ECE2150J Introduction to Circuits

Chapter 12. Three-Phase Circuits

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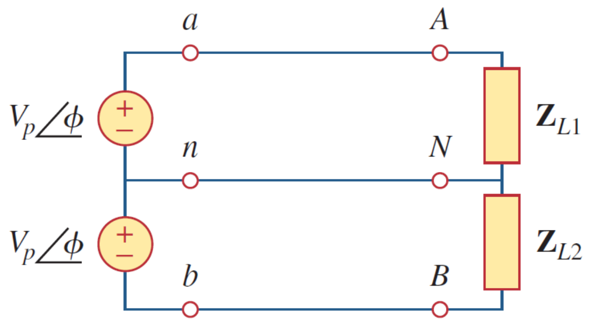
12.1 Introduction

Single phase two wire

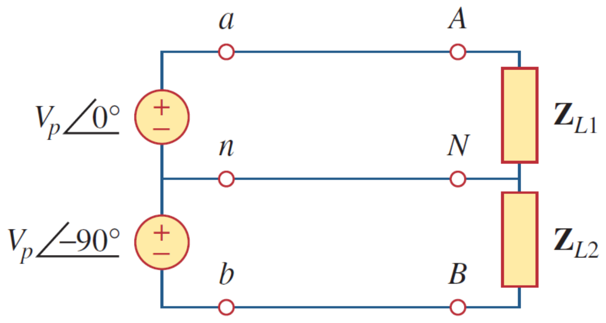


Single phase three wire

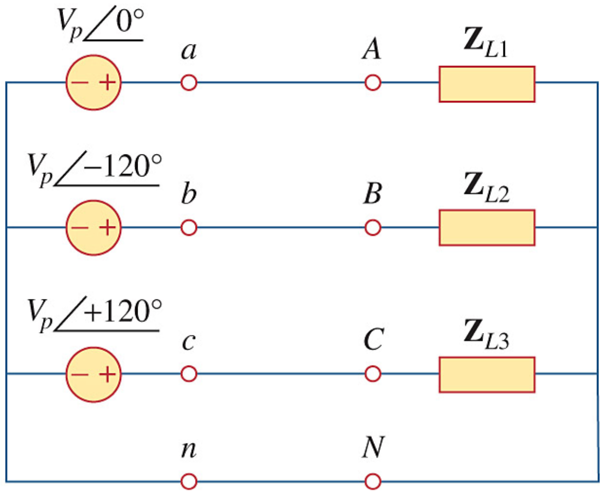
A single-phase ac power system: a generator connected through a pair of wires to a load.

A single-phase three wire system: two identical sources (equal magnitude and the same phase) connected to two loads by two outer wires and the neutral.

Polyphase: AC sources operate at the same frequency but different phases.



Two-phase three-wire system



Three-phase four-wire system

A three-phase system: Three sources having the same amplitude and frequency but out of phase with each other by 120°. The three-phase system is **by far the most prevalent and most economical polyphase system**.

Three-phase systems are important for at least three reasons.

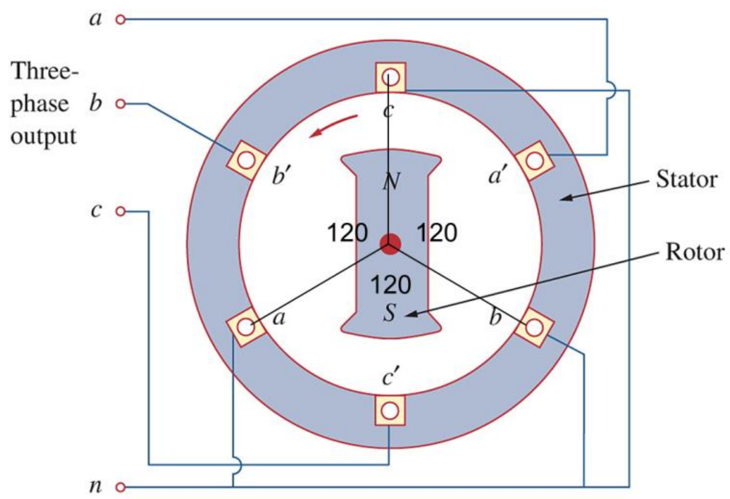
First, nearly all electric power is generated and distributed in three-phase, at the operating frequency of 50 or 60 Hz. **Multiphases** can be generated: from 1-2 to even 48 phases.

Second, the instantaneous power in a three-phase system can be **constant** (not pulsating). This results in uniform power transmission and less vibration of three-phase machines.

Third, for the same amount of power, the three-phase system is **more economical** than the single phase.

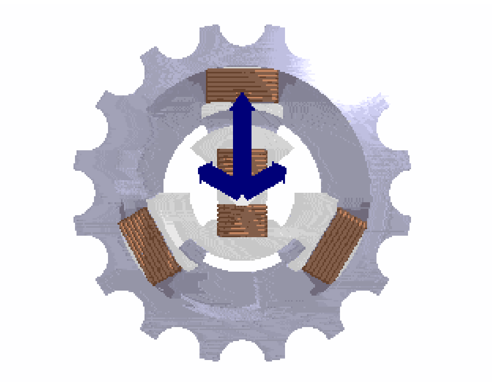
12.2 Balanced Three-Phase Voltages

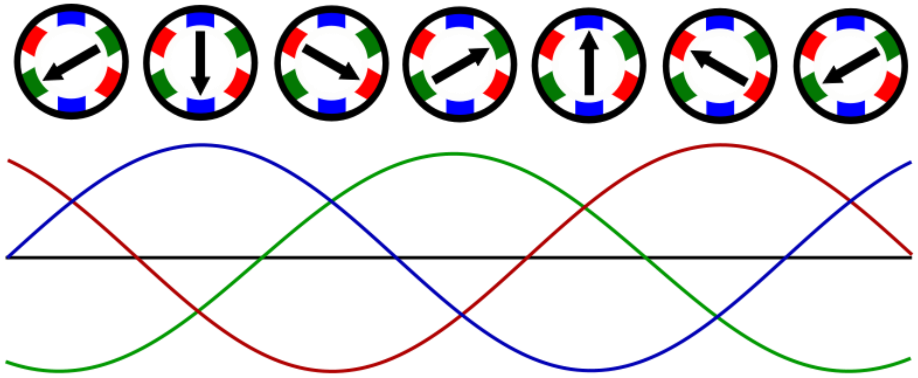
Three-phase voltages are often produced with a three-phase ac generator.

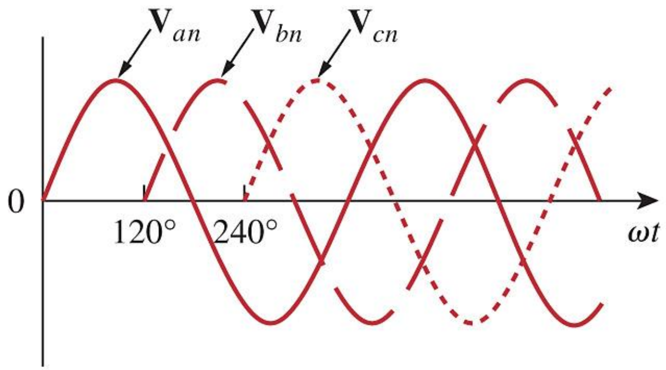


https://www.tme.eu/

Rotor: a rotating magnet
Stator: a stationary winding



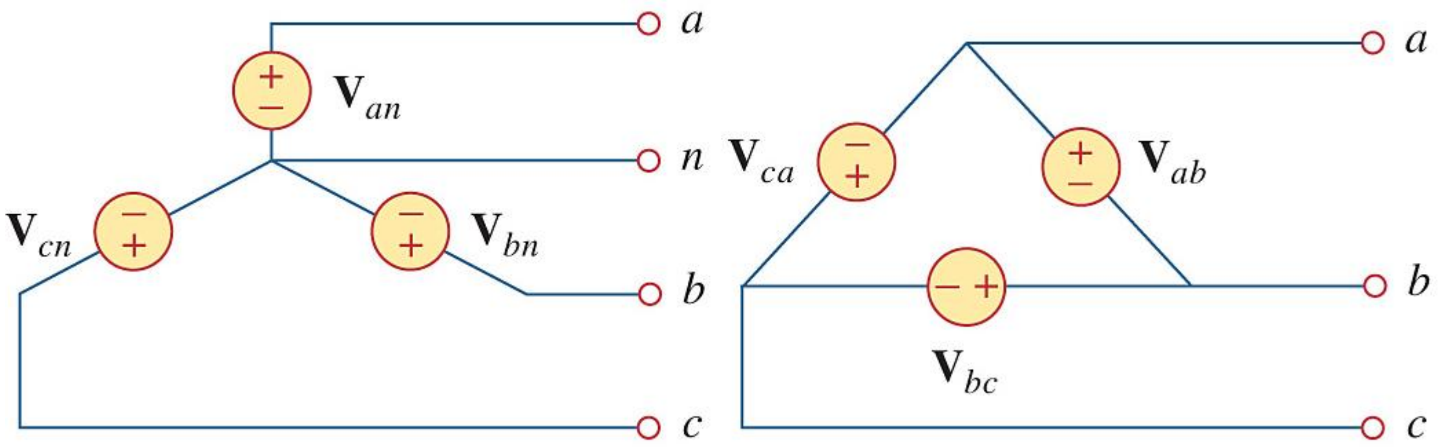


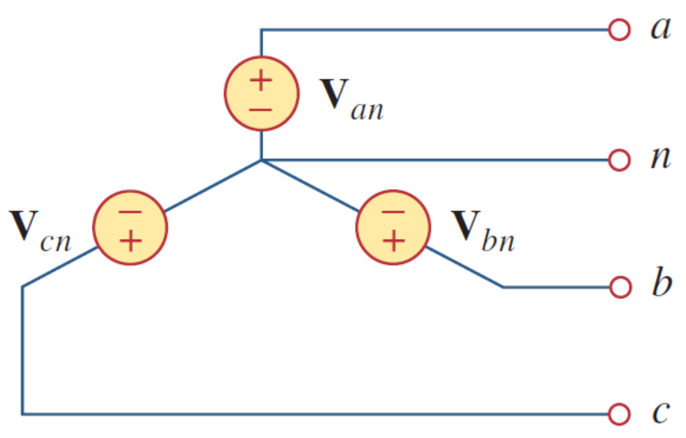


As the rotator rotates, the flux through each coil varies sinusoidally with time, inducing a sinusoidal voltage. Because the coils are placed **120°** apart, the induced voltages are out of phase by **120°**.

Voltage sources

A typical three-phase system consists of three voltage sources connected to loads by three or four wires. A three-phase system is equivalent to three single-phase system. **The voltage sources** can be either **wye- or delta-connected**.





V_{an}, V_{bn}, and V_{cn} are phase voltages.

Balanced phase voltages: Equal in magnitude and frequency and are out of phase with each other by 120°. This implies that

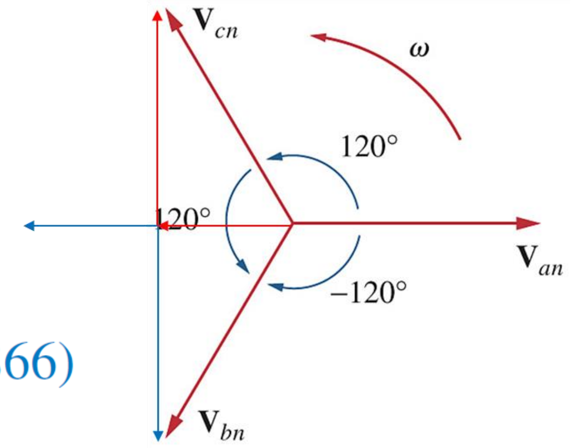
(i) $|\mathbf{V}_{an}| = |\mathbf{V}_{bn}| = |\mathbf{V}_{cn}|$

(ii) $\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} = 0$

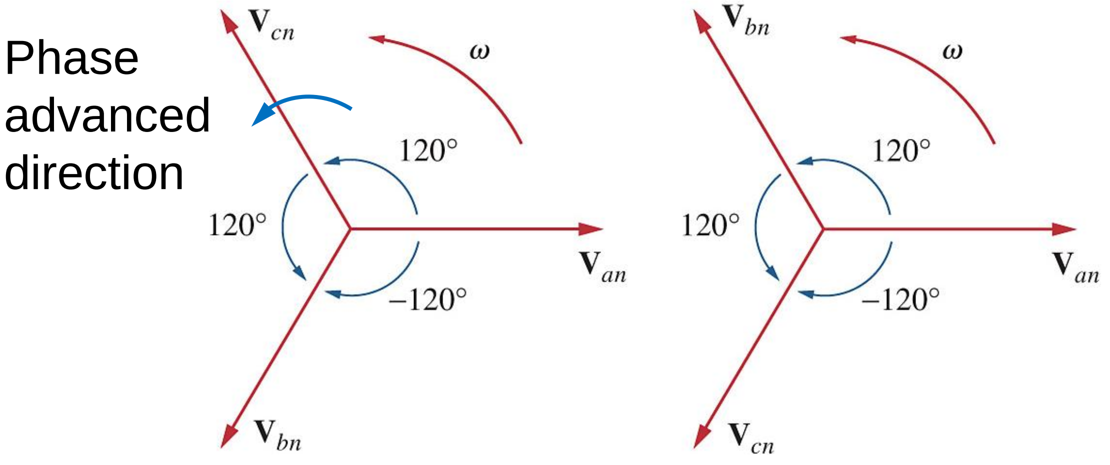
$$= V_p\angle 0^\circ + V_p\angle -120^\circ + V_p\angle +120^\circ$$

$$= V_p(1.0 - 0.5 - j0.866 - 0.5 + j0.866)$$

$$= 0$$



Phase sequence

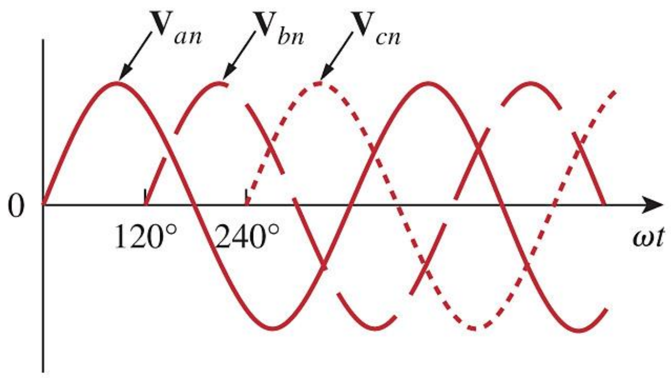


$$\mathbf{V}_{an} = V_p\angle 0^\circ$$
$$\mathbf{V}_{bn} = V_p\angle -120^\circ$$
$$\mathbf{V}_{cn} = V_p\angle -240^\circ = V_p\angle +120^\circ$$

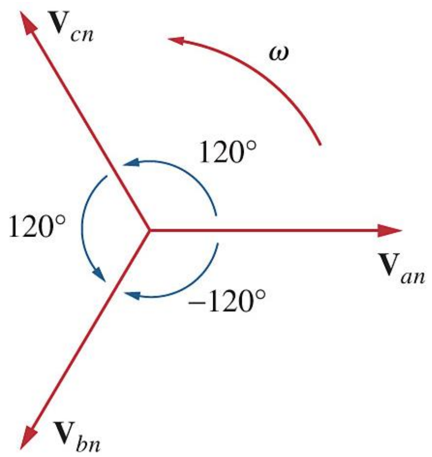
$$\mathbf{V}_{an} = V_p\angle 0^\circ$$
$$\mathbf{V}_{cn} = V_p\angle -120^\circ$$
$$\mathbf{V}_{bn} = V_p\angle -240^\circ = V_p\angle +120^\circ$$

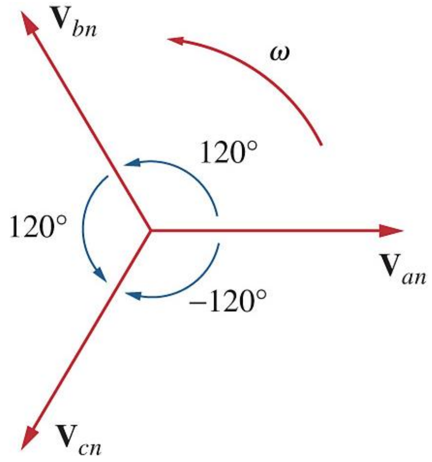
***V_p* is the effective (rms)** value of the phase voltage

The phase sequence is the time order in which the voltages pass through their respective maximum values.



abc: a leads b, b leads c
(i.e., $\angle$a > $\angle$ b > $\angle$c)

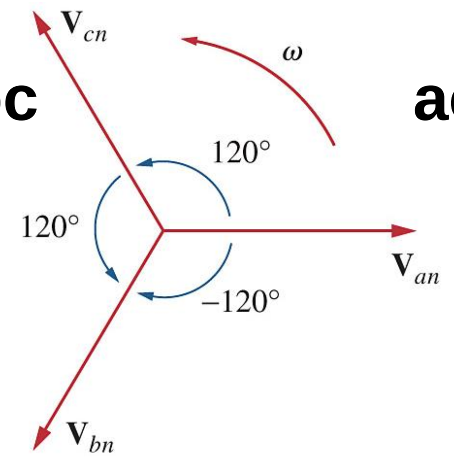




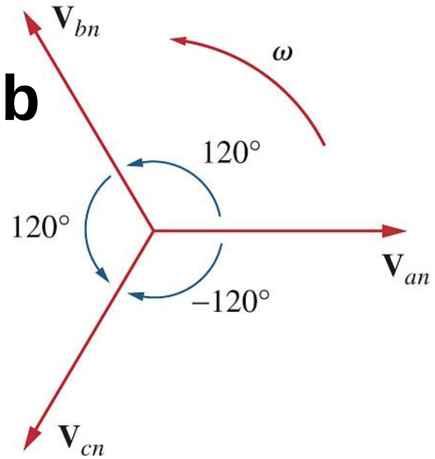
From which phase the voltage curve above is generated?
abc or acb sequence?

