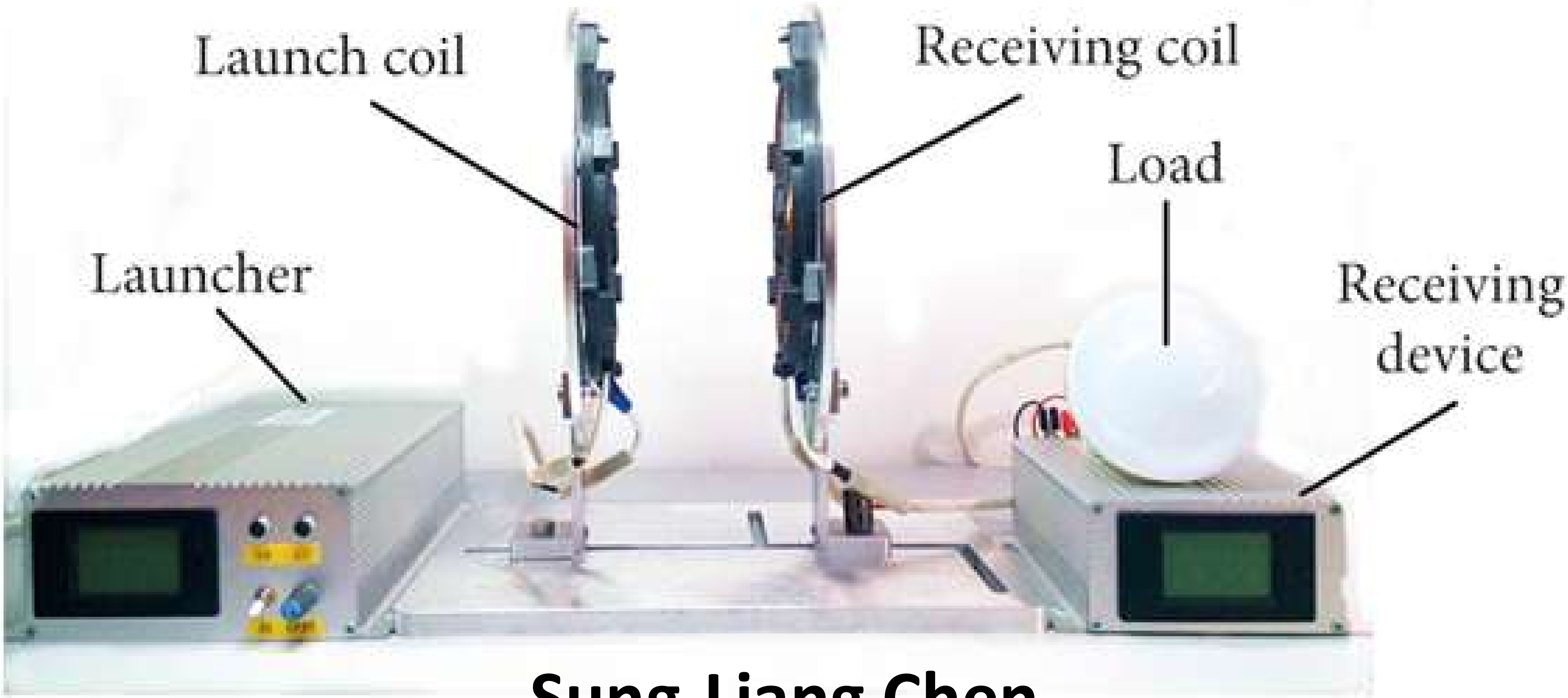
Chapter 13 Magnetically Coupled Circuits



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13.1 Introduction

The circuits we have considered so far may be regarded as *conductively coupled*, because one loop affects the neighboring loop through current conduction.

When two loops (with or without contacts between them) affect each other through the magnetic field generated by one of them, they are said to be *magnetically coupled*. In this chapter, we study magnetically coupled circuits.

13.2 Mutual Inductance

When two coils are in a close proximity to each other, the magnetic flux caused by current in one coil links with the other coil, thereby inducing voltage in the latter. This phenomenon is known as *mutual inductance.*

Chapter 6:

$i_1 \rightarrow \psi_1 = L_1 i_1 \rightarrow v_1 = d\psi_1/dt$

ψ_1: total magnetic flux

Self-inductance, L

Let us first consider a single coil with N turns. When current i flows through the coil, a magnetic flux ϕ is produced around it (Fig. 13.1). According to Faraday's law, the voltage v induced in the coil is

$$v = N\frac{d\phi}{dt}$$

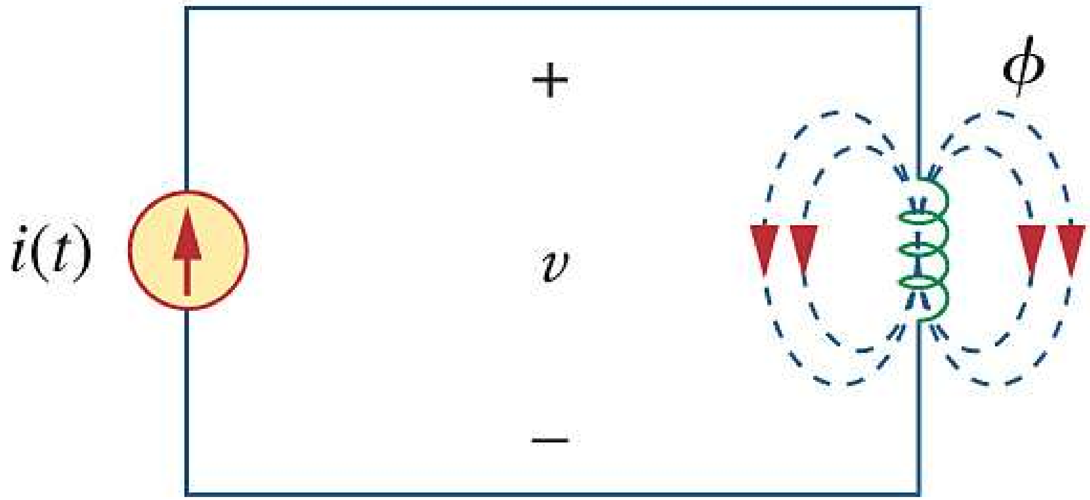


Figure 13.1 Magnetic flux produced by a single coil with N turns.

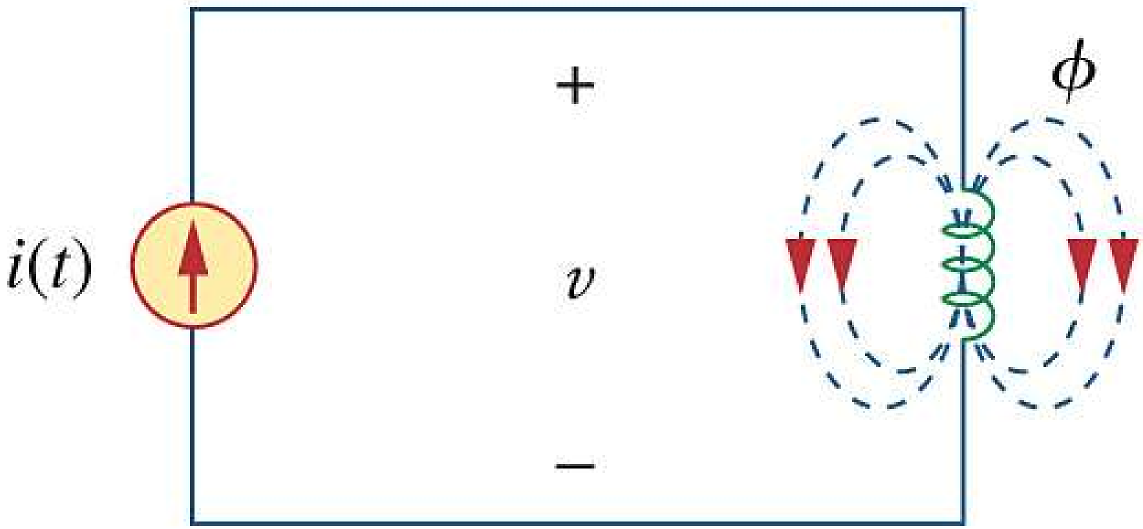


Figure 13.1 Magnetic flux produced by a single coil with N turns.

But the flux is produced by current i so that any change in ϕ is caused by a change in i. Hence,

$$v = N\frac{d\phi}{dt} = N\frac{d\phi}{di}\frac{di}{dt} = \underline{L\frac{di}{dt}}$$

The *v*-*i* relation of an inductor

where $L = N\dfrac{d\phi}{di}$ is commonly called the *self-inductance* of the coil.

Mutual-inductance, L

Now consider two coils that are in close proximity with each other (Fig. 13.2). Assume that coil 2 carries no current. The magnetic flux ϕ_1 emanating from coil 1 has two components, i.e.,

$$\phi_1 = \phi_{11} + \phi_{12}$$

where ϕ_{11} links only coil 1, and ϕ_{12} links both coils.

ϕ_{ab}: flux emanating from coil *a*

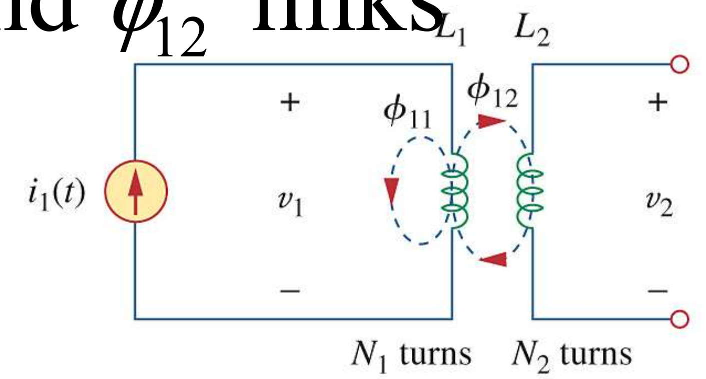


Figure 13.2 Mutual inductance M_{21} of coil 2 with respect to coil 1.

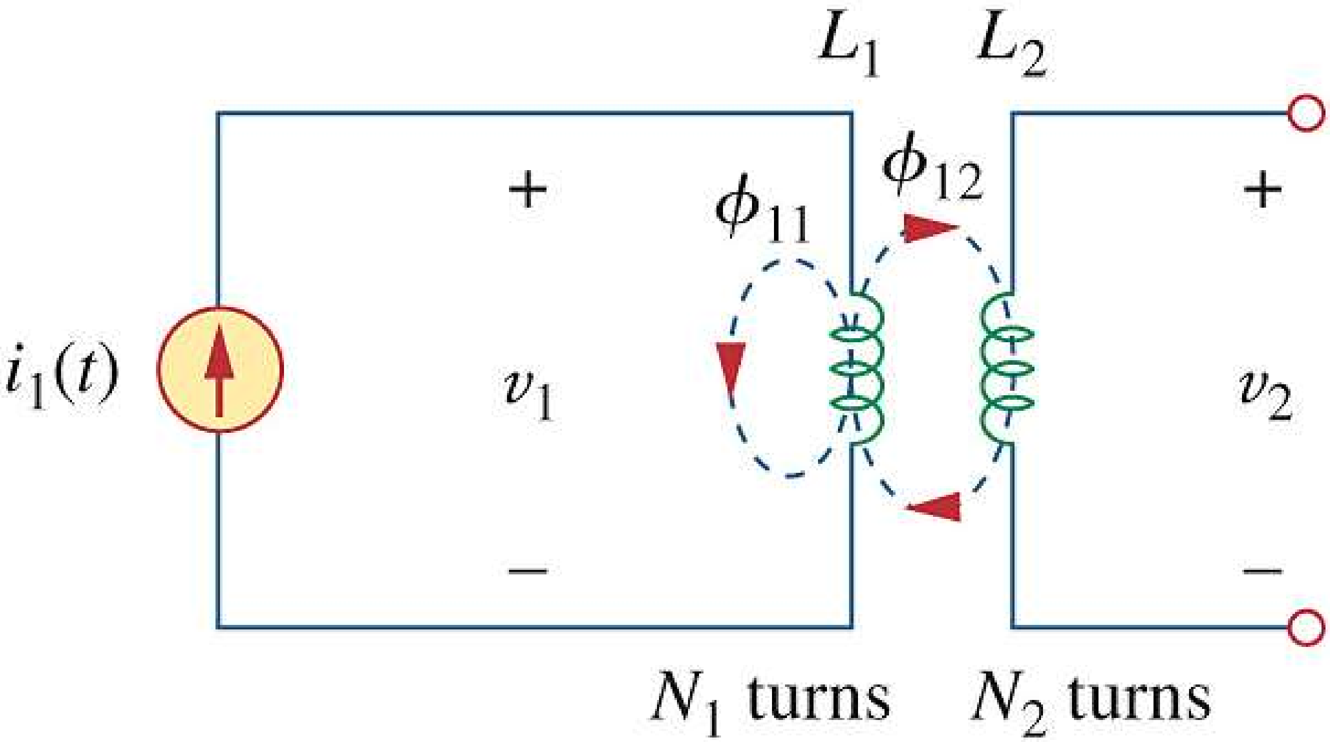


Figure 13.2 Mutual inductance M_{21} of coil 2 with respect to coil 1.

Hence, $\boxed{\phi_1 = \phi_{11} + \phi_{12}}$

$$v_1 = N_1 \frac{d\phi_1}{dt} = N_1 \frac{d\phi_1}{di_1}\frac{di_1}{dt} = L_1 \frac{di_1}{dt}$$

$$v_2 = N_2 \frac{d\phi_{12}}{dt} = N_2 \frac{d\phi_{12}}{di_1}\frac{di_1}{dt} = M_{21} \frac{di_1}{dt}$$

where $M_{21} = N_2 \frac{d\phi_{12}}{di_1}$ is known as the *mutual inductance* of coil 2 with respect to coil 1, measured in henrys (H).

ϕ_{12}

Current in coil 1

M_{21}

Induced inductance in coil 2 **Current in coil 1**

Suppose we now let current i_2 flow in coil 2, while coil 1 carries no current (Fig. 13.3). We have

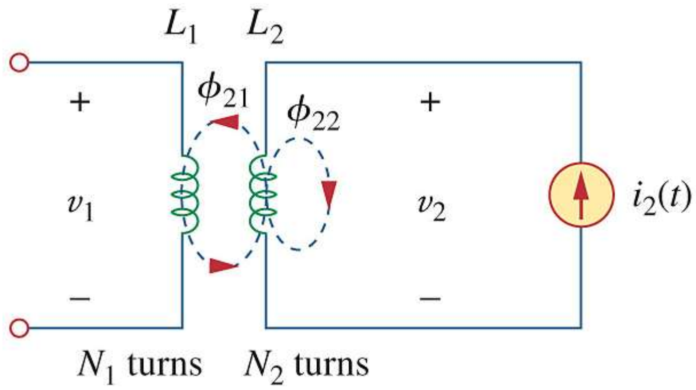


Figure 13.3 Mutual inductance M_{12} of coil 1 with respect to coil 2.

$$\phi_2 = \phi_{21} + \phi_{22}$$

$$v_2 = N_2 \frac{d\phi_2}{dt} = N_2 \frac{d\phi_2}{di_2}\frac{di_2}{dt} = L_2 \frac{di_2}{dt}$$

$$v_1 = N_1 \frac{d\phi_{21}}{dt} = N_1 \frac{d\phi_{21}}{di_2}\frac{di_2}{dt} = M_{12} \frac{di_2}{dt}$$

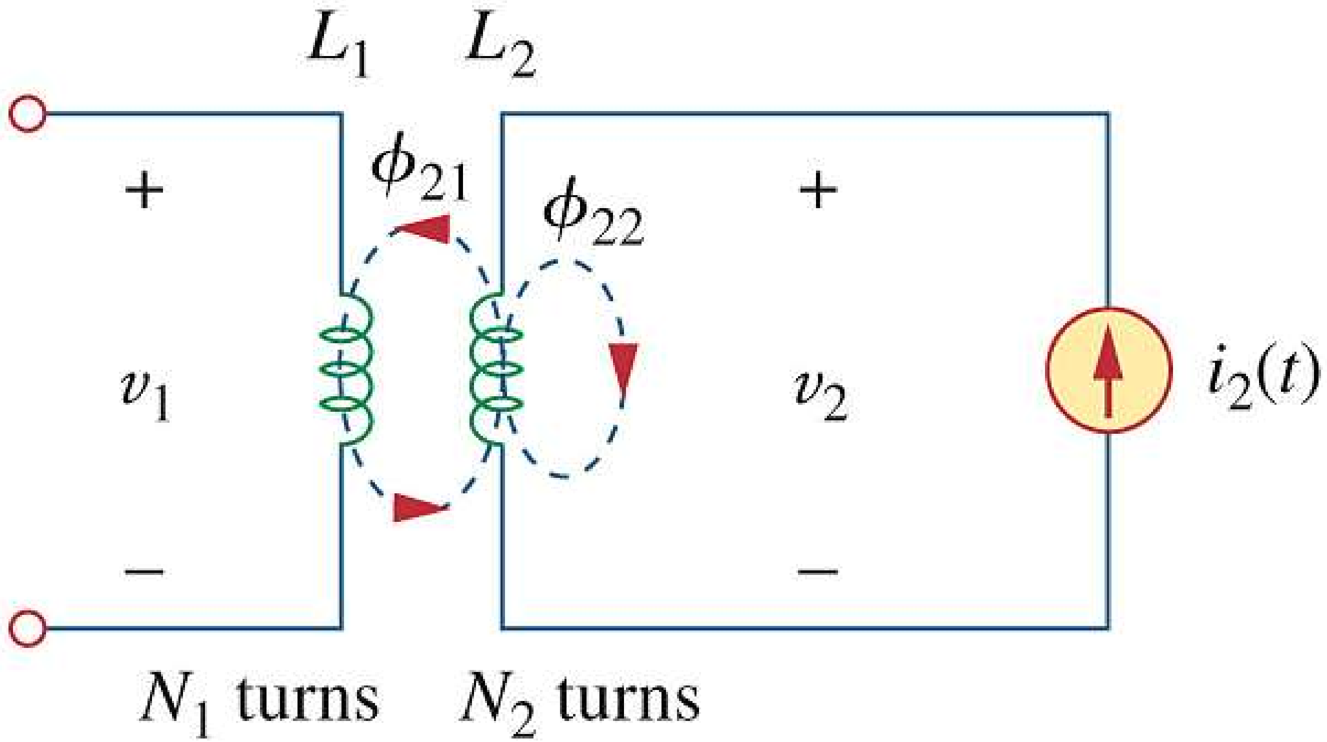


Figure 13.3 Mutual inductance M_{12} of coil 1 with respect to coil 2.

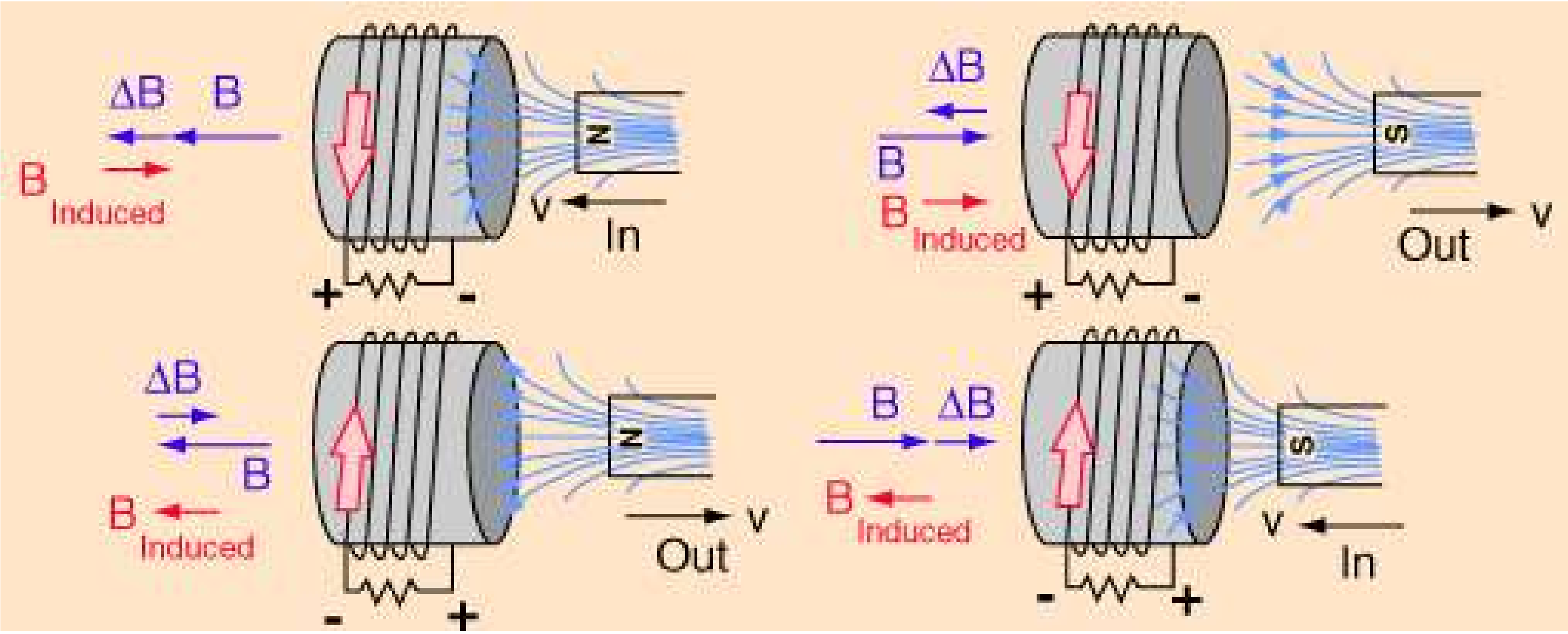
We will see in the next section that $M_{12} = M_{21} = M$, and we refer to M as the mutual inductance between the two coils.

