Some estimates for weighted Hardy-Littlewood averages *

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Abstract: The authors studied the boundedness of the weighted Hardy-Littlewood averages on λ-center Campanoto type spaces and weighted λ-center Morrey type spaces, and then obtained the responding operator norms.

Keywords Weighted Hardy-Littlewood average; λ-center Campanoto spaces; weighted λ-center Morrey spaces.

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1 Introduction

A celebrated Hardy integral inequality is formulated as follows [1],

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx\right)^{\frac{1}{p}} \leq \frac{p}{p-1}\left(\int_0^\infty |f(t)|^p dt\right)^{\frac{1}{p}},\ 1<p<\infty, \tag{1}$$

and the constant $\frac{p}{p-1}$ is best possible. In 1995, a n-dimensional version of Hardy operator was defined by Christ and Grafakos [2],

$$\mathcal{H}f(x) = \frac{1}{v_n|x|^n}\int_{|y|\leq|x|} f(y)dy, \quad x\in\mathbb{R}^n\backslash\{0\},$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$ and it also bounds on $L^p(\mathbb{R}^n)$ with the norm

$$\|\mathcal{H}f\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \frac{p}{p-1}, \quad 1<p<\infty.$$

In 1984, Carton-Lebrun and Fosset [3] introduced a class of weighted Hardy-Littlewood average U_φ. Let $\varphi : [0,1] \to [0,\infty]$ be a function. If f is a measurable complex-valued function on $\mathbb{R}^n$, one then defines the weighted Hardy-Littlewood average U_φ as

$$\big(U_\varphi f\big)(x) = \int_0^1 f(tx)\varphi(t)dt, \qquad x\in\mathbb{R}^n.$$

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When $\varphi \equiv 1$ and $n = 1$, U_φ may be reduced to the Hardy operator H,

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt, \qquad x \neq 0.$$

It is noticed that U_φ is different from $\mathcal{H}$ when $n > 1$ even though in the case $\varphi \equiv 1$.

In [4], Xiao obtained that the weighted Hardy-Littlewood average U_φ is bounded on $L^p(\mathbb{R}^n), 1 \leq p \leq \infty$, if and only if

$$\int_0^1 t^{-\frac{n}{p}}\varphi(t)dt < \infty.$$

Moreover,

$$\|U_\varphi f\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \int_0^1 t^{-\frac{n}{p}}\varphi(t)dt.$$

Xiao also showed that U_φ is bounded on $BMO(\mathbb{R}^n)$, if and only if

$$\int_0^1 \varphi(t)dt < \infty$$

Moreover,

$$\|U_\varphi f\|_{BMO(\mathbb{R}^n)\to BMO(\mathbb{R}^n)} = \int_0^1 \varphi(t)dt.$$

Definition 1 Let $\lambda < 1$ and $1 \leq q < \infty$. For all functions $f \in L^q_{loc}(\mathbb{R}^n)$, we can define the spaces of Campanato types $\mathcal{E}^{q,\lambda}(\mathbb{R}^n)$ and $\mathcal{E}^{q,\lambda}_*(\mathbb{R}^n)$ as follows respectively,

$$\|f\|_{\mathcal{E}^{q,\lambda}(\mathbb{R}^n)} = \sup_{B\subset\mathbb{R}^n}\frac{1}{|B|^{\lambda/n}}\left(\frac{1}{|B|}\int_B |f(x) - f_B|^q dx\right)^{\frac{1}{q}} < \infty,$$

and

$$\|f\|_{\mathcal{E}^{q,\lambda}_*(\mathbb{R}^n)} = \sup_{B\subset\mathbb{R}^n}\frac{1}{|B|^{\lambda/n}}\left(\frac{1}{|B|}\int_B [f(x) - \operatorname*{ess\,inf}_{y\in B} f(y)]^q dx\right)^{\frac{1}{q}} < \infty.$$

In [5], Zhao et.al considered the sharp bounds for weighted Hardy-Littlewood average operators on the spaces of Campanato types, i.e., U_φ is bounded operator on $\mathcal{E}^{q,\lambda}(\mathbb{R}^n)$ if and only if

$$\int_0^1 t^\lambda\varphi(t)dt < \infty.$$

Moreover,

$$\|U_\varphi f\|_{\mathcal{E}^{q,\lambda}(\mathbb{R}^n)\to\mathcal{E}^{q,\lambda}(\mathbb{R}^n)} = \int_0^1 t^\lambda\varphi(t)dt.$$

where $1 \leq q < \infty, -\frac{n}{q} < \lambda < 1$. So is on $\mathcal{E}^{q,\lambda}_*(\mathbb{R}^n)$.

