Collatz Theorem

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Dagnachew Jenber
Addis Ababa Science and Technology University

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Dagnachew Jenber Negash

Addis Ababa Science and Technology University
Addis Ababa, Ethiopia
Email: djdm_101979@yahoo.com

Abstract

This paper studies the proof of Collatz conjecture for some set of sequence of odd numbers with infinite number of elements. These set generalized to the set which contains all positive odd integers. This extension assumed to be the proof of the full conjecture, using the concept of mathematical induction.

Keywords: Collatz conjecture; Recurrence relations; Concept of mathematical induction.
MSC2010: 65Q30, 97N30, 39A12, 11B75

1. Introduction

The Collatz conjecture is an unsolved conjecture in mathematics. It is named after Lothar Collatz, who first proposed it in 1937. The conjecture is also known as the $3n + 1$ conjecture, the Ulam conjecture (after Stanislaw Ulam), the Syracuse problem, as the hailstone sequence or hailstone numbers, or as Wondrous numbers per Godel, Escher, Bach. It asks whether a certain kind of number sequence always ends in the same way, regardless of the starting number. The problem is related to a wide range of topics in mathematics, including number theory, computability theory, and the analysis of dynamic systems. Paul Erdos said about the Collatz conjecture, "Mathematics is not yet ready for such problems." He offered $500 for its solution. (Lagarias 1985)

2. Statement of the problem

Consider the following operation on an arbitrary positive integer[2][9]:

- If the number is even, divide it by two.
- If the number is odd, triple it and add one.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ 3n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Form a sequence by performing this operation repeatedly, beginning with any positive integer.

- Example: n = 6 produces the sequence

$$6,\ 3,\ 10,\ 5,\ 16,\ 8,\ 4,\ 2,\ 1,\ 4,\ 2,\ 1,\ \cdots$$

- The Collatz conjecture is:
  This process will eventually reach the number 1, regardless of which positive integer is chosen initially.

3. Some Examples

- n = 11 produces the sequence

$$11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.$$

which means we operated Collatz function 14 times to reach to the number 1.

- n = 27 produces the sequence

$$27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121,$$
$$364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175,$$
$$526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502,$$
$$251, 754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438,$$
$$719, 2158, 1079, 3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734,$$
$$1367, 4102, 2051, 6154, 3077, 9232, 4616, 2308, 1154, 577, 1732, 866,$$
$$433, 1300, 650, 325, 976, 488, 244, 122, 61, 184, 92, 46, 23, 70, 35, 106,$$
$$53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1.$$

that is, we operated or used Collatz function repeatdely 111 times to reach to the number 1.

4. Supporting arguments for the conjecture

- Experimental evidence[1][7]:

The conjecture has been checked by computer for all starting values up to $19\times2^{58} \approx 5.48\times10^{18}$

- A probabilistic argument:

One can show that each odd number in a sequence is on average $3/4$ of the previous one, so every sequence should decrease in the long run[5].

This is not a proof because Collatz sequences are not produced by random events.

5. One Definition

Through out this paper we will use the definition

Definition 1. *Define N_k as the number of collatz function operation for the number k to get* 1.

6. Proof of collatz analysis for some sequence of numbers

We know that Collatz analysis is true for all numbers in the set $\{a_n : a_n = 2^{2n}$ or $a_n = 2^{2n-1}, n \in \mathbb{N}\}$. Here I want to show that Collatz analysis is true for all numbers in the set $\{a_n : a_n = (2k+1)2^{2n}$ or $a_n = (2k+1)2^{2n-1}, n, k \in \mathbb{N}\}$, here we have to be sure that Collatz conjecture becomes collatz Theorem if it is proved for this set. Before the conclusion let's see some sort of sequence of numbers and Collatz analysis is definitely true for these set of sequence of numbers. Let's get started from the following figure and then we will discuss, give some analysis based on our objective on the figure and form one sequence function which contains all numbers in the sequence that shows Collatz analysis is true. The following theorem shows the truth of collatz conjecture on some recurrence relations. The recurrence relations are inter-related, which means one recurrence relation formed from the previous one. Therefore if we proved that one recurrence relation satisfies Collatz conjecture then so the next, and then so on. Hence from the concept of mathematical induction, we will prove Collatz conjecture for all odd natural numbers, then the proof for all even natural number is straight forward, because all even number have the form $(2k+1)2^n$. To do this let's get started from **Theorem 1**.

Theorem 1. *Collatz conjecture is true for all odd numbers in the set of the recurrence relation*

$$\{a_n : a_n - a_{n-1} = 2^{2n}, a_0 = 1 \text{ and } n \in \mathbb{N}\}$$

$$\Rightarrow a_n = \frac{1}{3}\left[4^{n+1} - 1\right],\ n \geq 0$$

and for each a_n, there are $2n+3$ number of steps (or Collatz function operation) needed to get 1.

Proof.

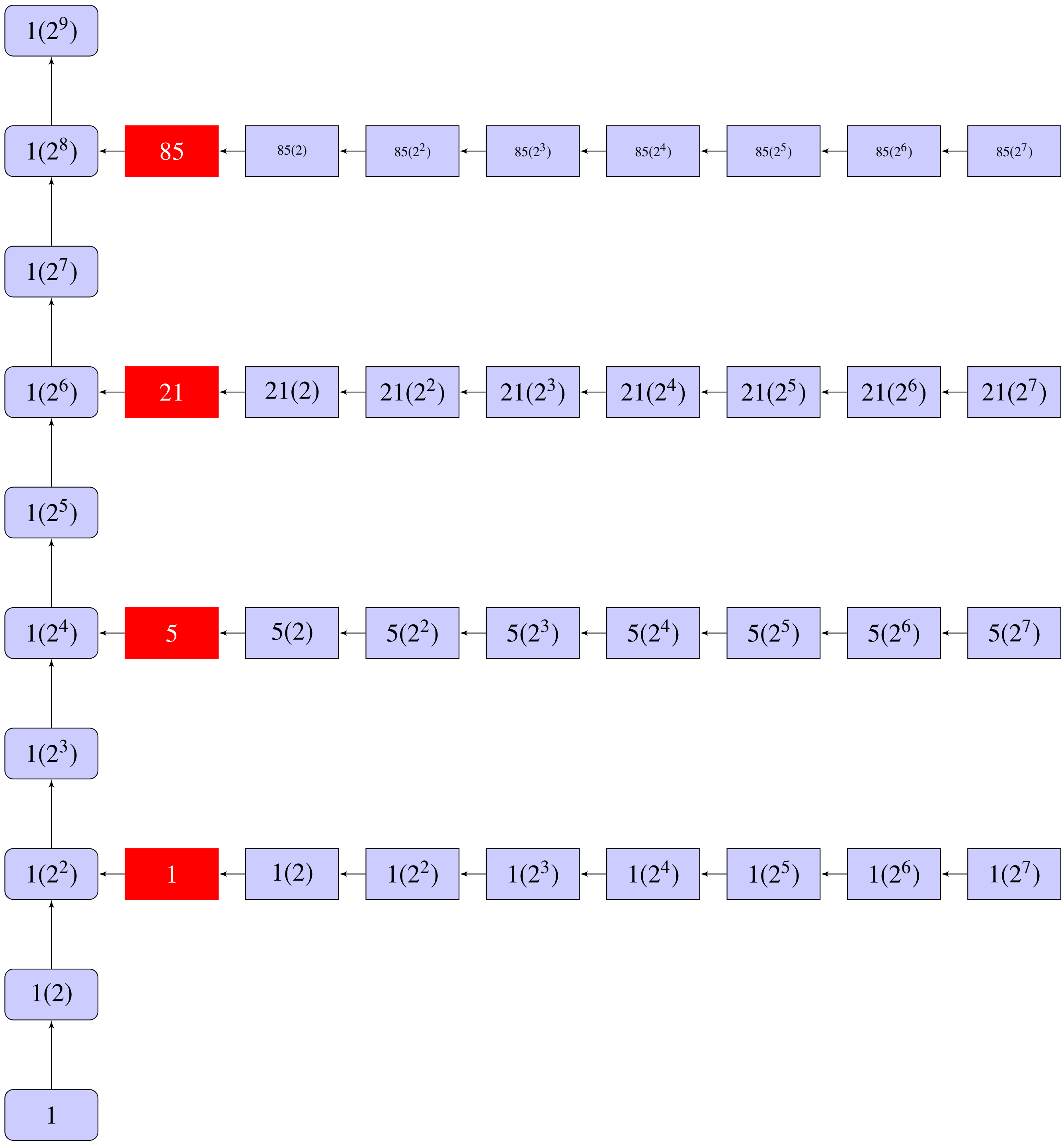

Figure 1: Proof of Collatz conjecture for the sequence of numbers $\{1, 5, 21, 85, \cdots\}$

As you see from the figure we have sequence of numbers $\{1, 5, 21, 85, \cdots\}$. All elements of this sequence are odd numbers with $a_0 = 1, a_1 = 5, a_2 = 21, a_3 = 85, \cdots$, and $3(1) + 1 = 2^2, 3(5) + 1 =$