Although mutual inductance $\underline{M}$ is always a <u>positive</u> quantity, the mutual voltage may be negative or positive, just like the self-induced voltage.

$v = Ldi/dt$
L: always a positive quantity
v: +/- depending on di/dt

However, the polarity of the mutual voltage is determined by examining the orientation in which both coils are physically wound and applying Lenz's law in conjunction with the right-hand rule. Since it is inconvenient to show the construction details of coils on a circuit schematic, we apply the *dot convention* in circuit analysis.

Lenz’s law



ΔB
B
B
Induced
N
v
In
+
-
ΔB
B
B
Induced
S
Out
v
+
-
ΔB
B
B
Induced
N
v
Out
-
+
B
ΔB
B
Induced
S
v
In
-
+

By this convention, a dot is placed in the circuit at one end of each of the two magnetically coupled coils to indicate the direction of the magnetic flux if current enters that dotted terminal. This is illustrated in Fig. 13.4.

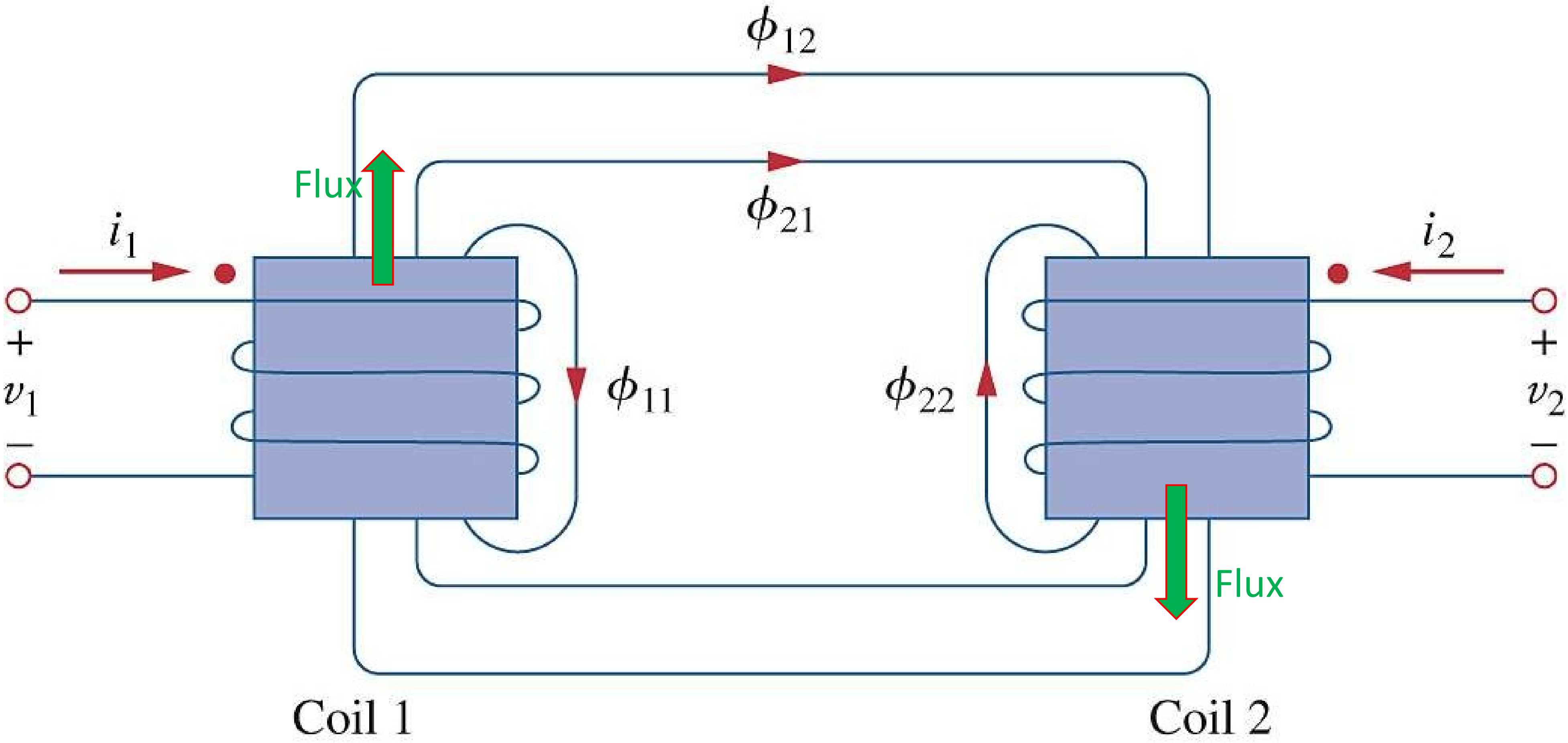


Figure 13.4 Illustration of the dot convention.

Current entering the dotted end of one winding produces flux in the same direction as the flux produced by current entering the dotted end of the other winding.

In other words, i_1 enters the dotted end of coil 1, and i_2 enters the dotted end of coil 2. They produce the flux in the same direction (clockwise in the figure).

1. Determined the construction details of the coils.

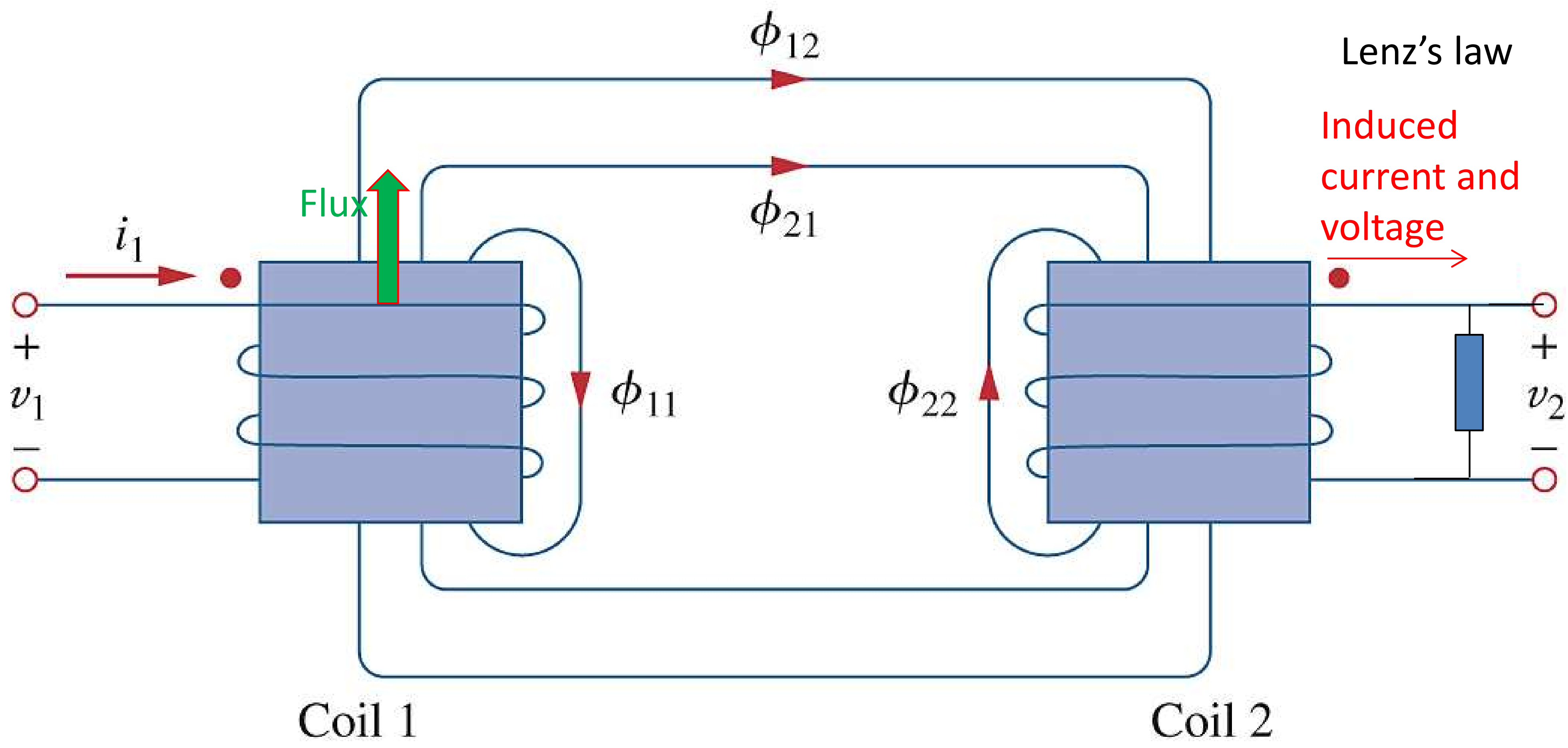


Figure 13.4 Illustration of the dot convention.

Now consider only the current entering the coil 1.
(neglect ϕ_{21} and ϕ_{22} in the figure)

2. Can determine the polarity of the induced voltage.

The dot convention is stated as follows:
If a current enters the dotted terminal of one coil, the reference polarity of the mutual voltage of the second coil is positive at the dotted terminal of the second coil.
Application of the dot convention is illustrated in Fig. 13.5.

Entering current in dot 1 ➔ + voltage in dot 2
Leaving current in dot 1 ➔ – voltage in dot 2

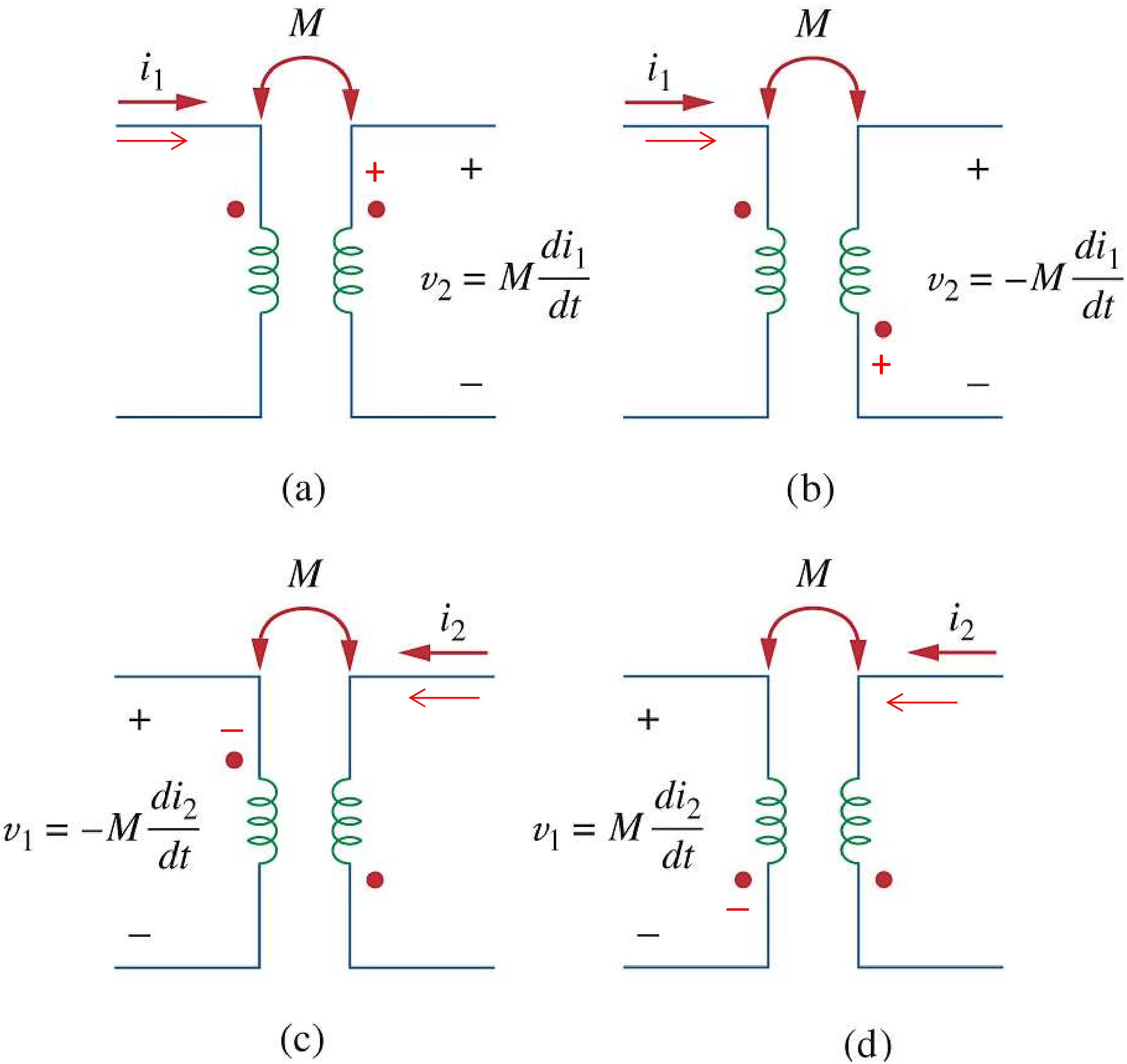


Figure 13.5 Examples illustrating how to apply the dot convention.

Figure 13.6 shows two coupled coils in series. For the coils in Fig. 13.6(a), the total inductance is

$$L = L_1 + L_2 + 2M$$

(series-aiding connection)

For the coils in Fig. 13.6(b),

$$L = L_1 + L_2 - 2M$$

(series-opposing connection)

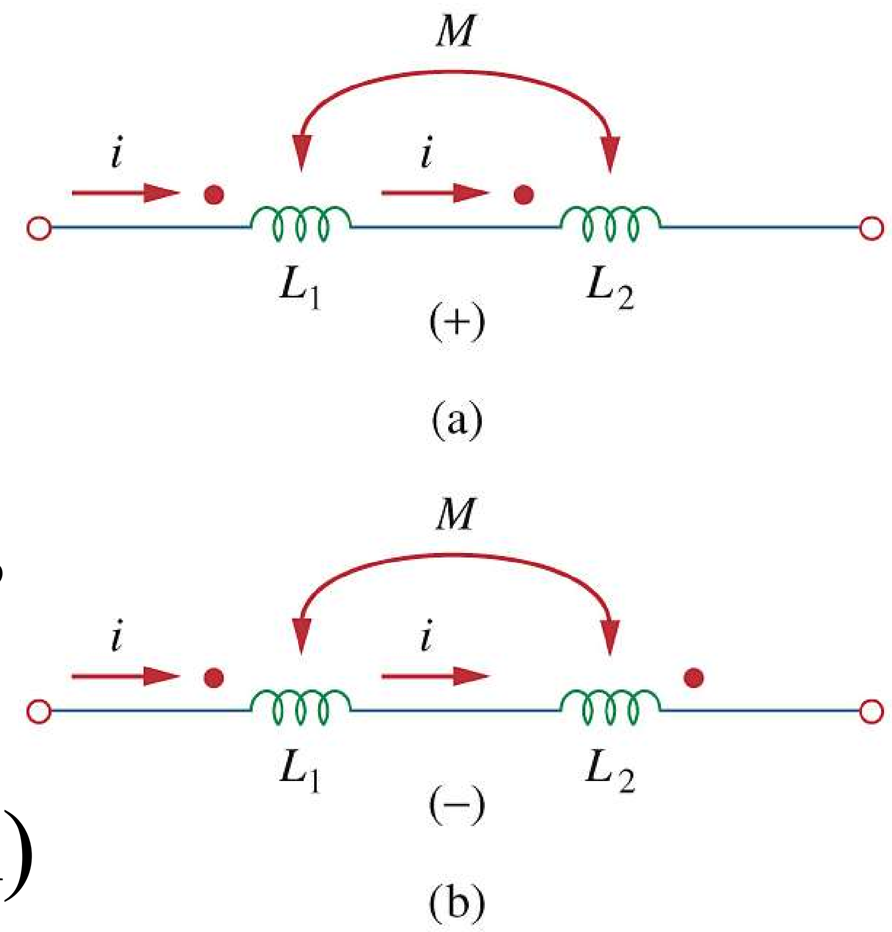
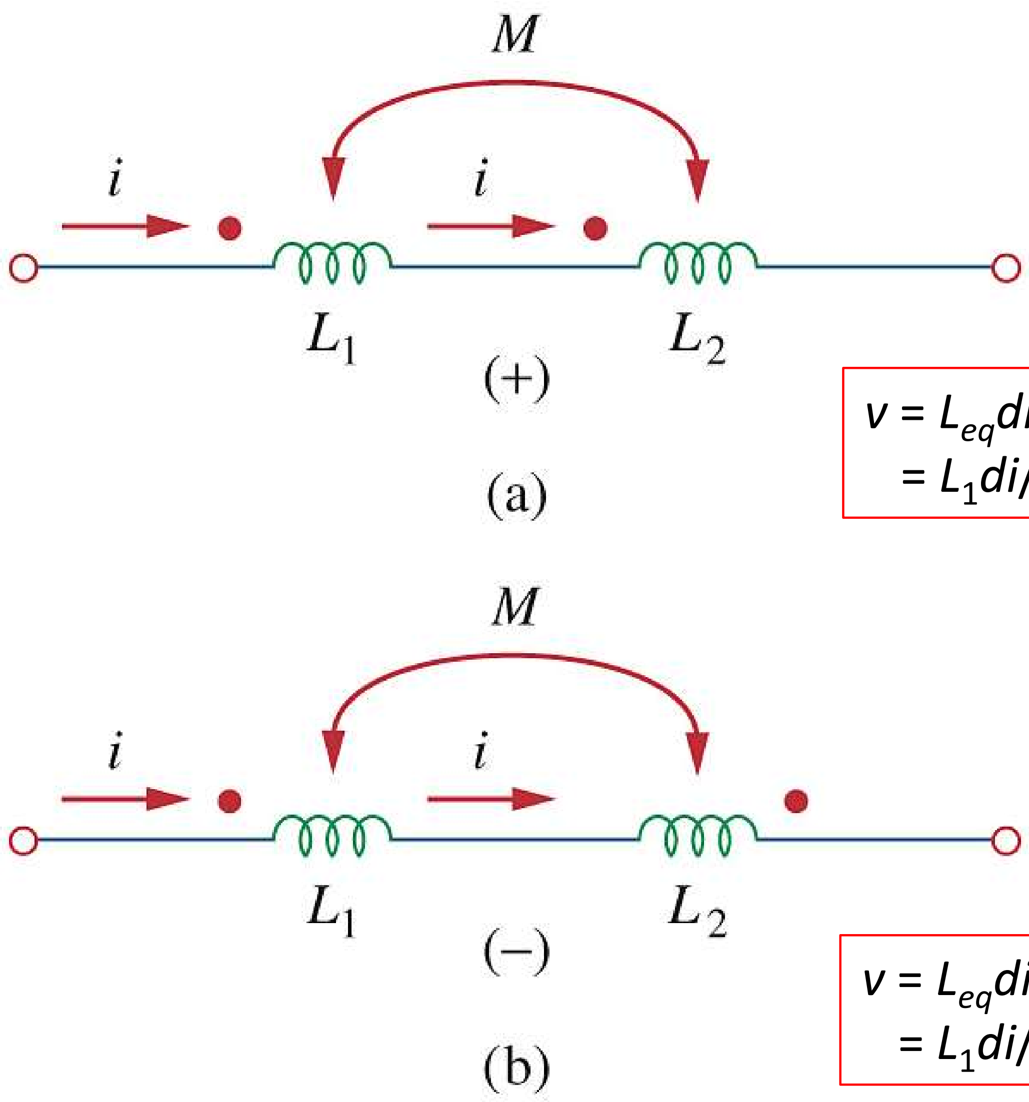


Figure 13.6 Dot convention for coils in series; the sign indicates the polarity of the mutual voltage: (a) series-aiding connection, (b) series-opposing connection.



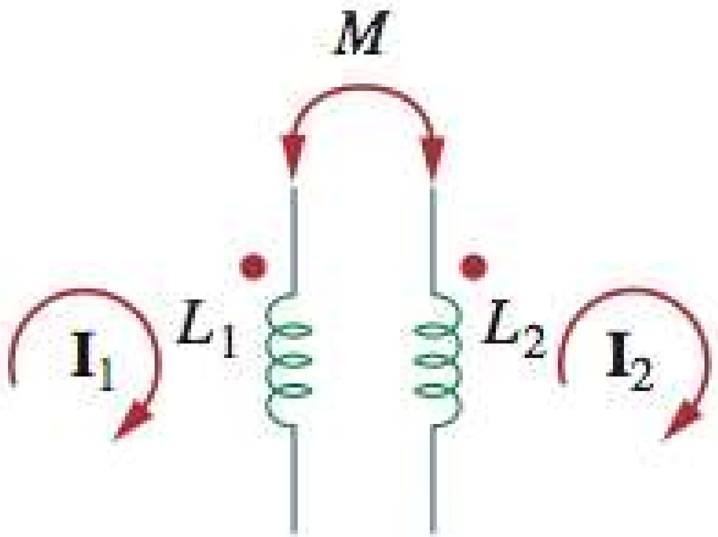
$$L = L_1 + L_2 + 2M$$

$$v = L_{eq} di/dt$$
$$= L_1 di/dt + L_2 di/dt + M_{21} di/dt + M_{12} di/dt$$

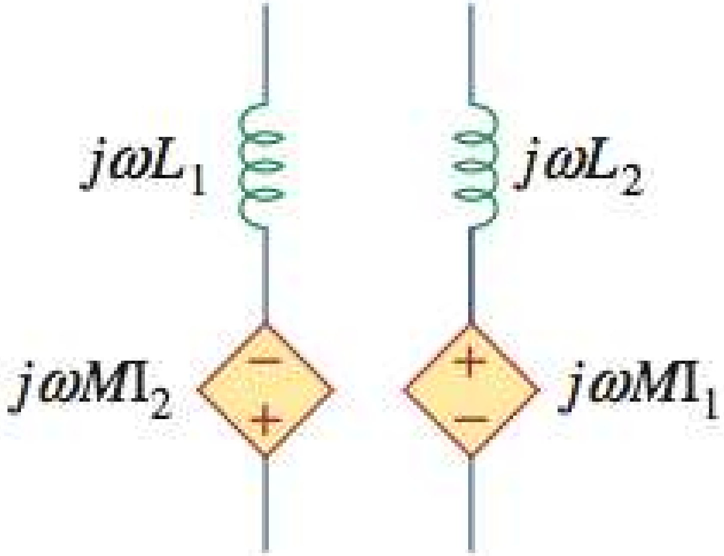
$$L = L_1 + L_2 - 2M$$

$$v = L_{eq} di/dt$$
$$= L_1 di/dt + L_2 di/dt - M_{21} di/dt - M_{12} di/dt$$

Figure 13.6 Dot convention for coils in series; the sign indicates the polarity of the mutual voltage: (a) series-aiding connection, (b) series-opposing connection.

