



JOINT INSTITUTE  
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# ECE2150J Introduction to Circuits

## Chapter 7. First-Order Circuits

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## 7.1 Introduction

- Until now, we considered passive elements (R, L, and C), and an active element (op amp) individually.
- Now, we will see circuits with the combination of R, L, and C: Chapter 7: RC and RL; Chapter 8: RLC.
- **RC and RL circuits** are characterized by **first-order differential equations**. Hence, the circuits are collectively called **first-order circuits**.

# Source of Energy

A circuit needs energy, either potential or current. In RL or RC circuits, there are two ways to provide energy to the circuits.

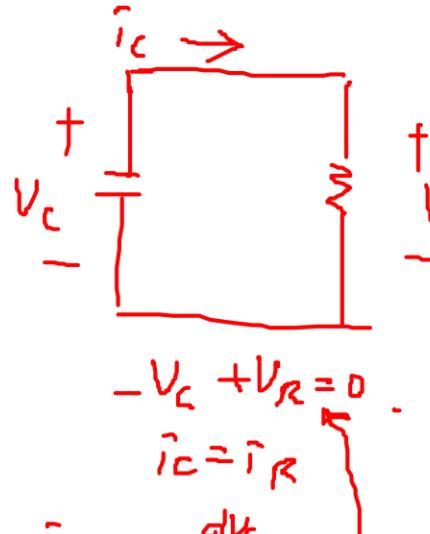
- (i) By energy **initially stored** in the capacitive or inductive element.
- (ii) By **independent sources**.

<b>Source of Energy</b>	<b>Case (i)</b>	<b>Case (ii)</b>
RC	7.2 The Source-Free <i>RC</i> Circuit: $v_c(t=0) = V_0$	7.5 An <i>RC</i> Circuit with Step Input
RL	7.3 The Source-Free <i>RL</i> Circuit: $i_L(t=0) = I_0$	7.6 An <i>RL</i> Circuit with Step Input

## 7.2 The Source-Free *RC* Circuit

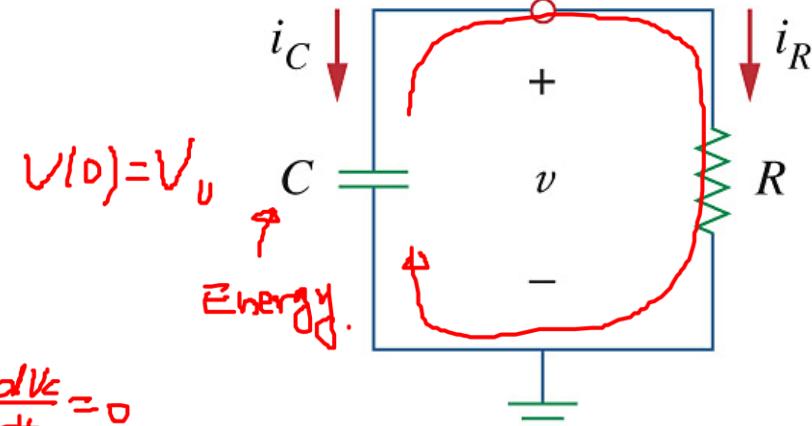
- A source-free RC circuit occurs when its dc source is suddenly disconnected. The **energy already stored in the capacitor** is released to the resistor.

# Source-Free RC Circuit Response



$$i_C = -C \frac{dv}{dt}$$

$$V_R = -RC \frac{dv_0}{dt}$$

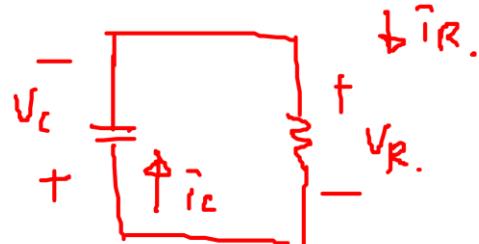


$$RC = \tau$$

The capacitor is initially charged. We assume that at time  $t=0$ , the initial voltage is  $v(0) = V_0$  with the corresponding value of energy stored:

$$w_C(0) = \frac{1}{2} CV_0^2$$

The circuit response, the capacitor voltage  $v$  in this case is

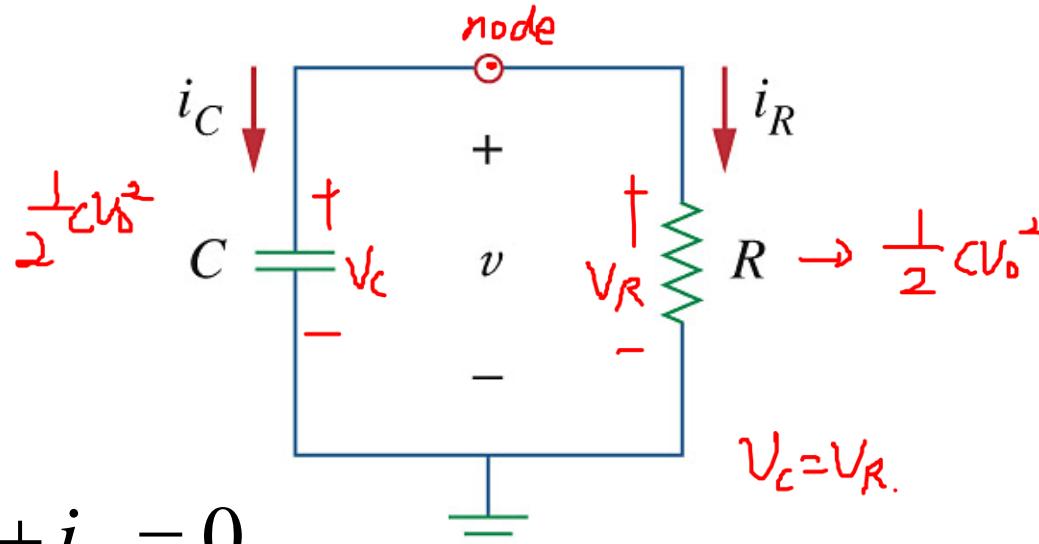


$$KVL: V_C + V_R = 0.$$

$$i = C \frac{dV_C}{dt}.$$

$$V_R = R i_C [= R i_R]$$

$$V_C + R \cdot C \frac{dV_C}{dt} = 0.$$



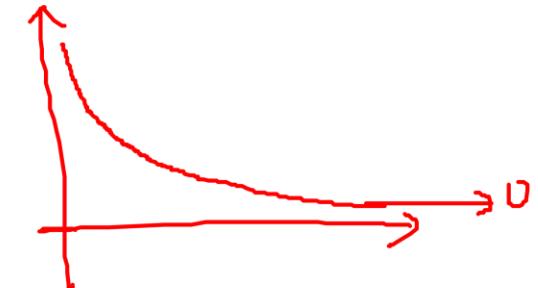
$$V_C = V_R.$$

$$KCL: i_C + i_{R_p} = 0$$

$$\dot{i}_C = C \frac{dv}{dt}, i_R = \frac{v}{R}$$

→ first-order.

$$C \frac{dv}{dt} + \frac{v}{R} = 0 \quad \Rightarrow \quad \frac{dv}{dt} + \frac{v}{RC} = 0$$



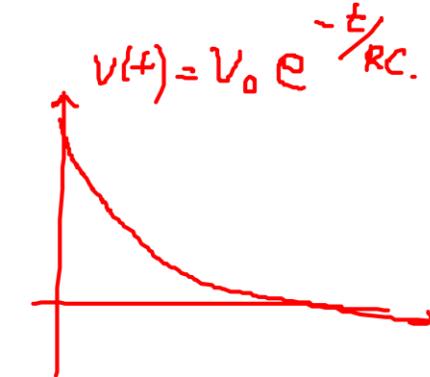
$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

$\frac{dv}{v} = -\frac{1}{RC} dt$  integrate both sides

$$\ln v = -\frac{t}{RC} + C \rightarrow v = Ae^{-\frac{t}{RC}}$$

Solution.

$$v(0) = A = V_0$$



Using the initial condition  $v(0) = V_0$ , we get  $v(t) = V_0 e^{-\frac{t}{RC}}$

The response is due to the initial energy stored and the physical characteristics of the circuit. We call it **Natural Response, or zero-input**, of the circuit.

$$v(t) = V_0 e^{-\frac{t}{RC}}$$

$RL = 1s$   
↳

$$t = RC.$$

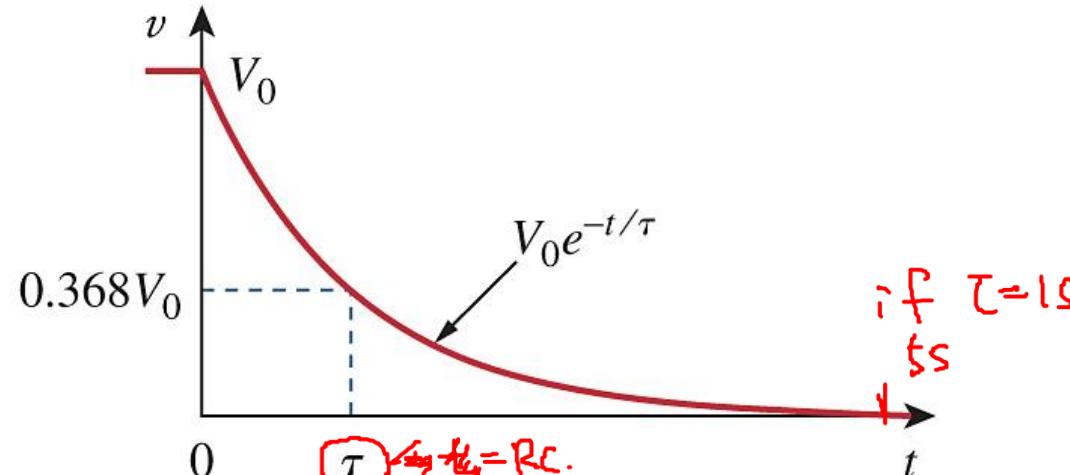
$$\begin{aligned} v(RC) &= V_0 e^{-1} \\ &= 0.368V_0 \end{aligned}$$

$$2\tau = 2RC.$$

$$v(2\tau) = V_0 e^{-2}$$

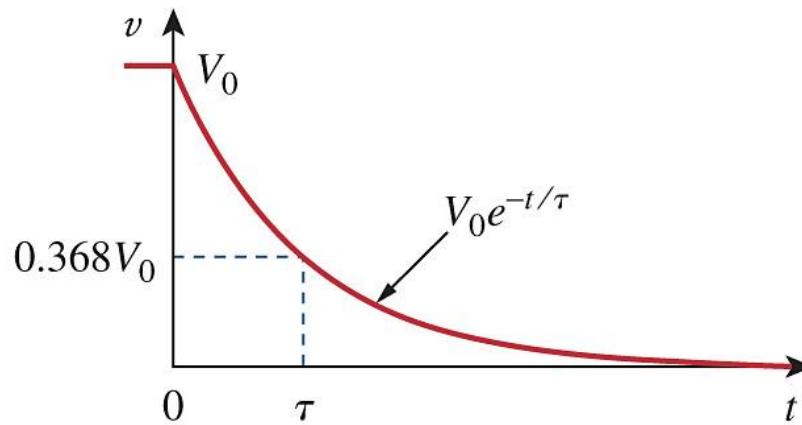
$$v(3\tau) = V_0 e^{-3}$$

$$v(5\tau) = V_0 e^{-5} \Rightarrow \text{less than } 1\%.$$



The response of the source-free RC circuit: at  $t=0$ ,  $v=V_0$ , and as  $t$  increases,  $v$  decreases toward zero. The rapidity with which  $v$  decreases is expressed in terms of **the time constant  $\tau$** .

## Time constant $\tau$



The time constant of a circuit is the time required for the response to decay to a factor of  $1/e$  or 36.8% of its initial value. This implies that at  $t = \tau$ ,

$$v = V_0 e^{-\frac{\tau}{RC}} = V_0 e^{-1}$$

or

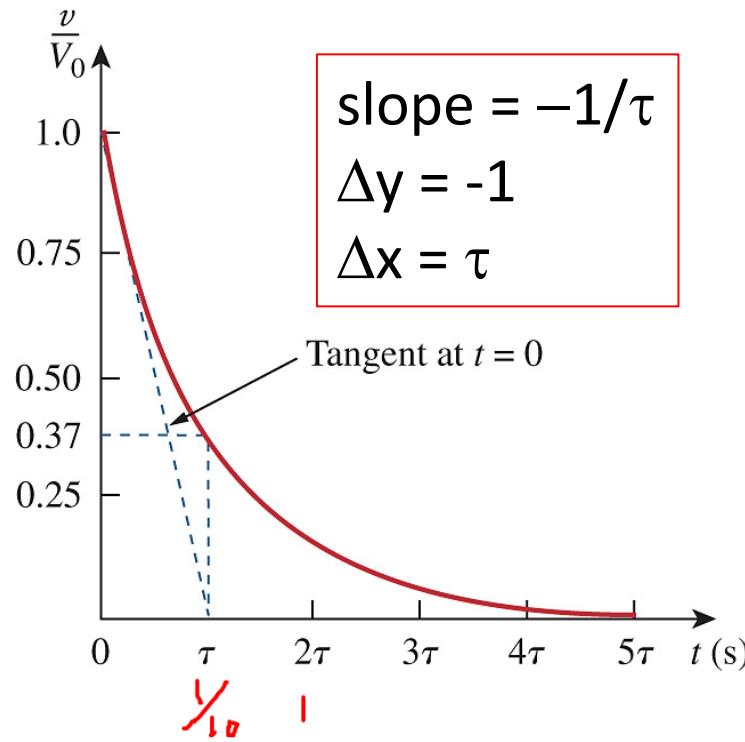
$$\tau = RC$$

The time constant may be viewed from another perspective.  
Evaluating the derivative of  $v/V_0$  at  $t = 0$ , we obtain

$$v(t) = V_0 e^{-\frac{t}{\tau}}$$

$$\left. \frac{d}{dt} \left( \frac{v}{V_0} \right) \right|_{t=0} = -\frac{1}{\tau} e^{-t/\tau} \Big|_{t=0} = -\frac{1}{\tau}$$

**slope at t=0**

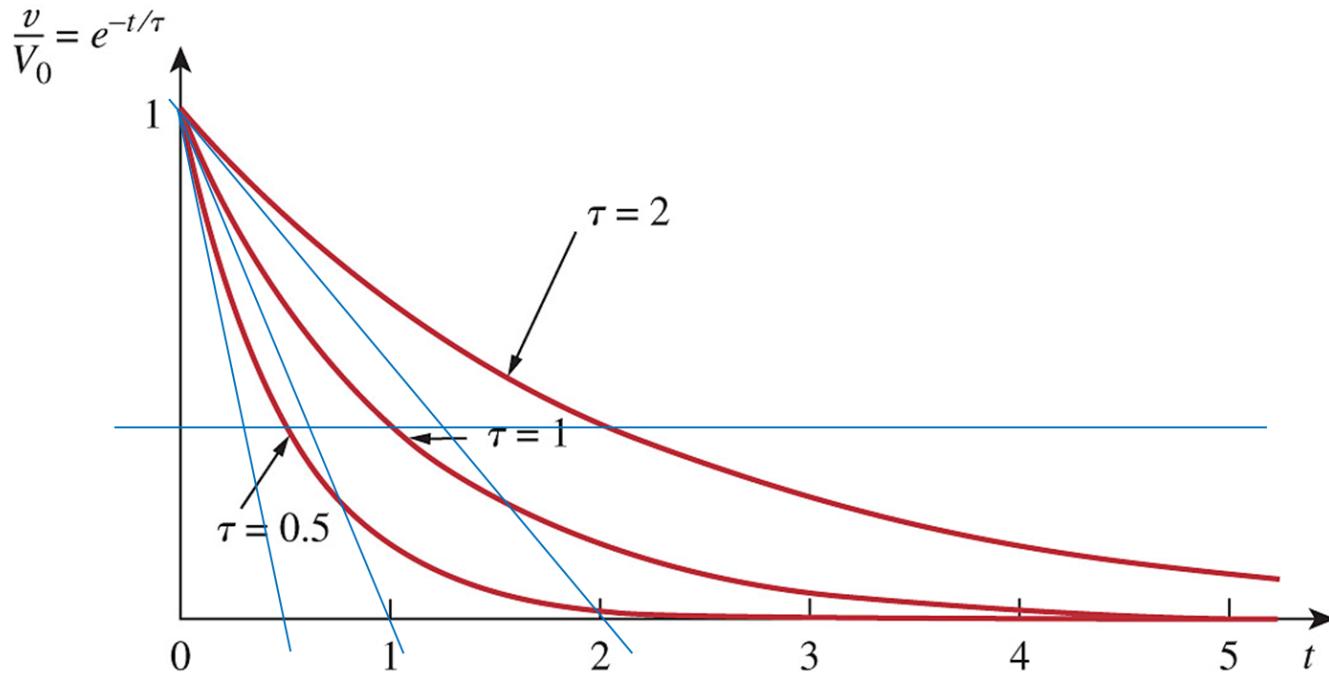


Thus, the time constant is the **initial rate of decay**. To find  $\tau$  from the response curve, draw the tangent to the curve at  $t = 0$ . The tangent intercepts with the time axis at  $t = \tau$ .

**TABLE 7.1 Values of  $v / V_0 = e^{-t/\tau}$**

$t$	$\tau$	$2\tau$	$3\tau$	$4\tau$	$5\tau$
$v / V_0$	0.36788	0.13534	0.04979	0.01832	0.00674 0.7%

After  $5\tau$ ,  $v$  is less than 1% of the initial value  $V_0$ . It is customary to assume that the capacitor is fully discharged after five time constants, i.e. the circuit reaches its final state or steady state when no changes take place with time.



**The smaller the time constant, the more rapidly the voltage decreases**, i.e. the faster response  $\rightarrow$  quick dissipation of energy stored.

## Circuit response at R

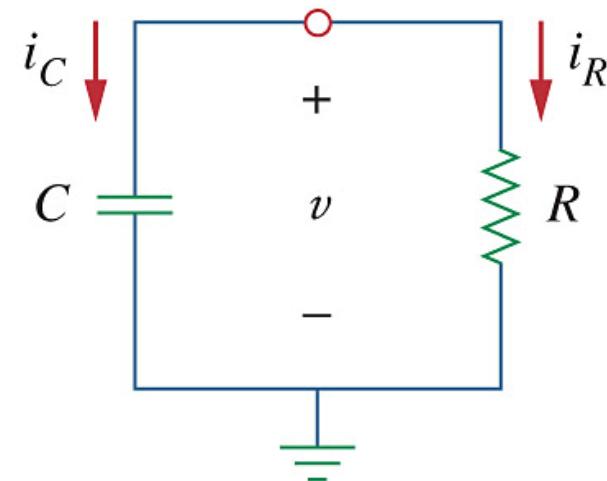
Go back to the **circuit response**, from the voltage in capacitor  $v(t) = V_0 e^{-\frac{t}{RC}}$  we can find the resistor current

$$i_R = \frac{v}{R} = \frac{V_0}{R} e^{-t/\tau}$$

$$\left\{ \begin{array}{l} i_C = C \frac{dv}{dt} \\ i_R = \frac{v}{R} \end{array} \right.$$

The power dissipated in the resistor is

$$p = vi_R = \frac{V_0^2}{R} e^{-2t/\tau}$$



The energy absorbed by the resistor up to time  $t$  is

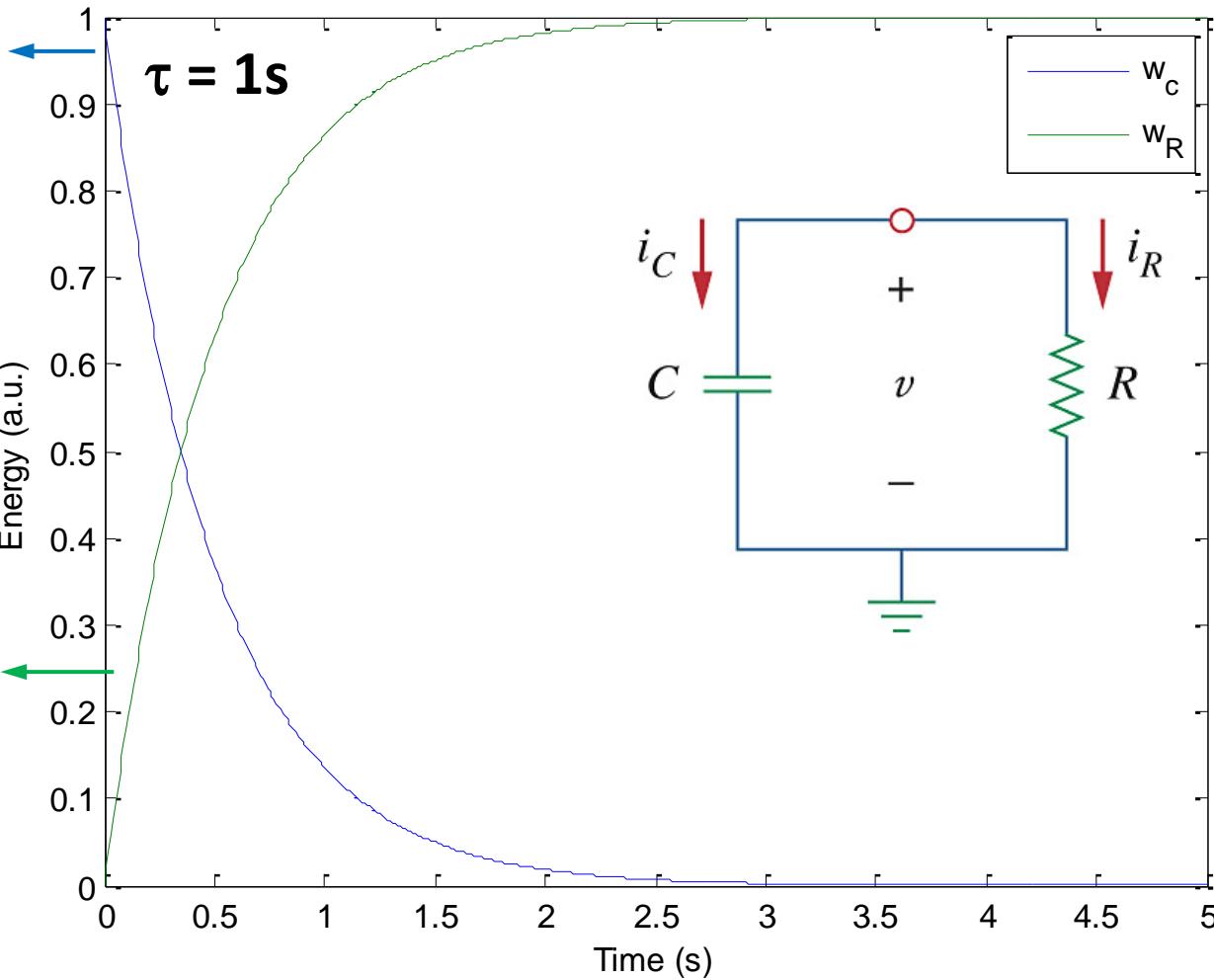
$$\begin{aligned} w_R &= \int_0^t \left( \frac{V_0^2}{R} e^{-2t/\tau} \right) dt = \frac{V_0^2}{R} \frac{e^{-2t/\tau}}{-2/\tau} \Big|_0^t \\ &= \frac{1}{2} C V_0^2 \left( 1 - e^{-2t/\tau} \right)^{\square} \end{aligned}$$

Notice that as  $t \rightarrow \infty$ ,  $w_R \rightarrow \frac{1}{2} C V_0^2 = w_C(0)$ ,

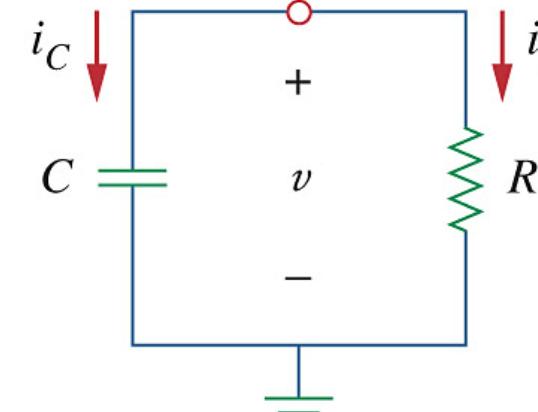
the energy initially stored in the capacitor

# Energy in Circuit

Energy in Capacitor



Energy in Resistor



## Summary: RC source free response

To find the RC source free response

1. The initial voltage  $v(0) = V_0$  across the capacitor.
2. The time constant  $\tau = RC$  where  $R$  is often the equivalent resistance at the terminals of  $C$ .

