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February 2023

1 Introduction

2 Problem 3

2.1 b

Find all polynomials g that satisfy the equation:

$$g(2x) = 2g(x)$$

Suppose that g is a k-th degree polynomial. g(x) can be written as the following with $x_i, c_i \in \mathbb{R}$ and $c_k \neq 0$:

$$g(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \cdots + c_{k-1} x^{k-1} + c_k x^k \tag{1}$$

Given that $g(2x) = 2g(x)$ is true for all real number x, $g(0) = 2g(0)$. Therefore $g(0) = 0$, combining with equation 1, it's easy to get $c_0 = 0$. Therefore, g(x) can be simplified to the following equation 2.

$$g(x) = c_1 x + c_2 x^2 + c_3 x^3 \cdots + c_{k-1} x^{k-1} + c_k x^k = \sum_{n=1}^{k} c_n x^n \tag{2}$$

The given condition can be rewritten and simplified:

$$g(2x) = \sum_{n=1}^{k} c_n (2x)^n = 2g(x) = 2 \sum_{n=1}^{k} c_n x^n$$

$$\sum_{n=1}^{k} 2^n c_n x^n - \sum_{n=1}^{k} 2 c_n x^n = 0$$

$$\sum_{n=1}^{k} (2^{n-1} - 1) c_n x^n = 0$$

Since this given condition works for all $x \in \mathbb{R}$, any integer values of x will satisfy this equation. Take $x = 1$, the condition will be simplified into:

$$\sum_{n=1}^{k} (2^{n-1} - 1)c_n = 0$$

When x=2, the condition becomes:

$$\sum_{n=1}^{k} (2^{n-1} - 1)2^n c_n = 0$$

First looking at these two situations of x, the equations can be combined into a system of equations since the coefficients are the same in both situations.

$$\begin{cases} 0 + \quad 1 \times c_2 + \quad 3 \times c_3 + 7 \times c_4 \quad + \cdots = 0 \\ 0 + 1 \times 2^2 c_2 + 3 \times 2^3 c_3 + 7 \times 2^4 c_4 + \cdots = 0 \end{cases}$$

Here, notice that each term with the same degree of x has a different coefficient in the two equations. The ratio between each term will be 2^n, which isn't a constant and changes each term. Therefore, this two equations are not the same function. This means that one cannot be simplified into the other using multiplication or division.

Using the process of elimination, the term c_2 can be eliminated, through multiplying the first equation by -4 and adding to the second equation. The second equation will become:

$$0 + (1 \times 2^2 - 4)c_2 + (3 \times 2^3 - 4)c_3 + (7 \times 2^4 - 4)c_4 + \cdots = 0$$

$$0 + 0 + (3 \times 2^3 - 4)c_3 + (7 \times 2^4 - 4)c_4 + \cdots = 0 \tag{3}$$

Since the ratio between terms changes as previously mentioned, no other terms of equation 1 and 2 have the coefficient ratio of 4 again, except the second term containing c_2. Therefore, no other terms, except the ones containing c_2, will be eliminated from this process, and all will have non-zero coefficients.

This process can be repeated using the equation when $x = 3$. Using the system of equations containing the equation when $x = 1$ and $x = 3$, the term with c_2 can also be eliminated by the same process, the resulting equation will be:

$$0 + 0 + (3 \times 3^3 - 9)c_3 + (7 \times 3^4 - 9)c_4 + \cdots = 0 \tag{4}$$

Now, the process of elimination shown above means that with k different values of x (one being $x = 1$), the equations can undergo the process of elimination as described before, and the term c_2 will be eliminated in the remaining k-1 equations (minus one since each equation uses the base equation when $x = 1$ to eliminate and each elimination with two equations only yields one result).

Observing equation 3 and 4, the coefficient of each term will be $(2^{n-1} - 1)3^n c_n - 9$. Generalizing this pattern, for any $x \neq 1$, the equation obtained from substitution can be simplified using the process of elimination and each term will have a coefficient of $(2^{n-1} - 1)x^n c_n - x^2$, since the coefficient of c_2 to be eliminated is x^2.

This is a rather useful generalization. This means that for any two different but known values of x, the difference between the coefficients will be different as well. This is because x^2 is a constant for known values of x. Therefore the difference between the coefficients will be

$$(2^{n-1} - 1)x_1^n c_n - x_1^2 - (2^{n-1} - 1)x_2^n c_n + x_2^2$$

Let m be the difference between x^2, denote this difference of coefficient as d,

$$d = (2^{n-1} - 1)x_1^n c_n - (2^{n-1} - 1)x_2^n c_n + m$$

With $x_1 \neq x_2$, this difference will not be constant because $x_1^n - x_2^n$ changes with n.

Recall that the term c_2 is already eliminated and turned k equations into k-1 equations in terms of c_3 to c_k. The difference in coefficient between any two equations are not the same as shown above, therefore the current k-1 equations can be considered as k_1 equations where $k_1 = k - 1$.

With changing difference in coefficients, the current system of equation can be treated as the system before applying the process of elimination. Therefore, with the same steps shown above, the k_1 equations can be reduced to $k_1 - 1$ equations where terms from c_4 to c_k have non-zero coefficients and c_1 to c_3 are eliminated.

Let $k_2 = k_1 - 1$, the process can be repeated again and c_4 will be eliminated. Repeating this process, the original system of k equations will be eliminated until the only term is c_k with some non-zero coefficient. Since the right side of equation is also zero, the process of elimination only adds or subtracts some multiple of 0 to 0, which will still be 0. So the final equation will be:

$$p \times c_k = 0$$

where p is some non-zero real number. Therefore $c_k = 0$, which contradicts with the definition when composing $g(x)$: $c_k \neq 0$. This means that $g(x)$ cannot be a k-th degree polynomial.

However, recall that the first process of elimination only eliminates c_2. c_1 was never eliminated since $(2^{1-1} - 1)xc_1$ will always be zero for all x. Therefore when $k < 1$, no process of elimination will be done and there is no contradiction. The only polynomial that satisfies the given condition will be a linear polynomial or a 0-th degree polynomial.

If $g(x)$ is 0-th degree polynomial – a constant – the condition given would be: $g(2x) = c = 2g(x) = 2c$, so $c = 0$.

If $g(x)$ is a linear function, let it be $g(x) = mx + c_0$. Since $c_0 = 0$ as previously stated, $g(x) = mx$ would suffice.

Therefore, all polynomials satisfying the given conditions will be:

$$g(x) = 0 \tag{5}$$

$$g(x) = mx(m \in \mathbb{R}) \tag{6}$$

PS: The process of elimination can be more easily understood in the field of linear algebra. Select k values of x to write k equations with c_1 to c_k to for a system of equation. The system can be written as a product between a coefficient matrix M and a vector X containing c_1 to c_k, and the product is 0: $MX = 0$. Since the each vector in the coefficient matrix are not equal as mentioned, each two are linearly independent so M has full rank k. The expanded matrix $(M, 0)$ will also have full rank k. Therefore, there must be one solution for X. It's easy to tell that $c_1 = c_2 = \cdots = c_k = 0$ is one solution, so it will also be the only solution.