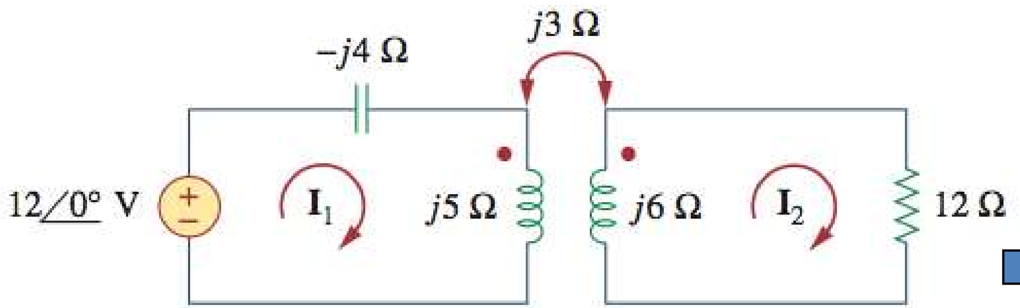


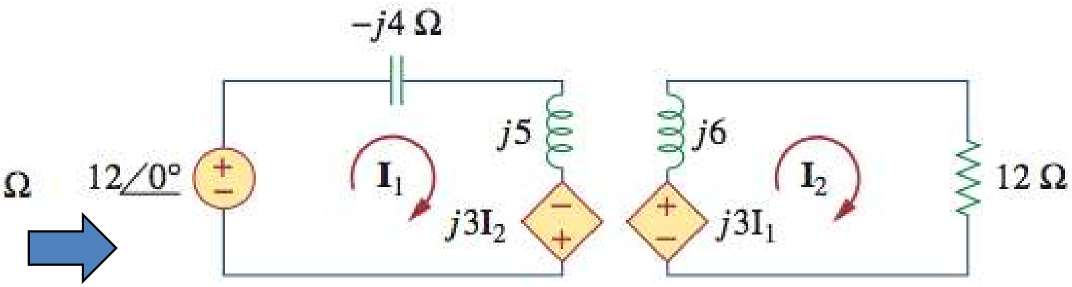
Phasor representation



Calculate the phasor currents $\mathbf{I}_1$ and $\mathbf{I}_2$ in the circuit of Fig. 13.9.

Example 13.1





(b)

Solution:

For loop 1, KVL gives

$$-12 + (-j4 + j5)\mathbf{I}_1 - j3\mathbf{I}_2 = 0$$

or

$$j\mathbf{I}_1 - j3\mathbf{I}_2 = 12 \qquad \textbf{(13.1.1)}$$

For loop 2, KVL gives

$$-j3\mathbf{I}_1 + (12 + j6)\mathbf{I}_2 = 0$$

or

$$\mathbf{I}_1 = \frac{(12 + j6)\mathbf{I}_2}{j3} = (2 - j4)\mathbf{I}_2 \qquad \textbf{(13.1.2)}$$

Substituting this in Eq. (13.1.1), we get

$$(j2 + 4 - j3)\mathbf{I}_2 = (4 - j)\mathbf{I}_2 = 12$$

or

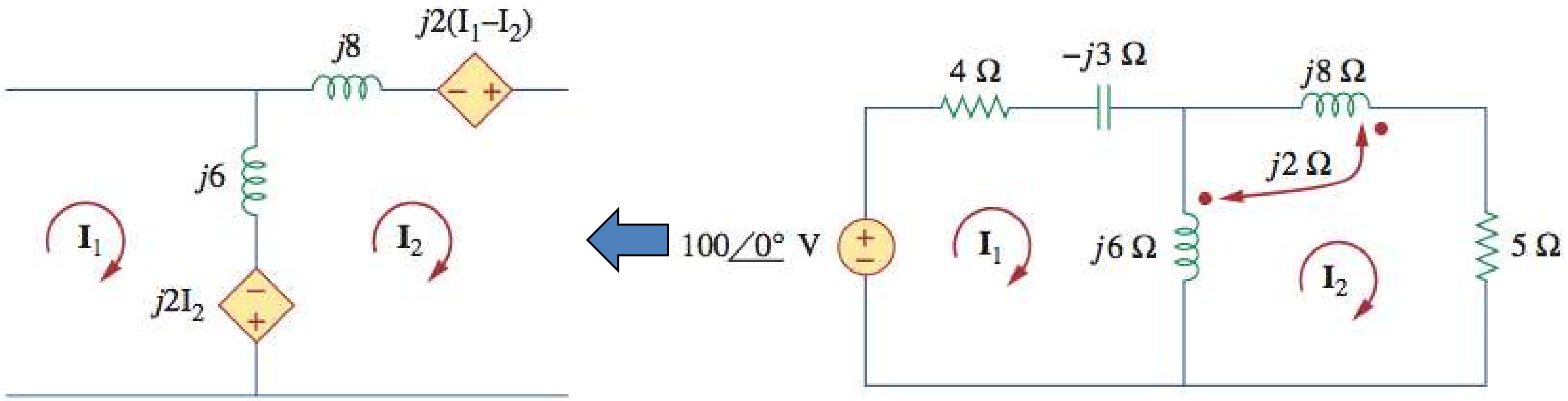
$$\mathbf{I}_2 = \frac{12}{4 - j} = 2.91\underline{/14.04^\circ} \text{ A} \qquad \textbf{(13.1.3)}$$

From Eqs. (13.1.2) and (13.1.3),

$$\mathbf{I}_1 = (2 - j4)\mathbf{I}_2 = (4.472\underline{/-63.43^\circ})(2.91\underline{/14.04^\circ})$$
$$= 13.01\underline{/-49.39^\circ} \text{ A}$$

Example 13.2

Calculate the mesh currents in the circuit of Fig. 13.11.



$$-100 + \mathbf{I}_1(4 - j3 + j6) - j6\mathbf{I}_2 - j2\mathbf{I}_2 = 0 \qquad 0 = -2j\mathbf{I}_1 - j6\mathbf{I}_1 + (j6 + j8 + j2 \times 2 + 5)\mathbf{I}_2$$

$$100 = (4 + j3)\mathbf{I}_1 - j8\mathbf{I}_2 \qquad 0 = -j8\mathbf{I}_1 + (5 + j18)\mathbf{I}_2$$

$$\begin{bmatrix} 100 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 + j3 & -j8 \\ -j8 & 5 + j18 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

$$\mathbf{I}_1 = \frac{\Delta_1}{\Delta} = \frac{100(5 + j18)}{30 + j87} = \frac{1{,}868.2\angle 74.5^\circ}{92.03\angle 71^\circ} = 20.3\angle 3.5^\circ \text{ A}$$

$$\mathbf{I}_2 = \frac{\Delta_2}{\Delta} = \frac{j800}{30 + j87} = \frac{800\angle 90^\circ}{92.03\angle 71^\circ} = 8.693\angle 19^\circ \text{ A}$$

13.3 Energy in a Coupled Circuit

Consider the circuit in Fig. 13.14. We assume that currents i_1 and i_2 are zero initially, so that the energy stored in the coils is zero.

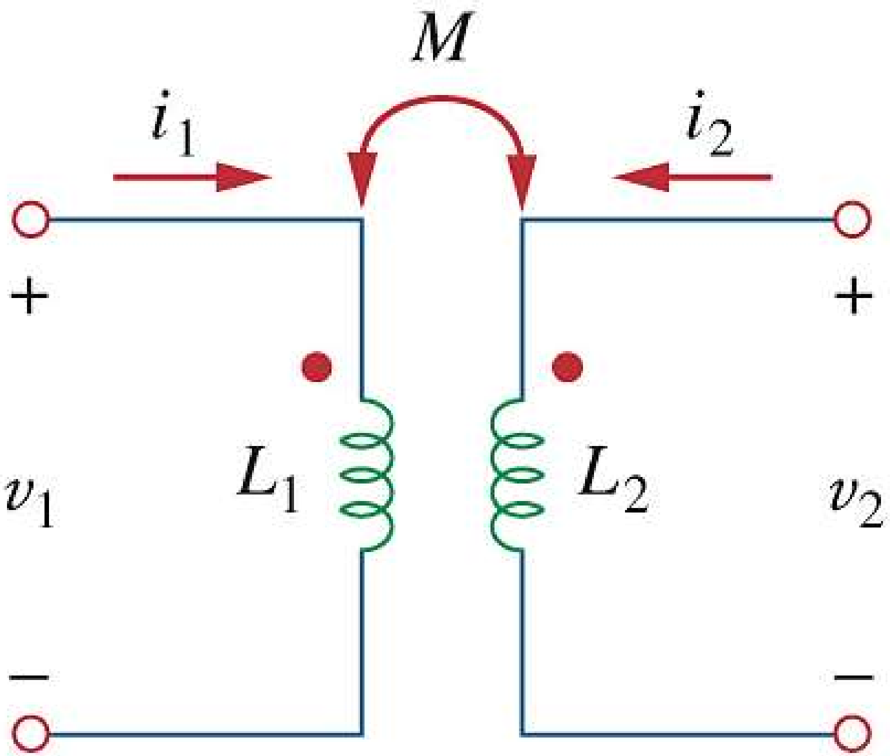


Figure 13.14 The circuit for deriving energy stored in a coupled circuit.

Step1: i_1 from 0 to I_1

If we let i_1 increase from zero to I_1 while maintaing $i_2 = 0,$ the power in the circuit is

$$p_1 = v_1 i_1 = L_1 \frac{di_1}{dt} i_1$$

and the energy stored in the circuit is

$$w_1 = \int p_1 dt = L_1 \int_0^{I_1} i_1 di_1 = \frac{1}{2} L_1 I_1^2$$

Step2: i_2 from 0 to I_2

If we now maintain $i_1 = I_1$ and increase i_2 from zero to I_2, the power in the coils is now

$p_2 = v_1 i_1 + v_2 i_2$

$$p_2 = \left(M_{12} \frac{di_2}{dt} \right) I_1 + \left(L_2 \frac{di_2}{dt} \right) i_2$$

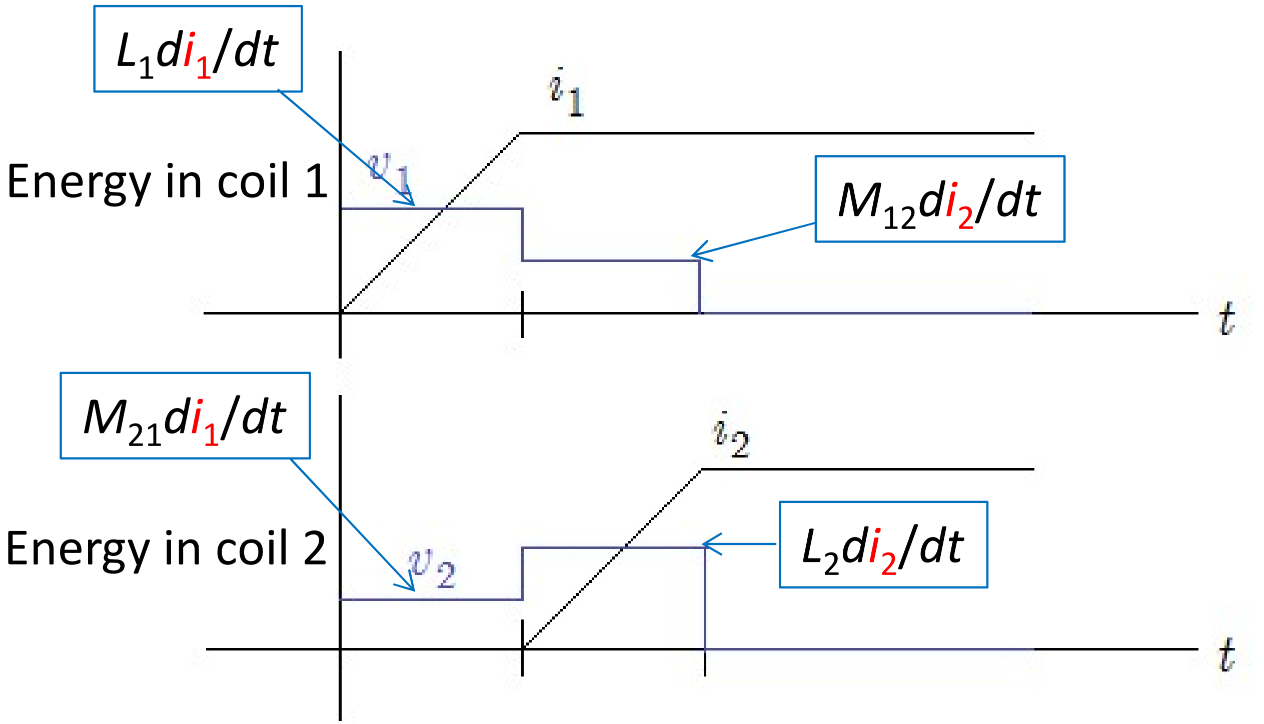
and the energy stored in the circuit is

$$w_2 = \int p_2 dt = M_{12} I_1 \int_0^{I_2} di_2 + L_2 \int_0^{I_2} i_2 di_2$$

$$= M_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2$$

The total energy stored in the coils when both i_1 and i_2 have reached constant values is

$$w = w_1 + w_2 = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M_{12} I_1 I_2$$



Black: currents
Blue: voltages

Step1: i_1 from 0 to I_1

Step2: i_2 from 0 to I_2

If we reverse the order by which the currents reach their final values, that is, if we first increase i_2 from zero to I_2 and later increase i_1 from zero to I_1, the total energy stored in the coils is

$$w = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M_{21} I_1 I_2$$

Since the total energy stored should be the same regardless of how we reach the final conditions, comparing the two total energy expressions leads us to conclude that

$$M_{12} = M_{21} = M$$

$$w = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2$$

This equation was derived based on the assumption that the coil currents both entered the dotted terminals. If one current enters one dotted terminal while the other current leaves the other dotted terminal, the total energy is

$$w = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 \underline{-} M I_1 I_2$$

In the derivation, it changes to *v* = –*Mdi*/*dt*

Also, since I_1 and I_2 are arbitrary values, they may be replaced by i_1 and i_2, which gives

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 \pm M i_1 i_2$$

The positive sign is selected for the mutual term if both currents enter or leave the dotted terminals of the coils; the negative sign is selected otherwise.

The derivation of the case "+" will be similar.

We use $w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 \pm M i_1 i_2$ to show that M cannot exceed $\sqrt{L_1 L_2}$.

The magnetically coupled coils are passive elements, so the total energy stored can never be negative; specifically,

$$\frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 - M i_1 i_2 \geq 0$$

If $i_1 i_2<0$, $\frac{1}{2}L_1 i_1^2+\frac{1}{2}L_2 i_2^2 \underbrace{-M i_1 i_2}_{>0} \geq 0$ is always satisfied.

when i_1 and i_2 are either both positive or both negative.

$$\frac{1}{2}\left(\sqrt{L_1}i_1-\sqrt{L_2}i_2\right)^2+i_1 i_2\left(\sqrt{L_1 L_2}-M\right)\geq 0$$

The square term can never be negative, but it can be zero. Therefore, $w(t)\geq 0$ only if

$\sqrt{L_1 L_2}\geq M$.

The extent to which M approaches $\sqrt{L_1 L_2}$ is specified by the *coefficient of coupling* k, given by

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

$$(0 \le k \le 1)$$

Coils are said to be *loosely coupled* when $k < 0.5$. If $k > 0.5$, they are *tightly coupled*.

Distance is one factor to affect the value *k*

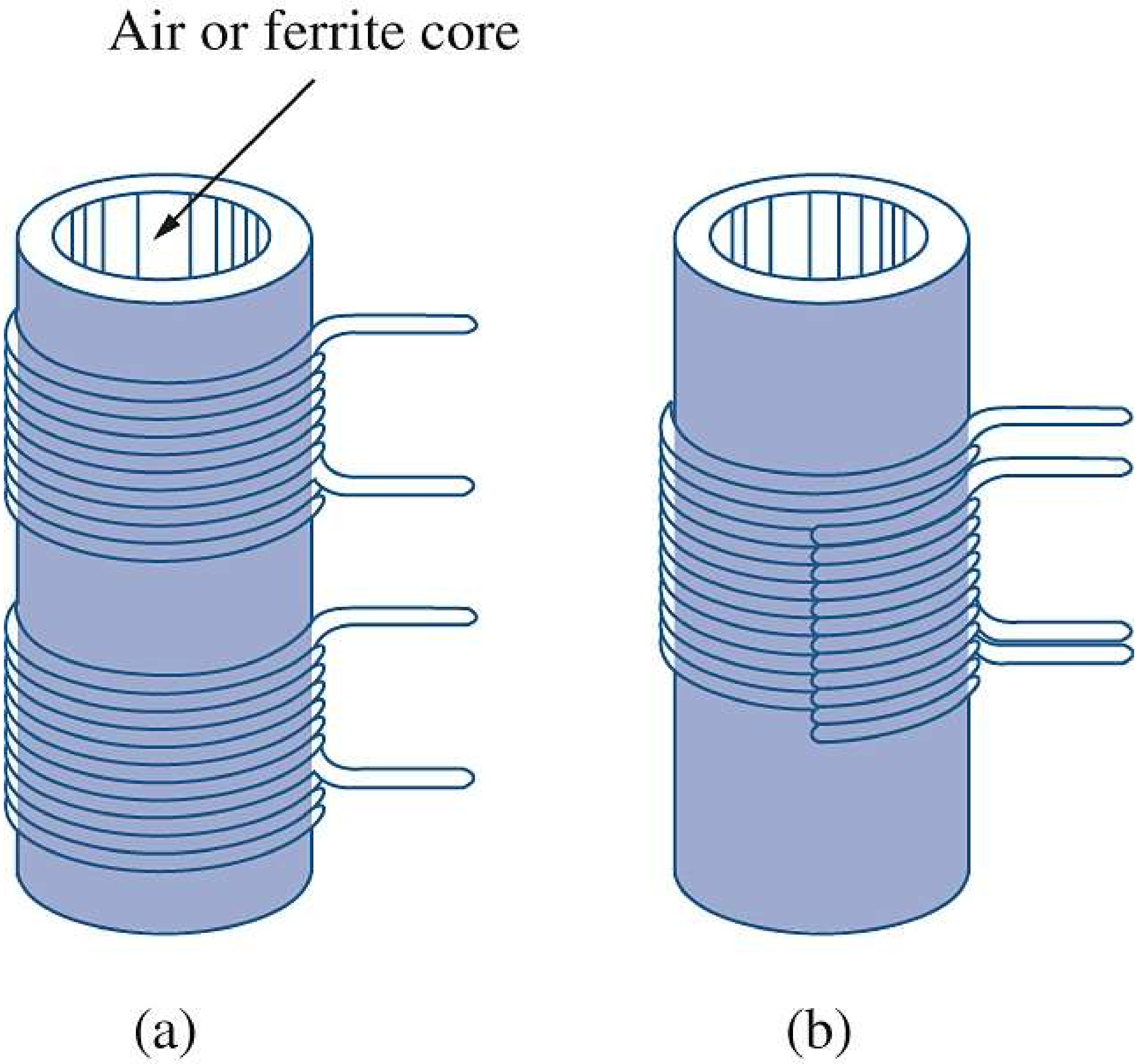
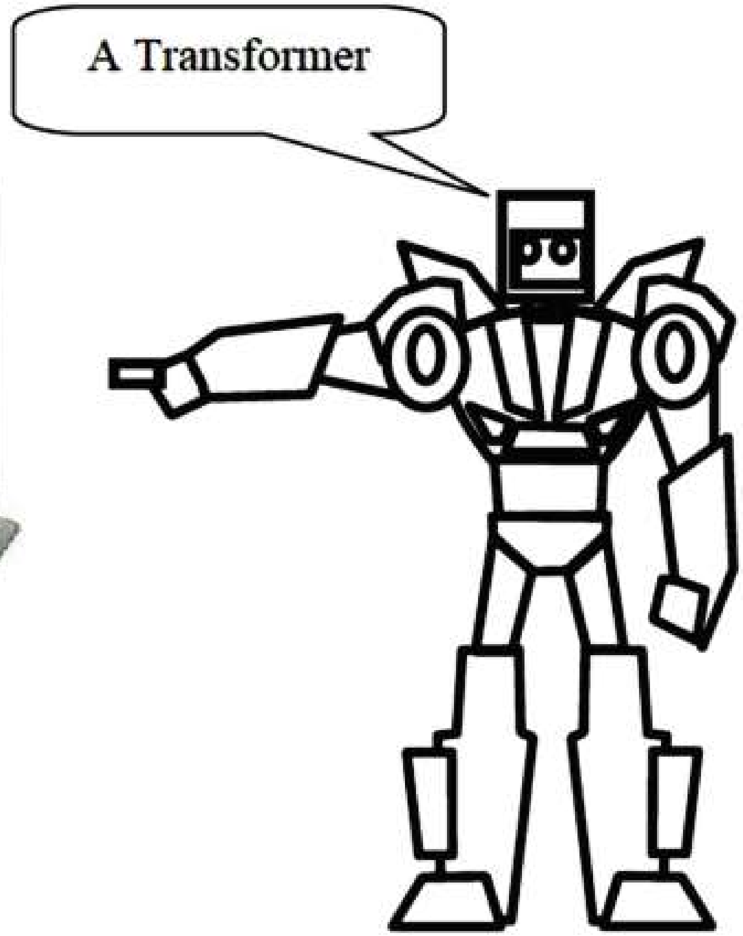
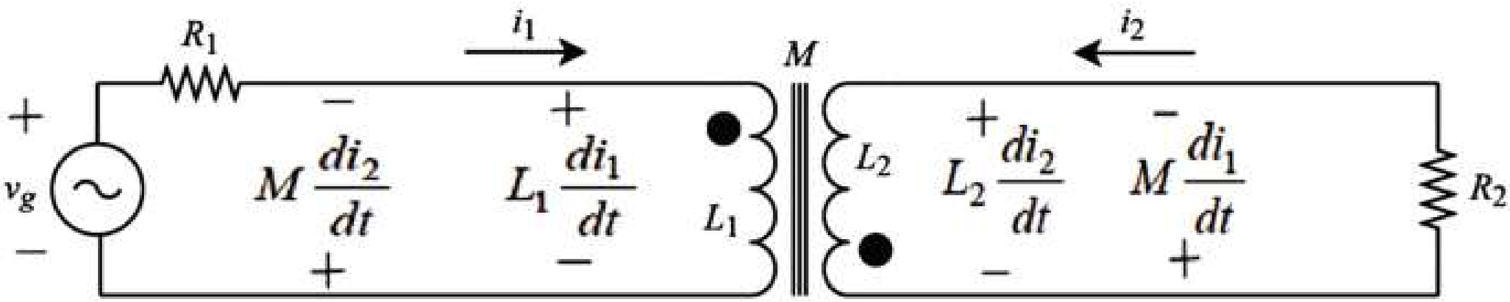


