Heat and Mass Transfer in Practice Using Mathematica. [1]

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January 26, 2015

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Contents

Part I

Tutorial

Chapter 1

Introduction to Mathematica

Mathematica, developed by Wolfram Research[1], is a computational software program used in many scientfic, engineering, mathematical and computing fields, based on symbolic mathematics. The programming language using in Mathematica is called Wolfram Language.

1.1 Overview

1.1.1 Installation

Students with a CSE Labs account could download and use *Mathematica* free of charge from Mathematica download page[2]. By following the instructions from that page, you could install *Mathematica* on Windows, Mac or Linux.

1.2 Tutorial

There are many online tutorials that could help you learning *Wolfram Language* and *Mathematica*. You could find detailed tutorial at Wolfram Language & System Documentation Center[3]. Those topics include:

- Core Language & Structure
- Symbolic & Numeric Computation
- Data Manipulation & Analysis

[1]Wolfram Research is a private company makes computation software
[2]`https://wwws.cs.umn.edu/download_software/mathematica`
[3]`http://reference.wolfram.com/language/?source=nav`

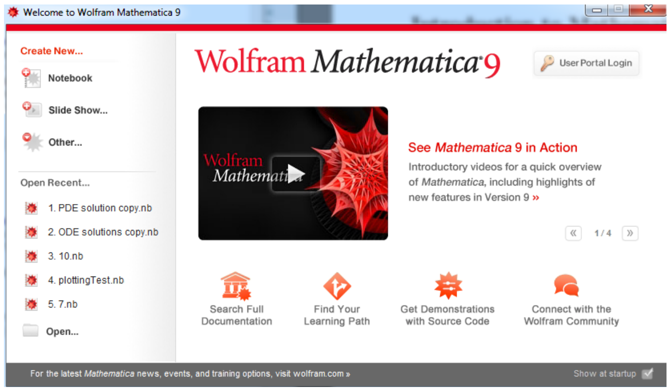


Figure 1.1: Initialization interface of Mathematica

- Visualization & Graphics
- Images
- ...

There are also many free video courses for educators and researchers which can be found at Mathematica for Teaching and Education.[4] and Mathematica for University Research[5]. In the following section, we will look through some basics of *Wolfram Language*.

1.3 Basics

1.3.1 Simple Calculations

In this section, we will see some examples on some basic arithmetic operations. The Wolfram Language is case sensitive. So we should use **Sin** rather than **sin** to represent a sinusoidal function.

n = 3 + 6

9

n = 2 + 7 - 8/6

$\frac{23}{3}$

[4] `http://www.wolfram.com/training/courses/edu001.html`

[5] `http://www.wolfram.com/training/courses/edu002.html`

Sin[Pi/6]
$\frac{1}{2}$

n = Sin[30 Degree]
$\frac{1}{2}$

N [Sin[Pi/6]]
0.5

1.3.2 Matrix Operations

Vectors and matrices in the Wolfram Language are simply represented by lists and by lists of lists, respectively. Some basic matrix operations are as shown in Table 1.1.

Expressions of a 3×3 matrix:
m={{-9,19,3},{-3,7,1},{-7,17,2}}
{{-9, 19, 3}, {-3, 7, 1}, {-7, 17, 2}}

Transposing matrix **m**:
Transpose[m]
{{-9, -3, -7}, {19, 7, 17}, {3, 1, 2}}

Expressing **m** in matrix form**m**:
MatrixForm[m]

$$\begin{pmatrix} -9 & 19 & 4 \\ -3 & 7 & 1 \\ -7 & 17 & 2 \end{pmatrix}$$

Inversing matrix **m**:
Inverse[m]

$$\begin{pmatrix} -\frac{3}{2} & \frac{13}{2} & -1 \\ -\frac{1}{2} & \frac{3}{2} & 0 \\ -1 & 10 & -3 \end{pmatrix}$$

Table 1.1: Basic matrix operations

Function	Purpose
Transpose[m]	Transpose m^T
ConjugateTranspose[m]	Conjugate transpose m^* (Hermitian conjugate)
Inverse[m]	Matrix inverse
Det[m]	Determinant
Tr[m]	Trace
MatrixRank[m]	Rank of matrix

1.3.3 Plotting

The Wolfram Language has many ways to plot functions and data by using function **Plot**. And it has many options on what the scales should be, how the axes should be draw and so on. In this section, we will see some basic examples on data plotting. **Plot**[**Sin**[**x**], **x, 0, 10**] gives result shown in Figure 1.2. **Plot3D**[**Sin**[**x**]**Cos**[**y**], **{x, 0, 5}, {y, 1, 10}**] produces result shown in Figure 1.3.
Plot[**Sin**[**x^2**],**{x,0,3},AxesLabel->{"x value",Sin**[**x^2**]**},GridLines->Automatic**] adds label and grid line to the plotting shown in Figure 1.4.