The response, the capacitor voltage  $V_C$ , is

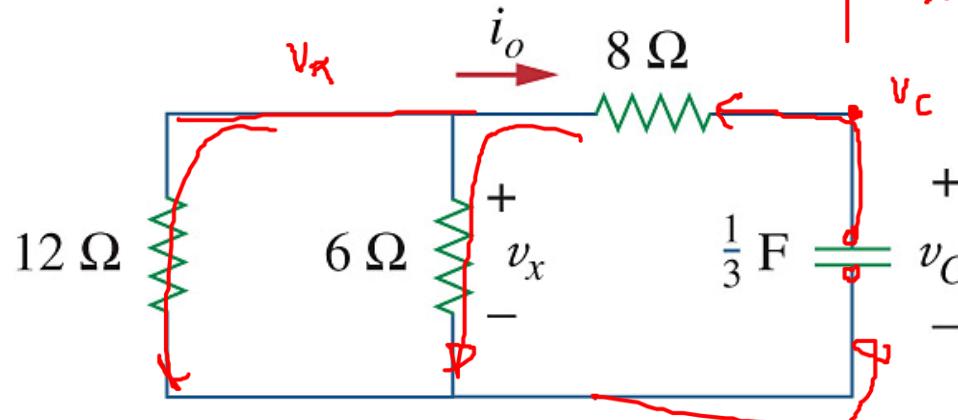
$$V_C = V = V(0)e^{-t/\tau}$$

and then, we can find **other circuit responses**,  $V_R$ ,  $i_R$ ,  $I_C$  etc.

## Practice Problem 7.1 Refer to the circuit

in Fig. 7.7. Let  $v_C(0) = 45$  V. Determine  $v_C$ ,

$v_x$ , and  $i_o$  for  $t \geq 0$ .



$$\text{Req} \Rightarrow RC \Rightarrow V(+)=V(0)e^{-t/\tau} = 45e^{-t/4}.$$

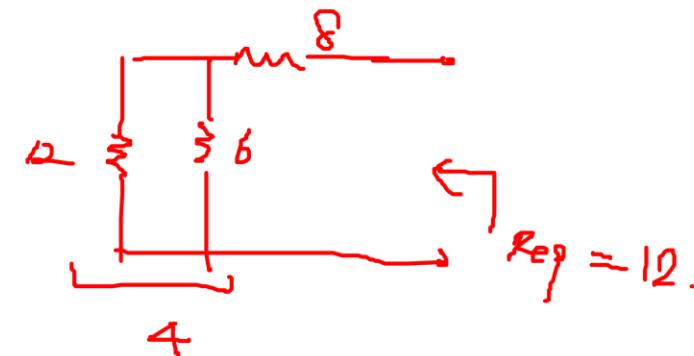


Figure 7.7 An  $RC$  circuit.

$$\tau = RC = \frac{1}{3} \times 12 = 4 \text{ s}$$

$$\left\{ \begin{array}{l} \tilde{i}_0 = C \cdot \frac{dv_C}{dt}, \\ v_T = 8\tilde{i}_0 + v_C. \end{array} \right.$$

## Solution :

The equivalent resistance seen at the terminals of the capacitor is

$$R_{eq} = 8 + 12 \parallel 6 = 12 \text{ } (\Omega)$$

The time constant is

$$\tau = R_{eq}C = 12 \times \frac{1}{3} = 4 \text{ } (\text{s})$$

The capacitor voltage is

$$v_C = v_C(0)e^{-t/\tau} = 45e^{-t/4} = 45e^{-0.25t} \text{ } (\text{V})$$

$$i_o = C \frac{dv_C}{dt} = \frac{1}{3} \times \frac{d}{dt} (45e^{-0.25t})$$

$$= 15 \times (-0.25e^{-t/4}) = -3.75e^{-t/4} \text{ } (\text{A})$$

$$\begin{aligned} v_x &= 8i_o + v_C = 8 \times (-3.75e^{-0.25t}) + 45e^{-0.24t} \\ &= 15e^{-0.25t} \text{ } (\text{V}) \end{aligned}$$

**Practice Problem 7.12** If the switch in Fig. 7.10 opens at  $t = 0$ , find  $v(t)$  for  $t \geq 0$  and  $w_C(0)$ .

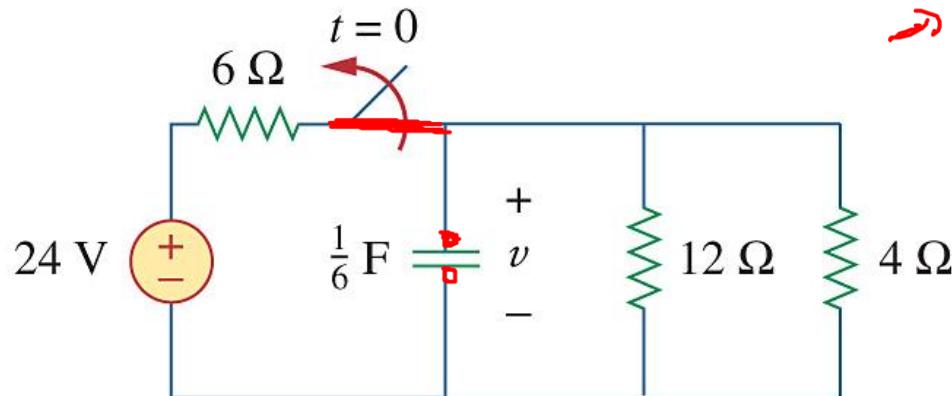
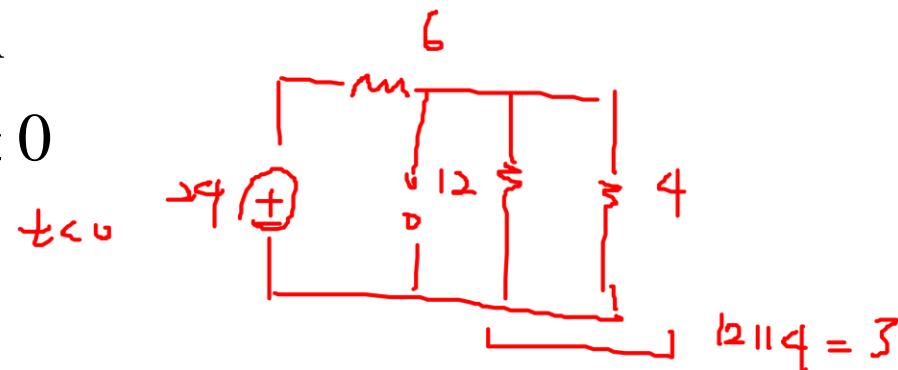


Figure 7.10

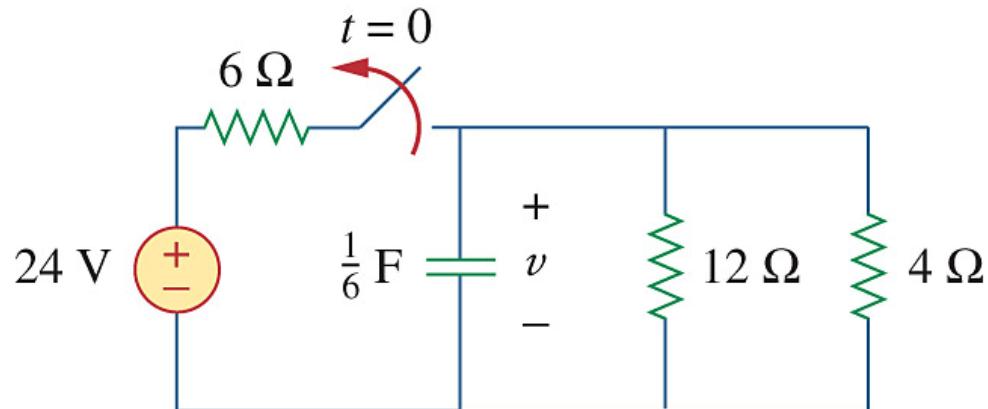


$$V_C(0-) = 24 \times \frac{3}{9} = 8 \text{ V}$$

$t > 0$

$$\frac{1}{6} \text{ F} \quad \frac{1}{3 \Omega} \quad \tau = R_C = \frac{1}{2}.$$

$$V_C(t) = 8 e^{-\frac{t}{2}}$$



**Solution :**

When  $t \leq 0$ , the capacitor voltage

$$v(t) = 24 \times \frac{12 \parallel 4}{6 + 12 \parallel 4} = 8 \text{ (V)}$$

Hence,  $v(0) = 8 \text{ V}$ .

When  $t \geq 0$ , the circuit becomes a source-free  $RC$  circuit with

$$\tau = R_{eq}C = (12 \parallel 4) \times \frac{1}{6} = \frac{1}{2} \text{ (s)}$$

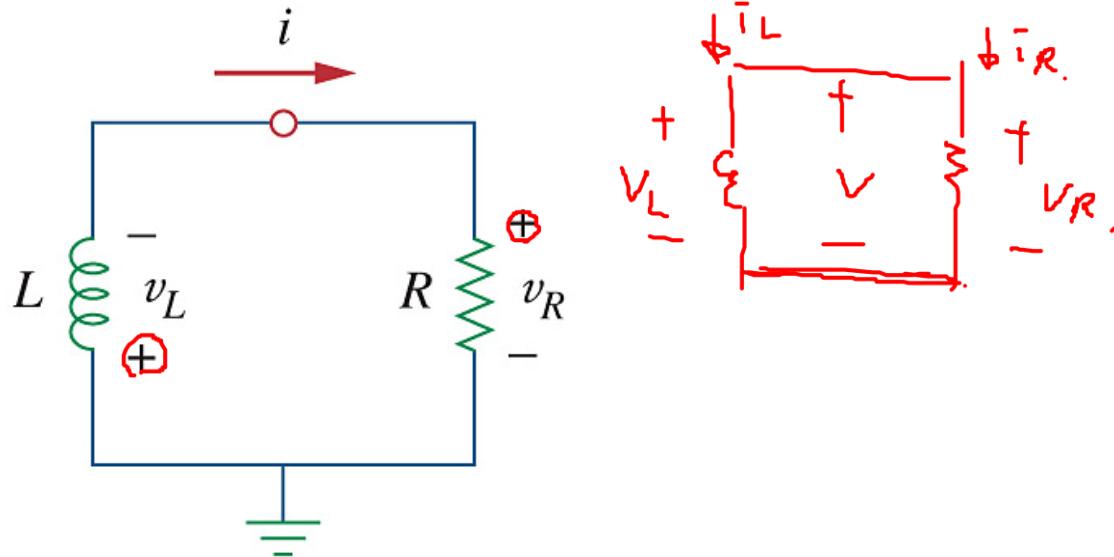
Therefore,

$$v(t) = v(0)e^{-t/\tau} = 8e^{-t/(1/2)} = 8e^{-2t} \text{ (V)}$$

$$w_C(0) = \frac{1}{2}C(v(0))^2 = \frac{1}{2} \times \frac{1}{6} \times 8^2$$

$$= \frac{16}{3} \approx 5.33 \text{ (J)}$$

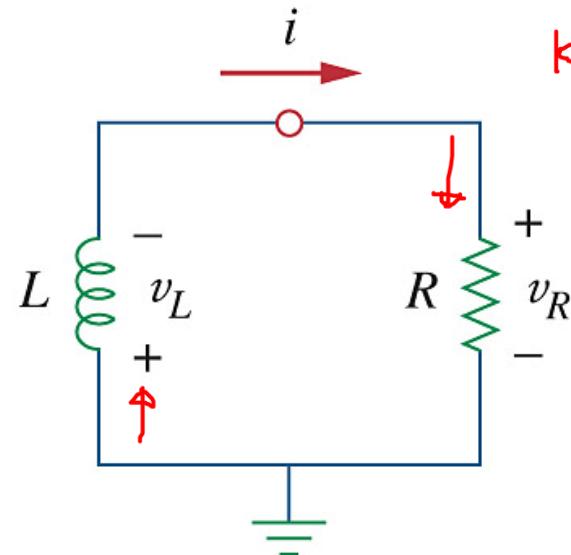
## 7.3 The Source-Free *RL* Circuit



Now, we think about the RL circuit with the initial current  $i(0) = I_0$  through the inductor, and corresponding energy

$$w_L(0) = \frac{1}{2} L I_0^2$$

# Source-Free RC Circuit Response



$$KVL: v_L + v_R = 0$$

$$v_L = L \frac{di}{dt}, v_R = iR$$

$$L \frac{di}{dt} + iR = 0 \quad \frac{di}{dt} + \frac{R}{L} i = 0$$

Solve the 1<sup>st</sup> order differential equation

$$i = Be^{rt} = Be^{-\frac{R}{L}t} = Be^{-\frac{t}{(L/R)}}$$

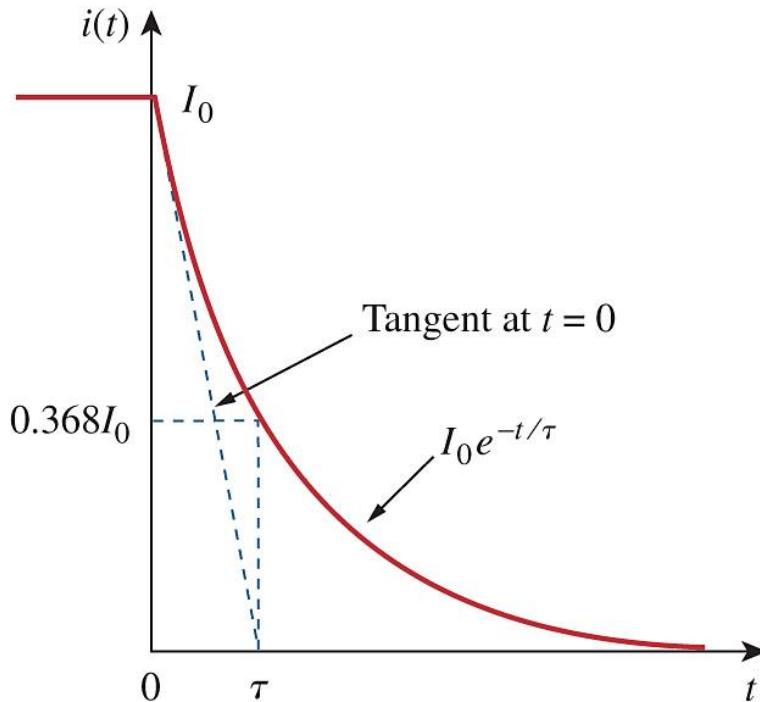
$$i(0) = B = I_0 \quad -\tau_{L,R} \quad \tau = \frac{L}{R}$$

$$i = I_0 e^{-\frac{t}{(L/R)}} = I_0 e^{-\frac{t}{\tau}}$$

$$i = I_0 e^{-\frac{t}{(L/R)}}$$

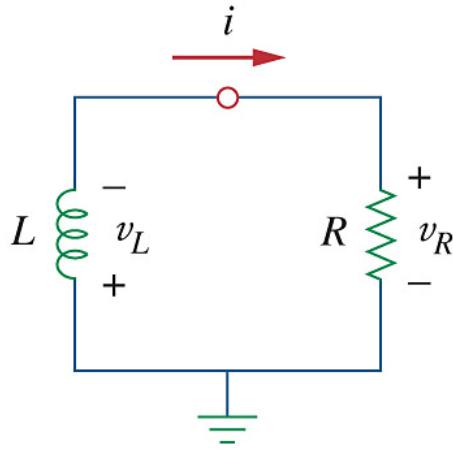
Let  $\tau = L / R$ , we have

$$i = I_0 e^{-t/\tau}$$



Because the response is due only to  $I_0$ , it is called **natural response, or zero-input**, of the circuit.

# Circuit response and Energy at R



**Voltage at the resistor**

$$v_R = iR = I_0 Re^{-t/\tau}$$

**Power at the resistor**

$$p = v_R i = I_0^2 R e^{-2t/\tau}$$

**Energy at the resistor up to time t**

$$w_R = \int_0^t (I_0^2 R e^{-2t/\tau}) dt = I_0^2 R \frac{e^{-2t/\tau}}{-2/\tau} \Big|_0^t$$

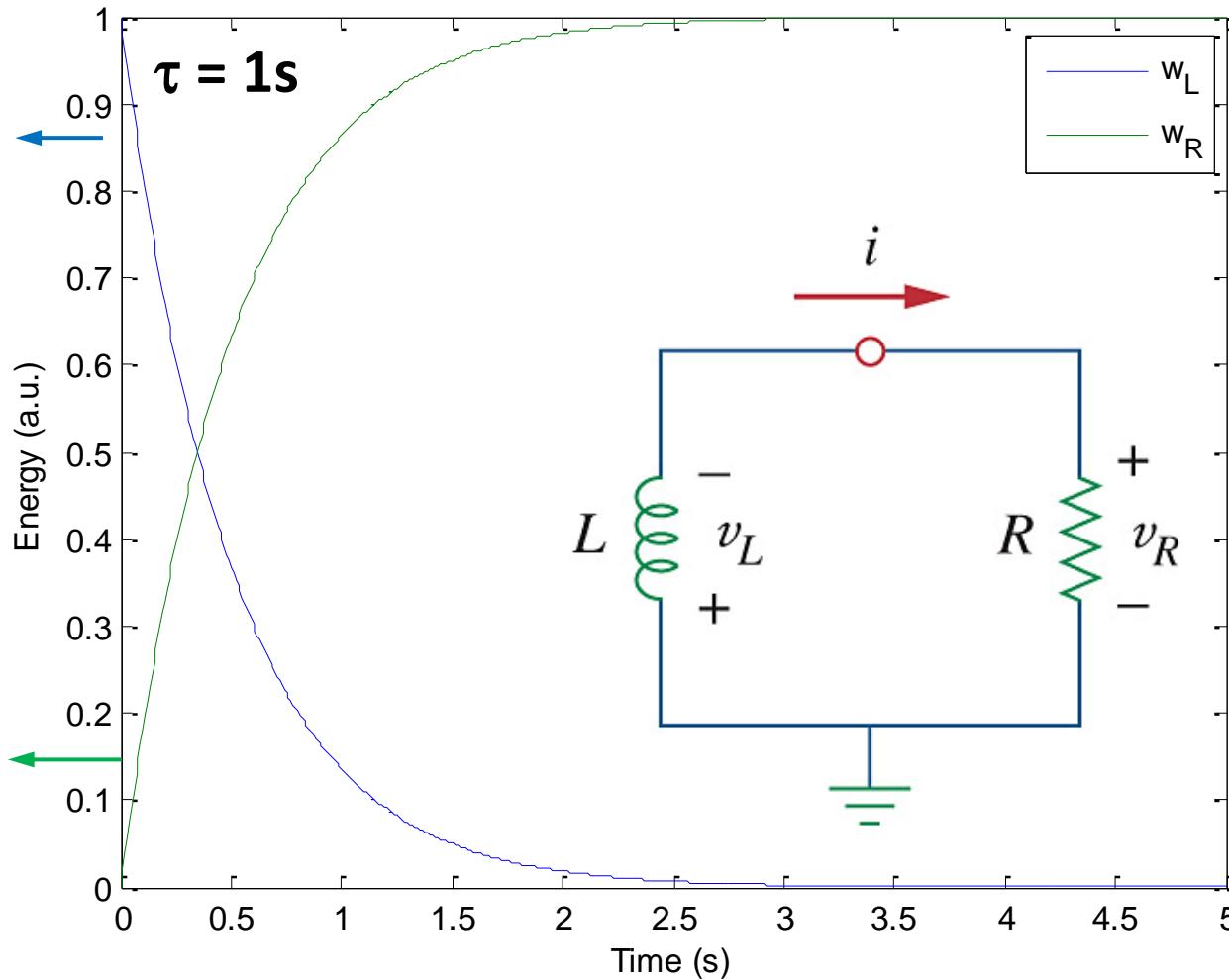
$$= \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau})$$

Notice that as  $t \rightarrow \infty$ ,  $w_R \rightarrow \frac{1}{2} L I_0^2 = w_L(0)$ , the energy initially stored in the inductor.

# Energy in Circuit

Energy in Inductor

Energy in Resistor



## Summary: RL source free response

To find the RL source free response

1. The initial current  $i(0) = I_0$  through the inductor.
2. The time constant  $\tau = L/R_{eq}$  where  $R_{eq}$  is often the equivalent resistance at the terminals of L.

The response, the inductor current i, is

$$i_L = i = i(0)e^{-t/\tau}$$

and then, we can find **other circuit responses**,  $V_L$ ,  $V_R$ ,  $i_R$  etc.