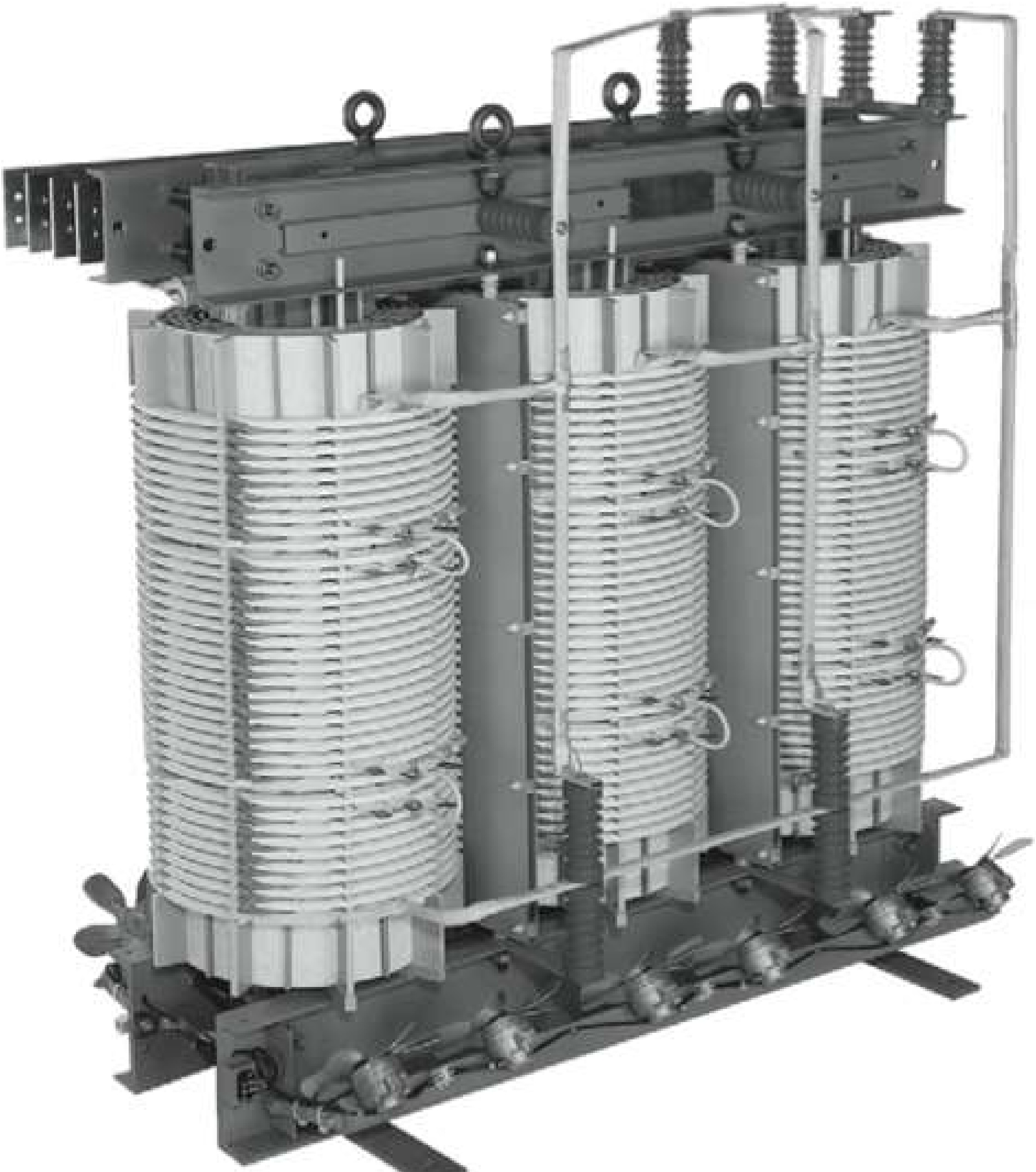
Figure 13.15 Windings: (a) loosely coupled, (b) tightly coupled.

13.4 Linear Transformer

Here we introduce the transformer as a new circuit element. A transformer is generally a four-terminal device comprising two (or more) magnetically coupled coils.


A Transformer


R_1
i_1
i_2
M
v_g
$M\frac{di_2}{dt}$
$L_1\frac{di_1}{dt}$
L_1
L_2
$L_2\frac{di_2}{dt}$
$M\frac{di_1}{dt}$
R_2



As shown in Fig. 13.19, the coil that is directly connected to the voltage source is called the *primary winding*. The coil connected to the load is called the *secondary winding*. The resistances R_1 and R_2 are included to account for the losses (power dissipation) in the coils.

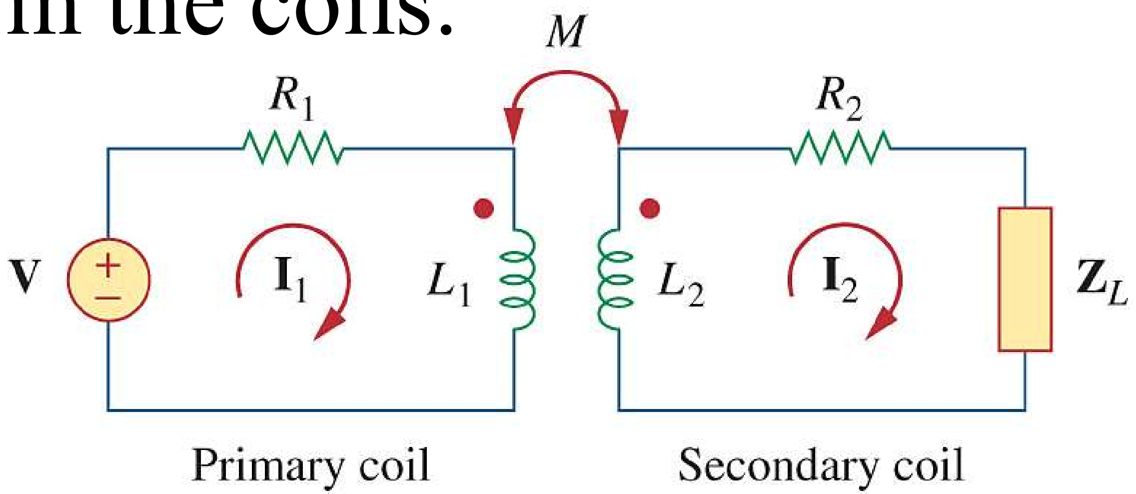


Figure 13.19 A linear transformer.

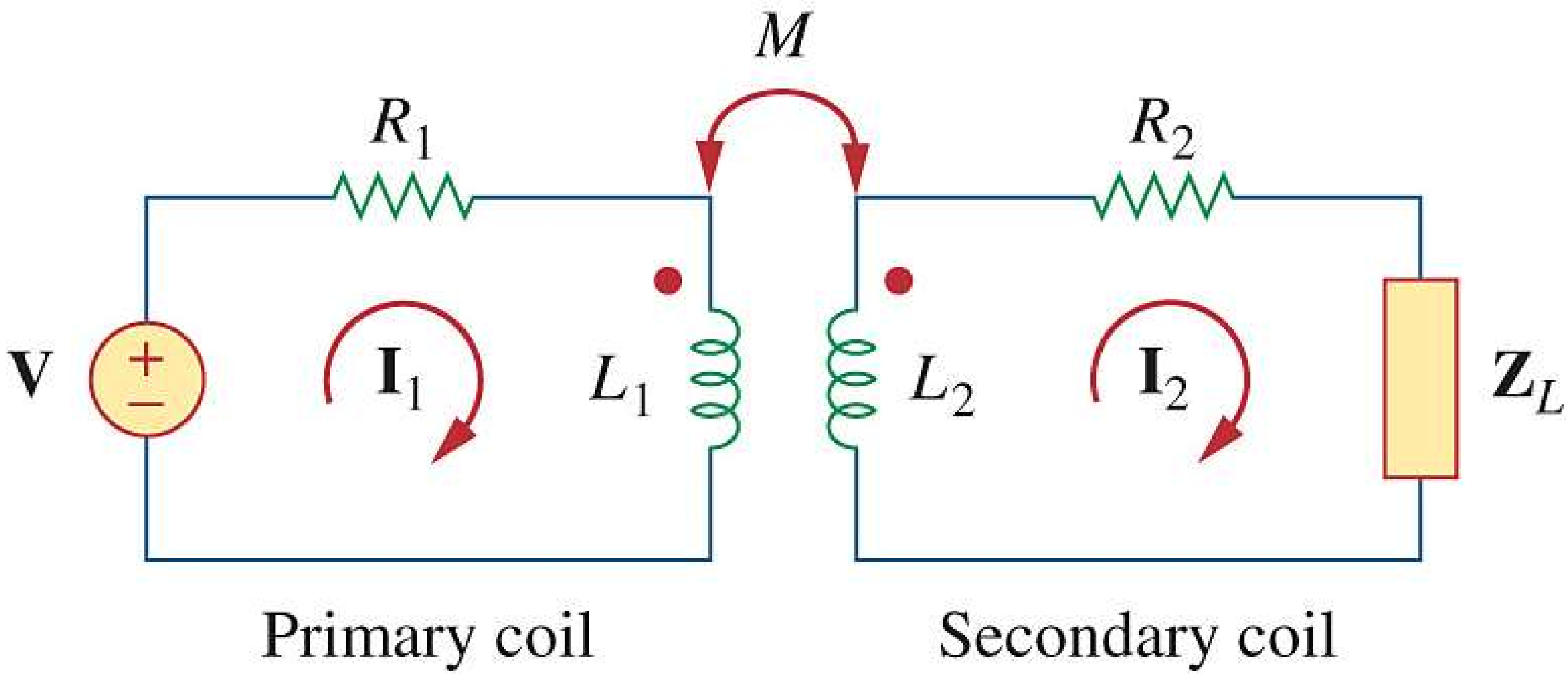


Figure 13.19 A linear transformer.

The transformer is said to be *linear* if the coils are wound on a magnetically linear material – a material for which the magnetic permeability is constant. Such materials include air, plastic, Bakelite, and wood. Linear transformers are sometimes called *air-core transformers*, although not all of them are necessarily air-core.

μ = constant

$i \Rightarrow \psi$ (linear)

For the linear transformer in Fig. 13.19,

$$\begin{cases} v_1 = L_1 \dfrac{di_1}{dt} - M \dfrac{di_2}{dt} \\ v_2 = M \dfrac{di_1}{dt} - L_2 \dfrac{di_2}{dt} \end{cases} \Rightarrow \begin{cases} \tilde{V}_1 = j\omega L_1 \tilde{I}_1 - j\omega M \tilde{I}_2 \\ \tilde{V}_2 = j\omega M \tilde{I}_1 - j\omega L_2 \tilde{I}_2 \end{cases}$$

where v_1 and v_2 denote the primary and secondary voltages, respectively.

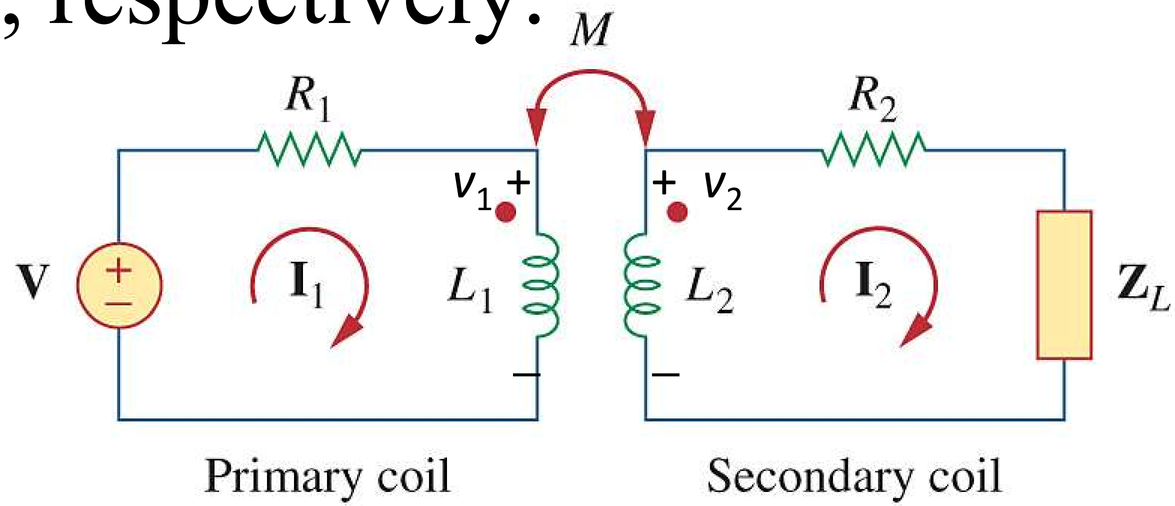


Figure 13.19 A linear transformer.

We would like to obtain the input impedance Z_{in} as seen from the source, because Z_{in} governs the behavior of the primary circuit.

$$Z_{in} = \frac{\tilde{V}}{\tilde{I}_1}$$

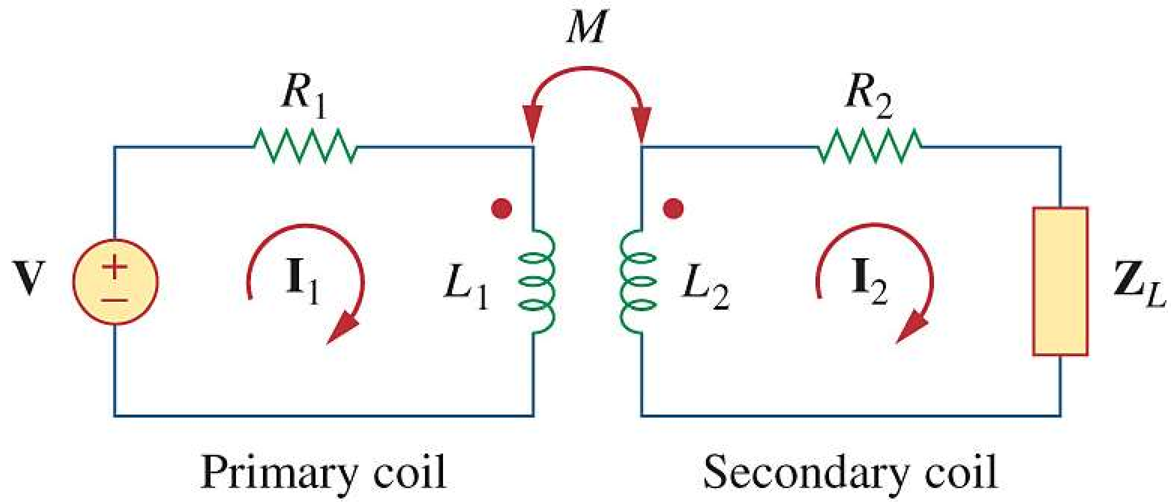


Figure 13.19 A linear transformer.

$$\begin{cases}\tilde{V}=\tilde{I}_1R_1+\tilde{V}_1=\tilde{I}_1R_1+j\omega L_1\tilde{I}_1-j\omega M\tilde{I}_2 \\ \tilde{V}_2=-j\omega L_2\tilde{I}_2+j\omega M\tilde{I}_1=\tilde{I}_2R_2+\tilde{I}_2Z_L\end{cases}$$

Mesh Eq. 1

Mesh Eq. 2

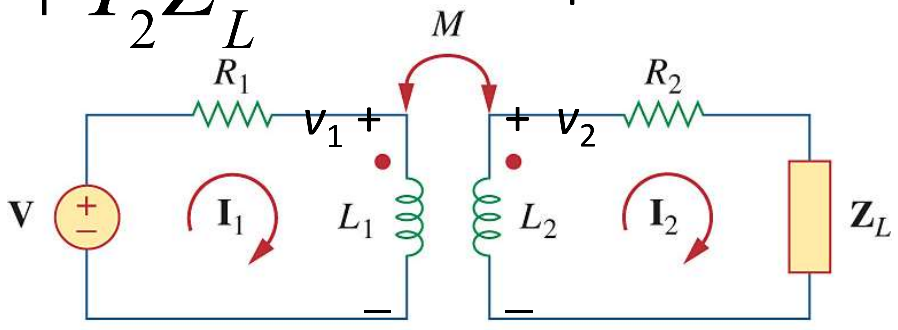


Figure 13.19 A linear transformer.

From Mesh Eq. 2

$$\tilde{I}_2=\frac{j\omega M\tilde{I}_1}{R_2+j\omega L_2+Z_L}$$

$$\tilde{V}=\tilde{I}_1R_1+j\omega L_1\tilde{I}_1-j\omega M\frac{j\omega M\tilde{I}_1}{R_2+j\omega L_2+Z_L}$$

$$Z_{in}=\frac{\tilde{V}}{\tilde{I}_1}=R_1+j\omega L_1+\frac{\omega^2M^2}{R_2+j\omega L_2+Z_L}$$

Notice that Z_{in} comprises two terms. The first term, $R_1 + j\omega L_1$, is the primary impedance. The second term, known as the *reflected impedance* Z_R, is due to the coupling between the primary and secondary windings.

$$Z_R = \frac{\omega^2 M^2}{R_2 + j\omega L_2 + Z_L}$$

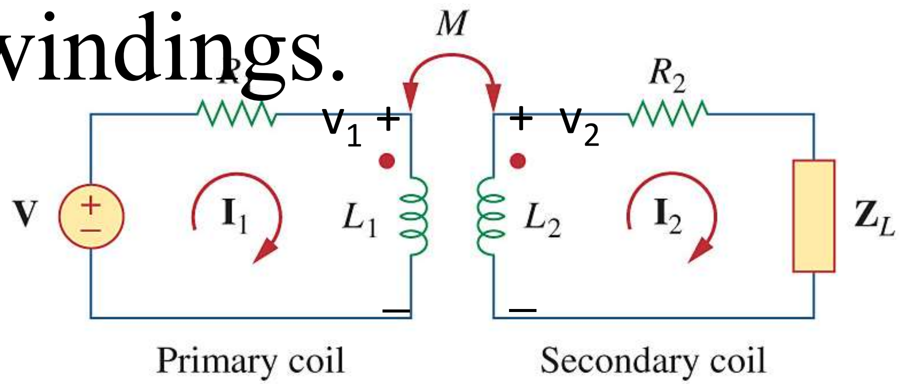


Figure 13.19 A linear transformer.

This equation is not affected by the location of the dots on the transformer. $\boxed{(\pm M)^2 = M^2}$

It is sometimes convenient to replace a magnetically coupled circuit by an equivalent circuit with no magnetic coupling. The linear transformer in Fig. 13. 21 can be replaced by an equivalent T circuit (Fig. 13.22) or Π circuit (Fig. 13.23).