The phase sequence is determined **by the order in which the phasors pass through a fixed point** in the phase diagram.

abc



acb



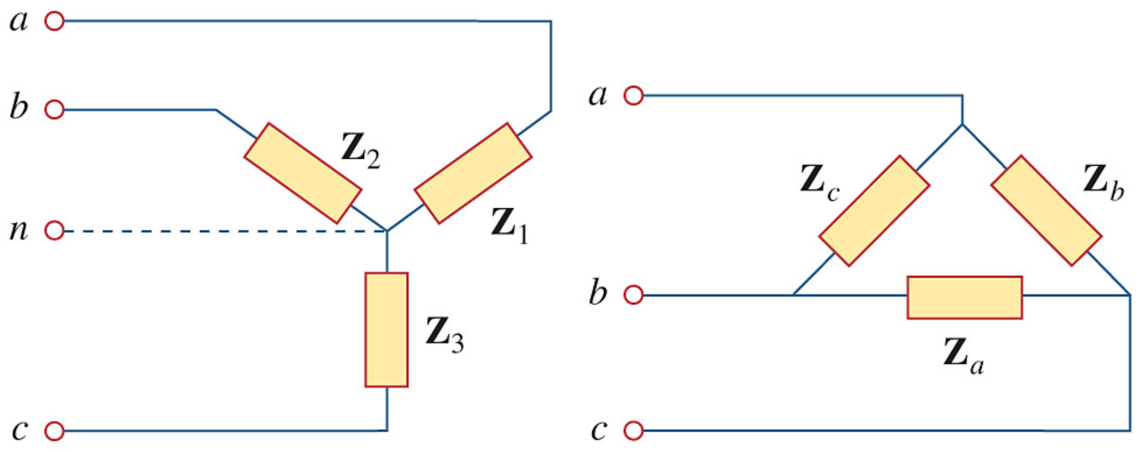
$$\mathbf{V}_{an} = V_p\underline{/0^\circ}$$
$$\mathbf{V}_{bn} = V_p\underline{/-120^\circ}$$
$$\mathbf{V}_{cn} = V_p\underline{/-240^\circ} = V_p\underline{/+120^\circ}$$

$$\mathbf{V}_{an} = V_p\underline{/0^\circ}$$
$$\mathbf{V}_{cn} = V_p\underline{/-120^\circ}$$
$$\mathbf{V}_{bn} = V_p\underline{/-240^\circ} = V_p\underline{/+120^\circ}$$

*V_p is the effective (rms) value of the phase voltage

Phase load connection

Similar to the source connections, a three-phase load can be either wye-connected or delta-connected



Balanced Load

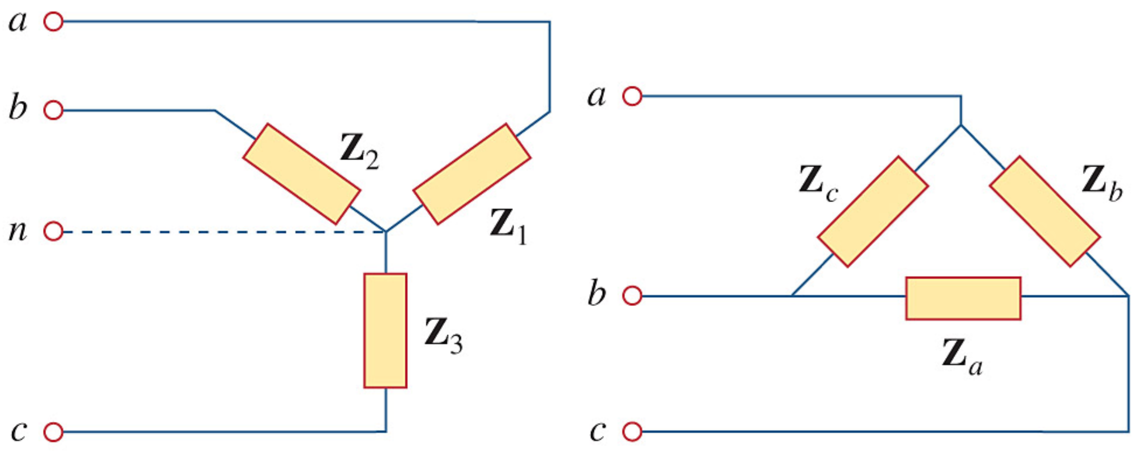
A balanced load is one in which the phase impedances are **equal in magnitude and in phase**.

(i) Balanced wye-connected load:

$$\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 = \mathbf{Z}_Y$$

(i) Balanced delta-connected load:

$$\mathbf{Z}_a = \mathbf{Z}_b = \mathbf{Z}_c = \mathbf{Z}_\Delta$$

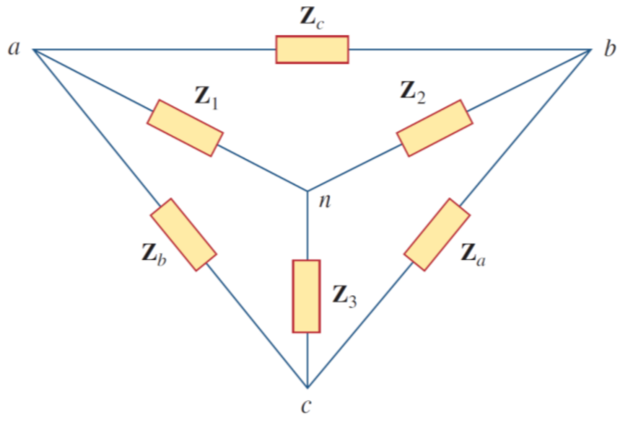


A wye-connected load can be transformed into a delta-connected load, or vice versa.

$$\boxed{Z_\Delta = 3Z_Y \text{ or } Z_Y = \frac{1}{3}Z_\Delta}$$

Y-Δ Conversion:

$$\mathbf{Z}_a = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_1}$$
$$\mathbf{Z}_b = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_2}$$
$$\mathbf{Z}_c = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_3}$$



Δ-Y Conversion:

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_b\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$
$$\mathbf{Z}_2 = \frac{\mathbf{Z}_c\mathbf{Z}_a}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$
$$\mathbf{Z}_3 = \frac{\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

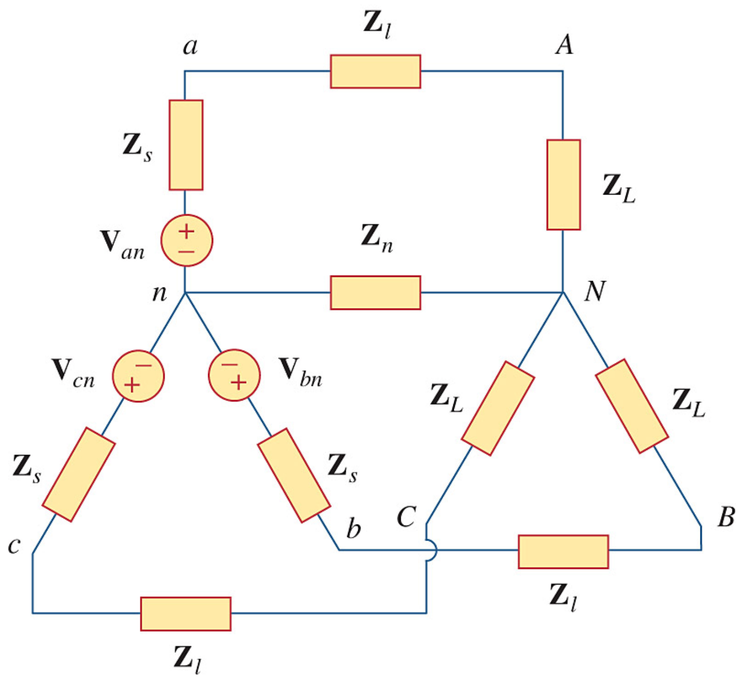
$$\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 = \mathbf{Z}_Y$$
$$\mathbf{Z}_a = \mathbf{Z}_b = \mathbf{Z}_c = \mathbf{Z}_\Delta$$

Since both the three-phase source and the three-phase load can be either wye- or delta-connected, we have four possible connections:

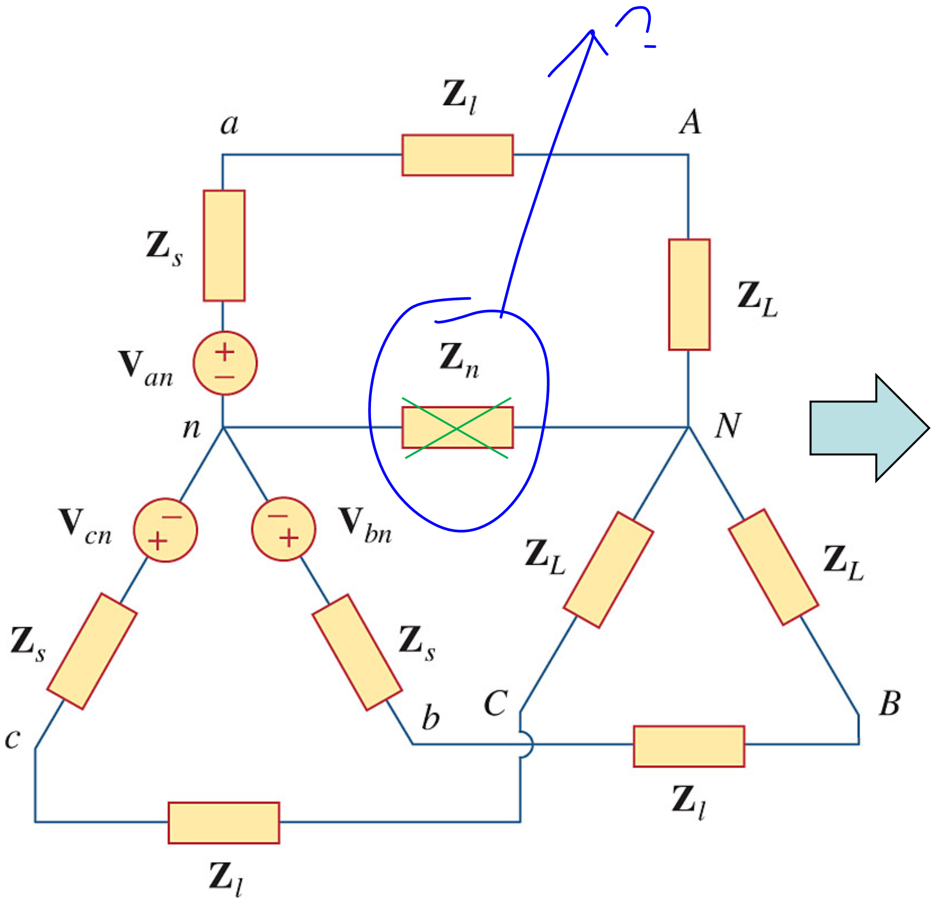
1. Y-Y connection (i.e., Y-connected source with a Y-connected load).
2. Y-Δ connection.
3. Δ-Δ connection.
4. Δ-Y connection.

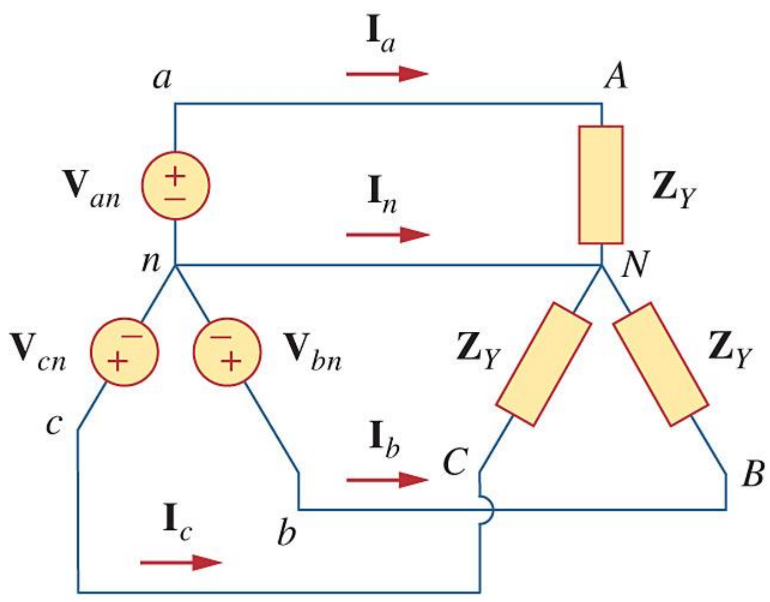
12.3 Balanced Wye-Wye Connection

We begin with the Y-Y system, because any balanced three-phase system can be reduced to an equivalent Y-Y system. A balanced Y-Y system is a three-phase system with a balanced Y-connected source and a balanced Y-connected load.



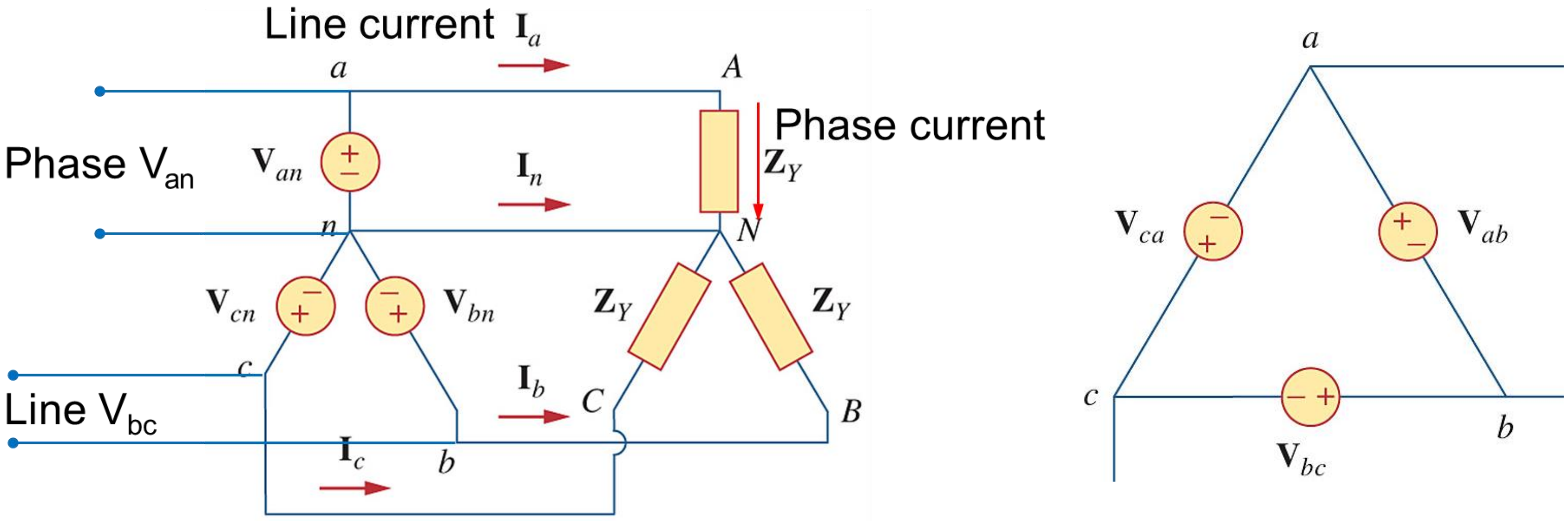
If balanced, the system can be simplified.





$$\mathbf{Z}_Y = \mathbf{Z}_s + \mathbf{Z}_\ell + \mathbf{Z}_L$$

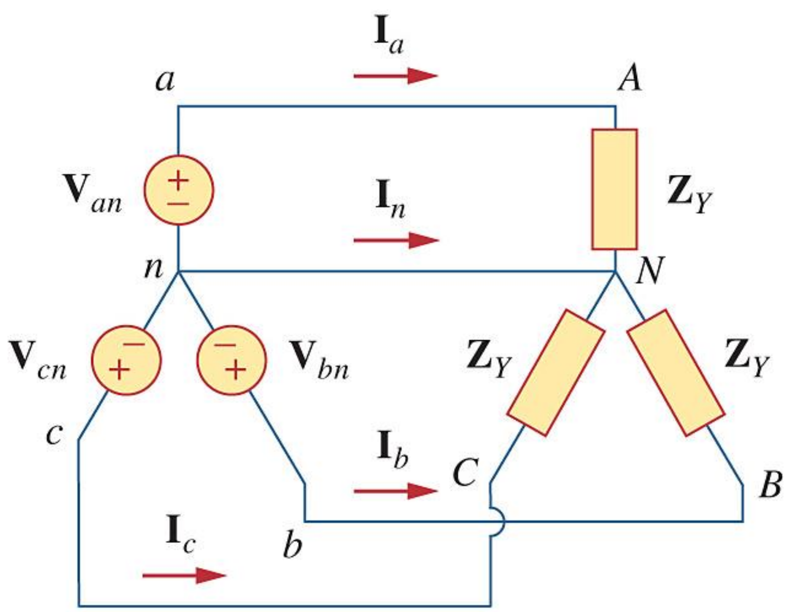
Phases and Lines?