$2^4, 3(21) + 1 = 2^6, 3(85) + 1 = 2^8, \cdots$. The question is how this sequence of numbers formed?. The answer is simple. You can observe that $5 = 1 + 2^2, 21 = 5 + 2^4, 85 = 21 + 2^6, \cdots$. Therefore the recurrence relation is $a_n = a_{n-1} + 2^{2n}$ or $a_n - a_{n-1} = 2^{2n}$ with $a_0 = 1$ and $n \in \mathbb{N}$.
Hence we can conclude that collatz analysis is true for all numbers in the set

$$\{a_n : a_n - a_{n-1} = 2^{2n}, a_0 = 1 \text{ and } n \in \mathbb{N}\}$$

If we solve this non-homogeneous recurrence relation we get,

$$a_n = \frac{1}{3}\left[4^{n+1} - 1\right],\ n \geq 0$$

This is the general term for the above sequence of numbers. We might ask "How many number of steps we need to reach to 1 for each number in the above sequence by applying Collatz function?". Here is the answer. For example for $n = 0$, we have $a_0 = 1$, therefore according to Collatz conjecture $3(a_0) + 1 = 2^2$ since a_0 is odd. As you can see from the above figure that we need to have 3 steps, which means $2 + 1$ steps. Similarly for $a_1 = 5$ we need to have 5 steps to reach to 1 since $3(5) + 1 = 2^4$, that is take the exponent of 2 on the right hand side of $3(5) + 1 = 2^4$ and add 1, that is, $4 + 1$ steps. Thus for $a_2 = 21$ we need to have 7 steps since $3(21) + 1 = 2^6$ and $a_3 = 85$ needs to have 9 steps since $3(85) + 1 = 2^8$. You can check that this is true for the rest of numbers in the sequence by choosing randomly. □

Next what I'm going to do is track another sequence function. The question is how to get this sequence function?. Well, the answer is simple if I draw another figure just like the previous but I have to use the previous figure to draw the next. First of all, for $k \in \mathbb{N}$, let me find the least $\beta_1 = a_k 2^{2n}$ or $= a_k 2^{2n-1}$ for $n = 1$ such that $\beta_1 - 1$ should be divisible by 3, where $a_k \in \{a_n : a_n - a_{n-1} = 2^{2n}, a_0 = 1 \text{ and } n \in \mathbb{N}\}$ and $(\beta_1 - 1)/3$ will be the initial term for the general term of the next sequence of numbers. So after two or three simple trials you can get that $a_k = 5$ and $\beta_1 = 5(2) = 10$ which is the least among others and you can verify that $\beta_1 - 1 = 10 - 1 = 9$ which is divisible by 3 and $(\beta_1 - 1)/3 = 3 = a_0$ is the initial term for the next sequence function. Now let's see the following figure to be more clear for the next and the previous work.

Theorem 2. *Collatz conjecture is true for all odd numbers in the set of the recurrence relation*

$$\{b_n : b_n - b_{n-1} = 5(2^{2n-1}), b_0 = 3 \text{ and } n \in \mathbb{N}\}$$

$$\Rightarrow b_n = \frac{1}{3}\left[5(2^{2n+1}) - 1\right],\ n \geq 0$$

and for each b_n, there are $N_5 + 2n + 2$ number of steps (or Collatz function operation) needed to get 1.

Proof.

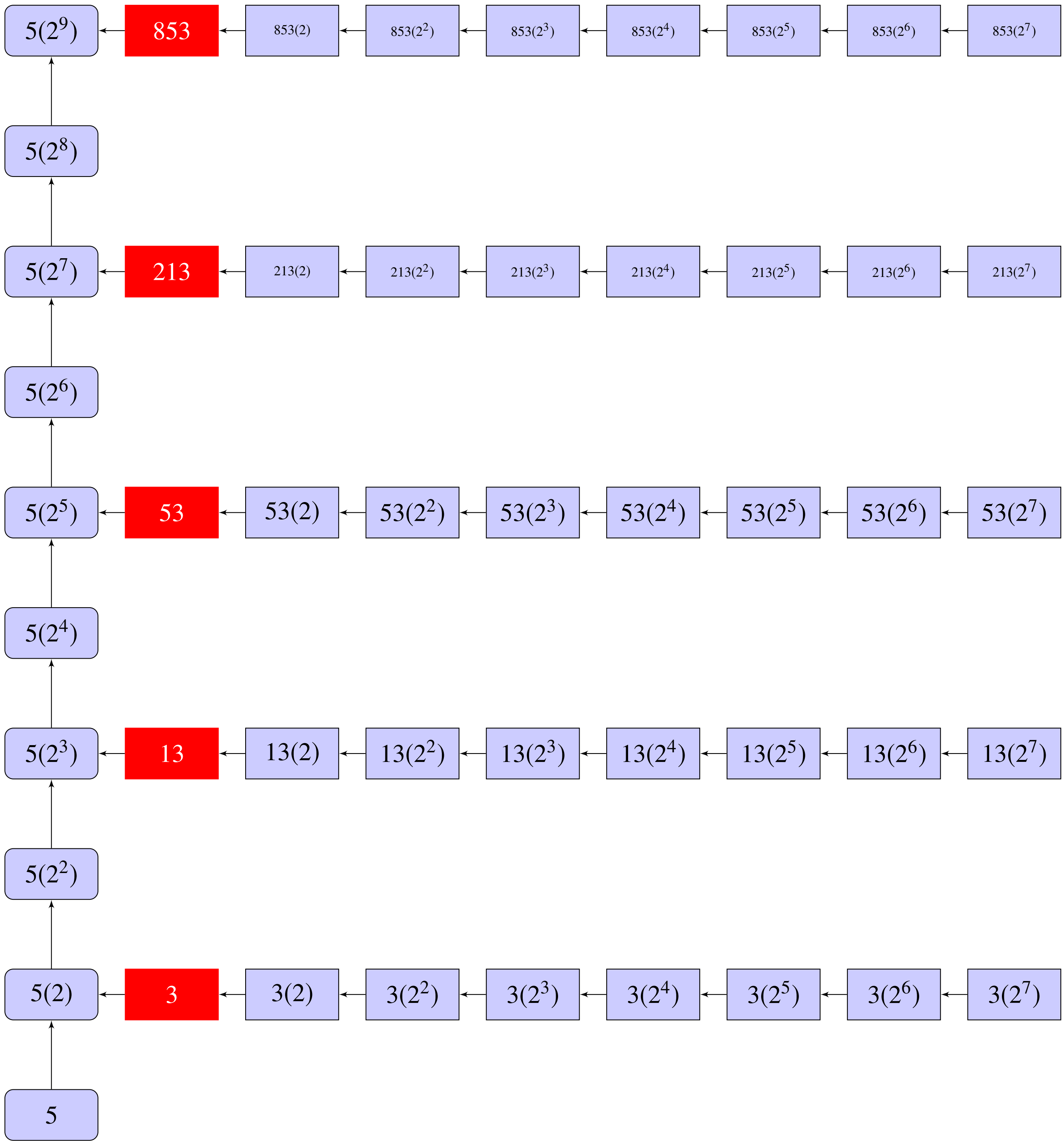

Figure 2: Proof of Collatz conjecture for the sequence of numbers $\{3, 13, 53, 213, 853 \cdots\}$ combined with Fig 1.

As you can see from figure 2, we have sequence of numbers $\{3, 13, 53, 213, 853 \cdots\}$. All elements

of this sequence are odd numbers with $b_0 = 3, b_1 = 13, b_2 = 53, b_3 = 213, b_4 = 853\cdots$, and $3(3)+1 = 5(2^1), 3(13)+1 = 5(2^3), 3(53)+1 = 5(2^5), 3(213)+1 = 5(2^7), 3(853)+1 = 5(2^9)\cdots$. The question is how this sequence of numbers formed?. The answer is simple. You can observe that $13 = 3 + 5(2^1), 53 = 13 + 5(2^3), 213 = 53 + 5(2^5), 853 = 213 + 5(2^7)\cdots$. Therefore the recurrence relation is $b_n = a_{n-1} + 5(2^{2n-1})$ or $b_n - b_{n-1} = 5(2^{2n-1})$ with $b_0 = 3$ and $n \in \mathbb{N}$.
Hence we can conclude that Collatz analysis is true for all numbers in the set

$$\{b_n : b_n - b_{n-1} = 5(2^{2n-1}), b_0 = 3 \text{ and } n \in \mathbb{N}\}$$

If we solve this non-homogeneous recurrence relation we get,

$$b_n = \frac{1}{3}\left[5(2^{2n+1}) - 1\right],\ n \geq 0$$

This is the general term for the above sequence of numbers. We might ask "How many number of steps we need to reach to 1 for each number in the above sequence by applying Collatz function?". Here is the answer. For example for $n = 0$, we have $b_0 = 3$, therefore according to Collatz conjecture $3(b_0) + 1 = 10 = 5(2)$ since b_0 is odd. As you can see from the above figure that we need to have 7 steps, which means $2 + 5$ steps, where $2 = 1 + 1$ is the exponent of 2 plus 1 from the equation on the right side of $3(b_0) + 1 = 10 = 5(2)$ and 5 is the number steps needed for 5, so $b_0 = 3$ needs to have $7 = 5 + 2$ number of steps to reach to 1 by applying Collatz function repeatedly. Similarly for $b_1 = 13$ we need to have $5 + 4 = 9$ steps to reach to 1 since $3(13) + 1 = 5(2^3)$, that is take the exponent of 2 on the right hand side of $3(13) + 1 = 5(2^3)$ and add 1, that is, $3 + 1$ steps plus 5 steps. Thus for $b_2 = 53$ we need to have 11 steps since $3(53) + 1 = 5(2^5)$ and $b_3 = 213$ needs to have 13 steps since $3(213) + 1 = 5(2^7)$. You can check that this is true for the rest of numbers in the sequence by choosing randomly. □

Next, what I'm going to do is, track another sequence function. The question is how to get this sequence function?. Well, the answer is simple if I draw another figure 3 just like the previous but I have to use the previous figures to draw the next. First of all, for $k \in \mathbb{N}$, let me find the least $\beta_2 = b_k 2^{2n}$ or $= b_k 2^{2n-1}$ for $n = 1$ such that $\beta_2 - 1$ should be divisible by 3, where $b_k \in \{b_n : b_n - b_{n-1} = 5(2^{2n-1}), b_0 = 3 \text{ and } n \in \mathbb{N}\}$ and $(\beta_2 - 1)/3$ will be the initial term for the general term of the next sequence of numbers. So after two or three simple trials you can get that $b_k = 13$ and $\beta_2 = 13(2^2) = 52$ which is the least among others and you can verify that $\beta_2 - 1 = 52 - 1 = 51$ which is divisible by 3 and $(\beta_2 - 1)/3 = 17 = b_0$ is the initial term for the next sequence function. Now let's see the following figure 3 to be more clear for the next and the previous work.