**Practice Problem 7.4** For the circuit in Fig. 7.18, find  $i(t)$  for  $t > 0$ .

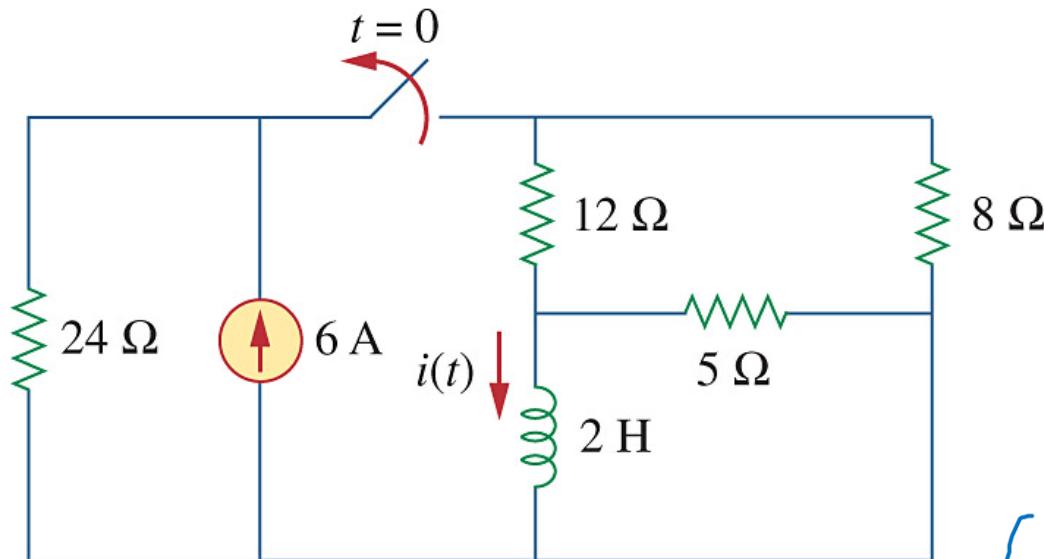
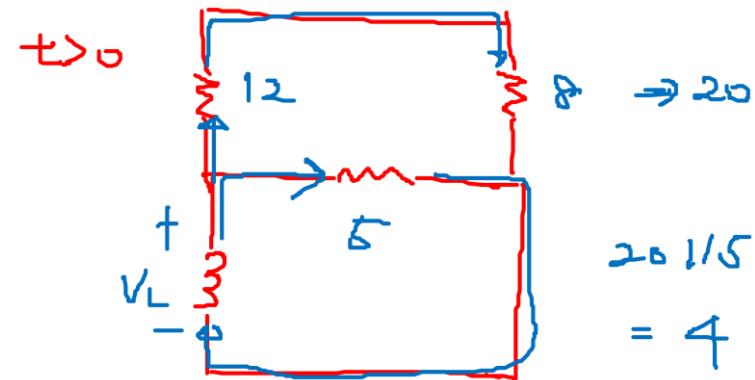


Figure 7.18

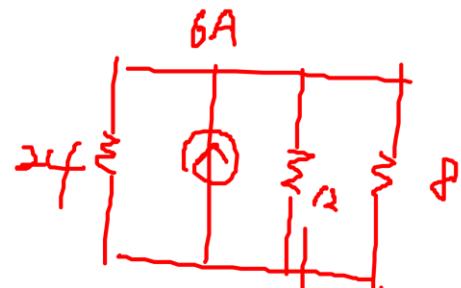


$$\tau = \frac{V_L}{R} = \frac{2}{\frac{1}{4}} = \frac{1}{2}$$

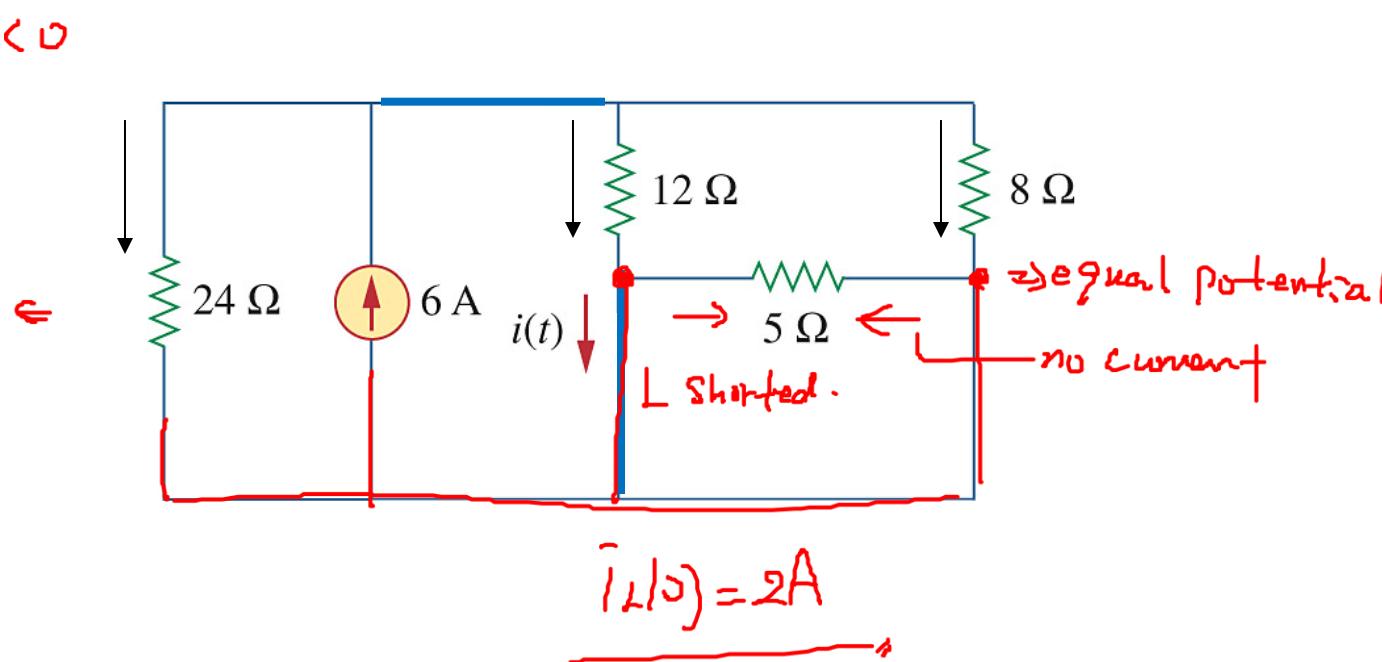
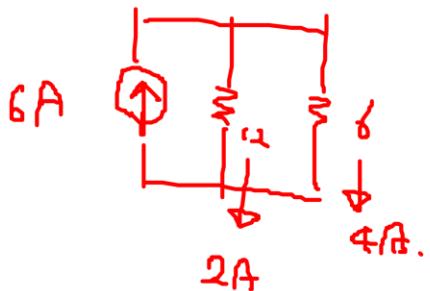
$$\hat{i}(t) = 2e^{-2t}$$

$$(\hat{i}(0) = \hat{i}(0)e^{-\frac{t}{\tau}})$$

$t < 0$



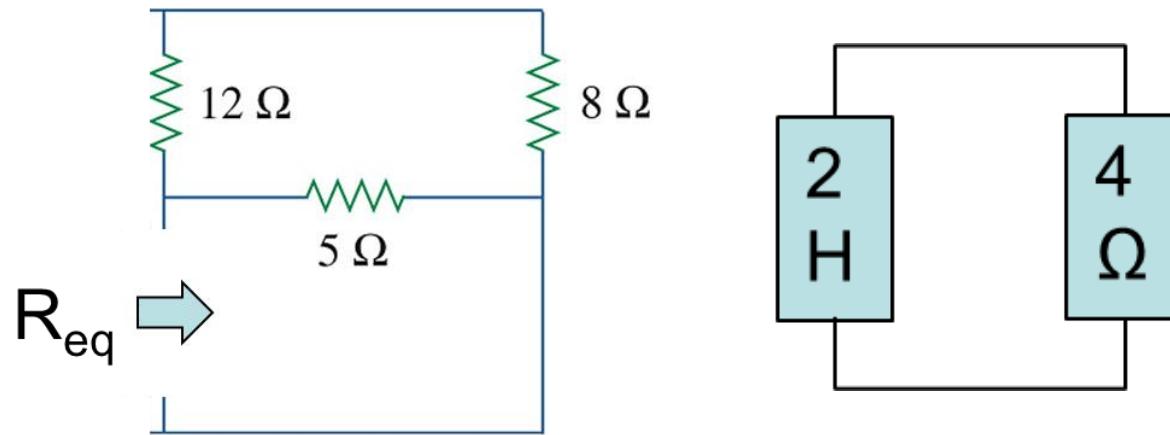
$$\left( \frac{24}{10} \right) = \frac{24 \times 8}{32} = 6.$$



### Solution :

When  $t < 0$ , the current through the 5-Ω resistor is zero.

$$i(t) = 6 \times \frac{24 \parallel 8}{24 \parallel 8 + 12} = 2 \text{ (A)}$$



When  $t > 0$ ,

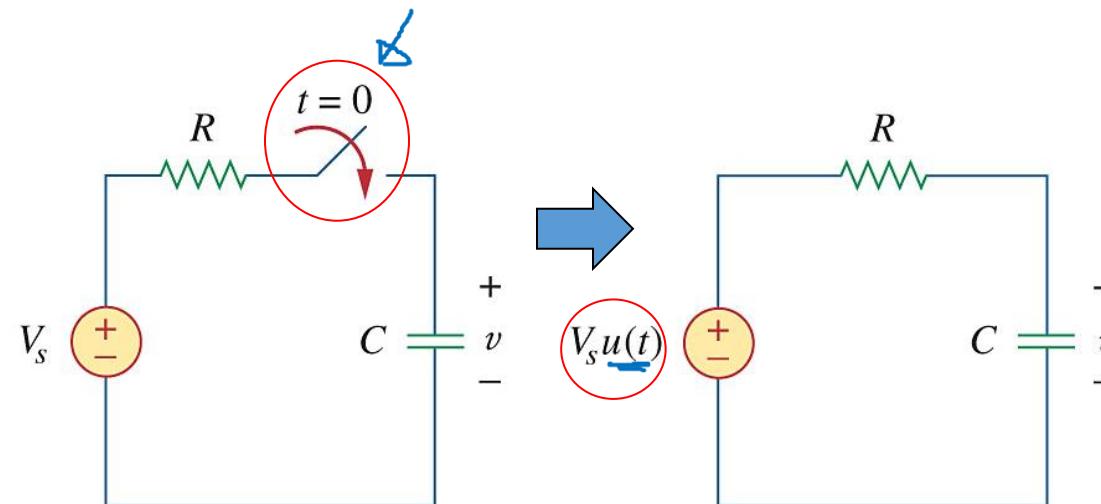
$$R_{eq} = 5 \parallel (12 + 8) = 4 \text{ } (\Omega)$$

$$\tau = L / R_{eq} = 2 / 4 = 0.5 \text{ } (\text{s})$$

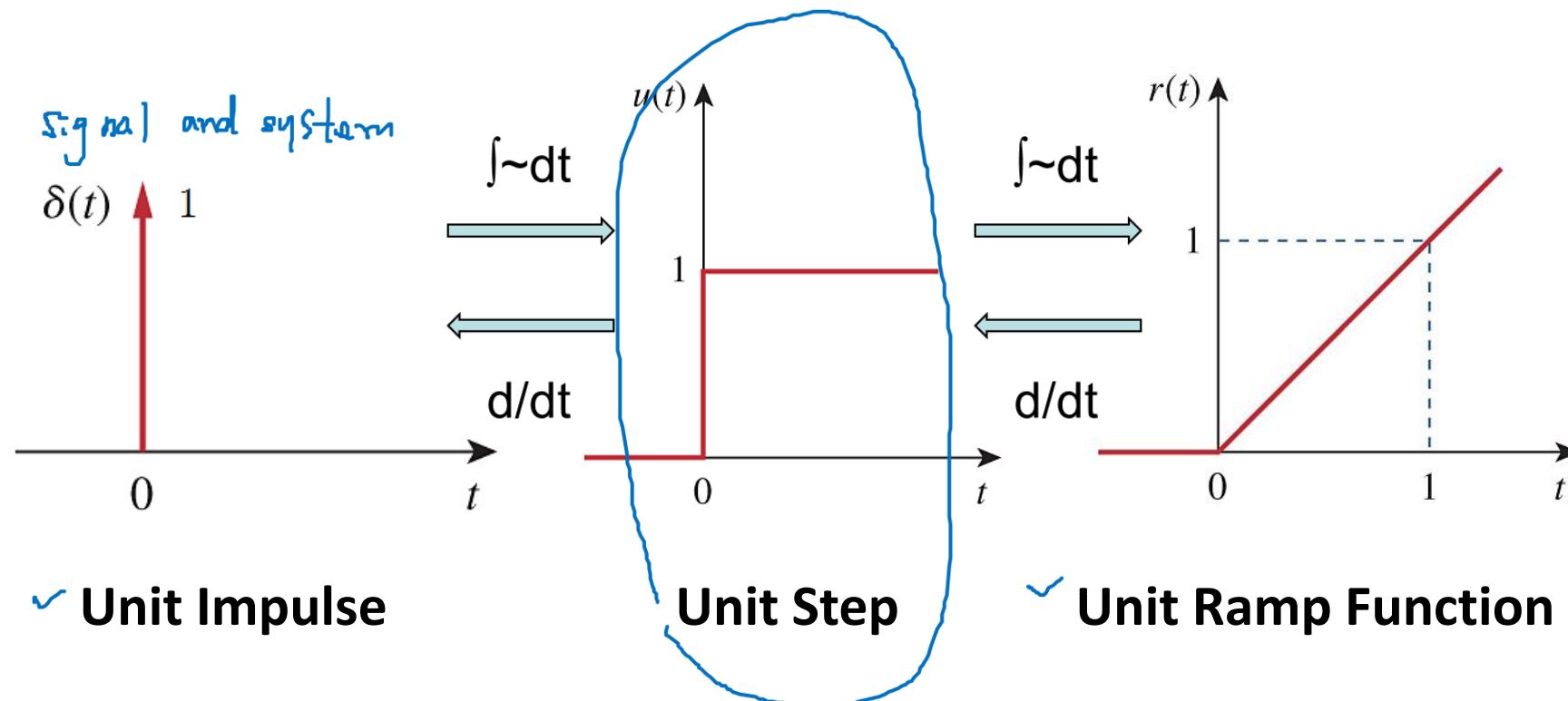
$$i(t) = i(0)e^{-t/\tau} = 2e^{-t/0.5} = 2e^{-2t} \text{ } (\text{A})$$

## 7.4 Singularity Functions

- Singularity functions (also called switching functions) are functions that either are **discontinuous** or have **discontinuous derivatives**. They serve as good approximations to switching operations.



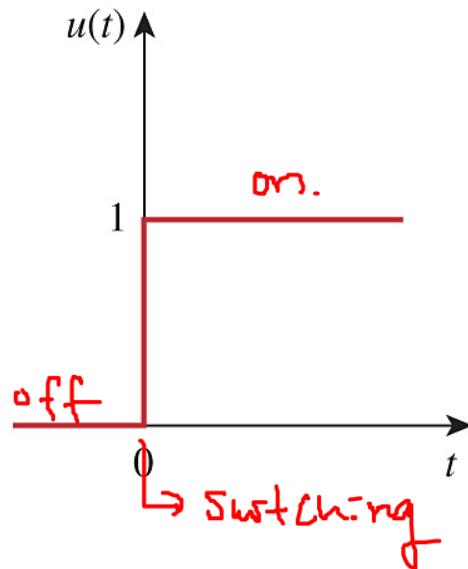
The three most widely used singularity functions in circuit analysis are the *unit step*, the *unit impulse*, and the *unit ramp* functions.



## (i) Unit Step

The unit step function  $u(t)$  is 0 for negative values of  $t$  and 1 for positive values of  $t$ . In mathematical terms,

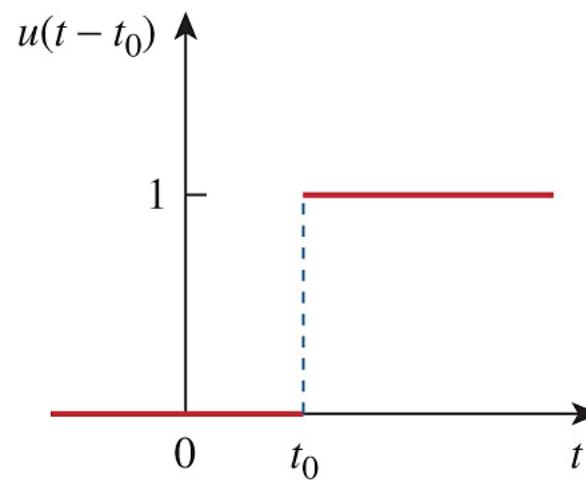
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



The unit step function is not defined at  $t = 0$ , where it changes abruptly from 0 to 1.

If the abrupt change occurs at  $t = t_0$  (where  $t_0 > 0$ ) instead of  $t = 0$ , the mathematical representation becomes

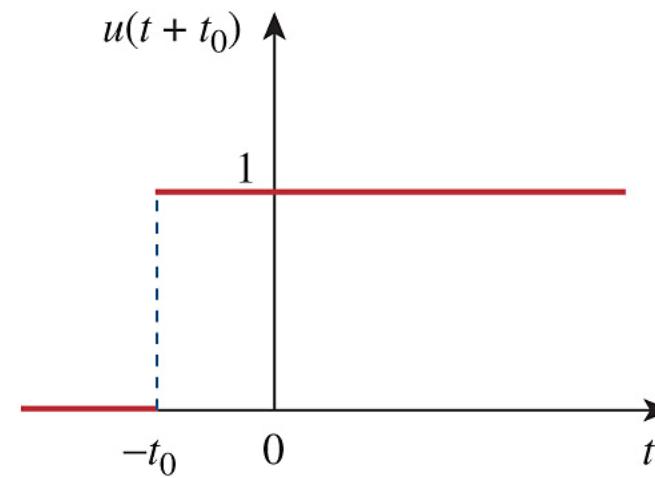
$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$



meaning that  $u(t)$  is delayed by  $t_0$  seconds.

If the abrupt change occurs at  $t = -t_0$  (where  $t_0 > 0$ ) instead of  $t = 0$ , the mathematical representation becomes

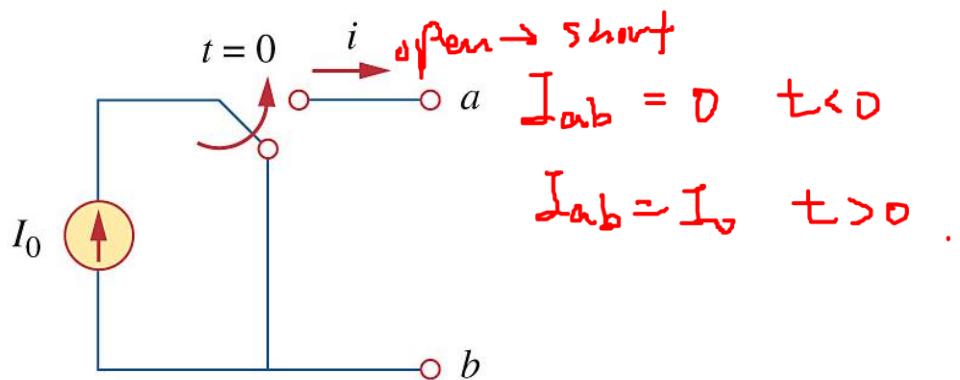
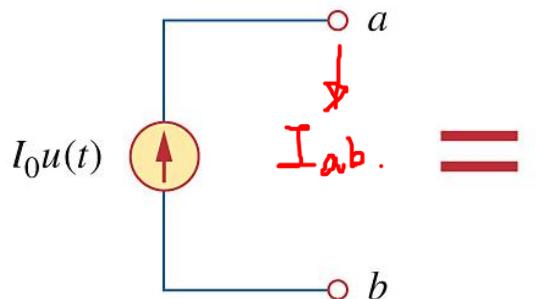
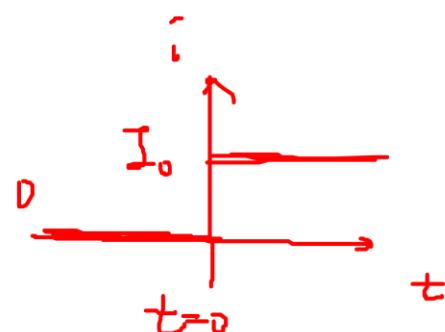
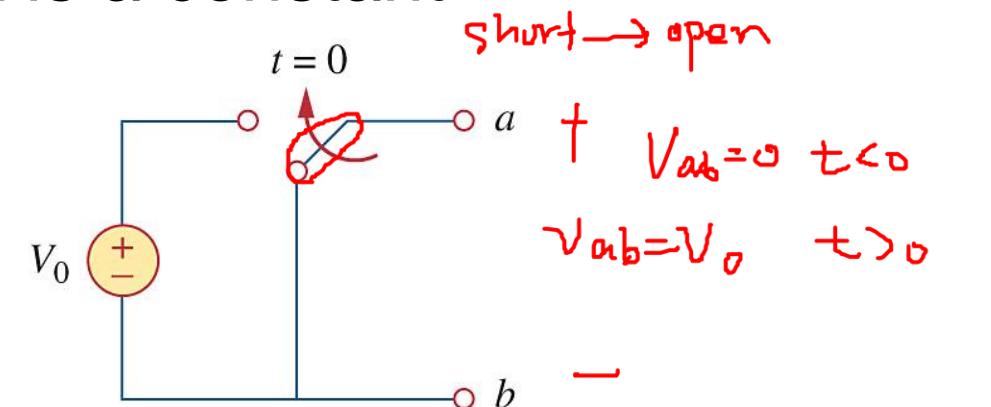
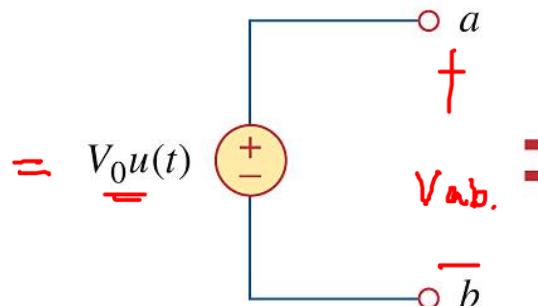
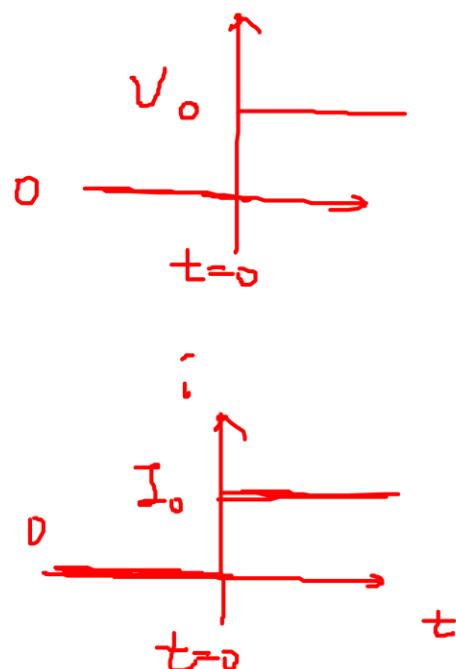
$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases}$$



meaning that  $u(t)$  is advanced by  $t_0$  seconds.