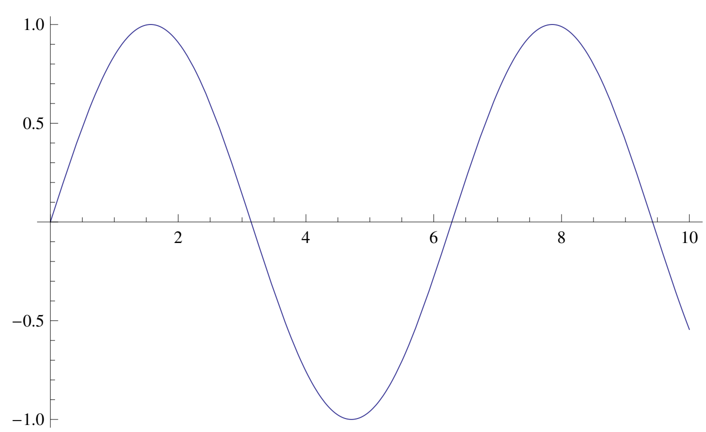


Figure 1.2: Plot of $y = sin(x)$

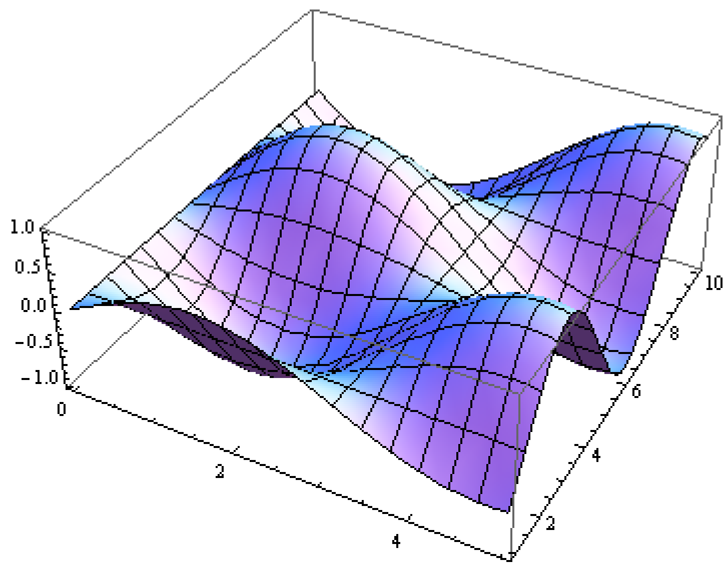


Figure 1.3: Plot of $z = sin(x)cos(y)$

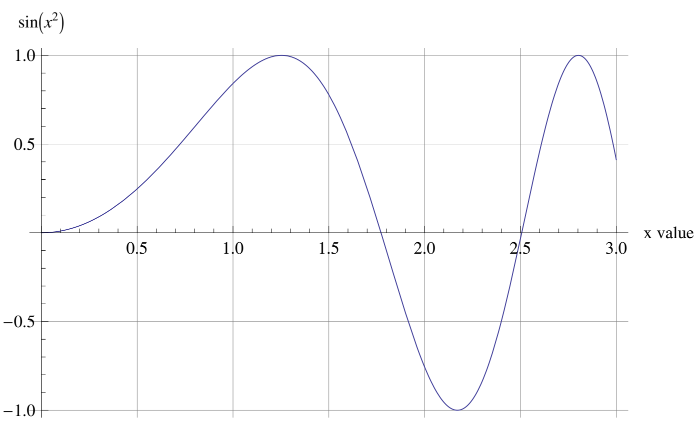


Figure 1.4: Plot of $y = sin(x^2)$ with label and axis description

1.3.4 Solving Differential Equation

Differential equations have three basic types of equations including:

- *Ordinary Differential Equations* (ODEs), in which there is a single independent variable t and one or more dependent variables $x_i(t)$.

- *Partial Differential Equations* (PDEs), in which there are two or more independent variables and one dependent variable.

- *Differential-Algebraic Equations* DAEs, in which some members of the system are differential equations and the others are purely algebraic, having no derivatives in them.

The function *DSolve* gives symbolic solutions to the differential equations while function *NDSolve* generates a general numerical differential equation solver. *DSolve* is powerful in solving ODEs and most first-order PDEs as well as a limited number of the second-order PDEs. For DAEs, it is difficult to find the exact solutions, but *DSolve* can solve many examples of such systems that occur in application.

Example of solving ODE:

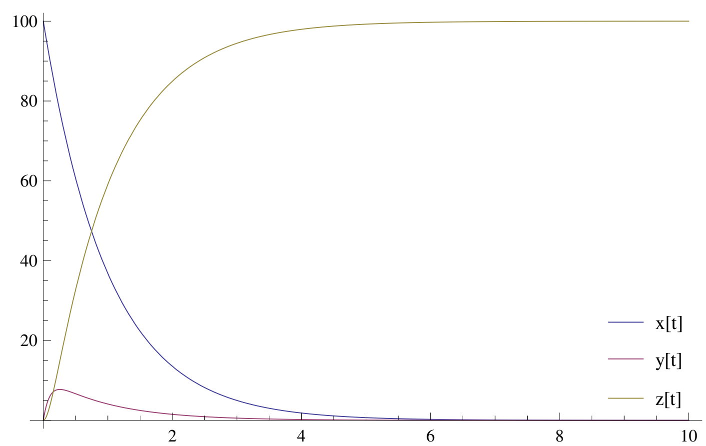


Figure 1.5: Plot of ODE solution

Listing 1.1: ODE solution

```
eqns = Join[
{x'[t]== −x[t], y'[t]== x[t] − 10y[t],
z'[t]== 10y[t],
x[0]==100,y[0]==0, z[0]==0},
{x,y,z}, {t,100}];
sol = NDSolve[eqns];
Plot[
Evaluate[{x[t],y[t],z[t]} /. sol],
```

```
{t,0,10},
PlotLegends->Placed[{"x[t]", "y[t]", "z[t]"} , {Right, Bottom}]]
```

In this example, we will solve the following ODEs.

$$\begin{cases} x'[t] = -x[t], \\ y'[t] = x[t] - 10y[t], \\ z'[t] = y[t] \end{cases}$$

where initial conditions are x[0]=100, y[0]=0, z[0]=0 and $t \in [0, 100]$. First, we would use function Join[$list_1, list_2$, ...] to concatenates lists of equations and initial conditions into a new list. Then, we use function NDSolve[*eqns*, u, x, x_{min}, x_{max}] to compute the numerical solution from the list gotten from Join. Finally, we use function Plot[f_1, f_2, ..., x, x_{min}, x_{max}] to draw all the curves of x[t], y[t] and z[t]. The code is as shown in Listing 1.1 and the curves are shown as Figure 1.5. For detailed usages of function Join, NDSolve and Plot, please refer to the on-line *Mathematica reference*.

Chapter 2

One-dimensional Steady State Problems

2.1 Overview

This chapter mainly discusses different kinds of one-dimensional steady state problems. Including the use of Fourier's law on constant cross sectional and non uniform cross sectional problem, the one dimensional steady state problem with constant heat generation problem in both catesian and cylindrical coodinate system, and the fin design under steady state.

2.2 Fourier's law

Example 2.2.1 Fourier's law, plane wall, constant internal heat generation. *Refer to tutorial 2-2-1 Fourier Plane Wall.nb* Consider a one-dimensional plane wall in steady state and without heat generation. Temperature on one side is $T_0 = 50\ ^\circ C$, other side is $T_1 = 30\ ^\circ C$, wall width is $L = 0.01\ m$, area is $A = 1\ m^2$, thermal conductivity $k = 0.5\ W/m.k$. Sketch the heat distribution on T-x coordinate, what is the heat flux through the wall? (As shown in figure 2.1)

Solution.
For the plan wall, the heat resistance is

$$R = \frac{L}{kA}$$

So the heat rate

$$Q = \frac{\Delta T}{R} = kA\frac{T_0 - T_1}{L} = 1000\ W$$

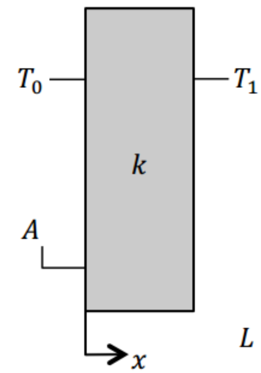


Figure 2.1: Model of example 2.2.1

The heat flux

$$q = \frac{Q}{A} = 1000\ W/m^2$$

Based on Fourier's equation, the heat rate

$$Q = -kA\frac{dT}{dx}$$
$$dT = -\frac{Q}{ka}dx$$

Integral

$$\int dT = -\int \frac{Q}{ka}dx$$

k and A are constant, so the heat distribution along the x direction is

$$T(x) = C_1x + C_2$$

As given, $T(0) = T_0 = 50\ °C, T(0.01) = T_1 = 30\ °C$, we can get

$$T(x) = \frac{T_1 - T_0}{L}x + T_0 = -2000x + 50$$

And the sketch of is shown as figure 2.2.