Theorem 3. *Collatz conjecture is true for all odd numbers in the set of the recurrence relation*

$$\{c_n : c_n - c_{n-1} = 13(2^{2n}), c_0 = 17 \text{ and } n \in \mathbb{N}\}$$

$$c_n = \frac{1}{3}\left[13(2^{2n+2}) - 1\right],\ n \geq 0$$

and for each c_n, there are $N_{13} + 2n + 3$ number of steps (or Collatz function operation) needed to get 1.

Proof.

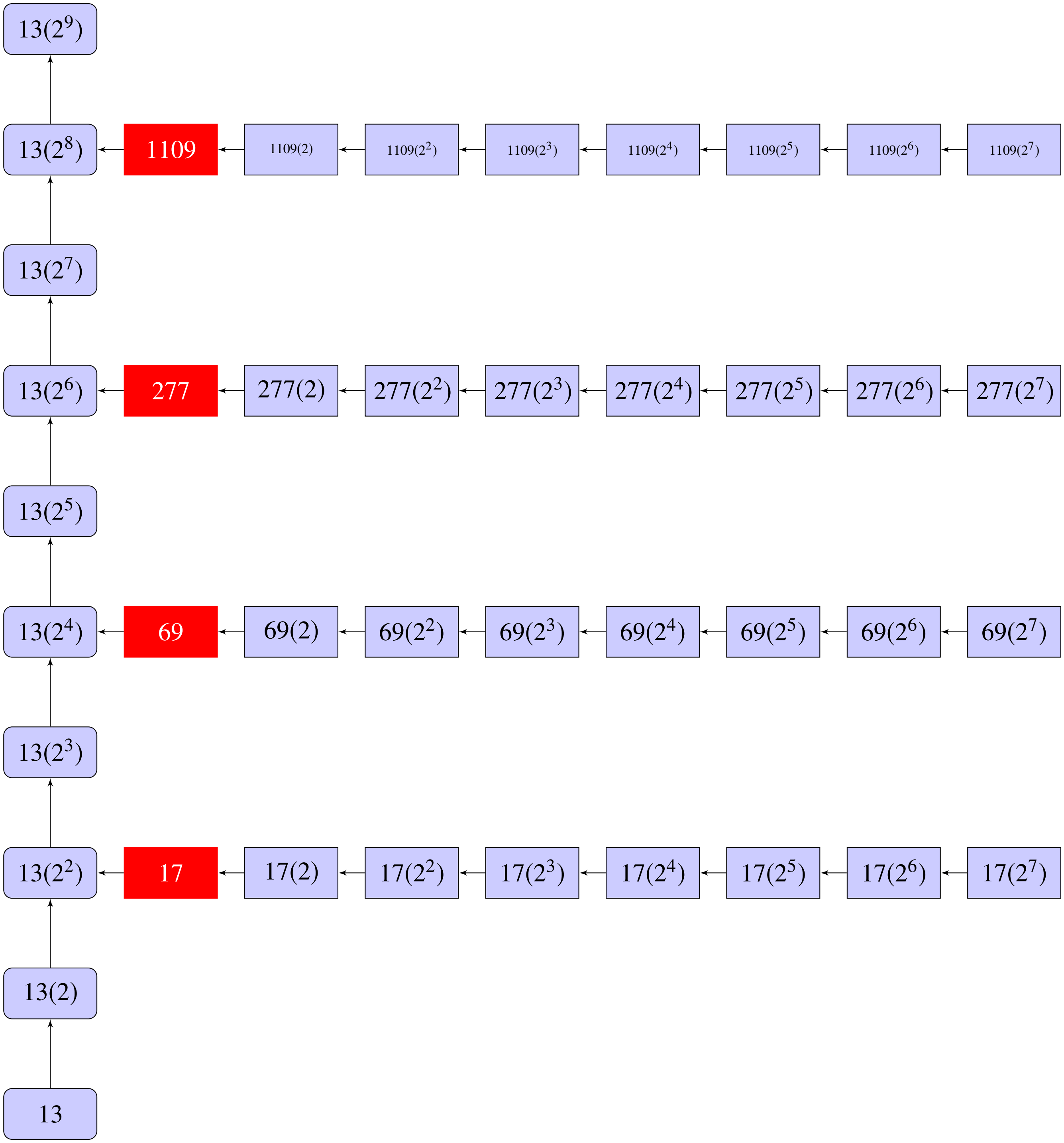

Figure 3: Proof of Collatz conjecture for the sequence of numbers $\{17, 69, 277, 1109 \cdots\}$ combined with Fig 1. and Fig 2.

As you can see from figure 3 we have sequence of numbers $\{17, 69, 277, 1109 \cdots\}$. All elements

of this sequence are odd numbers with $c_0 = 17, c_1 = 69, c_2 = 277, c_3 = 1109\cdots$, and $3(17)+1 = 13(2^2), 3(69)+1 = 13(2^4), 3(277)+1 = 13(2^6), 3(1109)+1 = 13(2^8), \cdots$. The question is how this sequence of numbers formed?. The answer is simple. You can observe that $69 = 17+13(2^2), 277 = 69 + 13(2^4), 1109 = 277 + 13(2^6), \cdots$. Therefore the general term is $c_n = a_{n-1} + 13(2^{2n})$ or $c_n - a_{n-1} = 13(2^{2n})$ with $c_0 = 17$ and $n \in \mathbb{N}$.
Hence we can conclude that Collatz analysis is true for all numbers in the set

$$\{c_n : c_n - c_{n-1} = 13(2^{2n}), c_0 = 17 \text{ and } n \in \mathbb{N}\}$$

If we solve this non-homogeneous recurrence relation we get,

$$c_n = \frac{1}{3}\left[13(2^{2n+2}) - 1\right],\ n \geq 0$$

This is the general term for the above sequence of numbers. We might ask "How many number of steps we need to reach to 1 for each number in the above sequence by applying Collatz function?". Here is the answer. For example, for $n = 0$, we have $c_0 = 17$, therefore according to Collatz conjecture $3(17) + 1 = 52 = 13(2^2)$ since c_0 is odd. As you can see from the above figure that we need to have 12 steps, which means $3 + 9$ steps, where $3 = 2 + 1$ is the exponent of 2 plus 1 from the equation on the right side of $3(c_0) + 1 = 52 = 13(2^2)$ and 9 is the number steps needed for 13, so $c_0 = 17$ needs to have $12 = 9 + 3$ number of steps to reach to 1 by applying Collatz function repeatedly. Similarly for $c_1 = 69$ we need to have $9 + 5 = 14$ steps to reach to 1 since $3(69) + 1 = 13(2^4)$, that is take the exponent of 2 on the right hand side of $3(69) + 1 = 13(2^4)$ and add 1, that is, $4 + 1$ steps plus 9 steps. Thus for $a_2 = 277$ we need to have 16 steps since $3(277) + 1 = 13(2^6)$ and $a_3 = 1109$ needs to have 18 steps since $3(1109) + 1 = 13(2^8)$. You can check that this is true for the rest of numbers in the sequence by choosing randomly. □

Next, what I'm going to do is track another sequence function. The question is how to get this sequence function?. Well, the answer is simple if I draw another picture just like the previous but I have to use the previous pictures to draw the next. First of all, for $k \in \mathbb{N}$, let me find the least $\beta_3 = c_k 2^{2n}$ or $= c_k 2^{2n-1}$ for $n = 1$ such that $\beta_3 - 1$ should be divisible by 3, where $c_k \in \{c_n : c_n - c_{n-1} = 13(2^{2n}), c_0 = 17 \text{ and } n \in \mathbb{N}\}$ and $(\beta_3 - 1)/3$ will be the initial term for the general term of the next sequence of numbers. So after two or three simple trials you can get that $c_k = 17$ and $\beta_3 = 17(2^1) = 34$ which is the least among others and you can verify that $\beta_3 - 1 = 34 - 1 = 33$ which is divisible by 3 and $(\beta_3 - 1)/3 = 11 = c_0$ is the initial term for the next sequence function. Now let's see the following figure 4 to be more clear for the next and the previous work.