# Switching operations

**Switching operations** create abrupt changes in voltages and currents. An abrupt change can be represented by the **step function**,  $V_0 u(t)$ , where  $V_0$  is a constant.

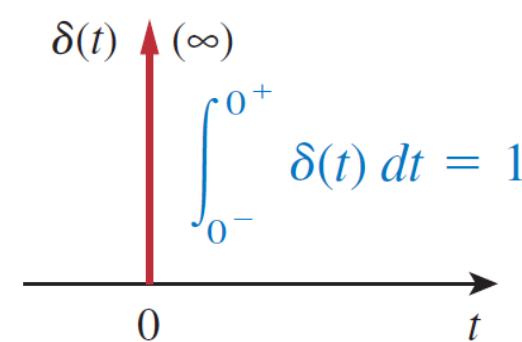


## (ii) Unit Impulse

The derivative of the unit step function  $u(t)$  is the unit impulse function  $\delta(t)$ , or delta function.

$$\delta(t) = \frac{d}{dt} u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$

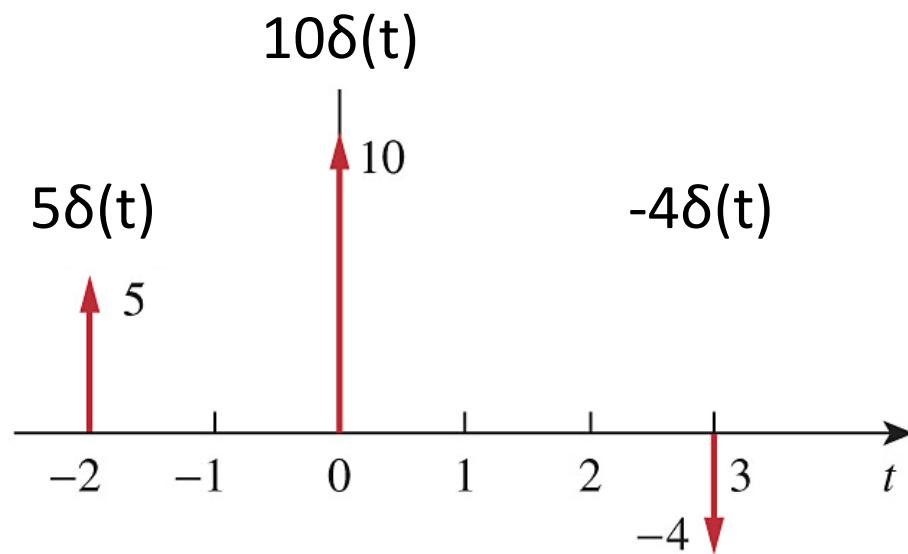
$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau$$



The **unit impulse function**  $\delta(t)$  is zero everywhere except at  $t = 0$ , where it is undefined, i.e. a signal of infinite amplitude and zero duration.

The **area** is known as the **strength** of the function.

e.g.  $10\delta(t)$  has an area of 10.



The unit impulse function has a property:

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

where  $f(t)$  is a function that is continuous at  $t = t_0$ .

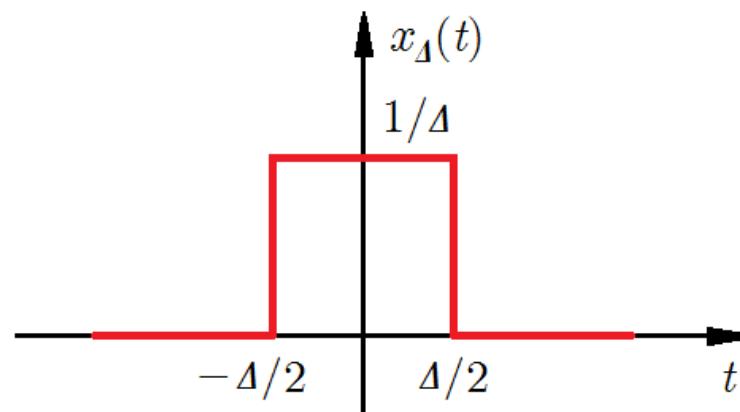
This means: when a function is integrated with the impulse function, we obtain the value of the function at the point where the impulse occurs → **sampling or sifting property**.

**Proof :**

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt &= \int_{-\infty}^{\infty} f(t_0)\delta(t-t_0)dt \\ &= f(t_0) \int_{-\infty}^{\infty} \delta(t-t_0)dt = f(t_0)\end{aligned}$$

# Unit Impulse Another Approach

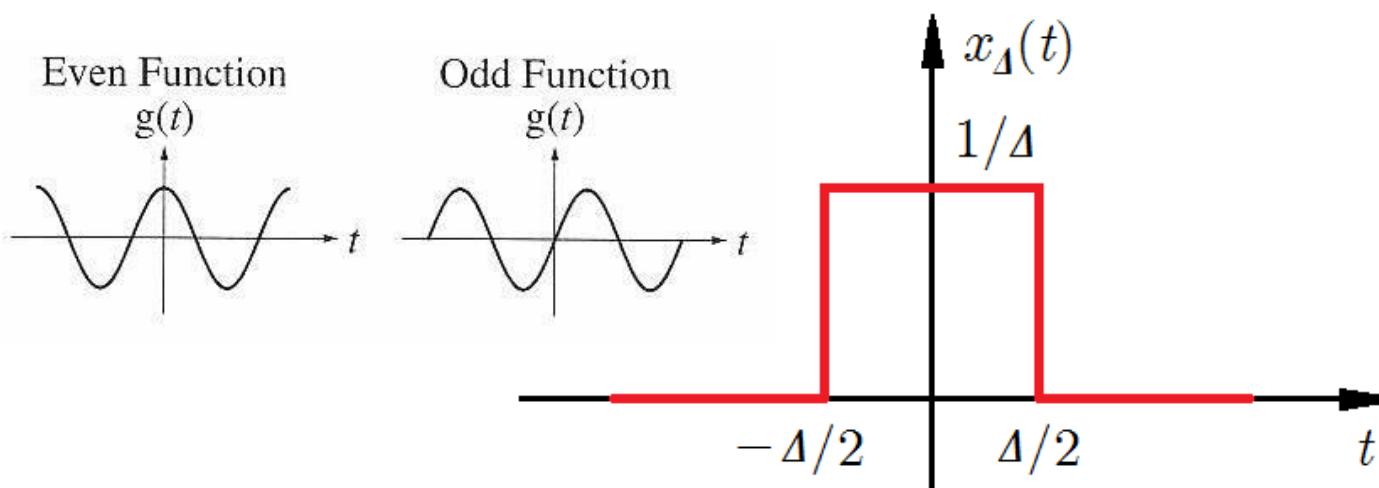
Singularity functions can be used to write mathematical expressions for signals. For example, the rectangular pulse



$$x_{\Delta}(t) = \frac{1}{\Delta} (u(t + \Delta/2) - u(t - \Delta/2))$$

We view the unit impulse function  $\delta(t)$  as the limiting form of any pulse, say the rectangular pulse  $x_\Delta(t)$  (shown below), that is an even function of time  $t$  with unit area:

$$\delta(t) = \lim_{\Delta \rightarrow 0} x_\Delta(t)$$



## Proof

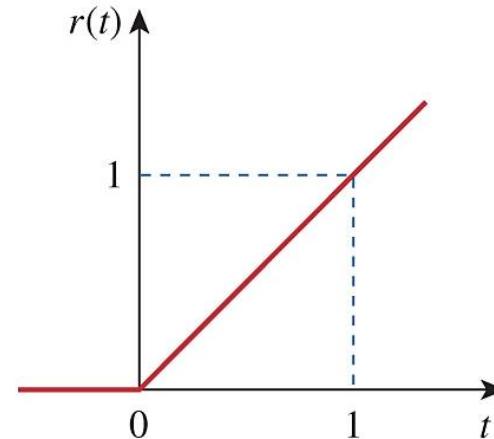
Without loss of generality, we take the rectangular pulse as an example.

$$\begin{aligned}\delta(t) &= \lim_{\Delta \rightarrow 0} x_\Delta(t) \\ &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} (u(t + \Delta/2) - u(t - \Delta/2)) \\ &= \lim_{\Delta \rightarrow 0} \frac{u(t + \Delta/2) - u(t - \Delta/2)}{(t + \Delta/2) - (t - \Delta/2)} = \frac{d}{dt} u(t)\end{aligned}$$

### (iii) Unit Ramp

The unit ramp function is zero for negative values of  $t$  and has a unit slope for positive values of  $t$ .

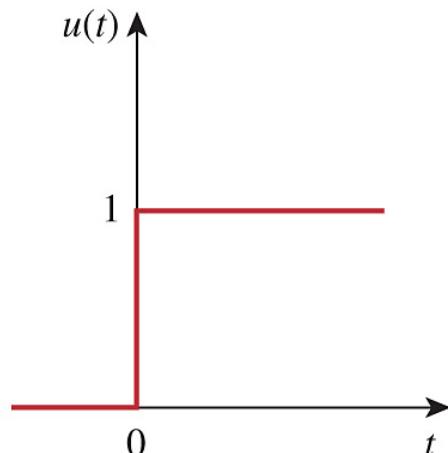
$$r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases}$$



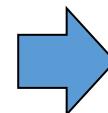
Integrating the unit step function  $u(t)$  results in the unit ramp function  $r(t)$ .

$$r(t) = \int_{-\infty}^t u(\tau) d\tau = tu(t)$$

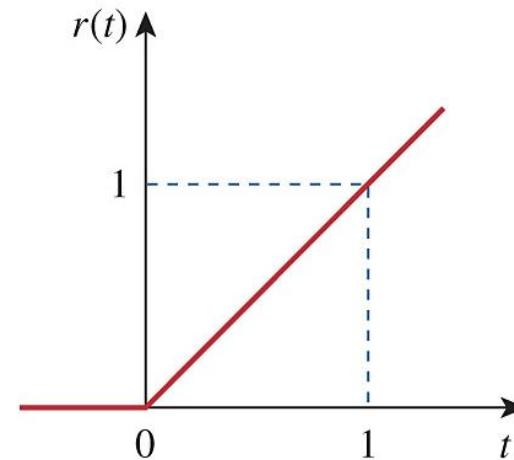
Unit step



$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



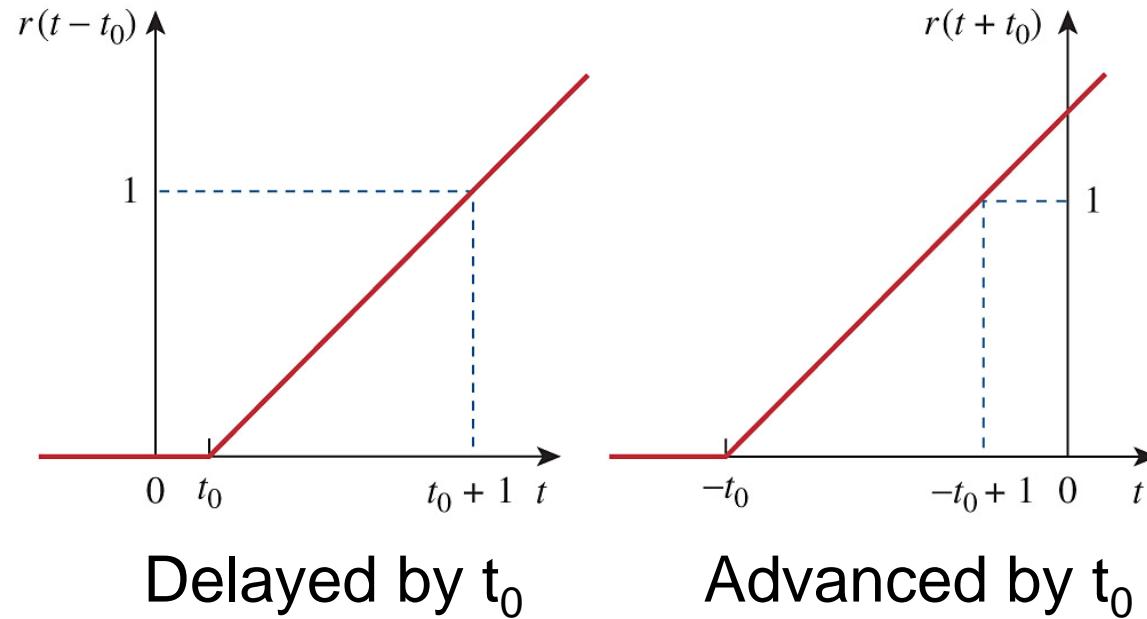
Unit ramp



$$r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases}$$

# Time delay in the unit ramp function

The unit ramp function may be delayed or advanced by time  $t_0$ .



**Example 7.6** Express the voltage pulse in Fig. 7.31 in terms of the unit step. Calculate its derivative and sketch it.

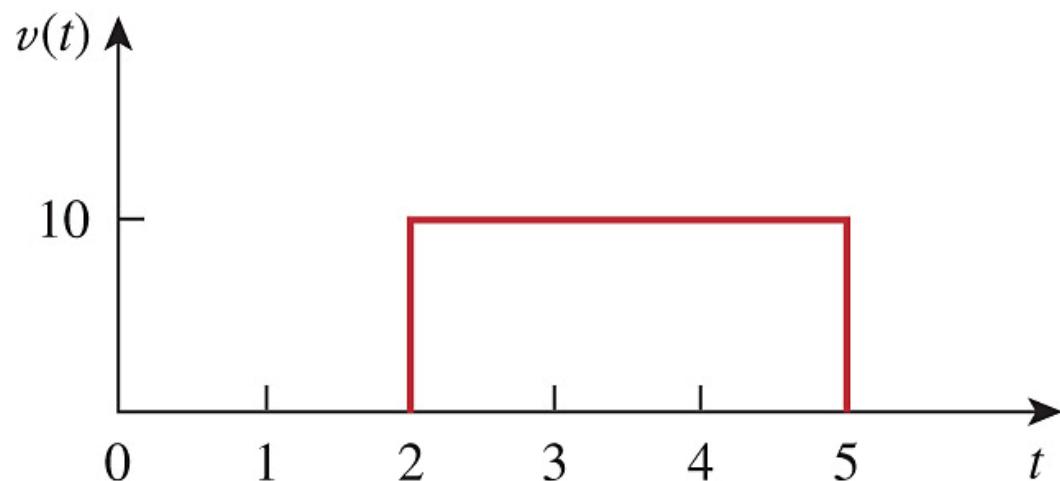


Figure 7.31

**Solution :**

$$v(t) = 10[u(t-2) - u(t-5)]$$

$$\frac{dv}{dt} = 10[\delta(t-2) - \delta(t-5)], \text{ see Fig. 7.32(b).}$$

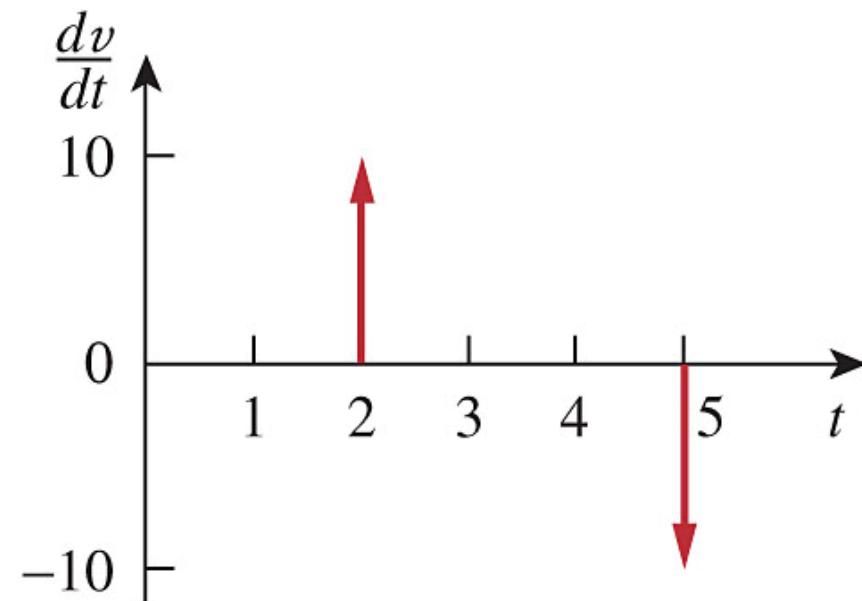


Figure 7.32(b)

**Example 7.7** Express the *sawtooth* function shown in Fig. 7.35 in terms of singularity functions.

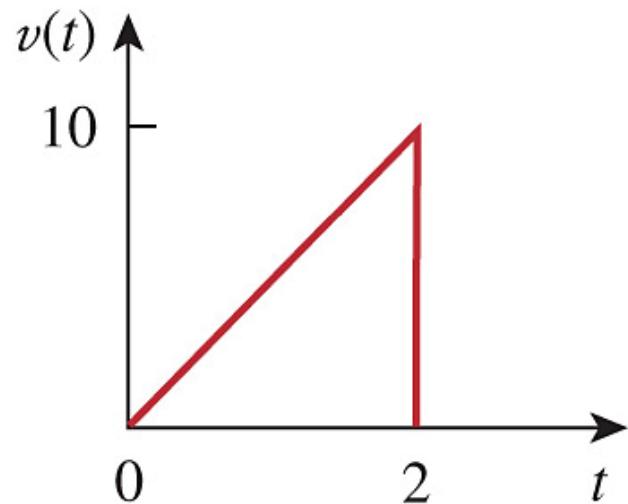
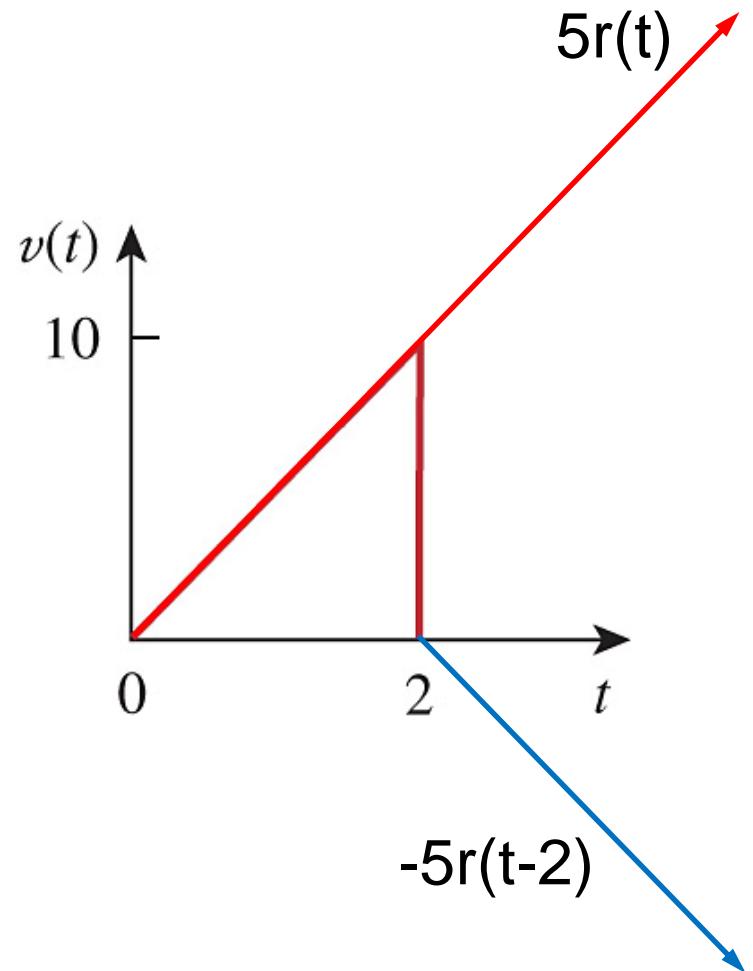
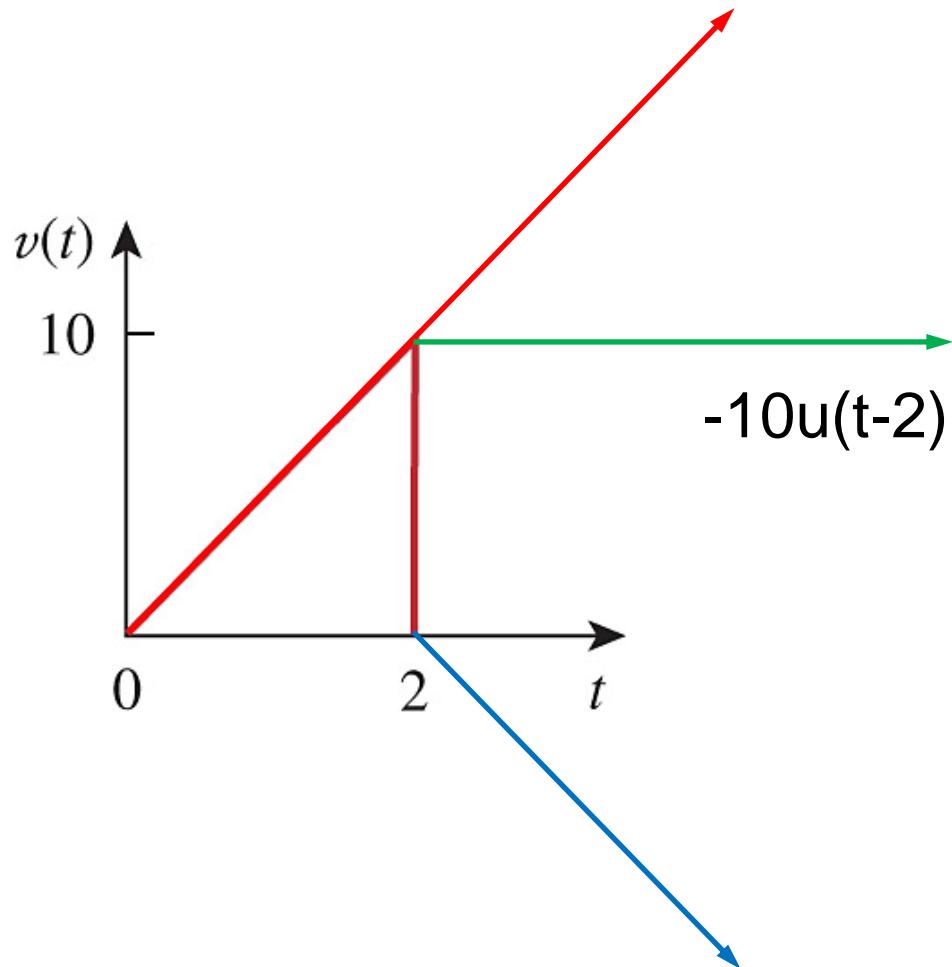
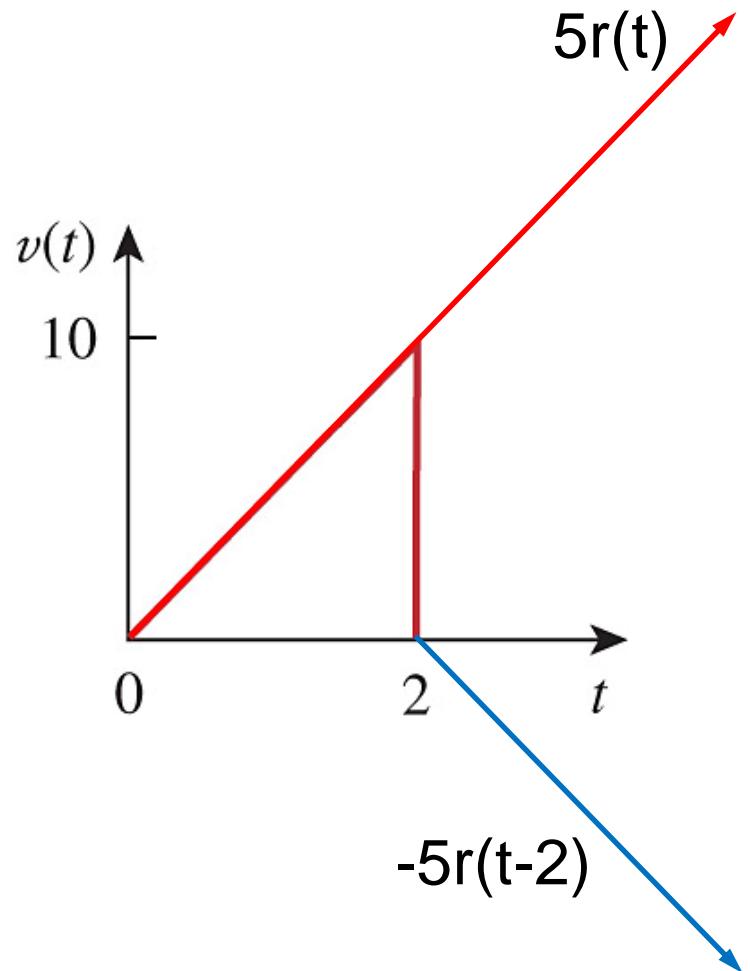


