf{A}}; \mathbf{A}]) L \Big(F_0(\mathbf{x}^{(0)}) - \inf_{\mathbf{x}} F_0(\mathbf{x}) \Big)}{18000\pi^2} \epsilon^{-2} \right\rceil$$

oracles are needed by any algorithm in the considered class to obtain a point that is ω-close to an ϵ-KKT point of the instance

- The lower complexity bound almost the same as the previous one

Worst-case instance inspired by [Sun-Hong'19]

- objective: let $m = 3m_1m_2$ be even for two integers $m_1 \geq 2, m_2 \geq 1$. Set $\mathbf{x} = (\mathbf{x}_1; \ldots; \mathbf{x}_m)$ and

$$f_0(\mathbf{x}) := \sum_{i=1}^{m} f_i(\mathbf{x}_i), \text{ with } f_i(\mathbf{z}) := \frac{300\pi\epsilon^2}{mL} h_i\left(\frac{\sqrt{m}L\mathbf{z}}{150\pi\epsilon}\right)$$

  where $\{h_i\}$ are designed to satisfy certain properties

- linear constraint: let $\mathcal{M} := \{im_1\}_{i=1}^{3m_2-1}$ and $\mathcal{M}^C := \{1, 2, \ldots, m-1\} \backslash \mathcal{M}$

$$\bar{\mathbf{A}} := mL \cdot \mathbf{J}_{\mathcal{M}} \otimes \mathbf{I}_{\bar{d}}, \ \mathbf{A} := mL \cdot \mathbf{J}_{\mathcal{M}^C} \otimes \mathbf{I}_{\bar{d}}, \ \bar{\mathbf{b}} = \mathbf{0}, \ \mathbf{b} = \mathbf{0}$$

  where $\mathbf{J}_{\mathcal{M}}$ and $\mathbf{J}_{\mathcal{M}^C}$ are the rows of $\mathbf{J}_m$ indexed by $\mathcal{M}$ and $\mathcal{M}^C$ and

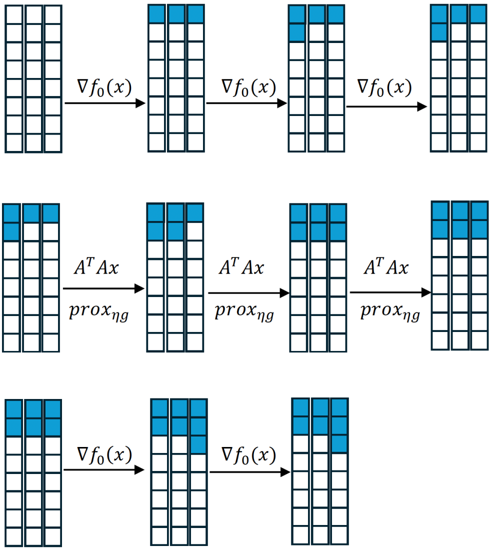
$$\mathbf{J}_m := \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix} \in \mathbb{R}^{(m-1)\times m}$$

- regularizer: $\bar{g}(\mathbf{y}) := \frac{\beta}{mL}\|\mathbf{y}\|_1$ and $g(\mathbf{x}) := \bar{g}(\bar{\mathbf{A}}\mathbf{x})$

Properties

- $\kappa(\mathbf{A}) = O(1), \kappa(\bar{\mathbf{A}}) = O(1)$ but $\kappa([\bar{\mathbf{A}}; \mathbf{A}]) = \Theta(m)$
- a KKT point $\mathbf{x}$ satisfies $\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_m$
- KKT violation$(\mathbf{x}) \geq \varepsilon$, if some $|[\bar{\mathbf{x}}]_j| < \frac{150\pi\epsilon}{\sqrt{m}L}$, where $\bar{\mathbf{x}} := \frac{1}{m}\sum_{i=1}^m \mathbf{x}_i$
- From $\mathbf{x}^{(0)} = \mathbf{0}$, it takes $\Theta(m)$ oracles to kill one zero in the block average

Zero-respecting sequences



Recap

- With a regularizer, lower complexity bounds become higher
- Hence, a regularizer makes the problem harder
- But still gap between the new lower bounds and existing upper bounds

Are the lower complexity bounds tight?

