EC708 Discussion 11
Trinity of Tests

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Outline

1. Trinity of Tests in Gaussian Linear Models
2. LM Test for Heteroskedasticity
3. LM Test for Serial Correlation

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Consider the linear model

$$Y_t = X_t'\beta + U_t, \quad U_t|X_t \overset{i.i.d.}{\sim} N(0, \sigma^2),$$

where $\beta \in \mathbb{R}^k$. We are interested in testing linear restrictions

$$H_0 : R\beta - r = 0,$$

where $R \in \mathbb{R}^{q\times k}$ and $r \in \mathbb{R}^q$.

- The unrestricted MLE of (θ, σ^2) is given by

$$\hat{\beta}_T = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'Y \quad \text{and} \quad \hat{\sigma}_T^2 = \frac{1}{T}(Y - \mathbf{X}\hat{\beta}_T)'(Y - \mathbf{X}\hat{\beta}_T).$$

- We can show $\sqrt{T}(\hat{\beta}_T - \beta) \xrightarrow{d} N(0, \sigma^2 E[X_t X_t']^{-1})$ and $\hat{\sigma}_T^2 \xrightarrow{p} \sigma^2$.
- Under H_0, $\sqrt{T}(R\hat{\beta}_T - r) \xrightarrow{d} N(0, \sigma^2 R E[X_t X_t']^{-1} R')$.
- The Wald statistic is

$$W_T = (R\hat{\beta}_T - r)'[\hat{\sigma}_T^2 R(X'X)^{-1}R']^{-1}(R\hat{\beta}_T - r) \xrightarrow{d} \chi_q^2 \text{ under } H_0.$$

- FOC for the unrestricted MLE of β:

$$\frac{1}{\hat{\sigma}_T^2}\mathbf{X}'(Y - \mathbf{X}\hat{\beta}_T) = 0. \tag{1}$$

- FOC for the restricted MLE of β:

$$\frac{1}{\tilde{\sigma}_T^2}\mathbf{X}'(Y - \mathbf{X}\tilde{\beta}_T) - R'\tilde{\lambda}_T = 0. \tag{2}$$

- Combining (1) and (2),

$$\begin{aligned} & \frac{1}{\tilde{\sigma}_T^2}\mathbf{X}'\mathbf{X}(\hat{\beta}_T - \tilde{\beta}_T) = R'\tilde{\lambda}_T \\ \Rightarrow\ & \frac{1}{\tilde{\sigma}_T^2}R(\hat{\beta}_T - \tilde{\beta}_T) = R(\mathbf{X}'\mathbf{X})^{-1}R'\tilde{\lambda}_T \\ \Rightarrow\ & \tilde{\lambda}_T = \frac{1}{\tilde{\sigma}_T^2}[R(\mathbf{X}'\mathbf{X})^{-1}R']^{-1}(R\hat{\beta}_T - r). \end{aligned}$$

Under H_0:

- The restricted MLE is consistent: $\tilde{\sigma}^2_T \xrightarrow{p} \sigma^2$.
- Hence, $T^{-1/2}\tilde{\lambda}_T \xrightarrow{d} N(0, (\sigma^2 R E[X_t X_t']^{-1} R')^{-1})$.
- The LM statistic is

$$LM_T = (R\hat{\beta}_T - r)'[\tilde{\sigma}^2_T R(\mathbf{X}'\mathbf{X})^{-1}R']^{-1}(R\hat{\beta}_T - r) \xrightarrow{d} \chi^2_q.$$

Note that

$$\frac{W_T}{LM_T} = \frac{\tilde{\sigma}^2_T}{\hat{\sigma}^2_T}$$

Trinity of Tests in Gaussian Linear Models

Comparison of Wald and LM

- The log-likelihood function evaluated at $(\hat{\beta}_T, \hat{\sigma}_T^2)$ is

$$\begin{aligned}\bar{\ell}_T(\hat{\beta}_T, \hat{\sigma}_T^2) &= -\frac{T}{2}\ln 2\pi - \frac{T}{2}\ln \hat{\sigma}_T^2 - \frac{1}{2\hat{\sigma}_T^2}(Y - \mathbf{X}\hat{\beta}_T)'(Y - \mathbf{X}\hat{\beta}_T) \\ &= -\frac{T}{2}(\ln 2\pi + 1) - \frac{T}{2}\ln \hat{\sigma}_T^2.\end{aligned}$$

