A Bilinear Version of weighted Hardy Operator *

LIU Xia[†], JIANG Yin-sheng[‡]

(College of Mathematics and System Sciences, Xinjiang University,Urumqi, Xinjiang 830046,China)

Abstract: The authors introduce a class bilinear weighted Hardy operator and obtain that the operator is bounded from $L^{p_1} \times L^{p_2}$ to L^p with $p, p_1, p_2 \in [1, \infty]$.

Keywords: Bilinear Weighted Hardy operator, $L^{p_1} \times L^{p_2}$.

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双线性型 Hardy 算子

刘霞，江寅生

(新疆大学 数学与系统科学学院, 新疆 乌鲁木齐 830046)

摘　要: 介绍了一类双线性型加权 Hardy 算子，对于 $p, p_1, p_2 \in [1, \infty]$， 证明了 Hardy 算子是映$L^{p_1} \times L^{p_2}$到 L^p 有界的.

关键词: 双线性型 Hardy 算子, $L^{p_1} \times L^{p_2}$.

0 Introduction

Let f be in $L^p(\mathbb{R}^1)$ and the Hardy operator U be defined by

$$U(f)(x) = \frac{1}{x}\int_0^x f(t)\mathrm{d}t,\ x \neq 0.$$

A celebrated Hardy integral inequality [1] can be formulated as

$$\|Uf\|_{L^p(\mathbb{R}^1)} \leq \frac{p}{p-1}\|f\|_{L^p(\mathbb{R}^1)}. \tag{1}$$

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†**Biography:** LIU Xia(1986-), female, master.

‡**Corresponding author:**JIANG Yin-sheng, ysjiang@xju.edu.cn

where $1 < p < \infty$ and the constant $\frac{p}{p-1}$ is the best possible. In 1995, Christ and Grafakos[2] generalize this operators to n-dimensional version,

$$U(f)(x) = \frac{1}{|x|} \int_{|y| \le |x|} f(y) \mathrm{d}y, \ x \in \mathbb{R}^n \backslash \{0\},$$

and they also obtained the inequality and the best constant $\frac{p}{p-1}$.

Let $\psi : [0,1] \to [0, \infty)$ be a function, and let f be a measurable complex-valued function on $\mathbb{R}^n$. Carton-Lebrun and Fosset [3] defined the weighted Hardy operator U_ψ by

$$U_\psi(f)(x) = \int_0^1 f(tx)\psi(t)\mathrm{d}t, \ x \in \mathbb{R}^n.$$

and they asserted that if $t^{1-n}\psi(t)$ is bounded on $[0, 1]$ then U_ψ is bounded on BMO. In 2001, Xiao [4] sharped and extended this result, more precisely.

Theorem A Let $\psi : [0,1] \to [0, \infty)$ be a function and let $p \in (1, \infty]$. Then

(i) $U_\psi : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ exists as a bounded operator if and only if

$$\int_0^1 t^{-n/p}\psi(t)\mathrm{d}t < \infty. \tag{2}$$

Moreover, when (2) holds, the operator norm of U_ψ on $L^p(\mathbb{R}^n)$ is given by

$$\|U_\psi\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} = \int_0^1 t^{-n/p}\psi(t)\mathrm{d}t.$$

(ii) $U_\psi : BMO(\mathbb{R}^n) \to BMO(\mathbb{R}^n)$ exists as a bounded operator if and only if

$$\int_0^1 \psi(t)\mathrm{d}t < \infty. \tag{3}$$

Moreover, when (3) holds, the operator norm of U_ψ on $BMO(\mathbb{R}^n)$ is given by

$$\|U_\psi\|_{BMO(\mathbb{R}^n) \to BMO(\mathbb{R}^n)} = \int_0^1 \psi(t)\mathrm{d}t.$$

Obviously, U_ψ is just the Hardy operator if $\psi \equiv 1$. This operator seems to be of interest as it is related closely to the Hardy-Littlewood maximal operators in harmonic analysis.

Next, we will introduce a bilinear version of this operator. Let $\Psi : [0,1] \times [0,1] \to [0, \infty)$ be a function, and let f_1 and f_2 be measurable complex-valued functions on $\mathbb{R}^n$. We can define the bilinear operator by

$$U_\Psi(f_1, f_2)(x) = \int_0^1 \int_0^1 f_1(sx) f_2(tx)\Psi(s,t)\mathrm{d}s\mathrm{d}t. \tag{4}$$

For this operator, we have the following theorem as our main result.