Example 2.2.2 Conical section *Refer to tutorial 2-2-2 Fourier Variable Cross Area.nb.*A diagram shows a conical section fabricated from pyroceram, $k = 3.46W/m.K$. It is of circular cross section with the diameter $D = ax$,

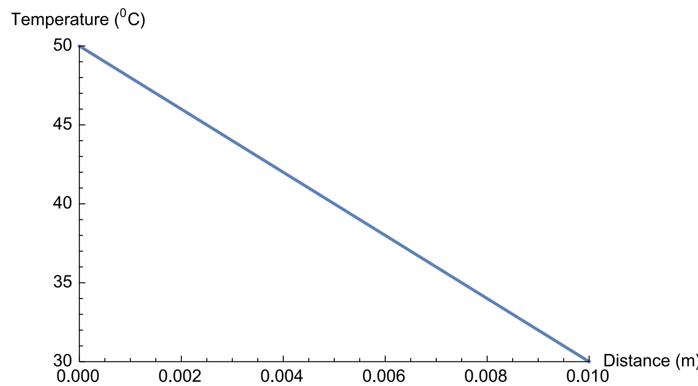


Figure 2.2: Temperature distribution in plane wall

where $a = 0.25$.The small end is at $x_1 = 50\ mm$ and the large end at $x_2 = 250\ mm$. The end temperature are $T_1 = 400\ K$ and $T_2 = 600\ K$, while the lateral surface is well insulated.

1. Derive and expression for the temperature distribution $T(x)$, assuming one-dimensional conditions. Sketch the temperature distribution.

2. Calculate the heat rate Q through the cone.

3. If a changes from 0.001 to 1, sketch change of Q.

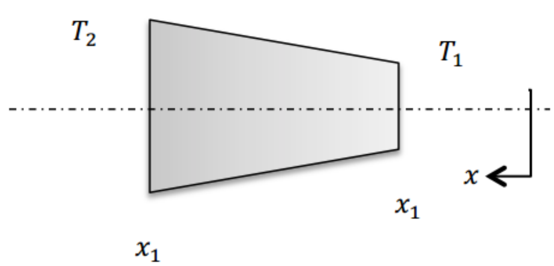


Figure 2.3: Model of example 2.2.2

Solution.

1. Consider the heat conduction is under steady state, one-dimensional coordinate, without internal heat generation, the heat transfer rate

is a constant independent of x. Use Fourier's Law to determine the temperature distribution.

$$Q = -kA\frac{dT}{dx}$$

Where $A = \pi D^2/4 = \pi a^2 x^2/4$. If $T(x_1) = T_1$, then we have

$$\begin{cases} \dfrac{4Qdx}{\pi a^2 x^2} = -kdT \\ T(x_1) = T_1 \end{cases}$$

Integrating from x_1 to any x within the cone, and recalling that Q and k are constant, it follows that

$$\frac{4Q}{\pi a^2}\int_x^{x_1} dx/x^2 = -k\int dT$$

Hence

$$\frac{4Q}{\pi a^2}\left(-\frac{1}{x} + \frac{1}{x_1}\right) = -k\left(T - T_1\right)$$

and

$$T(x) = T_1 - \frac{4Q}{\pi a^2}\left(\frac{1}{x_1} - \frac{1}{x}\right)$$

Although Q is a constant, it is as yet an unknown. However, it may be determined by evaluating the above expression at $x = x_2$, where $T(x_2) = T_2$. Hence

$$Q = \frac{\pi a^2 k(T_1 - T_2)}{4[(1/x_1) - (1/x_2)]}$$

and solving for Q Substituting for Q into the expression for $T(x)$, the temperature distribution becomes

$$T_2 = T_1 + (T_1 - T_2)\left[\frac{(1/x) - (1/x_2)}{(1/x_1) - (1/x_2)}\right]$$

From the result, temperature may be calculated as a function of x and the distribution is as shown in figure 2.4.

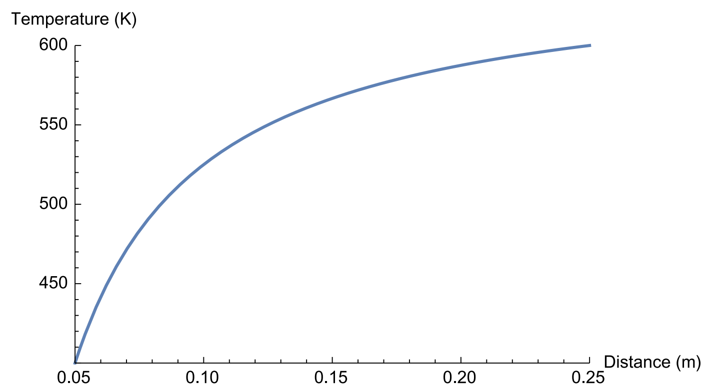


Figure 2.4: Temperature distribution in conical

2. Substituting numerical values into the foregoing result for the heat transfer rate, it follows that

$$Q = \frac{\pi 0.25^2 \times 3.46 \text{ W/m.K} \times (400 - 600)\ K}{4(1/0.05\ m - 1/0.25\ m)} = -2.12W$$

3. If a changes from 0.001 to 1, as Q has expression changes with a, we can sketch Q's changes with a as shown in figure 2.5.

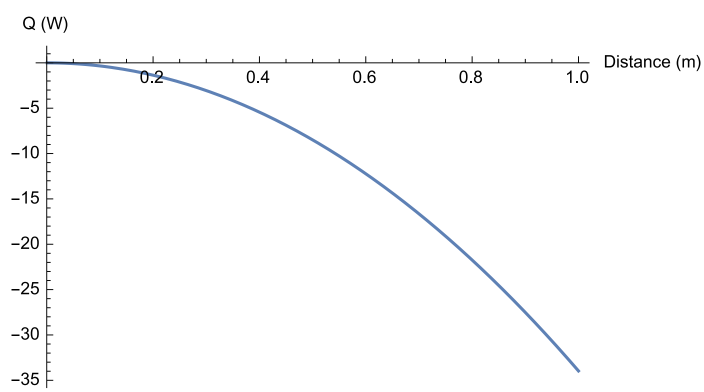


Figure 2.5: Heat rate cross conical section

2.3 1-D Steady State, constant internal heat generation problems

Example 2.3.1 *Refer to tutorial 2-3-1 2-4-1 and 2-3-2 1D ss heat generation.nb* A large thin slab of thickness $L = 0.1\ m$ is "setting." Setting is an exothermic process that releases $\dot{q} = 100W/m^3$. Here the slab heat conductivity is in steady state. Set the x-axis along with the wall thickness. At position $x = 0m$, temperature is $T_0 = 37\ °C$, at position $x = L$, $T_1 = 33\ °C$, thermal conductivity $k = 0.4W/m.K$. What's the temperature distribution in along the length of the slab?