- Similarly, the log-likelihood function evaluated at $(\tilde{\beta}_T, \tilde{\sigma}_T^2)$ is

$$\bar{\ell}_T(\tilde{\beta}_T, \tilde{\sigma}_T^2) = -\frac{T}{2}(\ln 2\pi + 1) - \frac{T}{2}\ln \tilde{\sigma}_T^2.$$

Hence,

$$\bar{\ell}_T(\hat{\beta}_T, \hat{\sigma}_T^2) \geq \bar{\ell}_T(\tilde{\beta}_T, \tilde{\sigma}_T^2) \;\Rightarrow\; \tilde{\sigma}_T^2 \geq \hat{\sigma}_T^2 \;\Rightarrow\; LM_T \leq W_T.$$

The LR statistic is

$$LR_T = 2(\bar{\ell}_T(\hat{\beta}_T, \hat{\sigma}_T^2) - \bar{\ell}_T(\tilde{\beta}_T, \tilde{\sigma}_T^2)) = T \ln \left(\frac{\tilde{\sigma}_T^2}{\hat{\sigma}_T^2} \right) \xrightarrow{d} \chi_q^2 \text{ under } H_0.$$

To make W_T, LM_T, and LR_T comparable, we need a reparameterization.

Write the model as

$$Y^*|\mathbf{X}^* \sim N(\mathbf{X}^*\beta, \sigma^2 I_T).$$

If R has rank q, then the model can be reparameterized as

$$Y|\mathbf{X} \sim N(\mathbf{X}\theta, \sigma^2 I_T).$$

- $\theta = (\theta_1', \theta_2')'$ with $\theta_1 \in \mathbb{R}^q$, and the null hypothesis becomes

$$H_0 : \theta_1 = 0 \Rightarrow \text{ test for omitted variables.}$$

- Y and $\mathbf{X}$ are linear combinations of Y^* and $\mathbf{X}^*$.

Trinity of Tests in Gaussian Linear Models

Comparison of Wald, LM, and LR

Partition $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$. The Wald and LM test statistics are

$$W_T = \hat{\theta}'_{1T}[\mathbf{X}'_1\mathbf{X}_1 - \mathbf{X}'_1\mathbf{X}_2(\mathbf{X}'_2\mathbf{X}_2)^{-1}\mathbf{X}'_2\mathbf{X}_1]\hat{\theta}_{1T}/\hat{\sigma}^2_T,$$
$$LM_T = \hat{\theta}'_{1T}[\mathbf{X}'_1\mathbf{X}_1 - \mathbf{X}'_1\mathbf{X}_2(\mathbf{X}'_2\mathbf{X}_2)^{-1}\mathbf{X}'_2\mathbf{X}_1]\hat{\theta}_{1T}/\tilde{\sigma}^2_T.$$

Let $M_{X_2} = I_T - \mathbf{X}_2(\mathbf{X}'_2\mathbf{X}_2)^{-1}\mathbf{X}'_2$. Then,

$$\tilde{U} - \hat{U} = M_{X_2}Y - (M_{X_2}Y - M_{X_2}\mathbf{X}_1\hat{\theta}_{1T}) = M_{X_2}\mathbf{X}_1\hat{\theta}_{1T}$$

and $\hat{U}'(\tilde{U} - \hat{U}) = 0$. It follows that

$$\begin{aligned}
\hat{\theta}'_{1T}[\mathbf{X}'_1\mathbf{X}_1 - \mathbf{X}'_1\mathbf{X}_2(\mathbf{X}'_2\mathbf{X}_2)^{-1}\mathbf{X}'_2\mathbf{X}_1]\hat{\theta}_{1T} &= \hat{\theta}'_{1T}\mathbf{X}'_1 M_{X_2}\mathbf{X}_1\hat{\theta}_{1T} \\
&= (\tilde{U} - \hat{U})'(\tilde{U} - \hat{U}) \\
&= \tilde{U}'\tilde{U} - \hat{U}'\hat{U}.
\end{aligned}$$