Theorem 1 Let $\Psi : [0,1] \times [0,1] \to [0, \infty)$ be a function and $1 \le p_1, p_2, p \le \infty$. U_Ψ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ if and only if

$$\int_0^1 \int_0^1 s^{-n/p_1} t^{-n/p_2} \Psi(s,t)\mathrm{d}s\mathrm{d}t < \infty. \tag{5}$$

meanwhile, the norm $\|U_\Psi\|_{L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)}$ is just the integration in (5).

Remark 1 When $\frac{1}{2} \leq p < 1$, we cann't assure that whether U_Ψ is also bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. So this is an open problem.

Remark 2 We can also obtain that U_Ψ is bounded from $L^\infty(\mathbb{R}^n)\times L^\infty(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$ if and only if

$$\int_0^1 \int_0^1 \Psi(s,t)\mathrm{d}s\mathrm{d}t < \infty.$$

and $\|U_\Psi\|_{L^\infty(\mathbb{R}^n)\times L^\infty(\mathbb{R}^n)\to BMO(\mathbb{R}^n)} = \int_0^1 \int_0^1 \Psi(s,t)\mathrm{d}s\mathrm{d}t$. Here the details left to readers.

Remark 3 We can also define V_Ψ as the bilinear weighted Cesàro average:

$$V_\Psi(f_1,f_2)(x) = \int_0^1 \int_0^1 f_1(x/s)f_2(x/t)(st)^{-n}\Psi(s,t)\mathrm{d}s\mathrm{d}t.$$

We also assert that V_Ψ bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 \leq p, p_1, p_2 \leq \infty$. But the adjoint relation will be not valid between V_Ψ and U_Ψ.

1 Proof of main result

It is suffice to prove the case $p_1 < \infty$ and $p_2 < \infty$. On the one hand, we suppose (2) holds, Minkowski's inequality and Hölder inequality tell us

$$\begin{aligned}
\|U_\Psi(f_1,f_2)\|_{L^p(\mathbb{R}^n)} &\leq \int_0^1 \int_0^1 \left(\int_{\mathbb{R}^n} |f_1(sx)f_2(tx)|^p \mathrm{d}x\right)^{1/p} \Psi(s,t)\mathrm{d}s\mathrm{d}t \\
&\leq \int_0^1 \int_0^1 \left(\int_{\mathbb{R}^n} |f_1(sx)|^{p_1} \mathrm{d}x\right)^{1/p_1} \left(\int_{\mathbb{R}^n} |f_2(tx)|^{p_2} \mathrm{d}x\right)^{1/p_2} \Psi(s,t)\mathrm{d}s\mathrm{d}t \\
&= \|f_1\|_{L^{p_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2}(\mathbb{R}^n)} \int_0^1 \int_0^1 s^{-n/p_1}t^{-n/p_2}\Psi(s,t)\mathrm{d}s\mathrm{d}t.
\end{aligned}$$

Thus U_Ψ is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, and $\|U_\Psi\|_{L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \leq \int_0^1 \int_0^1 s^{-n/p_1}t^{-n/p_2}\Psi(s,t)\mathrm{d}s\mathrm{d}t$.

On the other hand, we assume that there exist a positive constant C such that

$$\|U_\Psi(f_1,f_2)\|_{L^p(\mathbb{R}^n)} \leq C\|f_1\|_{L^{p_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2}(\mathbb{R}^n)}. \tag{6}$$

For any $\epsilon > 0$, take

$$f_1^\epsilon(x) = \begin{cases} 0, & |x| \leq 1, \\ |x|^{-\frac{n}{p_1}-\epsilon}, & |x| > 1. \end{cases}$$

and

$$f_2^\epsilon(x) = \begin{cases} 0, & |x| \leq 1, \\ |x|^{-\frac{n}{p_2}-\epsilon}, & |x| > 1. \end{cases}$$

Then $\|f_1^\epsilon\|^{p_1}_{L^{p_1}(\mathbb{R}^n)} = \frac{c_n}{p_1\epsilon}$ and $\|f_2^\epsilon\|^{p_2}_{L^{p_2}(\mathbb{R}^n)} = \frac{c_n}{p_2\epsilon}$, where $c_n = \frac{n\pi^{n/2}}{\Gamma(1+\frac{n}{2})}$. A trivial computation leads to that

$$U_\Psi(f_1^\epsilon, f_2^\epsilon)(x) = \begin{cases} 0, & |x| \leq 1, \\ |x|^{-\frac{n}{p}-2\epsilon} \int_{1/|x|}^1 \int_{1/|x|}^1 s^{-\frac{n}{p_1}-\epsilon}t^{-\frac{n}{p_2}-\epsilon}\Psi(s,t)\mathrm{d}s\mathrm{d}t, & |x| > 1. \end{cases}$$

