# ON RANDOM $\pm 1$ MATRICES: SINGULARITY AND DETERMINANT

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ABSTRACT. This papers contains two results concerning random  $n \times n$  Bernoulli matrices. First, we show that with probability tending to one the determinant has absolute value  $\sqrt{n!} \exp(O(\sqrt{n \ln n}))$ . Next, we prove a new upper bound .939<sup>n</sup> on the probability that the matrix is singular.

### 1. Introduction

Let n be a large integer parameter, and let  $M_n$  denote a random  $n \times n \pm 1$  matrix ("random" meaning with respect to the uniform distribution, i.e., the entries of  $M_n$  are i.i.d. Bernoulli random variables). Throughout the paper, we assume that n is sufficiently large, whenever needed. We use o(1) to denote any quantity which goes to zero as  $n \to \infty$ , keeping other parameters (such as  $\epsilon$ ) fixed.

This model of random matrices is of considerable interest in many areas, including combinatorics, theoretical computer science and mathematical physics. On the other hand, many basic questions concerning this model have been open for a long time. In this paper, we focus on the following two questions:

**Question 1.** What is the typical value of the determinant of  $M_n$ ?

Question 2. What is the probability that  $M_n$  is singular?

Let us first discuss Question 1. From Hadamard's inequality, we have the bound  $|\det(M_n)| \leq n^{n/2}$ , with equality if and only if  $M_n$  is an Hadamard matrix. However, in general we expect  $|\det(M_n)|$  to be somewhat smaller than  $n^{n/2}$ . Indeed, from the simple estimate<sup>1</sup>

$$\mathbf{E}((\det M_n)^2) = n! \tag{1}$$

(first observed by Turán [13]), one is led to conjecture that  $|\det(M_n)|$  should be of the order of  $\sqrt{n!} = e^{-n/2 + o(n)} n^{n/2}$  with high probability. On the other hand, even proving that  $|\det M_n|$  is typically positive (or equivalently, that  $M_n$  is typically

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<sup>&</sup>lt;sup>1</sup>Indeed, one can prove (1) by expanding det  $M_n$  as the sum of n! signs, and observing that all the covariances vanish.

non-singular) is already a non-trivial task. This task was first done by Komlós [7] (see Theorem 1.2 below).

The first main result of this paper shows that with probability tending to one (as n tends to infinity), the absolute value of the determinant is very close to  $\sqrt{n!}$ .

#### Theorem 1.1.

$$\mathbf{P}(|\det M_n| \ge \sqrt{n!} \exp(-29n^{1/2} \ln^{1/2} n)) = 1 - o(1).$$

The constant 29 is generous but we do not try to optimize it.

Note that from (1) and Chebyshev's inequality that

$$\mathbf{P}(|\det M_n| \le \omega(n)\sqrt{n!})) = 1 - o(1)$$

for any function  $\omega(n)$  which goes to infinity as  $n \to \infty$ . Combining this with the preceding theorem and the observation that  $\det M_n$  is symmetric around the origin, it follows that for each sign  $\pm$ , we have the concentration inequality.

$$\det(M_n) = \pm \sqrt{n!} \exp(O(n^{1/2} \ln^{1/2} n))$$

with probability 1/2 - o(1).

Let us now turn to the problem of determining the probability that  $M_n$  is singular. As mentioned above, Komlos showed, in 1967, that

**Theorem 1.2.** [7] 
$$P(\det M_n = 0) = o(1)$$
.

The task here is to give a precise formula for o(1) in the right hand side. Since a matrix  $M_n$  with two identical (or opposite) rows or two identical (or opposite) columns is necessarily singular, it is easy to see that

$$\mathbf{P}(\det M_n = 0) \ge (1 - o(1))n^2 2^{1-n}.$$

It has often been conjectured (see e.g. [10], [6]) that this is the dominant source of singularity. More precisely,

### Conjecture 1.3.

$$P(\det M_n = 0) = (1 - o(1))n^2 2^{1-n}.$$

Prior to this paper, the best partial result concerning this conjecture is the following, due to Kahn, Komlós and Szemerédi [6]:

**Theorem 1.4.** [6] We have 
$$\mathbf{P}(\det M_n = 0) \leq (1 - \varepsilon + o(1))^n$$
, where  $\varepsilon := .001$ .

Our second main result is the following improvement of this theorem:

**Theorem 1.5.** We have 
$$\mathbf{P}(\det M_n = 0) \le (1 - \varepsilon + o(1))^n$$
, where  $\varepsilon := .06191...$ 

This value of  $\varepsilon$  is the unique solution in the interval (0, 1/2) to the equation

$$h(\varepsilon) + \frac{\varepsilon}{\log_2 16/15} = 1,\tag{2}$$

where h is the entropy function

$$h(\varepsilon) := \varepsilon \log_2 \frac{1}{\varepsilon} + (1 - \varepsilon) \log_2 \frac{1}{1 - \varepsilon}.$$
 (3)

We prove Theorem 1.5 in Sections 5-7. Our argument uses several key ideas from the original proof of Theorem 1.4 in [6], but invoked in a simpler and more direct fashion. In a sequel to this paper [11] we shall use more complicated arguments to improve this value of  $\varepsilon$  further, to  $\varepsilon = \frac{1}{4}$ .

This paper is organized as follows. In Section 2 we establish some basic estimates for the distance between a randomly selected point on the unit cube  $\{-1,1\}^n$  and a fixed subspace, and in Section 3 we obtain similar types of estimates in the case when the subspace is also random. In Section 4 we then apply those estimates to prove Theorem 1.1. As a by-product, we also obtain a short proof of Theorem 1.2. We then give the proof of Theorem 1.5 in Sections 5-7.

In this paper we shall try to emphasize simplicity. Several results obtained in these parts can be extended or refined considerably with more technical arguments. In last part of the paper (Section 8), we will consider some of these extensions/refinements. In particular, we prove an extension of Theorem 1.2 and Theorem 1.1 for more general models of random matrices.

# 2. The distance between a random vector and a deterministic subspace

Let X be a random vector chosen uniformly at random from  $\{-1,1\}^n$ , thus  $X = (\epsilon_1, \ldots, \epsilon_n)$  where  $\epsilon_1, \ldots, \epsilon_n$  are i.i.d. Bernoulli signs. Let W be a (deterministic) d-dimensional subspace of  $\mathbf{R}^n$  for some  $0 \le d < n$ . In this section we collect a number of estimates concerning the distribution of the distance  $\mathrm{dist}(X, W)$  from X to W, which we will then combine to prove Theorem 1.2 and Theorem 1.1.

We have the crude estimate

$$0 \leq \operatorname{dist}(X, W) \leq \operatorname{dist}(X, 0) = \sqrt{n};$$

later we shall see that  $\operatorname{dist}(X, W)$  is in fact concentrated around  $\sqrt{n-d}$  (see Lemma 2.2).

We next recall a simple observation of Odlyzko.

**Lemma 2.1.** [10] 
$$P(\text{dist}(X, W) = 0) \le 2^{d-n}$$
.

**Proof** Since W has dimension d in  $\mathbb{R}^n$ , there is a set of d coordinates which determines all other n-d coordinates of an element of W. But the corresponding

n-d coordinates of X are distributed uniformly in  $\{-1,1\}^{n-d}$  (thinking of the other k coordinates of X as fixed). Thus the constraint  $\operatorname{dist}(X,W)=0$  can only be obeyed with probability at most  $2^{d-n}$ , as desired.

For a variant of Lemma 2.1 which gives a lower bound on  $\operatorname{dist}(X, W)$  with high probability, see, Lemma 8.10. Next, we establish that  $\operatorname{dist}(X, W)$  concentrates near  $\sqrt{n-d}$ .