The voltage between **any two line** conductors are termed **line voltages** and **the current** that passes along **each line** conductor is termed **a line current**.

Phases are connected **between any pair of line terminals**. **The voltage** appearing across **any phase** is termed a **phase voltage**, and **the current** passing through **any phase** is termed a **phase current**.

Assuming the positive sequence, **the phase voltages** (or line-to-neutral voltages) are



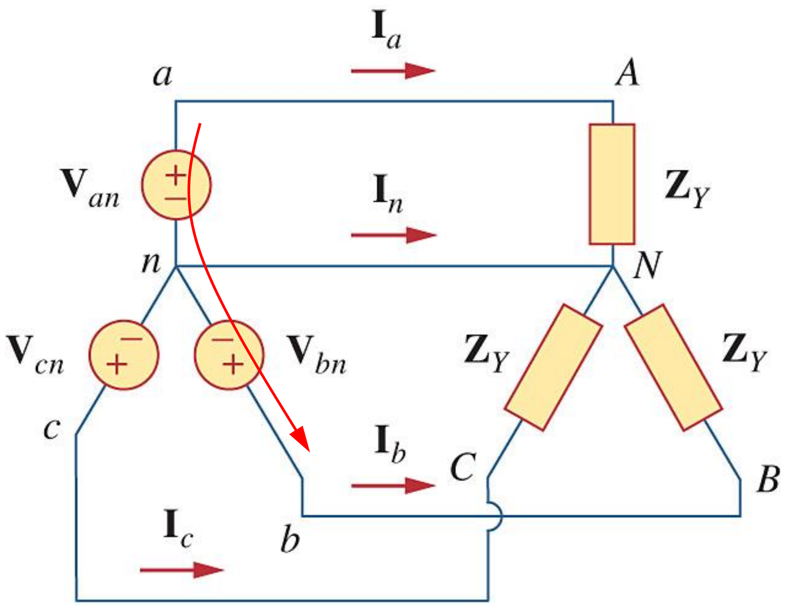
rms value phase voltage

$$\tilde{V}_{an} = V_p \angle 0^\circ$$
$$\tilde{V}_{bn} = V_p \angle -120^\circ$$
$$\tilde{V}_{cn} = V_p \angle -240^\circ$$

Balanced source:
Same amplitude and frequency
Out of phase with each other by 120°

Line voltages

The line voltages (or line-to-line voltages) are

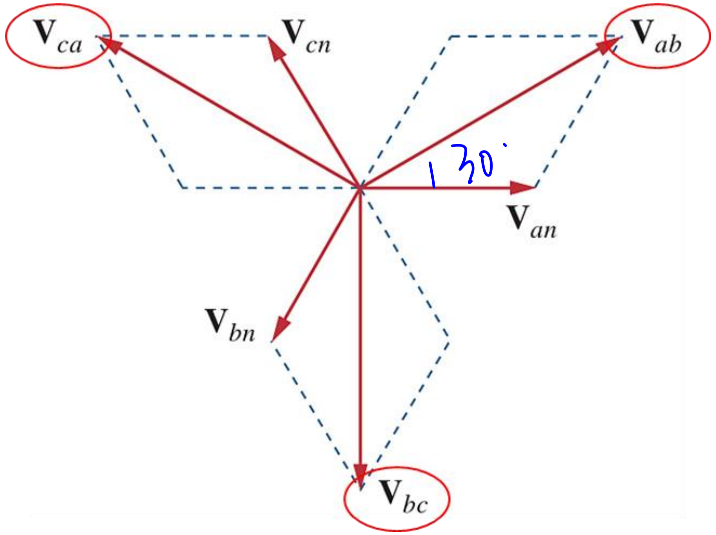


$$\tilde{V}_{ab} = \tilde{V}_{an} - \tilde{V}_{bn} = \boxed{\sqrt{3}V_p\angle 30°}$$

$$\tilde{V}_{bc} = \tilde{V}_{bn} - \tilde{V}_{cn} = \boxed{\sqrt{3}V_p\angle -90° = \tilde{V}_{ab}\angle -120°}$$

$$\tilde{V}_{ca} = \tilde{V}_{cn} - \tilde{V}_{an} = \boxed{\sqrt{3}V_p\angle -210° = \tilde{V}_{ab}\angle -240°}$$

$$\tilde{V}_{ab} + \tilde{V}_{bc} + \tilde{V}_{ca} = 0$$



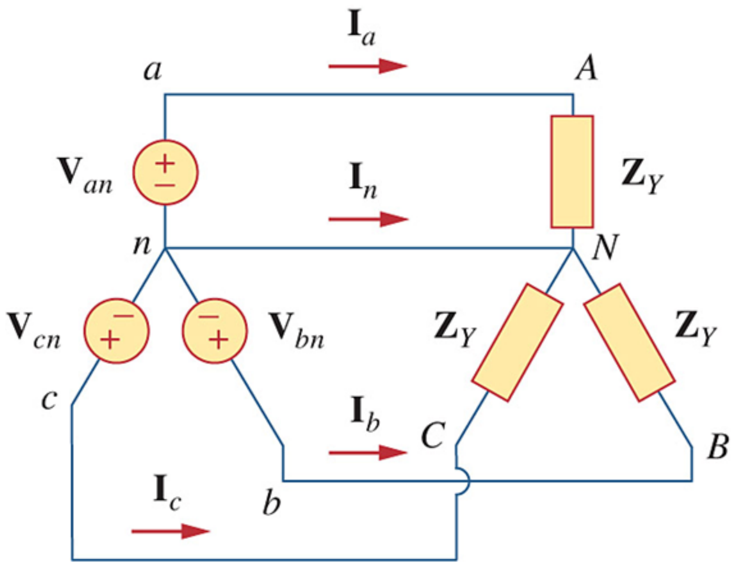
Line voltages are also balanced.

The magnitude of the line voltages is $\sqrt{3}$ times the magnitude of the phase voltages.

$$V_L = \sqrt{3} V_p$$

$$V_p = \left|\tilde{V}_{an}\right| = \left|\tilde{V}_{bn}\right| = \left|\tilde{V}_{cn}\right|$$

$$V_L = \left|\tilde{V}_{ab}\right| = \left|\tilde{V}_{bc}\right| = \left|\tilde{V}_{ca}\right|$$



The line voltages lead their corresponding phase voltages by 30°

$$\tilde{V}_{an} = V_p \angle 0° \qquad \tilde{V}_{ab} = \tilde{V}_{an} - \tilde{V}_{bn} = \sqrt{3} V_p \angle 30°$$

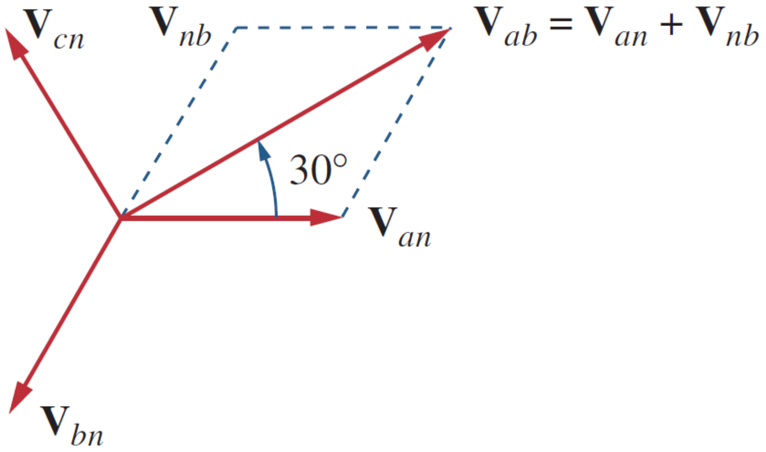
$$\tilde{V}_{bn} = V_p \angle -120° \qquad \tilde{V}_{bc} = \tilde{V}_{bn} - \tilde{V}_{cn} = \sqrt{3} V_p \angle -90°$$

$$\tilde{V}_{cn} = V_p \angle -240° \qquad \tilde{V}_{ca} = \tilde{V}_{cn} - \tilde{V}_{an} = \sqrt{3} V_p \angle -210°$$

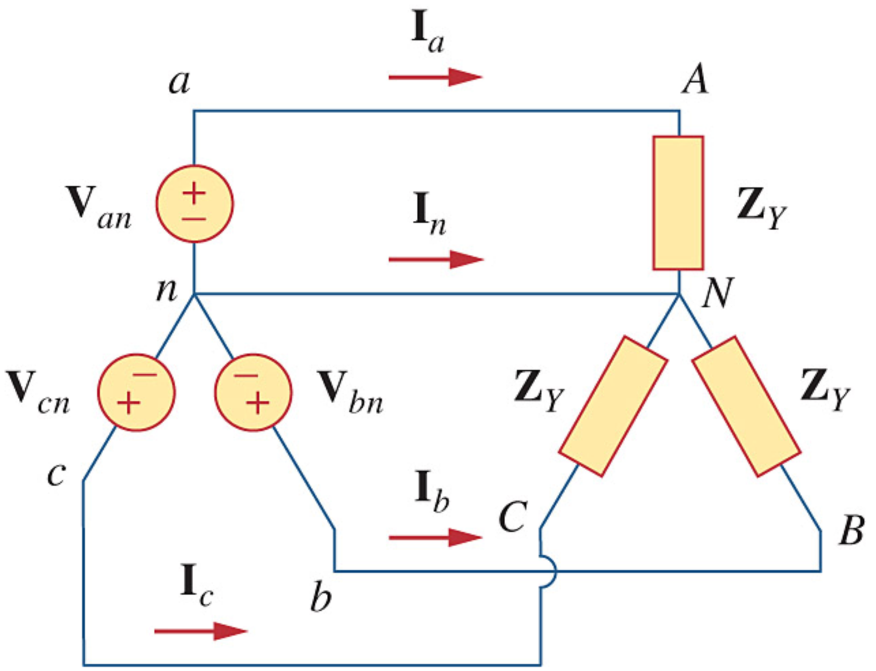
$$\angle \tilde{V}_{ab} = \angle \tilde{V}_{an} + \angle 30°$$

$$\angle \tilde{V}_{bc} = \angle \tilde{V}_{bn} + \angle 30°$$

$$\angle \tilde{V}_{ca} = \angle \tilde{V}_{cn} + \angle 30°$$



Line currents



By KVL at each phase

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y},$$

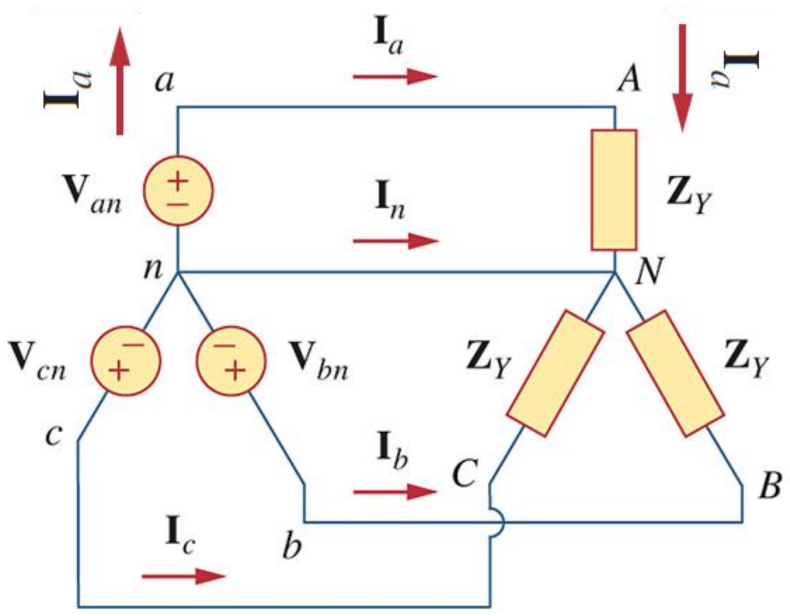
$$\mathbf{I}_b = \frac{\mathbf{V}_{bn}}{\mathbf{Z}_Y} = \frac{\mathbf{V}_{an}\angle{-120^\circ}}{\mathbf{Z}_Y} = \mathbf{I}_a\angle{-120^\circ}$$

$$\mathbf{I}_c = \frac{\mathbf{V}_{cn}}{\mathbf{Z}_Y} = \frac{\mathbf{V}_{an}\angle{-240^\circ}}{\mathbf{Z}_Y} = \mathbf{I}_a\angle{-240^\circ}$$

Line currents add up to zero: $\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c = 0$

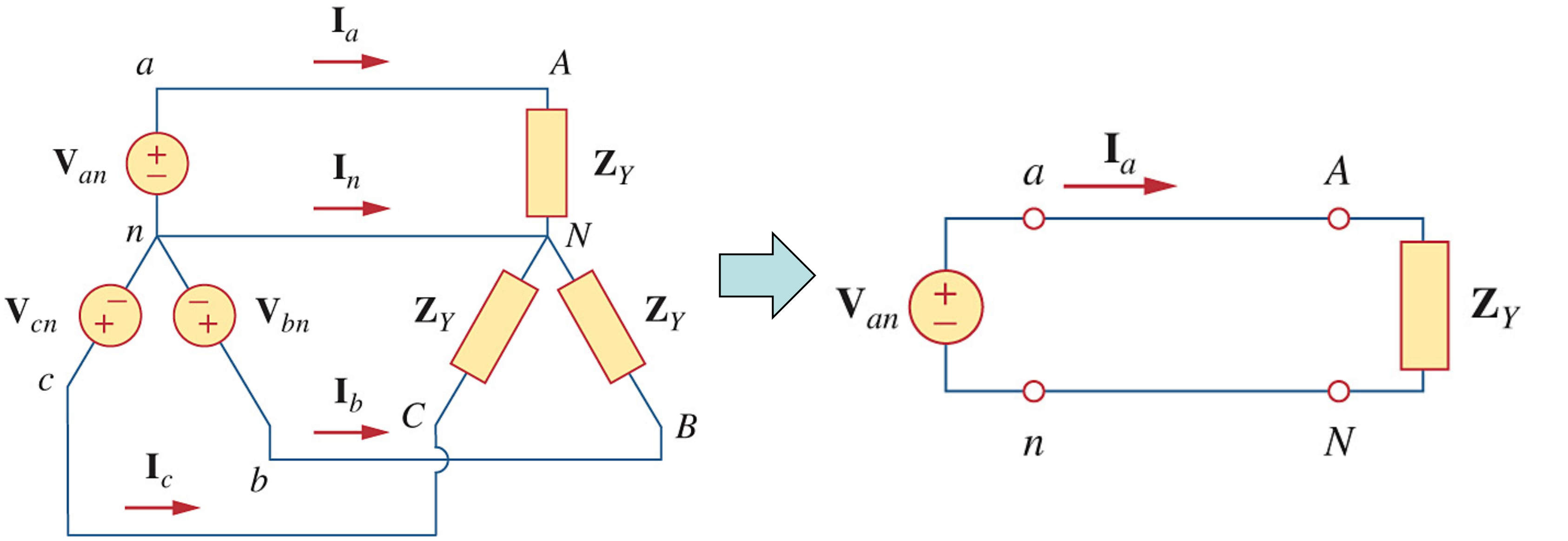
Thus, $\mathbf{I}_n = -(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c) = 0$ and $\mathbf{V}_{nN} = \mathbf{Z}_n\mathbf{I}_n = 0$

While **the line current** is the current in each line, **the phase current** is the current in each phase of the source or load.



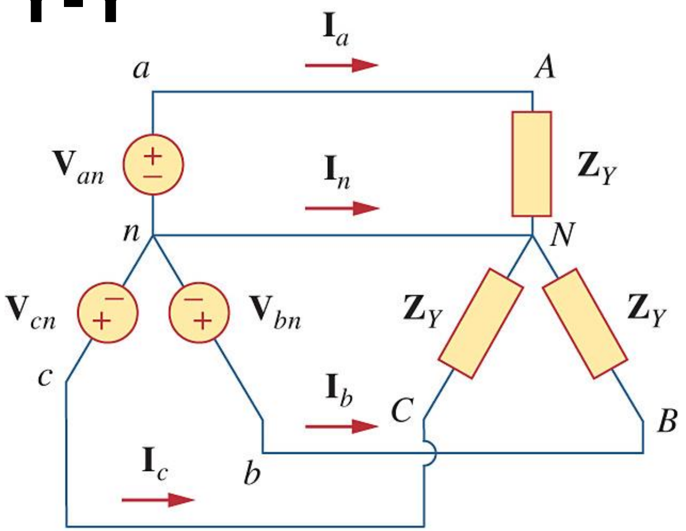
In Y-Y connected system, the line current is the same as the phase current.

An alternative way of analyzing **a balanced Y-Y system** is to do so on a **per phase basis**.



The single-phase analysis yields the line current $\mathbf{I}_a = \dfrac{\mathbf{V}_{an}}{\mathbf{Z}_Y}$

Summary of balanced Y-Y



Summary of phase and line voltages/currents for balanced three-phase systems.[1]

Connection	Phase voltages/currents	Line voltages/currents
Y-Y	$\mathbf{V}_{an} = V_p\angle 0^\circ$	$\mathbf{V}_{ab} = \sqrt{3}V_p\angle 30^\circ$
	$\mathbf{V}_{bn} = V_p\angle -120^\circ$	$\mathbf{V}_{bc} = \mathbf{V}_{ab}\angle -120^\circ$
	$\mathbf{V}_{cn} = V_p\angle +120^\circ$	$\mathbf{V}_{ca} = \mathbf{V}_{ab}\angle +120^\circ$
	Same as line currents	$\mathbf{I}_a = \mathbf{V}_{an}/\mathbf{Z}_Y$
		$\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$
		$\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$

Positive or abc sequence is assumed.

