1 Dirichlet Problem

$$\begin{aligned} \nabla^2 u(s,t) &= f(s,t,u) \quad \forall (s,t) \in \Omega \\ u(s,t) &= \varphi(s,t) \quad \forall (s,t) \in \partial\Omega \end{aligned}$$

where Ω is a simply-connected, bounded open region in the plane, and φ is a given function defined on the boundary $\partial\Omega$ of Ω. It is known that if $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function which satisifies

$$f(s,t,u) \geq 0, \quad \forall (s,t) \in \Omega, \, u \in \mathbb{R}$$

then under mild condition on Ω and φ, the problem has a unique solution.

For illustrative purpose, let the domain Ω be a unique square $(0,1) \times (0,1)$ and impose a uniform square mesh on Ω by defining the grid points

$$U_{ij} = u(ih, jh), \; h = 1/(m+1), \, i, j = 0, \ldots, m+1$$

where there are m^2 interior points. At each interior grid point U_{ij}, the partial derivatives $u_{ss}(U_{ij})$ and $u_{tt}(U_{ij})$ are now approximated by

$$\begin{aligned} u_{ss}(ih, jh) &\approx h^{-2}(U_{i+1,j} - 2U_{ij} + U_{i-1,j}) \\ u_{tt}(ih, jh) &\approx h^{-2}(U_{i,j+1} - 2U_{ij} + U_{i,j-1}) \end{aligned}$$

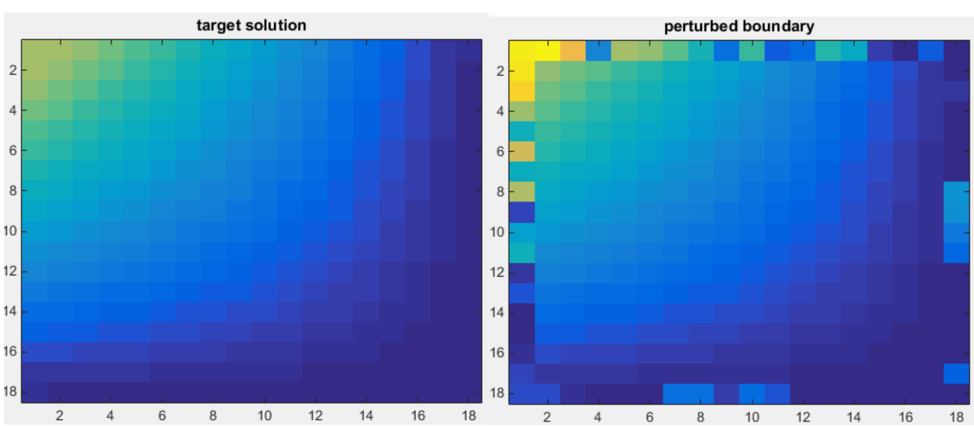
thus the partial differential equation is approximated as

$$4U_{ij} - U_{i-1,j} - U_{i+1,j} - U_{i,j+1} - U_{i,j-1} + h^2 f(ih, jh, U_{ij}) = 0$$

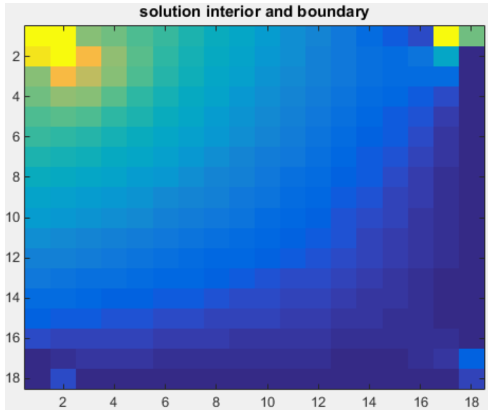
where boundary values are given by

$$\begin{aligned} U_{0,j} &= \varphi(0, jh), \, U_{1,j} = \varphi(1, jh) \\ U_{i,0} &= \varphi(ih, 0), \, U_{i,1} = \varphi(ih, 1) \end{aligned}$$

Target solution with continous boundary vs. perturbed boundary.