**Lemma 2.2.** Let W be a fixed subspace of dimension  $1 \le d \le n-4$  and X a random  $\pm 1$  vector. Then

$$\mathbf{E}(\operatorname{dist}(X,W)^2) = n - d. \tag{4}$$

Furthermore, for any t > 0

$$\mathbf{P}(|\text{dist}(X, W) - \sqrt{n - d}| \ge t + 2) \le 4\exp(-t^2/16). \tag{5}$$

**Proof** Let  $P = (p_{jk})_{1 \leq j,k \leq n}$  be the  $n \times n$  orthogonal projection matrix from  $\mathbf{R}^n$  to W. Let  $D = \operatorname{diag}(p_{11}, \ldots, p_{nn})$  be the diagonal component of P, and let  $A := P - D = (a_{jk})_{1 \leq j,k \leq n}$  be the off-diagonal component of P. Since P is an orthogonal projection matrix, we see that A is real symmetric with zero diagonal. If we write  $X = (\epsilon_1, \ldots, \epsilon_n)$ , then from Pythagoras's theorem we have

$$\operatorname{dist}(X, W)^{2} = |X|^{2} - |PX|^{2}$$

$$= n - \sum_{j=1}^{n} \sum_{k=1}^{n} \epsilon_{j} \epsilon_{k} p_{jk}$$

$$= n - \operatorname{tr}(P) - \sum_{j=1}^{n} \sum_{k=1}^{n} \epsilon_{j} \epsilon_{k} a_{jk}$$

$$= n - d - \sum_{j=1}^{n} \sum_{k=1}^{n} \epsilon_{j} \epsilon_{k} a_{jk}.$$

This already gives (4), since  $a_{jk}$  vanishes on the diagonal. Set  $Y = \sum_{j=1}^{n} \sum_{k=1}^{n} \epsilon_{j} \epsilon_{k} a_{jk}$ . It is easy to see that

$$\mathbf{E}(Y^2) = 2\sum_{1 \le j,k \le n} a_{jk}^2 = 2\text{tr}(A^2).$$

Observe that as P is a projection matrix, the coefficients  $p_{jk}$  are bounded in magnitude by 1, and we have

$$\sum_{j=1}^{n} \sum_{k=1}^{n} p_{jk}^{2} = \operatorname{tr}(P^{2}) = \operatorname{tr}(P) = d.$$

On the other hand

$$\sum_{j=1}^{n} p_{jj} = \operatorname{tr}(P) = d$$

so by Cauchy-Schwartz

$$\sum_{j=1}^{n} p_{jj}^2 \ge d^2/n.$$

This implies that

$$\operatorname{tr}(A^2) = \sum_{i=1}^n \sum_{k=1}^n p_{jk}^2 - \sum_{i=1}^n p_{jj}^2 \le d - d^2/n \le \min\{d, n - d\}.$$

Consider the event  $\operatorname{dist}(X,W) \geq \sqrt{n-d} + 2$ . This probability of this event is bounded from above by

$$\mathbf{P}(\mathrm{dist}^2(X, W) \ge (n - d) + 4\sqrt{n - d}) = \mathbf{P}(Y \ge 4\sqrt{n - d}) \le \mathbf{P}(Y^2 \ge 16(n - d)).$$

By Markov's inequality

$$\mathbf{P}(Y^2 \ge 16(n-d)) \le \frac{\mathbf{E}(Y^2)}{16(n-d)} \le \frac{2(n-d)}{16(n-d)} = \frac{1}{8},$$

which implies that the median M of  $\operatorname{dist}(X,W)$  is at most  $\sqrt{n-d}+2$ . To bound M from below, consider the event  $\operatorname{dist}(X,W) \geq \sqrt{n-d}-2$ . By a similar argument, the probability of this event is at most

$$\mathbf{P}(Y \le -4\sqrt{n-d} + 4) \le \mathbf{P}(Y^2 \ge 16(n-d) - 32\sqrt{n-d} + 16).$$

By Markov's inequality, the last probability is at most

$$\frac{2(n-d)}{16(n-d)-32\sqrt{n-d}+16}<\frac{1}{2},$$

for all  $d \le n - 4$ . Thus, we can conclude that  $|M - \sqrt{n - d}| \le 2$ .

Since dist(X, W) is a convex function on  $\{-1, 1\}^n$  with Lipschitz coefficient 1, Talagrand's [12] inequality implies that

$$\mathbf{P}(|\text{dist}(X, W) - M| \ge t) \le 4 \exp(-t^2/16),$$

for any t > 0. Since  $|M - \sqrt{n-d}| \le 2$ , Lemma 2.2 follows.

Remarks 2.3. One can deduce a concentration result similar to Lemma 2.2 using the high moment method; there is also a slightly weaker statement that can be obtained from Bonami's inequality [2].

One can have a similar statement for the case d=n-3, n-2 and n-1. In these cases  $\sqrt{n-d} < 2$ , so the event  $\operatorname{dist}(X,W) \leq \sqrt{n-d} - 2$  holds with probability zero. So the median M is between 0 and 3. Therefore, in these cases

$$\mathbf{P}(\text{dist}(X, W) \ge 3 + t) \le 4 \exp(-t^2/16).$$

#### 3. The distance between a random vector and a random subspace

The estimates in the last section are quite accurate when n-d is sufficiently large, but do not provide much useful information when n-d is small (e.g., n-d=2). For instance, it does not show that the distance is (with high probability) not zero in this case. Indeed, there are some exceptional spaces W (e.g. the hyperplane of points  $(x_1, \ldots, x_n)$  with  $x_1 = x_2$ ) which capture a very large fraction of the points in  $\{-1,1\}^n$ . However, in our applications W is a subspace spanned by random vectors and will thus "typically" not be of the exceptional form described above, in which the unit normal contains many zero coordinates. In such a case we can still recover good lower bounds on  $\operatorname{dist}(X,W)$  with high probability. More precisely, we have

**Lemma 3.1.** Let X be a random vector in  $\{-1,1\}^n$ , let  $1 \le d \le n-1$  and W a space spanned by d random vectors in  $\{-1,1\}^n$ , chosen independently of each other and with X. Then we have

$$\mathbf{P}(\operatorname{dist}(X, W) \le \frac{1}{4n}) = O(1/\sqrt{\ln n}).$$

Remark 3.2. In fact, as W is spanned by random vectors, we can fix X. The above formulation is, however, more convenient for the proof.

The remainder of this section will be devoted to the proof of Lemma 3.1.

Let  $1 \leq l \leq n$ . We say that W is *l*-typical if any unit vector  $(w_1, \ldots, w_n) \in W^{\perp}$  has at least l coordinates whose absolute values are at least  $\frac{1}{2n}$ . In order to prove Lemma 3.1, we need the following

**Lemma 3.3** (dist(X, W) is large for typical W). Let W be a (deterministic) subspace which is l-typical for some  $1 \le l \le n$ . Then

$$\mathbf{P}(\operatorname{dist}(X, W) \le \frac{1}{4n}) \le O(\frac{1}{\sqrt{l}}).$$

**Proof** By hypothesis and symmetry, we may assume without loss of generality that there is a unit normal  $(w_1, \ldots, w_n) \in W^{\perp}$  such that  $|w_1|, \ldots, |w_l| \geq \frac{1}{2n}$ . We

then see that

$$\mathbf{P}(\operatorname{dist}(X, W) \leq \frac{1}{4n}) = \mathbf{P}(|\epsilon_1 w_1 + \ldots + \epsilon_n w_n| \leq \frac{1}{4n})$$

$$\leq \sup_{x \in \mathbf{R}} \mathbf{P}(|\epsilon_1 w_1 + \ldots + \epsilon_l w_l - x| \leq \frac{1}{4n})$$

$$= \sup_{y \in \mathbf{R}} \mathbf{P}(\epsilon_1 2nw_1 + \ldots + \epsilon_l 2nw_l \in [y, y + 1])$$

where we have made the substitutions  $x := \sum_{l < j \le n} \epsilon_j w_j$  and  $y := 2nx - \frac{1}{2}$  respectively. To conclude the claim, we invoke the following variant of the Littlewood-Offord lemma, due to Erdös [3]:

**Lemma 3.4.** [3] Let  $a_1, \ldots, a_k$  be real numbers with absolute values larger than one. Then for any interval I of length at most one

$$\mathbf{P}(\sum_{i=1}^{k} a_i \epsilon_i \in I) = O(1/\sqrt{k}).$$

This lemma was proved by Erdös using Sperner's lemma. The reader may want to check Remark 7.2 for a different argument. Lemma 3.3 immediately follows.

We are now ready to prove Lemma 3.1.