Answer: The lower complexity bounds under Algorithm Class II is tight, see the second part of this talk

Conclusions and Open Questions

- established two lower complexity bounds of first-order methods for linear-constrained regularized nonconvex problems
  - under two different oracles
  - lower bounds higher than that for linear-constrained non-regularied nonconvex problems
  - **Take-away message:** regularizer makes linear-constrained nonconvex problems harder even if its proximal mapping is used

Open question:

- What is the "tight" lower bound if g is a general convex function?
- If high-order derivative is available, what does the lower bound look like?

Extension

If high-order derivative is available

- when f_0 has L_f-Lipschitz continuous p-th order derivatives, the lower bound would be
$$\Omega\left(\frac{\kappa([\bar{\mathbf{A}};\mathbf{A}])^{\frac{p+1}{2p}} L_f^{\frac{1}{p}} \Delta_{F_0}}{\epsilon^{\frac{p+1}{p}}}\right)$$
- compared to problem (P) without the regularization term, the lower bound would be [Carmon et. al'20]
$$\Omega\left(\frac{L_f^{\frac{1}{p}} \Delta_{F_0}}{\epsilon^{\frac{p+1}{p}}}\right)$$
- Answer: existence of g makes a more challenging problem in some cases
  - the latter lower bound is tight
  - The upper bound for solving problem (P) without the regularization term is **lower than the former lower bound**

Relax for a while

New upper complexity bounds by a new algorithm

Problem formulation

$$\min_{\mathbf{x}} F(\mathbf{x}) := f(\mathbf{x}) + g(\bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{b}}),$$
$$\text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \; \mathbf{h}(\mathbf{x}) := [h_1(\mathbf{x}), \ldots, h_m(\mathbf{x})] \leq \mathbf{0} \tag{QCP}$$

- f is smooth and ρ-weakly convex, i.e., $f(\cdot) + \frac{\rho}{2}\|\cdot\|^2$ is convex for some $\rho > 0$
- $h_i(\mathbf{x}) := \frac{1}{2}\mathbf{x}^\top \mathbf{Q}_i \mathbf{x} + \mathbf{r}_i^\top \mathbf{x} + c_i$ is convex quadratic
- $\mathbf{A}, \bar{\mathbf{A}}$ and $\mathbf{b}, \bar{\mathbf{b}}$ are given matrix and vector; g is convex

Two cases of f to be explored

1. $m = 0$ (reduce to Problem (P))
2. $m \geq 1$

Remark: these cases will be handled differently

Many applications

- all linear (equality or inequality) constrained and/or quadratically-constrained problems
- linear constrained nonlinear least squares [Orban-Siqueira'20]
- constrained machine learning: Neyman-Pearson classification [Scott'07], logical neural network [Riegel al et.'20]
- robust phase retrieval [Davis-Drusvyatskiy'19] and linear-constrained variant

Key assumptions

- Continuity of F, $\inf F(\mathbf{x}) > \infty$
- Lipschitz continuity of g
- Slater-type condition
- It holds that either (a) $\widetilde{\mathbf{A}} = \left[\overline{\mathbf{A}}^\top, \mathbf{A}^\top\right]^\top$ has a full-row rank; or (b) $g(\mathbf{y}) = \max\left\{\mathbf{u}^\top \mathbf{y} \,:\, \mathbf{C}\mathbf{u} \leq \mathbf{d}, \mathbf{u} \in \mathbb{R}^{\bar{n}}\right\}$ for some $\mathbf{C}$ and $\mathbf{d}$

Instances

- the assumption on g includes ℓ_1 norm
- the worst-case instance given in page 16 meets all the assumptions