Figure 7.35



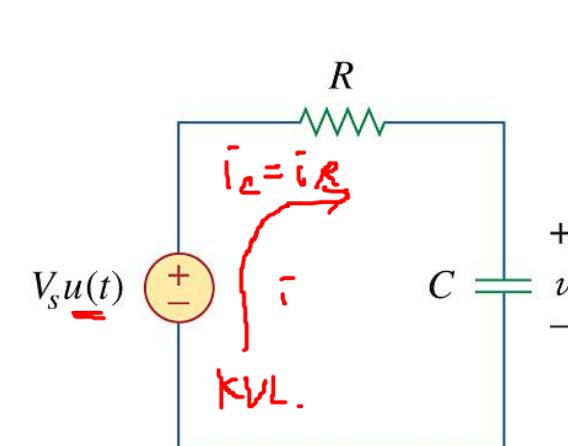
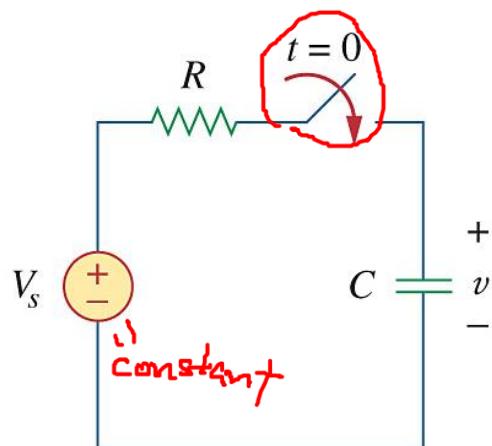


$$v(t) = 5r(t) - 5r(t-2) - 10u(t-2)$$

## 7.5 Step Response of an RC Circuit

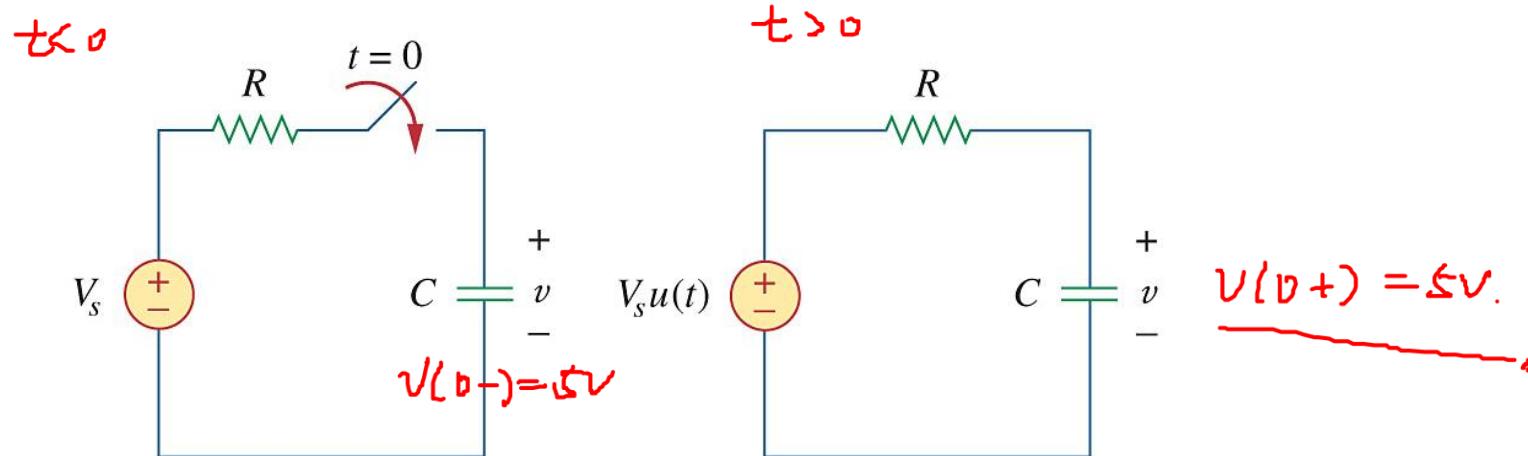
The **step response** of a circuit is its behavior due to a sudden application of the **step function**, either a voltage or a current source.

$$RC \frac{dV_C}{dt} + V_C = V_S$$
$$\frac{dV_C}{dt} + \frac{1}{RC} V_C = \frac{1}{RC} V_S$$



$$t > 0$$
$$-V_s + RI + V_c = 0 .$$
$$I = I_c = I_R$$
$$= C \frac{dV_c}{dt} .$$

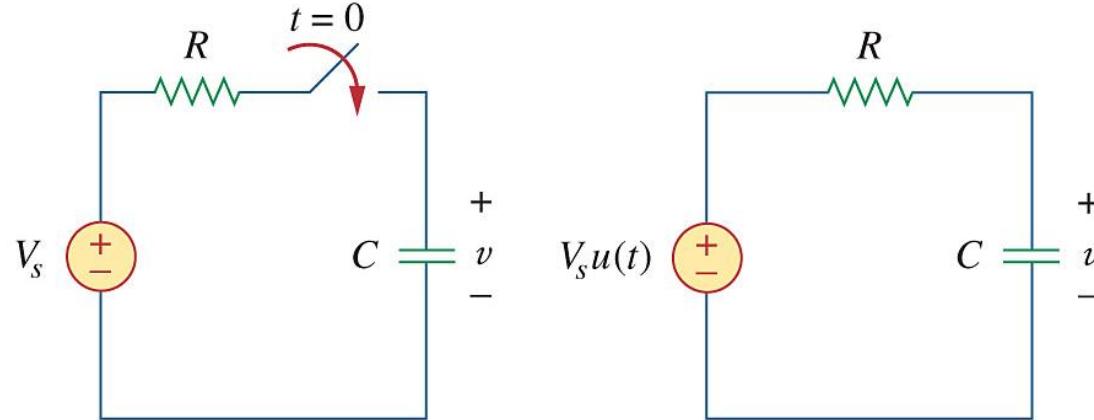
We select **the capacitor voltage as the circuit response** to be determined. We assume an initial voltage  $V_0$  on the capacitor.



Because the voltage across the capacitor cannot change instantaneously

$$v(0^+) = v(0^-) = V_0$$

**Initial condition**



For  $t > 0$ ,

$$\left( C \frac{dv}{dt} \right) R + v = V_s \rightarrow \frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}$$

or

$$\boxed{\frac{dv}{dt} + \frac{1}{\tau} v = \frac{V_s}{\tau}}, \text{ where } \tau = RC$$

**Solve this differential equation**

## Method 1

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}$$



$$\frac{dv}{v - V_s} = -\frac{dt}{RC}$$

Integrate both sides

$$\ln(v - V_s) \Big|_{V_0}^{v(t)} = -\frac{t}{RC} \Big|_0^t \rightarrow \ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

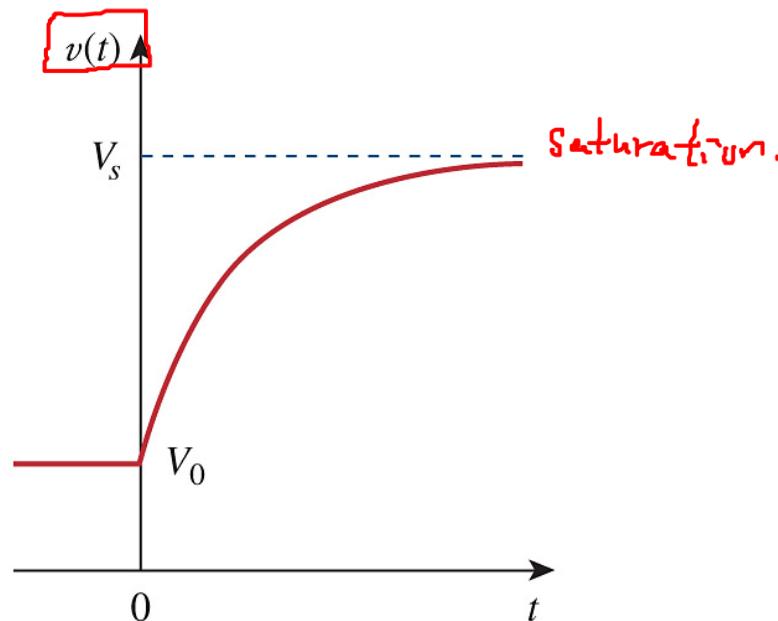
$$\ln\left(\frac{v - V_s}{V_0 - V_s}\right) = -\frac{t}{RC} \rightarrow \frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \quad \tau = RC$$

Finally we get  $v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \quad t > 0$

Thus, the complete response (or total response) is

$$v(t) = \begin{cases} V_0, & t \rightarrow 0^+ \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$

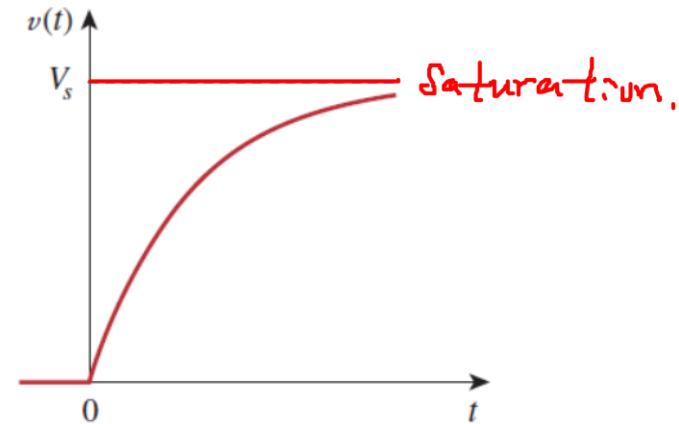
Assuming that  $V_s > V_0$  a plot of  $v(t)$  is



If we assume that the capacitor is **uncharged initially**

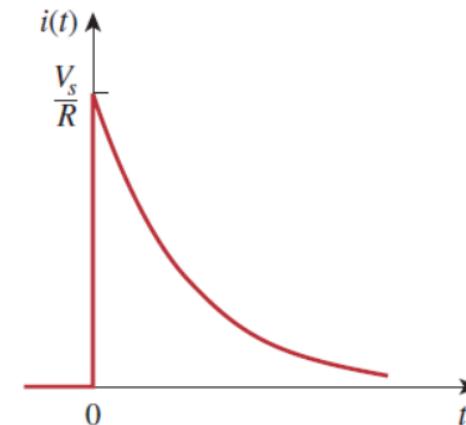
$$v(t) = \begin{cases} 0, & V(0) = 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

Or  $v(t) = V_s(1 - e^{-t/\tau})u(t)$   
 $t \rightarrow \infty \quad V(\infty) = V_s$



The current through the capacitor is  $i(t) = C \frac{dv}{dt}$  thus we get

$$i(t) = \frac{V_s}{R} e^{-t/\tau} u(t)$$



## Method 2

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} \quad \text{use a test solution } v(t) = Ae^{rt}$$

$$r + \frac{1}{\tau} = 0 \Rightarrow r = -\frac{1}{\tau}$$

(i) The homogeneous solution or *natural response*

$$v_n(t) = Ae^{-t/\tau}$$

(ii) Suppose the particular solution or force response

$$v_f(t) = B$$

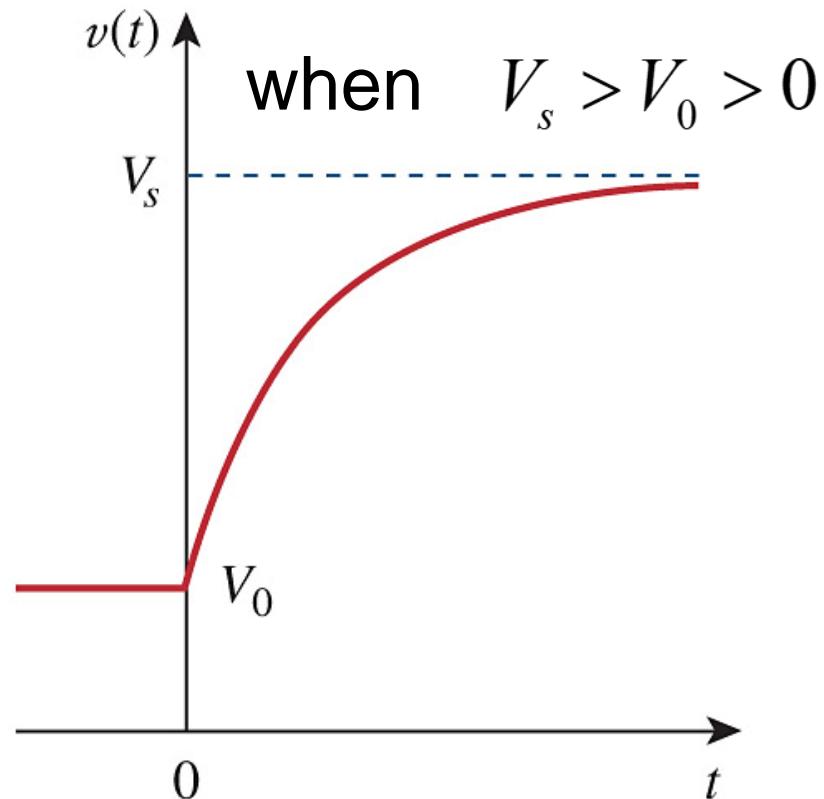
$$\frac{dB}{dt} + \frac{1}{\tau}B = \frac{V_s}{\tau} \Rightarrow B = V_s$$

## The complete solution (= complete response or total response)

$$v(t) = v_n(t) + v_f(t) = Ae^{-t/\tau} + V_s \quad \left. \begin{array}{l} \\ \text{When } t = 0^+, \\ v(0^+) = A + V_s = V_0 \Rightarrow A = V_0 - V_s \end{array} \right\} v(t) = (V_0 - V_s)e^{-t/\tau} + V_s$$

This is **the response of the RC circuit** to a sudden application of a constant dc source, assuming the capacitor is initially charged.

$$v(t) = (V_0 - V_s)e^{-t/\tau} + V_s$$



If the capacitor is **uncharged** initially, i.e.,  $V_0 = 0$ , then

$$v(t) = V_s(1 - e^{t/\tau}), \quad t > 0 \quad \text{or} \quad v(t) = V_s(1 - e^{t/\tau})u(t)$$

$v_{(0)} = V_s$

This is the **zero-state response**. The zero-state response corresponding to a unit-step input is called the unit-step response.

## Interpretations of the response

From the equation  $v(t) = V_s + (V_o - V_s)e^{-t/\tau}, \quad t > 0$

**We can have two interpretations:**

(i) Complete response = natural ( $v_n$ ) + forced response( $v_f$ )

$$v_n = V_o e^{-t/\tau} \quad v_f = V_s(1 - e^{-t/\tau})$$

(ii) Complete response = transient ( $v_t$ ) + steady-state ( $v_{ss}$ )

$$v_t = (V_o - V_s)e^{-t/\tau} \quad v_{ss} = V_s$$

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \quad t > 0$$

(i) Complete response = natural ( $v_n$ ) + forced response( $v_f$ )

$$v_n = V_o e^{-t/\tau} \quad v_f = V_s(1 - e^{-t/\tau})$$

The natural response ( $v_n$ ) is from the energy initially stored.

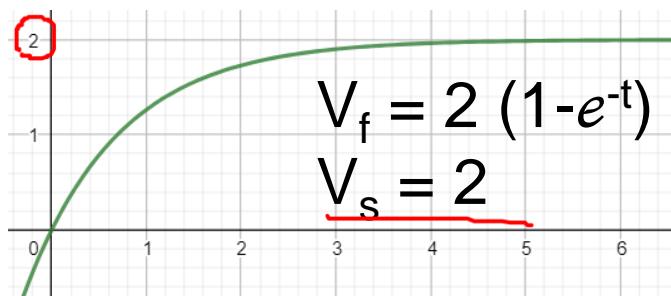
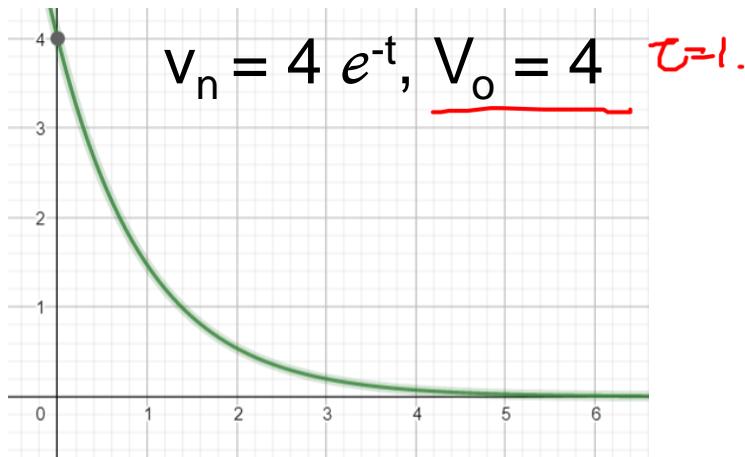
The forced response ( $v_f$ ) is by an external force (power sources).

Exponential signals die out eventually, leaving only the steady-state component of the forced response.

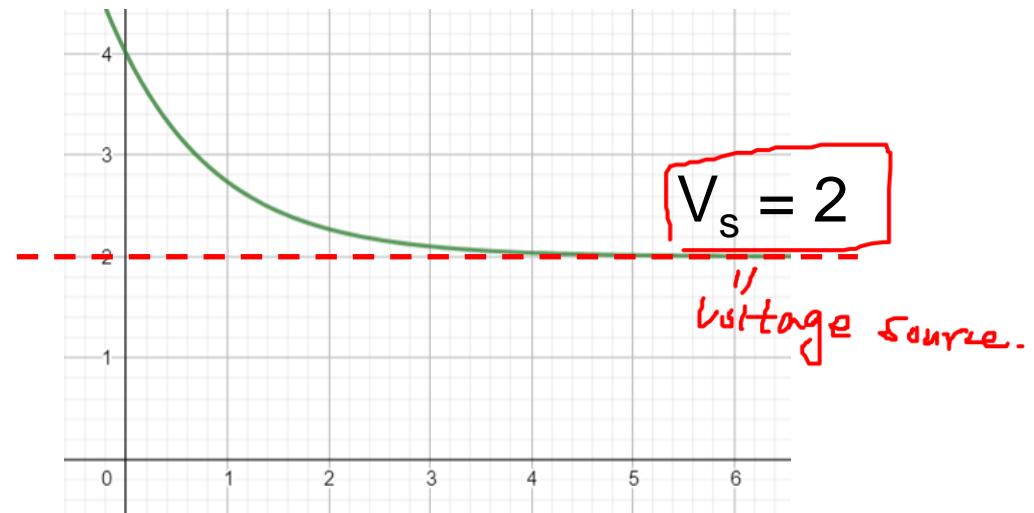
(i) Complete response = natural ( $v_n$ ) + forced response( $v_f$ )

$$v_n = V_o e^{-t/\tau} \quad v_f = V_s(1 - e^{-t/\tau})$$

$V_0 > V_s > 0$



$$\text{Response} = 2 (1 - e^{-t}) + 4 e^{-t}$$



$$v(t) = V_s + (V_o - V_s)e^{-t/\tau}, \quad t > 0$$

(ii) Complete response = transient ( $v_t$ ) + steady-state ( $v_{ss}$ )

$$v_t = (V_o - V_s)e^{-t/\tau} \quad v_{ss} = V_s$$

The **transient response** is the temporary response that will die out with time.

$t = \infty$ .

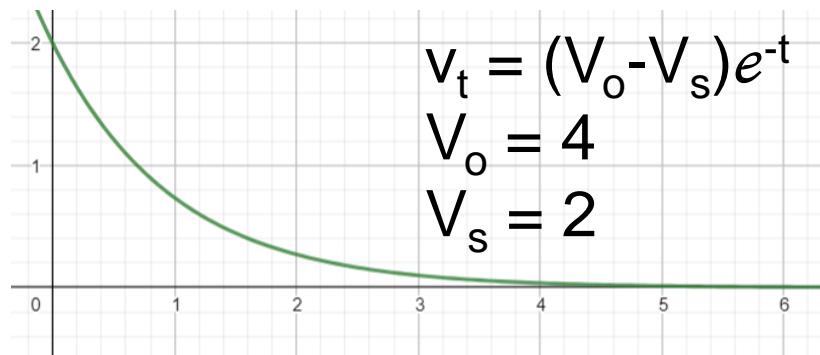
The **steady-state response** is the behavior of the circuit a long time after an external excitation is applied.