In 2000, Alvarez, Guzman-Partida and Lakey [6] introduced the λ central bounded mean oscillation spaces and central Morrey spaces.

Definition 2 Let $\lambda < 1$ and $1 \leq q < \infty$. We say that a function $f \in L^q_{loc}(\mathbb{R}^n)$ belongs to λ central bounded mean oscillation space $CBMO^{q,\lambda}(\mathbb{R}^n)$, if it satisfies

$$\|f\|_{CBMO^{q,\lambda}(\mathbb{R}^n)} = \sup_{R>0} \frac{1}{|B(0,R)|^{\lambda/n}} \left(\frac{1}{|B(0,R)|} \int_{B(0.R)} |f(x) - f_{B(0,R)}|^q dx \right)^{\frac{1}{q}} < \infty.$$

Remark 1 $\mathcal{E}^{q,\lambda}(\mathbb{R}^n) \subset CBMO^{q,\lambda}(\mathbb{R}^n)$.

We will introduce λ center BLO spaces $CBLO^{q,\lambda}(\mathbb{R}^n)$.

Definition 3 For $1 \leq q < \infty$ and $\lambda \in \mathbb{R}$. A locally integrable function f is said to belong to $CBLO^{q,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{CBLO^{q,\lambda}(\mathbb{R}^n)} = \sup_{R>0} \frac{1}{|B(0,R)|^{\lambda/n}} \left(\frac{1}{|B(0,R)|} \int_{B(0.R)} |f(x) - \inf_{y \in B(0,R)} f(y)|^q dx \right)^{\frac{1}{q}} < \infty.$$

Remark 2 *$\mathcal{E}^{q,\lambda}_*(\mathbb{R}^n) \subset CBLO^{q,\lambda}(\mathbb{R}^n)$ and $CBLO^{q,\lambda}(\mathbb{R}^n) \subset CBMO^{q,\lambda}(\mathbb{R}^n)$.*

Definition 4 For a weight ω, let $1 \leq q < \infty$ and $\lambda \in \mathbb{R}$, the weighted central Morrey space $B^{q,\lambda}(\omega)$ is defined by

$$B^{q,\lambda}(\omega) = \left\{ f \in L^q_{loc}(\omega) : \|f\|_{B^{q,\lambda}(\omega)} < \infty \right\}$$

where

$$\|f\|_{B^{q,\lambda}(\omega)} = \sup_{R>0} \frac{1}{\omega(B(0,R))^{\lambda/n}} \left(\frac{1}{\omega(B(0,R))} \int_{B(0.R)} |f(x)|^q \omega(x) dx \right)^{\frac{1}{q}}$$

Our main theorems are formulated as follows.

Theorem 1 *Let $\varphi : [0,1] \to [0,\infty]$ be a function and $1 \leq q < \infty, -\frac{n}{q} \leq \lambda < 1$. Then U_φ is a bounded operator on $CBMO^{q,\lambda}(\mathbb{R}^n)$ if and only if*

$$\int_0^1 t^\lambda \varphi(t) dt < \infty.$$

Moreover,

$$\|U_\varphi f\|_{CBMO^{q,\lambda}(\mathbb{R}^n) \to CBMO^{q,\lambda}(\mathbb{R}^n)} = \int_0^1 t^\lambda \varphi(t) dt.$$

Remark 3 *The case $0 \leq \lambda < 1$ is proved in the paper [7].*

Theorem 2 *Let $\varphi : [0,1] \to [0,\infty]$ be a function and $1 \leq q < \infty, -\frac{n}{q} \leq \lambda < 1$. Then U_φ is a bounded operator on $CBLO^{q,\lambda}(\mathbb{R}^n)$ if and only if*

$$\int_0^1 t^\lambda \varphi(t) dt < \infty.$$

Moreover,

$$\|U_\varphi\|_{CBLO^{q,\lambda}(\mathbb{R}^n) \to CBLO^{q,\lambda}(\mathbb{R}^n)} = \int_0^1 t^\lambda \varphi(t) dt.$$