Practice Problem 12.2 A Y-connected balanced three-phase generator with an impedance of $0.4 + j0.3\ \Omega$ per phase is connected to a Y-connected balanced load with an impedance of $24 + j19\ \Omega$ per phase. The line joining the generator and the load has an impedance of $0.6 + j0.7\ \Omega$ per phase. Assuming a positive sequence for the source voltages and that $\tilde{V}_{an} = 120\angle 30°$ V, find the line voltages and the line currents.

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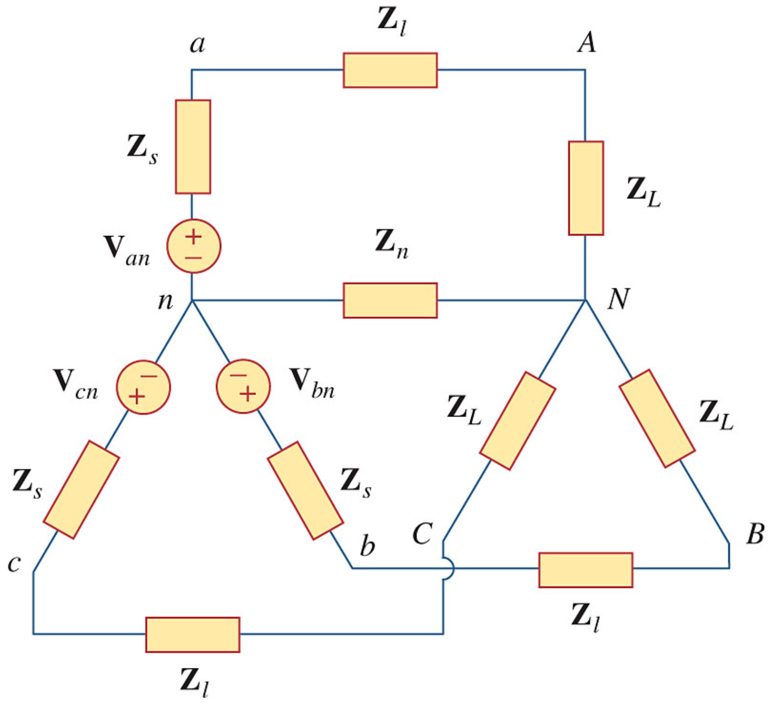
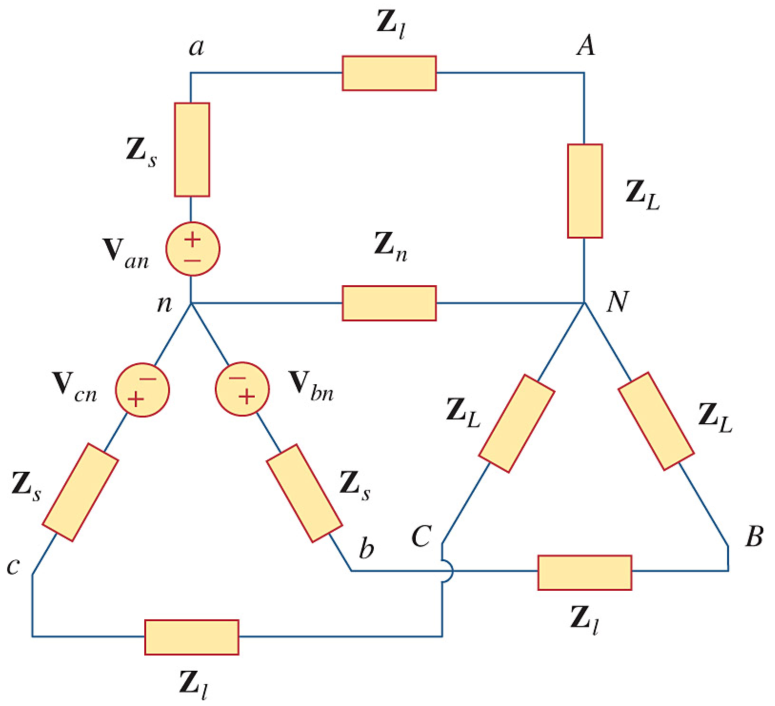


Line voltages by follow the definition

Practice Problem 12.2 A Y-connected balanced three-phase generator with an impedance of $0.4 + j0.3\ \Omega$ per phase is connected to a Y-connected balanced load with an impedance of $24 + j19\ \Omega$ per phase. The line joining the generator and the load has an impedance of $0.6 + j0.7\ \Omega$ per phase. Assuming a positive sequence for the source voltages and that $\tilde{V}_{an} = 120\angle 30°$ V, find the line voltages and the line currents.



The "ideal" line voltages are

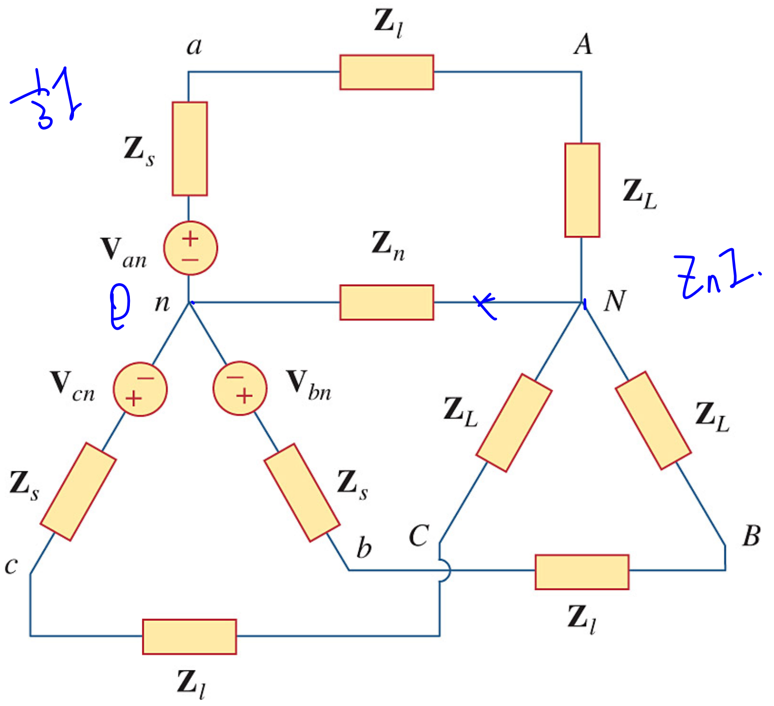
$$\tilde{V}_{ab} = \tilde{V}_{an}\sqrt{3}\angle 30° = 120\angle 30° \times \sqrt{3}\angle 30°$$

$$\approx 207.8461\angle 60°\ (\text{V})$$

$$\tilde{V}_{bc} = \tilde{V}_{ab}\angle -120° = 207.8461\angle -60°\ (\text{V})$$

$$\tilde{V}_{ba} = \tilde{V}_{ab}\angle -240° = 207.8461\angle -180°\ (\text{V})$$

Line currents



$V_{an} - Z_n I = \frac{1}{3} I \ (Z_{总})$

$$Z_Y = Z_s + Z_l + Z_L = (0.4 + j0.3) + (24 + j19)$$
$$+ (0.6 + j0.7) = 25 + j20 \ (\Omega)$$

$$\tilde{I}_a = \frac{\tilde{V}_{an}}{Z_Y} = \frac{120\angle 30^\circ}{25 + j20} \approx \frac{120\angle 30^\circ}{32.0156\angle 38.66^\circ}$$
$$\approx 3.7482\angle -8.66^\circ \ (\mathrm{A})$$

$$\tilde{I}_b = \tilde{I}_a \angle -120^\circ = 3.7482\angle -128.66^\circ \ (\mathrm{A})$$
$$\tilde{I}_c = \tilde{I}_a \angle -240^\circ = 3.7482\angle -248.66^\circ \ (\mathrm{A})$$

Alternatively, by the equations

The "ideal" line voltages are

$$\widetilde{V}_{ab} = \widetilde{V}_{an}\sqrt{3}\angle 30° = 120\angle 30° \times \sqrt{3}\angle 30°$$

$$\approx 207.8461\angle 60° \text{ (V)}$$

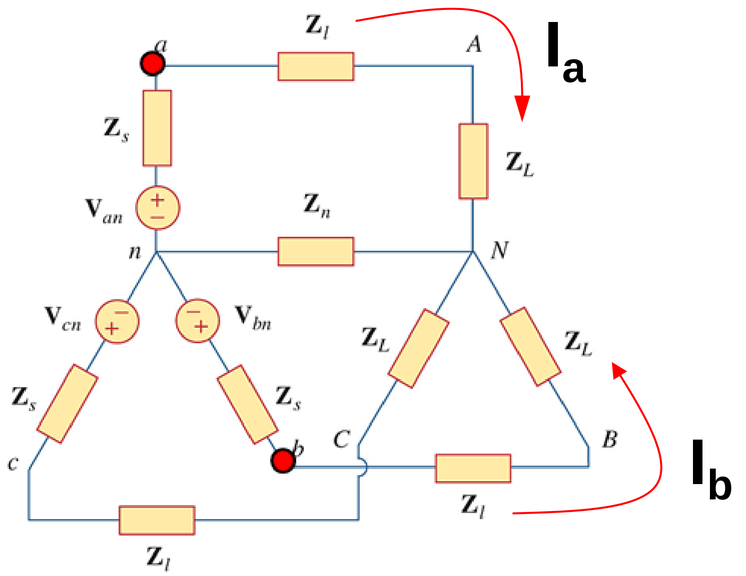
$$\widetilde{V}_{bc} = \widetilde{V}_{ab}\angle -120° = 207.8461\angle -60° \text{ (V)}$$

$$\widetilde{V}_{ba} = \widetilde{V}_{ab}\angle -240° = 207.8461\angle -180° \text{ (V)}$$

Note that for a balanced system, the other two voltages/currents can be quickly obtained by their **phase relation**.

Line current

$$Z_Y = Z_s + Z_l + Z_L = (0.4 + j0.3) + (24 + j19)$$
$$+(0.6 + j0.7) = 25 + j20\ (\Omega)$$
$$\tilde{I}_a = \frac{\tilde{V}_{an}}{Z_Y} = \frac{120\angle 30^\circ}{25 + j20} \approx \frac{120\angle 30^\circ}{32.0156\angle 38.66^\circ}$$
$$\approx 3.7482\angle -8.66^\circ\ (\mathrm{A})$$
$$\tilde{I}_b = \tilde{I}_a\angle -120^\circ = 3.7482\angle -128.66^\circ\ (\mathrm{A})$$
$$\tilde{I}_c = \tilde{I}_a\angle -240^\circ = 3.7482\angle -248.66^\circ\ (\mathrm{A})$$



The line voltages seen from the generator terminals are

$$\tilde{V}_{ab} = \tilde{I}_a(Z_l + Z_L) - \tilde{I}_b(Z_l + Z_L)$$

$$= \tilde{I}_a\sqrt{3}\angle 30°(Z_l + Z_L)$$

$$= \frac{\tilde{V}_{an}}{Z_Y}\sqrt{3}\angle 30°(Z_l + Z_L) = \tilde{V}_{ab}\left(\frac{Z_l + Z_L}{Z_Y}\right) \quad \text{Line voltage } V_{ab}$$

$$= 207.8461\angle 60^\circ \times \left(\frac{24.6 + j19.7}{25 + j20} \right)$$

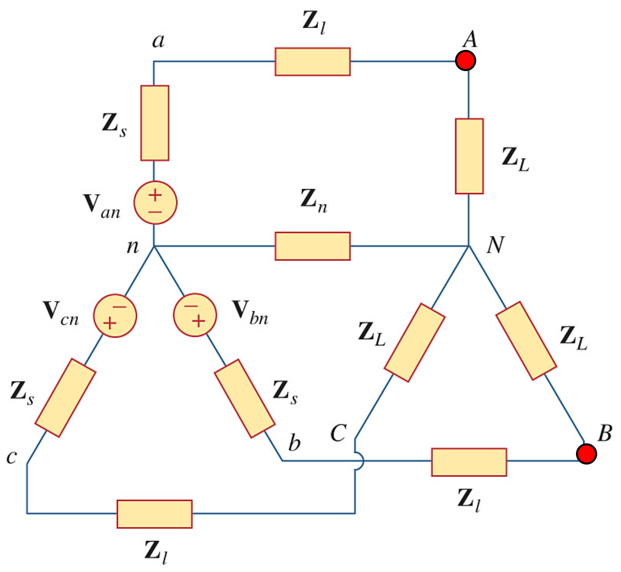
$$\approx 207.8461\angle 60^\circ \times \frac{31.5159\angle 38.69^\circ}{32.0156\angle 38.66^\circ}$$

$$\approx 204.6020\angle 60.03^\circ \text{ (V)}$$

$$\widetilde{V}_{bc} = 204.6020\angle -59.97^\circ \text{ (V)}$$

$$\widetilde{V}_{ca} = 204.6020\angle -179.97^\circ \text{ (V)}$$

Slightly lower magnitude and angles compared to the ideal line voltages.



The line voltages seen from the load terminals are

$$\tilde{V}_{AB} = \tilde{I}_a Z_L - \tilde{I}_b Z_L$$

$$I_a = 3.7482\angle -8.66$$
$$I_b = 3.7482\angle -8.66$$
$$Z_L = 14 + j19 = 30.61\angle 38.37$$

$$V_{AB} = 198.72\angle 59.71$$
$$V_{BC} = 198.72\angle -60.29$$
$$V_{CA} = 198.72\angle -180.29$$

The line voltages seen from the load terminals are

$\boxed{V_{AN}-V_{BN}}$ $\boxed{I_a-I_b=I_a\sqrt{3}\angle 30°=V_{an}/Z_Y\sqrt{3}\angle 30°=V_{ab}/Z_Y}$

$$\tilde{V}_{AB}=\tilde{I}_a Z_L-\tilde{I}_b Z_L=\tilde{V}_{ab}\frac{Z_L}{Z_Y}$$

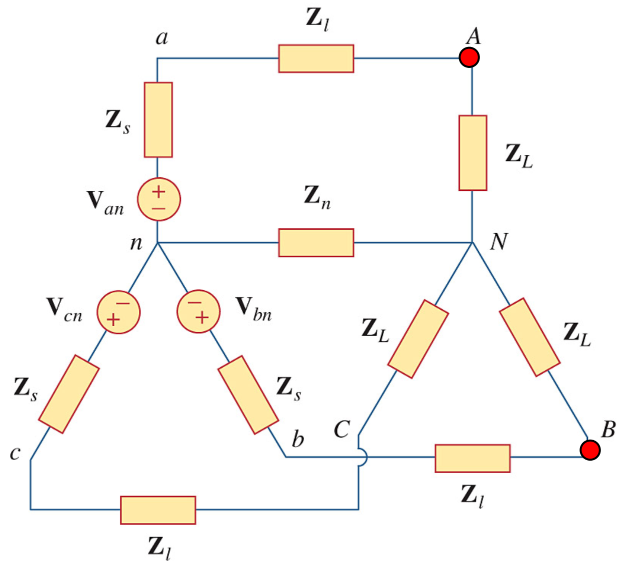
$$=207.8461\angle 60°\times\left(\frac{24+19}{25+j20}\right)$$

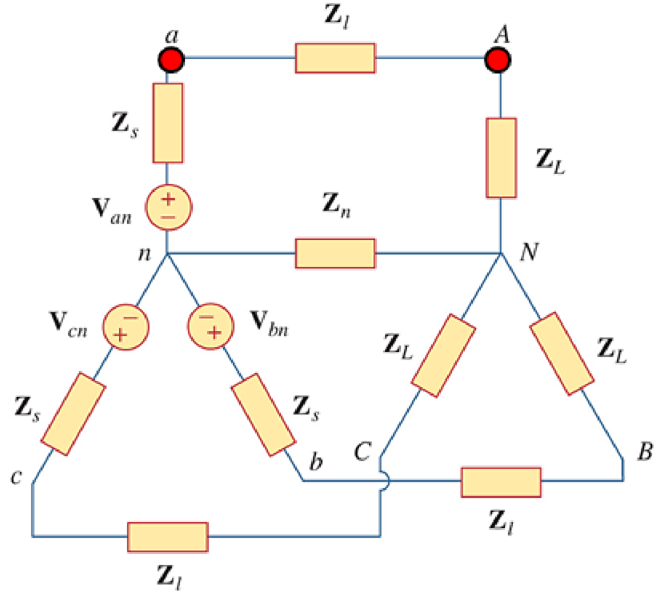
$$\approx 207.8461\angle 60°\times\frac{30.6105\angle 38.37°}{32.0156\angle 38.66°}$$

$$\approx 198.7242\angle 59.71°\ (\mathrm{V})$$

$$\tilde{V}_{BC}\approx 198.7242\angle -60.29°\ (\mathrm{V})$$

$$\tilde{V}_{CA}\approx 198.7242\angle -180.29°\ (\mathrm{V})$$





The line voltage drops are

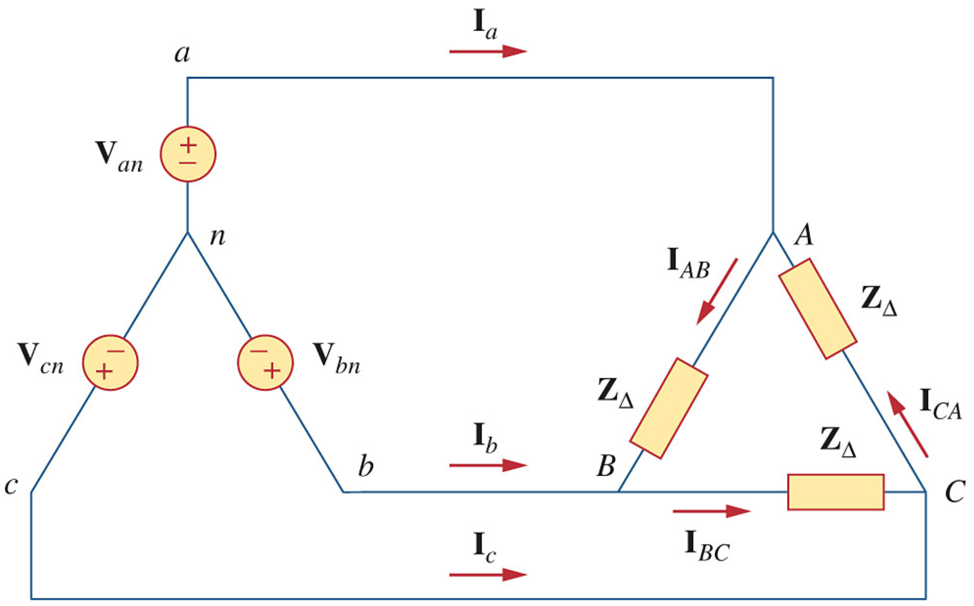
$$\tilde{I}_a Z_l = 3.7482\angle -8.66° \times (0.6 + j0.7)$$

$$\approx 3.4558\angle 40.74° \text{ (V)}$$

$$\tilde{I}_b Z_l = 3.4558\angle -79.26° \text{ (V)}$$

$$\tilde{I}_c Z_l = 3.4558\angle -199.26° \text{ (V)}$$

12.4 Balanced Wye-Delta Connection



A balanced Y-Δ system consists of a balanced Y-connected source feeding a balanced Δ-connected load.