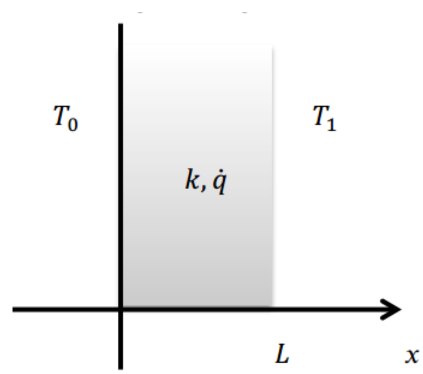


Figure 2.6: Model of example 2.3.1

Solution.
Based on the heat diffusion equation

$$\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\propto}\frac{\partial T}{\partial t}$$

where

$$\nabla^2 T \equiv \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

In this problem, the large thin slab could be considered as a one-dimensional problem with only x dimension, and consider the slab is in steady state with no change with t, so the one-dimensional heat diffusion equation for the slab could be write as

$$\frac{d^2 T}{dx^2} = -\frac{\dot{q}}{k}$$

Integration twice

$$T(x) = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

By evaluating $T(0) = T_0$, and $T(L) = T_1$, hence

$$T(x) = -\frac{\dot{q}}{2k}x^2 + \left(\frac{T_1 - T_0}{L} + \frac{\dot{q}L}{2k}\right)x + T_0$$

From the result, temperature distribution could be expressed as a quadratic curve as shown in figure 2.7.

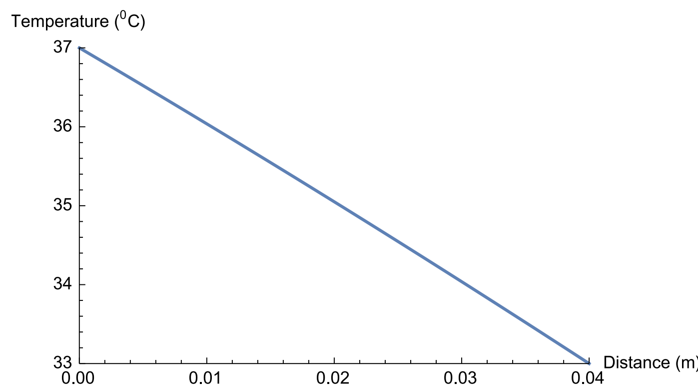


Figure 2.7: 1D steady state temperature distribution in plane wall with constant heat generation

Example 2.3.2 *Refer to tutorial 2-3-1 and 2-3-2 1D ss heat generation.nb.* A steady state long tube generating thermal energy at a uniform volumetric rate $\dot{q} = 1000W/m^3$, the thermal conductivity $k = 0.4$W/m.K. At radius $r_1 = 0.1368m$, temperature $T_0 = 37\ °C$, at position $r_2 = 0.1768m$, temperature $T_1 = 33\ °C$, the two end of the rod are well insulated. What is the temperature distribution along the radius of the tube?

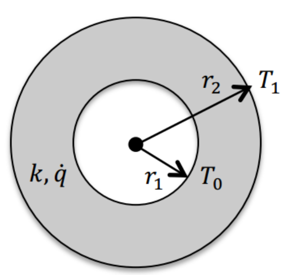


Figure 2.8: Model of example 2.3.2

Solution.
The heat diffusion equation for cylindrical system is

$$\frac{1}{r}\frac{\partial}{r}(kr\frac{\partial T}{\partial r}) + \frac{1}{r^2}\frac{\partial}{\phi}(k\frac{\partial T}{\partial \phi}) + \frac{\partial}{z}(k\frac{\partial T}{\partial z}) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

In this problem, consider the heat distribution change only on r direction, and since the tube is in steady state, the temperature distribution would not change with time t. The heat distribution equation could be written as

$$\frac{1}{r}\frac{\partial}{r}(kr\frac{\partial T}{\partial r}) + \dot{q} = 0$$

Integration twice

$$T(r) = -\frac{\dot{q}}{4k}r^2 + C_1 \ln r + C_2$$

By evaluating $r_1 = 0.1368\ m$, $T_0 = 37\ K, r_2 = 0.1768\ m$ and$T_1 = 33\ K$, sketch the $T(r)$ as shown in figure 2.9

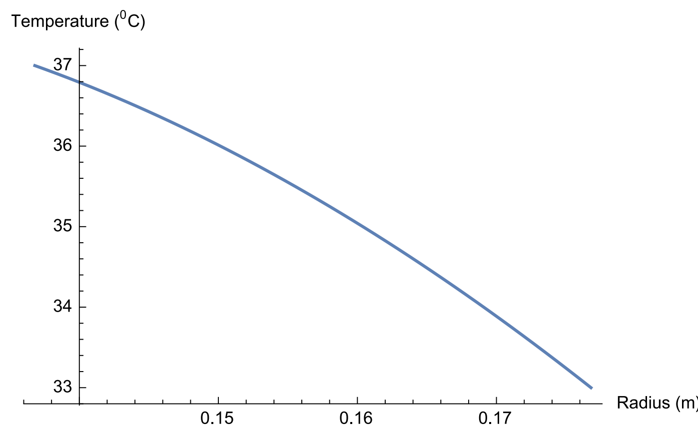


Figure 2.9: 1D steady state temperature distribution in tube with constant heat generation

2.4 Fin Problems

In advanced problems the fin problem becomes a way to solve for perfusion. The Bioheat Equation described in IntroHT.pdf can be solved in Steady State like a fin, and will also be discussed later in the advanced problems.

Example 2.4.1 Constant k and A. *Refer to tutorial 2-5-1 and 2-5-2 Fin Solutions.nb.* A rectangular fin with uniform cross-sectional area has constant heat conductivity $k = 3W/m.K$, width is $W = 0.01m$, Thickness is $L = 1m$. The surface of the fin is exposed to ambient air at 10 $°C$ with a convection heat transfer coefficient $h = 10W/m^2K$. Plot the temperature distribution in the fin under below conditions. (Shown in figure 2.10)

1. Prescribed tip temperature: at start position $x_1 = 0m$, temperature is $T_1 = 30°C$, at the end position $x_2 = 0.04m, T_2 = 40°C$
2. Adiabatic tip condition: at start position $x_1 = 0m$, temperature is $T_1 = 30°C$, at the end position, $h[T(L) - T_\infty] = -k\frac{dT}{dx}|_{x=L} = 0$.

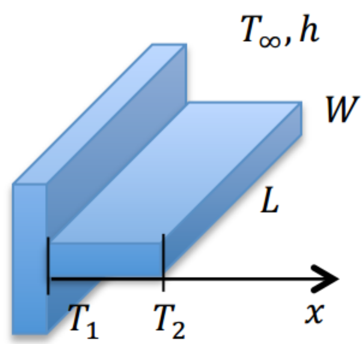


Figure 2.10: Model of example 2.4.1

Solution.