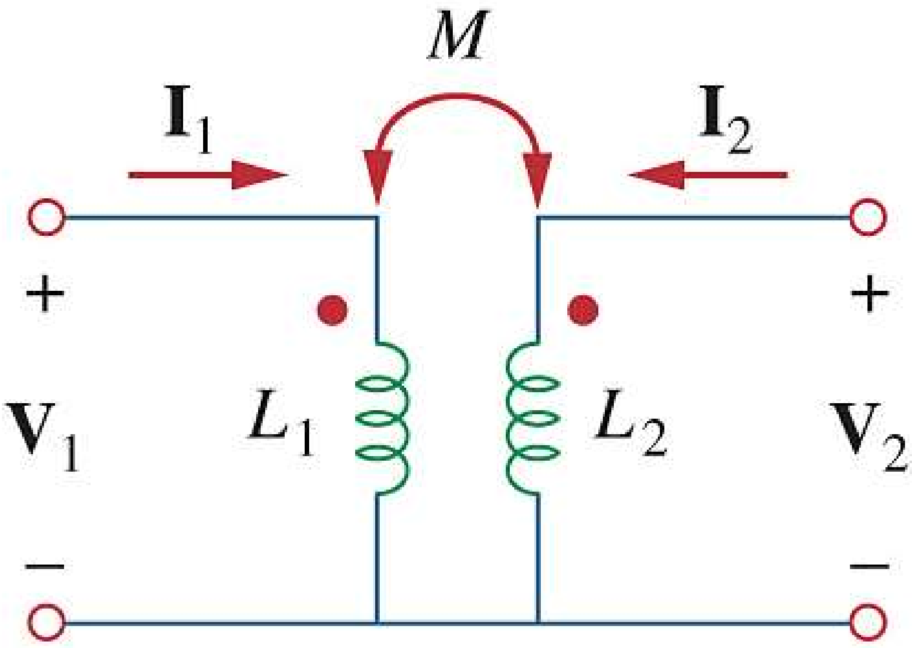


Figure 13.21 Determining the equivalent circuit of a linear transformer.

$$\boxed{\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}}$$

2 KVL equations

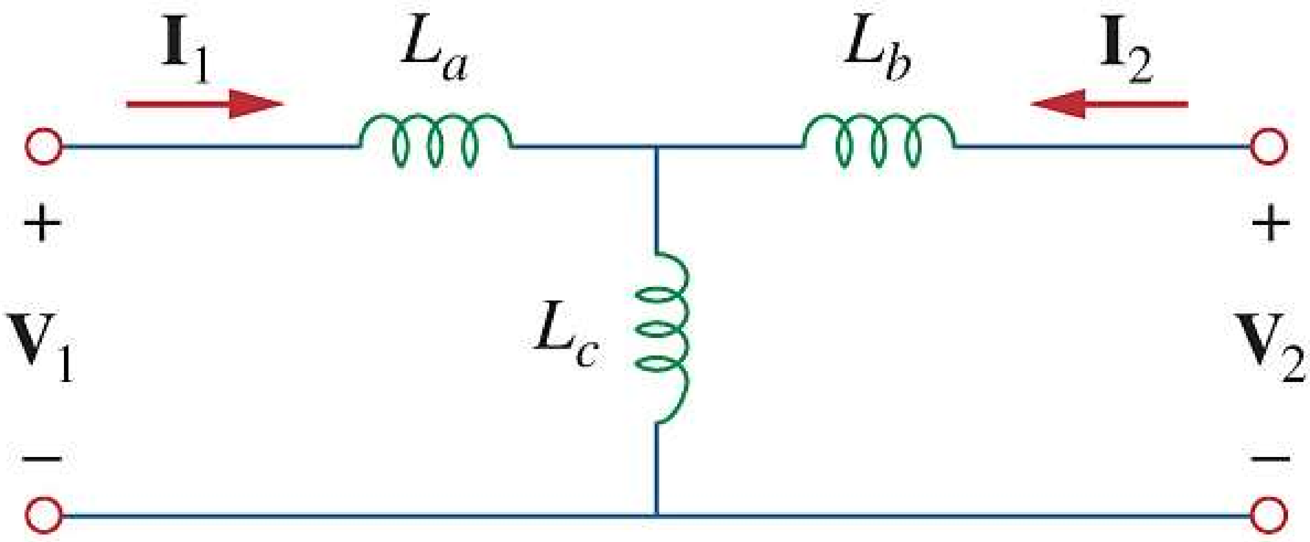


Figure 13.22 An equivalent T circuit.

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega(L_a + L_c) & j\omega L_c \\ j\omega L_c & j\omega(L_b + L_c) \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

2 KVL equations

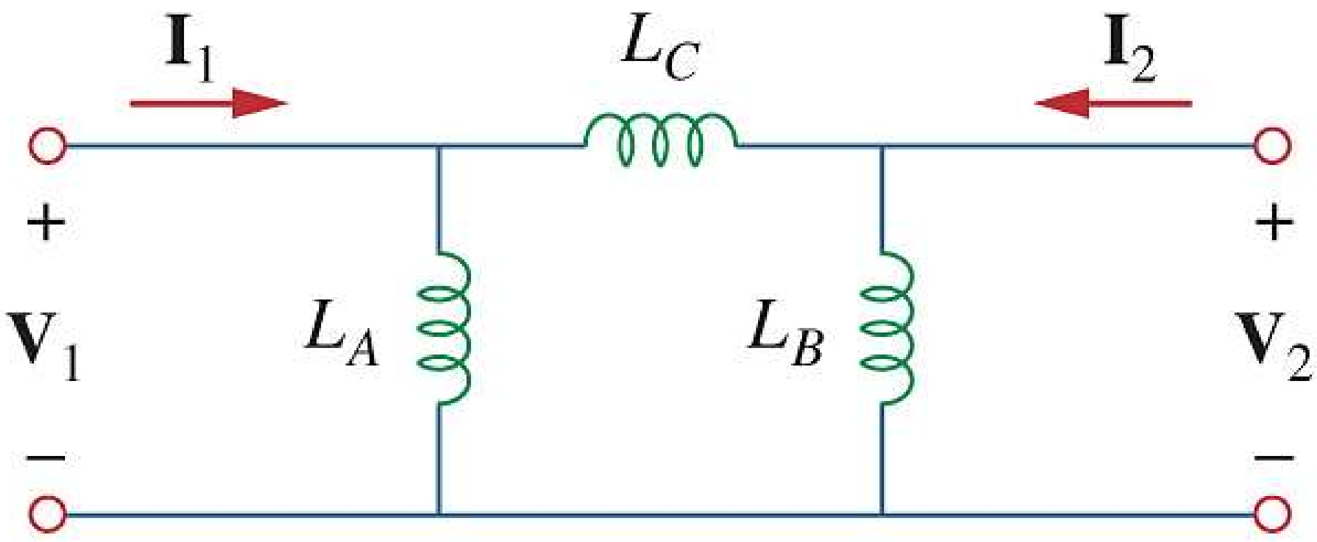


Figure 13.23 An equivalent Π circuit.

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{j\omega L_A} + \dfrac{1}{j\omega L_C} & -\dfrac{1}{j\omega L_C} \\ -\dfrac{1}{j\omega L_C} & \dfrac{1}{j\omega L_B} + \dfrac{1}{j\omega L_C} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

2 KCL equations

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

T $$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega(L_a + L_c) & j\omega L_c \\ j\omega L_c & j\omega(L_b + L_c) \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

Π $$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{j\omega L_A} + \frac{1}{j\omega L_C} & -\frac{1}{j\omega L_C} \\ -\frac{1}{j\omega L_C} & \frac{1}{j\omega L_B} + \frac{1}{j\omega L_C} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

It can be shown that

$$L_a = L_1 - M,\ L_b = L_2 - M,\ L_c = M$$

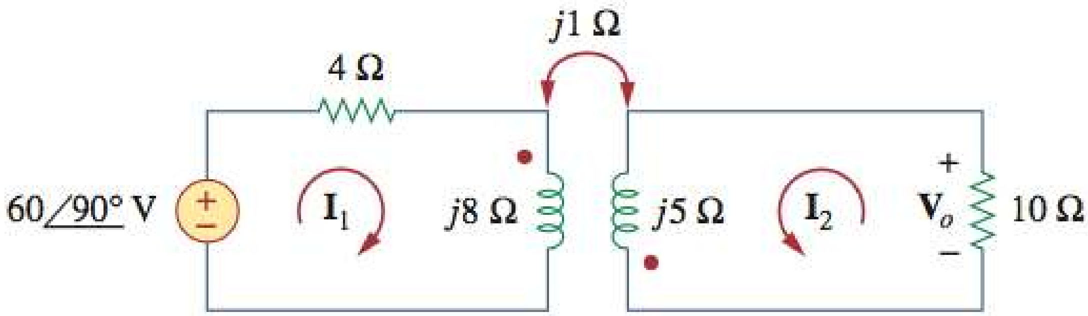
and

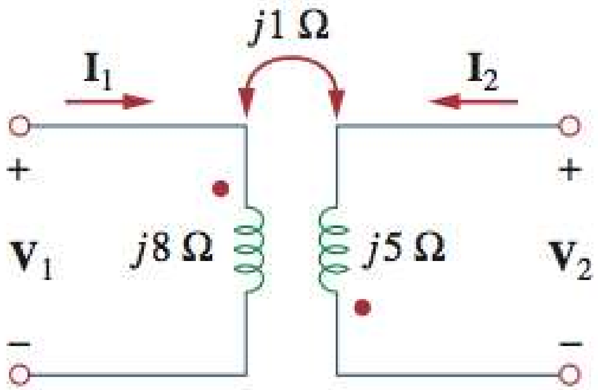
$$L_A = \frac{L_1 L_2 - M^2}{L_2 - M},\ L_B = \frac{L_1 L_2 - M^2}{L_1 - M}$$

$$L_C = \frac{L_1 L_2 - M^2}{M}$$

Example 13.6

Solve for $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{V}_o$ in Fig. 13.27 (the same circuit as for Practice Prob. 13.1) using the T-equivalent circuit for the linear transformer.

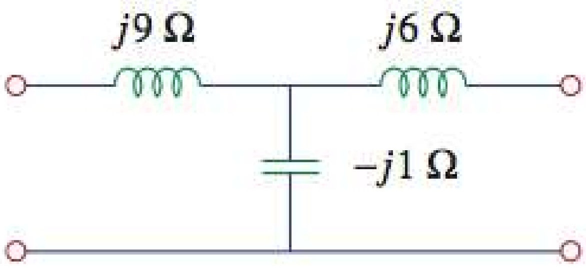




(a)

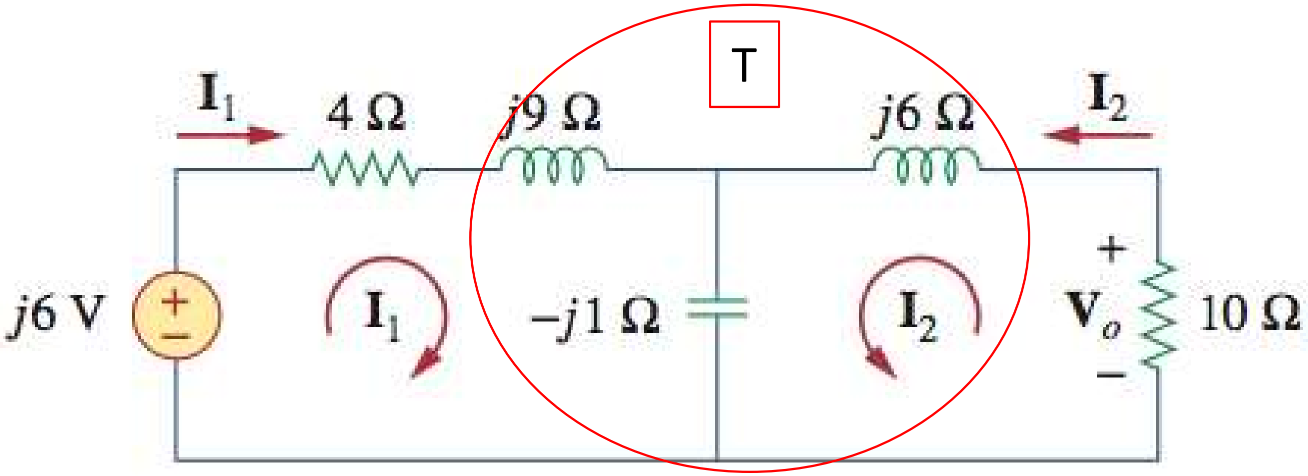
$$L_a = L_1 - (-M) = 8 + 1 = 9\text{ H}$$

$$L_b = L_2 - (-M) = 5 + 1 = 6\text{ H}, \qquad L_c = -M = -1\text{ H}$$



(b)

Dots on different sides ➔ *M* changes to –*M*



Apply mesh analysis, then:

$$j6 = \mathbf{I}_1(4 + j9 - j1) + \mathbf{I}_2(-j1)$$

$$0 = \mathbf{I}_1(-j1) + \mathbf{I}_2(10 + j6 - j1)$$

$$\mathbf{I}_2 = \frac{j6}{100} = j0.06 = 0.06\underline{/90^\circ} \text{ A}$$

$$\mathbf{I}_1 = (5 - j10)j0.06 = 0.6 + j0.3 \text{ A}$$

$$\mathbf{V}_o = -10\mathbf{I}_2 = -j0.6 = 0.6\underline{/-90^\circ} \text{ V}$$

13.5 Ideal Transformers

An ideal transformer is a unity-coupled ($k = 1$), lossless ($R_1 = R_2 = 0$) transformer in which the primary and secondary coils have infinite inductances ($L_1 = \infty$, $L_2 = \infty$, $M = \infty$). Iron-core transformers are close approximations to ideal transformers.

1. $k = 1$
2. $R_1 = R_2 = 0$
3. $L_1, L_2, M \rightarrow \infty$

Figure 13.30(a) shows a typical ideal transformer; the circuit symbol is in Fig. 13.30(b). The vertical lines between the coils indicate an iron core as distinct from the air core used in linear transformers. The primary winding has N_1 turns; the secondary winding has N_2 turns.

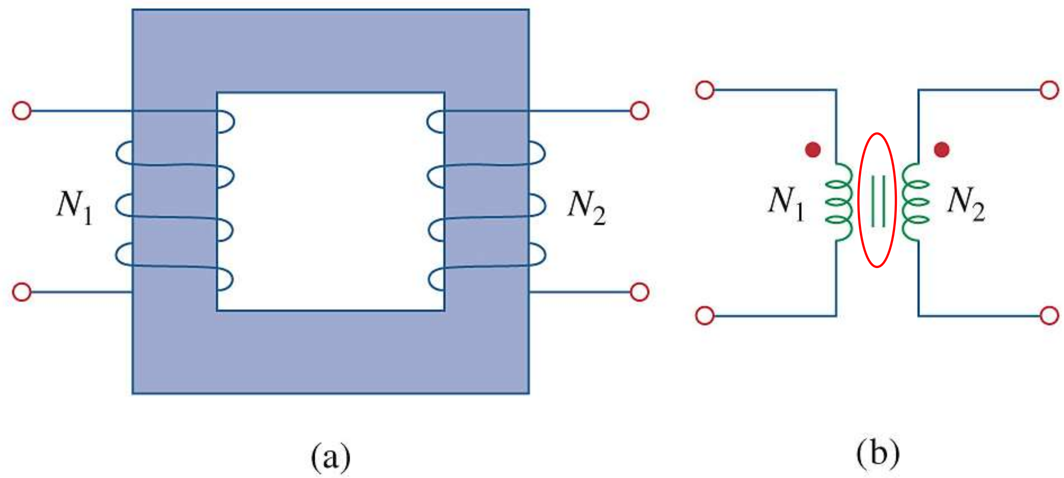


Figure 13.30 (a) Ideal transformer, (b) circuit symbol for ideal transformers.

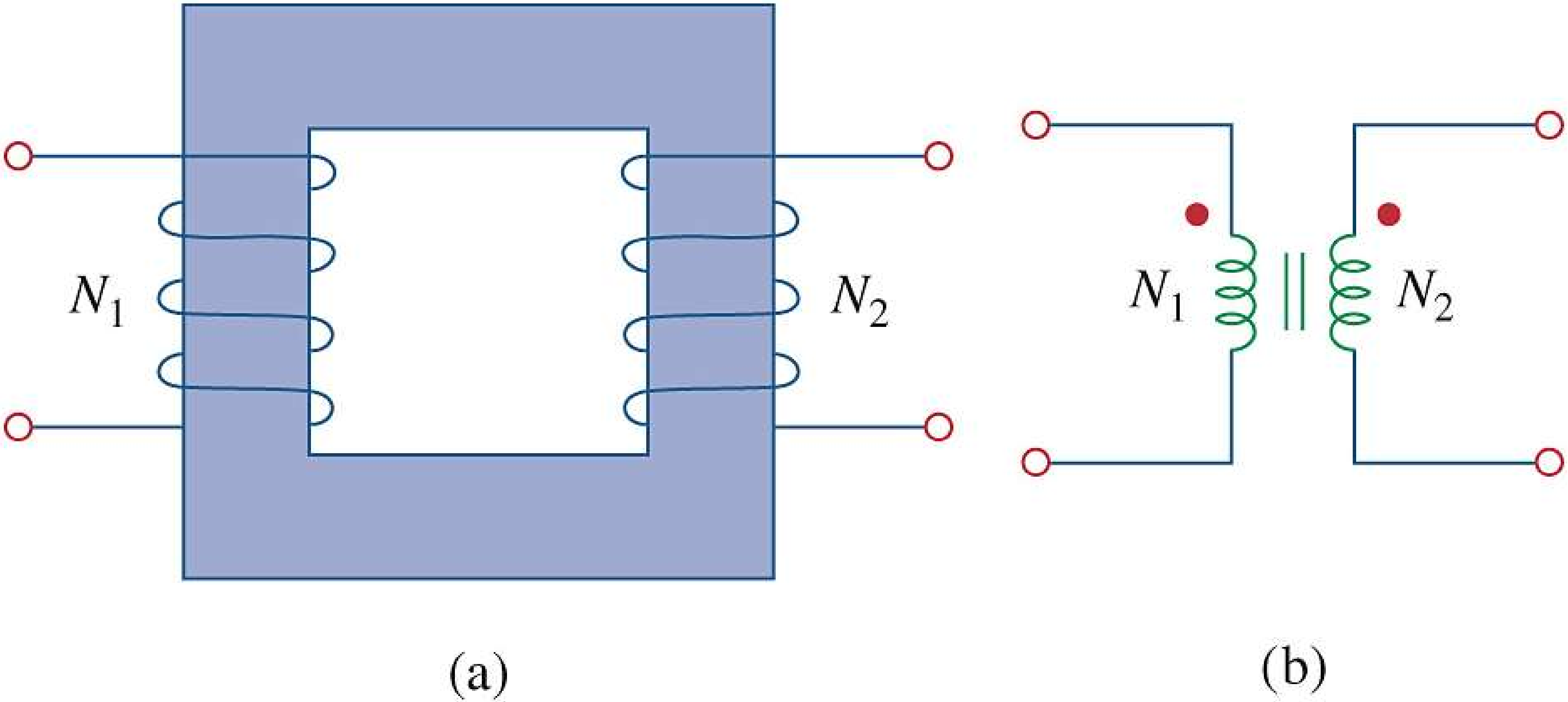


Figure 13.30 (a) Ideal transformer, (b) circuit symbol for ideal transformers.

When a sinusoidal voltage is applied to the primary winding as shown in Fig. 13.31, the same magnetic flux ϕ goes through both windings. According to Faraday's law,

$$v_1 = N_1 \frac{d\phi}{dt}$$

$$v_2 = N_2 \frac{d\phi}{dt}$$

- Ideal transformer:
  Iron coils enable the same ϕ
- Non-ideal: $\phi_1 = \phi_{11} + \phi_{12}$
  A portion of magnetic flux (ϕ_{12}) is coupled.

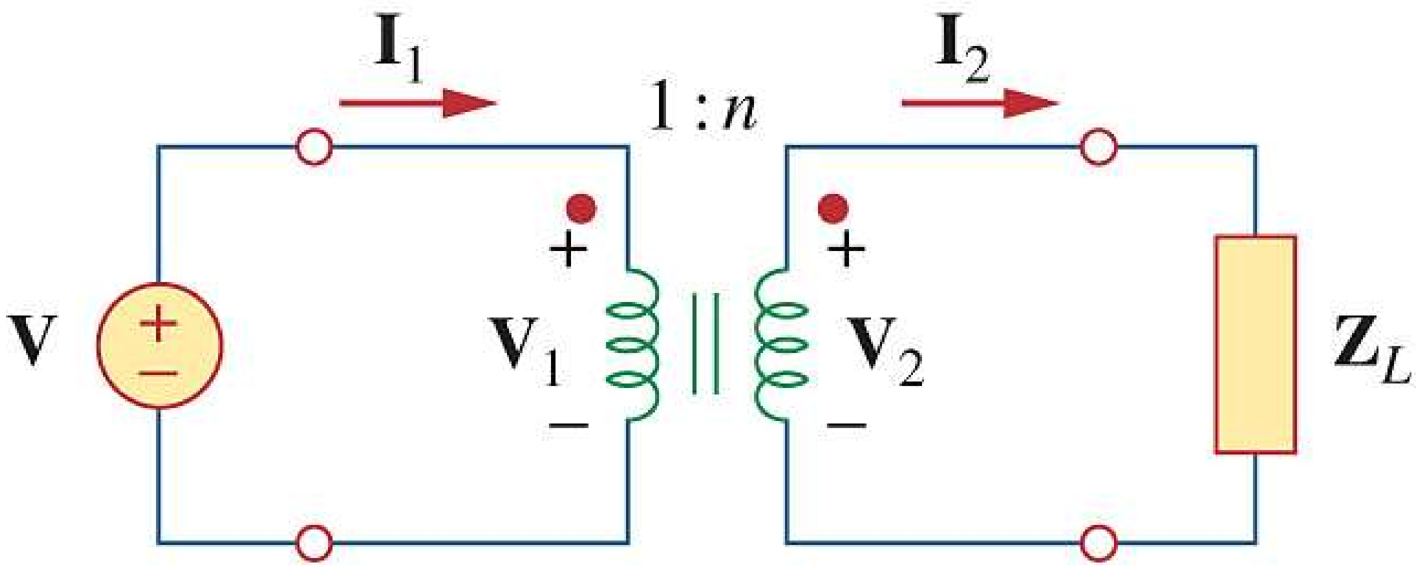


Figure 13.31 Relating primary and secondary quantities in an ideal transformer.