Under certain conditions,  $v_n$  and  $v_t$  response are the same. The same can be said about the  $v_f$  and  $v_{ss}$  response.

(ii) Complete response = transient ( $v_t$ ) + steady-state ( $v_{ss}$ )

$$v_t = (V_o - V_s)e^{-t/\tau} \quad v_{ss} = V_s$$

$$V_o > V_s > 0$$



$$\text{Response} = V_s + (V_o - V_s) e^{\lambda(-t)}$$



The complete response

$$v(t) = V_S + (V_0 - V_S)e^{-t/\tau}, \quad t > 0$$

may be written as

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

$v(0)$  is the initial voltage at  $t = 0^+$

$v(\infty)$  is the final or steady-state value

Thus, to find the step response of an *RC* circuit requires **three parameters**:

1. The initial capacitor voltage  $v(0)$
2. The final capacitor voltage  $v(\infty)$
3. The time constant  $\tau$ .

**Example 7.10** The switch in Fig.7.43 has been in position *A* for a long time. At  $t = 0$ , the switch moves to *B*. Determine  $v(t)$  for  $t > 0$  and calculate its value at  $t = 1$  s and 4s.

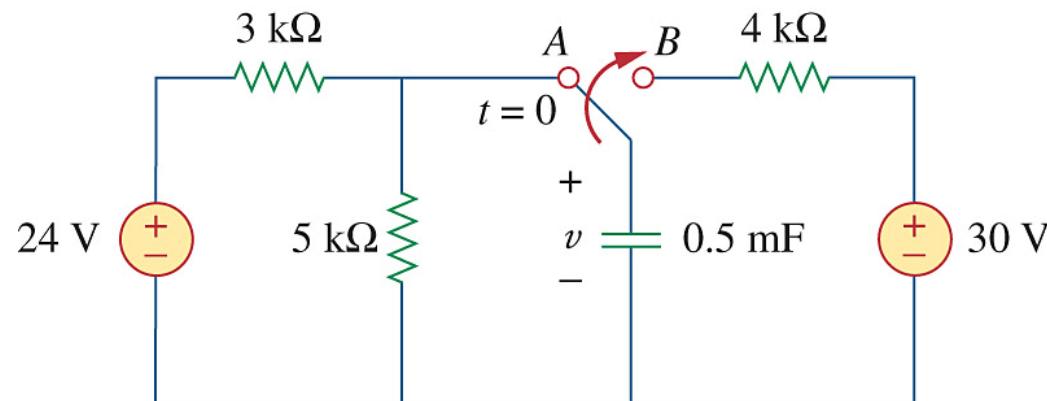
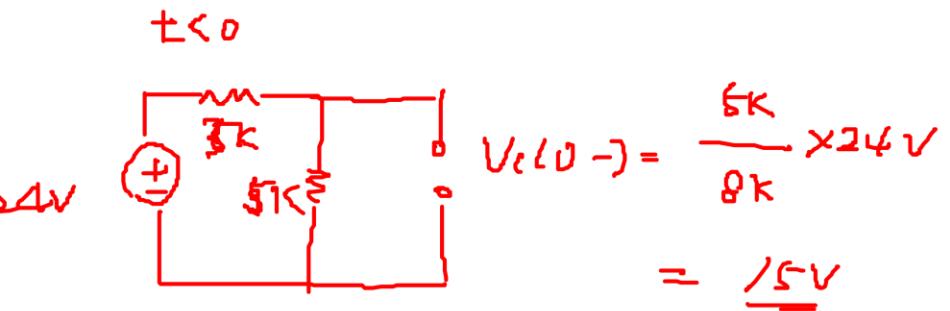


Figure 7.43

$$v(t) = 30 + (15 - 30)e^{-\frac{t}{2}}$$

$$= 30 - 15e^{-\frac{t}{2}}.$$



KVL:  $-30 + R_i i + v = 0$ .

$t > 0$

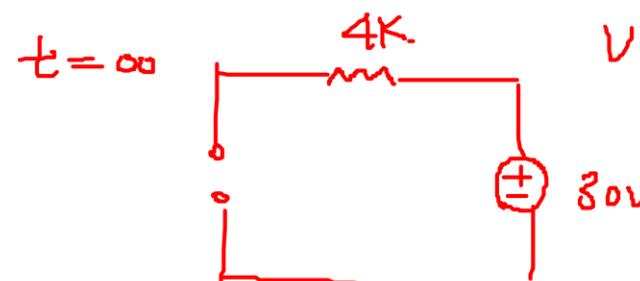
$$RC \frac{dv}{dt} + v = 30$$

$$RC \frac{dv}{dt} + \frac{1}{C} v = 15$$

$$v(t) = e^{\frac{-t}{2}} \left( \int 15 e^{\frac{t}{2}} dt + C \right)$$

$$v(0) = 30V$$

$$RC = 2s$$



**Example 7.10** The switch in Fig.7.43 has been in position *A* for a long time. At  $t = 0$ , the switch moves to *B*. Determine  $v(t)$  for  $t > 0$  and calculate its value at  $t = 1$  s and 4s.

**Solution :**

For  $t < 0$ ,

$$v(t) = 24 \times \frac{5}{3+5} = 15 \text{ (V)}$$

$$v(0^-) = 15 \text{ V}$$

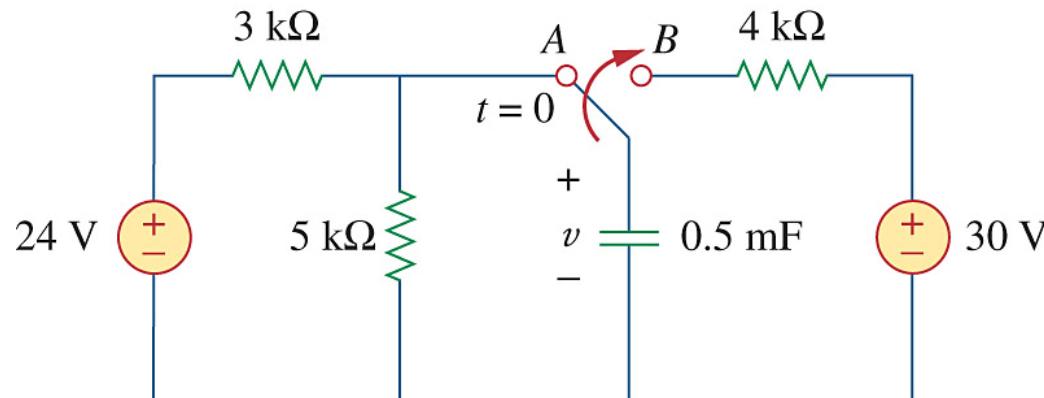


Figure 7.43

For  $t > 0$ ,

$$v(0^+) = v(0^-) = 15 \text{ V}$$

$$v(\infty) = 30 \text{ V}$$

$$\tau = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2 \text{ (s)}$$

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

$$= 30 + (15 - 30)e^{-t/2} = 30 - 15e^{-0.5t} \text{ (V)}$$

$$v(1) = 30 - 15e^{-0.5 \times 1} \approx 20.90 \text{ (V)}$$

$$v(4) = 30 - 15e^{-0.5 \times 4} \approx 27.97 \text{ (V)}$$

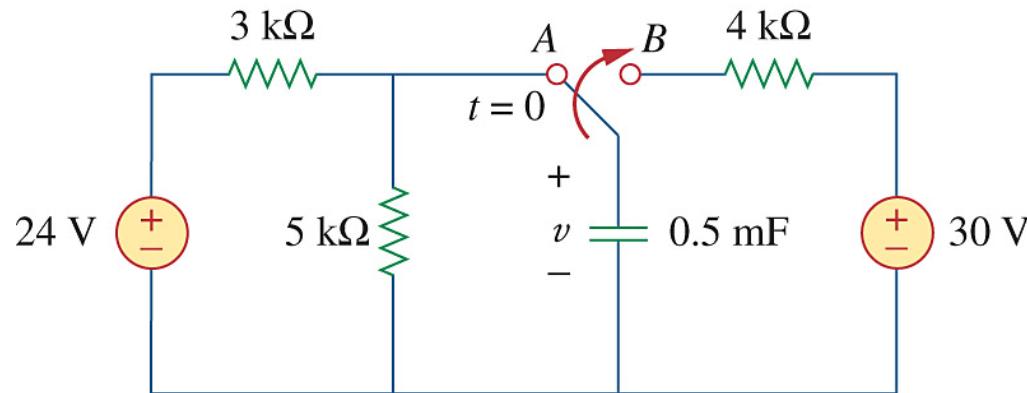


Figure 7.43

## 1<sup>st</sup> Order ODE (Optional)

$$y' + p(x)y = r(x) \rightarrow Fy' + Fp(x)y = Fr(x)$$

Left side:  $(Fy)' = Fy' + F'y$   
where  $F'y = Fp(x)y$ , then  $F' = Fp(x)$

$$F' = Fp(x) \rightarrow \frac{1}{F}F' = p(x) \rightarrow \ln F = \int p(x)dx$$

Thus,  $F = e^{\int p(x)dx}$  where we set  $\int p(x)dx = h$

$F = e^h$  then,  $p = h'$  due to  $h = \int p(x)dx$

$$F = e^h \quad p(x) = h'$$

$$\begin{aligned} Fy' + Fp(x)y &= Fr(x) \rightarrow e^h y' + e^h p(x)y = e^h r(x) \\ \rightarrow \underline{e^h y' + e^h h' y} &= e^h r(x) \end{aligned}$$

$(e^h y)' = e^h r(x)$  integral both sides

$$e^h y = \int e^h r(x) dx + c$$

$$\underline{y = e^{-h} \left( \int e^h r(x) dx + c \right)}$$
 where  $h = \int p(x) dx$

$$r_{(y)} = 0 \quad \text{natural response} \Rightarrow y = e^{-h} \times c.$$

## 1<sup>st</sup> Order ODE where $r(x) = 0$

$$y' + p(x)y = r(x), \text{ where } r(x) = 0$$

$$\frac{dy}{dx} + p(x)y = 0$$

$\rightarrow \frac{1}{y} dy = -p(x)dx$  integrate both sides

$$\ln y = - \int p(x)dx + c \rightarrow y = c \cdot e^{- \int p(x)dx}$$

e.g.  $\frac{di}{dt} + 5i = 0 \rightarrow i(t) = c \cdot e^{-5t}$

**Practice Problem 7.1** Refer to the circuit in Fig. 7.7. Let  $v_C(0) = 45$  V. Determine  $v_C$ ,  $v_x$ , and  $i_o$  for  $t \geq 0$ .

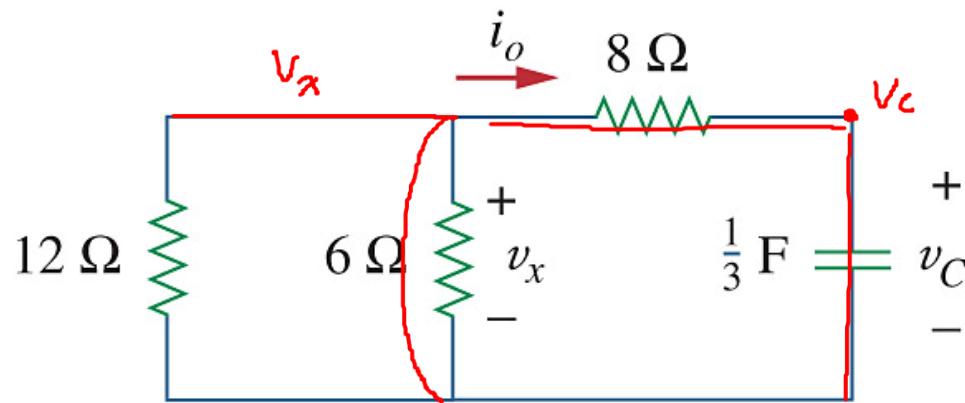


Figure 7.7 An  $RC$  circuit.

$$KCL: \frac{v_x}{12} + \frac{v_x}{6} + i_o = 0$$

$$V_T = 8i_o + v_C = RC \frac{dv_C}{dt} + v_L$$

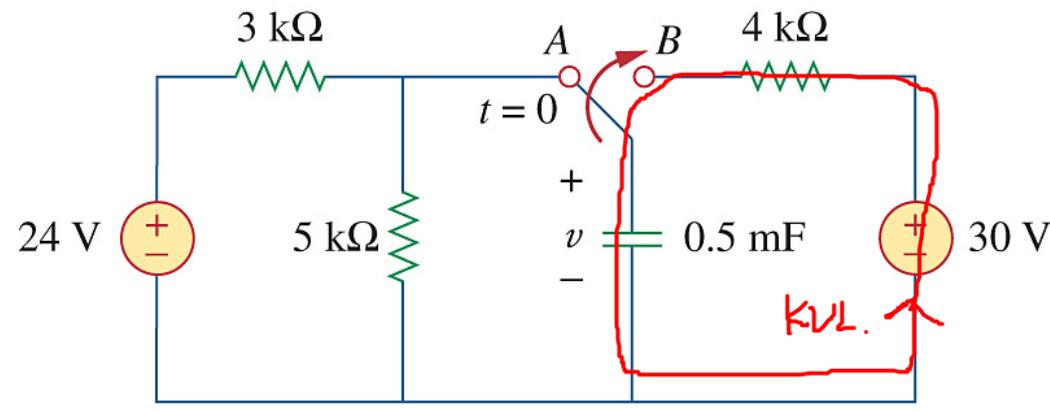
$$i_o = C \frac{dv}{dt}$$

$$\frac{1}{12} \left( RC \frac{dv_C}{dt} + v_C \right) + \frac{1}{6} \left( RC \frac{dv_C}{dt} + v_C \right) + C \frac{dv_C}{dt} = 0$$

$$H(\gamma) = 0$$

$$v_C = v_C(0)e^{-t/\tau} = 45e^{-t/4} = 45e^{-0.25t} \text{ (V)}$$

**Example 7.10** The switch in Fig.7.43 has been in position *A* for a long time. At  $t = 0$ , the switch moves to *B*. Determine  $v(t)$  for  $t > 0$  and calculate its value at  $t = 1$  s and 4s.



$$v(t) = 20 - 15e^{-\frac{1}{2}t}$$

$$v(0) = C + 30 = 15 \quad C = -15$$

$$\frac{dv}{dt} + \frac{1}{2}v = 15$$

$$e^{-\frac{1}{2}t} \left( \int e^{\frac{1}{2}t} \times 15 dt + C \right)$$

$$\Rightarrow (30e^{\frac{1}{2}t} + C) e^{-\frac{1}{2}t} \\ = C \cdot e^{-\frac{1}{2}t} + 30.$$

For  $t < 0$ ,

$$v(t) = 24 \times \frac{5}{3+5} = 15 \text{ (V)}$$

$$v(0^-) = \underline{15 \text{ V}}$$

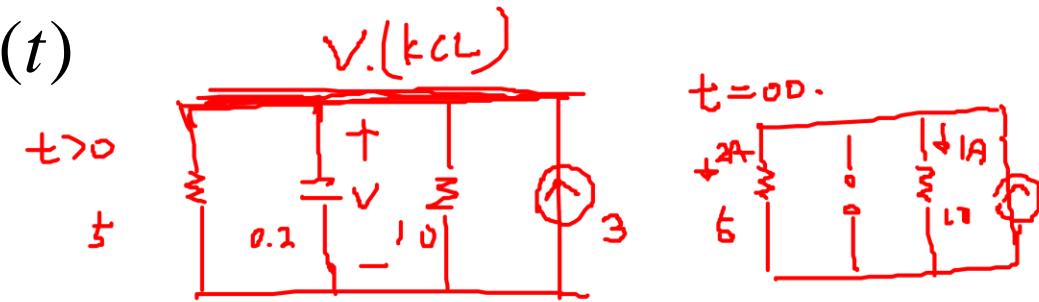
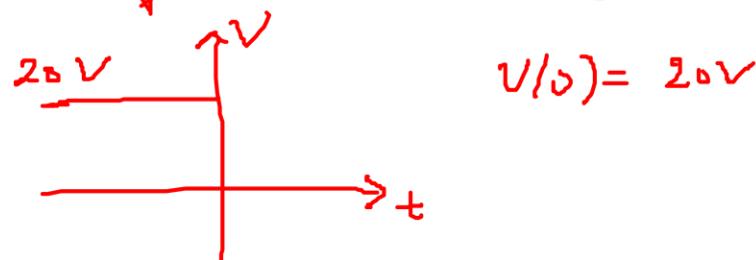
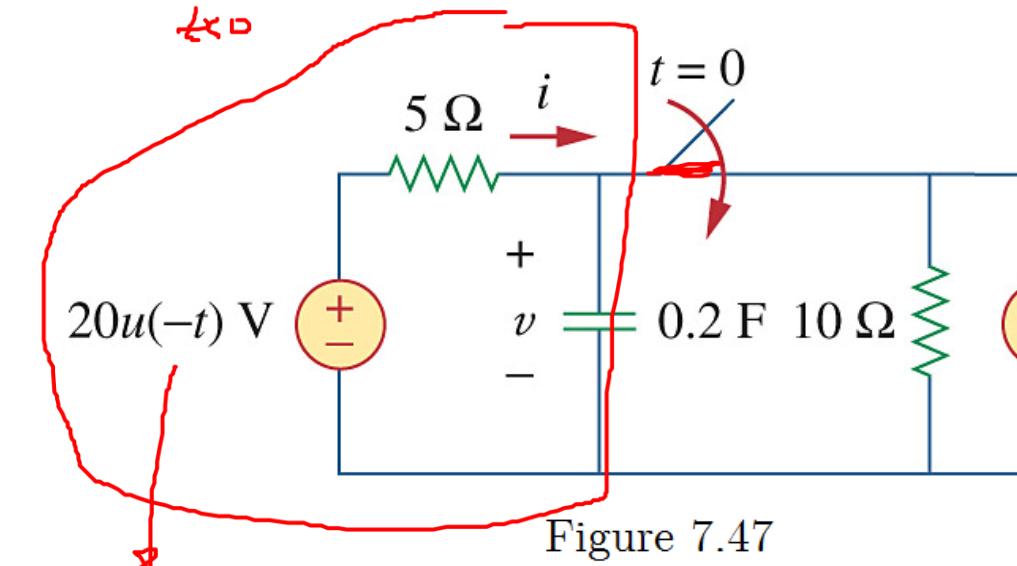
$$C = -15$$

## Practice Problem 7.11

The switch in Fig.

7.43 is closed at  $t = 0$ . Find  $i(t)$  and  $v(t)$

for all time.



$$R_{\text{eq}} = \frac{1}{\frac{1}{0.2} + \frac{1}{10}} = \frac{1}{\frac{1}{5} + \frac{1}{10}} = \frac{10}{3}$$

$$T = RC = \frac{10}{3} \times \frac{2}{10} = \frac{2}{3}$$

$$v(t) = 10 + [20 - 10] e^{-\frac{3}{2}t}$$

$$\text{kcl. } \frac{v}{5} + \frac{2}{10} \frac{dv}{dt} + \frac{v}{10} = 3 \Rightarrow \frac{dv}{dt} + \frac{3}{2}v = 15$$

$$v(t) = e^{-\frac{3}{2}t} \left[ \int 15e^{\frac{3}{2}t} dt + C \right] \Rightarrow 10 + 10e^{-\frac{3}{2}t}$$

$$v(0) = 10 + C = 20 \quad C = 10$$

For  $t > 0$ ,

$$v(0^+) = v(0^-) = 20 \text{ V}$$

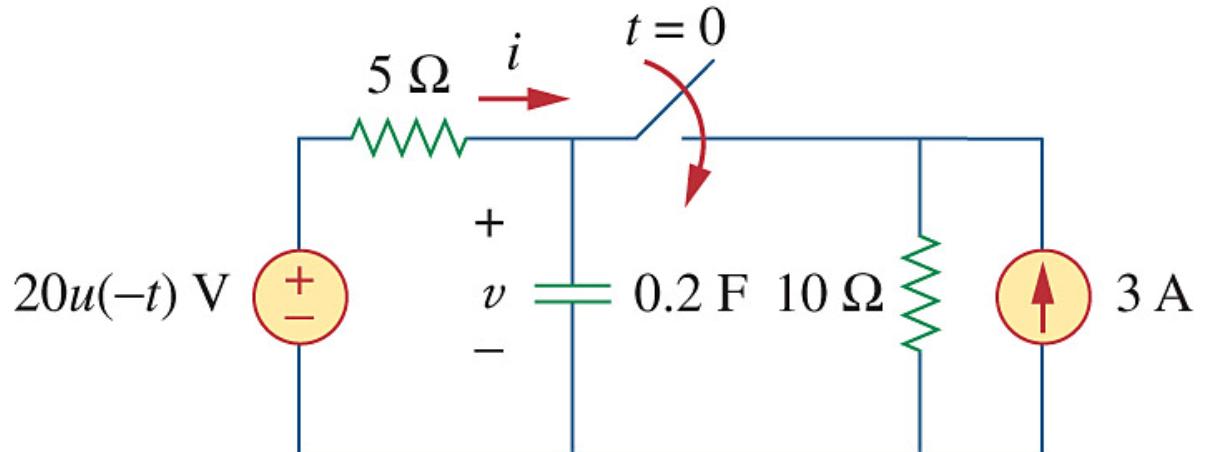
$$v(\infty) = 3 \times (5 \parallel 10) = 10 \text{ (V)}$$