1. The fin energy balance equation

$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{x}\right)\frac{dT}{dx} - \left(\frac{1}{A_c}\frac{h}{k}\frac{dA_s}{x}\right)(T - T_\infty) = 0$$

For the proscribed fin problem with constant cross section area $A_c = WL$, and surface area, we have $dA_c/dx = 0$, $dAs/dx = p$. Hence

$$\frac{d^2T}{dx^2} - \frac{hp}{kA_c}(T - T_\infty) = 0$$

To simplify the form of this equation, we transform the dependent variable by defining an excess temperature θ as

$$\theta(x) \equiv T(x) - T_\infty$$

Where since T_∞ is constant, $d\theta/dx = dT/dx$. Then we obtain

$$\frac{d^2T}{dx^2} - m^2\theta = 0$$

Where

$$m^2 = \frac{hp}{kA_c}$$

With prescribed boundary condition

$$\theta(0) = T_1 - T_\infty \equiv \theta_1$$

$$\theta(0.04) = T_2 - T_\infty \equiv \theta_2$$

Then for prescribed condition the fin temperature distribution is as below equation, and the sketch is shown below in figure 2.11

$$\frac{\theta}{\theta_1} = \frac{(\theta/\theta_1)\sinh mx + \sinh m(L-x)}{\sinh mL}$$

And for the total heat rate transfer from the fin, we can get it by calculate the fin heat rate at fin base $x = 0$

$$Q_f = -kA_cT(x)\,\mathrm{d}x|_{x=0} = -kA_c\theta\,\mathrm{d}x|_{x=0}$$

For prescribed tip, fin heat rate is

$$Q = M\frac{\cosh mL - \theta_2/\theta_1}{\sinh mL}$$

Where $M = \sqrt{hpkA_c\theta_0}$.

2. For adiabatic tip condition, the heat convection rate at the tip is considered negligible

$$hA_c\theta(L) = -kA_c\frac{d\theta}{dx}|_{x=L} = 0$$

And

$$\frac{d\theta}{dx}|_{x=L} = 0$$

Then the heat distribution equation of adiabatic tip condition could be written as

$$\frac{\theta}{\theta_1} = \frac{\cosh m(L-x)}{\cosh mL}$$

The temperature distribution along x direction is shown below in figure 2.12

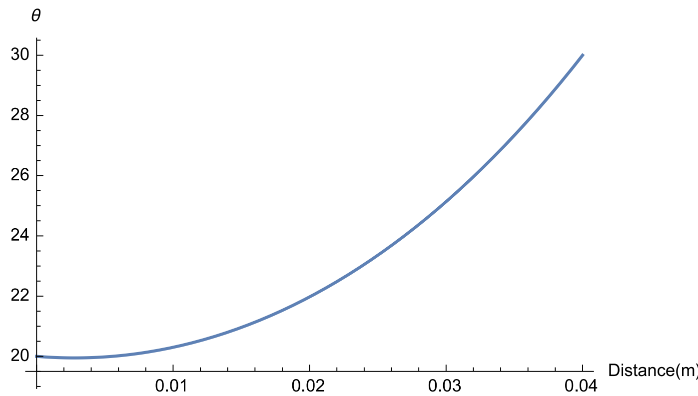


Figure 2.11: Temperature distribution in rectangular fin under prescribed tip condition

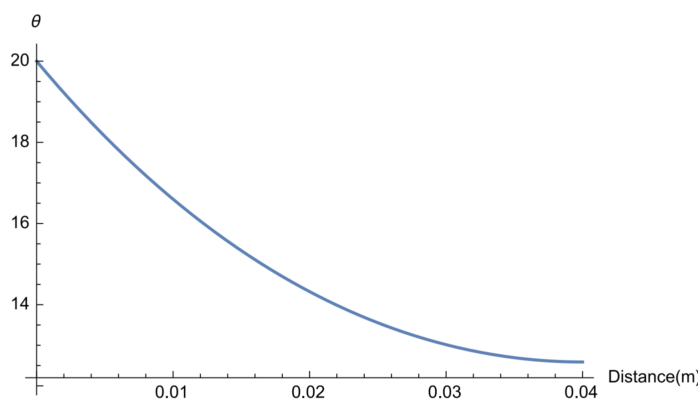


Figure 2.12: Temperature distribution in rectangular fin under adiabatic tip condition

Example 2.4.2 Non constant cross sectional problem *Refer to tutorial 2-4-1 and 2-4-2 Fin Solutions.nb.* A cylindrical fin with has constant heat conductivity $k = 3W/m.K$, width is $W = 0.01m$. The surface of the fin is exposed to ambient air at $10°C$ with a convection heat transfer coefficient $h = 10W/m^2K$. Plot the temperature distribution in the fin under prescribed tip condition, At start position $r_1 = 0.01m$, temperature is $T_1 = 30°C$. at the end $r_2 = 0.04m$, temperature $T_2 = 40°C$. (Shown in figure 2.13)

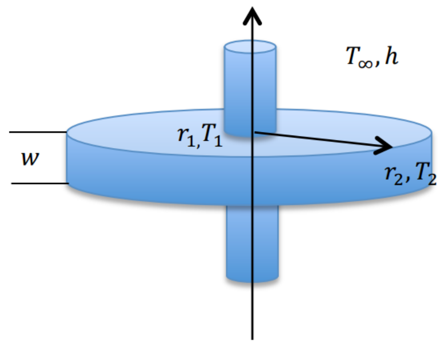


Figure 2.13: Cylindrical fin 2.4.2

Solution. For a cylindrical fin, the cross section area $A_c = 2\pi r w$, surface area $A_s = 2\pi(r^2 - r_1^2)$, replace x by r in

$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{x}\right)\frac{dT}{dx} - \left(\frac{1}{A_c}\frac{h}{k}\frac{dA_s}{x}\right)(T - T_\infty) = 0$$

and get

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} - \frac{2h}{kw}(T - T_\infty) = 0$$

with $m^2 \equiv 2h/kw and \theta = T - T_\infty$

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - m^2\theta = 0$$

For proscribed tip problem, boundary condition is

$$\theta(0.01) = T_1 - T_\infty \equiv \theta_1$$

$$\theta(0.04) = T_2 - T_\infty \equiv \theta_2$$

Solve this two step differential equation with Mathematica and get the sketch of temperature distribution as below in figure 2.14

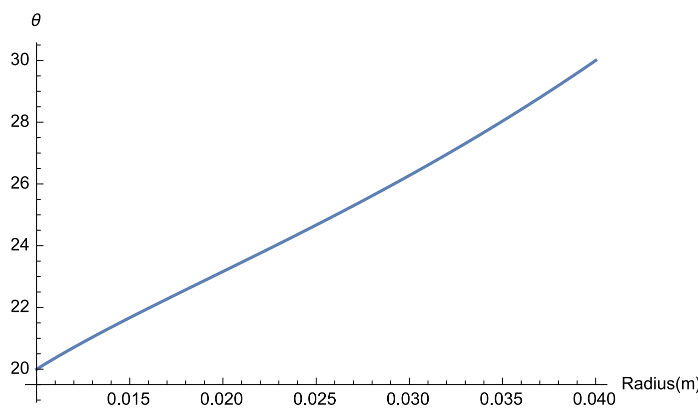


Figure 2.14: Temperature distribution in cylindrical fin under prescribed tip condition

Chapter 3

Two-dimensional Steady State problems

3.1 Overview

This section we mainly introduce the method to solve two dimensional steady state problems, the example given introduces how to calculate the temperature distribution in square steady heat conduction problem without internal heat generation, and how to calculate the heat flux in the square.