Framework of inexact proximal method when $m = 0$

$$(\mathbf{x}^{(k+1)}, \mathbf{y}^{(k+1)}) \approx \arg\min_{\mathbf{x},\mathbf{y}} \ \left\langle \nabla f(\mathbf{x}^{(k)}), \mathbf{x} - \mathbf{x}^{(k)} \right\rangle + \frac{\tau}{2} \left\| \mathbf{x} - \mathbf{x}^{(k)} \right\|^2 + g(\mathbf{y})$$

$$\text{s.t.} \ \ \mathbf{y} = \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{b}}, \ \mathbf{A}\mathbf{x} + \mathbf{b} = \mathbf{0}$$

where $\tau > L$ to have a strongly convex problem

- Let $\mathbf{z}^{k+1}$ be an approximate solution of the Lagrangian dual problem

$$\min_{\mathbf{z}} \frac{1}{2\tau} \|\bar{\mathbf{A}}^\top \mathbf{z}_1 + \mathbf{A}^\top \mathbf{z}_2 + \nabla f(\mathbf{x}^{(k)}) - \tau \mathbf{x}^{(k)}\|^2 + g^\star(\mathbf{z}_1) - \mathbf{z}_1^\top \bar{\mathbf{b}} - \mathbf{z}_2^\top \mathbf{b}$$

- Set $(\mathbf{x}^{(k+1)}, \mathbf{y}^{(k+1)})$ to

$$\begin{aligned} \mathbf{x}^{(k+1)} &= \mathbf{x}^{(k)} - \frac{1}{\tau} \left(\bar{\mathbf{A}}^\top \mathbf{z}_1^{(k+1)} + \mathbf{A}^\top \mathbf{z}_2^{(k+1)} + \nabla f(\mathbf{x}^{(k)}) \right) \\ \mathbf{y}^{(k+1)} &= \mathbf{prox}_{\sigma^{-1}\bar{g}} \left(\sigma^{-1} \mathbf{z}_1^{(k+1)} + \bar{\mathbf{A}}\mathbf{x}^{(k+1)} + \bar{\mathbf{b}} \right) \end{aligned}$$

Worst-case complexity

- Assumption: Lipschitz continuity of ∇f
- A (restarted) APG has linear convergence
- Total oracle complexity to find a point that is ω-close to an ϵ-KKT point of problem (P) is

  $$O\left(\kappa([\bar{\mathbf{A}};\mathbf{A}])\log\left(\tfrac{\Delta_F}{\epsilon}\right)\frac{L\Delta_F}{\epsilon^2}\right)$$

  where $\Delta_F = F(\mathbf{x}^{(0)}) - \inf_{\mathbf{x}} F(\mathbf{x})$, and $\omega > 0$ arbitrarily close to 0
- Upper bound almost matches the lower bound
  This answers Q2