$$\tau = (5 \parallel 10) \times 0.2 = \frac{2}{3} \text{ (s)}$$

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

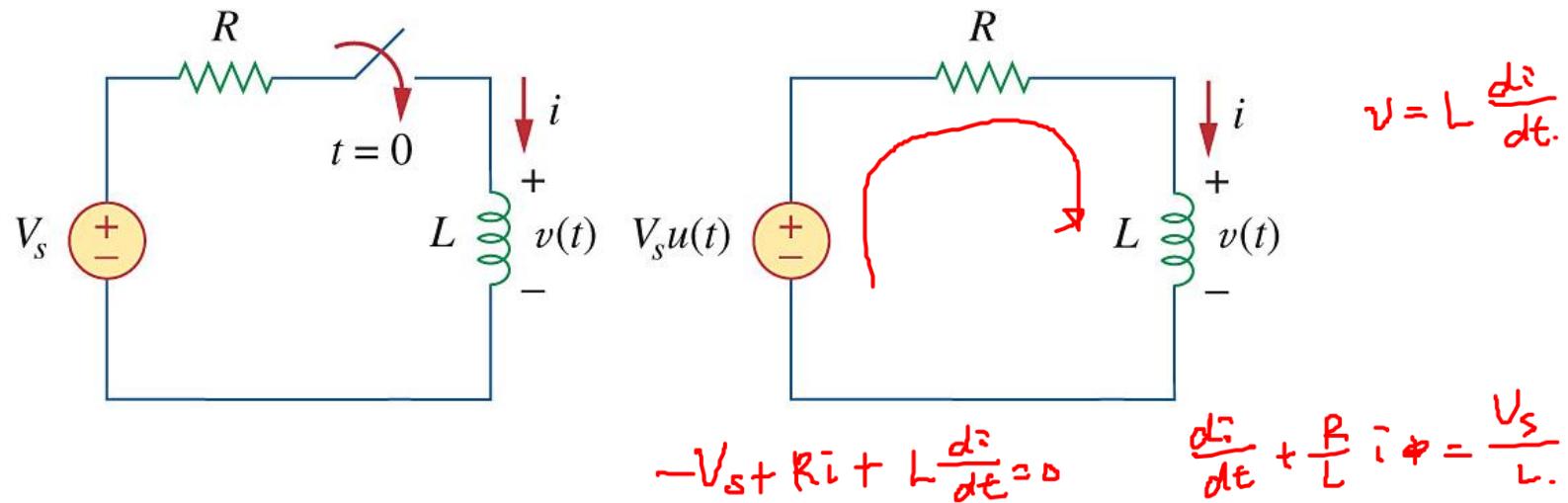
$$= 10 + (20 - 10)e^{-t/(2/3)} = 10(1 + e^{-1.5t}) \text{ (V)}$$

$$i(t) = -\frac{v(t)}{5} = -2(1 + e^{-1.5t}) \text{ (A)}$$

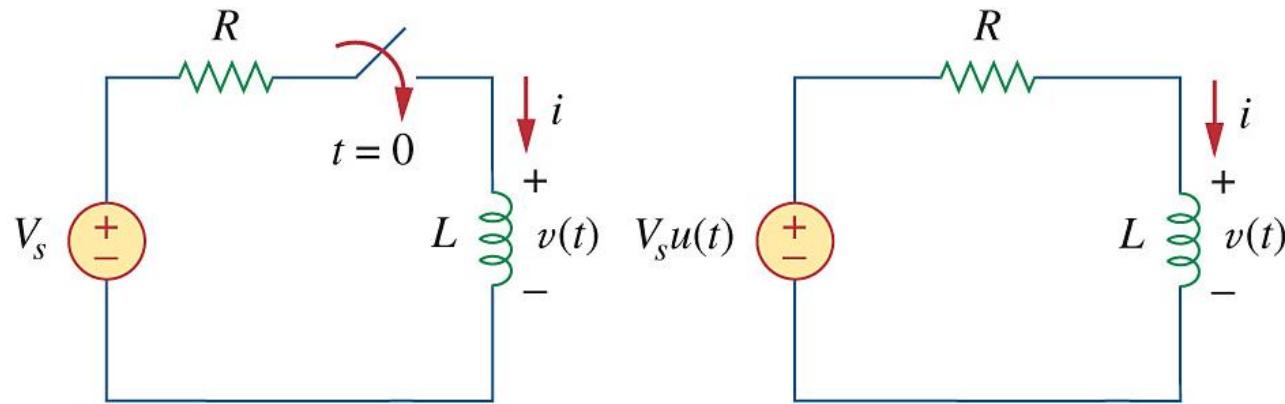


$v(0^+)$ : by the value of  $v(0^-)$   
 $v(\infty)$ : by the circuit at  $t > 0$

## 7.6 Step Response of an RL Circuit



We select the inductor current  $i$  as the circuit response to be determined. We assume an initial current  $I_0$  in the inductor.



Because the current through the inductor cannot change instantaneously, the initial condition is  $i(0^+) = i(0^-) = I_0$

For  $t > 0$ ,

$$iR + L \frac{di}{dt} = V_s$$

or

$$\frac{di}{dt} + \frac{1}{\tau} i = \frac{V_s / R}{\tau}, \text{ where } \tau = \frac{L}{R}$$

**RL**

$$\frac{di}{dt} + \frac{1}{\tau} i = \frac{V_s / R}{\tau}, \text{ where } \tau = \frac{L}{R}$$

**RC**

$$\frac{dv}{dt} + \frac{1}{\tau} v = \frac{V_s}{\tau}, \text{ where } \tau = RC$$

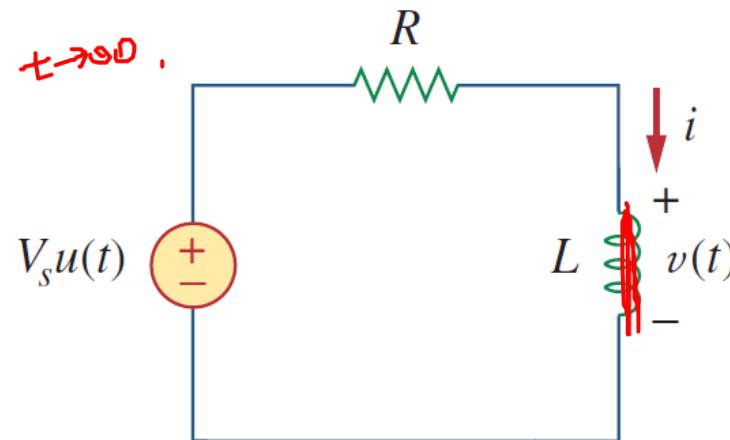
$$v(t) = (V_0 - V_s)e^{-t/\tau} + V_s$$

Since this differential equation has the same form as that describing the RC circuit,

$$\Rightarrow i(t) = \frac{V_s}{R} + \left( I_0 - \frac{V_s}{R} \right) e^{-t/\tau}$$

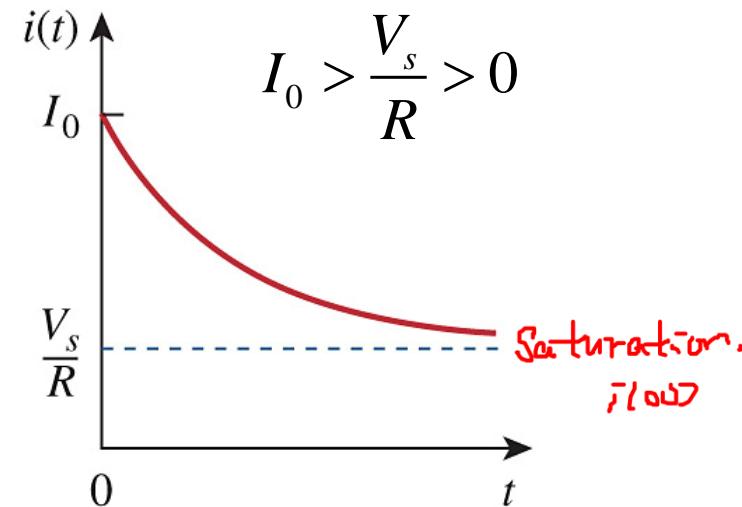
It is evident that

$$i(\infty) = \frac{V_s}{R}$$

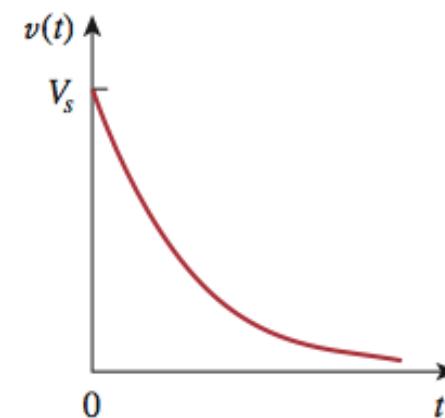
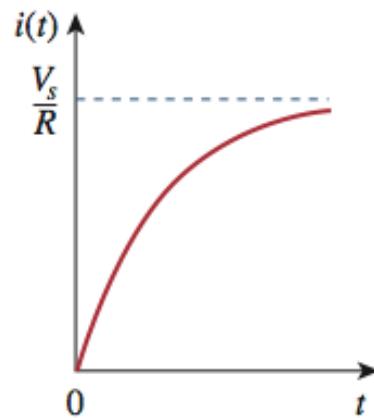


The plot of the RL step response is thus

$$i(t) = \frac{V_s}{R} + \left( I_0 - \frac{V_s}{R} \right) e^{-t/\tau}$$



If  $I_0 = 0$ ,  $i(t) = \frac{V_s}{R} (1 - e^{-t/\tau}) u(t)$ ,  $v(t) = V_s e^{-t/\tau} u(t)$



The complete response can be written as

$$i(t) = i(\infty) + [i(0^+) - i(\infty)] e^{-t/\tau}$$

Thus, to find the response of a first-order RL circuit requires three conditions:

1. The initial inductor current  $i(0^+)$ .
2. The final inductor current  $i(\infty)$ .
3. The time constant  $\tau = L / R$ .

**Source-free RL circuit is a special case when  $i(\infty) = 0$**

**Example 7.13** At  $t = 0$ , switch 1 in Fig. 7.53 is closed, and switch 2 is closed 4 s later. Find  $i(t)$  for  $t > 0$ . Calculate  $i$  for  $t = 2$  s and  $t = 5$  s.

**Solution :**

For  $t < 0$ ,

$$i(t) = 0$$

$$\underline{i(0^-) = 0}$$

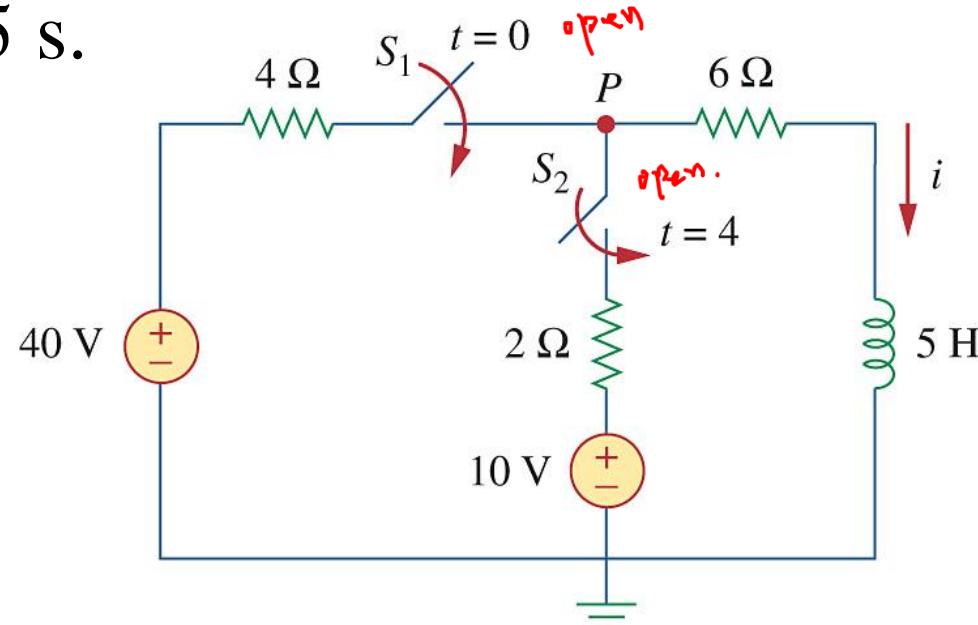
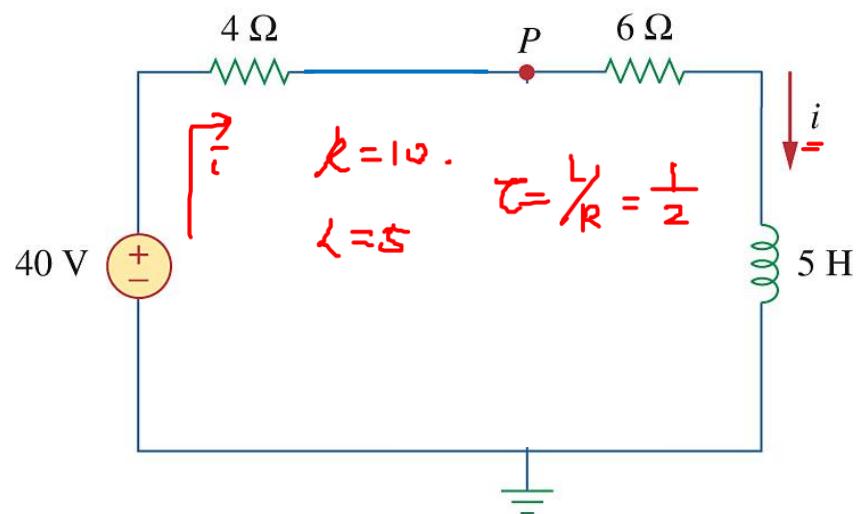


Figure 7.53

ii)  $0 < t < 4$



$$\vec{i}(0^+) = \frac{40V}{10\Omega} = 4A.$$

$$i(t) = 4 + (0 - 4)e^{-2t}$$

$$kVL: -40 + 10\vec{i} + 5 \frac{d\vec{i}}{dt} = 0$$

$$\frac{d\vec{i}}{dt} + 2\vec{i} = 8$$

$$\vec{i}(t) = e^{-2t} \left( \int 8e^{2t} dt + C \right)$$

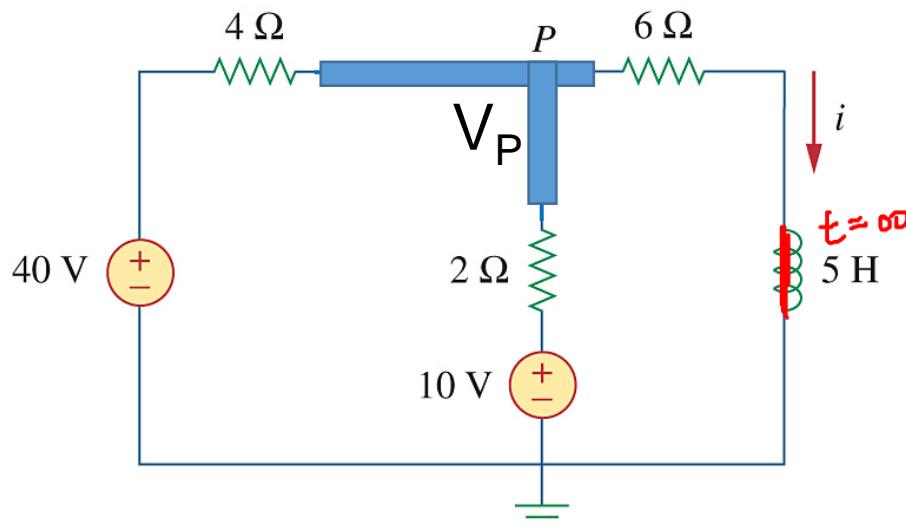
$$= 4 + Ce^{-2t} \Rightarrow 4 - 4e^{-2t}.$$

$$i(0) = 4 + C = 0 \quad C = -4$$

$$\text{At } t=4 \quad 4 - 4e^{-8} \approx 4$$

iii)  $t > 4$

$$i(4) = 4 \text{ A.}$$

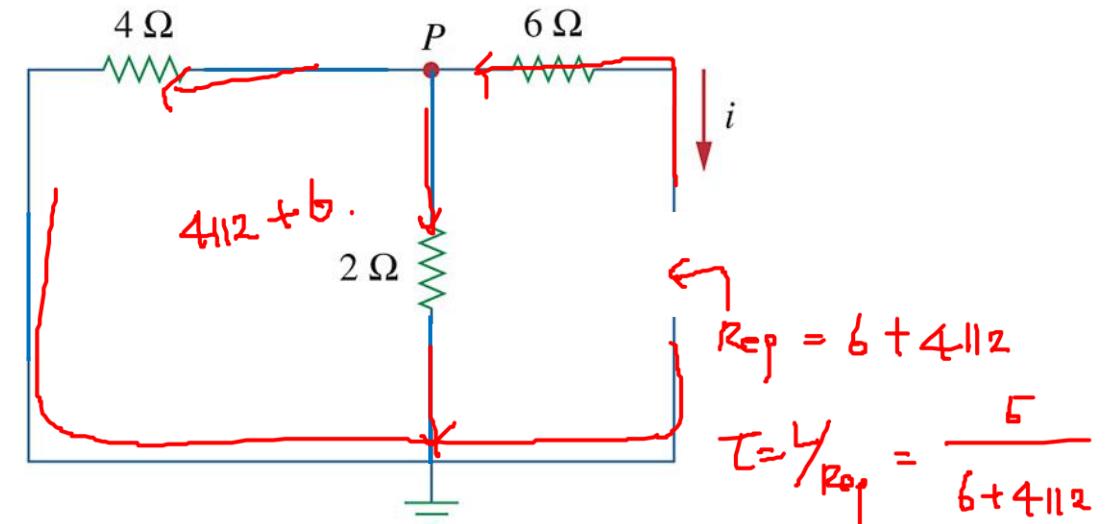


$$\frac{V_p - 40}{4} + \frac{V_p - 10}{2} + \frac{V_p}{6} = 0.$$

$$V_p = \frac{180}{11}$$

$$i(\infty) = \frac{\frac{180}{11}}{6} = \frac{30}{11}$$

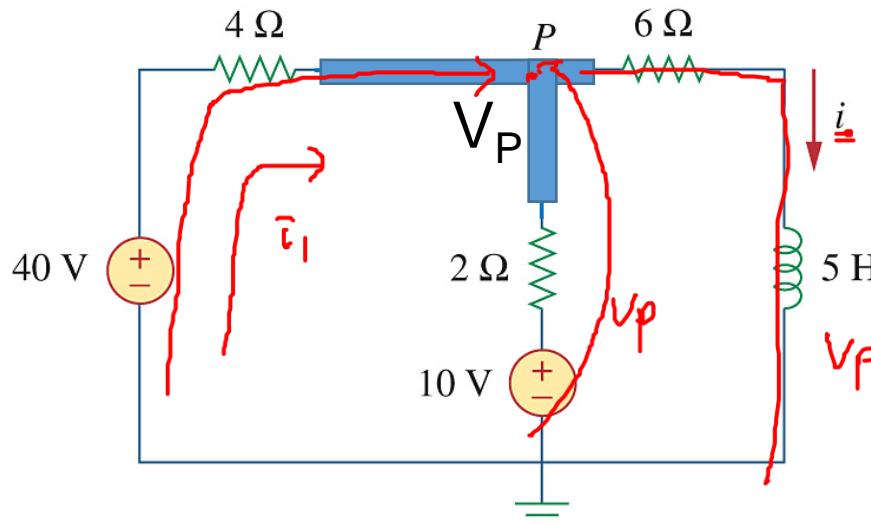
$$R_{eq} = 2 \parallel 4 + 6 = 22/3$$



$$i(t) = \frac{30}{11} + \left(4 - \frac{30}{11}\right) e^{-\frac{22}{3} \frac{t}{15} (t-4)}$$

$$= 15/32.$$

iii)  $t > 4$



(1) KCL at node P

$$(2) V_P = 6i + L \frac{di}{dt}$$

$$\text{KVL: } -40 + 4i_1 + V_P = 0$$

$$V_P = 6i + L \frac{di}{dt}$$

$$\text{Node P; KCL: } i_1 = \frac{V_P - 10}{2} + i$$

1st order eq. with  $i$

For  $t > 0$ ,

$$i(0^+) = i(0^-) = 0$$

For  $0 < t \leq 4$ ,

$$i(\infty) = \frac{40}{4+6} = 4 \text{ (A)}$$

$$\tau = \frac{5}{4+6} = 0.5 \text{ (s)}$$

$$i(t) = 4 + (0 - 4)e^{-t/0.5} = 4(1 - e^{-2t}) \text{ (A)}$$

$$i(2) = 4(1 - e^{-2 \times 2}) \approx 3.93 \text{ (A)}$$

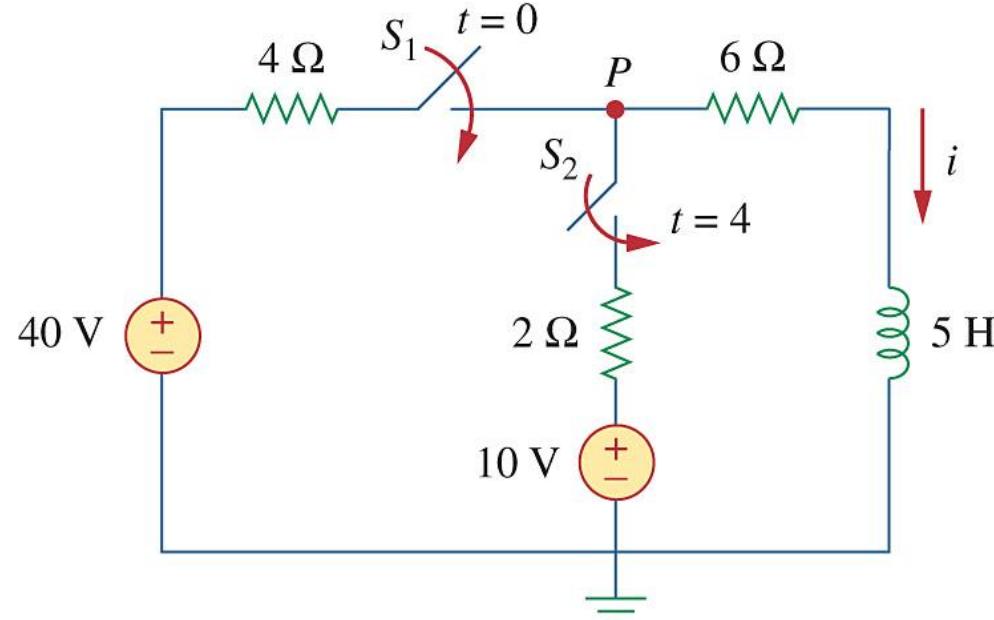


Figure 7.53

For  $t > 4$ ,

$$i(4^+) = i(4^-) = 4(1 - e^{-2 \times 4}) = 4(1 - e^{-8}) \text{ (A)}$$

$$v_P(\infty) = 40 \times \frac{2 \parallel 6}{4 + 2 \parallel 6} + 10 \times \frac{4 \parallel 6}{2 + 4 \parallel 6}$$