3.2 Exact Solution

Example 3.2.1 *Refer to 3-2-1Two dimenstional steady state not heat generation.nb.* A two-dimensional rectangular plate length and width is $L = 2$,$W = 1$, and on one boundary temperature $T_2 = 150\ K$, other three boundaries $T_2 = 50\ K$. The square is in steady state, sketch the temperature distribution and temperature at position $x = 1$ and $y = 0.5$, Calculate the heat flux when $y = 0$. (As shown in figure 3.1)

Solution. To simplify the solution, use below transformation

$$\theta = \frac{T - T_1}{T_2 - T_1}$$

According to heat diffusion equation, the two-dimensional, steady state with no internal heat generation equation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

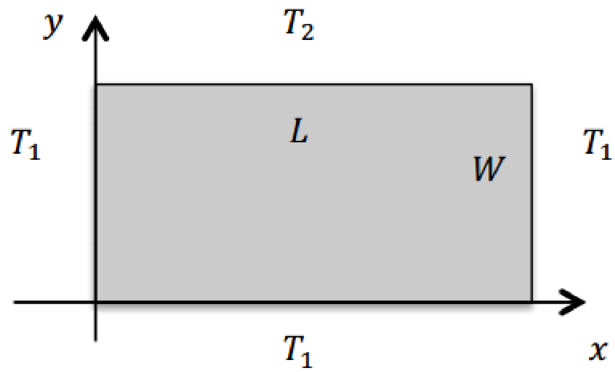


Figure 3.1: Model of Example 3.2.1

the transformed differential equation is then

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

Since the equation is second order in both x and y, two boundary conditions are needed for each of the coordinates, they are

$$\theta(0, y) = 0, \text{and} \theta(x, 0) = 0$$

$$\theta(L, y) = 0, \text{and} \theta(x, W) = 1$$

The $\theta(x, y)$ can be solved by SOV. The solution (also using Table from Osizik) can be shown to be:

$$\theta(x, y) = \frac{2}{\pi}\sum_{i=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\frac{n\pi x}{L}\frac{\sinh n\pi y/L}{\sinh n\pi W/L}$$

So that

$$T(x, y) = (T_2 - T_1)\frac{2}{\pi}\sum_{i=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\frac{n\pi x}{L}\frac{\sinh n\pi y/L}{\sinh n\pi W/L} + T_1$$

Plot 3D sketch of the temperature distribution in the rectangular is shown as Figure3.2

And by fixing x and y separately we can get the temperature distribution $T(1, y)$ as Figure 3.3 and $T(x, 0.5)$ as Figure 3.4 To calculate the heat flux passing through a unique section line, we use Fourier's equation

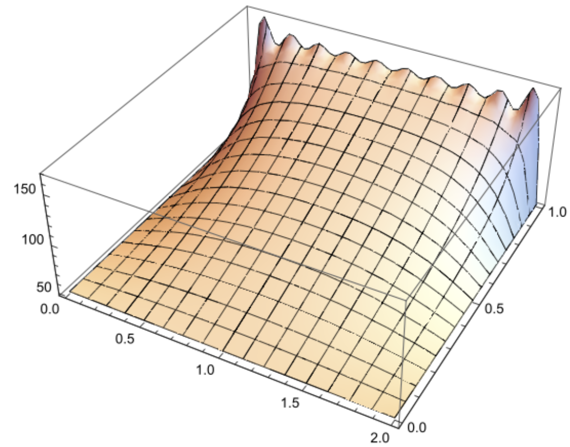


Figure 3.2: Temperature distribution in square with series method

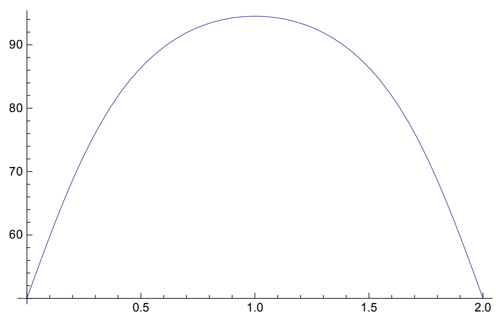


Figure 3.3: Temperature distribution in square at $x = 1$

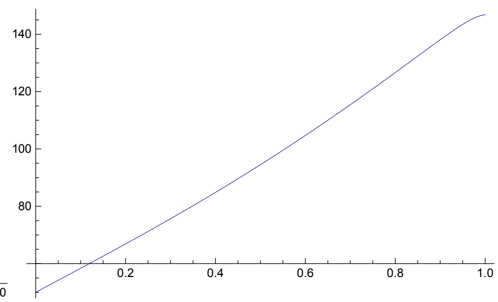


Figure 3.4: Temperature distribution in square at $y = 0.5$

$q = -kA(\partial T/\partial x \partial y)$ for heat rate at point (x, y). When $y = 0$, the heat flux passing through x direction in the square could be represented as

$$
\begin{aligned}
Q &= \int_0^L Aq \, \mathrm{d}x|_{y=0} \\
&= 2kA(T_2 - T_1) \sum_{n=1}^{\infty} \cosh \frac{(2n-1)\pi y}{2} \operatorname{csch} \frac{(2n-1)\pi}{2} \sin \frac{(2n-1)\pi x}{2}|_{y=0} \\
&= 8346.27
\end{aligned}
$$

Chapter 4

Transient conduction problems

4.1 Overview

This chapter mainly discussed how to solve transient problems, including lumped method, the SOV method and one term method.

4.2 The Lumped Method

Example 4.2.1 Lumped method and steady state *Refer to tutorial 4-2-1 Lumped and General Solution.nb* Consider a plane wall with consist internal temperature T_i was quickly put into an environment surrounded with symmetrically fluids with constant temperature T_∞. Wall thickness is $2L_c$. The density of the wall is $\rho = 921 kg/m^3$, the specific heat capacity is $c = 2100 J/kg \cdot K, k = 2.0 W/m \cdot K$. The covection coefficient between the junction surface and the gas is $h = 200 W/m^2 \cdot K$. Parameters are given below, calculate the temperature difference changed with time compared to the initial temperature difference. $L_c = 0.001m, Bi = 0.1$ (As shown in figure 4.1)