**Proof** It suffices to prove the extremal case when W is spanned by n-1 random vectors. Set  $l := \lfloor \frac{\ln n}{10} \rfloor$ . In light of Lemma 3.3, we see that it suffices to show that

$$\mathbf{P}(W \text{ is not } l\text{-typical}) = O(1/\sqrt{\ln n}). \tag{6}$$

If W is not l-typical, then there exists a unit vector w orthogonal to W with at least n-l coordinates which are less than  $\frac{1}{2n}$  in magnitude. There are  $\binom{n}{n-l}=\binom{n}{l}$  such possibilities for these coordinates. Thus by symmetry we have

$$\mathbf{P}(W \text{ is not } l\text{-typical}) \leq \binom{n}{l} \mathbf{P}(W \perp w \text{ for some } w \in \Omega)$$

where  $\Omega$  is the space of all unit vectors  $w=(w_1,\ldots,w_n)$  such that  $|w_j|<\frac{1}{2n}$  for all  $l< j\leq n$ .

Suppose that  $w \in \Omega$  was such that  $W \perp w$ , then  $X_i \perp w$  for all  $1 \leq i \leq n-1$ . Write  $X_i = (\epsilon_{i,1}, \ldots, \epsilon_{i,n})$ , then

$$\sum_{j=1}^{n} \epsilon_{i,j} w_j = 0.$$

Since  $\epsilon_{i,j} = \pm 1$ , and  $|w_j| < 1/2n$  for j > l, we thus conclude from the triangle inequality that

$$\left| \sum_{i=l+1}^{n} \epsilon_{i,j} w_{j} \right| \le (n-l) \frac{1}{2n} \le \frac{1}{2}.$$

On the other hand, we have

$$\sum_{j=1}^{l} |w_j| \ge \sum_{j=1}^{l} |w_j|^2$$

$$= 1 - \sum_{j=l+1}^{n} |w_j|^2$$

$$\ge 1 - (n-l)(\frac{2}{n})^2$$

$$\ge 1 - \frac{4}{n}.$$

Comparing these two inequalities, we see that (for n > 8; the cases  $n \le 8$  are of course trivial) that for each  $1 \le i \le n-1$ , at least one of the  $\epsilon_{i,j}w_j$  has to be negative. Thus, if we let  $\epsilon_1, \ldots, \epsilon_l$  be signs such that  $\epsilon_j w_j$  is positive for all  $1 \le j \le l$ , we thus have

$$(\epsilon_{i,j})_{1 \le j \le l} \ne (\epsilon_j)_{1 \le j \le l}$$
 for all  $1 \le i \le n-1$ .

Thus we have

$$\mathbf{P}(W \perp w \text{ for some } w \in \Omega)$$

$$\leq \sum_{\epsilon_1, \dots, \epsilon_l \in \{-1, 1\}} \mathbf{P}((\epsilon_{i,j})_{1 \leq j \leq l} \neq (\epsilon_j)_{1 \leq j \leq l} \text{ for all } 1 \leq i \leq n-1).$$

Since the  $\epsilon_{i,j}$  are i.i.d. Bernoulli variables, we have

$$\mathbf{P}((\epsilon_{i,j})_{1 \le j \le l} \ne (\epsilon_j)_{1 \le j \le l} \text{ for all } 1 \le i \le n-1) = (1-2^{-l})^{n-1}.$$

Putting this all together, we obtain

$$\mathbf{P}(W \text{ is not } l - \text{typical}) \le \binom{n}{l} 2^l (1 - 2^{-l})^{n-1}$$
  
  $\le n^{l+1} 2^l e^{-2^l (n-1)},$ 

and (6) follows by choice of l. This proves Lemma 3.1.

As a consequence of this lemma, we derive a short proof of Theorem 1.2. Let  $X_1, \ldots, X_n$  be the row vectors of  $M_n$  and  $W_j$  be the subspace spanned by  $X_1, \ldots, X_j$ . Observe that if  $M_n$  is singular, then  $X_1, \ldots, X_n$  are linearly dependent, and thus we have  $\operatorname{dist}(X_{j+1}, W_j) = 0$  for some  $1 \leq j \leq n-1$ . Thus we have

$$\mathbf{P}(\det(M_n) = 0) \le \sum_{j=1}^{n-1} \mathbf{P}(\operatorname{dist}(X_{j+1}, W_j) = 0)$$
$$= \sum_{j=1}^{n-1} \mathbf{P}(\operatorname{dist}(X, W_j) = 0).$$

From Lemma 2.1 we have  $\mathbf{P}(\operatorname{dist}(X, W_j) = 0) \leq 2^{j-n}$ . Since  $\mathbf{P}(\operatorname{dist}(X, W_j) = 0)$  is clearly monotone increasing in j, we obtain the inequality

$$\mathbf{P}(\det(M_n) = 0) \le 2^{-k} + k\mathbf{P}(\det(X, W_{n-1}) = 0)$$

for any  $1 \le k < n$ . By the lemma just proved,  $\mathbf{P}(\operatorname{dist}(X, W_{n-1}) = 0) = O(1/\sqrt{\ln n})$ . By choosing  $k = \ln^{1/4} n$ 

$$2^{-k} + O(k/\sqrt{\ln n}) = o(1)$$

completing the proof.

### 4. Proof of Theorem 1.1

For an  $n \times n$  matrix A,  $|\det A|$  is the volume of the parallelepiped spanned by the row vectors of A. If one instead expresses this volume in terms of base times height, we obtain the factorization

$$|\det(M_n)| = \prod_{0 \le j \le n-1} \operatorname{dist}(X_{j+1}, W_j).$$

To estimate this quantity, we shall simply control each of the factors  $dist(X_{j+1}, W_j)$  separately, using the estimates obtained in the previous two sections.

We may assume n is large. Set  $d_0 = n - \ln^{1/4} n$ . For  $1 \le j \le d_0$ 

$$\gamma_j := 7\sqrt{\frac{\ln(n-j)}{n-j}}.$$

It is trivial that all  $\gamma_j$  are bounded from above by 1/2 if n is sufficiently large. Consider  $1 \leq j \leq d_0$ . Assuming that  $W_j$  has dimension j, by Lemma 2.2 we have that the probability that the distance

$$\operatorname{dist}(X_{j+1}, W_j) \le (1 - \gamma_j) \sqrt{n - j}$$

is at most

$$4\exp(-\gamma_j^2(n-j)/16) = 4\exp(-\frac{49}{16}\ln(n-j)) \le (n-j)^{-2}$$

provided that n-j is sufficiently large. This implies that with probability at least

$$1 - \sum_{j=1}^{d_0} (n-j)^{-2} = 1 - o(1)$$

the distance  $\operatorname{dist}(X_{j+1},W_j)$  is at least  $(1-\gamma_j)\sqrt{n-j}$ , for every  $1\leq j\leq d_0$ . (Notice that if  $\operatorname{dist}(X_{j+1},W_j)>0$  then  $W_{j+1}$  has full dimension j+1.)

For  $d_0 < j \le n-1$ , we are going to use Lemma 3.1 to estimate the distances. By this lemma, we have that with probability at least

$$1 - \sum_{d_0 < j \le n} O(\frac{1}{\sqrt{\ln n}}) = 1 - o(1)$$

the distance  $\operatorname{dist}(X_{j+1},W_j)$  is at least  $\frac{1}{4n}$  for every  $d_0 < j \leq n-1$ . (In fact, the bound holds for all  $1 \leq j \leq n-1$ .)

Combining the two estimates on distances, we see that with probability 1 - o(1),

$$\prod_{0 \le j \le n-1} \operatorname{dist}(X_{j+1}, W_j) \ge \frac{\sqrt{n!}}{\sqrt{(n-d_0)!}} (\frac{1}{4n})^{n-d_0} \prod_{j=0}^{d_0} (1-\gamma_j).$$

Since  $n-d_0=o(\ln n)$ , the error term  $\frac{1}{\sqrt{(n-d_0)!}}(\frac{1}{4n})^{n-d_0}$  is only  $\exp(-o(\ln^2 n))$ .

