Error Propagation for 3D Registration Problems

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Following the write up of Haralick, let

$$
\begin{aligned}
f(X,\Theta) &= \sum_{j=1}^{n} \|\mathbf{R}\mathbf{x}_i + \mathbf{t} - \hat{\mathbf{x}}_i\|^2 \\
X &= [\mathbf{x}_1; \mathbf{x}_2; \ldots; \mathbf{x}_n; \hat{\mathbf{x}}_1; \hat{\mathbf{x}}_2; \ldots; \hat{\mathbf{x}}_n] \\
\Theta &= [\theta, \phi, \rho, x, y, z]
\end{aligned}
$$

where rotation

$$\mathbf{R} = \mathbf{R}(\theta,\phi,\rho) = \cos\rho\mathbf{I} + \sin\rho[\mathbf{u}]_S + (1-\cos\rho)\ \mathbf{u}\mathbf{u}^T$$

where $\mathbf{u} = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)^T$ is the axis of rotation of $\mathbf{R}$, ρ is the angle of rotation of $\mathbf{R}$, and $[\mathbf{u}]_S$ is the skew-symmetric matrix representation of the vector $\mathbf{u}$. And, translation

$$\mathbf{t} = [x,\ y,\ z]^T.$$

Note that each component of f can be be decomposed as

$$
\begin{aligned}
\|\mathbf{R}\mathbf{x}_i + \mathbf{t} - \hat{\mathbf{x}}_i\|^2 &= (\mathbf{R}\mathbf{x}_i + \mathbf{t} - \hat{\mathbf{x}}_i)^T(\mathbf{R}\mathbf{x}_i + \mathbf{t} - \hat{\mathbf{x}}_i) \\
&= \mathbf{x}_i^T\mathbf{x}_i + 2\mathbf{t}^T\mathbf{R}\mathbf{x}_i - 2\hat{\mathbf{x}}_i^T\mathbf{R}\mathbf{x}_i - 2\mathbf{t}^T\hat{\mathbf{x}}_i + \mathbf{t}^T\mathbf{t} + \hat{\mathbf{x}}_i^T\hat{\mathbf{x}}_i.
\end{aligned}
$$

Using this formulation, the gradient of f can be decomposed as the partials

$$
g(X,\Theta) = \begin{pmatrix} f_\theta \\ f_\phi \\ f_\rho \\ f_x \\ f_y \\ f_z \end{pmatrix} = \sum_{j=1}^{n} \begin{pmatrix} 2\mathbf{t}^T\mathbf{R}_\theta\mathbf{x}_i - 2\hat{\mathbf{x}}_i^T\mathbf{R}_\theta\mathbf{x}_i \\ 2\mathbf{t}^T\mathbf{R}_\phi\mathbf{x}_i - 2\hat{\mathbf{x}}_i^T\mathbf{R}_\phi\mathbf{x}_i \\ 2\mathbf{t}^T\mathbf{R}_\rho\mathbf{x}_i - 2\hat{\mathbf{x}}_i^T\mathbf{R}_\rho\mathbf{x}_i \\ 2\mathbf{R}\mathbf{x}_i - 2\hat{\mathbf{x}}_i + 2\mathbf{t} \end{pmatrix} = \sum_{j=1}^{n} \begin{pmatrix} 2(\mathbf{R}_\theta\mathbf{x}_i)^T(\mathbf{t} - \hat{\mathbf{x}}_i) \\ 2(\mathbf{R}_\phi\mathbf{x}_i)^T(\mathbf{t} - \hat{\mathbf{x}}_i) \\ 2(\mathbf{R}_\rho\mathbf{x}_i)^T(\mathbf{t} - \hat{\mathbf{x}}_i) \\ 2\mathbf{R}\mathbf{x}_i - 2\hat{\mathbf{x}}_i + 2\mathbf{t} \end{pmatrix}
$$

where

$$
\begin{aligned}
\mathbf{R}_\theta &= \sin\rho\left[\frac{\partial\mathbf{u}}{\partial\theta}\right]_S + (1-\cos\rho)\ \left(\frac{\partial\mathbf{u}}{\partial\theta}\mathbf{u}^T + \mathbf{u}\frac{\partial\mathbf{u}}{\partial\theta}^T\right) \\
\mathbf{R}_\phi &= \sin\rho\left[\frac{\partial\mathbf{u}}{\partial\phi}\right]_S + (1-\cos\rho)\ \left(\frac{\partial\mathbf{u}}{\partial\phi}\mathbf{u}^T + \mathbf{u}\frac{\partial\mathbf{u}}{\partial\phi}^T\right) \\
\mathbf{R}_\rho &= -\sin\rho\mathbf{I} + \cos\rho[\mathbf{u}]_S + \sin\rho\ \mathbf{u}\mathbf{u}^T
\end{aligned}
$$