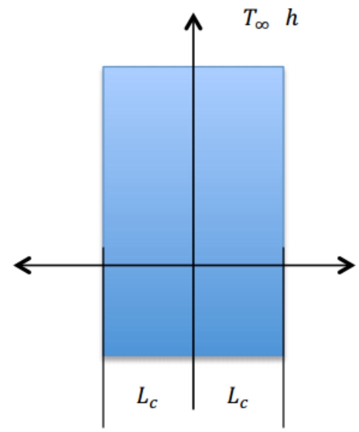


Figure 4.1: case 1 when $L_{c1} = 0.001m$

Solution.
As in this case,first caculateBi, $Bi \leq 0.1$, meet the requirment for using lump method. Based on lumped capacitance method, internal temperature of the wall is considered consistant; the plane wall with convection's temperature distribution is expressed as

$$-hA_s(T - T_\infty) = \rho V c \frac{dT}{dt}$$

Then get solution for lumped method

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp(-Bi \cdot Fo)$$

Where

$$Fo = \frac{\alpha t}{{L_c}^2}$$

$$\alpha = \frac{k}{\rho c}$$

Sketch the wall internal temperature change with time as figure 4.2

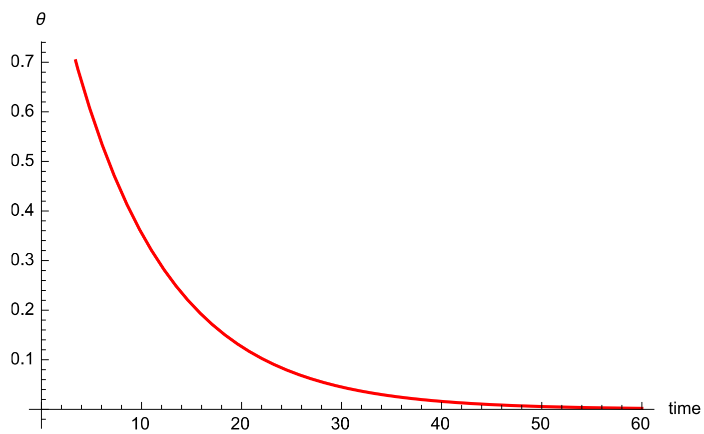


Figure 4.2: Plane wall lumped

4.3 Overview of SOV Method

Consider the one dimensioanl heat diffusion equation:

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$$

Based on the boundary conditions, select proper solution of

$$\text{Eigenfxn: } X(\beta_m, x)$$
$$\text{Eigenvalue: } \beta_m$$
$$\text{Norm: } N(\beta_m)$$

from the table listed in figure 4.3[1]. Apply initial condition

$$C(x,0) = \sum_m^\infty C_m X_n(\beta_m, x) = F(x)$$

Use orthogonality to get

$$C_m = \frac{\int_0^L F(x) X_m(x) dx}{N(\beta_m)}$$

[1]M. Necati Ozisik: Heat Conduction, 2nd Edition (John Wiley & Sons, 1993), 49.

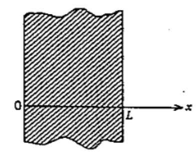
Exact solution

$$C(x,t) = \sum_{m}^{\infty} C_m X_m(\beta_m, x) \exp\left(-D{\beta_m}^2 t\right)$$

in table listed in 4.3

TABLE 2-2 The Solution $X(\beta_m, x)$, the Norm $N(\beta_m)$ and the Eigenvalues β_m of the Differential Equation

$$\frac{d^2X(x)}{dx^2} + \beta^2 X(x) = 0 \quad \text{in} \quad 0 < x < L$$



Subject to the Boundary Conditions Shown in the Table Below

No.	Boundary Condition at $x=0$	Boundary Condition at $x=L$	$X(\beta_m, x)$	$1/N(\beta_m)$	Eigenvalues β_m's are Positive Roots of
1	$-\frac{dX}{dx} + H_1 X = 0$	$\frac{dX}{dx} + H_2 X = 0$	$\beta_m \cos\beta_m x + H_1 \sin\beta_m x$	$2\left[(\beta_m^2 + H_1^2)\left(L + \frac{H_2}{\beta_m^2 + H_2^2}\right) + H_1\right]^{-1}$	$\tan\beta_m L = \frac{\beta_m(H_1 + H_2)}{\beta_m^2 - H_1 H_2}$
2	$-\frac{dX}{dx} + H_1 X = 0$	$\frac{dX}{dx} = 0$	$\cos\beta_m(L - x)$	$2\frac{\beta_m^2 + H_1^2}{L(\beta_m^2 + H_1^2) + H_1}$	$\beta_m \tan\beta_m L = H_1$
3	$-\frac{dX}{dx} + H_1 X = 0$	$X = 0$	$\sin\beta_m(L - x)$	$2\frac{\beta_m^2 + H_1^2}{L(\beta_m^2 + H_1^2) + H_1}$	$\beta_m \cot\beta_m L = -H_1$
4	$\frac{dX}{dx} = 0$	$\frac{dX}{dx} + H_2 X = 0$	$\cos\beta_m x$	$2\frac{\beta_m^2 + H_2^2}{L(\beta_m^2 + H_2^2) + H_2}$	$\beta_m \tan\beta_m L = H_2$
5	$\frac{dX}{dx} = 0$	$\frac{dX}{dx} = 0$	$^*\cos\beta_m x$	$\frac{2}{L}$ for $\beta_m \neq 0$; $\frac{1}{L}$ for $\beta_0 = 0^*$	$\sin\beta_m L = 0^*$
6	$\frac{dX}{dx} = 0$	$X = 0$	$\cos\beta_m x$	$\frac{2}{L}$	$\cos\beta_m L = 0$
7	$X = 0$	$\frac{dX}{dx} + H_2 X = 0$	$\sin\beta_m x$	$2\frac{\beta_m^2 + H_2^2}{L(\beta_m^2 + H_2^2) + H_2}$	$\beta_m \cot\beta_m L = -H_2$
8	$X = 0$	$\frac{dX}{dx} = 0$	$\sin\beta_m x$	$\frac{2}{L}$	$\cos\beta_m L = 0$
9	$X = 0$	$X = 0$	$\sin\beta_m x$	$\frac{2}{L}$	$\sin\beta_m L = 0$

*For this particular case $\beta_0 = 0$ is also an eigenvalue corresponding to $X = 1$.

Figure 4.3: SOV slab solution with different boundary condition

Example 4.3.1 SOV for slab *Refer to tutorial 4-3-1 SOV method.nb.* Consdider a slab under homogenous dirichlet boundary condition.$T_\infty = 0$,

$\alpha = 1$, length L, $T(x,0) = F(x) = x^2$, $T(0,t) = 0$, and $T(L,t) = 0$. Please use Ozisik Table in figure 4.3

Solution.
First start with the general solution for 1-dimensional homogenuous boundary condition solution. The homogenuous boudary condition for the fod is as below:

$$\begin{cases} X = 0 & , x = 0 \\ X = 0 & , x = L \end{cases} \tag{4.1}$$

So that from the table in the figure below as figure 4.3
It is condition 9, based on the exact solution of SOV method

$$C(x,t) = \sum_{m}^{\infty} C_m X_m(\beta_m, x) \exp\left(-D{\beta_m}^2 t\right)$$