Theorem 4. *Collatz conjecture is true for all odd numbers in the set of the recurrence relation*

$$\{d_n : d_n - d_{n-1} = 17(2^{2n-1}), d_0 = 11 \textit{ and } n \in \mathbb{N}\}$$

$$\Rightarrow d_n = \frac{1}{3}\left[17(2^{2n+1}) - 1\right],\ n \geq 0$$

and for each d_n, there are $N_{17} + 2n + 2$ number of steps (or Collatz function operation) needed to get 1.

Proof.

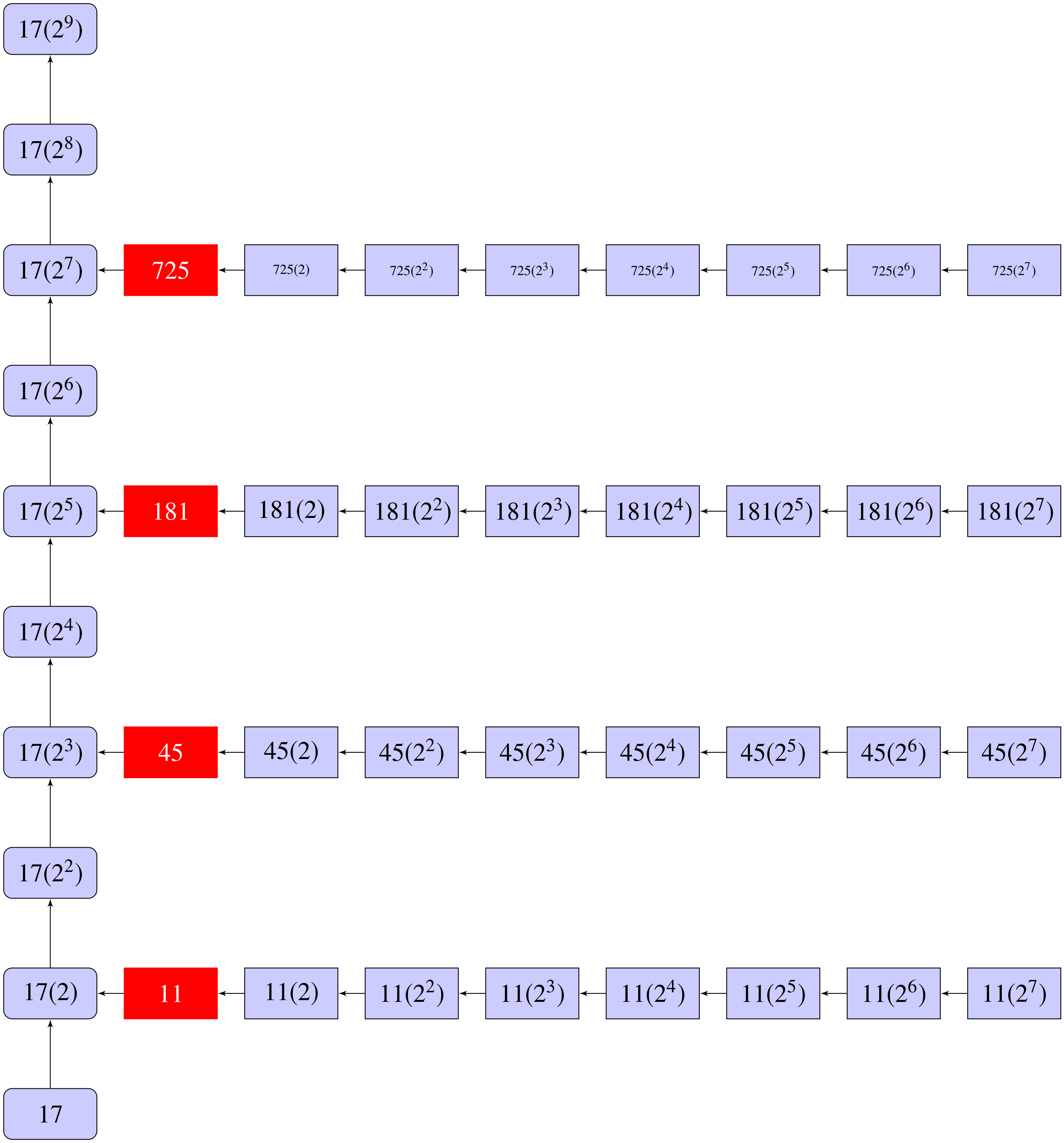

Figure 4: Proof of Collatz conjecture for the sequence of numbers $\{11, 45, 181, 725\cdots\}$ combined with the previous figures.

As you see from the picture we have sequence of numbers $\{11, 45, 181, 725\cdots\}$. All elements of

this sequence are odd numbers with $d_0 = 11, d_1 = 45, d_2 = 181, d_3 = 725 \cdots$, and $3(11) + 1 = 17(2^1), 3(45) + 1 = 17(2^3), 3(181) + 1 = 17(2^5), 3(725) + 1 = 17(2^7), \cdots$. The question is how this sequence of numbers formed?. The answer is simple. You can observe that $45 = 11 + 17(2^1), 181 = 45 + 17(2^3), 725 = 181 + 17(2^5), \cdots$. Therefore the recurrence relation is $d_n = d_{n-1} + 17(2^{2n-1})$ or $d_n - d_{n-1} = 17(2^{2n-1})$ with $d_0 = 11$ and $n \in \mathbb{N}$.
Hence we can conclude that Collatz analysis is true for all numbers in the set

$$\{d_n : d_n - d_{n-1} = 17(2^{2n-1}), d_0 = 11 \text{ and } n \in \mathbb{N}\}$$

If we solve this non-homogeneous recurrence relation we get,

$$d_n = \frac{1}{3}\left[17(2^{2n+1}) - 1\right],\ n \geq 0$$

This is the general term for the above sequence of numbers. We might ask "How many number of steps we need to reach to 1 for each number in the above sequence by applying Collatz function?". Here is the answer. For example for $n = 0$, we have $d_0 = 11$, therefore according to Collatz conjecture $3(11) + 1 = 34 = 17(2^1)$ since d_0 is odd. As you can see from the above figure that we need to have 14 steps, which means $2 + 12$ steps, where $2 = 1 + 1$ is the exponent of 2 plus 1 from the equation on the right side of $3(d_0) + 1 = 34 = 17(2^1)$ and 12 is the number steps needed for 17, so $d_0 = 11$ needs to have $14 = 2 + 12$ number of steps to reach to 1 by applying Collatz function repeatedly. Similarly for $a_1 = 45$ we need to have $12 + 4 = 16$ steps to reach to 1 since $3(45) + 1 = 17(2^3)$, that is take the exponent of 2 on the right hand side of $3(45) + 1 = 17(2^3)$ and add 1, that is, $3 + 1$ steps plus 12 steps. Thus for $d_2 = 181$ we need to have 18 steps since $3(181) + 1 = 17(2^5)$ and $d_3 = 725$ needs to have 20 steps since $3(725) + 1 = 17(2^7)$. You can check that this is true for the rest of numbers in the sequence by choosing randomly. □

Next, what I'm going to do is track another sequence function. The question is how to get this sequence function?. Well, the answer is simple if I draw another figure just like the previous but I have to use the previous figures to draw the next. First of all, for $k \in \mathbb{N}$, let me find the least $\beta_4 = d_k 2^{2n}$ or $= d_k 2^{2n-1}$ for $n = 1$ such that $\beta_4 - 1$ should be divisible by 3, where $d_k \in \{d_n : d_n - d_{n-1} = 17(2^{2n-1}), d_0 = 11 \text{ and } n \in \mathbb{N}\}$ and $(\beta_4 - 1)/3$ will be the initial term for the general term of the next sequence of numbers. So after two or three simple trials you can get that $d_k = 11$ and $\beta_4 = 11(2^1) = 22$ which is the least among others and you can verify that $\beta_4 - 1 = 22 - 1 = 21$ which is divisible by 3 and $(\beta_4 - 1)/3 = 7 = d_0$ is the initial term for the next sequence function. Now let's see the following figure 5 to be more clear for the next and the previous work.

Theorem 5. *Collatz conjecture is true for all odd numbers in the set of the recurrence relation*

$$\{e_n : e_n - e_{n-1} = 11(2^{2n-1}), e_0 = 7 \text{ and } n \in \mathbb{N}\}$$

$$\Rightarrow e_n = \frac{1}{3}\left[11(2^{2n+1}) - 1\right],\ n \geq 0$$

and for each e_n, there are $N_{11} + 2n + 2$ number of steps (or Collatz function operation) needed to get 1.

Proof.