The main error term comes from the product  $\prod_{j=0}^{d_0} (1 - \gamma_j)$ . By, the definition of  $\gamma_j$  and the fact that all  $\gamma_j$  are less than 1/2, we have

$$\prod_{j=1}^{d_0} (1 - \gamma_j) \ge \exp(-2\sum_{j=1}^{d_0} \gamma_j) \ge \exp(-14\sum_{j=1}^{d_0} \sqrt{\frac{\ln(n-j)}{n-j}}).$$

We use a rough estimate that

$$\sum_{j=1}^{d_0} \sqrt{\frac{\ln(n-j)}{n-j}} \le \sqrt{\ln n} \int_0^n x^{-1/2} dx = 2\sqrt{n \ln n}.$$

Putting these together, we obtain, with probability 1 - o(1), that

$$\prod_{0 \le j \le n-1} \operatorname{dist}(X_{j+1}, W_j) \ge \sqrt{n!} \exp(-28n^{1/2} \ln^{1/2} n + o(\ln^2 n))$$

$$> \sqrt{n!} \exp(-29n^{1/2} \ln^{1/2} n)$$

proving the theorem.

#### 5. Proof of Theorem 1.5

In this section, we denote  $N := 2^n$ . Our goal is to prove that  $\mathbf{P}(\det M_n = 0) \le N^{-(1+o(1))\varepsilon}$ , where  $\varepsilon$  is as in Theorem 1.5.

Notice that if  $M_n$  is singular, then  $X_1, \ldots, X_n$  span a proper subspace V of  $\mathbb{R}^n$ . The first (fairly simple) observation is that we can restrict to the case V is a hyperplane, thanks to the following lemma:

Lemma 5.1. [6] We have

$$\mathbf{P}(X_1,\ldots,X_n \text{ linearly dependent}) \leq N^{o(1)}\mathbf{P}(X_1,\ldots,X_n \text{ span a hyperplane}).$$

Remark 5.2. One can replace  $N^{o(1)}$  by 1 + o(1), but this refinement has no significance in the current situation.

**Proof** If  $X_1, \ldots, X_n$  are linearly dependent, then there must exist  $0 \le d \le n-1$  such that  $X_1, \ldots, X_{d+1}$  span a space of dimension exactly d. Since the number of possible d is at most  $n = N^{o(1)}$ , it thus suffices to show that

$$\mathbf{P}(X_1, \dots, X_{d+1} \text{ span a space of dimension exactly } d)$$
  
  $\leq \operatorname{const} \times \mathbf{P}(X_1, \dots, X_n \text{ span a hyperplane})$ 

for each fixed d. However, from Lemma 2.1 we see that

 $\mathbf{P}(X_1,\ldots,X_{d+2})$  span a space of dimension exactly d+1

$$|X_1,\ldots,X_{d+1}|$$
 span a space of dimension exactly  $d \ge 1 - 2^{d-n}$ ,

and so the claim follows from n-d-1 applications of Bayes' identity.

In view of this lemma, it suffices to show

$$\sum_{V,V \text{ hyperplane}} \mathbf{P}(X_1, \dots, X_n \text{ span } V) \leq N^{-\varepsilon + o(1)}.$$

Clearly, we may restrict our attention to those hyperplanes V which are spanned by their intersection with  $\{-1,1\}^n$ . Let us call such hyperplanes non-trivial. Furthermore, we call a hyperplane H degenerate if there is a vector v orthogonal to H and at most  $\log \log n$  coordinates of v are non-zero.

Fix a hyperplane V. Clearly we have

$$\mathbf{P}(X_1, \dots, X_n \text{ span } V) \le \mathbf{P}(X_1, \dots, X_n \in V) = \mathbf{P}(X \in V)^n. \tag{7}$$

The contribution of the degenerate hyperplanes is negligible, thanks to the following easy lemma (cf. the proof of (6)):

**Lemma 5.3.** The number of degenerate non-trivial hyperplanes is at most  $N^{o(1)}$ .

**Proof** If V is degenerate, then there is an integer normal vector  $v = (v_1, \ldots, v_n)$  with at most  $\log \log n$  non-zero entries. There are  $\sum_{k \leq \log \log n} \binom{n}{k} \leq \log \log n n^{\log \log n} \leq N^{o(1)}$  possible places for the non-zero entries. By relabeling if necessary we may assume that it is  $v_1, \ldots, v_k$  which are non-zero for some  $1 \leq k \leq \log \log n$ . Let  $\pi: \{-1,1\}^n \to \{-1,1\}^k$  be the obvious projection map. Then V is then determined by the projections  $\{\pi(X_1), \ldots, \pi(X_n)\}$ , which are a subset of  $\{-1,1\}^k$ . The number of such subsets is at most  $2^{2^k} \leq 2^{2^{\log \log n}} = N^{o(1)}$ , and the claim follows<sup>2</sup>.

By Lemma 2.1,  $\mathbf{P}(X \in V)$  is at most 1/2 for any hyperplane V, so the contribution of the degenerate non-trivial hyperplanes to  $\mathbf{P}(\det M_n = 0)$  is only  $N^{-1+o(1)}$ .

Following [6], it will be useful to specify the magnitude of  $\mathbf{P}(X \in V)$ . For each non-trivial hyperplane V, define the discrete codimension d(V) of V to be the unique integer multiple of 1/n such that

$$N^{-\frac{d(V)}{n} - \frac{1}{n^2}} < \mathbf{P}(X \in V) \le N^{-\frac{d(V)}{n}}.$$
 (8)

Thus d(V) is large when V contains few elements from  $\{-1,1\}^n$ , and conversely.

We define by  $\Omega_d$  the set of all non-degenerate, non-trivial hyperplanes with discrete codimension d. It is simple to see that  $1 \leq d(V) \leq n$  for all non-trivial V. In particular, there are at most  $O(n^2) = N^{o(1)}$  possible values of d, so to prove our theorem it suffices to prove that

$$\sum_{V \in \Omega_d} \mathbf{P}(X_1, \dots, X_n \text{ span } V) \le N^{-\varepsilon + o(1)}$$
(9)

for all  $1 \le d \le n$ . (Our errors o(1) shall decay to zero as  $n \to \infty$  uniformly in the choice of d.)

We first handle the (simpler) case when d is large. Note that if  $X_1, \ldots, X_n$  span V, then some subset of n-1 vectors already spans V. By symmetry, we have

$$\sum_{V \in \Omega_d} \mathbf{P}(X_1, \dots, X_n \text{ span } V) \le n \sum_{V \in \Omega_d} \mathbf{P}(X_1, \dots, X_{n-1} \text{ span } V) \mathbf{P}(X_n \in V)$$

$$\le n N^{-\frac{d}{n}} \sum_{V \in \Omega_d} \mathbf{P}(X_1, \dots, X_{n-1} \text{ span } V)$$

$$\le n N^{-\frac{d}{n}} = N^{-\frac{d}{n} + o(1)}$$

This disposes of the case when  $d \geq (\varepsilon - o(1))n$ . Thus to prove Theorem 1.5 it will now suffice to prove

<sup>&</sup>lt;sup>2</sup>The above estimates were extremely crude. In fact, as shown in [6], one can replace  $\log \log n$  with a quantity as high as  $n-3\log_2 n$  and still achieve the same result.

**Lemma 5.4.** If d is any integer multiple of 1/n such that

$$1 \le d \le (\varepsilon - o(1))n \tag{10}$$

then we have

$$\sum_{V \in \Omega_d} \mathbf{P}(X_1, \dots, X_n \ span \ V) \le N^{-\varepsilon + o(1)}.$$

This is the objective of the next section.

### 6. Proof of Lemma 5.4

The key idea in [6] is to find a new kind of random vectors which are more concentrated on hyperplanes in  $\Omega_d$  (with small d) than ( $\pm 1$ ) vectors. Roughly speaking, if we can find a random vector Y such that for any  $V \in \Omega_d$ 

$$\mathbf{P}(X \in V) \le c\mathbf{P}(Y \in V)$$

for some 0 < c < 1, then, intuitively, one may expect that

$$\mathbf{P}(X_1, \dots, X_n \text{ span } V) < c^n \mathbf{P}(Y_1, \dots, Y_n \text{ span } V)$$
 (11)

where  $X_i$  and  $Y_i$  are independent samples of X and Y, respectively. Since  $Y_1, \ldots, Y_n$  can only span at most one hyperplane V, one can then hope to conclude a bound of  $O(c^n)$  for the probability that  $X_1, \ldots, X_n$  span a hyperplane.