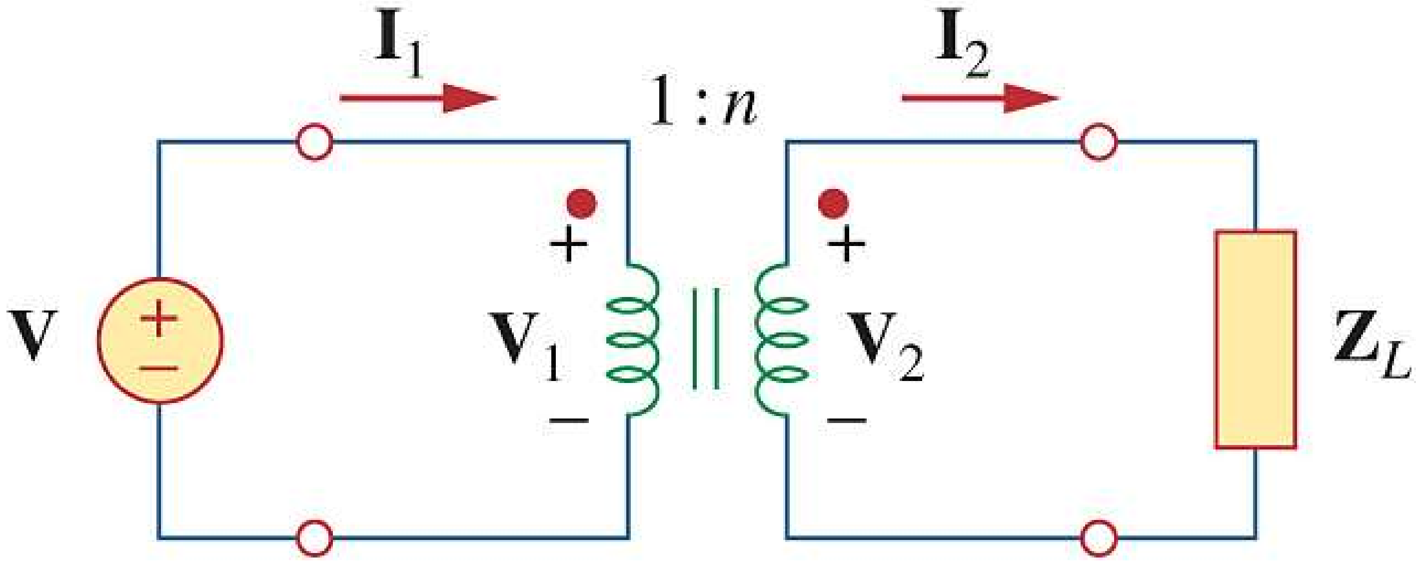


Figure 13.31 Relating primary and secondary quantities in an ideal transformer.

We have

$$\frac{v_2}{v_1} = \frac{N_2}{N_1} = n$$

where n is the *turns ratio* or *transformation ratio*. Using the phasor voltages,

$$\frac{\tilde{V}_2}{\tilde{V}_1} = \frac{N_2}{N_1} = n$$

For the reason of power conservation,

$$v_1 i_1 = v_2 i_2$$

$$\frac{i_1}{i_2} = \frac{v_2}{v_1} = n$$

$$\frac{\tilde{I}_1}{\tilde{I}_2} = \frac{\tilde{V}_2}{\tilde{V}_1} = n$$

If $n > 1$, we have a *step-up transformer*, as $V_2 > V_1$. On the other hand, if $n < 1$, the transformer is a *step-down transformer*, since $V_2 < V_1$. The ratings of transformers are usually specified as V_1 / V_2.

e.g., 9V/3V

When $n = 1$, the transformer is generally called an *isolation transformer* (Section 13.9).

Section 13.9 not included

If the polarity of $\tilde{V}_1$ or $\tilde{V}_2$ or the direction of $\tilde{I}_1$ or $\tilde{I}_2$ is changed, n in the above equations may need to be replaced by $-n$.

Two simple rules to follow are:

1. If $\tilde{V}_1$ and $\tilde{V}_2$ are *both* positive or negative at the dotted terminals, $\tilde{V}_2 / \tilde{V}_1 = n$. Otherwise, $\tilde{V}_2 / \tilde{V}_1 = -n$.

2. If $\tilde{I}_1$ and $\tilde{I}_2$ *both* enter into or both leave the dotted terminals, $\tilde{I}_1 / \tilde{I}_2 = -n$. Otherwise, $\tilde{I}_1 / \tilde{I}_2 = n$.

The rules are demonstrated with the four circuits in Fig. 13.32.

For *V*, same polarity at dotted terminals ➔ $+n$
For *I*, same flowing direction (entering/leaving) to the dotted terminals ➔ $-n$

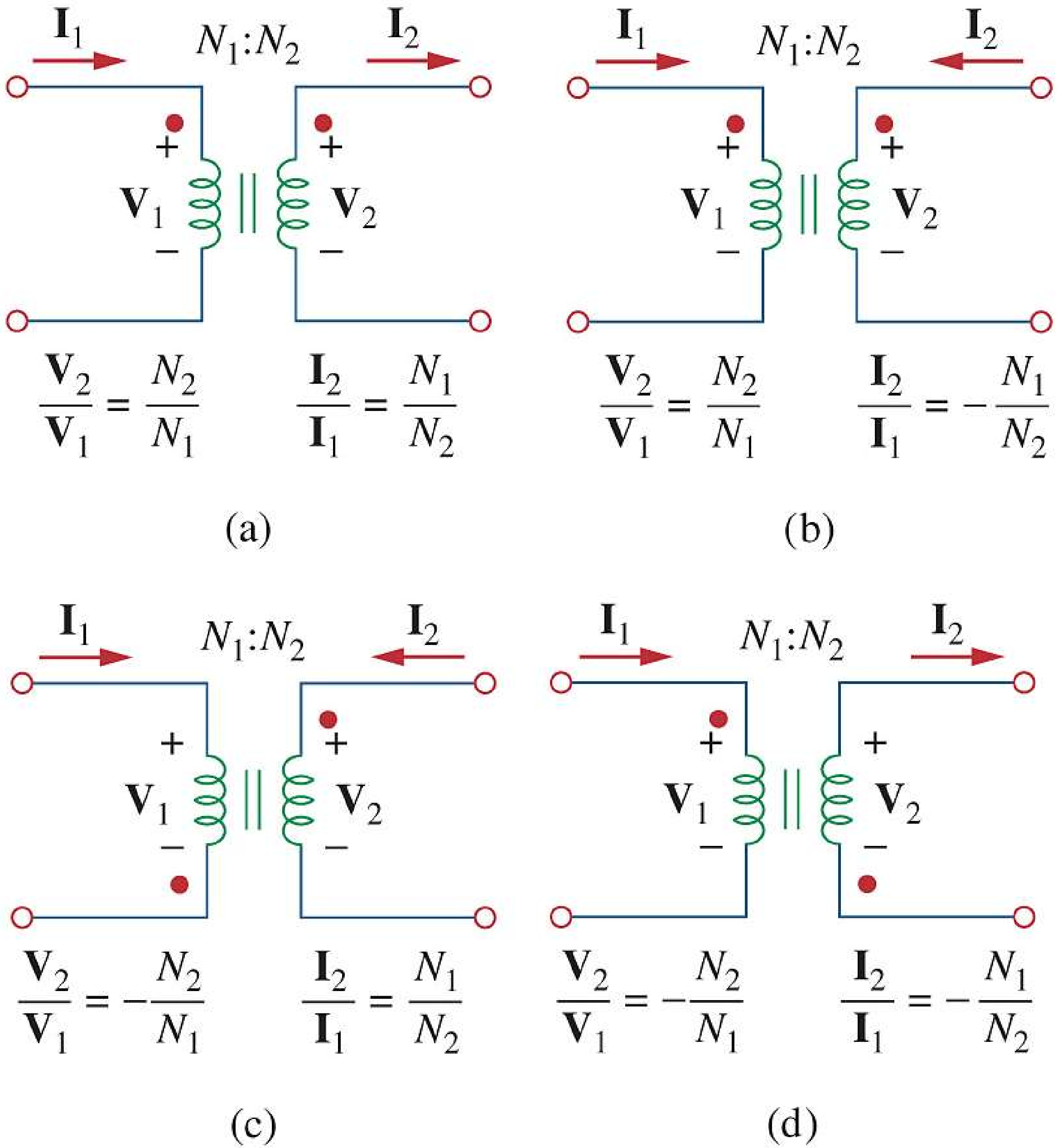
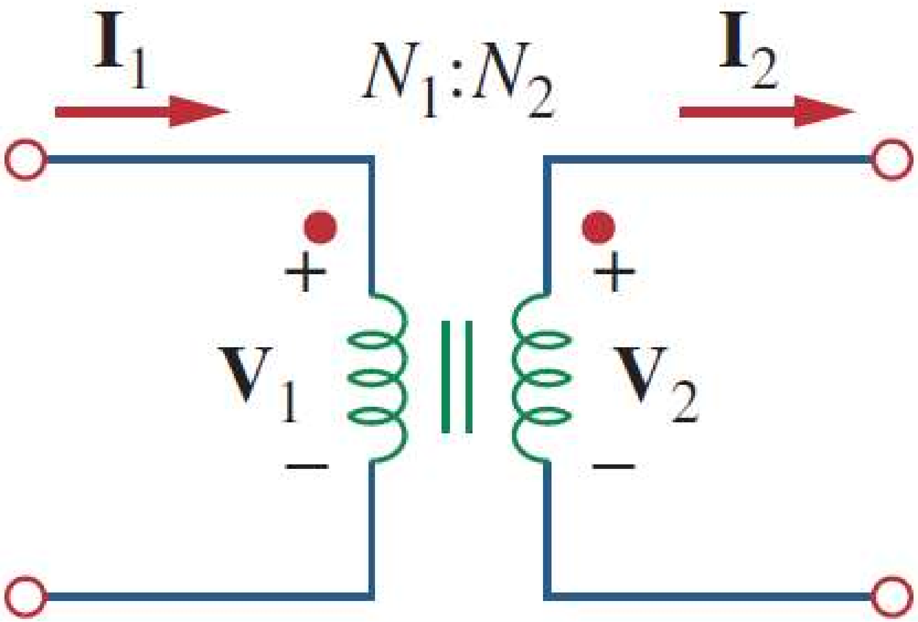


Figure 13.32 Typical circuits illustrating proper voltage polarities and circuit directions in an ideal transformer.

For *V*, same polarity at dotted terminals ➔ +*n*
For *I*, same flowing direction (entering/leaving) the dotted terminals ➔ –*n*

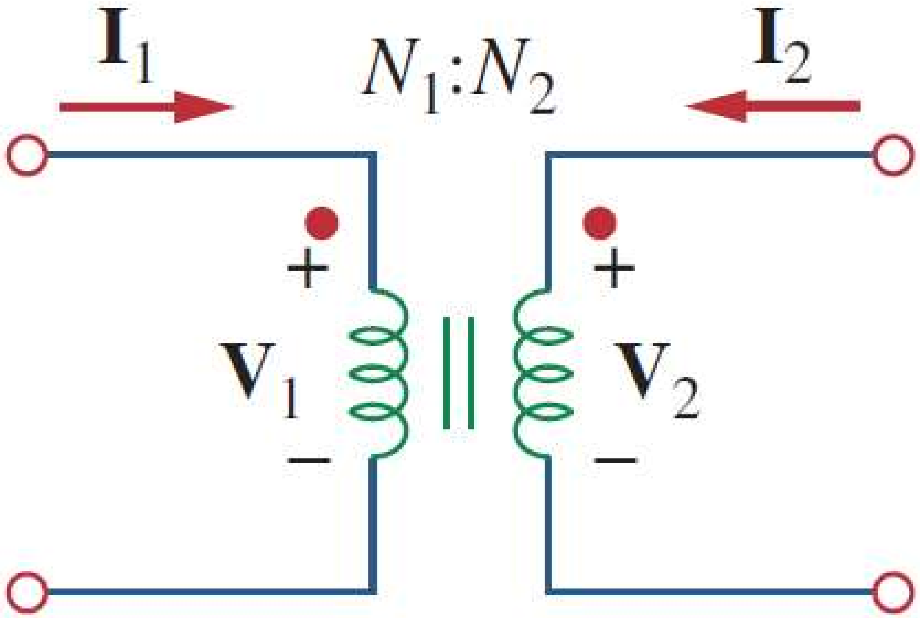


$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{N_2}{N_1} \qquad \frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{N_1}{N_2}$$

(a)

Method 2:
For V, by dotted convention
For I, by power conservation

$V_1 I_1 + (-V_2 I_2) = 0$

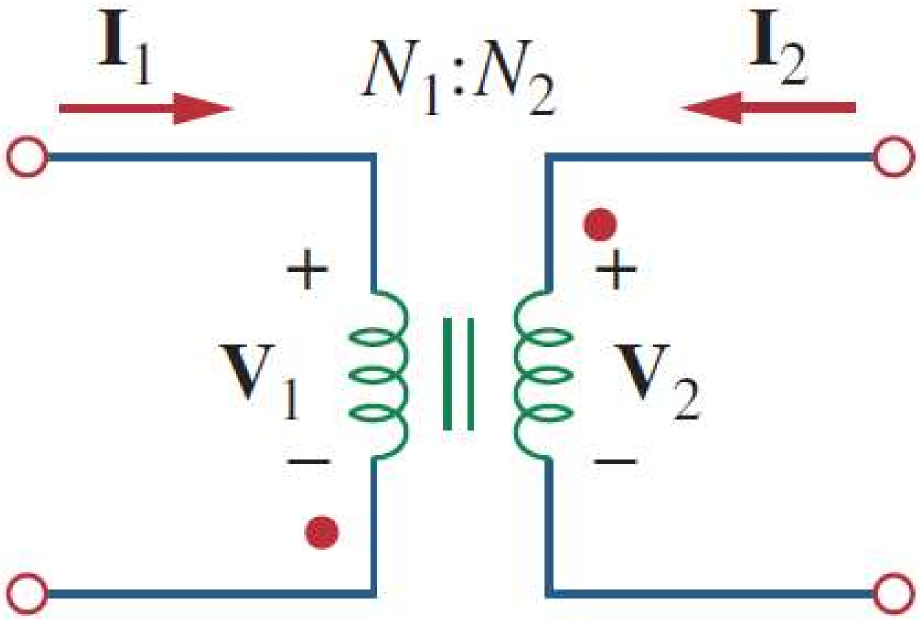


$$\frac{V_2}{V_1} = \frac{N_2}{N_1} \qquad \frac{I_2}{I_1} = -\frac{N_1}{N_2}$$

(b)

Method 2:
For *V*, by dotted convention
For *I*, by power conservation

$V_1 I_1 + V_2 I_2 = 0$

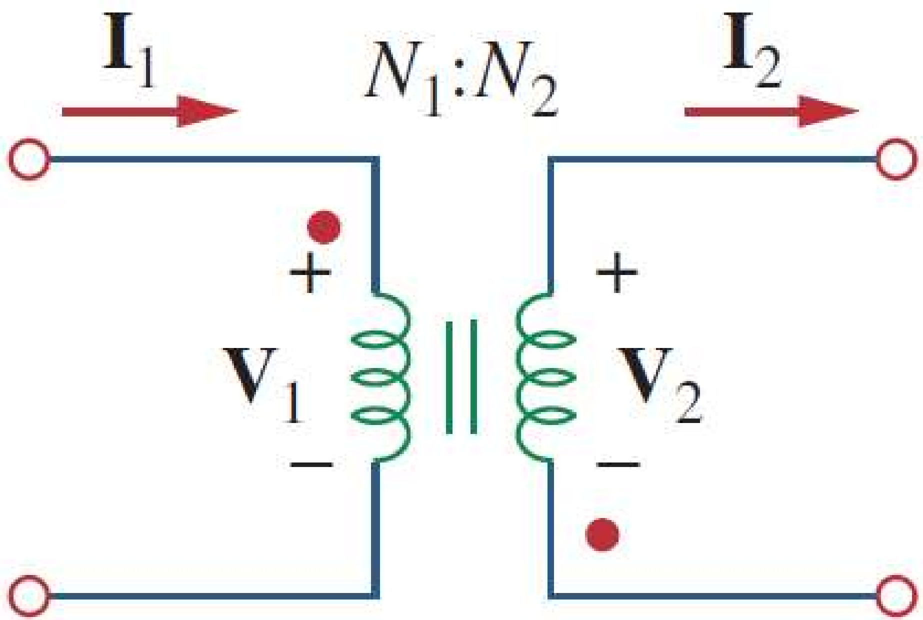


$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = -\frac{N_2}{N_1} \qquad \frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{N_1}{N_2}$$

(c)

Method 2:
For *V*, by dotted convention
For *I*, by power conservation

$V_1 I_1 + V_2 I_2 = 0$



$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = -\frac{N_2}{N_1} \qquad \frac{\mathbf{I}_2}{\mathbf{I}_1} = -\frac{N_1}{N_2}$$

(d)

Method 2:
For V, by dotted convention
For I, by power conservation

$V_1 I_1 + (-V_2 I_2) = 0$

The complex power in the primary winding is equal to the complex power in the secondary winding:

$$S_1 = \tilde{V}_1 \tilde{I}_1^* = \frac{\tilde{V}_2}{n}\left(n\tilde{I}_2\right)^* = \tilde{V}_2 \tilde{I}_2^* = S_2$$

showing that the ideal transformer absorbs no power.

Ideal transformer:
$R_1 = R_2 = 0$

The input impedance as seen by the source is

$$Z_{in} = \frac{\tilde{V}_1}{\tilde{I}_1} = \frac{\tilde{V}_2 / n}{n\tilde{I}_2} = \frac{1}{n^2}\frac{\tilde{V}_2}{\tilde{I}_2} = \frac{1}{n^2} Z_L$$

It is also called the *reflected impedance*.

This ability of the transformer provides us a means of *impedance matching* to ensure maximum power transfer (Section 13.9).

Section 13.9 not included

Recall: $Z_L = Z_{Th}{}^*$

In analyzing a circuit containing an ideal transformer, it is common practice to eliminate the transformer by reflecting impedances and sources from one side of the transformer to the other. Figures 13.35 and 13.36 show the equivalent circuits for Fig. 13.33.

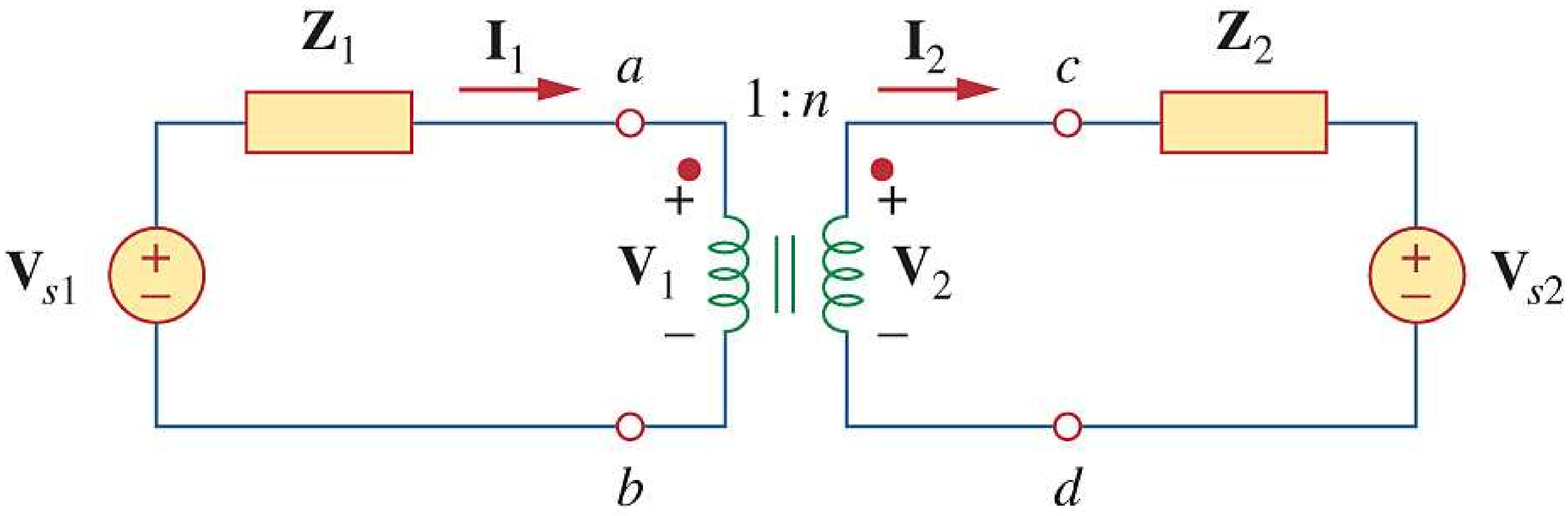


Figure 13.33 Ideal transformer circuit whose equivalent circuits are to be found.

