EC708 Discussion 2
Weak IV

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February 5, 2021

Outline

1. Problems of IV and 2SLS Estimators with Weak Instruments
2. Weak Instruments Asymptotics
3. Detecting Weak Instruments
4. Robust Inference against Weak Instruments

Notational remark: For a random scalar or vector w_t, let $w = (w_1, \ldots, w_T)'$.

Problems with Weak Instruments

Toy Example: Totally Irrelevant Instruments

$$
\begin{aligned}
y_t &= x_t\beta + u_t, \\
x_t &= Z_t\pi + v_t,
\end{aligned}
$$

where x_t and Z_t are scalars. Suppose

- one observes i.i.d. data $\{(y_t, x_t, Z_t)\}$;
- $\pi = 0$;
- conditional homoskedasticity:

$$
\mathsf{Var}\left(\begin{pmatrix} u_t \\ v_t \end{pmatrix} \middle| Z_t \right) = \begin{pmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{pmatrix}.
$$

Problems with Weak Instruments

Toy Example: Totally Irrelevant Instruments

OLS estimator is inconsistent:

$$\hat{\beta}_{OLS} - \beta = \frac{\frac{1}{T}\sum_{t=1}^{T} u_t v_t}{\frac{1}{T}\sum_{t=1}^{T} v_t^2} \xrightarrow{p} \frac{\sigma_{uv}}{\sigma_v^2} \neq 0.$$

On the other hand, by CLT,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \begin{pmatrix} Z_t u_t \\ Z_t v_t \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \xi_u \\ \xi_v \end{pmatrix} \sim N\left(0, E(Z_t^2) \begin{pmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{pmatrix}\right).$$

We can decompose $\xi_u = \frac{\sigma_{uv}}{\sigma_v^2}\xi_v + \xi$ with $\xi \perp\!\!\!\perp \xi_v$, and

$$\hat{\beta}_{IV} - \beta = \frac{\frac{1}{\sqrt{T}}\sum_{t=1}^{T} Z_t u_t}{\frac{1}{\sqrt{T}}\sum_{t=1}^{T} Z_t v_t} \xrightarrow{d} \frac{\xi_u}{\xi_v} = \frac{\sigma_{uv}}{\sigma_v^2} + \frac{\xi}{\xi_v}$$

where $\frac{\xi}{\xi_v}$ has Cauchy distribution.

Problems with Weak Instruments

Toy Example: Totally Irrelevant Instruments

Observations:

- $\hat{\beta}_{IV}$ is inconsistent and centered around the limit of $\hat{\beta}_{OLS}$.
- Asymptotically $\hat{\beta}_{IV}$ has heavy tails.
- The t-statistic is

$$t_{\hat{\beta}_{IV}} = \frac{\hat{\beta}_{IV} - \beta}{\sqrt{\hat{\sigma}_u^2 \sum_{t=1}^T Z_t^2 / (\sum_{t=1}^T Z_t x_t)^2}}$$

where

$$\hat{\sigma}_u^2 = \frac{1}{T} \sum_{t=1}^{T} (y_t - \hat{\beta}_{IV} x_t)^2 \xrightarrow{d} \sigma_u^2 - 2\sigma_{uv} \frac{\xi_u}{\xi_v} + \sigma_v^2 \left(\frac{\xi_u}{\xi_v} \right)^2 .$$

As $\frac{\sigma_{uv}}{\sigma_u \sigma_v} \to 1$, $\frac{\xi_u}{\xi_v} \to \frac{\sigma_u}{\sigma_v}$, $\hat{\sigma}_u^2 \xrightarrow{d} 0$, and $t_{\hat{\beta}_{IV}} \to \infty$. (rejects very often!)

Talk about 2SLS:

$$y_t = x_t\beta + u_t,$$
$$x_t = Z_t'\pi + v_t,$$

where x_t is scalar and Z_t is $k \times 1$. Suppose

- one observes i.i.d. data $\{(y_t, x_t, Z_t)\}$;
- conditional homoskedasticity:

$$\mathsf{Var}\left(\begin{pmatrix} u_t \\ v_t \end{pmatrix} \middle| Z_t \right) = \begin{pmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{pmatrix}.$$

Let $P_Z = Z(Z'Z)^{-1}Z'$. Then

$$\hat{\beta}_{2SLS} - \beta = \frac{x'P_Z u}{x'P_Z x} = \frac{\pi'Z'u + v'P_Z u}{\pi'Z'Z\pi + 2\pi'Z'v + v'P_Z v}.$$

Define the concentration parameter $\mu^2 = \pi'Z'Z\pi/\sigma_v^2$ (Rothenberg, 1984).