While (11) may be too optimistic (because the samples of Y on V may be too linearly dependent), it has turned out that something little bit weaker can be obtained, with a proper definition of Y. We next present this important definition.

**Definition 6.1.** For any  $0 \le \mu \le 1$ , let  $\eta^{(\mu)} \in \{-1,0,1\}$  be a random variable which takes +1 or -1 with probabilities  $\frac{\mu}{2}$ , and 0 with probability  $1 - \mu$ . Let  $X^{(\mu)} \in \{-1,0,1\}^n$  be a random variable of the form  $X^{(\mu)} = (\eta_1^{(\mu)}, \ldots, \eta_n^{(\mu)})$ , where the  $\eta_i^{(\mu)}$  are iid random variables with the same distribution as  $\eta^{(\mu)}$ .

Thus  $X^{(1)}$  has the same distribution as X, while  $X^{(0)}$  is concentrated purely at the origin. The other random variables  $X^{(\mu)}$  have an intermediate behavior. We shall work with  $X^{(\mu)}$  for  $\mu := 1/16$ ; this is not the optimal value of  $\mu$  but is the cleanest to work with. For this value of  $\mu$  we have the crucial inequality, following a Fourier-analytic argument of Halász [5] (see also [6]).

**Lemma 6.2.** Let V be a non-degenerate non-trivial hyperplane. Then we have

$$\mathbf{P}(X \in V) \le (\frac{1}{2} + o(1))\mathbf{P}(X^{(1/16)} \in V).$$

This lemma can be viewed as an assertion that any subspace which contains many points from  $\{-1,1\}^n$ , must necessarily contain several further points from  $\{-1,0,1\}^n$  (weighted appropriately). We will prove this lemma in the next section.

Remark 6.3. One can obtain similar results for smaller values of  $\mu$  than 1/16; for instance this was achieved in [6] for the value  $\mu := \frac{1}{108}e^{-1/108}$ , eventually resulting in their final gain  $\varepsilon := .001$  in Theorem 1.4. However the smaller one makes  $\mu$ , the smaller the final bound on  $\varepsilon$ ; indeed, most of the improvement in our bounds over those in [6] comes from increasing the value of  $\mu$ . One can increase the 1/16 parameter somewhat at the expense of worsening the  $\frac{1}{2}$  factor; in fact one can increase 1/16 all the way to 1/4 but at the cost of replacing 1/2 with 1. This shows that  $(3/4 + o(1))^n$  is the limit of our method. We have actually been able to attain this limit; see [11].

Let V be a hyperplane in  $\Omega_d$  for some d obeying the bound in Lemma 5.4. Let  $\gamma$  denote the quantity

$$\gamma := \frac{d}{n \log_2 16/15};\tag{12}$$

note from (2) and (10) that  $0 < \gamma < 1$ . Let  $\varepsilon' := \min(\varepsilon, \gamma)$ .

Consider the event that the i.i.d random vectors  $X_1, \ldots, X_{(1-\gamma)n}, X'_1, \ldots, X'_{(\gamma-\varepsilon')n}$  are linearly independent in V (we omit the rounding which plays no significant role). One can lower bound the probability of this event by the probability that all  $X_i$  and all  $X'_i$  belong to V, which is

$$\mathbf{P}(X \in V)^{(1-\varepsilon')n} = N^{-(1-\varepsilon')d - o(1)}.$$

Let us replace  $X_j$  by  $X^{(1/16)}$  for  $1 \leq j \leq (1-\gamma)n$  and consider the event  $A_V$  that  $X_1^{(1/16)}, \ldots, X_{(1-\gamma)n}^{(1/16)}, X_1', \ldots, X_{(\gamma-\varepsilon')n}'$  are linearly independent in V. Using Lemma 6.2, we are able to give a much better lower bound for this event:

$$\mathbf{P}(A_V) \ge N^{(1-\gamma)-(1-\varepsilon')d-o(1)}. \tag{13}$$

The critical gain is the term  $N^{(1-\gamma)}$ . In a sense, this gain is expected since  $X^{(1/16)}$  is much more concentrated on V then X. We will prove (13) at the end of the section. Let us now use it to conclude the proof<sup>3</sup> of Lemma 5.4.

Fix  $V \in \Omega_d$ . Let us denote by  $B_V$  the event that  $X_1, \ldots, X_n$  span V. To prove Lemma 5.4, we thus need to show

$$\sum_{V \in \Omega_d} \mathbf{P}(B_V) \le N^{-\varepsilon + o(1)}.$$

<sup>&</sup>lt;sup>3</sup>The argument below is a simplified version of a hypergraph covering argument used in [6].

The idea is to use (13) to "replace" some of the  $X_1, \ldots, X_n$  with the random variables  $X_1^{(1/16)}, \ldots, X_{(1-\gamma)n}^{(1/16)}$ , which are more concentrated on V, in order to obtain an exponential type gain. Since  $A_V$  and  $B_V$  are independent, we have, by (13) that

$$\mathbf{P}(B_V) = \frac{\mathbf{P}(A_V \wedge B_V)}{\mathbf{P}(A_V)} \le N^{-(1-\gamma)+(1-\varepsilon')d+o(1)} \mathbf{P}(A_V \wedge B_V).$$

Consider a set

$$X_1^{(1/16)}, \ldots, X_{(1-\gamma)n}^{(1/16)}, X_1', \ldots, X_{(\gamma-\varepsilon')n}', X_1, \ldots, X_n$$

of vectors satisfying  $A_V \wedge B_V$ . Then there exists  $\varepsilon' n - 1$  vectors  $X_{j_1}, \ldots, X_{j_{\varepsilon' n - 1}}$  inside  $X_1, \ldots, X_n$  which, together with  $X_1^{(1/16)}, \ldots, X_{(1-\gamma)n}^{(1/16)}, X_1', \ldots, X_{(\gamma-\varepsilon')n}'$ , span V. Since the number of possible indices  $(j_1, \ldots, j_{\varepsilon' n - 1})$  is  $\binom{n}{\varepsilon' n - 1} = N^{h(\varepsilon') + o(1)}$ , by conceding a factor of  $N^{h(\varepsilon') + o(1)}$ , we can assume that  $j_i = i$  for all relevant i. Let  $C_V$  be the event that  $X_1^{(1/16)}, \ldots, X_{(1-\gamma)n}^{(1/16)}, X_1', \ldots, X_{(\gamma-\varepsilon')n}', X_1, \ldots, X_{\varepsilon' n - 1}$  span V. Then we have

$$\mathbf{P}(B_V) \le N^{-(1-\gamma)+(1-\varepsilon')d+h(\varepsilon')+o(1)} \mathbf{P}\Big(C_V \wedge (X_{\varepsilon'n}, \dots, X_n \text{ in } V)\Big).$$

On the other hand,  $C_V$  and the event  $(X_{\varepsilon'n}, \ldots, X_n \text{ in } V)$  are independent, so

$$\mathbf{P}\Big(C_V \wedge (X_{\varepsilon'n}, \dots, X_n \text{ in } V)\Big) = \mathbf{P}(C_V)\mathbf{P}(X \in V)^{(1-\epsilon')n+1}.$$

Putting the last two estimates together we obtain

$$\mathbf{P}(B_V) \le N^{-(1-\gamma)+(1-\varepsilon')d+h(\varepsilon')+o(1)} N^{-((1-\varepsilon')n+1)d/n} \mathbf{P}(C_V)$$
$$= N^{-(1-\gamma)+h(\varepsilon')-\epsilon+o(1)} \mathbf{P}(C_V).$$

Since any set of vectors can only span a single space V, we have  $\sum_{V \in \Omega_d} \mathbf{P}(C_V) \leq 1$ . Thus, by summing over  $\Omega_d$ , we have

$$\sum_{V \in \Omega_d} \mathbf{P}(B_V) \le N^{-(1-\gamma)+h(\epsilon')-\epsilon+o(1)}.$$

We can rewrite the right hand side using (12) as  $N^{h(\varepsilon')+\frac{d}{n}(\frac{1}{\log_2 16/15}-1)-1+o(1)}$ . Since  $\frac{1}{\log_2 16/15}-1>0,\ d/n\leq \epsilon$ , and h is monotone in the interval  $0<\varepsilon'\leq \varepsilon<1/2$  we obtain

$$\sum_{V \in \Omega_d} \mathbf{P}(B_V) \le N^{h(\varepsilon) + \varepsilon \left(\frac{1}{\log_2 16/15} - 1\right) - 1 + o(1)}.$$

and the claim follows from the definition of  $\epsilon$  in (2).