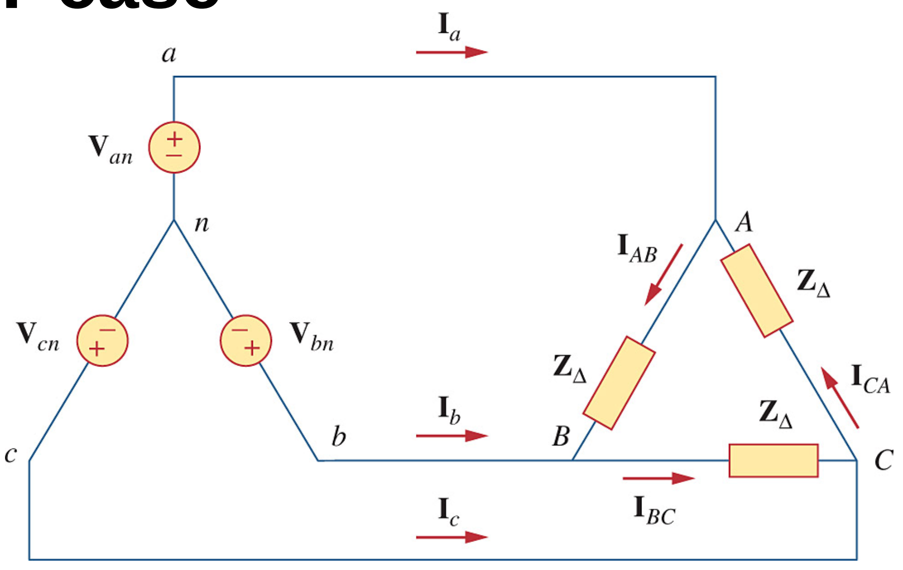
Phase voltages: same as Y-Y case

Assume that

$$\tilde{V}_{an} = V_p \angle 0°$$

$$\tilde{V}_{bn} = V_p \angle -120°$$

$$\tilde{V}_{cn} = V_p \angle -240°$$



The line voltages are

$$\tilde{V}_{ab} = \sqrt{3} V_p \angle 30° = \tilde{V}_{an} \sqrt{3} \angle 30° = \tilde{V}_{AB}$$

$$\tilde{V}_{bc} = \tilde{V}_{ab} \angle -120° = \tilde{V}_{BC}$$

$$\tilde{V}_{ca} = \tilde{V}_{ab} \angle -240° = \tilde{V}_{CA}$$

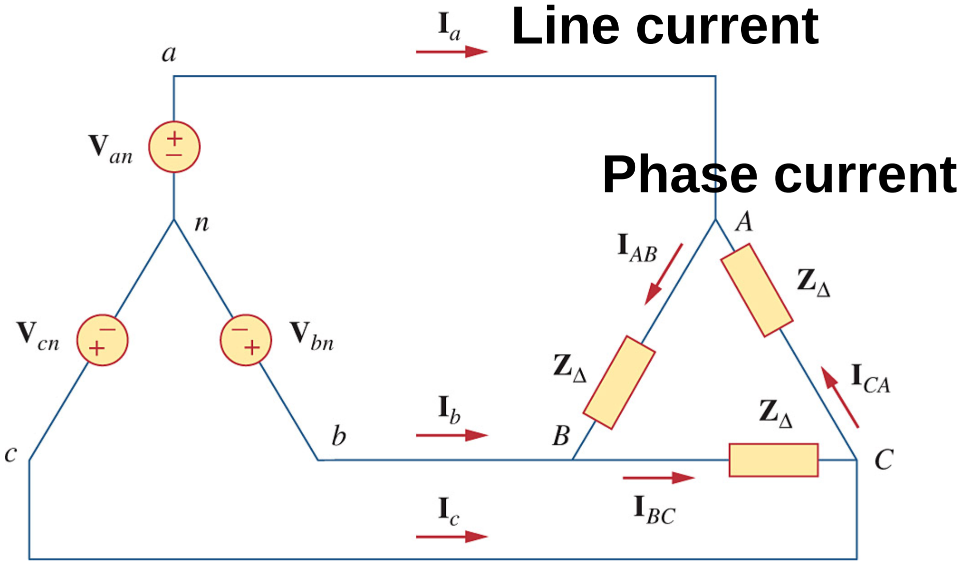
Phase currents

The phase currents are

$$\tilde{I}_{AB} = \frac{\tilde{V}_{AB}}{Z_{\Delta}}$$

$$\tilde{I}_{BC} = \frac{\tilde{V}_{BC}}{Z_{\Delta}} = \tilde{I}_{AB}\angle -120°$$

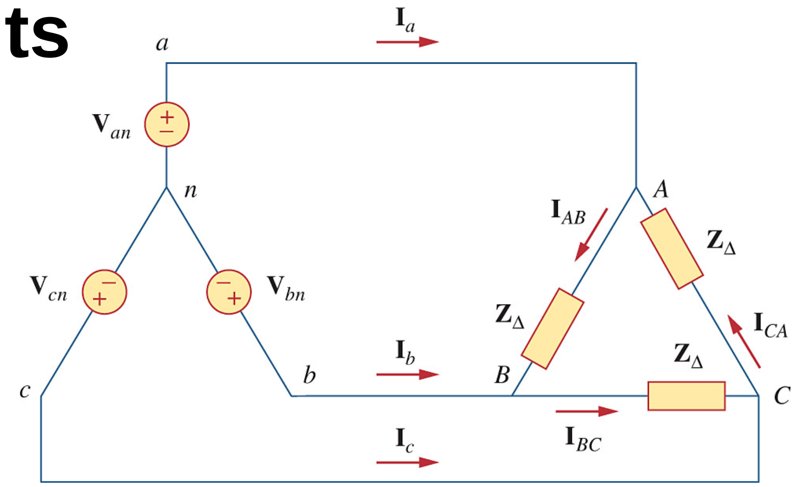
$$\tilde{I}_{CA} = \frac{\tilde{V}_{CA}}{Z_{\Delta}} = \tilde{I}_{AB}\angle -240°$$

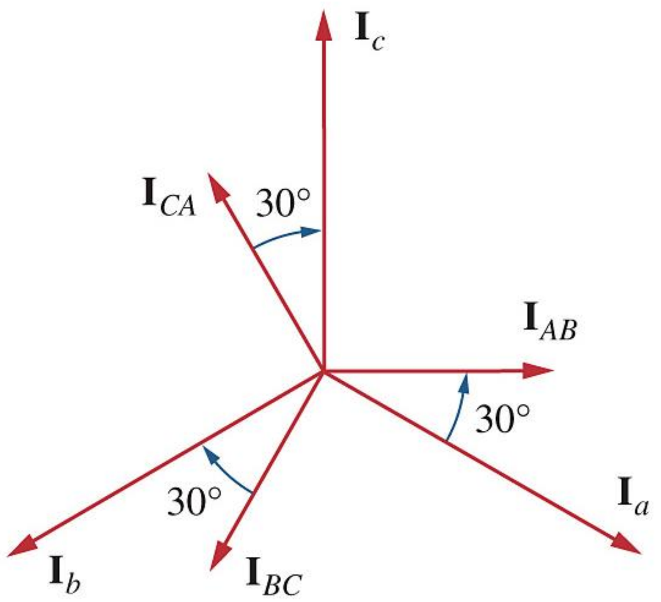


$$\mathbf{Z}_Y = \frac{\mathbf{Z}_\Delta}{3}$$

While **the line current** is the current in each line, **the phase current** is the current in each phase of the source or load.

Line currents





The line currents are

$$\tilde{I}_a = \tilde{I}_{AB} - \tilde{I}_{CA} = \tilde{I}_{AB}\sqrt{3}\angle -30^\circ$$

$$\tilde{I}_b = \tilde{I}_{BC} - \tilde{I}_{AB} = \tilde{I}_{BC}\sqrt{3}\angle -30^\circ = \tilde{I}_a\angle -120^\circ$$

$$\tilde{I}_c = \tilde{I}_{CA} - \tilde{I}_{BC} = \tilde{I}_{CA}\sqrt{3}\angle -30^\circ = \tilde{I}_a\angle -240^\circ$$

$$\tilde{I}_{AB} = \frac{\tilde{V}_{AB}}{Z_\Delta}$$

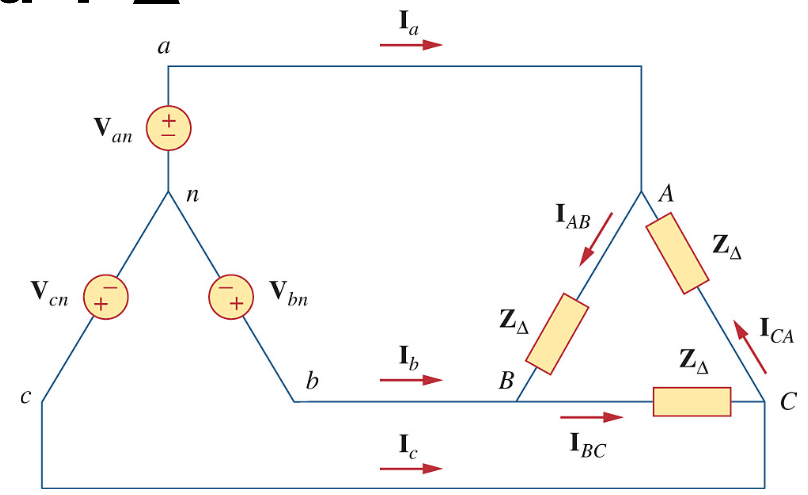
$$\tilde{I}_{BC} = \frac{\tilde{V}_{BC}}{Z_\Delta} = \tilde{I}_{AB}\angle -120^\circ$$

$$\tilde{I}_{CA} = \frac{\tilde{V}_{CA}}{Z_\Delta} = \tilde{I}_{AB}\angle -240^\circ$$

showing that

$$I_L = \sqrt{3}I_p \qquad I_L = \left|\tilde{I}_a\right| = \left|\tilde{I}_b\right| = \left|\tilde{I}_c\right|,\ I_p = \left|\tilde{I}_{AB}\right| = \left|\tilde{I}_{BC}\right| = \left|\tilde{I}_{CA}\right|$$

Summary of balanced Y-Δ

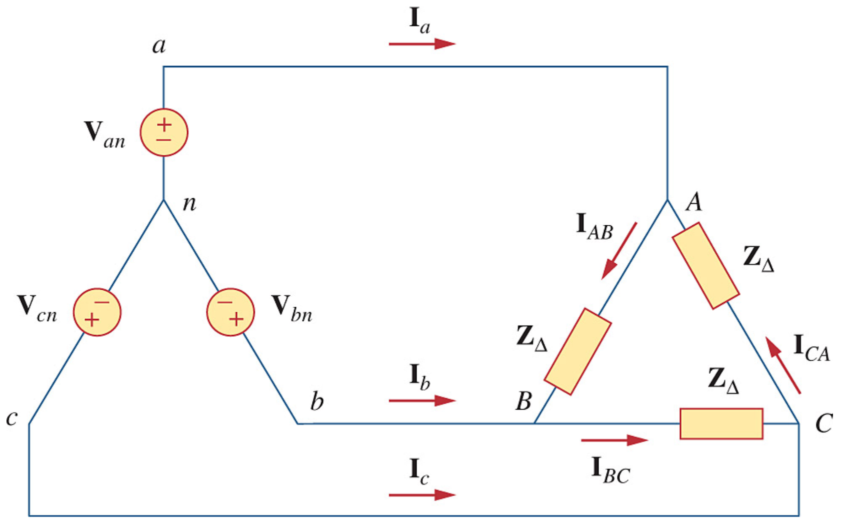


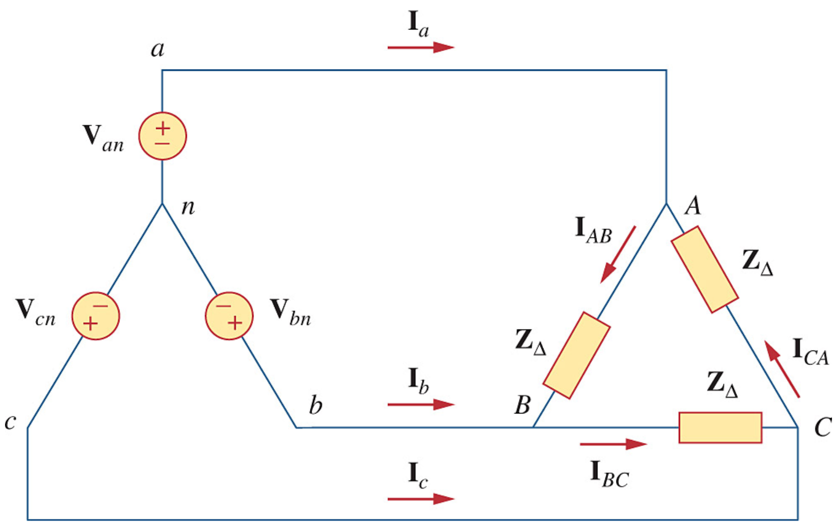
Summary of phase and line voltages/currents for balanced three-phase systems.[1]

Connection	Phase voltages/currents	Line voltages/currents
Y-Δ	$\mathbf{V}_{an} = V_p\angle 0°$	$\mathbf{V}_{ab} = \mathbf{V}_{AB} = \sqrt{3}V_p\angle 30°$
	$\mathbf{V}_{bn} = V_p\angle -120°$	$\mathbf{V}_{bc} = \mathbf{V}_{BC} = \mathbf{V}_{ab}\angle -120°$
	$\mathbf{V}_{cn} = V_p\angle +120°$	$\mathbf{V}_{ca} = \mathbf{V}_{CA} = \mathbf{V}_{ab}\angle +120°$
	$\mathbf{I}_{AB} = \mathbf{V}_{AB}/\mathbf{Z}_\Delta$	$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle -30°$
	$\mathbf{I}_{BC} = \mathbf{V}_{BC}/\mathbf{Z}_\Delta$	$\mathbf{I}_b = \mathbf{I}_a\angle -120°$
	$\mathbf{I}_{CA} = \mathbf{V}_{CA}/\mathbf{Z}_\Delta$	$\mathbf{I}_c = \mathbf{I}_a\angle +120°$

Positive or abc sequence is assumed.