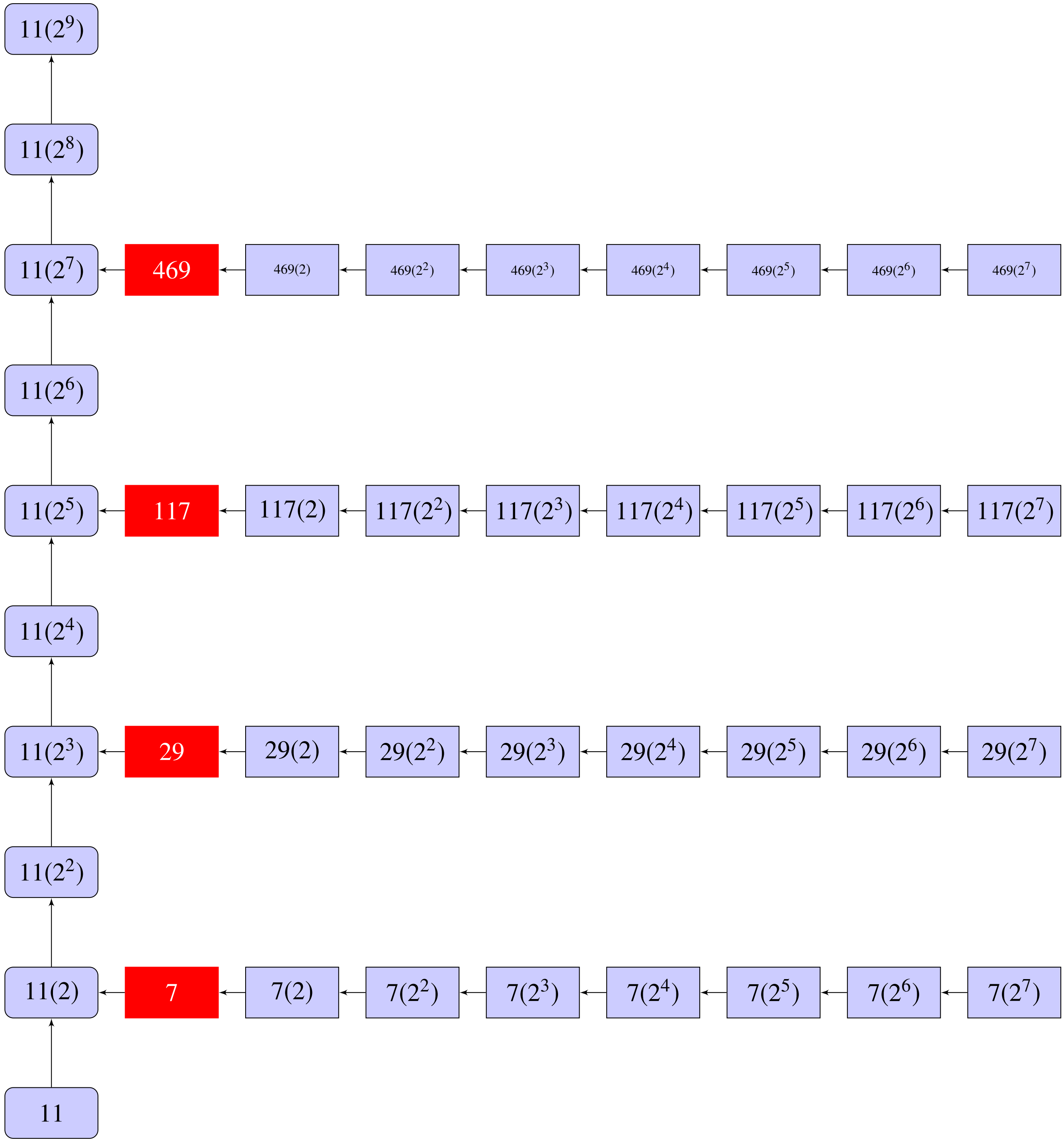

Figure 5: Proof of Collatz conjecture for the sequence of numbers $\{7, 29, 117, 469 \cdots\}$ combined with the previous figures.

As you can see from figure 5, we have sequence of numbers $\{7, 29, 117, 469 \cdots\}$. All elements

of this sequence are odd numbers with $e_0 = 7, e_1 = 29, e_2 = 117, e_3 = 469\cdots$, and $3(7)+1 = 11(2^1), 3(29)+1 = 11(2^3), 3(117)+1 = 11(2^5), 3(469)+1 = 11(2^7), \cdots$. The question is how this sequence of numbers formed?. The answer is simple. You can observe that $29 = 7 + 11(2^1), 117 = 29 + 11(2^3), 469 = 117 + 11(2^5), \cdots$. Therefore the recurrence relation is $e_n = e_{n-1} + 11(2^{2n-1})$ or $e_n - e_{n-1} = 11(2^{2n-1})$ with $a_0 = 7$ and $n \in \mathbb{N}$.
Hence we can conclude that Collatz analysis is true for all numbers in the set

$$\{e_n : e_n - e_{n-1} = 11(2^{2n-1}), e_0 = 7 \text{ and } n \in \mathbb{N}\}$$

If we solve this non-homogeneous recurrence relation we get,

$$e_n = \frac{1}{3}\left[11(2^{2n+1}) - 1\right],\ n \geq 0$$

This is the general term for the above sequence of numbers. We might ask "How many number of steps we need to reach to 1 for each number in the above sequence by applying Collatz function?". Here is the answer. For example for $n = 0$, we have $e_0 = 7$, therefore according to Collatz conjecture $3(7)+1 = 22 = 11(2^1)$ since e_0 is odd. As you can see from the above figures that we need to have 16 steps, which means $2 + 14$ steps, where $2 = 1 + 1$ is the exponent of 2 plus 1 from the equation on the right side of $3(e_0) + 1 = 22 = 11(2^1)$ and 14 is the number steps needed for 11, so $e_0 = 7$ needs to have $16 = 2 + 14$ number of steps to reach to 1 by applying Collatz function repeatedly. Similarly for $e_1 = 29$ we need to have $14 + 4 = 18$ steps to reach to 1 since $3(29)+1 = 11(2^3)$, that is take the exponent of 2 on the right hand side of $3(29)+1 = 11(2^3)$ and add 1, that is, $3+1$ steps plus 14 steps. Thus for $e_2 = 117$ we need to have 20 steps since $3(117)+1 = 11(2^5)$ and $e_3 = 469$ needs to have 22 steps since $3(469)+1 = 11(2^7)$. You can check that this is true for the rest of numbers in the sequence by choosing randomly. □

Next, what I'm going to do is track another sequence function. The question is how to get this sequence function?. Well, the answer is simple if I draw another figure just like the previous but I have to use the previous figures to draw the next. First of all, for $k \in \mathbb{N}$, let me find the least $\beta_5 = e_k 2^{2n}$ or $= e_k 2^{2n-1}$ for $n = 1$ such that $\beta_5 - 1$ should be divisible by 3, where $e_k \in \{e_n : e_n - e_{n-1} = 11(2^{2n-1}), e_0 = 7 \text{ and } n \in \mathbb{N}\}$ and $(\beta_5 - 1)/3$ will be the initial term for the general term of the next sequence of numbers. So after two or three simple trials you can get that $e_k = 7$ and $\beta_5 = 7(2^2) = 28$ which is the least among others and you can verify that $\beta_5 - 1 = 28 - 1 = 27$ which is divisible by 3 and $(\beta_5 - 1)/3 = 9 = e_0$ is the initial term for the next sequence function. Now let's see the following figure 6 to be more clear for the next and the previous work.

Theorem 6. *Collatz conjecture is true for all odd numbers in the set of the recurrence relation*

$$\{f_n : f_n - f_{n-1} = 7(2^{2n}), f_0 = 9 \text{ and } n \in \mathbb{N}\}$$

$$\Rightarrow f_n = \frac{1}{3}\left[7(2^{2n+2}) - 1\right],\ n \geq 0$$

and for each f_n, there are $N_7 + 2n + 3$ number of steps (or Collatz function operation) needed to get 1.

Proof.