We find

$$
\begin{aligned}
W_T &= \frac{\tilde{U}'\tilde{U} - \hat{U}'\hat{U}}{\hat{\sigma}_T^2} = 2(\bar{\ell}_T(\hat{\theta}_T, \hat{\sigma}_T^2) - \bar{\ell}_T(\tilde{\theta}_T, \hat{\sigma}_T^2)),\\
LM_T &= \frac{\tilde{U}'\tilde{U} - \hat{U}'\hat{U}}{\tilde{\sigma}_T^2} = 2(\bar{\ell}_T(\hat{\theta}_T, \tilde{\sigma}_T^2) - \bar{\ell}_T(\tilde{\theta}_T, \tilde{\sigma}_T^2)),\\
LR_T &= 2(\bar{\ell}_T(\hat{\theta}_T, \hat{\sigma}_T^2) - \bar{\ell}_T(\tilde{\theta}_T, \tilde{\sigma}_T^2)).
\end{aligned}
$$

Since

$$
\bar{\ell}_T(\hat{\theta}_T, \hat{\sigma}_T^2) \geq \bar{\ell}_T(\hat{\theta}_T, \tilde{\sigma}_T^2) \text{ and } \bar{\ell}_T(\tilde{\theta}_T, \tilde{\sigma}_T^2) \geq \bar{\ell}_T(\tilde{\theta}_T, \hat{\sigma}_T^2),
$$

we have $LM_T \leq LR_T \leq W_T$.

Trinity of Tests in Gaussian Linear Models

When Are LM, LR, and Wald the Same?

Suppose the log-likelihood function has the form

$$\bar{\ell}_T(\theta) = c - \frac{1}{2}(\theta - \hat{\theta}_T)' A(\theta - \hat{\theta}_T),$$

where A is symmetric & positive definite and $\hat{\theta}_T$ is a function of data. Then,

$$LM_T = LR_T = W_T = (\theta_0 - \hat{\theta}_T)' A(\theta_0 - \hat{\theta}_T),$$

where θ_0 is subject to H_0.

- It holds in Gaussian linear models when σ^2 is known.
- It approximates the log-likelihood function in the neighborhood of θ_0 for large $T \Rightarrow$ asymptotic equivalence of W_T, LM_T, LR_T.

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LM Test for Heteroskedasticity

Setup

Consider the following model:

$$Y_t = X_t'\beta + U_t, \quad U_t|X_t, Z_t \sim N(0, \sigma_t^2), \quad t = 1, \ldots, T,$$
$$E[U_t U_s] = 0 \text{ for } t \neq s,$$

where $Z_t = (1, Z_{1t}, \ldots, Z_{qt})' \in \mathbb{R}^{q+1}$ is a vector function of X_t. We are interested in testing

$$H_0 : \sigma_t^2 = \sigma^2 \ \forall t \text{ v.s. } H_1 : \sigma_t^2 = h(Z_t'\alpha).$$

The null can be rewritten as

$$H_0 : \alpha_1 = \cdots = \alpha_q = 0.$$

LM Test for Heteroskedasticity

Breush-Pagan Test Statistic

Log-likelihood function:

$$\bar{\ell}_T(\beta,\alpha) = -\frac{T}{2}\ln 2\pi - \frac{1}{2}\sum_{t=1}^{T}\ln h(Z_t'\alpha) - \frac{1}{2}\sum_{t=1}^{T}\frac{(Y_t - X_t'\beta)^2}{h(Z_t'\alpha)}.$$