Theorem 3 *Let $1 \leq q < \infty$, $-\frac{n}{q} \leq \lambda < 0$ and $-n < \alpha$. Then*

$$\|U_\varphi\|_{B^{q,\lambda}(|x|^\alpha dx)} \leq C\|f\|_{B^{q,\lambda}(|x|^\alpha dx)},$$

if and only if

$$\int_0^1 t^{\frac{n+\alpha}{n}\lambda}\varphi(t)dt < \infty,$$

Therefore,

$$\|U_\varphi\|_{B^{q,\lambda}(|x|^\alpha dx)\to B^{q,\lambda}(|x|^\alpha dx)} = \int_0^1 t^{\frac{n+\alpha}{n}\lambda}\varphi(t)dt.$$

Remark 4 *If $\varphi(t) = 1$ and $n = 1$, U_φ is just reduced to the classical Hardy operator $\mathcal{H}$. If $\varphi(t) = nt^{n-1}, n \geq 2$ and f is a radial function, then U_φ is just reduced to the n-dimensional Hardy operator $\mathcal{H}$*

$$\mathcal{H}f(x) = \int_0^1 f(tx)nt^{n-1}dt.$$

Therefore, denote by

$$CBMO_*^{q,\lambda}(\mathbb{R}^n) = \{f : f \text{ is radial and } f \in CBMO^{q,\lambda}(\mathbb{R}^n)\},$$

$$CBLO_*^{q,\lambda}(\mathbb{R}^n) = \{f : f \text{ is radial and } f \in CBLO^{q,\lambda}(\mathbb{R}^n)\},$$

and

$$B_*^{q,\lambda}(\omega) = \{f : f \text{ is radial and } f \in B^{q,\lambda}(\omega)\}.$$

Corollary 1 *.Let $1 \leq q < \infty, -\frac{n}{q} \leq \lambda < 1$. Then $\mathcal{H}$ is a bounded operator on $CBMO_*^{q,\lambda}(\mathbb{R}^n)$. Moreover,*

$$\|\mathcal{H}f\|_{CBMO_*^{q,\lambda}(\mathbb{R}^n)\to CBMO_*^{q,\lambda}(\mathbb{R}^n)} = \frac{n}{n+\lambda}.$$

Especially, for $\lambda = 0$, $\|\mathcal{H}f\|_{CBMO_*^{q,0}(\mathbb{R}^n)\to CBMO_*^{q,0}(\mathbb{R}^n)} = 1$.

Corollary 2 *.Let $1 \leq q < \infty, -\frac{n}{q} \leq \lambda < 1$. Then $\mathcal{H}$ is a bounded operator on $CBLO_*^{q,\lambda}(\mathbb{R}^n)$. Moreover,*

$$\|\mathcal{H}\|_{CBLO_*^{q,\lambda}(\mathbb{R}^n)\to CBLO_*^{q,\lambda}(\mathbb{R}^n)} = \frac{n}{n+\lambda}.$$

Especially, for $\lambda = 0$, $\|\mathcal{H}\|_{CBLO_*^{q,\lambda}(\mathbb{R}^n)\to CBLO_*^{q,\lambda}(\mathbb{R}^n)} = 1$.

Corollary 3 *For $1 \leq q < \infty$, $-n/q \leq \lambda \leq 0$ and $\alpha < -\frac{n^2}{\lambda} - n$, then*

$$\|\mathcal{H}f\|_{B_*^{q,\lambda}(|x|^\alpha dx)} \leq C\|f\|_{B_*^{q,\lambda}(|x|^\alpha dx)},$$

Moreover,

$$\|\mathcal{H}\|_{B_*^{q,\lambda}(|x|^\alpha dx)\to B_*^{q,\lambda}(|x|^\alpha dx)} = \frac{n^2}{n^2+(n+\alpha)\lambda}.$$

For the case $q = 1$, we have the following result for n-dimension Hardy operators.

Theorem 4 *Let $-\frac{n}{q} \leq \lambda < 1$. Hardy operator $\mathcal{H}$ is bounded on $CBMO^{1,\lambda}(\mathbb{R}^n)$ and $CBLO^{1,\lambda}(\mathbb{R}^n)$. And the best possible constant is $\frac{n}{n+\lambda}$.*

Theorem 5 *Let $-n \leq \lambda < 0$ and $-n < \alpha < -\frac{n^2}{\lambda} - n$, then*

$$\|\mathcal{H}f\|_{B^{1,\lambda}(|x|^\alpha dx)} \leq C\|f\|_{B^{1,\lambda}(|x|^\alpha dx)},$$