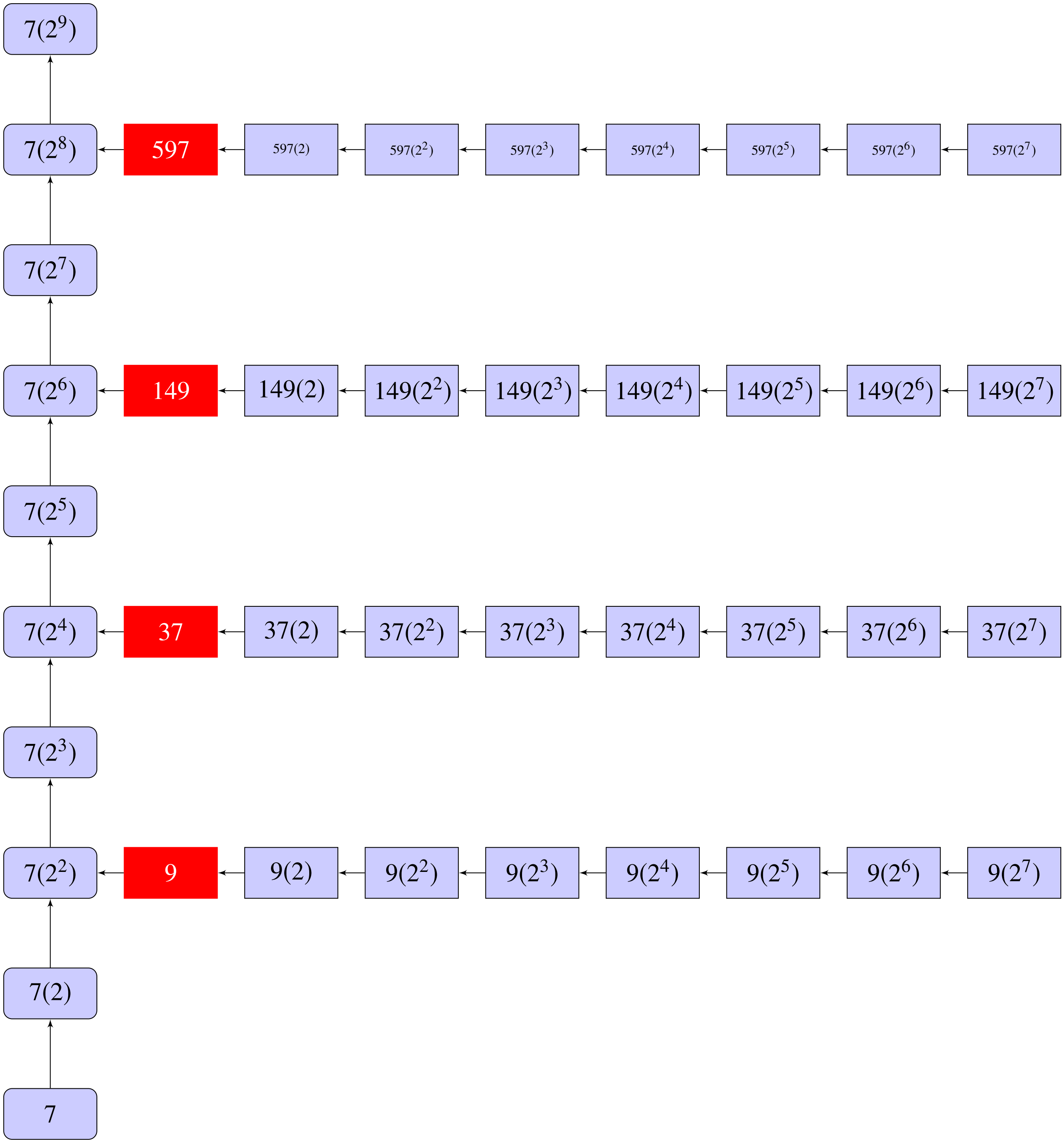

Figure 6: Proof of Collatz conjecture for the sequence of numbers $\{9, 37, 149, 597 \cdots\}$ combined with the previous figures.

As you can see from figure 6, we have sequence of numbers $\{9, 37, 149, 597 \cdots\}$. All elements

of this sequence are odd numbers with $f_0 = 9, f_1 = 37, f_2 = 149, f_3 = 597 \cdots$, and $3(9) + 1 = 7(2^2), 3(37) + 1 = 7(2^4), 3(149) + 1 = 7(2^6), 3(597) + 1 = 11(2^8), \cdots$. The question is how this sequence of numbers formed?. The answer is simple. You can observe that $37 = 9 + 7(2^2), 149 = 37 + 7(2^4), 597 = 149 + 7(2^6), \cdots$. Therefore the recurrence relation is $f_n = f_{n-1} + 7(2^{2n})$ or $f_n - f_{n-1} = 7(2^{2n})$ with $f_0 = 9$ and $n \in \mathbb{N}$.
Hence we can conclude that collatz analysis is true for all numbers in the set

$$\{f_n : f_n - f_{n-1} = 7(2^{2n}), f_0 = 9 \text{ and } n \in \mathbb{N}\}$$

If we solve this non-homogeneous recurrence relation we get,

$$f_n = \frac{1}{3}\left[7(2^{2n+2}) - 1\right],\ n \geq 0$$

This is the general term for the above sequence of numbers. We might ask "How many number of steps we need to reach to 1 for each number in the above sequence by applying Collatz function?". Here is the answer. For example for $n = 0$, we have $f_0 = 9$, therefore according to Collatz conjecture $3(9)+1 = 28 = 7(2^2)$ since f_0 is odd. As you can see from the above figure that we need to have 19 steps, which means $3 + 16$ steps, where $3 = 2 + 1$ is the exponent of 2 plus 1 from the equation on the right side of $3(f_0) + 1 = 28 = 7(2^2)$ and 16 is the number steps needed for 7, so $f_0 = 9$ needs to have $19 = 3 + 16$ number of steps to reach to 1 by applying Collatz function repeatedly. Similarly for $f_1 = 37$ we need to have $16 + 5 = 21$ steps to reach to 1 since $3(37) + 1 = 7(2^4)$, that is take the exponent of 2 on the right hand side of $3(37) + 1 = 7(2^4)$ and add 1, that is, $4 + 1$ steps plus 16 steps. Thus for $f_2 = 149$ we need to have 23 steps since $3(149) + 1 = 7(2^6)$ and $f_3 = 597$ needs to have 25 steps since $3(597) + 1 = 7(2^8)$. You can check that this is true for the rest of numbers in the sequence by choosing randomly. □

Next, what I'm going to do is track another sequence function. The question is how to get this sequence function?. Well, the answer is simple if I draw another figure just like the previous but I have to use the previous pictures to draw the next. First of all, for $k \in \mathbb{N}$, let me find the least $\beta_6 = f_k 2^{2n}$ or $= f_k 2^{2n-1}$ for $n = 1$ such that $\beta_6 - 1$ should be divisible by 3, where $f_k \in \{f_n : f_n - f_{n-1} = 11(2^{2n-1}), f_0 = 7 \text{ and } n \in \mathbb{N}\}$ and $(\beta_6 - 1)/3$ will be the initial term for the general term of the next sequence of numbers. So after two or three simple trials you can get that $f_k = 29$ and $\beta_6 = 29(2^1) = 58$ which is the least among others and you can verify that $\beta_6 - 1 = 58 - 1 = 57$ which is divisible by 3 and $(\beta_6 - 1)/3 = 19 = f_0$ is the initial term for the next sequence function. Now let's see the following figure 7 to be more clear for the next and the previous work.

Theorem 7. *Collatz conjecture is true for all odd numbers in the set of the recurrence relation*

$$\{g_n : g_n - g_{n-1} = 29(2^{2n-1}), g_0 = 19 \textit{ and } n \in \mathbb{N}\}$$

$$g_n = \frac{1}{3}\left[29(2^{2n+1}) - 1\right],\ n \geq 0$$

and for each g_n, there are $N_{29} + 2n + 2$ number of steps (or Collatz function operation) needed to get 1.

Proof.

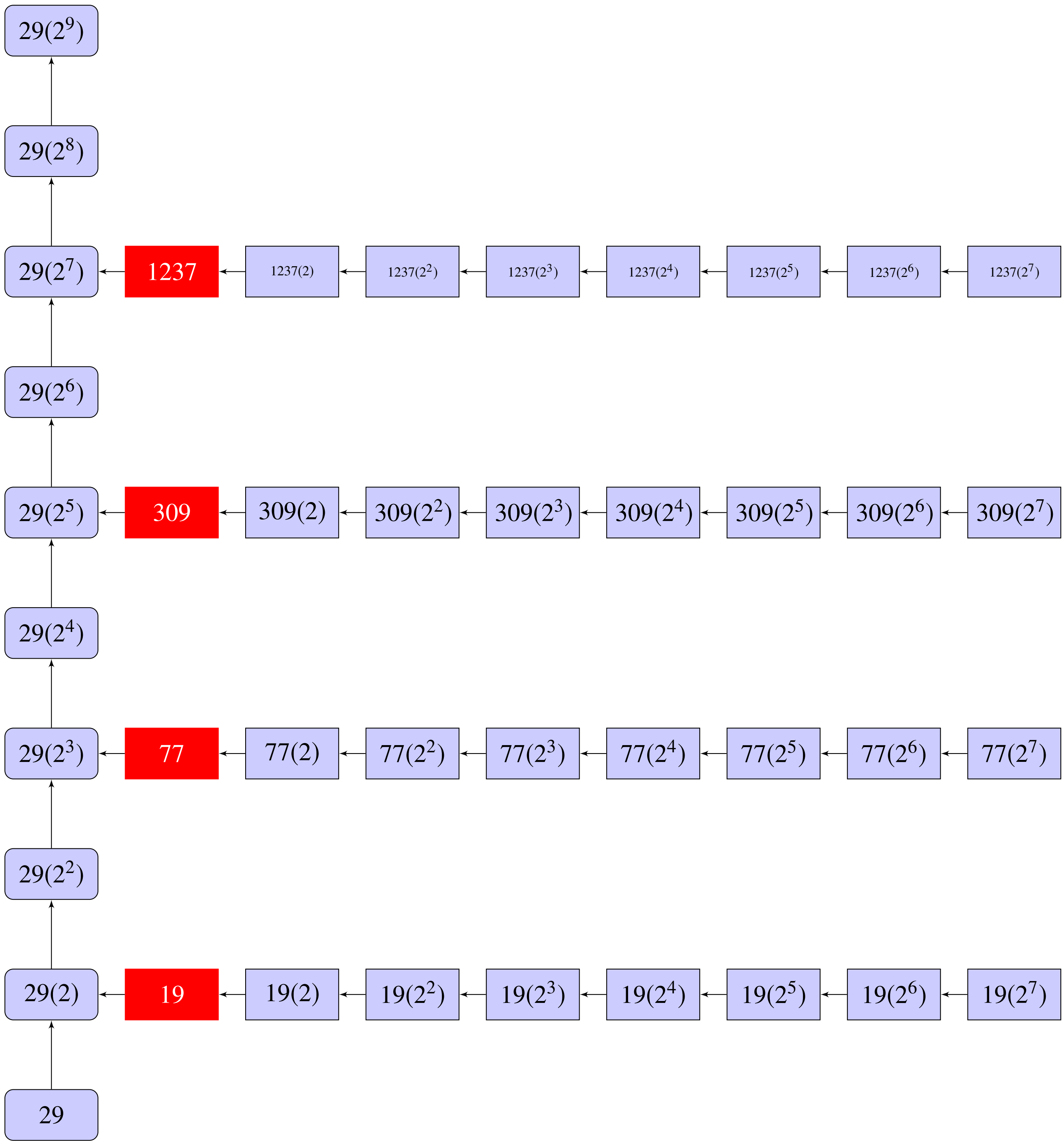

Figure 7: Proof of Collatz conjecture for the sequence of numbers $\{19, 77, 309, 1237 \cdots\}$ combined with the previous figures.