Where

$$C_m = \frac{\int_0^L F(x) X_m(x) dx}{N(\beta_m)}$$

And imply the initial condition that $T(x,0) = F(x) = x^2$

$$T(x,t) = \underbrace{\frac{2}{L}}_{\frac{1}{N\beta m}} \sum_{m=1}^{\infty} \exp^{-\alpha t\left(\frac{m\pi}{L}\right)} \underbrace{\sin\left(xm\frac{\pi}{L}\right)}_{X(\beta_m, x)} \int_0^L y^2 \sin\left(ym\frac{\pi}{L}\right) dy$$

And sketch the temperature distribution with time as figure 4.4

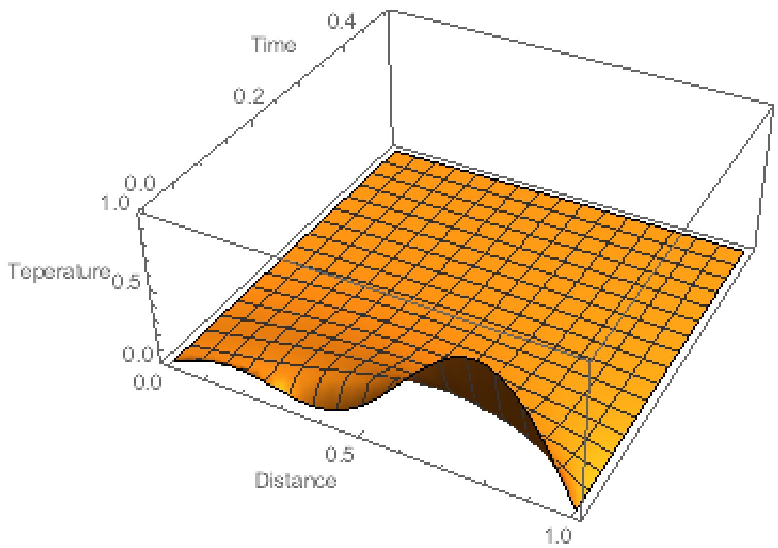


Figure 4.4: SOV solve result for slab under homogenuous dirichlet condition

4.4 The One Term Method

Example 4.4.1 Transient one term solution *Refer to tutorial 4-2-1 Lumped and General Solution.nb* Consider the wall thickness in Example 4.2.1, get the rest of the constants and properties for the previous problem, please answer the following case where the half thickness is changed to $L = 0.1$ m, and $Bi = 10$. Please plot the centerline solutions over time.

Solution. First calculate Bi, and F_o, $L_c = 0.1$.

$$Bi = 10$$

$$Fo = -\frac{\alpha t}{L_c^2} = 2.27496$$

Hence based on the table listed in figure 4.5[2] , the figure is also in IntroHT.pdf solving this problem with SOV method, one obtains the exact solution

$$\theta^* = \frac{T - T_\infty}{T_i - T_\infty} = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) \cos(\zeta_n x^*)$$

[2]Theodore L. Bergman, Adrienne S. Lavine, Frank P. Incropera and David P. Dewitt: Fundamentals of Heat and Mass Transfer, 7th Edition (Wiley 2002),301.

Where $x^* = x/L$. For values of $Fo > 0.2$,the infinite series solution, $\theta*$ could be approximated by the first term of the series, $n = 1$ the form of the temperature distribution becomes

$$\theta^* = C_1 \exp(-\zeta_1^2 Fo) \cos(\zeta_1 x^*)$$

TABLE 5.1 *Coefficients used in the one-term approximation to the series solutions for transient one-dimensional conduction*

	Plane Wall		*Infinite Cylinder*		*Sphere*	
Bi^a	ζ_1 (rad)	C_1	ζ_1 (rad)	C_1	ζ_1 (rad)	C_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.03	0.1732	1.0049	0.2439	1.0075	0.2989	1.0090
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.05	0.2217	1.0082	0.3142	1.0124	0.3852	1.0149
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.07	0.2615	1.0114	0.3708	1.0173	0.4550	1.0209
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.09	0.2956	1.0145	0.4195	1.0222	0.5150	1.0268
0.10	0.3111	1.0160	0.4417	1.0246	0.5423	1.0298
0.15	0.3779	1.0237	0.5376	1.0365	0.6608	1.0445
0.20	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.25	0.4801	1.0382	0.6856	1.0598	0.8448	1.0737
0.30	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0932	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0185	1.1346	1.2644	1.1713
0.7	0.7506	1.0919	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1725	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1795	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7201
5.0	1.3138	1.2402	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8674
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8921
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2881	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5707	1.2733	2.4050	1.6018	3.1415	2.0000

[a] $Bi = hL/k$ for the plane wall and hr_o/k for the infinite cylinder and sphere. See Figure 5.6.

Figure 4.5: One term coefficients

Hence based on the table listed in figure 4.5 , $\zeta_1 = 0.4801$, $C_1 = 1.0382$,sketch the centerline temperature with time θ_0^* as figure 4.6

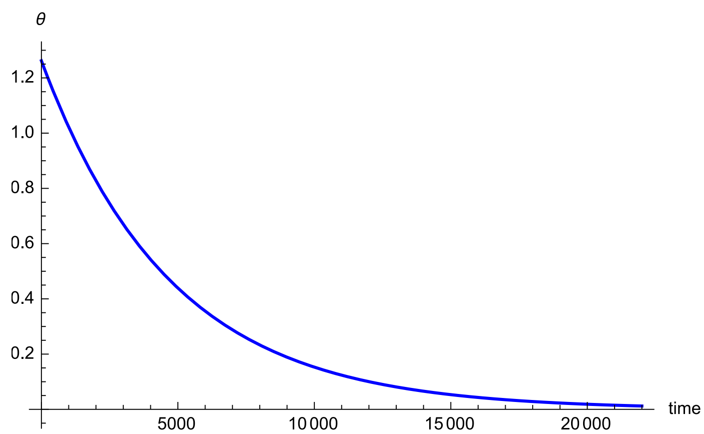


Figure 4.6: One term result on centerline

Use lumped method to solve the problem, and compare the sketch of lumped method result with one term result in figure 4.7. Clearly, the lumped solution no longer can capture the heat loss. It overpredicts the loss.

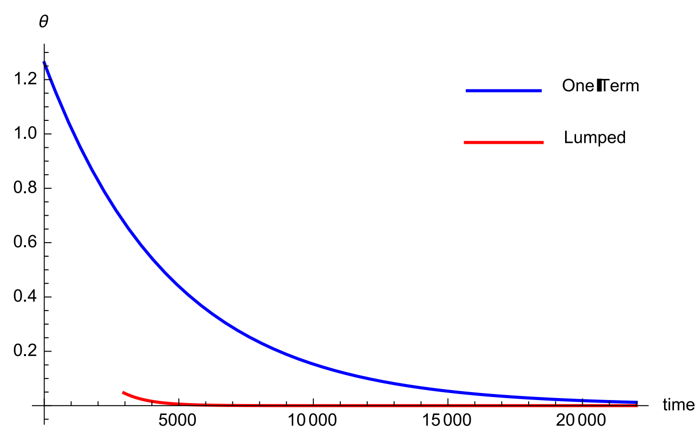


Figure 4.7: One term compare with lumped