Moreover,

$$\|\mathcal{H}\|_{B^{1,\lambda}(|x|^\alpha dx)\to B^{1,\lambda}(|x|^\alpha dx)} = \frac{n^2}{n^2 + (n+\alpha)\lambda}.$$

Remark 5 *In this case $q = 1$, our results base on the fact that the operator $\mathcal{H}$ and its restriction to radial functions have the same operator norm on $CBMO(\mathbb{R}^n)$, $CBLO(\mathbb{R}^n)$ and $B^{1,\lambda}(\omega)$. When $q > 1$ and $\alpha = 0$, the related result refers to [8]. But for $q > 1$, we cannot assure that whether the above fact is also true. An important gap will be showed in the last part of the next section.*

2 Proofs of main results

Proof of Theorem 1. The proof of this theorem is showed in [7] for the case $0 \leq \lambda < 1$. We still go to verify that this result is valid for $-n/q \leq \lambda < 0$.

We also take the function $f_0(x) = |x|^\lambda, x \in \mathbb{R}^n$, we have

$$\begin{aligned}
&\frac{1}{|B(0,R)|^{\frac{\lambda}{n}}}\left(\frac{1}{|B(0,R)|}\int_{B(0,R)}|x|^{\lambda q}dx\right)^{\frac{1}{q}} \\
&= \frac{1}{|B(0,R)|^{\frac{\lambda}{n}}}\left(\frac{1}{|B(0,R)|}\int_0^R \omega_{n-1}\cdot\rho^{\lambda q+n-1}d\rho\right)^{\frac{1}{q}} \\
&= \frac{1}{|B(0,R)|^{\frac{\lambda}{n}}}\frac{1}{|B(0,R)|^{\frac{1}{q}}}\left(\omega_{n-1}\cdot\frac{1}{n+\lambda q}R^{\lambda q+n}\right)^{\frac{1}{q}} \\
&= \left(\frac{n}{n+\lambda q}\right)^{\frac{1}{q}}(\omega_n)^{-\frac{\lambda}{n}},
\end{aligned}$$

where $\omega_{n-1} = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ and $\omega_{n-1} = n\omega_n$. Therefor, $f_0 \in CBMO^{q,\lambda}(\mathbb{R}^n)$. Moreover

$$U_\varphi(f_0)(x) = \int_0^1 t^\lambda\varphi(t)dtf_0(x),$$

so, $\|U_\varphi\|_{CBMO^{q,\lambda}(\mathbb{R}^n)\to CBMO^{q,\lambda}(\mathbb{R}^n)} = \int_0^1 t^\lambda\varphi(t)dt.$

Proof of Theorem 2. Similar to the proof of Theorem 1. In fact

$$\begin{aligned}
&\frac{1}{|B(0,R)|^{\frac{\lambda}{n}}}\left(\frac{1}{|B(0,R)|}\int_{B(0,R)}\left[U_\varphi f(y) - \inf_{z\in B(0,R)} U_\varphi f(z)\right]^q dy\right)^{\frac{1}{1}}\\
&\quad= \frac{1}{|B(0,R)|^{\frac{\lambda}{n}}}\left(\frac{1}{|B(0,R)|}\int_{B(0,R)}\left[\int_0^1 f(ty)\varphi(t)dt - \inf_{z\in B(0,R)}\int_0^1 f(tz)\varphi(t)dt\right]^q dy\right)^{\frac{1}{q}}\\
&\quad\le \frac{1}{|B(0,R)|^{\frac{\lambda}{n}}}\left(\frac{1}{|B(0,R)|}\left(\int_{B(0,R)}\int_0^1 (f(ty) - \inf_{z\in B(0,R)} f(tz))\varphi(t)dt\right)^q dy\right)^{\frac{1}{q}}\\
&\quad= \frac{1}{|B(0,R)|^{\frac{\lambda}{n}}}\left(\frac{1}{|B(0,tR)|}\int_{B(0,tR)}\left(\int_0^1 \left(f(y) - \inf_{z\in B(0,tR)} f(z)\right)\varphi(t)dt\right)^q dy\right)^{\frac{1}{q}}\\
&\quad\le \int_0^1 \frac{1}{|B(0,tR)|^{\frac{\lambda}{n}}}\left(\frac{1}{|B(0,tR)|}\int_{B(0,tR)}\left[f(y) - \inf_{z\in B(0,tR)} f(z)\right]^q dy\right)^{\frac{1}{q}} t^\lambda\varphi(t)dt\\
&\quad\le \int_0^1 t^\lambda\varphi(t)\|f\|_{CBLO^{q,\lambda}(\mathbb{R}^n)}.
\end{aligned}$$

The proof is completed. The best possible constant is taken similar to above.