In the rest of this section, we prove (13). The proof of Lemma 6.2, which uses entirely different arguments (based on Fourier analysis), will be presented in the next section.

To prove (13), first notice that the right hand side is the probability of the event  $A'_V$  that  $X_1^{(1/16)}, \ldots, X_{(1-\gamma)n}^{(1/16)}, X'_1, \ldots, X'_{(\gamma-\varepsilon)n}$  belong to V. Thus, by Bayes' identity it is sufficient to show that

$$\mathbf{P}(A_V|A_V') = N^{o(1)}.$$

This is an estimate similar to that in Lemma 5.1, and we prove it by similar arguments. From (8) we have

$$\mathbf{P}(X \in V) = (1 + O(1/n))2^{-d} \tag{14}$$

and hence by Lemma 6.2

$$\mathbf{P}(X^{(1/16)} \in V) \ge (2 + O(1/n))2^{-d}. \tag{15}$$

On the other hand, by a trivial modification of the proof of Lemma 2.1 we have

$$P(X^{(1/16)} \in W) \le (15/16)^{n - \dim(W)}$$

for any subspace W. By Bayes' identity we thus have the conditional probability bound

$$\mathbf{P}(X^{(1/16)} \in W | X^{(1/16)} \in V) \le (2 + O(1/n))2^d (15/16)^{n - \dim(W)}.$$

This is non-trivial when  $\dim(W) \leq (1 - \gamma)n$  thanks to (12).

Let  $E_k$  be the event that  $X_1^{(1/16)}, \ldots, X_k^{(1/16)}$  are independent. The above estimates imply that

$$\mathbf{P}(E_{k+1}|E_k \wedge A_V') \ge 1 - (2 + O(1/n))2^d (15/16)^{n-k}.$$

for all  $0 \le k \le (1 - \gamma)n$ . Applying Bayes' identity repeatedly (and (12)) we thus obtain

$$\mathbf{P}(E_{(1-\gamma)n}|A_V') \ge N^{-o(1)}.$$

If  $\gamma \leq \varepsilon$  then we are now done, so suppose  $\gamma > \varepsilon$  (so that  $\varepsilon' = \varepsilon$ ). From Lemma 2.1 we have

$$\mathbf{P}(X \in W) \le (1/2)^{n - \dim(W)}$$

for any subspace W, and hence by (14)

$$P(X \in W | X \in V) \le (1 + O(1/n))2^d (1/2)^{n - \dim(W)}$$
.

Let us assume  $E_{(1-\gamma n)}$  and denote by W the  $(1-\gamma n)$ -dimensional subspace spanned by  $X_1^{(1/16)}, \ldots, X_{(1-\gamma)n}^{(1/16)}$ . Let  $U_k$  denote the event that  $X_1', \ldots, X_k', W$  are independent. We have

$$p_k = \mathbf{P}(U_{k+1}|U_k \wedge A_V') \ge 1 - (1 + O(1/n))2^d(1/2)^{n-k-(1-\gamma)n} \ge 1 - \frac{1}{100}2^{(k+\varepsilon-\gamma)n}$$

for all  $0 \le k < (\gamma - \varepsilon)n$ , thanks to (10). Thus by Bayes' identity we obtain

$$\mathbf{P}(A_V|A_V') \ge N^{o(1)} \prod_{0 \le k < (\gamma - \varepsilon)n} p_k = N^{o(1)}$$

as desired.  $\Box$ 

# 7. Halász-type arguments

We now prove Lemma 6.2. The first step is to use Fourier analysis as in [5] to obtain usable formulae for  $\mathbf{P}(X \in V)$  and  $\mathbf{P}(X^{(\mu)} \in V)$ . Let  $v \in \mathbf{Z}^n \setminus \{0\}$  be an normal vector to V with integer coefficients (such a vector exists since V is spanned by the integer points  $V \cap \{-1,1\}^n$ ). By hypothesis, at least  $\log \log n$  of the coordinates of v are non-zero.

We first observe that the probability  $\mathbf{P}(X^{(\mu)} \in V)$  can be computed using the Fourier transform:

$$\begin{split} \mathbf{P}(X^{(\mu)} \in V) &= \mathbf{P}(X^{(\mu)} \cdot v = 0) \\ &= \mathbf{E}(\int_0^1 e^{2\pi i \xi X^{(\mu)} \cdot v} \ d\xi) \\ &= \int_0^1 \mathbf{E}(e^{2\pi i \xi \sum_{j=1}^n \epsilon_j^{(\mu)} v_j}) \ d\xi \\ &= \int_0^1 \prod_{j=1}^n ((1-\mu) + \mu \cos(2\pi \xi v_j)) \ d\xi. \end{split}$$

Applying this with  $\mu = 1/16$  we obtain

$$\mathbf{P}(X^{(1/16)} \in V) = \int_0^1 \prod_{i=1}^n \left(\frac{15}{16} + \frac{1}{16}\cos(2\pi\xi v_j)\right) d\xi.$$

Applying instead with  $\mu = 1$ , we obtain

$$\mathbf{P}(X \in V) = \int_{0}^{1} \prod_{j=1}^{n} \cos(2\pi \xi v_{j}) d\xi$$

$$\leq \int_{0}^{1} \prod_{j=1}^{n} |\cos(2\pi \xi v_{j})| d\xi$$

$$= \int_{0}^{1} \prod_{j=1}^{n} |\cos(\pi \xi v_{j})| d\xi,$$

where the latter identity follows from the change of variables  $\xi \mapsto \xi/2$  and noting that  $|\cos(\pi \xi v_i)|$  is still well-defined for  $\xi \in [0,1]$ . Thus if we set

$$F(\xi) := \prod_{j=1}^{n} |\cos(\pi \xi v_j)|; \quad G(\xi) := \prod_{j=1}^{n} (\frac{15}{16} + \frac{1}{16} \cos(2\pi \xi v_j)), \tag{16}$$

it will now suffice to show that

$$\int_0^1 F(\xi) \ d\xi \le \left(\frac{1}{2} + o(1)\right) \int_0^1 G(\xi) \ d\xi. \tag{17}$$

We now observe three estimates on F and G.

**Lemma 7.1.** For any  $\xi, \xi' \in [0,1]$ , we have the pointwise estimates

$$F(\xi) \le G(\xi)^4 \tag{18}$$

and

$$F(\xi)F(\xi') \le G(\xi + \xi')^2 \tag{19}$$

and the crude integral estimate

$$\int_0^1 G(\xi) \ d\xi \le o(1). \tag{20}$$

Of course, all operations on  $\xi$  and  $\xi'$  such as  $(\xi + \xi')$  in (19) are considered modulo 1.

**Proof of Lemma 7.1.** We first prove (18). From (16) it will suffice to prove the pointwise inequality

$$|\cos \theta| \le \left[\frac{15}{16} + \frac{1}{16}\cos 2\theta\right]^4$$

for all  $\theta \in \mathbf{R}$ . Writing  $\cos 2\theta = 1 - 2x$  for some  $0 \le x \le 1$ , then  $|\cos \theta| = (1 - x)^{1/2}$  and the inequality becomes

$$(1-x)^{1/2} \le (1-x/8)^4$$
.

Introducing the function  $f(x) := \log(\frac{1}{1-x})$ , this inequality is equivalent to

$$\frac{f(x) - f(0)}{x - 0} \ge \frac{f(x/8) - f(0)}{x/8 - 0}$$

but this is immediate from the convexity of f.