Taking the Taylor's series expansion of g around $(\hat{X},\hat{\Theta}) = (X + \Delta X, \Theta + \Delta\Theta)$ we obtain a first order approximation:

$$g(X,\Theta) = g(\hat{X},\hat{\Theta}) - \frac{\partial g}{\partial X}(\hat{X},\hat{\Theta})\Delta X - \frac{\partial g}{\partial \Theta}(\hat{X},\hat{\Theta})\Delta\Theta$$

Here

$$\frac{\partial g}{\partial X} = \begin{pmatrix} 2(\mathbf{t}-\hat{\mathbf{x}}_1)^T\mathbf{R}_\theta & \cdots & 2(\mathbf{t}-\hat{\mathbf{x}}_n)^T\mathbf{R}_\theta & -2(\mathbf{R}_\theta\mathbf{x}_1)^T & \cdots & -2(\mathbf{R}_\theta\mathbf{x}_n)^T \\ 2(\mathbf{t}-\hat{\mathbf{x}}_1)^T\mathbf{R}_\phi & \cdots & 2(\mathbf{t}-\hat{\mathbf{x}}_n)^T\mathbf{R}_\phi & -2(\mathbf{R}_\phi\mathbf{x}_1)^T & \cdots & -2(\mathbf{R}_\phi\mathbf{x}_n)^T \\ 2(\mathbf{t}-\hat{\mathbf{x}}_1)^T\mathbf{R}_\rho & \cdots & 2(\mathbf{t}-\hat{\mathbf{x}}_n)^T\mathbf{R}_\rho & -2(\mathbf{R}_\rho\mathbf{x}_1)^T & \cdots & -2(\mathbf{R}_\rho\mathbf{x}_n)^T \\ 2\mathbf{R} & \cdots & 2\mathbf{R} & -2\mathbf{I} & \cdots & -2\mathbf{I} \end{pmatrix}$$

$$\frac{\partial g}{\partial \Theta} = \sum_{j=1}^{n} \begin{pmatrix} 2(\mathbf{R}_{\theta\theta}\mathbf{x}_i)^T(\mathbf{t}-\hat{\mathbf{x}}_i) & 2(\mathbf{R}_{\phi\theta}\mathbf{x}_i)^T(\mathbf{t}-\hat{\mathbf{x}}_i) & 2(\mathbf{R}_{\rho\theta}\mathbf{x}_i)^T(\mathbf{t}-\hat{\mathbf{x}}_i) & 2(\mathbf{R}_\theta\mathbf{x}_i)^T \\ 2(\mathbf{R}_{\theta\phi}\mathbf{x}_i)^T(\mathbf{t}-\hat{\mathbf{x}}_i) & 2(\mathbf{R}_{\phi\phi}\mathbf{x}_i)^T(\mathbf{t}-\hat{\mathbf{x}}_i) & 2(\mathbf{R}_{\rho\phi}\mathbf{x}_i)^T(\mathbf{t}-\hat{\mathbf{x}}_i) & 2(\mathbf{R}_\phi\mathbf{x}_i)^T \\ 2(\mathbf{R}_{\theta\rho}\mathbf{x}_i)^T(\mathbf{t}-\hat{\mathbf{x}}_i) & 2(\mathbf{R}_{\phi\rho}\mathbf{x}_i)^T(\mathbf{t}-\hat{\mathbf{x}}_i) & 2(\mathbf{R}_{\rho\rho}\mathbf{x}_i)^T(\mathbf{t}-\hat{\mathbf{x}}_i) & 2(\mathbf{R}_\rho\mathbf{x}_i)^T \\ 2\mathbf{R}_\theta\mathbf{x}_i & 2\mathbf{R}_\phi\mathbf{x}_i & 2\mathbf{R}_\rho\mathbf{x}_i & 2\mathbf{I} \end{pmatrix}$$