APG: linear convergence

- an initial point $\boldsymbol{\gamma}^{\text{ini}} = ((\mathbf{y}^{(0)})^\top, (\mathbf{A}\mathbf{x}^{(0)})^\top)^\top$
- $\boldsymbol{\gamma}^{(0,k)} \leftarrow \boldsymbol{\gamma}^{\text{ini}}$, $\widehat{\boldsymbol{\gamma}}^{(0,k)} \leftarrow \boldsymbol{\gamma}^{(0,k)}$, $\alpha_0 \leftarrow 1$
- **For** $i = 0, \ldots, i_k - 1$
  - **For** $j = 0, \ldots, j_k - 1$
    - $\widehat{\mathcal{G}}_1^j \leftarrow \frac{1}{\tau}\bar{\mathbf{A}}\left(\bar{\mathbf{A}}^\top\widehat{\boldsymbol{\gamma}}_1^{(j,k)} + \mathbf{A}^\top\widehat{\boldsymbol{\gamma}}_2^{(j,k)} + \boldsymbol{\xi}_f^{(k)} - \tau\mathbf{x}^{(k)}\right) - \bar{\mathbf{b}}$
    - $\widehat{\mathcal{G}}_2^j \leftarrow \frac{1}{\tau}\mathbf{A}\left(\bar{\mathbf{A}}^\top\widehat{\boldsymbol{\gamma}}_1^{(j,k)} + \mathbf{A}^\top\widehat{\boldsymbol{\gamma}}_2^{(j,k)} + \boldsymbol{\xi}_f^{(k)} - \tau\mathbf{x}^{(k)}\right) - \mathbf{b}$
    - $\boldsymbol{\gamma}_1^{(j+1,k)} \leftarrow \mathbf{prox}_{L_{\mathcal{D}}^{-1}g^*}\left(\widehat{\boldsymbol{\gamma}}_1^{(j,k)} - \frac{1}{L_{\mathcal{D}}}\widehat{\mathcal{G}}_1^j\right)$, $\boldsymbol{\gamma}_2^{(j+1,k)} \leftarrow \widehat{\boldsymbol{\gamma}}_2^{(j,k)} - \frac{1}{L_{\mathcal{D}}}\widehat{\mathcal{G}}_2^j$
    - $\alpha_{j+1} \leftarrow \frac{1+\sqrt{1+4\alpha_j^2}}{2}$
    - $\widehat{\boldsymbol{\gamma}}^{(j+1,k)} \leftarrow \boldsymbol{\gamma}^{(j+1,k)} + \left(\frac{\alpha_j - 1}{\alpha_{j+1}}\right)\left(\boldsymbol{\gamma}^{(j+1,k)} - \boldsymbol{\gamma}^{(j,k)}\right)$
  - $\boldsymbol{\gamma}^{(0,k)} \leftarrow \boldsymbol{\gamma}^{(j_k,k)}$, $\widehat{\boldsymbol{\gamma}}^{(0,k)} \leftarrow \boldsymbol{\gamma}^{(0,k)}$, $\alpha_0 \leftarrow 1$
- **Output:** $\boldsymbol{\gamma}^{(k+1)} = \boldsymbol{\gamma}^{(0,k)}$

Framework of inexact proximal method when $m = O(1)$

$$
(\mathbf{x}^{(k+1)}, \mathbf{y}^{(k+1)}) \approx \arg\min_{\mathbf{x},\mathbf{y}} \; \left\langle \nabla f(\mathbf{x}^{(k)}), \mathbf{x} - \mathbf{x}^{(k)} \right\rangle + \frac{\tau}{2} \left\| \mathbf{x} - \mathbf{x}^{(k)} \right\|^2 + g(\mathbf{y})
$$

$$
\text{s.t.} \;\; \mathbf{y} = \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{b}}, \;\; \mathbf{A}\mathbf{x} + \mathbf{b} = \mathbf{0}, h_i(\mathbf{x}) \le 0
$$

where $\tau > L$ to have a strongly convex problem

Lagrangian function:

$$
\mathcal{L}_k(\mathbf{x}, \mathbf{y}; \boldsymbol{\gamma}, \boldsymbol{\lambda}) = \bar{f}^k(\mathbf{x}) + g(\mathbf{y}) - \boldsymbol{\gamma}_1^\top \left(\mathbf{y} - \left(\bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{b}} \right) \right) + \boldsymbol{\gamma}_2^\top \left(\mathbf{A}\mathbf{x} + \mathbf{b} \right) + \sum_{i=1}^{m} \lambda_i h_i(\mathbf{x})
$$

Let $\mathcal{D}_k(\boldsymbol{\gamma}, \boldsymbol{\lambda}) := -\min_{\mathbf{x},\mathbf{y}} \mathcal{L}_k(\mathbf{x}, \mathbf{y}; \boldsymbol{\gamma}, \boldsymbol{\lambda})$

$$
\Omega^{(k+1)} := \operatorname*{Arg\,min}_{\boldsymbol{\gamma}, \boldsymbol{\lambda} \ge 0} \mathcal{D}_k(\boldsymbol{\gamma}, \boldsymbol{\lambda}) \quad \text{and} \quad \mathcal{D}_k^* := \min_{\boldsymbol{\gamma}, \boldsymbol{\lambda} \ge 0} \mathcal{D}_k(\boldsymbol{\gamma}, \boldsymbol{\lambda})
$$