Proof of Theorem 3. Denote $\omega(x) = |x|^\alpha$. From Minkoski's inequality, it follows that

$$\begin{aligned}
&\left(\frac{1}{\omega(B(0,R))^{1+\lambda q/n}}\int_{B(0,R)}|U_\varphi f(x)|^q\omega(x)dx\right)^{\frac{1}{q}}\\
&\quad= \left(\frac{1}{\omega(B(0,R))^{1+\lambda q/n}}\int_{B(0,R)}\left(\int_0^1 |f(tx)\varphi(t)|dt\right)^q\omega(x)dx\right)^{\frac{1}{q}}\\
&\quad\le \int_0^1\left(\frac{1}{\omega(B(0,R))^{1+\lambda q/n}}\int_{B(0,R)}|f(tx)|^q\omega(x)dx\right)^{\frac{1}{q}}\varphi(t)dt\\
&\quad\le \int_0^1\left(\frac{1}{\omega(B(0,tR))^{1+\lambda q/n}}\int_{B(0,tR)}|f(x)|^q\omega(x/t)dx\right)^{\frac{1}{q}} t^{\lambda+\alpha/q+\alpha\lambda/q}\varphi(t)dt\\
&\quad= \int_0^1\left(\frac{1}{\omega(B(0,tR))^{1+\lambda q/n}}\int_{B(0,tR)}|f(x)|^q\omega(x)dx\right)^{\frac{1}{q}} t^{\frac{n+\alpha}{n}\lambda}\varphi(t)dt\\
&\quad\le \int_0^1 t^{\frac{n+\alpha}{n}\lambda}\varphi(t)dt\|f\|_{B^{q,\lambda}(\omega)}.
\end{aligned}$$

It implies that

$$\|U_\varphi\|_{B^{q,\lambda}(|x|^\alpha dx)\to B^{q,\lambda}(|x|^\alpha dx)} \le \int_0^1 t^{\frac{n+\alpha}{n}\lambda}\varphi(t)dt.$$

If we take $f_0(x) = |x|^{\frac{n+\alpha}{n}\lambda}$, it is easy to check that $f_0 \in B_*^{q,\lambda}(\omega)$ for $-n < \alpha$ and $U_\varphi(f_0)(x) = f_0\int_0^1 t^{\frac{n+\alpha}{n}\lambda}\varphi(t)dt$, then we have that

$$\|U_\varphi\|_{B^{q,\lambda}(|x|^\alpha dx)\to B^{q,\lambda}(|x|^\alpha dx)} \ge \int_0^1 t^{\frac{n+\alpha}{n}\lambda}\varphi(t)dt.$$

Here we complete the proof of Theorem 3.

Proof of Theorem 4 and 5. Theorem 4 and Theorem 5 follow from Theorem 1, Theorem2 and Theorem 3 as consequent, because the operator $\mathcal{H}$ and its restriction to radial functions have the same operator norm on $CBMO^{1,\lambda}(\mathbb{R}^n)$, $CBLO^{1,\lambda}(\mathbb{R}^n)$ and $B^{1,\lambda}(\omega)$. Here we give the details only for Theorem 5. For $q=1$, we set

$$g(x)=\frac{1}{\omega_{n-1}}\int_{|\xi|=1} f(|x|\xi)d\xi;\ x\in\mathbb{R}^n.$$

Then we can check that

$$\frac{\|\mathcal{H}f\|_{B^{1,\lambda}(\omega)}}{\|f\|_{B^{1,\lambda}(\omega)}}\le\frac{\|\mathcal{H}g\|_{B^{1,\lambda}(\omega)}}{\|g\|_{B^{1,\lambda}(\omega)}}.$$