As you see from the picture we have sequence of numbers $\{19, 77, 309, 1237 \cdots\}$. All elements of

this sequence are odd numbers with $g_0 = 19, g_1 = 77, g_2 = 309, g_3 = 1237\cdots$, and $3(19)+1 = 29(2^1), 3(77)+1 = 29(2^3), 3(309)+1 = 29(2^5), 3(1237)+1 = 29(2^7), \cdots$. The question is how this sequence of numbers formed?. The answer is simple. You can observe that $77 = 19+29(2^1), 309 = 77+29(2^3), 1237 = 309+29(2^5), \cdots$. Therefore the general term is $g_n = g_{n-1} + 29(2^{2n-1})$ or $g_n - g_{n-1} = 29(2^{2n-1})$ with $g_0 = 19$ and $n \in \mathbb{N}$.
Hence we can conclude that Collatz analysis is true for all numbers in the set

$$\{g_n : g_n - g_{n-1} = 29(2^{2n-1}), g_0 = 19 \text{ and } n \in \mathbb{N}\}$$

If we solve this non-homogeneous recurrence relation we get,

$$g_n = \frac{1}{3}\left[29(2^{2n+1}) - 1\right],\ n \geq 0$$

This is the general term for the above sequence of numbers. We might ask "How many number of steps we need to reach to 1 for each number in the above sequence by applying Collatz function?". Here is the answer. For example for $n = 0$, we have $g_0 = 19$, therefore according to Collatz conjecture $3(19)+1 = 58 = 29(2^1)$ since g_0 is odd. As you can see from the above figures that we need to have 20 steps, which means $2 + 18$ steps, where $2 = 1 + 1$ is the exponent of 2 plus 1 from the equation on the right side of $3(g_0)+1 = 58 = 29(2^1)$ and 18 is the number steps needed for 29, so $g_0 = 19$ needs to have $20 = 2 + 18$ number of steps to reach to 1 by applying Collatz function repeatedly. Similarly for $g_1 = 77$ we need to have $18 + 4 = 22$ steps to reach to 1 since $3(77)+1 = 29(2^3)$, that is take the exponent of 2 on the right hand side of $3(77)+1 = 29(2^3)$ and add 1, that is, $3 + 1$ steps plus 18 steps. Thus for $g_2 = 309$ we need to have 24 steps since $3(309)+1 = 29(2^5)$ and $g_3 = 1237$ needs to have 26 steps since $3(1237)+1 = 29(2^7)$. You can check that this is true for the rest of numbers in the sequence by choosing randomly. □

7. Proof of Collatz conjecture for all positive integers

I have proved Collatz conjecture for numbers in the recurrence relations:

$$a_n - a_{n-1} = 2^{2n}\ ;\ a_0 = 1$$

$$b_n - b_{n-1} = 5(2^{2n-1})\ ;\ b_0 = 3$$

$$c_n - c_{n-1} = 13(2^{2n-1})\ ;\ c_0 = 17$$

$$d_n - d_{n-1} = 17(2^{2n-1})\ ;\ d_0 = 11$$

$$e_n - e_{n-1} = 11(2^{2n-1})\ ;\ e_0 = 7$$

$$f_n - f_{n-1} = 7(2^{2n}) \; ; \; f_0 = 9$$

$$g_n - g_{n-1} = 29(2^{2n-1}) \; ; \; g_0 = 19$$

Hence we can propose the following recurrence relations such that Collatz conjecture is true for all $k \in \mathbb{Z}+$:

$$D_n - D_{n-1} = (3k-1)2^{2n-1} \; ; \; D_0 = 2k-1 \tag{1}$$

$$\Rightarrow D_n = \frac{1}{3}(2^{2n+1}(3k-1)-1)$$

$$J_n - J_{n-1} = (3k+2)2^{2n-1} \; ; \; J_0 = 2k+1 \tag{2}$$

$$\Rightarrow J_n = \frac{1}{3}(2^{2n+1}(3k+2)-1)$$

$$M_n - M_{n-1} = (6k-1)2^{2n-1} \; ; \; M_0 = 4k-1 \tag{3}$$

$$\Rightarrow M_n = \frac{1}{3}(2^{2n+1}(6k-1)-1)$$

$$K_n - K_{n-1} = (12k-1)2^{2n-1} \; ; \; K_0 = 8k-1 \tag{4}$$

$$\Rightarrow K_n = \frac{1}{3}(2^{2n+1}(12k-1)-1)$$

$$S_n - S_{n-1} = (3k+1)2^{2n} \; ; \; S_0 = 4k+1 \tag{5}$$

$$\Rightarrow S_n = \frac{1}{3}(2^{2n+2}(3k+1)-1)$$

You can easily understand that one recurrence relation is depends on the previous recurrence relation, that is to formulate the next recurrence relation formulae we have to use numbers in the previous recurrence relation formulas. Actually we have discussed this in detail earlier.

Here we have to ask a big question "Is it possible to generalize the above recurrence relations for all $\alpha_0 = 2k-1 = n$ such that Collatz analysis is true for all $k \in \mathbb{N}$, that is, the existence of $\alpha_0 = n = 2k-1$ for all $k \in \mathbb{N}$ where Collatz conjecture is true". If we can show the existence of $n = 2k-1$ for all $k \in \mathbb{N}$ and if we generalize the above recurrence relations that we have discussed above by one recurrence relation formula for all odd natural numbers $\alpha_0 = n$, then Collatz conjecture becomes Collatz Theorem. Let's get started from the proof of the Theorem that shows the existence of $n = 2k-1 = (\beta-1)/3,\; k \in \mathbb{N}$, where $\beta = 2\alpha_k$ or $\beta = 4\alpha_k$ such that $\beta - 1$ is multiple of 3 or divisible by 3 as we have discussed before.

Theorem 8. *If we can find the sequence α_k or $2\alpha_k$ for all $k \in \mathbb{N}$ where $\beta = 2\alpha_k$ or $\beta = 4\alpha_k$ such that $\beta - 1$ is a multiple of 3 or divisible by 3, then $n = 2k - 1$ existed for all $k \in \mathbb{N}$.*

Proof.

- **Case I:** When $\beta = 2\alpha_k$,
  Suppose $n = 2k - 1 = (\beta - 1)/3$ for all $k \in \mathbb{N}$

$$\Longrightarrow 2k - 1 = \frac{2\alpha_k - 1}{3}$$

$$\Longrightarrow 2k = \frac{2\alpha_k - 1}{3} + 1$$

$$\Longrightarrow \alpha_k = 3k - 1$$

  Therfore $\beta = 2\alpha_k = 2(3k - 1) = 6k - 2 \Longrightarrow \alpha_0 = (\beta - 1)/3 = 2k - 1$

- **Case II:** When $\beta = 4\alpha_k$,
  Suppose $n = 2k - 1 = (\beta - 1)/3$ for all $k \in \mathbb{N}$

$$\Longrightarrow 2k - 1 = \frac{4\alpha_k - 1}{3}$$

$$\Longrightarrow 2k = \frac{4\alpha_k - 1}{3} + 1$$

$$\Longrightarrow 2\alpha_k = 3k - 1$$

  Therfore $\beta = 4\alpha_k = 2(3k - 1) = 6k - 2 \Longrightarrow \alpha_0 = (\beta - 1)/3 = 2k - 1$

  We can also suppose $n = 4k + 1 = (\beta - 1)/3$ for all $k \in \mathbb{N}$

$$\Longrightarrow 4k + 1 = \frac{4\alpha_k - 1}{3}$$

$$\Longrightarrow 4k = \frac{4\alpha_k - 1}{3} - 1$$

$$\Longrightarrow \alpha_k = 3k + 1$$

  Therfore $\beta = 4\alpha_k = 4(3k + 1) = 12k + 4 \Longrightarrow \alpha_0 = (\beta - 1)/3 = 4k + 1$

□

Theorem 9. *For all $n = 2k - 1$, $k \in \mathbb{N}$, Using collatz analysis function, we can form the sequence function that generalizes the above recurrence relations, that is,*

$$a_m = \frac{1}{3}\Big[4^m(3n + 1) - 1\Big], \; m \geq 0$$

Proof. As we have discussed above, it is always true that for $a_0 = n$, we have the following recurrence relations

$$a_k - a_{k-1} = 3a_{k-1} + 1, a_0 = n$$
$$\implies a_k = 4a_{k-1} + 1, a_0 = n$$

which means

$$\text{when } k = 1, a_1 = 4a_0 + 1 = 4n + 1 = 4^1 n + 1$$

$$\text{when } k = 2, a_2 = 4(a_1) + 1 = 4(4n + 1) + 1 = 16n + 5 = 4^2 n + 5$$

$$\text{when } k = 3, a_3 = 4(a_2) + 1 = 4(16n + 5) + 1 = 64n + 21 = 4^3 n + 21$$

$$\text{when } k = 4, a_4 = 4(a_3) + 1 = 4(64n + 21) + 1 = 256n + 85 = 4^4 n + 85$$

$$\text{when } k = 5, a_5 = 4(a_4) + 1 = 4(256n + 85) + 1 = 1024n + 341 = 4^5 n + 341$$
$$\ldots$$

Therefore we have sequence of numbers for all $n = 2k - 1$, $k \in \mathbb{N}$,

$$n, 4^1 n + 1, 4^2 n + 5, 4^3 n + 21, 4^4 n + 85, 4^5 n + 341, \cdots$$

But we know that the sequence of numbers $0, 1, 5, 21, 85, 341, \cdots$ is equal to

$$\frac{1}{3}\left[4^m - 1\right], m \geq 0$$

Hence we obtain the general term of the sequence,

$$n, 4^1 n + 1, 4^2 n + 5, 4^3 n + 21, 4^4 n + 85, 4^5 n + 341, \cdots$$

becomes

$$a_m = 4^m n + \frac{1}{3}\left[4^m - 1\right], m \geq 0$$

$$\implies a_m = \frac{1}{3}\left[(3n + 1)4^m - 1\right], m \geq 0$$

□

Now we have to discuss on the generalization of the number of Collatz function operation needed for each given odd natural number n. Let's get started from the **Lemma 1**.