Now we prove (19). It suffices to prove that

$$|\cos \theta| |\cos \theta'| \le \left[\frac{15}{16} + \frac{1}{16}\cos(2(\theta + \theta'))\right]^2$$

for all  $\theta, \theta' \in \mathbf{R}$ . As this inequality is periodic with period  $\pi$  in both  $\theta$  and  $\theta'$  we may assume that  $|\theta|, |\theta'| < \pi/2$  (the cases when  $\theta = \pi/2$  or  $\theta' = \pi/2$  being trivial). Next we observe from the concavity of  $\log \cos(\theta)$  in the interval  $(-\pi/2, \pi/2)$  that

$$\cos\theta\cos\theta' \le \cos^2\frac{\theta+\theta'}{2} = \frac{1}{2} + \frac{1}{2}\cos(\theta+\theta').$$

Writing  $\cos(\theta + \theta') = 1 - 2x$  for some  $0 \le x \le 1$ , then  $\cos 2(\theta + \theta') = 2(1 - 2x)^2 - 1 = 1 - 8x + 8x^2$ , and our task is now to show that

$$1 - x \le (1 - (x - x^2)/2)^2 = 1 - x + x^2 + (x - x^2)^2/4,$$

but this is clearly true.

Now we prove (20). We know that at least  $\log \log n$  of the  $v_j$  are non-zero; without loss of generality we may assume that it is  $v_1, \ldots, v_K$  which are non-zero for some  $K > \log \log n$ . Then we have by Hölder's inequality, followed by a rescaling by  $v_j$ 

$$\int_{0}^{1} G(\xi) \ d\xi \le \int_{0}^{1} \prod_{j=1}^{K} \left(\frac{15}{16} + \frac{1}{16} \cos(2\pi \xi v_{j})\right) \ d\xi$$

$$\le \prod_{j=1}^{\log \log n} \left(\int_{0}^{1} \left(\frac{15}{16} + \frac{1}{16} \cos(2\pi \xi v_{j})\right)^{\log \log n} \ d\xi\right)^{1/\log \log n}$$

$$= \prod_{j=1}^{K} \left(\int_{0}^{1} \left(\frac{15}{16} + \frac{1}{16} \cos(2\pi \xi)\right)^{K} \ d\xi\right)^{1/K}$$

$$= \int_{0}^{1} \left(\frac{15}{16} + \frac{1}{16} \cos(2\pi \xi)\right)^{K} \ d\xi$$

$$= o(1)$$

as desired, since  $K \ge \log \log n$ .

Now we can quickly conclude the proof of (17). From (19) we have the sumset inclusion

$$\{\xi \in [0,1] : F(\xi) > \alpha\} + \{\xi \in [0,1] : F(\xi) > \alpha\} \subseteq \{\xi \in [0,1] : G(\xi) > \alpha\}$$

for any  $\alpha > 0$ . Taking measures of both sides and applying the Mann-Kneser-Macbeath " $\alpha + \beta$  inequality"  $|A + B| \ge \min(|A| + |B|, 1)$  (see [9]), we obtain

$$\min(2|\{\xi \in [0,1]: F(\xi) > \alpha\}|, 1) \le |\{\xi \in [0,1]: G(\xi) > \alpha\}|.$$

But from (20) we see that  $|\{\xi \in [0,1] : G(\xi) > \alpha\}|$  is strictly less than 1 if  $\alpha > o(1)$ . Thus we conclude that

$$|\{\xi \in [0,1]: F(\xi) > \alpha\}| \leq \frac{1}{2}|\{\xi \in [0,1]: G(\xi) > \alpha\}|$$

when  $\alpha > o(1)$ . Integrating this in  $\alpha$ , we obtain

$$\int_{[0,1]:F(\xi)>o(1)} F(\xi) \ d\xi \le \frac{1}{2} \int_0^1 G(\xi) \ d\xi.$$

On the other hand, from (18) we see that when  $F(\xi) \leq o(1)$ , then  $F(\xi) = o(F(\xi)^{1/4}) \leq G(\xi)$ , and thus

$$\int_{[0,1]:F(\xi)\leq o(1)} F(\xi) \ d\xi \leq o(1) \int_0^1 G \ d\xi.$$

Adding these two inequalities we obtain (17) as desired. This proves Lemma 6.2.  $\Box$ 

Remark 7.2. A similar Fourier-analytic argument can be used to prove Lemma 3.4. To see this, we first recall Esséen's concentration inequality [4]

$$\mathbf{P}(X \in I) \le C \int_{|t| < 1} |\mathbf{E}(e^{itX})| \ dt$$

for any random variable X and any interval I of length at most 1. Thus to prove Lemma 3.4 it would suffice to show that

$$\int_{|t| \le 1} |\mathbf{E}(\exp(it\sum_{j=1}^k a_j \epsilon_j)|) \ dt = O(1/\sqrt{k}).$$

But by the independence of the  $\epsilon_i$ , we have

$$|\mathbf{E}(\exp(it\sum_{j=1}^{k} a_j \epsilon_j))| = \prod_{j=1}^{k} |\mathbf{E}(e^{ita_j \epsilon_j})| = |\prod_{j=1}^{k} \cos(ta_j)|$$

and hence by Hölder's inequality

$$\int_{|t| \le 1} |\mathbf{E}(\exp(it \sum_{j=1}^k a_j \epsilon_j))| \ dt \le \prod_{j=1}^k (\int_{|t| \le 1} |\cos(t a_j)|^k \ dt)^{1/k}.$$

But since each  $a_j$  has magnitude at least 1, it is easy to check that  $\int_{|t| \le 1} |\cos(ta_j)|^k dt = O(1/\sqrt{k})$ , and the claim follows.

## 8. Extensions and Refinements

8.1. Singularity of more general random matrices. In [8], Komlós extended Theorem 1.2 by showing that the singularity probability is still o(1) for a random matrix whose entries are i.i.d. random variables with non-degenerate distribution. By slightly modifying our proof of Theorem 1.2, we are able to prove a different extension.

We say that a random variable  $\xi$  has  $(c, \rho)$ -property if

$$\min\{\mathbf{P}(\xi \ge c), \mathbf{P}(\xi \le -c)\} \ge \rho.$$

Let  $\xi_{ij}$ ,  $1 \leq i, j \leq n$  be independent random variables. Assume that there are positive constants c and  $\rho$  (not depending on n) such that for all  $1 \leq i, j \leq n$ ,  $\xi_{ij}$  has  $(c, \rho)$ -property. The new feature here is that we do not require  $\xi_{ij}$  be identical.

**Theorem 8.2.** Let  $\xi_{ij}$ ,  $1 \leq i, j \leq n$  be as above. Let  $M_n$  be the random matrix with entries  $\xi_{ij}$ . Then

$$\mathbf{P}(\det M_n = 0) = o(1).$$

We only sketch the proof, which follows the proof of Theorem 1.2 very closely and uses the same notation:  $X_1, \ldots, X_n$  are the row vectors of  $M_n$  and  $W_j$  is the subspace spanned by  $X_1, \ldots, X_j$ . We will show

$$\sum_{j=1}^{n-1} \mathbf{P}(X_{j+1} \in W_j) = o(1). \tag{21}$$

This estimate is a consequence of the following two lemmas, which are generalization of Lemmas 2.1 and 3.1.

**Lemma 8.3.** Let W be a k dimensional subspace of  $\mathbb{R}^n$ . Then for any  $1 \leq j \leq n$ 

$$\mathbf{P}(X_i \in W) \le (1 - \rho)^{n - k}.$$

**Lemma 8.4.** For any  $n/2 \le j \le n$ 

$$\mathbf{P}(X_j \in W_{j-1}) = O(1/\sqrt{\ln n}).$$

The proof of Lemma 8.3 is the same as that of Lemma 2.1. The only information we need is that for any fixed number x and any plausible i, j,  $\mathbf{P}(\xi_{ij} = x) \leq 1 - \rho$ .

To prove Lemma 8.4, let us consider the case j = n (the proof is the same for other cases). We need to modify the definition of universality as follows.

We call a subset V of n-dimensional vectors k-universal if for any set of k indices  $1 \le i_1 < i_2 < \cdots < i_k \le n$  and any sign sequence  $\epsilon_1, \ldots, \epsilon_k$ , one can find a vector  $v \in V$ , such that the  $i_j$  coordinate of v has sign  $\epsilon_j$  and absolute value at least c.