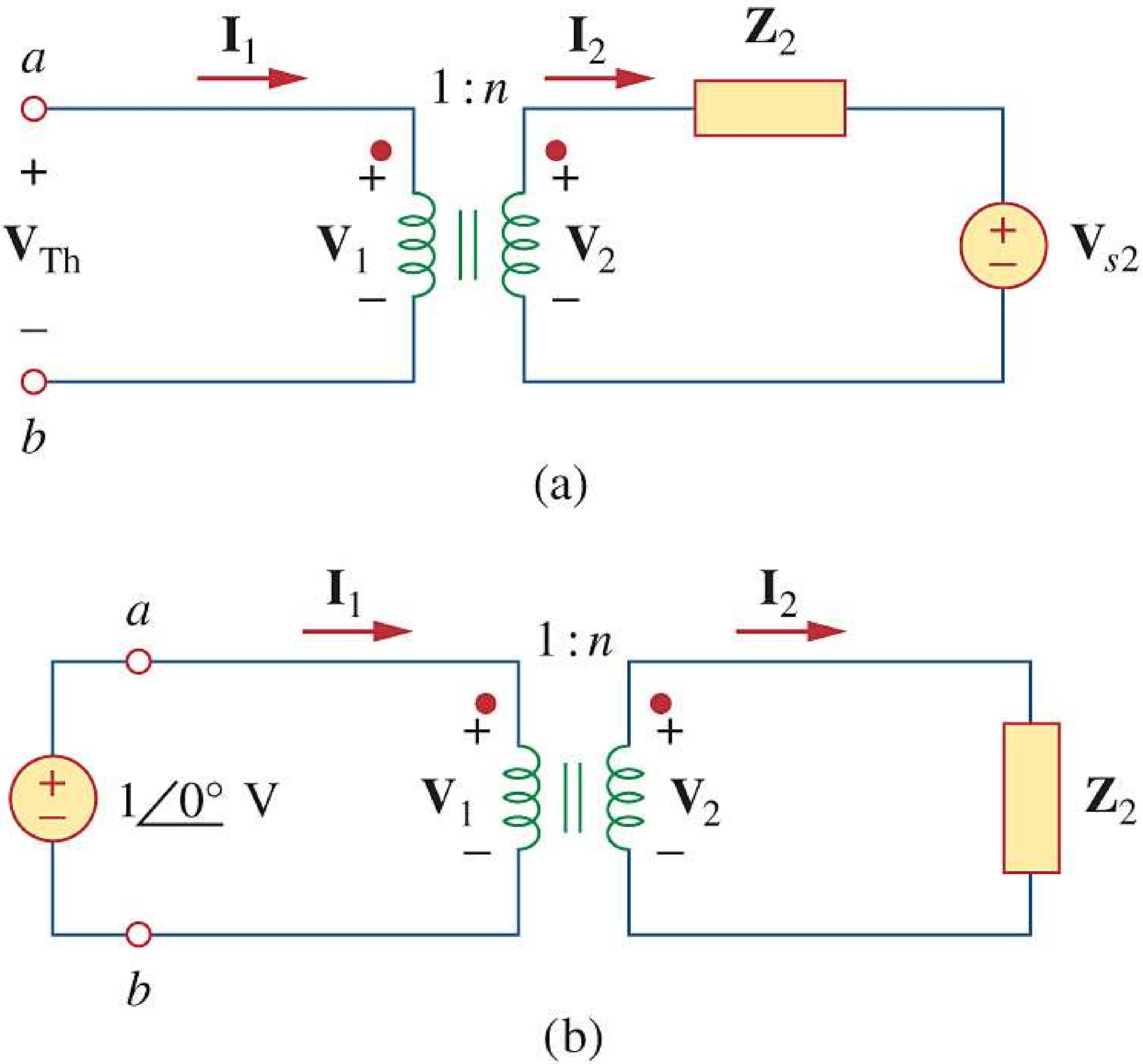


Figure 13.34 Obtaining the Thevenin equivalent for the circuit in Fig. 13.33.

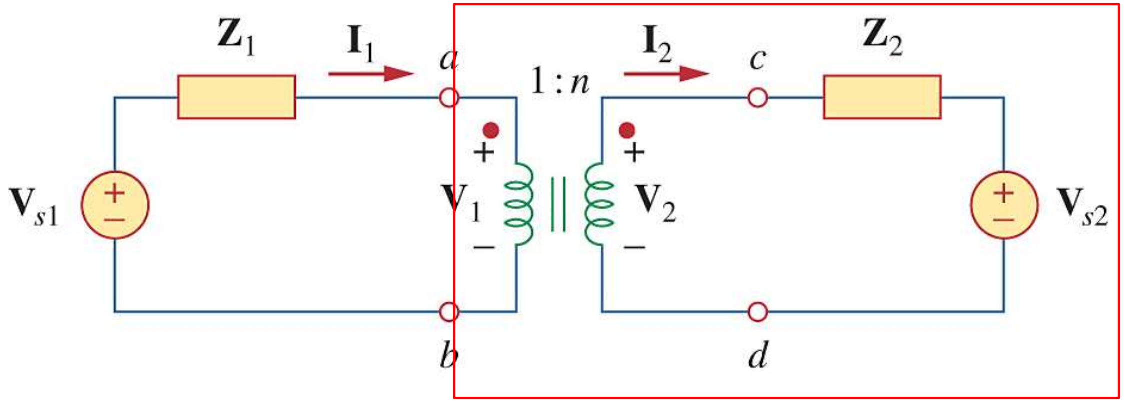


Figure 13.33 Ideal transformer circuit whose equivalent circuits are to be found.

Replace the circuit to the right of a-*b* with its Thevenin equivalent

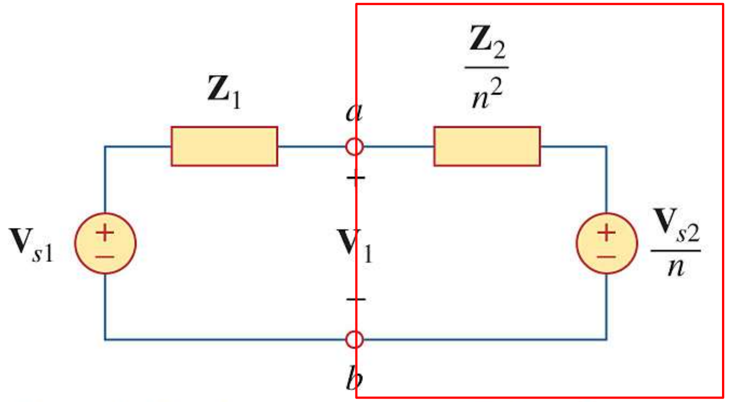


Figure 13.35 Equivalent circuit for Fig. 13.33 obtained by reflecting the secondary circuit to the primary side.

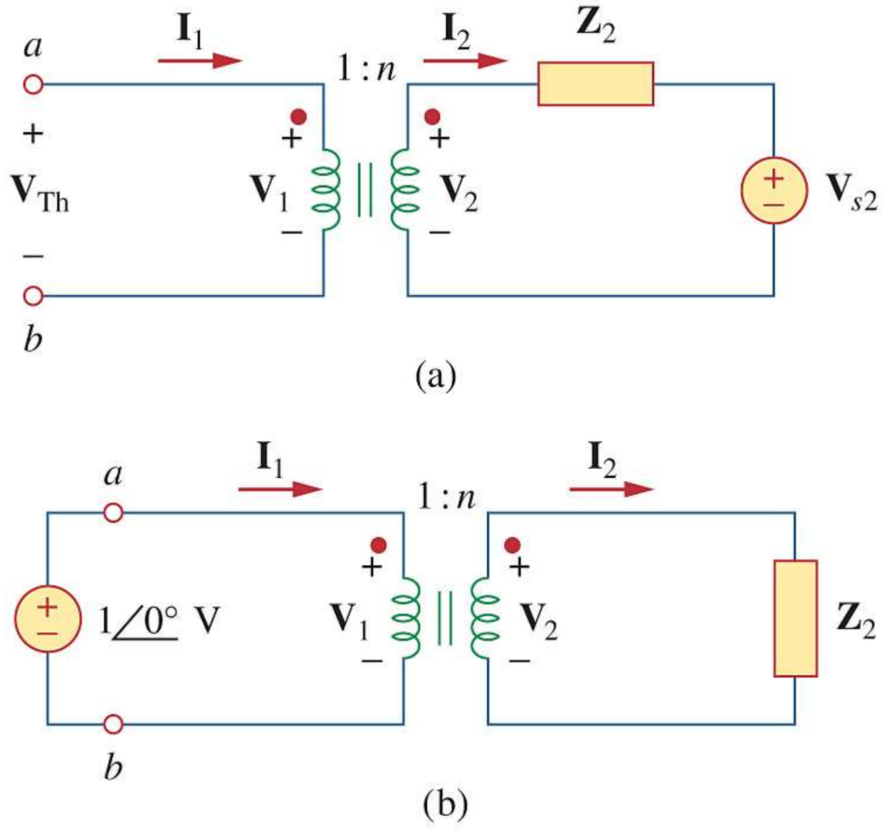


Figure 13.34 Obtaining the Thevenin equivalent for the circuit in Fig. 13.33.

$I_1 = 0 = I_2$

$V_{Th} = V_{oc}$ $$\mathbf{V}_{\text{Th}} = \mathbf{V}_1 = \frac{\mathbf{V}_2}{n} = \frac{\mathbf{V}_{s2}}{n} \quad (13.61)$$

Z_{Th}: turn off the source V_{s2} $$\mathbf{Z}_{\text{Th}} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{\mathbf{V}_2/n}{n\mathbf{I}_2} = \frac{\mathbf{Z}_2}{n^2}, \qquad \mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}_2 \quad (13.62)$$

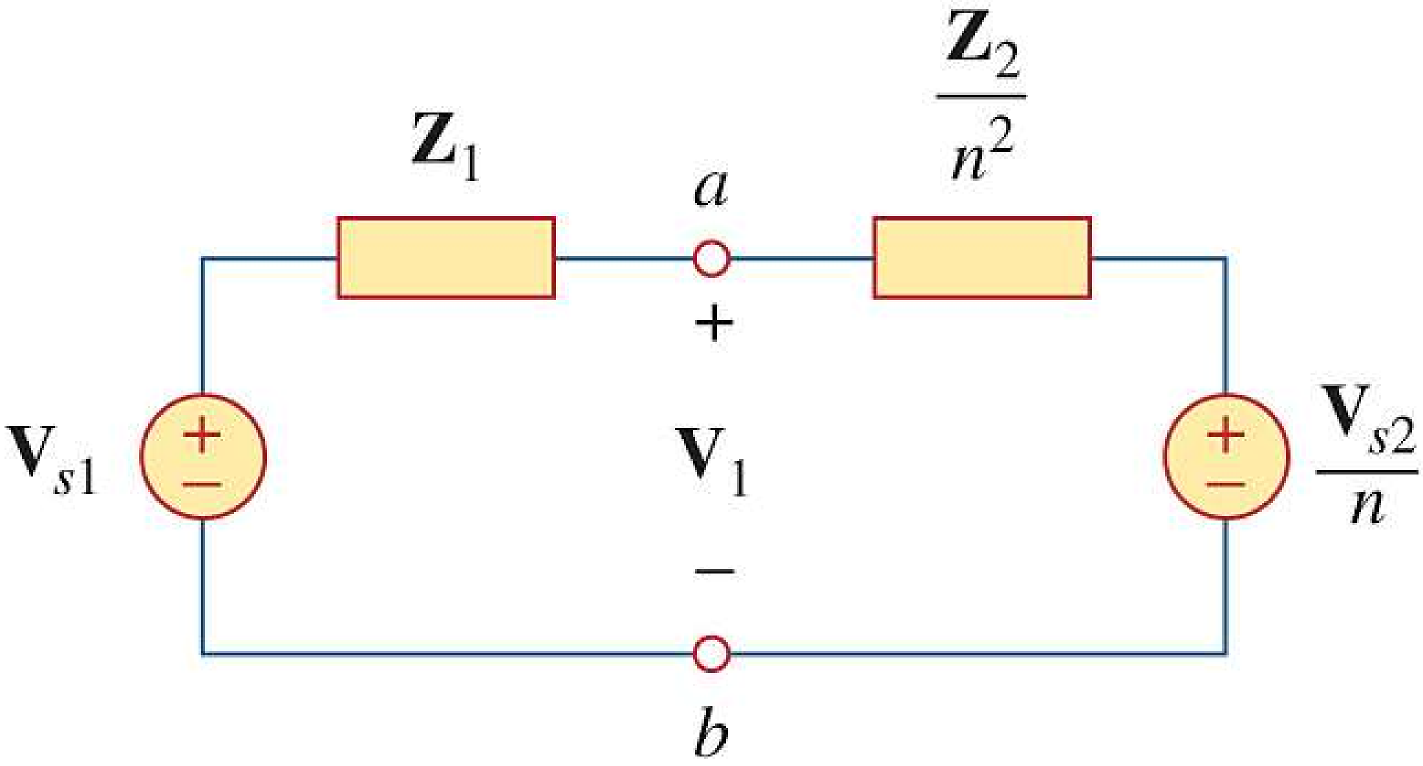


Figure 13.35 Equivalent circuit for Fig. 13.33 obtained by reflecting the secondary circuit to the primary side.

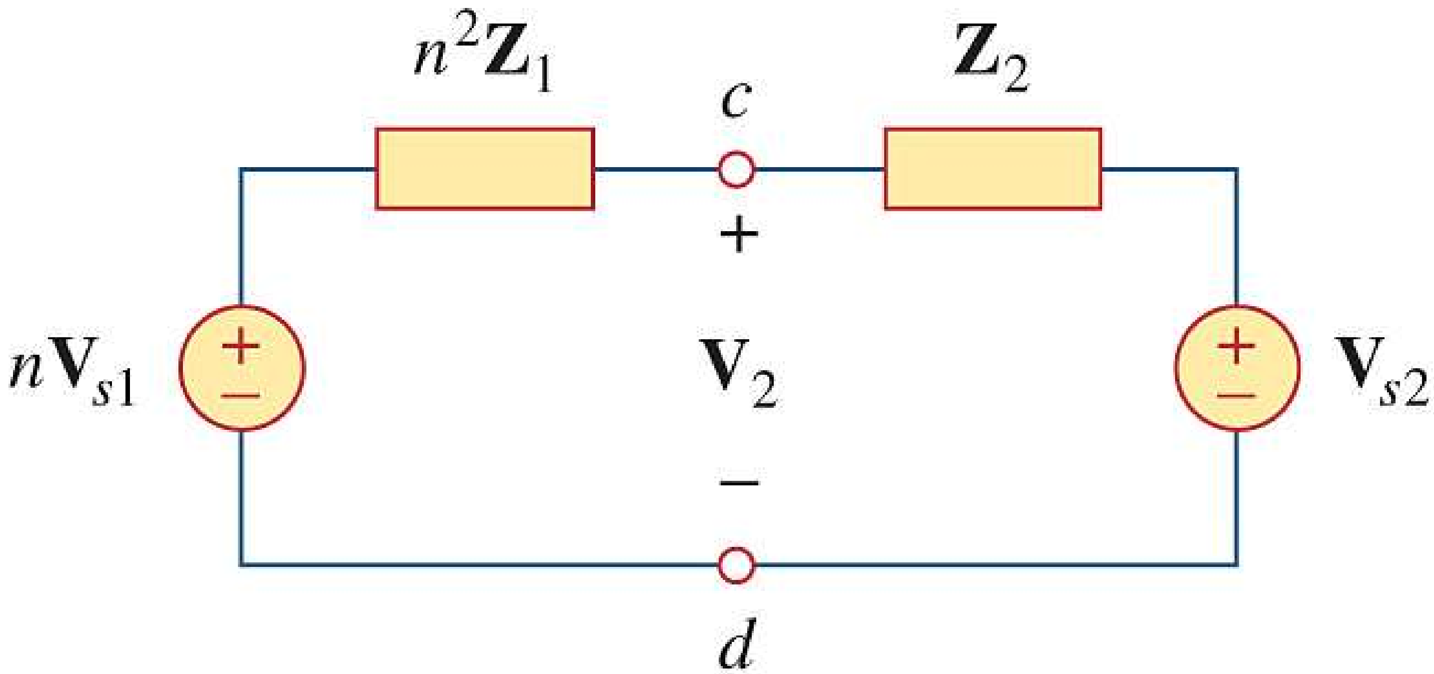


Figure 13.36 Equivalent circuit for Fig. 13.33 obtained by reflecting the primary circuit to the secondary side.

The general rule for eliminating the transformer by reflecting the secondary circuit to the primary side is: divide the secondary impedance by n^2, divide the secondary voltage by n, and multiply the secondary current by n.

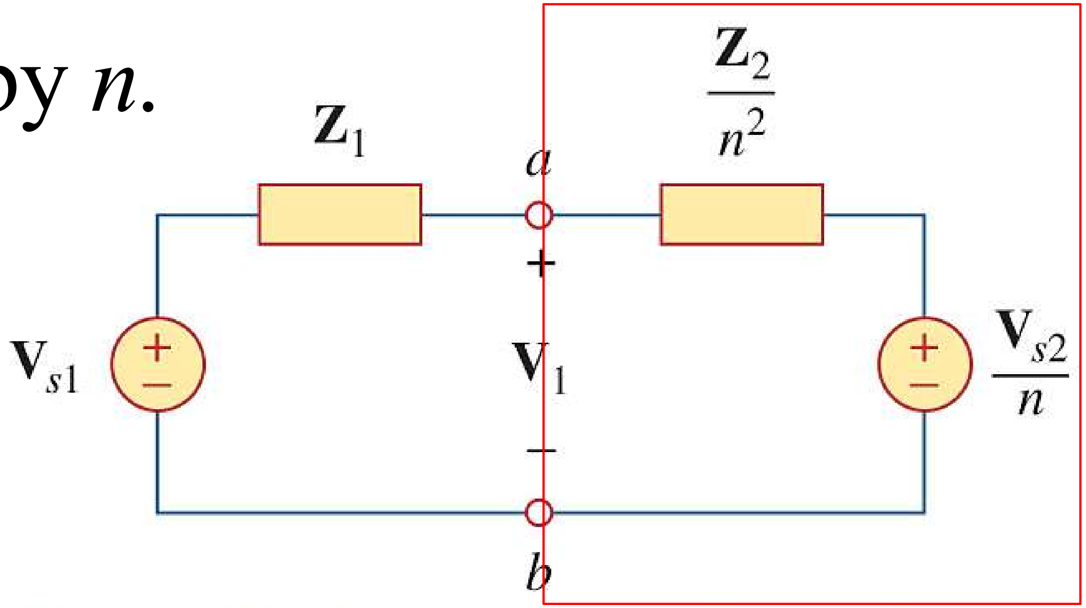


Figure 13.35 Equivalent circuit for Fig. 13.33 obtained by reflecting the secondary circuit to the primary side.

The general rule for eliminating the transformer by reflecting the primary circuit to the secondary side is: multiply the primary impedance by n^2, multiply the primary voltage by n, and divide the primary current by n.

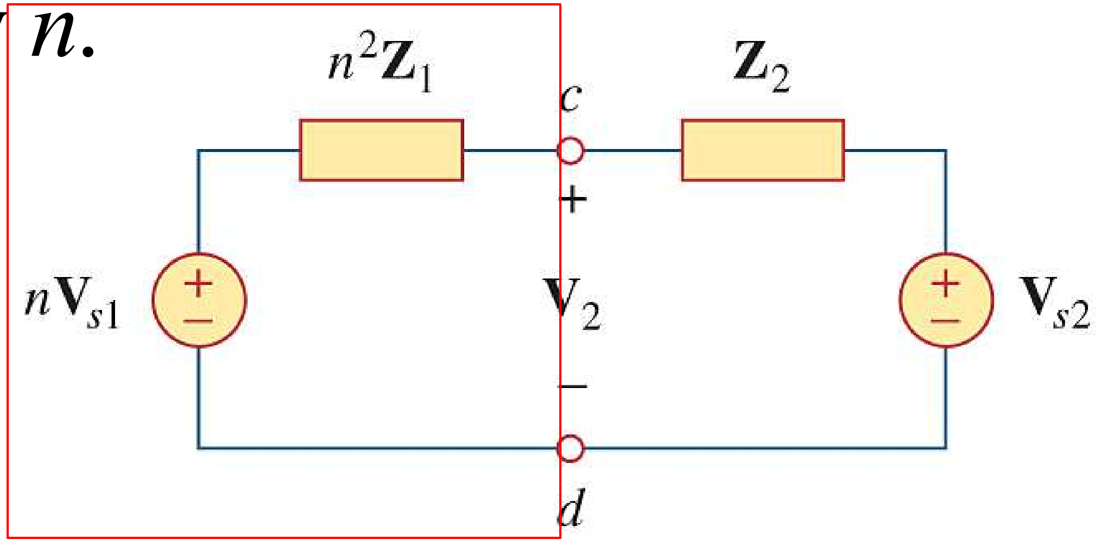
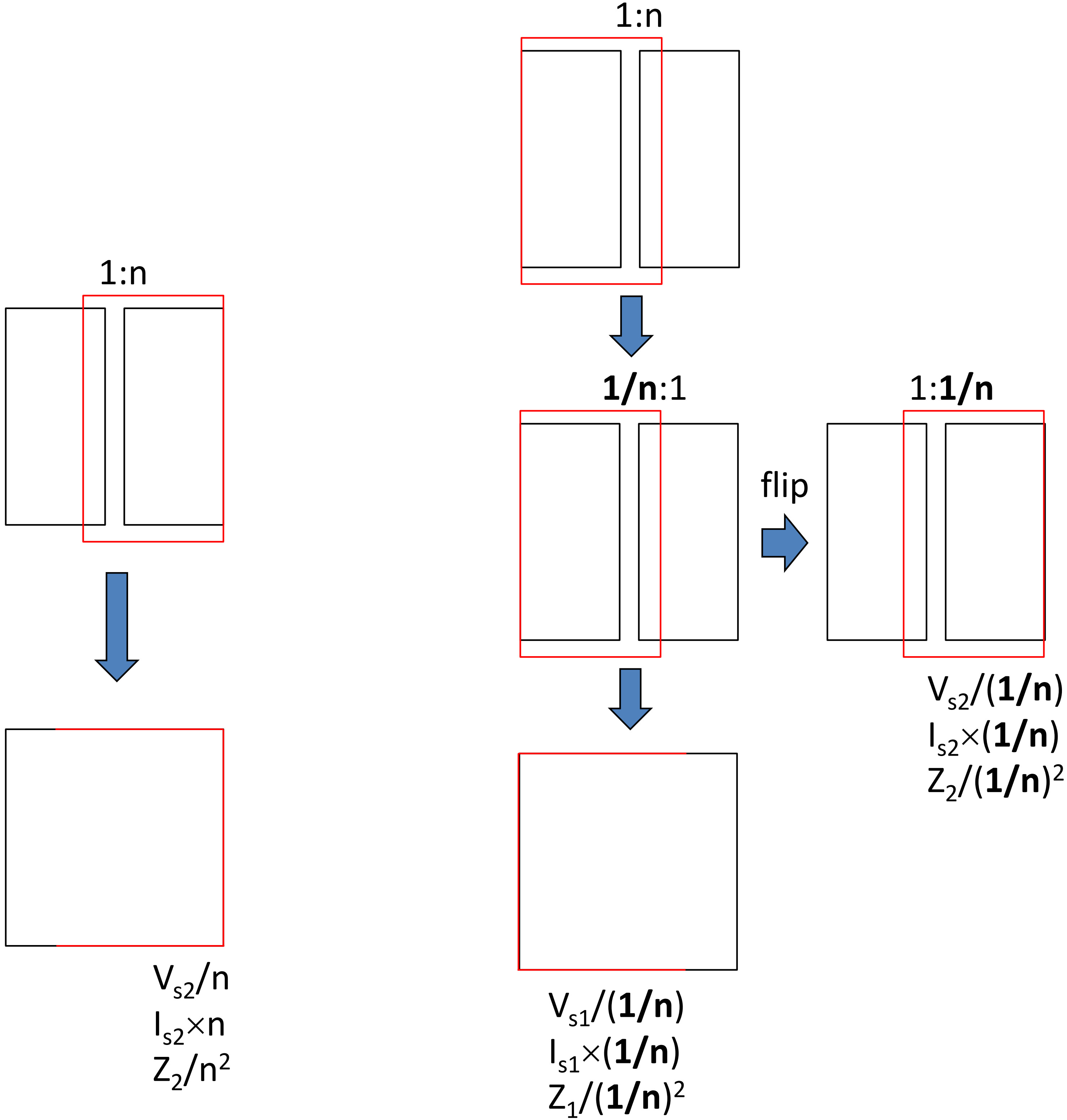


Figure 13.36 Equivalent circuit for Fig. 13.33 obtained by reflecting the primary circuit to the secondary side.