Recovered solution boundary and interior by minimizing $\|u - U\|^2$ using Gauss-Newton



2 Heat Conductivity Inverse Problem

1-D heat conduction is mathematically modeled by the following partial differential equation:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial}{\partial x}\left(s(x)\dfrac{\partial u(x,t)}{\partial x}\right), \quad x \in (0,1), \, t \in (0,T] \\ u(x,0) = u^0(x), \quad x \in (0,1) \\ u(0,t) = f(t), \quad u(1,t) = g(t), \quad t \in (0,T] \end{cases} . \tag{1}$$

Function $u(x,t)$ represents the temperature of a rod at position x and time t. Function $s(x)$ is the unknown conductivity of the rod. Our goal is to determine $s(x)$ given the rod's initial temperature $u^0(x)$ at time $t = 0$, boundary temperature $f(t)$ and $g(t)$, and temperature $u(x,T)$ measured at time $t = T$.

To obtain the numerical result, discretization of spatial and time domains is required to employ a finite difference method:

$$\begin{aligned} x_j &= j\Delta x, \quad \Delta x = \frac{1}{M+1} \\ t_n &= n\Delta t, \quad \Delta t = \frac{T}{N} \\ u_j^n &= u(x_j, t_n), \quad s_j = s(x_j), \end{aligned}$$

where M is the number of interior nodes in the discretized grid and N is the number of time steps. Denote $\lambda = \Delta t / \Delta x^2$. The finite difference operators that will be used to approximate equation (1) are

$$\begin{aligned} &\text{Forward Time} && D_t^+ u_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} \\ &\text{Forward Space} && D_x^+ u_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x} \\ &\text{Backward Space} && D_x^- u_j^n = \frac{u_j^n - u_{j-1}^n}{\Delta x} \\ &\text{Centered Space} && D_x^+ D_x^- u_j^n = D_x^- D_x^+ u_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}. \end{aligned}$$

The heat equation (1) is approximated as

$$D_t^+ u_j^n = \frac{1}{2}\left(D_x^+\left(s_j D_x^- u_j^n\right) + D_x^-\left(s_j D_x^+ u_j^n\right)\right). \tag{2}$$

Expanding (2) to obtain

$$u_j^{n+1} = a_j u_{j-1}^n + b_j u_j^n + c_j u_{j+1}^n \tag{3}$$

where for $j \in \{2, \ldots, M-1\}$

$$a_j = \frac{\lambda}{2}\left(s_{j-1} + s_j\right), \, b_j = 1 - \frac{\lambda}{2}\left(s_{j-1} + 2s_j + s_{j+1}\right), \, c_j = \frac{\lambda}{2}\left(s_j + s_{j+1}\right), \tag{4}$$

for $j = 1$

$$a_1 = \frac{\lambda}{2}\left(3s_1 - s_2\right), \, b_1 = 1 - 2\lambda s_1, \, c_1 = \frac{\lambda}{2}\left(s_1 + s_2\right), \tag{5}$$

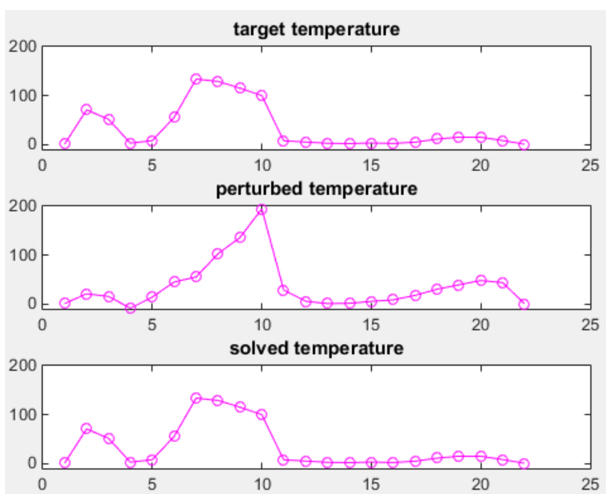
and for $j = M$

$$a_M = \frac{\lambda}{2}\left(s_{M-1} + s_M\right), \, b_M = 1 - 2\lambda s_M, \, c_M = \frac{\lambda}{2}\left(s_M - s_{M-1}\right). \tag{6}$$