Practice Problem 12.3 One line voltage of a balanced Y-connected source is $\tilde{V}_{AB} = 240\angle -20^\circ$ V. If the source is connected to a Δ-connected load of $20\angle 40^\circ\ \Omega$, find the phase and line currents. Assume the *abc* sequence.





$$\tilde{I}_{AB} = \frac{\tilde{V}_{AB}}{Z_\Delta} = \frac{240\angle -20^\circ}{20\angle 40^\circ} = 12\angle -60^\circ \text{ (A)}$$

$$\tilde{I}_{BC} = \tilde{I}_{AB}\angle -120^\circ = 12\angle -180^\circ \text{ (A)}$$

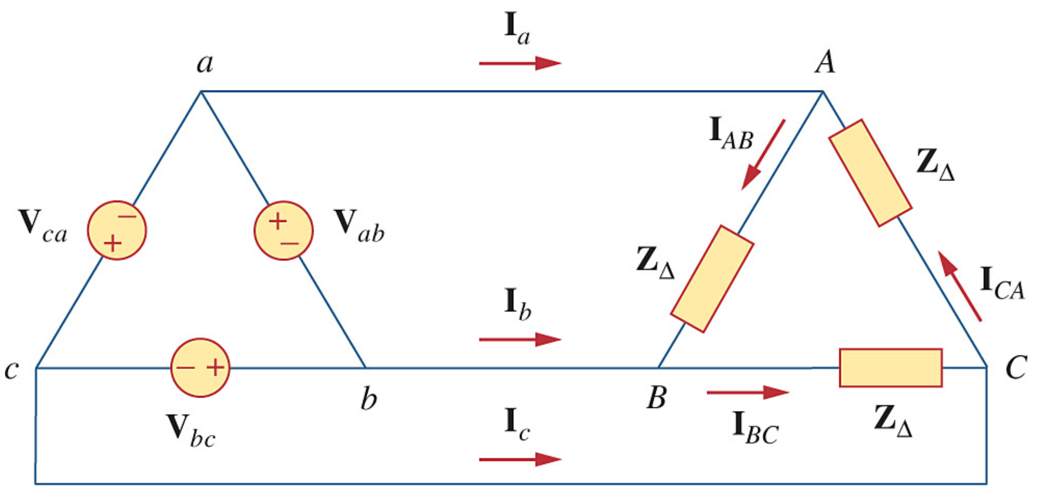
$$\tilde{I}_{CA} = \tilde{I}_{AB}\angle +120^\circ = 12\angle 60^\circ \text{ (A)}$$

$$\tilde{I}_a = \tilde{I}_{AB}\sqrt{3}\angle -30^\circ \approx 20.7846\angle -90^\circ \text{ (A)}$$

$$\tilde{I}_b = \tilde{I}_a\angle +240^\circ = 20.7846\angle 150^\circ \text{ (A)}$$

$$\tilde{I}_c = \tilde{I}_a\angle +120^\circ = 20.7846\angle 30^\circ \text{ (A)}$$

12.5 Balanced Delta-Delta Connection



A balanced Δ-Δ system is one in which both the balanced source and balanced load are Δ-connected.

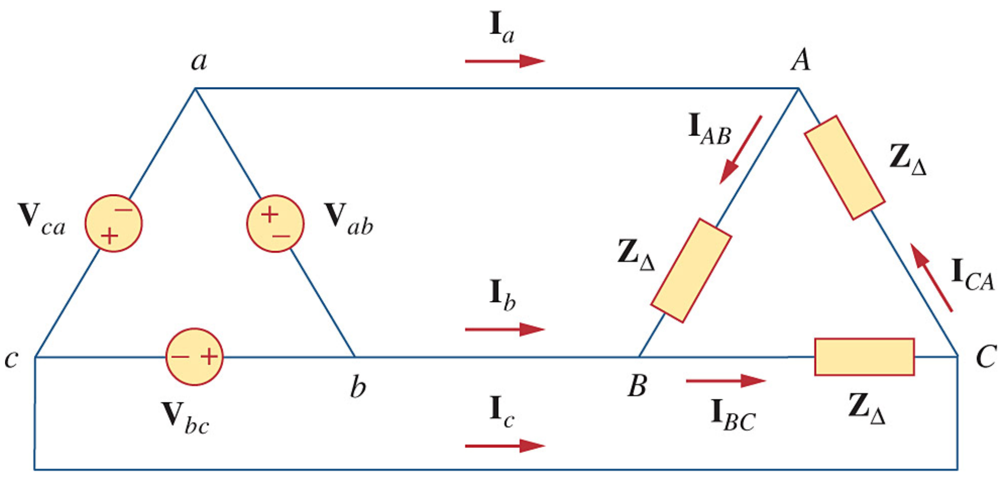
Phase and line voltages

Assume that

$$\tilde{V}_{ab} = V_p \angle 0°$$

$$\tilde{V}_{bc} = V_p \angle -120°$$

$$\tilde{V}_{ca} = V_p \angle +120°$$

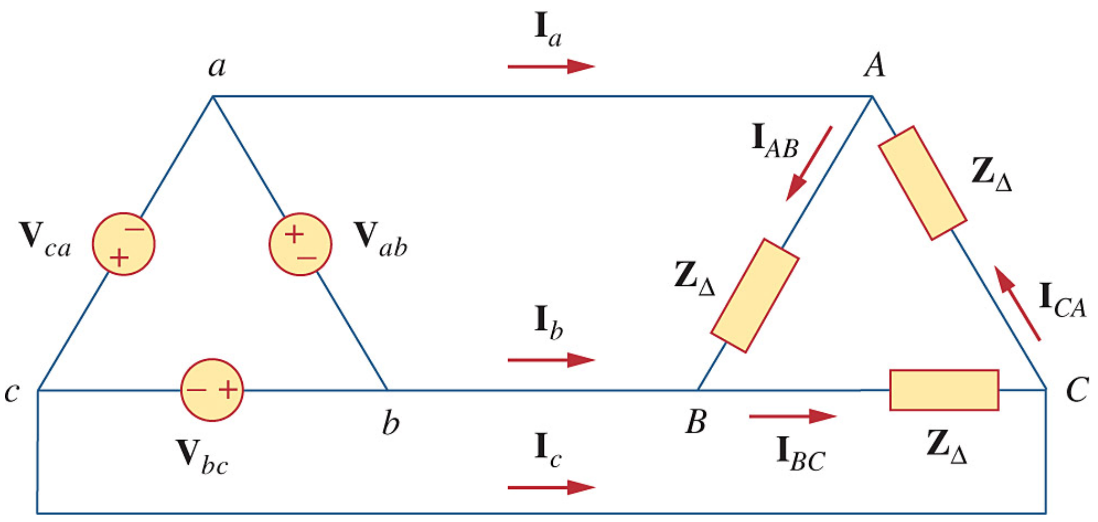


Phase voltage = Line voltage

Phase voltage = Voltage across the impedances

$$\tilde{V}_{ab} = \tilde{V}_{AB}, \quad \tilde{V}_{bc} = \tilde{V}_{BC}, \quad \tilde{V}_{ca} = \tilde{V}_{CA}$$

Phase current



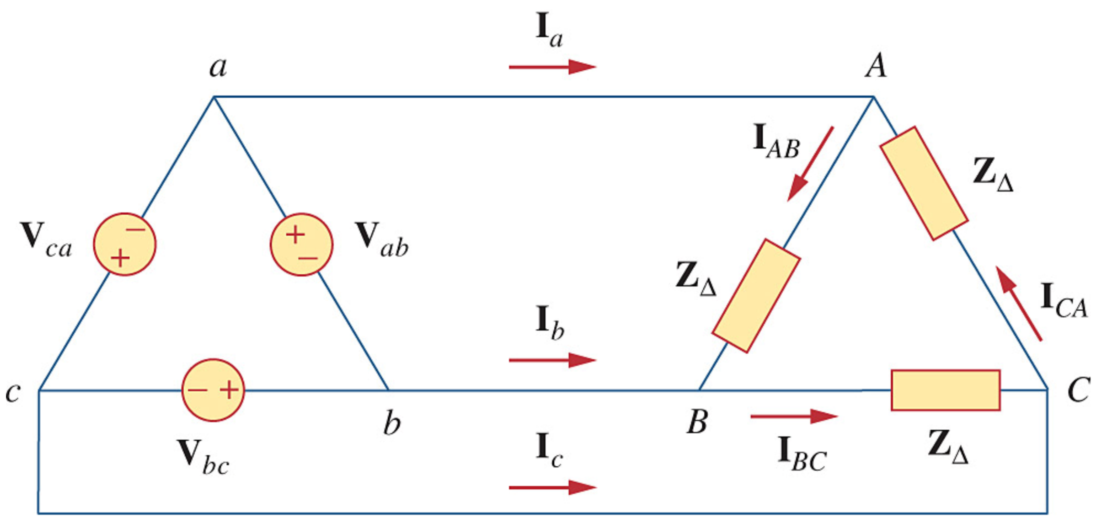
The phase currents are

$$\tilde{I}_{AB} = \frac{\tilde{V}_{AB}}{Z_\Delta}$$

$$\tilde{I}_{BC} = \frac{\tilde{V}_{BC}}{Z_\Delta} = \tilde{I}_{AB}\angle -120°$$

$$\tilde{I}_{CA} = \frac{\tilde{V}_{CA}}{Z_\Delta} = \tilde{I}_{AB}\angle +120°$$

Line current



The line currents are

$$\tilde{I}_a = \tilde{I}_{AB} - \tilde{I}_{CA} = \tilde{I}_{AB}\sqrt{3}\angle -30^\circ$$

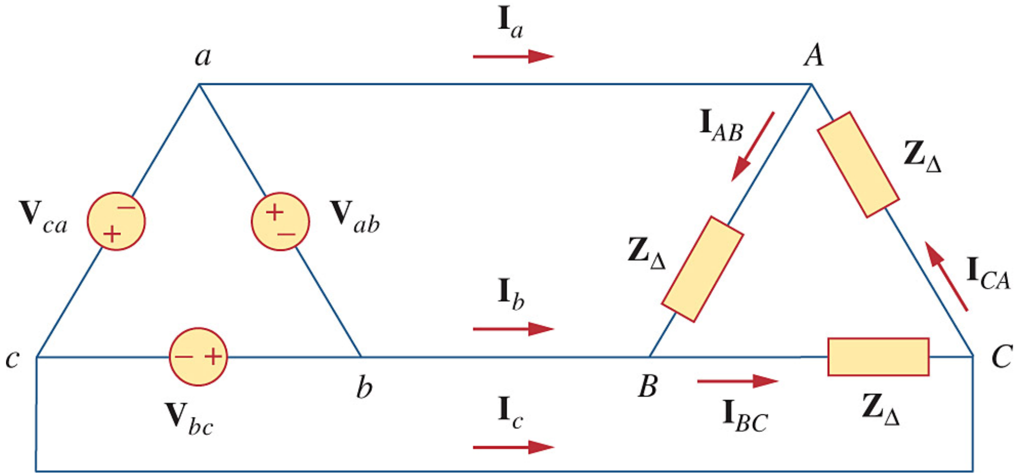
$$\tilde{I}_b = \tilde{I}_{BC} - \tilde{I}_{AB} = \tilde{I}_{BC}\sqrt{3}\angle -30^\circ = \tilde{I}_a \angle -120^\circ$$

$$\tilde{I}_c = \tilde{I}_{CA} - \tilde{I}_{BC} = \tilde{I}_{CA}\sqrt{3}\angle -30^\circ = \tilde{I}_a \angle -240^\circ$$

showing that

$$I_L = \sqrt{3} I_p \qquad I_L = |\tilde{I}_a| = |\tilde{I}_b| = |\tilde{I}_c|,\ I_p = |\tilde{I}_{AB}| = |\tilde{I}_{BC}| = |\tilde{I}_{CA}|$$

Summary of balanced Δ-Δ

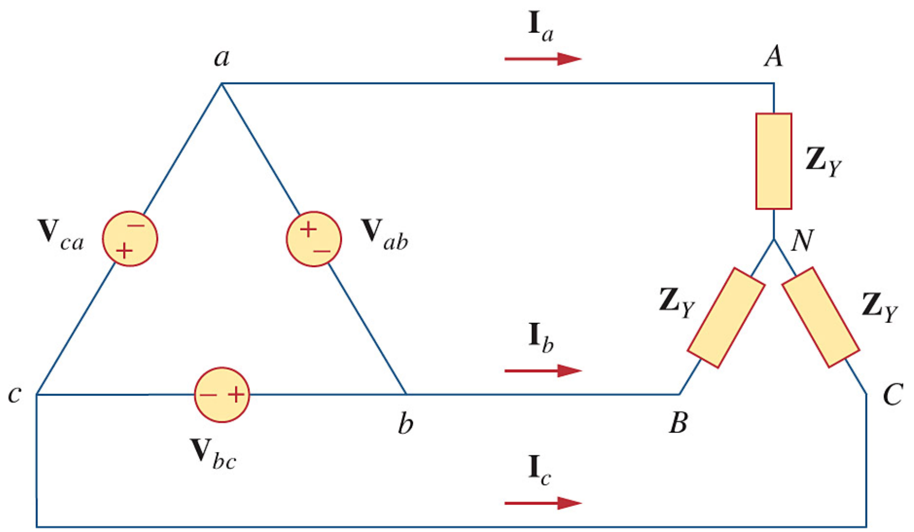


Summary of phase and line voltages/currents for balanced three-phase systems.[1]

Connection	Phase voltages/currents	Line voltages/currents
Δ-Δ	$\mathbf{V}_{ab} = V_p\angle 0°$	Same as phase voltages
	$\mathbf{V}_{bc} = V_p\angle -120°$	
	$\mathbf{V}_{ca} = V_p\angle +120°$	
	$\mathbf{I}_{AB} = \mathbf{V}_{ab}/\mathbf{Z}_\Delta$	$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle -30°$
	$\mathbf{I}_{BC} = \mathbf{V}_{bc}/\mathbf{Z}_\Delta$	$\mathbf{I}_b = \mathbf{I}_a\angle -120°$
	$\mathbf{I}_{CA} = \mathbf{V}_{ca}/\mathbf{Z}_\Delta$	$\mathbf{I}_c = \mathbf{I}_a\angle +120°$

Positive or abc sequence is assumed.

12.6 Balanced Delta-Wye Connection



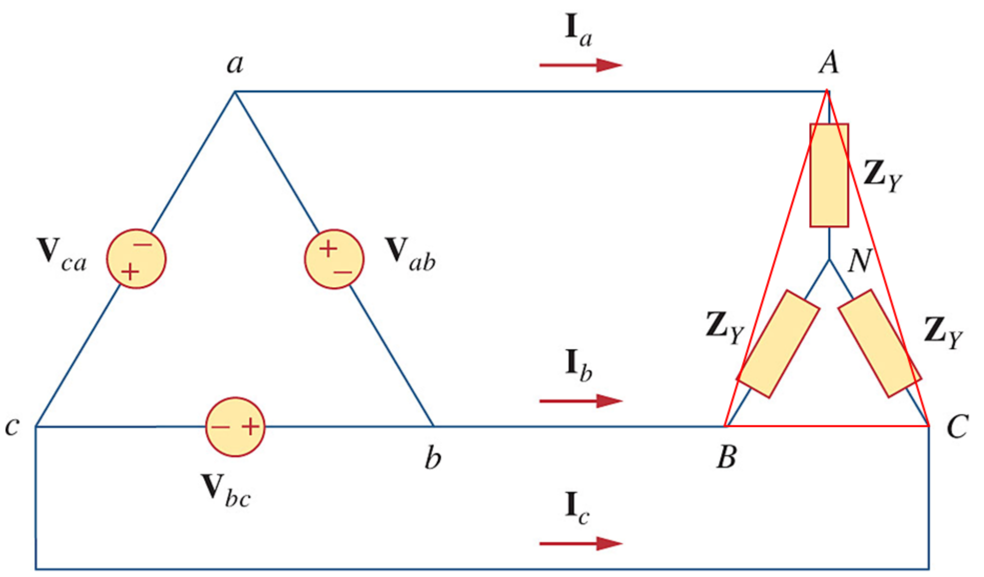
A balanced Δ-Y system consists of a balanced Δ-connected source feeding a balanced Y-connected load.