Framework of inexact proximal method when $m = O(1)$ (Cont'd)

A primal-dual step

Find a near-optimal point $(\boldsymbol{\gamma}^{(k+1)}, \boldsymbol{\lambda}^{(k+1)})$ by solving the convex problem $\min_{\boldsymbol{\gamma}, \boldsymbol{\lambda} \geq 0} \mathcal{D}_k(\boldsymbol{\gamma}, \boldsymbol{\lambda})$ inexactly

Let $\sigma > 0$ be a given scalar, $\widetilde{\mathbf{Q}}_i = (\mathbf{Q}_i + {\mathbf{Q}_i}^\top)/2$

Obtain $(\mathbf{x}^{(k+1)}, \mathbf{y}^{(k+1)})$ by solving

$$\left(\tau \mathbf{I}_d + \sum_{i=1}^m \lambda_i^{(k+1)} \widetilde{\mathbf{Q}}_i\right) \mathbf{x}^{(k+1)} = \tau \mathbf{x}^{(k)} - \left(\widetilde{\mathbf{A}}^\top \boldsymbol{\gamma}^{(k+1)} + \nabla f(\mathbf{x}^{(k)}) + \sum_{i=1}^m \lambda_i^{(k+1)} \mathbf{r}_i\right)$$

and letting

$$\mathbf{y}^{(k+1)} = \mathbf{prox}_{\sigma^{-1} g}\left(\sigma^{-1} \gamma_1^{(k+1)} + \bar{\mathbf{A}} \mathbf{x}^{(k+1)} + \bar{\mathbf{b}}\right)$$

The x-solution is

$$\mathbf{x}^{k+1}(\boldsymbol{\gamma}, \boldsymbol{\lambda}) := \left(\tau \mathbf{I}_d + \boldsymbol{\lambda} \widetilde{\mathbf{Q}}\right)^{-1} \left(\tau \mathbf{x}^{(k)} - \boldsymbol{\lambda} \mathbf{r} - \widetilde{\mathbf{A}}^\top \boldsymbol{\gamma} - \nabla f(\mathbf{x}^{(k)})\right)$$

Worst-case complexity

For outer loop

- Assumption: Lipschitz continuity of ∇f
- Outer oracle complexity to find a point that is ω-close to an ϵ-KKT point of problem (P) is

$$O\left(\frac{L\Delta_F}{\epsilon^2}\right)$$

  where $\Delta_F = F(\mathbf{x}^{(0)}) - \inf_{\mathbf{x}} F(\mathbf{x})$, and $\omega > 0$ arbitrarily close to 0

For inner loop: solve the convex problem $\min_{\boldsymbol{\gamma},\boldsymbol{\lambda}\geq 0} \mathcal{D}_k(\boldsymbol{\gamma},\boldsymbol{\lambda})$ inexactly

- use a bisection method on $\boldsymbol{\lambda}$
- $O(m)$ iterations

Comparison to existing methods

- For composite nonconvex optimization with affinely constraints and smooth f, $O\left(\kappa([\bar{\mathbf{A}};\mathbf{A}])\log\left(\frac{\Delta_F}{\epsilon}\right)\frac{L\Delta_F}{\epsilon^2}\right) \Rightarrow$ Near Optimal
- For composite nonconvex optimization with convex quadratic constraints and smooth f, $\tilde{O}(m/\varepsilon^2)$ V.S. $\tilde{O}(1/\varepsilon^{2.5})$

Conclusion and Future Work

Conclusion

- The lower complexity bound under Algorithm Class II is tight
- The upper complexity bound for solving problem (OCP) is improved

Future Work

- For Algorithm Class 1 on solving Problem (P), can the lower bound be achieved, i.e., is it tight?
- How to formulate a lower complexity bound of Algorithm Class 1 on solving problem (P) without $g = \bar{g}(\bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{b}})$?
- What will the lower bound look like if there are convex nonlinear inequality constraints?

Thank you!!!