Denote the solution of equation (2) at time $t = T$ as U^N , and the given temperature measurement at time $t = T$ as u^N. Both U^N and u^N are M-dimensional vectors. U^N can be viewed as a function of the conductivity s. s is also an M-dimensional vector after discretization. The inverse problem is to find conductivity s that matches the measured data u^N. That is, we want to solve the M-dimensional nonlinear system

$$F(s) = u^N - U^N = 0. \tag{7}$$

Sample solution



Residual w.r.t. to the number of iteration is

Iteration	1	2	3	4	5	6	7
Residual	1.55×10^2	1.19×10^2	2.61×10^1	1.50	1.65×10^{-2}	2.40×10^{-6}	9.54×10^{-13}

3 The Implied Volatility Surface Problem for Option Pricing

The value of the European option satisfies the generalized Black-Scholes equations

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{1}{2}\sigma^2\left(S,t\right)\dfrac{\partial^2 y}{\partial S^2} + rS\dfrac{\partial y}{\partial S} - ry = 0 \\ y(S,T) = \max\left\{K - S, 0\right\} \quad \forall S \\ y\left(S_{\max}, t\right) = 0, \quad y\left(0,t\right) = Ke^{-r(T-t)} \quad \forall t \end{cases} \tag{8}$$

where K is the strike price of the option, $\sigma\left(S,t\right)$ is a surface reflecting the volatility of underlying S, y is the fair value of the option, r is the risk-free ineterest rate with known boundary conditions and S is current stock price assumed the follow 1-factor continuous diffusion equation

$$\frac{dS\left(t\right)}{S\left(t\right)} = \mu\left(S\left(t\right), t\right)dt + \sigma\left(S\left(t\right), t\right)dW\left(t\right).$$

Similar with previous example of heat equation, equation (8) can be discretized according to

$$\begin{aligned} S_j &= j\Delta S,\ \Delta S = \frac{S_{\max}}{M+1} \\ t_n &= n\Delta t, \quad \Delta t = \frac{T}{N} \\ y_j^n &= y\left(S_j, t_n\right), \ \ \sigma_j^n = \sigma\left(S_j, t_n\right) \end{aligned}$$

so the B-S euqation approximated as

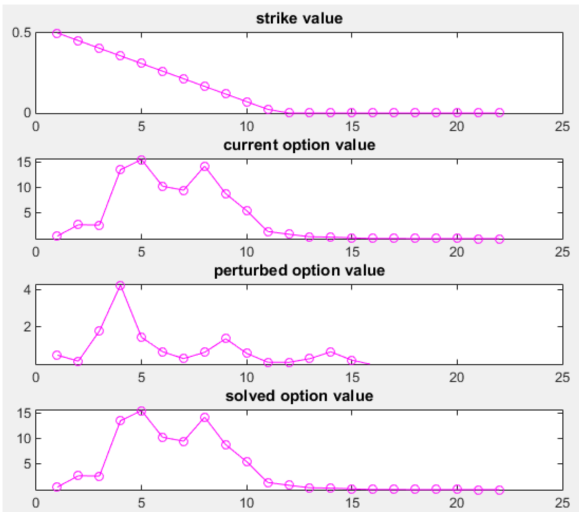
$$y_j^{n-1} = a_j^n y_{j-1}^n + b_j y_j^n + c_j y_{j+1}^n$$

where for $j, n \in \{1, \ldots, M\}$

$$a_j^n = \frac{1}{2}\Delta t\left(\left(\sigma_j^n\right)^2 j^2 - rj\right), \ b_j^n = 1 - \Delta t\left(\left(\sigma_j^n\right)^2 j^2 + r\right), \ c_j^n = \frac{1}{2}\Delta t\left(\left(\sigma_j^n\right)^2 j^2 + rj\right)$$

which is the same form with heat equation (3).

Sample solution



Residual w.r.t. to the number of iteration is

Iteration	1	2	3	4	5	6	7
Residual	27.161	25.09	10.35	7.623×10^{-1}	6.269×10^{-2}	4188×10^{-2}	4514×10^{-4}