In what follows, we set  $l = \ln n/10$ . We first show  $X_1, \ldots, X_n$  is very likely to be l-universal. (Notice that the  $X_j$  have different distribution.)

**Lemma 8.5.** With probability 1 - o(1/n),  $X_1, \ldots, X_n$  is l-universal.

**Proof of Lemma 8.5.** Fix a set of indices and a sequence of signs. For any  $1 \le j \le n$ , the probability that  $X_j$  fails is at most  $1 - \rho^l$ . The rest of the proof is the same.

It follows that

**Corollary 8.6.** Let H be a subspace spanned by n-1 random vectors. Then with probability 1-o(1/n), any unit vector perpendicular to H has at least l+1 coordinates whose absolute values are at least  $\frac{1}{Kn}$ , where K is a constant depending on c.

The last ingredient is the following generalization of Lemma 3.4.

**Lemma 8.7.** Let  $a_1, \ldots, a_k$  be real numbers with absolute values larger than one and  $\epsilon_1, \ldots, \epsilon_k$  be independent random variables satisfying the  $(c, \rho)$ -property. Then for any interval I of length one

$$\mathbf{P}(\sum_{i=1}^{k} a_i \epsilon_i \in I) = O(1/\sqrt{k}).$$

Theorem 8.2 follows from Corollary 8.6 and Lemma 8.7. To conclude, let us remark that statements more accurate than Lemma 8.7 are known (see e.g. [5]). However, this lemma can be proved using an argument similar to the one in Remark 7.2.

8.8. Determinants of more general random matrices. Let  $\xi_{ij}$ ,  $1 \le i, j \le n$ , be a set of independent (but not necessarily i.i.d.) r.v's with the following two properties:

- Each  $\xi_{ij}$  has mean zero and variance one.
- There is a constant K that  $|\xi_{ij}| \leq K$  with probability one.

These two properties imply the following property

• There are constants  $\delta > 0$  and  $\delta' > 0$  such that for any interval I of length  $2\delta$ ,  $\mathbf{P}(\xi_{ij} \in I) \leq 1 - \delta'$  for all  $1 \leq i, j \leq n$ .

**Theorem 8.9.** Consider the random matrix  $M_n$  with entries  $\xi_{ij}$  as above. Let  $\epsilon$  be an arbitrary positive constant. With probability 1 - o(1),

$$|\det M_n| \ge \sqrt{n!} \exp(-n^{1/2+\epsilon}).$$

Notice that Lemma 2.1 holds for this model of random matrices, since the last property of  $\xi_{ij}$  implies that  $\xi_{ij}$  has  $(c, \rho)$  property.

Next, consider Lemma 2.2. Consider a row vector, say,  $X = (\xi_{i1}, \dots, \xi_{in})$  and a fixed subspace W of dimension d. Again, we have (with the same notation as in Section 2)

$$\operatorname{dist}(X, W)^{2} = |X|^{2} - |PX|^{2} = |X|^{2} - \sum_{i=1}^{n} \sum_{k=1}^{n} \xi_{ij} \xi_{ik} p_{jk}.$$

However, it is no longer the case that the last formula equals

$$n - d - \sum_{i=1}^{n} \sum_{k=1}^{n} \xi_{ij} \xi_{ik} a_{jk}$$

since  $\xi_{ij}$  are not Bernoulli random variables. On the other hand, we can have something similar with an extra error term. It is easy to show, using Chernoff's bound, that

$$|X|^2 = \sum_{j=1}^n \xi_{ij}^2 \ge n - \frac{C}{2} n^{1/2} \ln n$$

holds with probability at least  $1 - 1/2n^2$ , for some sufficiently large C. Similarly,

$$\sum_{j=1}^{n} \xi_{ij}^2 p_{jj} \le d - \frac{C}{2} n^{1/2} \ln n$$

holds with probability at least  $1-1/2n^2$ . (The use of Chernoff's bound requires of random variables be bounded. One can of course, use some other method to remove this assumption.)

The probability  $1/n^2$  is negligible. Moreover, we can apply Talagrand's inequality the same way as before. However, because of the new error term  $Cn^{1/2} \ln n$ , we cannot set  $d_0 = n - \ln^{1/4} n$ , but have to stop at  $n - Cn^{1/2} \ln n$ . In order to handle the cases when  $\ln^{1/4} n \le n - d \le Cn^{1/2} \ln n$ , we need the following lemma, due to Bourgain (private conversation), which can be seen as an extension of Lemma 2.1.

**Lemma 8.10.** There are constants a > 0, 1 > b > 0 such that the following holds. Let W be a fixed subspace of dimension  $d \le n - 1$  and X a random (row) vector. Then

$$\mathbf{P}(\operatorname{dist}(X, W) \le \frac{a}{\sqrt{n}}) \le b^{d-n}.$$

**Proof of Lemma 8.10.** We construct unit vectors  $Z_1, \ldots, Z_{n-d}$  (not necessarily orthogonal) in the orthogonal complement  $W^{\perp}$  of W as follows. We let  $Z_1$  be an arbitrary unit vector in  $W^{\perp}$ ; since  $Z_1$  has unit length, at least one of its coordinates has magnitude at least  $1/\sqrt{n}$ . Without loss of generality we may assume that it is the first coordinate  $\langle Z_1, e_1 \rangle$  which has magnitude at least  $1/\sqrt{n}$ . Now we let  $Z_2$  be an arbitrary unit vector in  $W^{\perp} \cap e_1^{\perp}$  (which has dimension at least n-d-1); then  $Z_2$  is orthogonal to  $e_1$  and has a coordinate of magnitude at least  $1/\sqrt{n}$ . Without loss of generality we may take  $|\langle Z_2, e_2 \rangle| \geq 1/\sqrt{n}$ . Continuing in this fashion, we can (without loss of generality) find  $Z_1, \ldots, Z_{n-d} \in W^{\perp}$  such that each  $Z_j$  is orthogonal to  $e_1, \ldots, e_{j-1}$  and is such that  $|\langle Z_j, e_j \rangle| \geq 1/\sqrt{n}$ .

Now suppose that  $X = (\epsilon_1, \dots, \epsilon_n)$  is such that  $\operatorname{dist}(X, W) \leq \frac{a}{\sqrt{n}}$ , where a is a sufficiently small positive constant. Fix the last d coordinates  $\epsilon_{n-d+1}, \dots, \epsilon_n$  and let T denote the set of all vectors X with these fixed coordinates satisfying

$$\operatorname{dist}(X, W) \le \frac{a}{\sqrt{n}}.$$

Fix a vector  $X_0 = (g_1, \ldots, g_n)$  in T. It is easy to show that for any vector  $X = (g'_1, \ldots, g'_n) \in T$ ,  $|g'_i - g_i| \leq 2a$ , for all  $1 \leq i \leq n - d$ . On the other hand, if a is sufficiently small, then by the third property of the  $\xi_{ij}$ , there is a positive constant b < 1 such that the set of  $g'_i$  where  $|g'_i - g_i| \leq 2a$  has measure at most b for all  $1 \leq i \leq n - d$ . This proves the claim.

The rest of the proof is basically the same, with some minor and natural modification in the calculation. The error term obtained from Lemma 8.10 (in the determinant) is only

$$n^{-O(n^{1/2}\ln n)} = \exp(-o(n^{1/2+\epsilon}))$$

for any fixed  $\epsilon > 0$ .

In certain situations, we do not have the assumption that  $|\xi_{ij}|$  are bounded from above by a constant. We are going to consider the following model. Let  $\xi_{ij}, 1 \leq i, j \leq n$  be i.i.d. random variables with mean zero and variance one. Assume furthermore that their fourth moment is finite. Consider the random matrix  $M_n$  with  $\xi_{ij}$  as its entries.

By using Lemmas 2.1 and 3.1 and replacing Lemma 2.2 by a result of Bai and Yin [1], which asserts that the volume of the  $(1-\gamma)n$ -dimensional parallelepiped spanned by the first  $(1-\gamma)n$  row vectors is at least  $n^{(1/2-\gamma/2-o(1))n}$  with probability 1-o(1) for any fixed  $\gamma > 0$ , we can prove

**Theorem 8.11.** We have, with probability 1 - o(1), that

$$|\det M_n| \ge n^{(1/2 - o(1))n}$$
.

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