Lemma 1. *The number of Collatz function operation needed for* 1 *is* 3. *That is,*

$$N_1 = 3$$

Proof. Since 1 is odd, then we have to multiply it by 3 and add 1, that is, $3(1) + 1$ gives us an even number 4, then divide it by 2, that is, $4/2$ gives us an even number 2 again, then divide it by 2, that is, $2/2$ gives us 1. Therefore we have used Collatz function 3 times. Hence $N_1 = 3$. □

Theorem 10. *For each $k = 1, 2, 3, \cdots$, the number of Collatz function operation needed for $2k - 1$ is $2 + N_{(3k-1)}$. That is, we can state the recurrence relation:*

$$N_{(2k-1)} = 2 + N_{(3k-1)} \text{ ; } N_1 = 3$$

Proof. We can proof this Theorem using mathematical induction. For $k = 1$, we have

$$N_{(2(1)-1)} = 2 + N_{(3(1)-1)}$$

$$\Rightarrow N_1 = 2 + N_2$$

$$\Rightarrow 3 = 2 + N_2$$

$$\Rightarrow N_2 = 3 - 2 = 1$$

Therefore for $k = 1$, the recurrence relation is true.

Now let's prove for $k = n$, that is,

Since $2n - 1$ is odd, then we have to multiply it by 3 and add 1, that is, $3(2n - 1) + 1$ gives us an even number $2(3n - 1)$, then divide it by 2, that is, $2(3n - 1)/2$ gives us a number $3n - 1$. Therefore we have used Collatz function 2 times. Hence $N_{(2n-1)} = 2 + N(3n - 1)$. Therefore for $k = n$, the recurrence relation is true.

Now let's prove for $k = n + 1$, that is,

Since $2(n + 1) - 1 = 2n + 1$ is odd, then we have to multiply it by 3 and add 1, that is, $3(2n + 1) + 1$ gives us an even number $2(3n + 2)$, then divide it by 2, that is, $2(3n + 2)/2$ gives us a number $3n + 2$. Therefore we have used Collatz function 2 times. Hence $N_{(2n+1)} = 2 + N(3n + 2)$. Therefore for $k = n + 1$, the recurrence relation is true, that is,

$$N_{(2(n+1)-1)} = 2 + N_{(3(n+1)-1)}$$

$$\Rightarrow N_{(2n+1)} = 2 + N_{(3n+2)} \text{ ; } N_3 = 7$$

□

Theorem 11. *For each $k = 1, 2, 3, \cdots$, the number of Collatz function operation needed for D_n is $2n + N_{D_0}$. That is,*

$$N_{D_n} = 2n + N_{D_0}$$

where $D_0 = 2k - 1$ and D_n is as defined as in equation **(1)***, that is, for $k = 1, 2, 3, \cdots$*

$$D_n = \frac{1}{3}(2^{2n+1}(3k - 1) - 1)$$

Proof. We can proof this Theorem using mathematical induction. For $n = 1$, we have

$$N_{D_1} = 2 + N_{D_0}$$

$$\Rightarrow N_{\left(\frac{1}{3}(2^3(3k-1)-1)\right)} = 2 + N_{2k-1}$$

Since the number D_1 is odd, then we have to multiply by 3 and add 1, that is, $3D_1 + 1$ gives us an even number $2^3(3k-1)$, then divide it by 2 (three times), that is, $2^3(3k-1)/2^3$ gives us a number $3k-1$. Therefore we applied Collatz function 4 times. Hence $N_{D_1} = 4 + N_{3k-1}$. Therefore

$$N_{D_1} = 2 + N_{D_0}$$

$$\Rightarrow 4 + N_{(3k-1)} = 2 + N_{D_0} = 2 + N_{(2k-1)}$$

$$\Rightarrow 2 + N_{(3k-1)} = N_{(2k-1)}$$

Therefore for $n = 1$, the recurrence relation is true.

Now let's prove for $n = m$, that is,

Since D_m is odd, then we have to multiply it by 3 and add 1, that is, $3D_m + 1$ gives us an even number $2^{2m+1}(3k-1)$, then divide it by two ($2m+1$ times), that is, $2^{2m+1}(3k-1)/2^{2m+1}$ gives us a number $3k-1$. Therefore we have operated Collatz function $2m+2$ times. Hence $N_{D_m} = 2m + 2 + N(3k-1) = 2m + N_{(2k-1)} = 2m + N_{D_0}$. Therefore for $n = m$, the relation is true.

Now let's prove for $n = m + 1$, that is,

Since D_{m+1} is odd, then we have to multiply it by 3 and add 1, that is, $3D_{m+1} + 1$ gives us an even number $2^{2(m+1)+1}(3k-1)$, then divide it by two ($2(m+1)+1$ times), that is, $2^{2(m+1)+1}(3k-1)/2^{2(m+1)+1}$ gives us a number $3k-1$. Therefore we have operated Collatz function $2(m+1)+2$ times. Hence $N_{D_{m+1}} = 2(m+1) + 2 + N(3k-1) = 2m + 2 + 2 + N_{(3k-1)} = 2m + 2 + N_{(2k-1)} = 2(m+1) + N_{D_0}$. Therefore for $n = m + 1$, the relation is true, that is,

$$N_{D_n} = 2n + N_{D_0}$$

$$\Rightarrow N_{D_{m+1}} = 2(m+1) + N_{D_0}$$

□

Theorem 12. *If* $3n + 1 = 2^{s_1} b_1$*,* $3(b_1) + 1 = 2^{s_2} b_2$*,* $3(b_2) + 1 = 2^{s_3} b_3, \cdots, 3(b_{k-1}) + 1 = 2^{s_k}$*, then for the given odd natural number n, it takes* $s_1 + s_2 + \cdots + s_k + k$ *steps to reach to* 1 *by applying Collatz function*

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ 3n+1 & \text{if } n \text{ is odd.} \end{cases}$$

repeatedly.

Proof. Since $3n+1 = 2^{s_1} b_1$ is even, then divide $2^{s_1} b_1$ by 2, s_1 times to get b_1. So for the odd number n we used $s_1 + 1$ steps to get an odd number b_1 or we operate Collatz function $s_1 + 1$ times to get b_1. Now we have to multiply b_1 by 3 and add 1 since b_1 is an odd number, that is, $3(b_1) + 1 = 2^{s_2} b_2$. Now since $2^{s_2} b_2$ is an even number, therefore we have to operate by Collatz function s_2 times to get b_2 or divide $2^{s_2} b_2$ by 2, s_2 times. Thus we used $s_2 + 1$ steps for the odd number b_1 to get b_2. Similarly we have to operate Collatz function $s_3 + 1$ times to get b_3 starting from $3(b_2) + 1$, operate Collatz function $s_4 + 1$ times to get b_4 starting from $3(b_3) + 1$, operate Collatz function $s_5 + 1$ times to get b_5 starting from $3(b_4) + 1, \cdots$, operate Collatz function $s_k + 1$ times to get $b_k = 1$ starting from $3(b_{k-1}) + 1$. Hence we used $(s_1+1)+(s_2+1)+(s_3+1)+(s_4+1)+\cdots+(s_k+1) = s_1+s_2+s_3+\cdots+s_k+k$ steps to get 1 for the given number n, or we operated Collatz function $s_1 + s_2 + s_3 + \cdots + s_k + k$ times to get 1 starting from $3n + 1$. Hence

$$N_n = s_1 + s_2 + s_3 + \cdots + s_k + k$$

□

8. Examples

Example 1. *Find N_{8k-3}.*

Solution. Since $N_{D_n} = 2n + N_{D_0}$ this implies $N_{D_1} = 2 + N_{D_0}$, where

$$D_n = \frac{1}{3}(2^{2n+1}(3k-1) - 1)$$

$$\Rightarrow D_1 = \frac{1}{3}(2^3(3k-1) - 1) = 2^3 k - \frac{1}{3}(2^3 + 1) = 2^3 k - 3$$

Therefore $N_{8k-3} = 2 + N_{2k-1}$.

Example 2. *Find N_{109} and N_{437}.*

Solution. From $N_{8k-3} = 2 + N_{2k-1}$, take $k = 14$ and we know that $N_{27} = 111$. Therefore $N_{109} = 2 + N_{27} = 113$. Similarly, $N_{437} = 2 + 113 = 115$.

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