The reverse inequality is valid naturally. Denote $\omega(x)=|x|^\alpha$. By Fubini's theorem, we have that

$$\begin{aligned}
&\frac{1}{\omega(B(0,R))^{1+\lambda/n}}\int_{B(0,R)}|g(x)|\omega(x)dx\\
&=\frac{1}{\omega_{n-1}}\frac{1}{\omega(B(0,R))^{1+\lambda q/n}}\int_{B(0,R)}\left|\int_{|\xi|=1} f(|x|\xi)d\xi\right|\omega(x)dx\\
&\le\frac{1}{\omega_{n-1}}\int_{|\xi|=1}\frac{1}{\omega(B(0,R))^{1+\lambda/n}}\int_{B(0,R)}|f(|x|\xi)|\omega(x)d\xi\\
&=\frac{1}{\omega_{n-1}}\int_{|\xi|=1}\frac{1}{\omega(B(0,R))^{1+\lambda/n}}\int_0^R\int_{|\eta|=1}|f(\rho\xi)|\rho^{n-1}\omega(\rho)d\eta d\rho d\xi\\
&=\int_{|\xi|=1}\frac{1}{\omega(B(0,R))^{1+\lambda/n}}\int_0^R|f(\rho\xi)|\rho^{n-1}\omega(\rho)d\rho d\xi\\
&=\frac{1}{\omega(B(0,R))^{1+\lambda/n}}\int_0^R\int_{|\xi|=1}|f(\rho\xi)|\rho^{n-1}\omega(\rho)d\xi d\rho\\
&=\frac{1}{\omega(B(0,R))^{1+\lambda/n}}\int_{B(0,R)}|f(x)|\omega(x)dx.
\end{aligned}$$

Then by Minkoski's inequality, we have that

$$
\begin{aligned}
&\frac{1}{\omega(B(0,R))^{1+\lambda/n}}\int_{B(0,R)}|\mathcal{H}f(x)|\omega(x)dx\\
&=\bigg(\frac{1}{\omega(B(0,R))^{1+\lambda/n}}\int_{B(0,R)}\frac{1}{\omega_{n-1}|x|^n}\int_{B(0,|x|)}|f(y)|dy\omega(x)dx\\
&=\frac{1}{v_n}\frac{1}{\omega(B(0,R))^{1+\lambda/n}}\int_{B(0,R)}\bigg(\int_{B(0,1)}|f(|x|y)|dy\omega(x)dx\\
&\leq\frac{1}{\omega_{n-1}}\int_{B(0,1)}\frac{1}{\omega(B(0,R))^{1+\lambda/n}}\int_{B(0,R)}|f(x|y|)|\omega(x)dxdy\\
&\leq\frac{1}{\omega_{n-1}}\int_{B(0,1)}\frac{1}{\omega(B(0,|y|R))^{1+\lambda/n}}\int_{B(0,|y|R)}|f(x)|\omega(x/|y|)dx|y|^{\lambda+\alpha+\alpha\lambda}dy\\
&=\frac{1}{\omega_{n-1}}\int_{B(0,1)}\frac{1}{\omega(B(0,|y|R))^{1+\lambda/n}}\int_{B(0,|y|R)}|f(x)|\omega(x)dx|y|^{\frac{n+\alpha}{n}\lambda}dy\\
&\leq\frac{n^2}{n^2+(n+\alpha)\lambda}\|f\|_{B^{1,\lambda}(\omega)}.
\end{aligned}
$$

Then also take $f_0=|x|^{\frac{n+\alpha}{n}\lambda}$, we have that

$$\|\mathcal{H}\|_{B^{1,\lambda}(|x|^\alpha dx)\to B^{1,\lambda}(|x|^\alpha dx)}=\frac{n^2}{n^2+(n+\alpha)\lambda}.$$

Remark 6 *We will show a gap for the following inequality when $q>1$. By the Minkowski inequality, we have*