Assume instruments are fixed and errors are normal. Then

$$\mu(\hat{\beta}_{2SLS} - \beta) = \frac{\sigma_u}{\sigma_v}\frac{\xi_u + S_{uv}/\mu}{1 + 2\xi_v/\mu + S_{vv}/\mu^2},$$

where ξ_u, ξ_v are standard normal, and S_{uv}, S_{vv} are Wishart distributed. $\left(\xi_u = \frac{\pi'Z'u}{\sigma_u\sqrt{\pi'Z'Z\pi}}, \xi_v = \frac{\pi'Z'v}{\sigma_v\sqrt{\pi'Z'Z\pi}}, S_{uv} = \frac{v'P_Z u}{\sigma_u\sigma_v}, S_{vv} = \frac{v'P_Z v}{\sigma_v^2}\right)$.

Problems with Weak Instruments

Concentration Parameter

μ^2 plays the role of the sample size.

- If μ is large, $\mu(\hat{\beta}_{2SLS} - \beta)$ is well approximated by $N(0, \frac{\sigma_u^2}{\sigma_v^2})$
  $\Rightarrow$ "strong instruments asymptotics":
  $\mu \to \infty$ as $T \to \infty$;
- if μ is small, $\mu(\hat{\beta}_{2SLS} - \beta)$ is a non-normal random variable
  $\Rightarrow$ "weak instruments asymptotics":
  $\pi = C/\sqrt{T}$ so that $\mu = C' \left(\frac{Z'Z}{T}\right) C \xrightarrow{p} C' Q_{ZZ} C$.

Weak Instruments Asymptotics

Staiger and Stock (1997)

$$y_t = x_t\beta + u_t,$$
$$x_t = Z_t'\pi + v_t,$$

where x_t is scalar and Z_t is $k \times 1$. Suppose

- $\pi = C/\sqrt{T}$, where C is a fixed $k \times 1$ vector;
- $\left(\frac{u'u}{T}, \frac{u'v}{T}, \frac{v'v}{T}\right) \xrightarrow{p} (\sigma_u^2, \sigma_{uv}, \sigma_v^2)$;
- $\frac{Z'Z}{T} \xrightarrow{p} Q_{ZZ}$;
- $\left(\frac{Z'u}{\sqrt{T}}, \frac{Z'v}{\sqrt{T}}\right) \xrightarrow{d} (z_u, z_v)$, where $(z_u, z_v) \sim N\left(0, \begin{pmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{pmatrix} \otimes Q_{ZZ}\right)$.

  (conditional homoskedasticity)

Weak Instruments Asymptotics

Staiger and Stock (1997)

$$\begin{aligned}
\hat{\beta}_{2SLS} - \beta &= \frac{x' P_Z u}{x' P_Z x} \\
&= \left[\frac{x'Z}{\sqrt{T}} \left(\frac{Z'Z}{T} \right)^{-1} \frac{Z'x}{\sqrt{T}} \right]^{-1} \frac{x'Z}{\sqrt{T}} \left(\frac{Z'Z}{T} \right)^{-1} \frac{Z'u}{\sqrt{T}}.
\end{aligned}$$

Note that

$$\frac{Z'x}{\sqrt{T}} = \frac{Z'(Z \cdot C/\sqrt{T} + v)}{\sqrt{T}} = \frac{Z'Z}{T} C + \frac{Z'v}{\sqrt{T}} \xrightarrow{d} Q_{ZZ} C + z_v.$$

Weak Instruments Asymptotics

Staiger and Stock (1997)

Hence,

$$\hat{\beta}_{2SLS} - \beta \xrightarrow{d} \frac{(Q_{ZZ}C + z_v)' Q_{ZZ}^{-1} z_u}{(Q_{ZZ}C + z_v)' Q_{ZZ}^{-1} (Q_{ZZ}C + z_v)}.$$

- We can simulate it.
- We cannot use it for inference because the estimation of σ_u^2 and σ_{uv} depends on $\hat{\beta}_{2SLS}$, which is inconsistent:

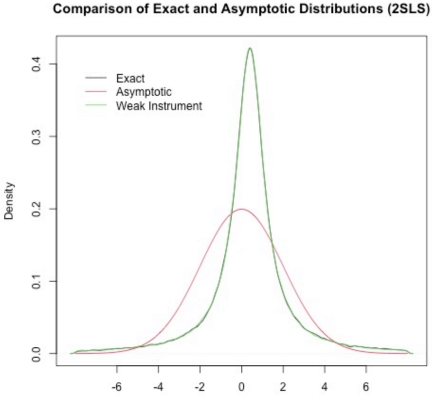
$$\hat{\sigma}_u^2 = \frac{\hat{u}'\hat{u}}{T}, \quad \hat{\sigma}_{uv} = \frac{\hat{u}'\hat{v}}{T}, \quad \hat{u}_t = y_t - x_t\hat{\beta}_{2SLS}.$$

Weak Instruments Asymptotics

A Simulation

$$n = 100, C = .5, \beta = 0, Z_t \overset{i.i.d.}{\sim} N(0,1),$$

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} \overset{i.i.d.}{\sim} N\left(0, \begin{pmatrix} 1 & .5 \\ .5 & 1 \end{pmatrix}\right), Z_t \perp\!\!\!\perp (u_t, v_t).$$



Detecting Weak Instrument

Stock and Yogo (2005)

Stock and Yogo (2005) provide two characterizations of a weak instrument set:

1. The squared bias of $\hat{\beta}_{2SLS}$ relative to the squared bias of $\hat{\beta}_{OLS}$ exceeds a certain threshold b, for example $b = 10\%$;
2. The conventional α-level Wald test based on $\hat{\beta}_{2SLS}$ has an actual size that exceeds a certain threshold r, for example $r = 15\%$ when $\alpha = 5\%$;

(Both characterizations look at the "worst case" with respect to the degree of simultaneity between u_t and v_t.)

They then develop tests for the null hypothesis that π lies in the weak instrument set.

Detecting Weak Instrument

Stock and Yogo (2005)

With a single endogenous regressor, Stock and Yogo (2005)'s test reduces to the first-stage F-statistic:

$$F = \frac{\hat{\pi}'[\hat{\sigma}_v^2(Z'Z)^{-1}]^{-1}\hat{\pi}}{k} = \frac{\hat{\pi}'Z'Z\hat{\pi}'}{\hat{\sigma}_v^2}\frac{1}{k}.$$

kF is asymptotically distributed as a non-central χ^2 with a non-centrality parameter $C'Q_{ZZ}C = \text{plim}_{T\to\infty}\mu^2$.

Detecting Weak Instrument

Stock and Yogo (2005)

The critical values depend on the number of instruments and how the weak instrument set is characterized.

- If we define instruments as weak when the worst-case bias of $\hat{\beta}_{2SLS}$ exceeds 10% of the worst-case bias of $\hat{\beta}_{OLS}$:
  for $3 \sim 30$ instruments, the critical value for a 5% test is $9 \sim 11.52$, which is close to the Staiger and Stock (1997) rule of thumb cutoff of 10.
- If we define instruments as weak when the worst-case size of a nominal 5% Wald test based on $\hat{\beta}_{2SLS}$ exceeds 15%:
  the critical value depends strongly on the number of instruments
  - a single instrument: 8.96;
  - 30 instruments: 44.78.

Detecting Weak Instrument

Stock and Yogo (2005)

With multiple endogenous regressors, Stock and Yogo (2005)'s test is based on the Cragg and Donald (1993) statistic:

$$g_{\min} = \text{mineval}\left(\frac{X'P_Z X}{\hat{\sigma}_v^2}\frac{1}{k}\right)$$

but with different critical values.

Detecting Weak Instrument

Non-Homoskedastic Errors

- Results of Stock and Yogo (2005) rely heavily on the homoskedasticity assumption.
- When run without assuming homoskedastic errors, the `ivreg2` command in Stata automatically reports a robust F-statistic $\frac{\hat{\pi}'\hat{\Sigma}_{\pi\pi}^{-1}\hat{\pi}}{k}$ with critical values based on Stock and Yogo (2005). (not justified!)
- Montiel Olea and Pflueger (2013) propose using the effective first-stage F-statistic:

$$F^{Eff} = \frac{\hat{\pi}'Z'Z\hat{\pi}'}{\text{tr}(\hat{\Sigma}_{\pi\pi}(Z'Z)/T)}$$

with their critical values or the rule-of-thumb value of 10. Stata package `weakivtest` implements this test.