The score is

$$\bar{s}_T(\tilde{\beta}_T,\tilde{\alpha}_T) = \begin{bmatrix} \frac{1}{T}\frac{\partial\bar{\ell}_T(\tilde{\beta}_T,\tilde{\alpha}_T)}{\partial\beta} \\ \frac{1}{T}\frac{\partial\bar{\ell}_T(\tilde{\beta}_T,\tilde{\alpha}_T)}{\partial\alpha} \end{bmatrix} = \begin{bmatrix} \frac{1}{T}\sum_{t=1}^{T}\frac{X_t\tilde{U}_t}{h(\tilde{\alpha}_{0T})}(=0) \\ \frac{1}{2T}\sum_{t=1}^{T} Z_t\Big(\frac{\tilde{U}_t^2}{h(\tilde{\alpha}_{0T})} - 1\Big)\frac{h'(\tilde{\alpha}_{0T})}{h(\tilde{\alpha}_{0T})} \end{bmatrix}.$$

A consistent estimator of the information matrix is

$$\hat{I}_T(\tilde{\beta}_T,\tilde{\alpha}_T) = \begin{bmatrix} \frac{1}{T}\sum_{t=1}^{T}\frac{X_tX_t}{h(\tilde{\alpha}_{0T})} & 0 \\ 0 & \frac{1}{2T}\sum_{t=1}^{T} Z_tZ_t'\frac{h'(\tilde{\alpha}_{0T})^2}{h(\tilde{\alpha}_{0T})^2} \end{bmatrix}$$

The LM test statistic is

$$\begin{aligned} LM_T &= T\bar{s}_T(\tilde{\beta}_T, \tilde{\alpha}_T)' \hat{I}_T(\tilde{\beta}_T, \tilde{\alpha}_T)^{-1} \bar{s}_T(\tilde{\beta}_T, \tilde{\alpha}_T) \\ &= \frac{1}{2}\Big[\sum_{t=1}^{T} Z_t\Big(\frac{\tilde{U}_t^2}{h(\tilde{\alpha}_{0T})} - 1\Big)\Big]'\Big(\sum_{t=1}^{T} Z_t Z_t'\Big)^{-1}\Big[\sum_{t=1}^{T} Z_t\Big(\frac{\tilde{U}_t^2}{h(\tilde{\alpha}_{0T})} - 1\Big)\Big]. \end{aligned}$$

Let $f = (f_1, \ldots, f_T)'$ with $f_t = \frac{\tilde{U}_t^2}{h(\tilde{\alpha}_{0T})} - 1$. Then,

$$LM_T = \frac{1}{2} f'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' f = \frac{1}{2} f' P_Z f.$$

LM Test for Heteroskedasticity

Implementation

1. Apply OLS to $Y_t = X_t'\beta + U_t$ and obtain residuals $\hat{U}_t$.
2. Compute $f_t = \frac{\hat{U}_t^2}{\hat{\sigma}_T^2} - 1$ where $\hat{\sigma}_T^2 = \frac{1}{T}\sum_{t=1}^{T}\hat{U}_t^2$.
3. Run OLS of f_t on Z_t and compute the LM statistic

$$LM_T = \frac{1}{2}f'P_Z f = \frac{1}{2}ESS.$$

Under H_0, $LM_T \xrightarrow{d} \chi_q^2$.

Remark: Since $\text{plim}_{T\to\infty} f'f/T = 2$ under H_0 and H_1, an asymptotically equivalent test statistic is

$$TR^2 = T\frac{f'P_Z f}{f'f}.$$

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LM Test for Serial Correlation

Setup

Consider the model with p lags in error:

$$Y_t = X_t'\beta + U_t$$
$$U_t = \psi_1 U_{t-1} + \cdots + \psi_p U_{t-p} + \varepsilon_t$$
$$\varepsilon_t \overset{i.i.d.}{\sim} N(0, \sigma^2)$$

- The null is $H_0 : \psi = (\psi_1, \ldots, \psi_p) = 0$.
- Ignore the first p observations and rewrite the model as

$$Y_t = X_t'\beta + \sum_{j=1}^{p} \psi_j (Y_{t-j} - X_{t-j}'\beta) + \varepsilon_t.$$