where

$$\begin{aligned}
\mathbf{R}_{\theta\theta} &= \sin\rho\left[\frac{\partial^2\mathbf{u}}{\partial\theta^2}\right]_S + (1-\cos\rho)\left(\frac{\partial^2\mathbf{u}}{\partial\theta^2}\mathbf{u}^T + 2\frac{\partial\mathbf{u}}{\partial\theta}\frac{\partial\mathbf{u}}{\partial\theta}^T + \mathbf{u}\frac{\partial^2\mathbf{u}}{\partial\theta^2}^T\right) \\
\mathbf{R}_{\phi\theta} &= \mathbf{R}_{\theta\phi} = \sin\rho\left[\frac{\partial^2\mathbf{u}}{\partial\phi\,\partial\theta}\right]_S + (1-\cos\rho)\left(\frac{\partial^2\mathbf{u}}{\partial\phi\,\partial\theta}\mathbf{u}^T + \frac{\partial\mathbf{u}}{\partial\theta}\frac{\partial\mathbf{u}}{\partial\phi}^T + \frac{\partial\mathbf{u}}{\partial\phi}\frac{\partial\mathbf{u}}{\partial\theta}^T + \mathbf{u}\frac{\partial^2\mathbf{u}}{\partial\phi\,\partial\theta}^T\right) \\
\mathbf{R}_{\rho\theta} &= \mathbf{R}_{\theta\rho} = \cos\rho\left[\frac{\partial\mathbf{u}}{\partial\theta}\right]_S + \sin\rho\left(\frac{\partial\mathbf{u}}{\partial\theta}\mathbf{u}^T + \mathbf{u}\frac{\partial\mathbf{u}}{\partial\theta}^T\right) \\
\mathbf{R}_{\phi\phi} &= \sin\rho\left[\frac{\partial^2\mathbf{u}}{\partial\phi^2}\right]_S + (1-\cos\rho)\left(\frac{\partial^2\mathbf{u}}{\partial\phi^2}\mathbf{u}^T + 2\frac{\partial\mathbf{u}}{\partial\phi}\frac{\partial\mathbf{u}}{\partial\phi}^T + \mathbf{u}\frac{\partial^2\mathbf{u}}{\partial\phi^2}^T\right) \\
\mathbf{R}_{\rho\phi} &= \mathbf{R}_{\phi\rho} = \cos\rho\left[\frac{\partial\mathbf{u}}{\partial\phi}\right]_S + \sin\rho\left(\frac{\partial\mathbf{u}}{\partial\phi}\mathbf{u}^T + \mathbf{u}\frac{\partial\mathbf{u}}{\partial\phi}^T\right) \\
\mathbf{R}_{\rho\rho} &= -\cos\rho\mathbf{I} - \sin\rho[\mathbf{u}]_S + \cos\rho\;\mathbf{u}\mathbf{u}^T
\end{aligned}$$

Since $\hat{\Theta}$ extremizes $F(\hat{X},\hat{\Theta})$,

$$g(\hat{X},\hat{\Theta}) = 0.$$

Similarly, Θ extremizes $F(X,\Theta)$, so

$$g(X,\Theta) = 0$$

Therefore,

$$0 = -\frac{\partial g}{\partial X}(\hat{X},\hat{\Theta})\Delta X - \frac{\partial g}{\partial \Theta}(\hat{X},\hat{\Theta})\Delta\Theta$$

Since the relative extremum of F is a relative minimum,

$$\frac{\partial g}{\partial \Theta}(\hat{X},\hat{\Theta}) = \frac{\partial f^2}{\partial^2 \Theta}(\hat{X},\hat{\Theta})$$

must be positive-definite and invertible. Consequently,

$$\boxed{\Delta\Theta = -\left(\frac{\partial g}{\partial \Theta}(\hat{X},\hat{\Theta})\right)^{-1}\frac{\partial g}{\partial X}(\hat{X},\hat{\Theta})\Delta X}$$

up to a first order approximation.