1:n
V_{s2}/n
$I_{s2} \times n$
Z_2/n^2
1:n
1/n:1
flip
1:**1/n**
$V_{s2}/(\mathbf{1/n})$
$I_{s2} \times (\mathbf{1/n})$
$Z_2/(\mathbf{1/n})^2$
$V_{s1}/(\mathbf{1/n})$
$I_{s1} \times (\mathbf{1/n})$
$Z_1/(\mathbf{1/n})^2$

Practice Problem 13.7 The primary current to an ideal transformer rated at 3300/110 V is 5 A. Calculate: (a) the turns ratio, (b) the kVA rating, (c) the secondary current.

Solution :

$$\text{(a) } n = \frac{V_2}{V_1} = \frac{110}{3300} = \frac{1}{30}$$

$$\text{(b) } |S| = V_1 I_1 = 3300 \times 5 = 16500 \text{ (VA)}$$

$$= 16.5 \text{ kVA}$$

$$\text{(c) } \frac{I_1}{I_2} = n \Rightarrow I_2 = \frac{I_1}{n} = \frac{5}{1/30} = 150 \text{ (A)}$$

Practice Problem 13.8 In the ideal transformer circuit of Fig. 13.38, find $\tilde{V}_o$ and the complex power supplied by the source.

Solution :

$$Z_R = \frac{16 - j24}{4^2} = 1 - j1.5\ (\Omega)$$

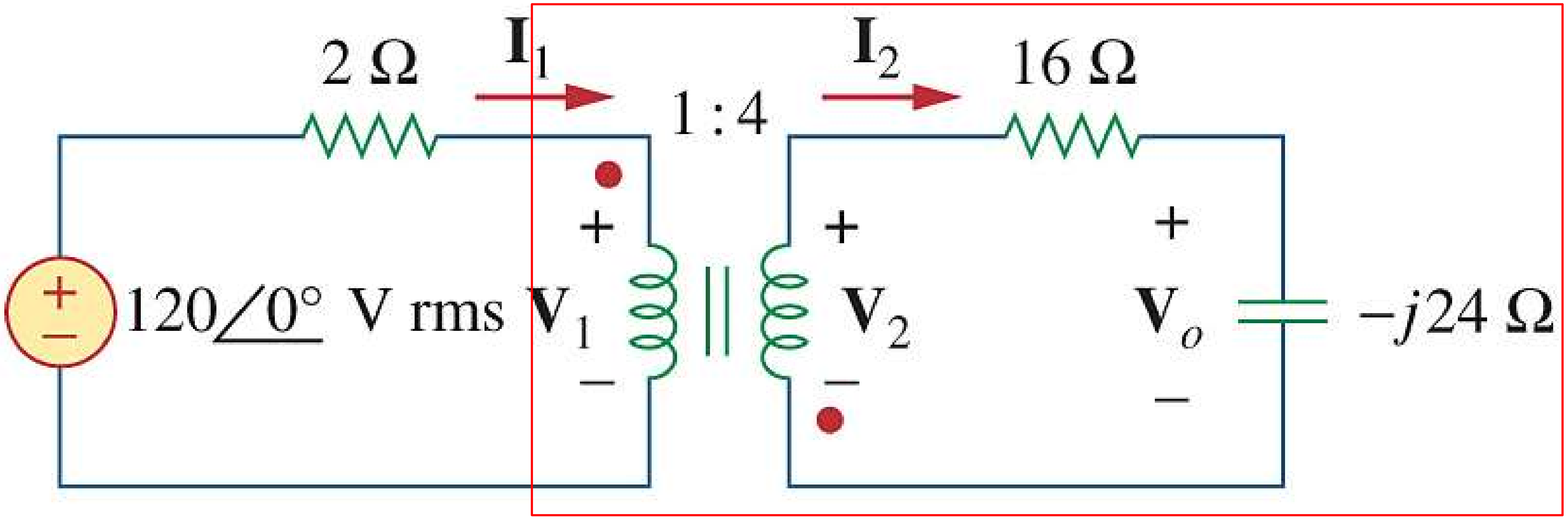
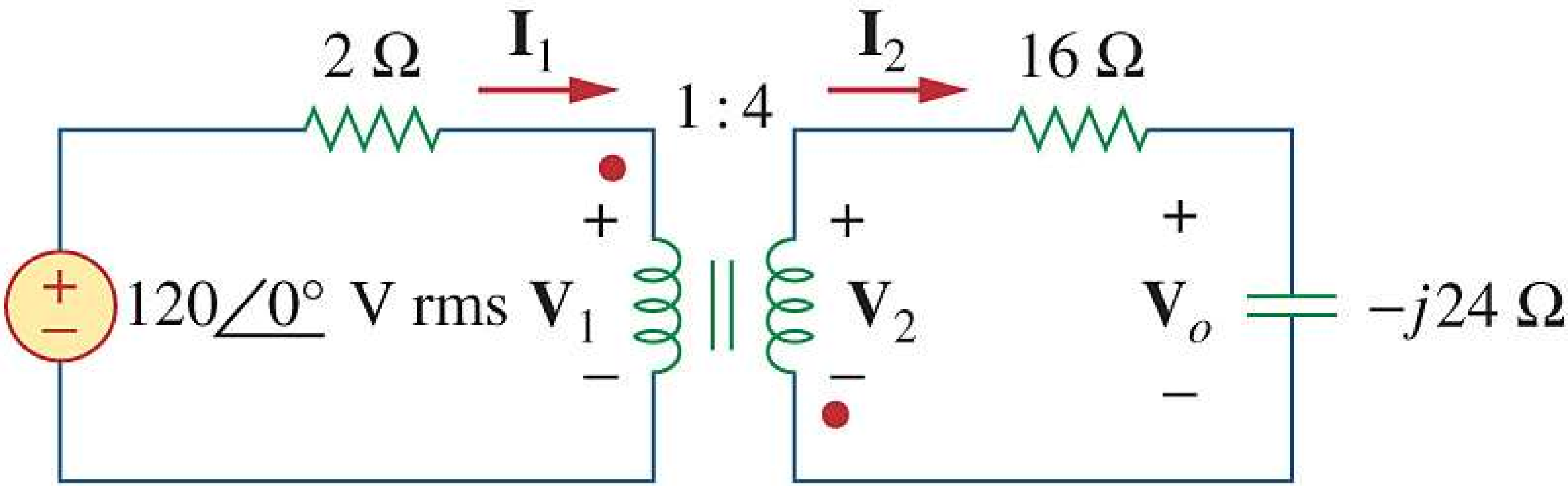


Figure 13.38



2 Ω
$\mathbf{I}_1$
1 : 4
$\mathbf{I}_2$
16 Ω
+
−
$120\angle 0°$ V rms
$\mathbf{V}_1$
+
−
$\mathbf{V}_2$
+
−
$\mathbf{V}_o$
+
−
$-j24\ \Omega$

Figure 13.38

$$\tilde{I}_1 = \frac{120\angle 0^\circ}{2+Z_R} = \frac{120\angle 0^\circ}{2+(1-j1.5)}$$

Thevenin equivalent used

$$\approx \frac{120\angle 0^\circ}{3.3541\angle -26.57^\circ}$$

$$\approx 35.7771\angle 26.57^\circ \text{ (A)}$$

$$\tilde{I}_2 = -\frac{\tilde{I}_1}{n} = -\frac{35.7771\angle 26.57^\circ}{4}$$

Original circuit

$$\approx 8.9443\angle 206.57^\circ \text{ (A)}$$

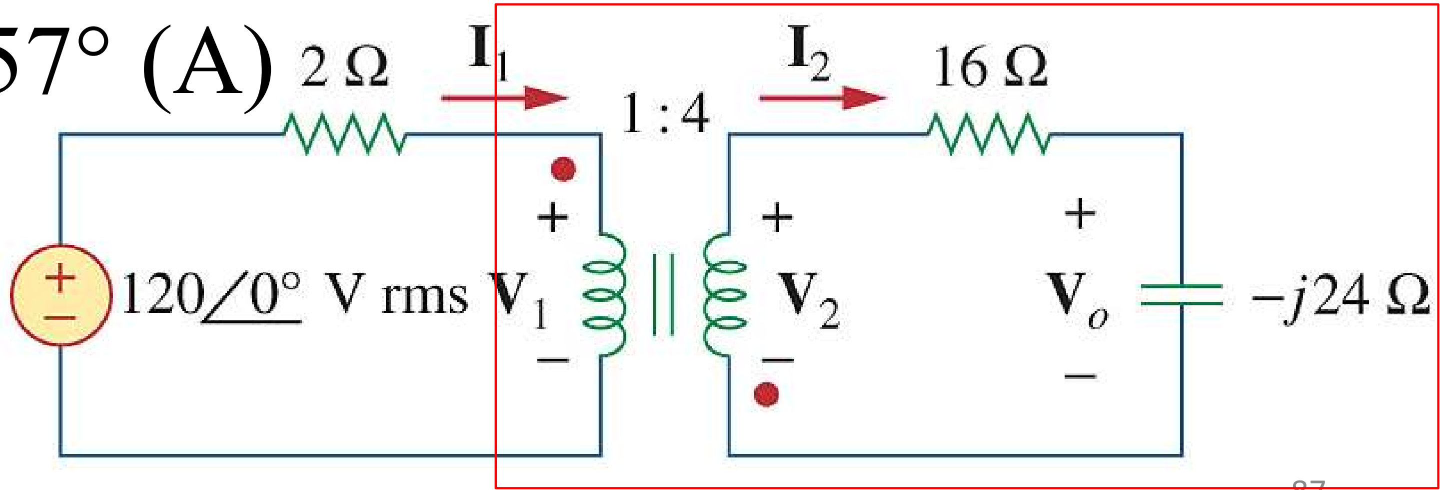


Figure 13.38

$$\tilde{V}_o = \tilde{I}_2 \times (-j24)$$

$$= 8.9443\angle 206.57^\circ \times 24\angle -90^\circ$$

$$= 214.6632\angle 116.57^\circ \text{ (V)}$$

$$S = 120\angle 0^\circ \times \tilde{I}_1^*$$

$$= 120\angle 0^\circ \times 35.7771\angle -26.57^\circ$$

$$\approx 4293.252\angle -26.57^\circ \text{ (VA)}$$

$$\approx 4.2933\angle -26.57^\circ \text{ kVA}$$

Practice Problem 13.9 Find $\tilde{V}_o$ in the circuit of Fig. 13.40.

Solution :

$$\frac{\tilde{V}_1 - 240\angle 0°}{4} + \frac{\tilde{V}_1 - \tilde{V}_3}{8} + \tilde{I}_1 = 0 \qquad (1)$$

KCL of node V_1

$$\frac{\tilde{V}_3 - \tilde{V}_1}{8} + \tilde{I}_2 + \frac{\tilde{V}_3}{8} = 0 \qquad (2)$$

KCL of node V_3

$$\tilde{I}_2 = \frac{\tilde{V}_3 - \tilde{V}_2}{2} \qquad (3)$$

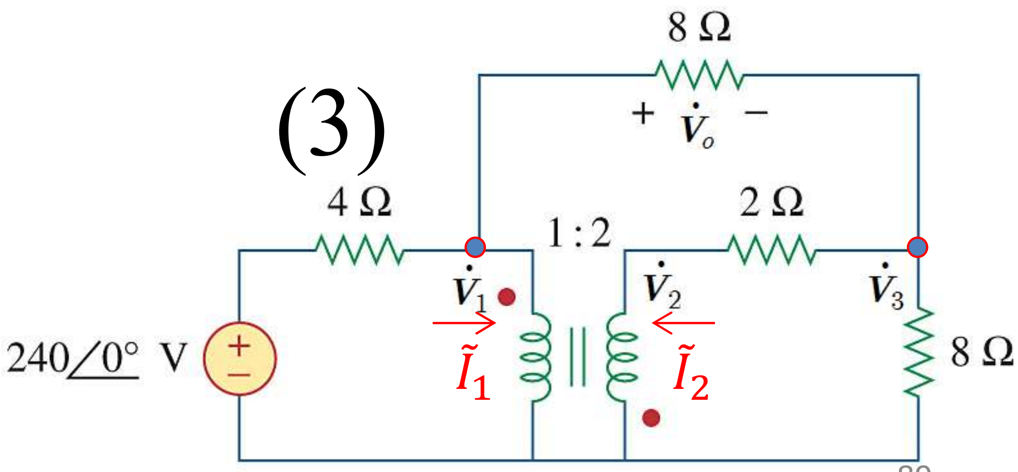


Figure 13.40

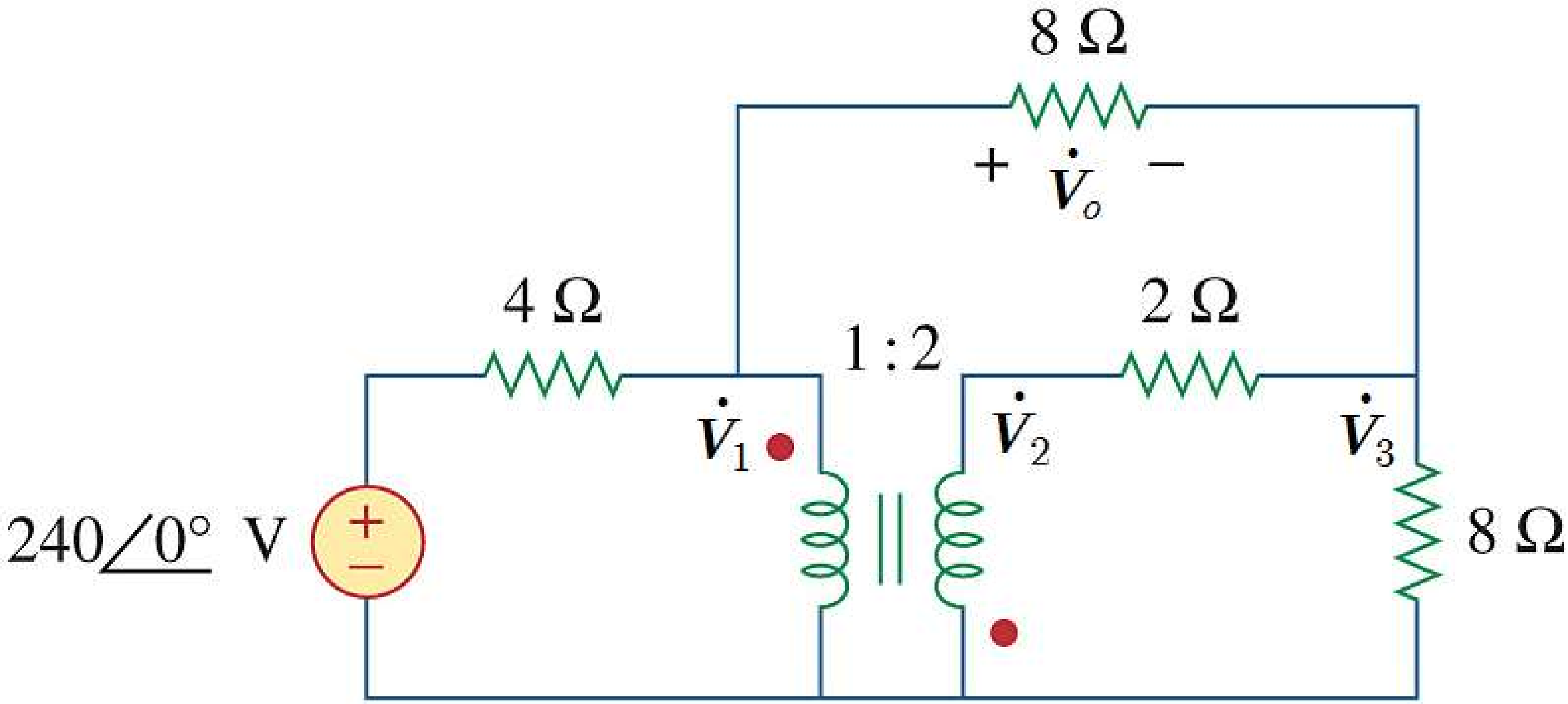


Figure 13.40

$$\left.\begin{array}{ll}\tilde{V}_2 = -2\tilde{V}_1 & (4) \\ \tilde{I}_1 = 2\tilde{I}_2 & (5)\end{array}\right\} \boxed{\text{Ideal transformer}}$$

From (3) and (4),

$$\tilde{I}_2 = \frac{\tilde{V}_3}{2} - \frac{\tilde{V}_2}{2} = \frac{\tilde{V}_3}{2} + \tilde{V}_1 \qquad (6)$$

From (5) and (6),

$$\tilde{I}_1 = \tilde{V}_3 + 2\tilde{V}_1 \qquad (7)$$

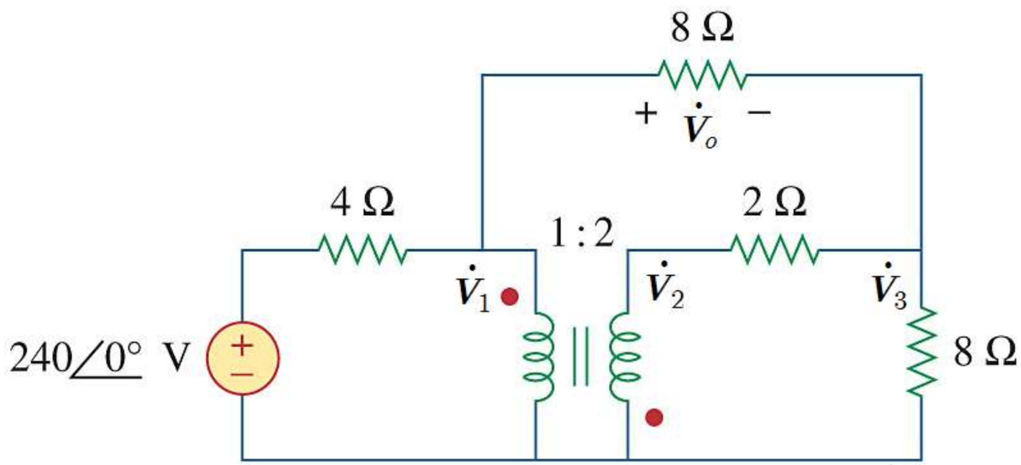


Figure 13.40

Substitute (7) in (1),

$$\frac{\tilde{V}_1 - 240}{4} + \frac{\tilde{V}_1 - \tilde{V}_3}{8} + \tilde{V}_3 + 2\tilde{V}_1 = 0 \Rightarrow$$

$$19\tilde{V}_1 + 7\tilde{V}_3 = 480 \qquad (8)$$

Substitute (6) in (2),

$$7\tilde{V}_1 + 6\tilde{V}_3 = 0 \qquad (9)$$

From (8) and (9),

$$\tilde{V}_1 = \frac{576}{13}\ \text{(V)},\ \tilde{V}_3 = -\frac{672}{13}\ \text{(V)} \Rightarrow$$

$$\tilde{V}_o = \tilde{V}_1 - \tilde{V}_3 = 96\ \text{(V)} = 96\angle 0^\circ\ \text{(V)}$$