LM Test for Serial Correlation

Testing for $AR(p)$ Errors

Denote $\theta = (\beta, \psi, \sigma^2)$. The log-likelihood function is

$$\bar{\ell}_T(\theta) = -\frac{T}{2}\ln 2\pi - \frac{T}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{t=1}^{T}\Big(U_t - \sum_{j=1}^{p}\psi_j U_{t-j}\Big)^2.$$

The score is

$$\bar{s}_T(\tilde{\theta}_T) = \begin{bmatrix} \frac{1}{T}\frac{\partial \bar{\ell}_T(\tilde{\theta}_T)}{\partial \beta} \\ \frac{1}{T}\frac{\partial \bar{\ell}_T(\tilde{\theta}_T)}{\partial \psi} \\ \frac{1}{T}\frac{\partial \bar{\ell}_T(\tilde{\theta}_T)}{\partial \sigma^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{T}\sum_{t=1}^{T}\frac{X_t\tilde{U}_t}{\tilde{\sigma}_T^2} \\ \frac{1}{T}\sum_{t=1}^{T}\frac{V_t\tilde{U}_t}{\tilde{\sigma}_T^2} \\ -\frac{1}{2\tilde{\sigma}_T^2} + \frac{1}{T}\sum_{t=1}^{T}\frac{\tilde{U}_t^2}{2\tilde{\sigma}_T^4} \end{bmatrix},$$

where $V_t = (\tilde{U}_{t-1}, \ldots, \tilde{U}_{t-p})'$.

LM Test for Serial Correlation

Testing for $AR(p)$ Errors

A consistent estimator of the information matrix is

$$\hat{I}_T(\tilde{\theta}_T) = \begin{bmatrix} \frac{1}{T}\sum_{t=1}^{T}\frac{X_tX_t'}{\tilde{\sigma}_T^2} & \frac{1}{T}\sum_{t=1}^{T}\frac{X_tV_t'}{\tilde{\sigma}_T^2} & 0 \\ \frac{1}{T}\sum_{t=1}^{T}\frac{V_tX_t'}{\tilde{\sigma}_T^2} & \frac{1}{T}\sum_{t=1}^{T}\frac{V_tV_t'}{\tilde{\sigma}_T^2} & 0 \\ 0 & 0 & \frac{1}{2\tilde{\sigma}_T^4} \end{bmatrix}.$$

Let $F_t = (X_t', V_t')'$. The LM test statistic is

$$\begin{aligned} LM_T &= T\bar{s}_T(\tilde{\theta}_T)'\hat{I}_T(\tilde{\theta}_T)^{-1}\bar{s}_T(\tilde{\theta}_T) \\ &= \frac{1}{\tilde{\sigma}_T^2}\tilde{U}'F(F'F)^{-1}F'\tilde{U}. \end{aligned}$$

LM Test for Serial Correlation

Implementation

1. Run OLS of Y_t on X_t and get $\hat{U}_t$.
2. Run OLS for the auxiliary regression

$$\hat{U}_t = X_t'\tau + V_t'\delta + \eta_t.$$

3. Compute R^2 from the auxiliary regression and construct

$$LM_T = TR^2.$$

Under the null, $LM_T \xrightarrow{d} \chi^2_p$.

Remarks:

- If X_t includes no lagged dependent variables, then $\text{plim}_{T\to\infty}\frac{\mathbf{X}'\mathbf{V}}{T} = 0$ and the auxiliary regression will be unaffected by leaving out X_t:

$$LM_T \approx \frac{1}{\tilde{\sigma}_T^2}\tilde{U}'\mathbf{V}(\mathbf{V}'\mathbf{V})^{-1}\mathbf{V}'\tilde{U}.$$

- For $p = 1$, LM_T is asymptotically equivalent to Durbin-Watson:

$$LM_T \approx \frac{1}{\tilde{\sigma}_T^2}\frac{(\sum_{t=2}^T \tilde{U}_t\tilde{U}_{t-1})^2}{\sum_{t=2}^T \tilde{U}_{t-1}^2} \approx T(1 - d_T/2)^2.$$