***In Δ-connected source:**
Phase voltages = Line voltages

Line current

Assume that

$$\tilde{V}_{ab} = V_p \angle 0°$$
$$\tilde{V}_{bc} = V_p \angle -120°$$
$$\tilde{V}_{ca} = V_p \angle +120°$$



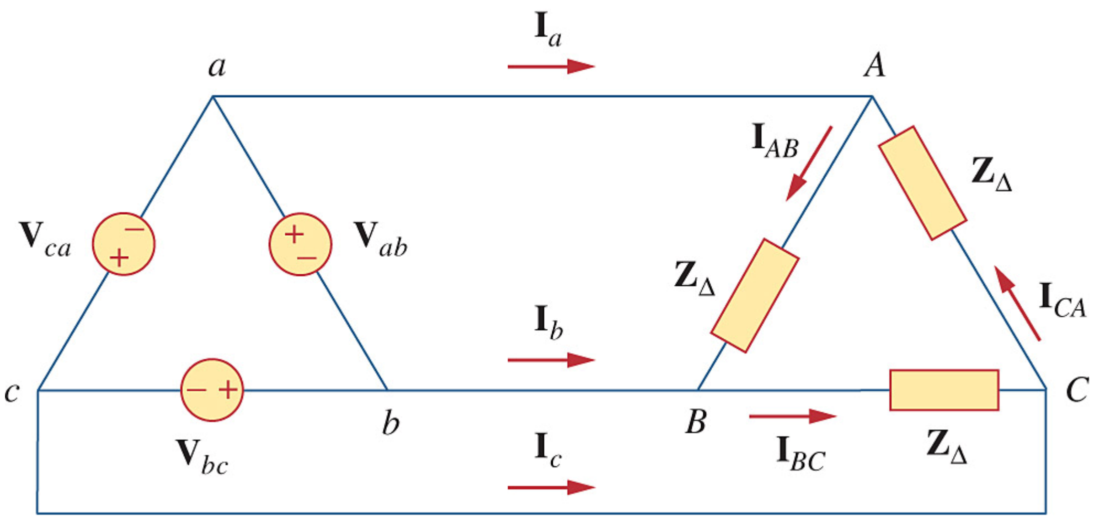
Replace the Y-connected load with its equivalent Δ-connected load. **Line current (e.g. I_a)** in a Δ-connected load is:

$$\tilde{I}_a = \tilde{I}_{AB}\sqrt{3}\angle -30° = \frac{\tilde{V}_{AB}}{Z_\Delta}\sqrt{3}\angle -30°$$

$$= \frac{\tilde{V}_{ab}}{3Z_Y}\sqrt{3}\angle -30° = \frac{\tilde{V}_{ab}}{\sqrt{3}Z_Y}\angle -30°$$

$Z_\Delta = 3Z_Y$
$Z_Y = Z_\Delta/3$

Recall: Δ-Δ Phase current



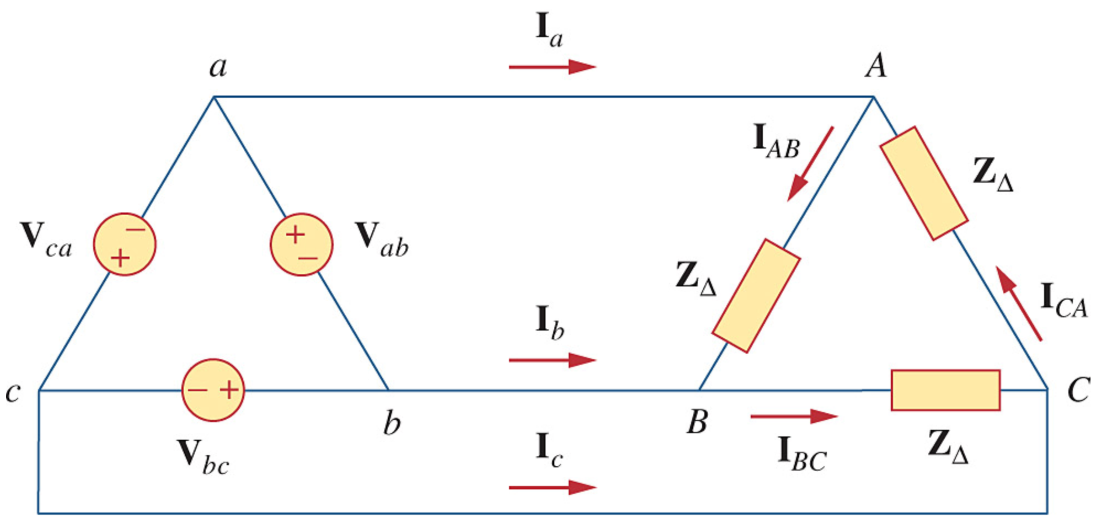
The phase currents are

$$\tilde{I}_{AB} = \frac{\tilde{V}_{AB}}{Z_\Delta}$$

$$\tilde{I}_{BC} = \frac{\tilde{V}_{BC}}{Z_\Delta} = \tilde{I}_{AB}\angle -120°$$

$$\tilde{I}_{CA} = \frac{\tilde{V}_{CA}}{Z_\Delta} = \tilde{I}_{AB}\angle +120°$$

Recall: Δ-Δ Line current



The line currents are

$$\tilde{I}_a = \tilde{I}_{AB} - \tilde{I}_{CA} = \tilde{I}_{AB}\sqrt{3}\angle -30°$$

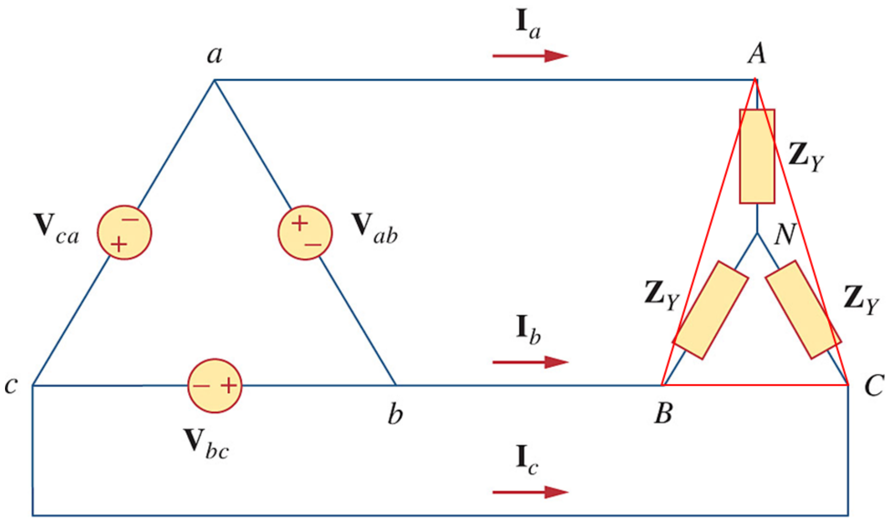
$$\tilde{I}_b = \tilde{I}_{BC} - \tilde{I}_{AB} = \tilde{I}_{BC}\sqrt{3}\angle -30° = \tilde{I}_a\angle -120°$$

$$\tilde{I}_c = \tilde{I}_{CA} - \tilde{I}_{BC} = \tilde{I}_{CA}\sqrt{3}\angle -30° = \tilde{I}_a\angle -240°$$

showing that

$$I_L = \sqrt{3}I_p \qquad I_L = \left|\tilde{I}_a\right| = \left|\tilde{I}_b\right| = \left|\tilde{I}_c\right|, I_p = \left|\tilde{I}_{AB}\right| = \left|\tilde{I}_{BC}\right| = \left|\tilde{I}_{CA}\right|$$

Line current



$$\tilde{I}_a = \frac{\tilde{V}_{ab}}{\sqrt{3}Z_Y}\angle -30^\circ$$

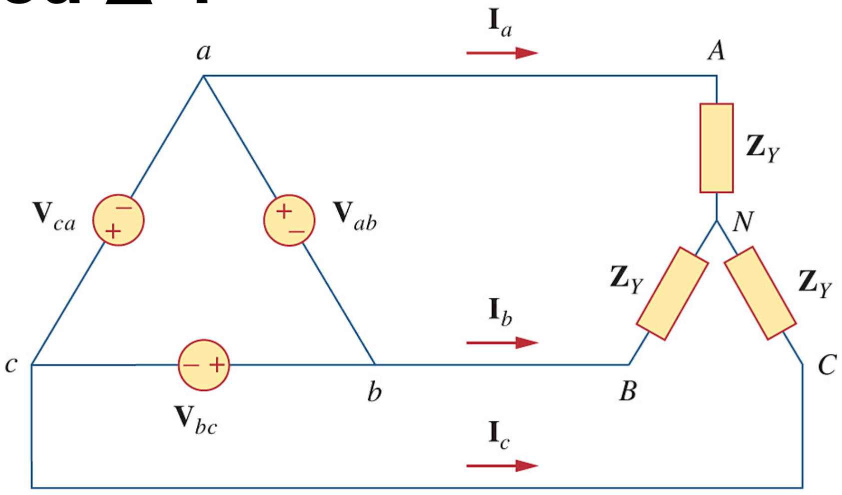
$$\tilde{I}_b = \frac{\tilde{V}_{bc}}{\sqrt{3}Z_Y}\angle -30^\circ = \tilde{I}_a\angle -120^\circ$$

$$\tilde{I}_c = \frac{\tilde{V}_{ca}}{\sqrt{3}Z_Y}\angle -30^\circ = \tilde{I}_a\angle +120^\circ$$

$$\boxed{\begin{aligned}\tilde{V}_{ab} &= V_p\angle 0^\circ\\ \tilde{V}_{bc} &= V_p\angle -120^\circ\\ \tilde{V}_{ca} &= V_p\angle +120^\circ\end{aligned}}$$

In Y-connected load,
Phase currents = Line currents

Summary of balanced Δ-Y



Summary of phase and line voltages/currents for balanced three-phase systems.[1]

Connection	Phase voltages/currents	Line voltages/currents
Δ-Y	$\mathbf{V}_{ab} = V_p\angle 0^\circ$ $\mathbf{V}_{bc} = V_p\angle -120^\circ$ $\mathbf{V}_{ca} = V_p\angle +120^\circ$	Same as phase voltages
	Same as line currents	$\mathbf{I}_a = \dfrac{V_p\angle -30^\circ}{\sqrt{3}\mathbf{Z}_Y}$ $\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$ $\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$

Positive or abc sequence is assumed.

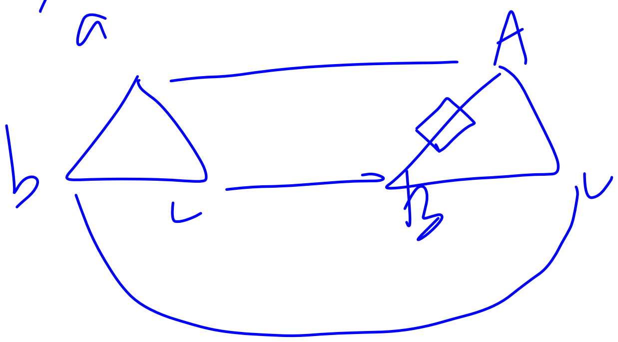
Practice Problem 12.5

In a balanced Δ-Y circuit, $\mathbf{V}_{ab} = 240\angle 15^\circ$ and $\mathbf{Z}_Y = (12 + j15)\ \Omega$. Calculate the line currents.

Answer: $7.21\angle -66.34^\circ$ A, $7.21\angle +173.66^\circ$ A, $7.21\angle 53.66^\circ$ A.

$Z_\Delta = 3Z_Y$
$Z_Y = Z_\Delta/3$

$\Delta - Y \Rightarrow \Delta - \Delta$



$Z_\Delta = 3Z_Y = 9(4+j5)$

$I_{AB} = \dfrac{V_{AB}}{Z_\Delta} = \dfrac{V_{ab}}{3Z_Y}$ ✓

I_{BC} ✓ I_{CA} ✓

$I_a = I_{AB} - I_{CA}$

12.7 Power in a Balanced System

Three phase system - Advantage 1:

We begin by examining the **instantaneous power** absorbed by the load (time domain). For a **Y-connected load**, the phase voltages in the abc sequence are

$$v_{AN} = \sqrt{2}V_p \cos \omega t$$

$$v_{BN} = \sqrt{2}V_p \cos(\omega t - 120°)$$

$$v_{CN} = \sqrt{2}V_p \cos(\omega t + 120°)$$

Phase voltage V_p is the rms value

➔ Peak value: $\sqrt{2}V_P$

If $Z_Y = Z\angle\theta$, the phase currents are

$$i_a = \sqrt{2} I_p \cos(\omega t - \theta)$$

$$i_b = \sqrt{2} I_p \cos(\omega t - \theta - 120°)$$

$$i_c = \sqrt{2} I_p \cos(\omega t - \theta + 120°)$$

I=V/Z

➔ ∠I=∠V−∠Z

The total instantaneous power is

$$p = p_a + p_b + p_c$$

$$= v_{AN} i_a + v_{BN} i_b + v_{CN} i_c$$

$$= 3 V_p I_p \cos\theta$$

$$= 2V_p I_p [\cos\omega t \cos(\omega t - \theta)$$
$$+ \cos(\omega t - 120°)\cos(\omega t - \theta - 120°)$$
$$+ \cos(\omega t + 120°)\cos(\omega t - \theta + 120°)]$$

$$= 2V_p I_p \{\frac{\cos\theta + \cos(2\omega t - \theta)}{2}$$
$$+ \frac{\cos\theta + \cos(2\omega t - \theta - 240°)}{2}$$
$$+ \frac{\cos\theta + \cos(2\omega t - \theta + 240°)}{2}]$$

$$= 3V_p I_p \cos\theta$$

$$p = p_a + p_b + p_c \; = 3V_p I_p \cos\theta$$

Thus, **the total instantaneous power** in a balanced three-phase system is **constant** – it does not change with time as the instantaneous power of each phase does.

This result is true whether the load is Y- or Δ-connected. This is one important reason for using a three-phase system to generate and distribute power.

The advantage over a single-phase AC system: **constant p(t)**

Since the total instantaneous power is independent of time,

$Z_Y = Z\angle\theta$ θ is the power factor angle

(i) **Average power** per phase P_p for the Y- or Δ-connected load is $P/_3$, or $P_p = V_P I_P cos\theta$

(ii) **Reactive power** per phase is $Q_p = V_P I_P sin\theta$

(iii) **Apparent power** per phase is $|S_P| = V_P I_P$

(iv) **Complex power** per phase is $S_P = P_P + jQ_P = \boldsymbol{V_P I_P^*}$

(v) **Total average power** by AC power conservation is $P = P_a + P_b + P_c = 3P_P = 3V_P I_P cos\theta$

The total average power:

For a Y-connected load (with Y-connected source),
$\boldsymbol{I_L = I_P \; but \; V_L = \sqrt{3} V_P}$

For a Δ-connected load (with Δ-connected source),
$\boldsymbol{I_L = \sqrt{3} I_P \; but \; V_L = V_P}$

Thus, $\boldsymbol{3 V_p I_P cos\theta = \sqrt{3} V_L I_L cos\theta}$

V_p: phase voltage I_p: phase current	V_L: line voltage I_L: line current

The total average power:

P = 3P_P = $3V_p I_P cos\theta = \sqrt{3}V_L I_L cos\theta$

The total reactive power:

Q = 3Q_P = $3V_p I_P sin\theta = \sqrt{3}V_L I_L sin\theta$

Recall:

$Z_Y = Z\angle\theta$ θ=$\angle\mathbf{V_p} - \angle\mathbf{I_p}$

θ is the power factor angle

The total complex power:

S = 3S_P = $3\boldsymbol{V_p I_P^*} = 3I_P^2 Z_P = \frac{3V_P^2}{Z_P^*}$

where Z_P is the load impedance per phase.

$S = P + jQ = 3V_P I_P \cos\theta + j3V_P I_P \sin\theta = \sqrt{3}V_L I_L \cos\theta + j\sqrt{3}V_L I_L \sin\theta$

$\mathbf{V_P}$, $\mathbf{I_P}$, $\mathbf{V_L}$, and $\mathbf{I_L}$ are **all rms values** and that θ is the angle of the load impedance or the angle between the phase voltage and the phase current.

The total average power:

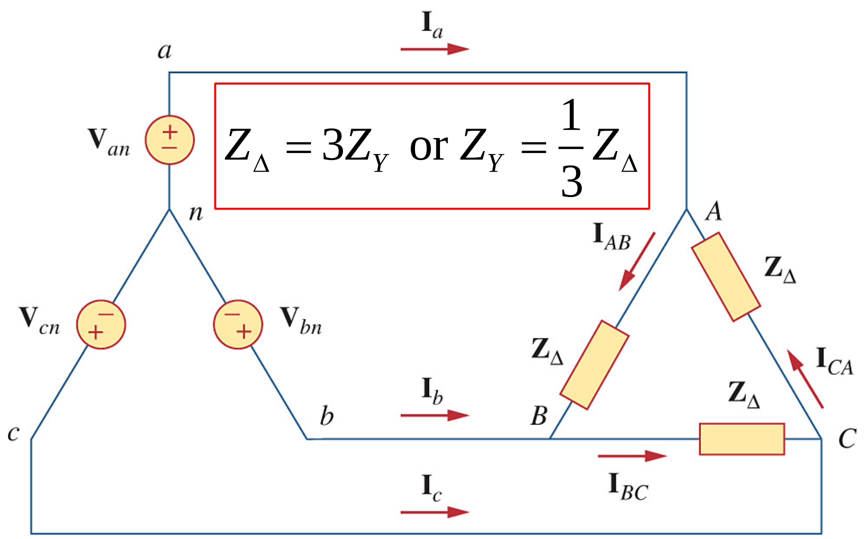
For a Y-Y connection, $\boldsymbol{I_L = I_P \ but \ V_L = \sqrt{3}V_P}$

For a Δ-Δ connection, $\boldsymbol{I_L = \sqrt{3}I_P \ but \ V_L = V_P}$

P = 3P_P = $3V_p I_P cos\theta = \sqrt{3}V_L I_L cos\theta$

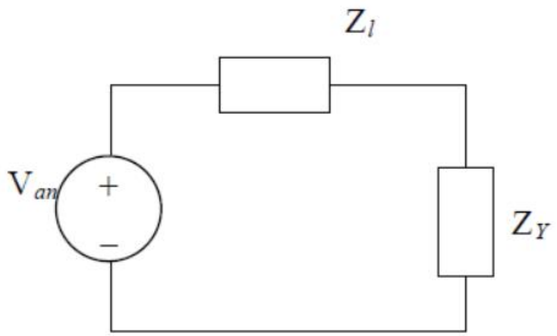
How about other connections?

12.29 A balanced three-phase Y-Δ system has $\mathbf{V}_{an} = 240\underline{/0^\circ}$ V rms and $\mathbf{Z}_\Delta = 51 + j45\ \Omega$. If the line impedance per phase is $0.4 + j1.2\ \Omega$, find the total complex power delivered to the load.



12.29 A balanced three-phase Y-Δ system has $\mathbf{V}_{an} = 240\angle 0°$ V rms and $\mathbf{Z}_{\Delta} = 51 + j45\ \Omega$. If the line impedance per phase is $0.4 + j1.2\ \Omega$, find the total complex power delivered to the load.