Putting $\delta = \frac{1}{\epsilon} > 1$. Applying the equality for the functions f_1^ϵ and f_2^ϵ, we have that

$$\begin{aligned}
[C\|f_1^\epsilon\|_{L^{p_1}(\mathbb{R}^n)}\|f_2^\epsilon\|_{L^{p_2}(\mathbb{R}^n)}]^p &\geq [\|U_\Psi(f_1^\epsilon, f_2^\epsilon)\|_{L^p(\mathbb{R}^n)}]^p \\
&= \int_{|x|>1}\left(|x|^{-\frac{n}{p}-2\epsilon}\int_{1/|x|}^1\int_{1/|x|}^1 s^{-\frac{n}{p_1}-\epsilon}t^{-\frac{n}{p_2}-\epsilon}\Psi(s,t)\mathrm{d}s\mathrm{d}t\right)^p\mathrm{d}x \\
&\geq \int_{|x|>\delta}\left(|x|^{-\frac{n}{p}-2\epsilon}\int_{1/\delta}^1\int_{1/\delta}^1 s^{-\frac{n}{p_1}-\epsilon}t^{-\frac{n}{p_2}-\epsilon}\Psi(s,t)\mathrm{d}s\mathrm{d}t\right)^p\mathrm{d}x \\
&= \int_{|x|>\delta}|x|^{-n-2p\epsilon}\mathrm{d}x\left(\int_{1/\delta}^1\int_{1/\delta}^1 s^{-\frac{n}{p_1}-\epsilon}t^{-\frac{n}{p_2}-\epsilon}\Psi(s,t)\mathrm{d}s\mathrm{d}t\right)^p \\
&= \frac{c_n}{2p\epsilon}\left(\delta^{-2\epsilon}\int_{1/\delta}^1\int_{1/\delta}^1 s^{-\frac{n}{p_1}-\epsilon}t^{-\frac{n}{p_2}-\epsilon}\Psi(s,t)\mathrm{d}s\mathrm{d}t\right)^p.
\end{aligned}$$

which implies that

$$\int_{1/\delta}^1\int_{1/\delta}^1 s^{-\frac{n}{p_1}-\epsilon}t^{-\frac{n}{p_2}-\epsilon}\Psi(s,t)\mathrm{d}s\mathrm{d}t \leq \frac{Cp^{1/p}}{p_1^{1/p_1}p_2^{1/p_2}}\epsilon^{2\epsilon}.$$

The limit with $\epsilon \to 0$ reaches (5).

The last thing is to verify that $\|U_\Psi\|_{L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \int_0^1\int_0^1 s^{-n/p_1}t^{-n/p_2}\Psi(s,t)\mathrm{d}s\mathrm{d}t$. If there exists a positive number K such that

$$\|U_\Psi\|_{L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \leq K < \int_0^1\int_0^1 s^{-n/p_1}t^{-n/p_2}\Psi(s,t)\mathrm{d}s\mathrm{d}t. \tag{7}$$

Using the same process with the second step, we obtain that

$$\begin{aligned}
[K\|f_1^\epsilon\|_{L^{p_1}(\mathbb{R}^n)}\|f_2^\epsilon\|_{L^{p_2}(\mathbb{R}^n)}]^p &\geq [\|U_\Psi(f_1^\epsilon, f_2^\epsilon)\|_{L^p(\mathbb{R}^n)}]^p & (8)\\
&\geq \frac{c_n}{2p\epsilon}\left(\delta^{-2\epsilon}\int_{1/\delta}^1\int_{1/\delta}^1 s^{-\frac{n}{p_1}-\epsilon}t^{-\frac{n}{p_2}-\epsilon}\Psi(s,t)\mathrm{d}s\mathrm{d}t\right)^p. & (9)
\end{aligned}$$

This implies

$$\int_0^1\int_0^1 s^{-\frac{n}{p_1}-\epsilon}t^{-\frac{n}{p_2}-\epsilon}\Psi(s,t)\mathrm{d}s\mathrm{d}t \leq \frac{Kp^{1/p_1}p^{1/p_2}}{p_1^{1/p_1}p_2^{1/p_2}} \leq K. \tag{10}$$

We know (7) contradicts (10). Therefore

$$\|U_\Psi\|_{L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \int_0^1\int_0^1 s^{-n/p_1}t^{-n/p_2}\Psi(s,t)\mathrm{d}s\mathrm{d}t.$$

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