$$
\begin{aligned}
&\left(\frac{1}{\omega(B(0,R))^{1+\lambda q/n}}\int_{B(0,R)}|g(x)|^q\omega(x)dx\right)^{\frac{1}{q}}\\
&=\frac{1}{\omega_{n-1}}\left(\frac{1}{\omega(B(0,R))^{1+\lambda q/n}}\int_{B(0,R)}\left|\int_{|\xi|=1}f(|x|\xi)d\xi\right|^q\omega(x)dx\right)^{\frac{1}{q}}\\
&\leq\frac{1}{\omega_{n-1}}\int_{|\xi|=1}\left(\frac{1}{\omega(B(0,R))^{1+\lambda q/n}}\int_{B(0,R)}|f(|x|\xi)|^q\omega(x)dx\right)^{\frac{1}{q}}d\xi\\
&=\frac{1}{\omega_{n-1}}\int_{|\xi|=1}\left(\frac{1}{\omega(B(0,R))^{1+\lambda q/n}}\int_0^R\int_{|\eta|=1}|f(\rho\xi)|^q\rho^{n-1}\omega(\rho)d\eta d\rho\right)^{\frac{1}{q}}d\xi\\
&=\int_{|\xi|=1}\left(\frac{1}{\omega(B(0,R))^{1+\lambda q/n}}\int_0^R|f(\rho\xi)|^q\rho^{n-1}\omega(\rho)d\rho\right)^{\frac{1}{q}}d\xi.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\left(\frac{1}{\omega(B(0,R))^{1+\lambda q/n}}\int_{B(0,R)}|f(x)|^q\omega(x)dx\right)^{\frac{1}{q}}\\
&\qquad=\left(\frac{1}{\omega(B(0,R))^{1+\lambda q/n}}\int_0^R\int_{|\xi|=1}|f(\rho\xi)|^q\rho^{n-1}\omega(\rho)d\xi d\rho\right)^{\frac{1}{q}}.
\end{aligned}
$$

So, to get

$$\left(\frac{1}{\omega(B(0,R))^{1+\lambda q/n}}\int_{B(0,R)}|g(x)|^q\omega(x)dx\right)^{\frac{1}{q}}$$
$$\leq\left(\frac{1}{\omega(B(0,R))^{1+\lambda q/n}}\int_{B(0,R)}|f(x)|^q\omega(x)dx\right)^{\frac{1}{q}},$$

one needs that

$$\int_{|\xi|=1}\left(\frac{1}{\omega(B(0,R))^{1+\lambda q/n}}\int_0^R|f(\rho\xi)|^q\rho^{n-1}\omega(\rho)d\rho\right)^{\frac{1}{q}}d\xi$$
$$\leq\left(\frac{1}{\omega(B(0,R))^{1+\lambda q/n}}\int_0^R\int_{|\xi|=1}|f(\rho\xi)|^q\rho^{n-1}\omega(\rho)d\xi d\rho\right)^{\frac{1}{q}}.$$

However, the above inequality is not true when f is not radial and $q>1$. So we can not deduce that

$$\frac{\|\mathcal{H}f\|_{B^{q,\lambda}(\omega)}}{\|f\|_{B^{q,\lambda}(\omega)}}\leq\frac{\|\mathcal{H}g\|_{B^{q,\lambda}(\omega)}}{\|g\|_{B^{q,\lambda}(\omega)}}$$

for f is not radial and $q>1$.

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References

[1] Hardy G. H., Littlewood J. E., Polya .G., Inequalities, 2nd ed., London, New York: Cambridge Univ. Press, 1934.

[2] Christ M., Grafakos L.. Best constants for two non-convolution inequalities, Proc. Amer. Math. Soc., 1995, 123, 1687-1693.

[3] Carton-Lebrun C. Fosset M. Moyennes et quotients de Taylor dans BMO, Bull Soc. Roy. Sci. Liège., 1984, 53, 85-87.

[4] Xiao J.. L^p and BMO bounds of weighted Hardy-Littlewood average. J. Math. Anal. Appl., 2001, 262, 660-666.

[5] Zhao F. Y., Fu Z. W., Lu S. Z.. Endpoint estimates for n-dimensional Hardy operators and their commutators, Scie. China. Math., 2012, 55 (10), 1977-1990.

[6] Alvararez J., Guzman-Partida M., Lakey J.. Spaces of bounded λ-center mean oscillation, Morrey spaces and λ-center Carleson measures, Collect. Math. 2000, 51(1), 1-47.

[7] Fu Z. W., Lu S. Z., Yuan W., A weighted variant of Riemann-Liouville fractional integral on $\mathbb{R}^n$. Abstr. Appl. Anal. (2012): Art. ID 780132, 18 pages.

[8] Fu Z. W., Lin Y., λ-center BMO estimates for commutators of higher dimensional fractional Hardy operators, Acta Mathmatica Sinica, Chinese series, 2010, 53 (5), 925-932.

加权 Hardy-Littlewood 平均的一些估计

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摘　要: 作者研究了加权 Hardy-Littlewood 平均在中心-λ Campanoto 型空间上以及加幂权的λ Morrey 型空间上的有界性，并且得到了它的算子范数。

关键词: 加权 Hardy-Littlewood 平均算子; λ-中心 Campanoto 型空间; 加权 λ-中心 Morrey 型空间.