Replace the delta load with a Y load, $Z_Y = Z_{\Delta}/3$



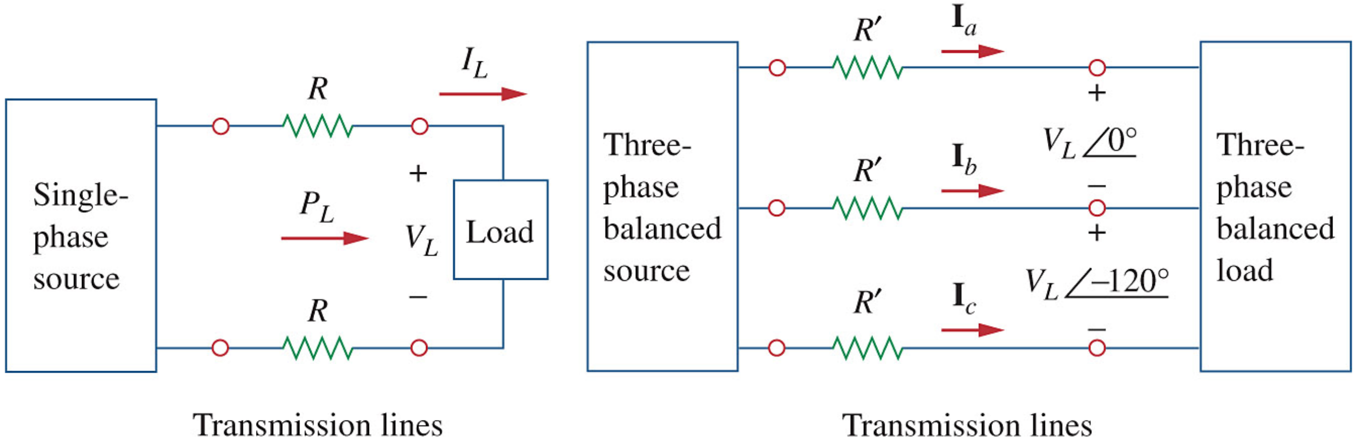
$\mathbf{I_a} = \mathbf{V_{an}}/|\mathbf{Z_Y} + \mathbf{Z_l}| = 240/|17+j15+0.4+j1.2| = 240/|17.4+j16.2| = 240/23.77 = 10.095$

$\mathbf{S} = 3[(I_a)^2(17+j15)] = 3x101.91(17+j15)$

$= \mathbf{[5.197+j4.586]\ kVA}$.

Three phase system - Advantage 2:

The three-phase system uses a lesser amount of wire than the single-phase system for the same line voltage V_L and **the same absorbed power P_L** → Economical

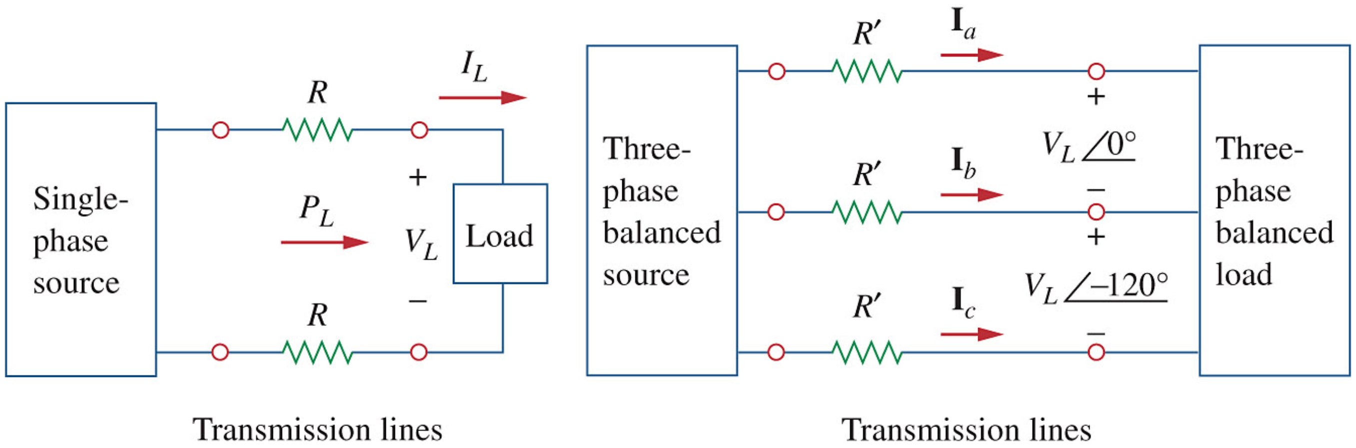


Single phase **Three phases**

1. The same total power delivered P_L
2. The same line voltage V_L

Recall: Average power in the rms form in Chapter 11

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i)$$



Assume that the load is purely resistive

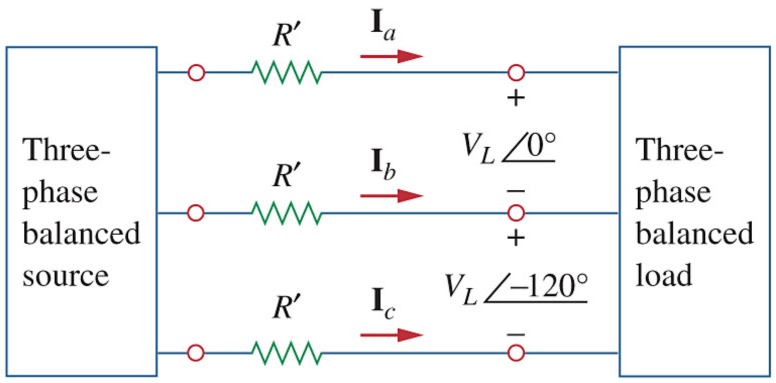
For the single-phase **two-wire system**, $P_L = V_L I_L$, the power loss in the two wires is

$$P_{loss} = 2 I_L^2 R = 2\left(\frac{P_L}{V_L}\right)^2 R$$

2 wires

Recall: Average power in the rms form in Chapter 11

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i)$$



Transmission lines

I'_L: line current
R': line resistance

$$I'_L = |\mathbf{I}_a| = |\mathbf{I}_b| = |\mathbf{I}_c| = P_L/\sqrt{3}V_L$$

For the three-phase wire system, $P_L = \sqrt{3} V_L I'_L$, the power loss in **the three wires** is

$$P'_{loss} = 3I_L'^2 R' = 3\left(\frac{P_L}{\sqrt{3}V_L}\right)^2 R' = \left(\frac{P_L}{V_L}\right)^2 R'$$

3 wires

The equations show that for **the same total power delivered P_L** and **the same line voltage V_L**,

$$P_{loss} = 2I_L^2 R = 2\left(\frac{P_L}{V_L}\right)^2 R$$ **2 wires**

$$P'_{loss} = 3I_L'^2 R' = 3\left(\frac{P_L}{\sqrt{3}V_L}\right)^2 R' = \left(\frac{P_L}{V_L}\right)^2 R'$$ **3 wires**

$$\frac{P_{loss}}{P'_{loss}} = \frac{2R}{R'} = \frac{2\rho l/(\pi r^2)}{\rho l/(\pi r'^2)} = \frac{2r'^2}{r^2}$$

$$R = \frac{\rho L}{A}$$

ρ = resistivity
L = length
A = cross sectional area

where ρ is resistivity; l is length of transmission line; and r is radius of the transmission line

If we consider **the same power loss** in both systems,

$$\frac{P_{loss}}{P'_{loss}} = \frac{2r'^2}{r^2} = 1 \quad \text{thus, } r^2 = 2r'^2$$

Two wires

Material volume needed

Two wires

Three wires

$$\frac{\text{Material for single-phase}}{\text{Material for three-phase}} = \frac{2(\pi r^2 l)}{3(\pi r'^2 l)} = \frac{2r^2}{3r'^2} = \frac{4}{3}$$

https://www.energy.gov/sites/prod/files/2015/12/f28/united-states-electricity-industry-primer.pdf

The second major advantage of three-phase systems for power distribution: **Economical**

Summary of the advantages of 3-ϕ system

1. The total instantaneous power in a balanced three-phase system is constant.

2. The material (transmission line) to deliver the same power and to tolerate the same loss needed is ¾ times less.

$$S = P + jQ = 3V_P I_P \cos\theta + j3V_P I_P \sin\theta$$
$$= \sqrt{3}V_L I_L \cos\theta + j\sqrt{3}V_L I_L \sin\theta$$

Example 12.8

Two balanced loads are connected to a 240-kV rms 60-Hz line, as shown in Fig. 12.22(a). Load 1 draws 30 kW at a power factor of 0.6 lagging, while load 2 draws 45 kVAR at a power factor of 0.8 lagging. Assuming the *abc* sequence, determine: (a) the complex, real, and reactive powers absorbed by the combined load, (b) the line currents, and (c) the kVAR rating of the three capacitors Δ-connected in parallel with the load that will raise the power factor to 0.9 lagging and the capacitance of each capacitor.

30kW
pf=0.6 lagging

45kVAR
pf=0.8 lagging

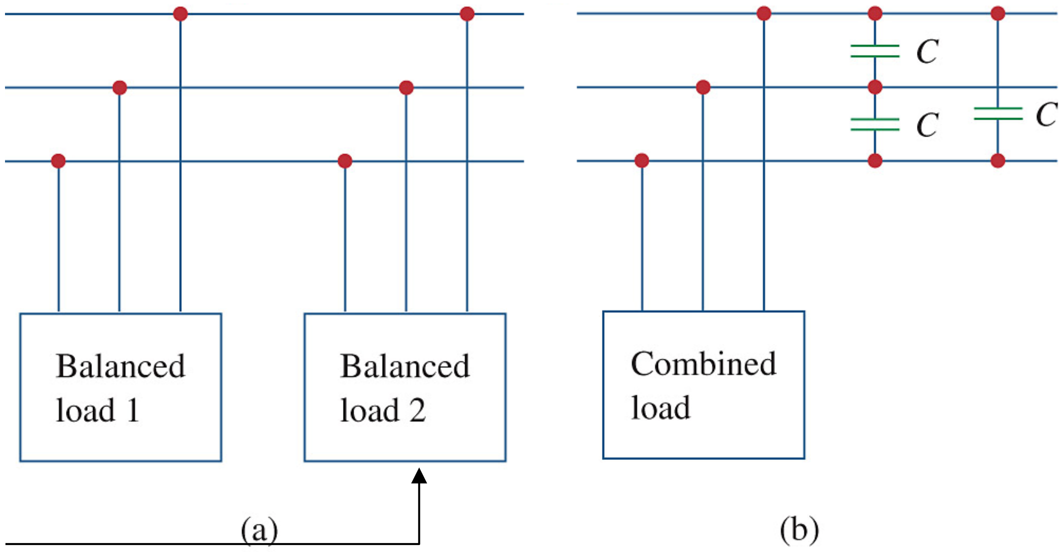


Figure 12.22 (a) The original balanced loads, (b) the combined load with improved power factor.

Load 1:
30kW
pf=0.6 lagging

Load 2:
45kVAR
pf=0.8 lagging

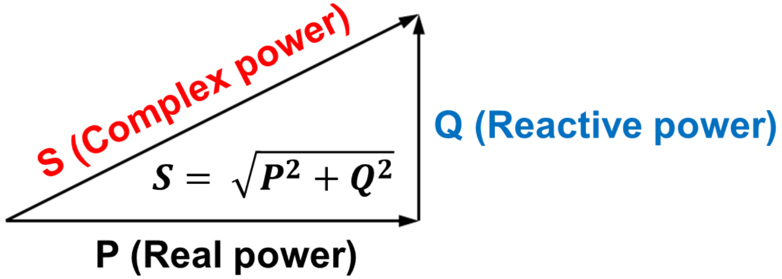
$\mathrm{V_L = 240kV, 60\ Hz}$

$S = P + jQ = 3V_P I_P \cos\theta + j3V_P I_P \sin\theta$

$= \sqrt{3}V_L I_L \cos\theta + j\sqrt{3}V_L I_L \sin\theta$

(a) S, P, Q

Load 1: 30kW, pf=0.6 lagging



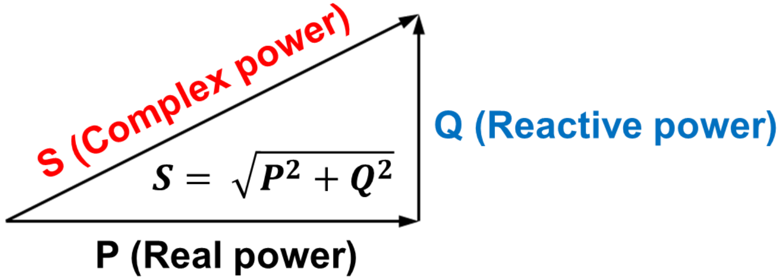
$\cos\theta_1 = 0.6 \rightarrow \theta_1 = 53.13^\circ$

$\tan 53.13^\circ = Q/P$

Thus, Q = 40k [VAR]

$\mathbf{S_1 = 30k + j40k\ [VA]}$

Load 2: 45kVAR, pf=0.8 lagging



$\cos\theta_2 = 0.8 \rightarrow \theta_2 = 36.87^\circ$

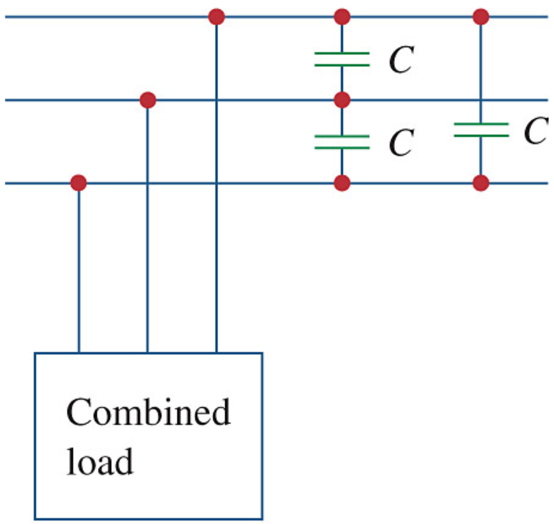
$\tan 36.87^\circ = Q/P$

Thus, P = 60k [W]

$\mathbf{S_1 = 60k + j45k\ [VA]}$

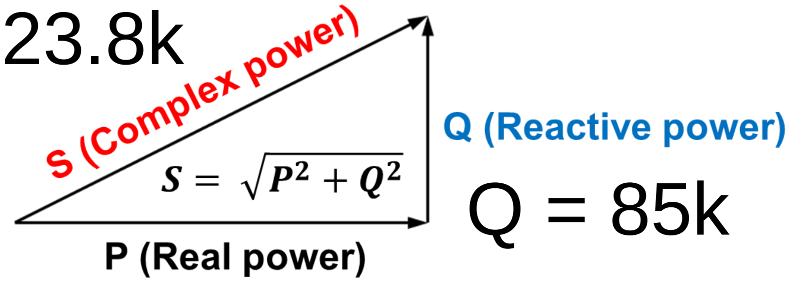
Complex power absorbed by the combined load, therefore, is
S = 90k + j85k [VA] where P = 90kW and Q = 85kVA

(b) Line currents



240 kV rms 60-Hz line → V_L = 240 kV
S from (a) is 90k + j85k

ISI = 123.8k



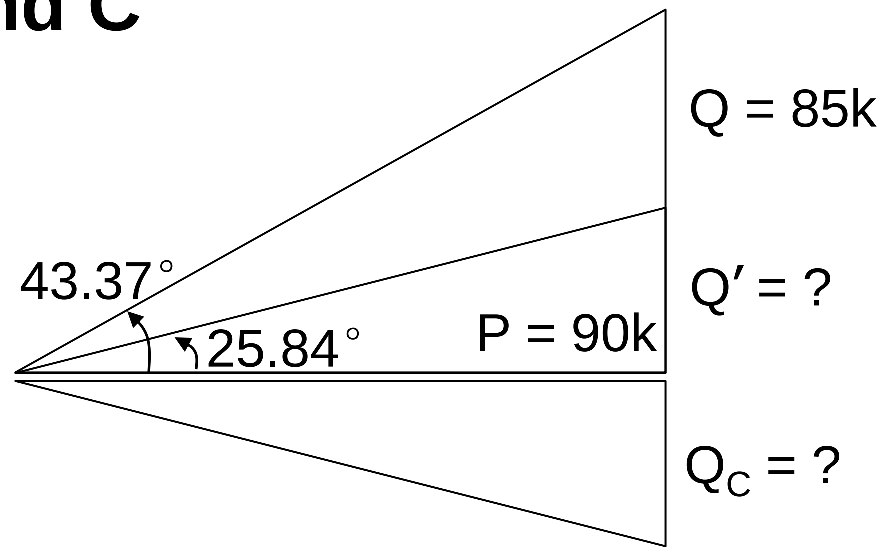
Q = 85k

P = 90k

$$P = \sqrt{3} V_L I_L \cos\theta \;\rightarrow 90k = \sqrt{3}\, 240k\, I_L\, {}^{90}/_{123.8}$$

Thus, I_L = 0.296 A or 0.296 mA

(c) Q_C and C



pf 0.9 lagging

$\cos\theta' = 0.9 \rightarrow \theta' = 25.84°$

$$Q' \rightarrow \tan 25.84 = \frac{Q'}{P(90k)} \rightarrow Q' = 43.56k$$

$\mathbf{Q_C}$ = Q – Q′ = 85k – 43.56k = **41.44k VAR**

→ Each capacitor has an effect of 13.81k VAR

$Q_C = I^2X = \omega CV^2$ for purely capacitive case, and thus,

$$C = \frac{Q_C}{\omega(=2\pi f)V^2} = \mathbf{6.36 \times 10^{-10}[F]}$$