Idea: Given a size-α test of $H_0 : \beta = \beta_0$, we can construct a level $1-\alpha$ confidence set for β by collecting the set of non-rejected values. Let

$$\phi(\beta_0) = \begin{cases} 1 & \text{reject} \\ 0 & \text{do not reject} \end{cases}.$$

$\phi(\beta_0)$ is a size-α test of $H_0 : \beta = \beta_0$ if

$$\sup_{\pi} \mathbb{E}_{\beta_0,\pi}[\phi(\beta_0) = 1] \leq \alpha.$$

$CS = \{\beta : \phi(\beta) = 0\}$ is a level $1-\alpha$ confidence set if

$$\inf_{\beta,\pi} \mathbb{P}_{\beta,\pi}\{\beta \in CS\} \geq 1-\alpha.$$

We focus on the case of one endogenous regressor.

Anderson-Rubin statistic:

$$AR(\beta) = \frac{(y - x\beta)' P_Z (y - x\beta)}{(y - x\beta)' M_Z (y - x\beta)/(T - k)}$$

where $M_Z = I - P_Z$ (annihilator). Under $H_0 : \beta = \beta_0$, $y - x\beta = u$, and so

$$AR(\beta) \xrightarrow{d} \frac{z_u' Q_{ZZ}^{-1} z_u}{\sigma_u^2} \sim \chi_k^2.$$

We can form a size-α test and a level $1 - \alpha$ confidence set as

$$\phi_{AR}(\beta_0) = 1\{AR(\beta_0) > \chi^2_{k,1-\alpha}\}, \quad CS_{AR}(\beta) = 1\{AR(\beta) \leq \chi^2_{k,1-\alpha}\}$$

where $\chi^2_{k,1-\alpha}$ is the $1 - \alpha$ quantile of a χ_k^2 distribution.

In just-identified models:

- $CS_{AR}(\beta)$ can take one of three forms: (i) $[a, b]$, (ii) $(-\infty, a] \cup [b, \infty)$, (iii) the real line $(-\infty, \infty)$ (iff a robust F-test cannot reject $\pi = 0$).
- AR test has (weakly) higher power than any other size-α unbiased test. (We say a size-α test ϕ is unbiased if $\mathbb{E}_{\beta,\pi}[\phi(\beta_0)] \geq \alpha$ for all $\beta \neq \beta_0$ and all π.) In other words, the AR test is efficient.

In over-identified models with homoskedastic errors:

- $CS_{AR}(\beta)$ can take one of four forms: (i) $[a, b]$, (ii) $(-\infty, a] \cup [b, \infty)$, (iii) the real line $(-\infty, \infty)$, (iv) empty set.
- AR test is inefficient under strong identification with $\|\pi\|$ large
  - Use k degrees of freedom for one parameter: loss of power
  - Failure of overidentifying restrictions can lead to empty $CS_{AR}(\beta)$

Kleibergen (2002) proposes a modified score test:

$$K(\beta) = \frac{(y - x\beta)' P_{\tilde{X}(\beta)} (y - x\beta)}{(y - x\beta)' M_Z (y - x\beta)/(T - k)} \xrightarrow{d} \chi^2_1 \quad \text{when } \beta = \beta_0,$$

where $\tilde{X}(\beta) = Z\tilde{\pi}(\beta)$. Under $H_0 : \beta = \beta_0$, $\tilde{\pi}(\beta)$ is both a consistent estimator of π and asymptotically independent of $(y - x\beta_0)'Z$.

- Take a linear combination of $(y - x\beta_0)'Z$ to reduce dimension and improve power.
- $K(\beta) = 0$ has an extraneous root, leading to non-monotonic power and disconnected confidence intervals in finite samples.

Robust Inference

Conditional Likelihood Ratio (CLR) Test

With homoskedastic errors, Moreira (2003) considers a test statistic ϕ conditional a sufficient statistic $Z'(y, x)$ for β and π.

- $Z'(y, x)$ can be represented by (S, T), where under $H_0 : \beta = \beta_0$, $S = Z'(y - x\beta)$ is independent of π while T depends on π, and $S \perp\!\!\!\perp T$.
- The conditional null distribution of ϕ given $T = t$ does not depend on π. Hence, one can use critical values depending on realization of T.
- In fact, any test that has exact size α for all π is a conditional test.
- Moreira (2003) recommends using the conditional test based on the LR statistic. It is implemented in Stata (command `condivreg`).

However, the literature has not yet converged on a recommendation in the non-homoskedastic case. See Andrews et al. (2019) for a survey.

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