$$= \frac{180}{11} \text{ (V)}$$

$$i(\infty) = \frac{v_P(\infty)}{6} = \frac{30}{11} \text{ (A)}$$

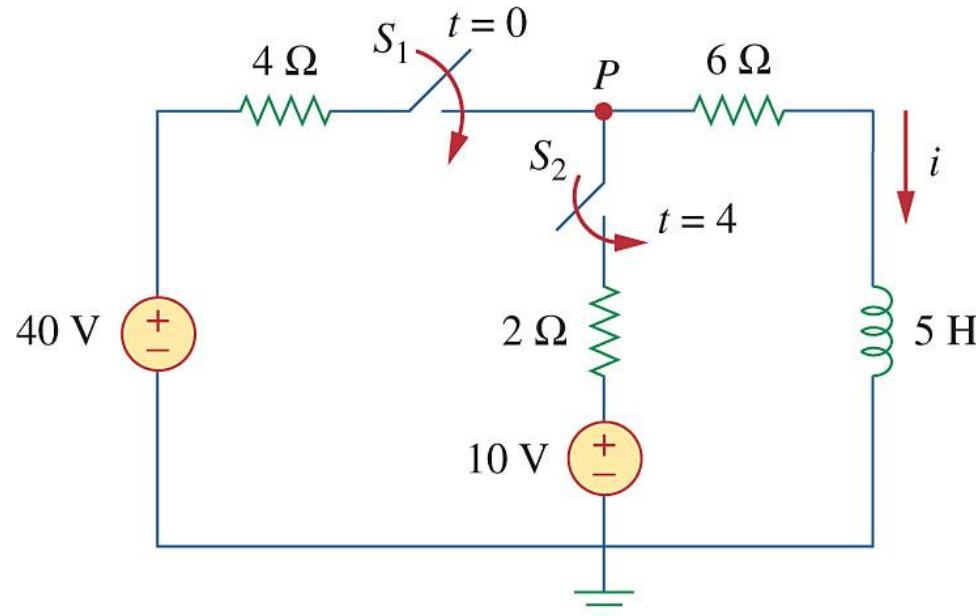


Figure 7.53

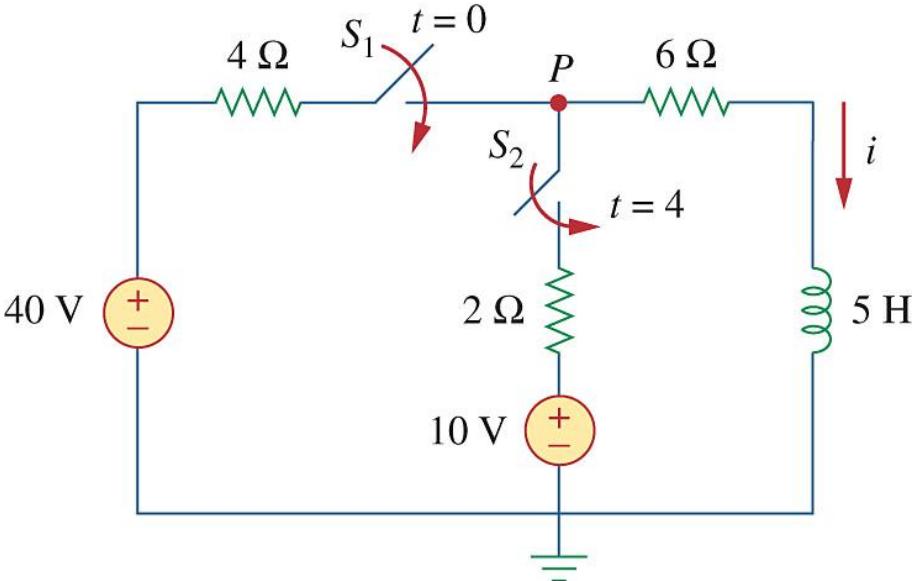


Figure 7.53

$$\tau = \frac{5}{6 + 4 \parallel 2} = \frac{15}{22} \text{ (s)}$$

$$i(t) = \frac{30}{11} + \left( 4(1 - e^{-8}) - \frac{30}{11} \right) e^{-(t-4)/(15/22)}$$

Time shift property

$$\approx 2.7273 + 1.2714e^{-1.4667(t-4)} \text{ (A)}$$

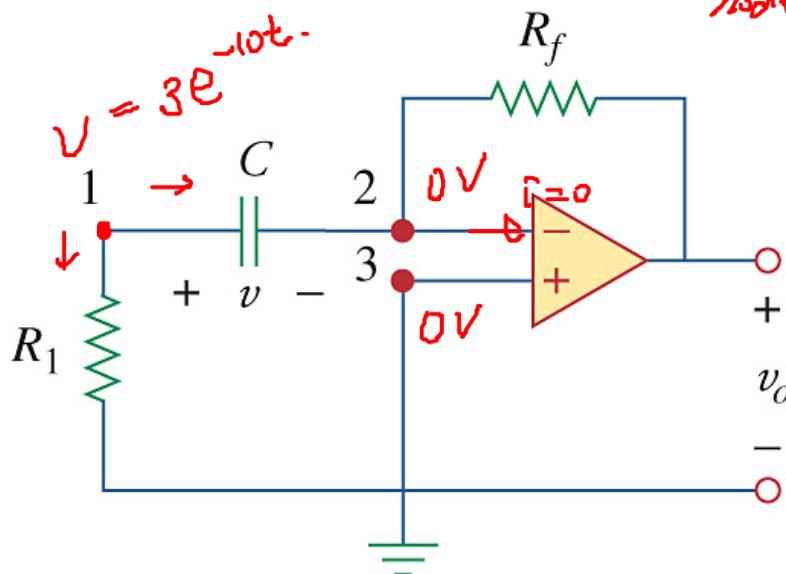
$$i(5) = 2.7273 + 1.2714e^{-1.4667(5-4)} \approx 3.02 \text{ (A)}$$

## 7.7 First-Order Op Amp Circuits

An op amp circuit containing a storage element will exhibit **first-order behavior**. For practical reasons, inductors are hardly ever used in op amp circuits; we will consider Op Amp circuits with RC.

- Possible location of the capacitor
  - (1) At the input
  - (2) At the output
  - (3) At the feedback loop

**Example 7.14** For the op amp circuit in Fig. 7.55(a), find  $v_o$  for  $t > 0$ , given that  $v(0) = 3 \text{ V}$ . Let  $R_f = 80 \text{ k}\Omega$ ,  $R_1 = 20 \text{ k}\Omega$ , and  $C = 5 \mu\text{F}$ .



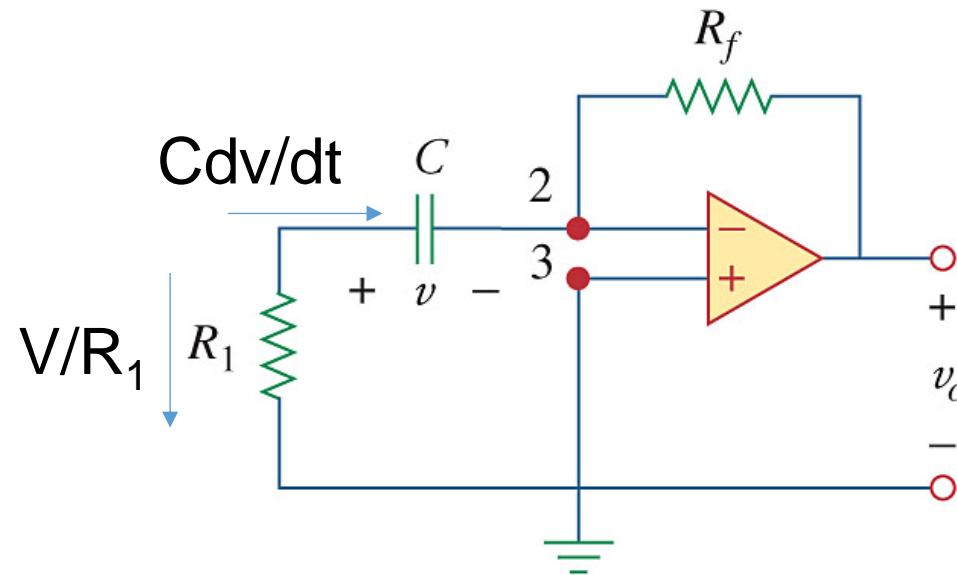
$$\text{node 1: } \frac{v}{R_1} + C \frac{dv}{dt} = 0$$

$$RC = T = 20\text{ k}\Omega \times 5\mu\text{F} \Rightarrow 3e^{-10t}, \\ v(0) = 3\text{ V}.$$

$$\text{node 2: } C \frac{d(0-v)}{dt} + \frac{0-v_o}{R_f} = 0.$$

$$v_o(t) = 12e^{-10t}.$$

Figure 7.55(a)

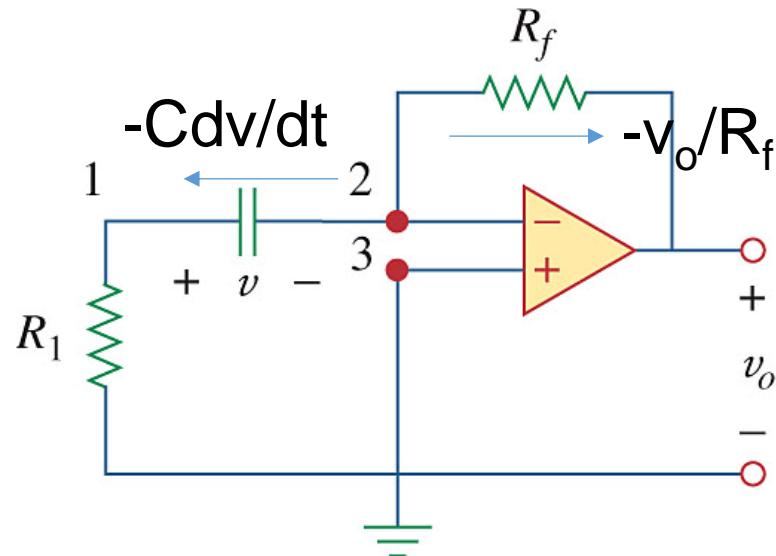


Node 1, by KCL

$$\frac{V}{R_1} + C \frac{dv}{dt} = 0 \rightarrow \frac{dV}{dt} + \frac{1}{RC_1} v = 0$$

$$\tau = R_1 C = 20 \times 10^3 \times 5 \times 10^{-6} = 0.1 \text{ (s)}$$

$$v = v(0)e^{-t/\tau} = 3e^{-t/0.1} = 3e^{-10t} \text{ (V)}$$



KCL from node 2 gives

$$\begin{aligned}
 -C \frac{dv}{dt} + \frac{0 - v_0}{R_f} &= 0 \\
 \rightarrow v_0 &= -R_f C \frac{dv}{dt} = -\left(5 \times 10^{-6} \frac{d}{dt} (3e^{-10t})\right) \times 80 \times 10^3 \\
 &= -0.4 \times \left(3 \times (-10e^{-10t})\right) \\
 &= 12e^{-10t} \text{ (V)}
 \end{aligned}$$

**Example 7.15** Determine  $v(t)$  and  $v_o(t)$

in the circuit of Fig. 7.57.

**Solution :**

For  $t < 0$ , **No energy at the circuit**

$$v_1(t) = 0, v_o(t) = 0, v(t) = 0$$

$$v(0^-) = 0$$

For  $t > 0$ ,

$$v_1(t) = 3 \times \frac{20}{10 + 20} = 2 \text{ (V)}$$

$$\left\{ \begin{array}{l} v(0) \\ v(0^+) \\ \tau = RC \end{array} \right.$$

$$\tau = 50 \text{ k}\Omega \times 1 \mu\text{F} = \frac{1}{20}.$$
$$R_{\text{eq}} = 50 \text{ k}\Omega.$$

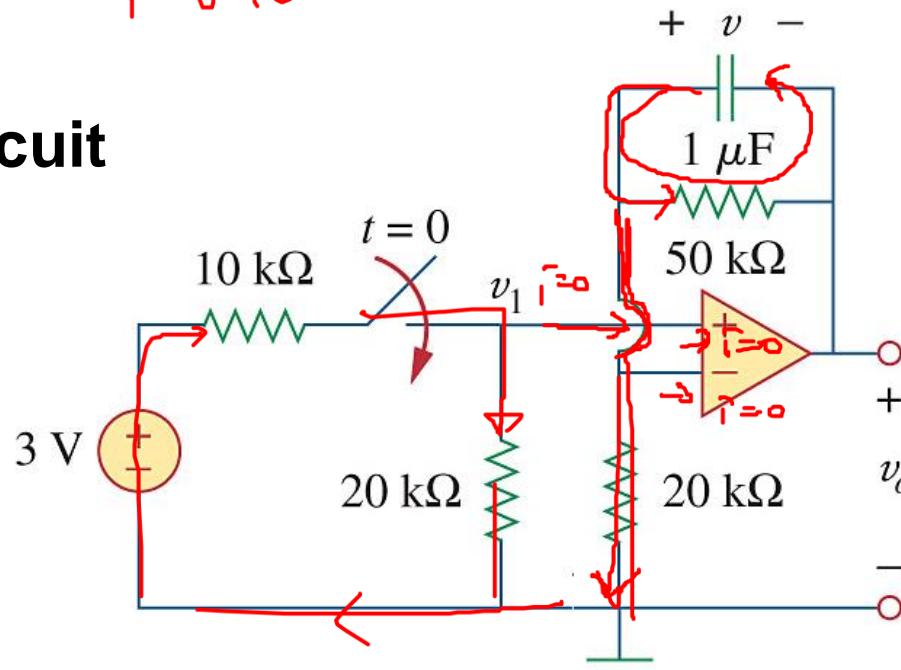
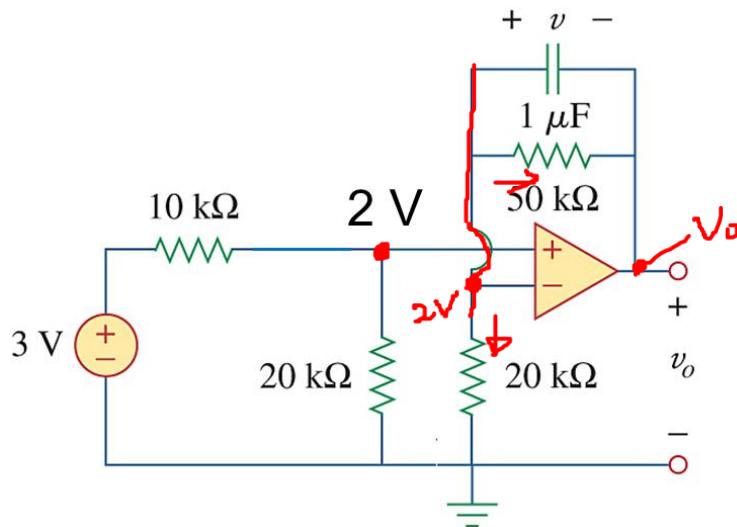


Figure 7.57

(ii)  $t > 0$

$$v = 2 - v_o$$

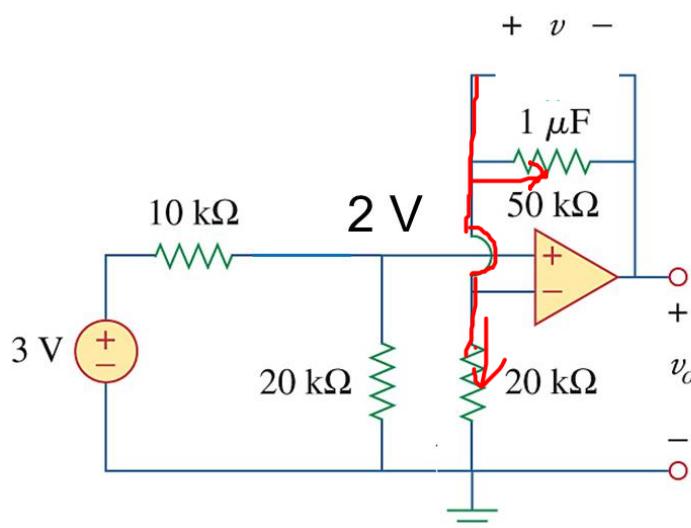


$$(1) RC = 10^{-6} \times 50k = 1/20$$

KCL.

$$\frac{2}{20k} + \left( \frac{2-v_o}{50k} \right) + 1\mu F \frac{dv}{dt} = 0 \quad \times 10^6$$

$$\frac{dv}{dt} + \frac{20}{2} v = -1000.$$



$$(2) v(\infty)$$

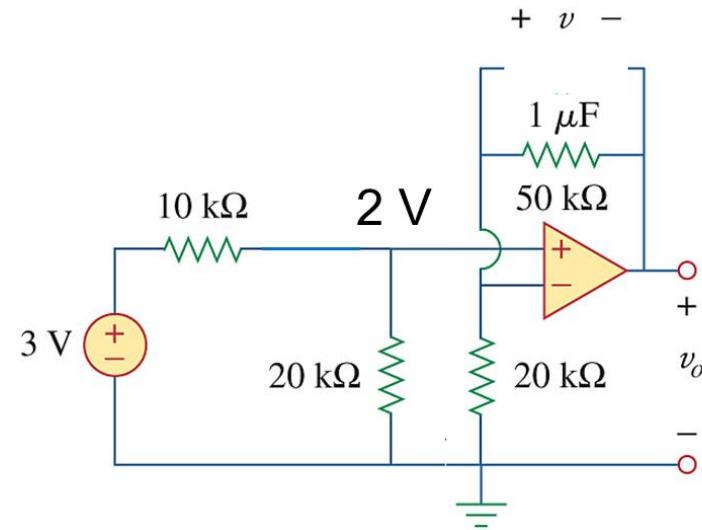
$$v = 2 - v_o \Rightarrow v(\infty) = 2 - 7 = -5$$

$$\text{By KCL: } \frac{2}{20k} + \frac{2-v_o}{50k} = 0, v_o = 7 \text{ V}$$

$$\text{Thus } v(\infty) = -5 \text{ V}$$

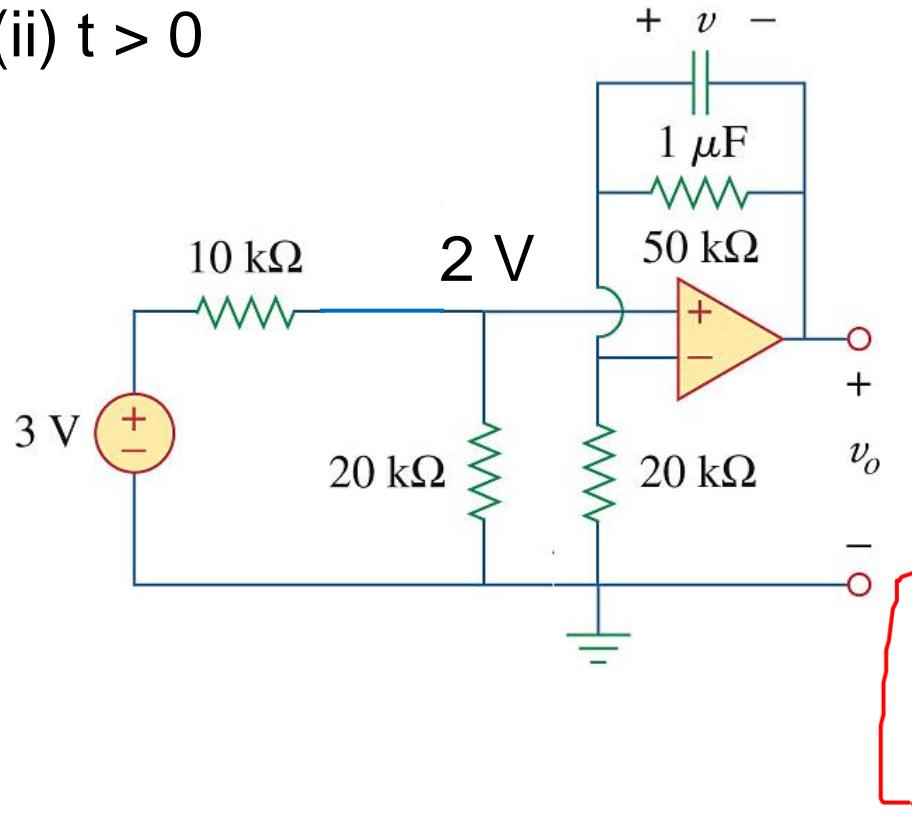
$$v(t) = -5 + [0 + 5]e^{-20t} = 5e^{-20t} - 5$$

(ii)  $t > 0$



Because  $2 - v_o = v$   
where  $v(t) = 5e^{-20t} - 5$  [V]  
 $v_o(t) = 7 - 5e^{-20t}$  [V]

(ii)  $t > 0$



$$(i) 2 - v_o = v$$

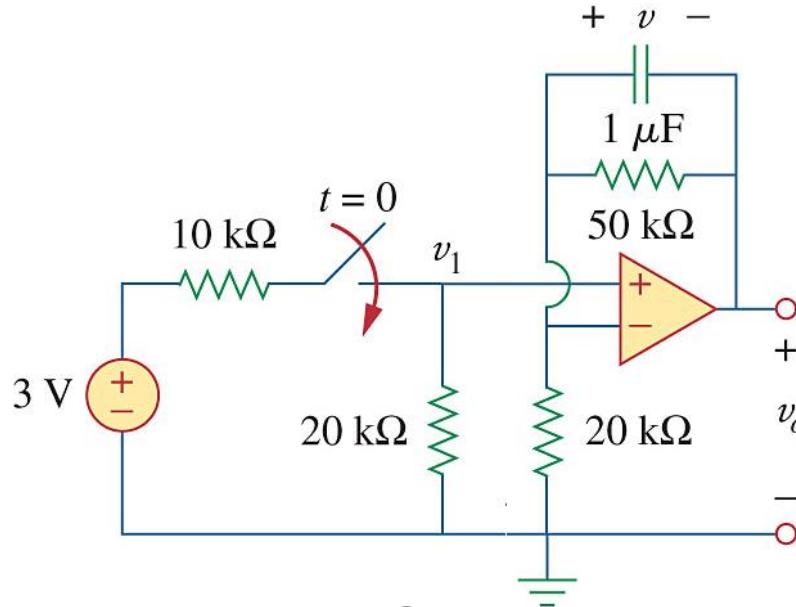
$$(ii) \text{ By KVL: } \frac{2}{20k} + \frac{2-v_o}{50k} + C \frac{dv}{dt} = 0$$

$$\frac{2}{20k} + \frac{v}{50k} + 10^{-6} \frac{dv}{dt} = 0 \times 10^6$$

$$\frac{dv}{dt} + 20v = -100$$

$$v[+] = e^{-20t} \left( \int_{-\infty}^t -100 \cdot e^{20t} dt + c \right)$$

$$v[0] = 0,$$



$$(1 \times 10^{-6}) \frac{dv}{dt} + \frac{v}{50 \times 10^3} = -\frac{2}{20 \times 10^3}$$

$$\frac{dv}{dt} + 20v = -100$$

$$v(t) = Ae^{-20t} + B = 5e^{-20t} - 5 \text{ (V)}$$

$$v_o(t) = -v(t) + 2 = 7 - 5e^{-20t} \text{ (V)}$$