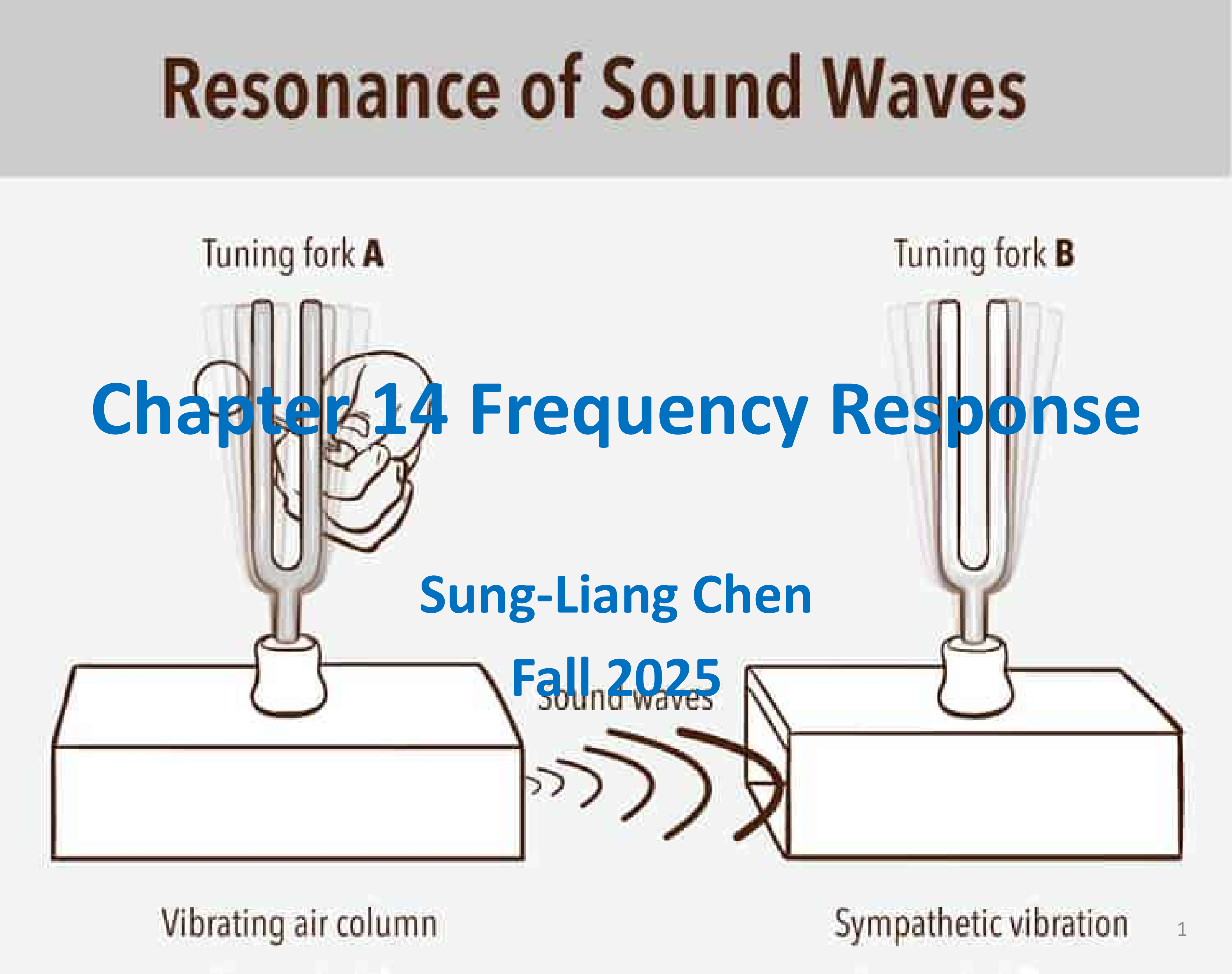

Chapter 14 Frequency Response

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14.1 Introduction

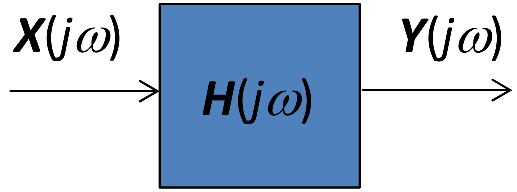
In our sinusoidal steady-state analysis, we have learned how to find voltages and currents in a circuit with a constant frequency source. $\boxed{\omega = \omega_0}$ If we let the amplitude and phase angle of the sinusoidal source remain constant and vary the frequency, we obtain the circuit's *frequency response.* $\boxed{f(\omega)}$

The frequency response may be regarded as a complete description of the sinusoidal steady-state behavior of a circuit as a function of frequency.

14.2 Frequency Response

The frequency response $H(j\omega)$ of a circuit is the frequency-dependent ratio of a phasor output $\tilde{Y}(j\omega)$ (an element voltage or current) to a phasor input $\tilde{X}(j\omega)$ (source voltage or current).

$$H(j\omega) = \frac{\tilde{Y}(j\omega)}{\tilde{X}(j\omega)}$$



There are four types of frequency response:

$$H(j\omega) = \frac{\tilde{V}_o(j\omega)}{\tilde{V}_i(j\omega)} \qquad \text{(Voltage gain)}$$

$$H(j\omega) = \frac{\tilde{I}_o(j\omega)}{\tilde{I}_i(j\omega)} \qquad \text{(Current gain)}$$

$$H(j\omega) = \frac{\tilde{V}_o(j\omega)}{\tilde{I}_i(j\omega)} \qquad \text{(Transfer impedance)}$$

$$H(j\omega) = \frac{\tilde{I}_o(j\omega)}{\tilde{V}_i(j\omega)} \qquad \text{(Transfer admittance)}$$

The frequency response can be expressed in terms of its numerator polynomial $N(j\omega)$ and denominator polynomial $D(j\omega)$ as

$$H(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{n=0}^{N} a_n (j\omega)^n}$$

Proof :

A linear network can be described by a linear constant-coefficient differential equation

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

Recall Chaps. 7 and 8:
First-order circuit
Second-order circuit

Transform the equation to the phasor domain,

$$\sum_{n=0}^{N} a_n (j\omega)^n \ \tilde{Y}(j\omega) = \sum_{m=0}^{M} b_m (j\omega)^m \ \tilde{X}(j\omega)$$

Recall Chap. 9:
d/dt ➔ $j\omega$

$$\tilde{Y}(j\omega)\sum_{n=0}^{N} a_n (j\omega)^n = \tilde{X}(j\omega)\sum_{m=0}^{M} b_m (j\omega)^m$$

$$H(j\omega) = \frac{\tilde{Y}(j\omega)}{\tilde{X}(j\omega)} = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{n=0}^{N} a_n (j\omega)^n}$$

E.g., Section 8.5 series RLC

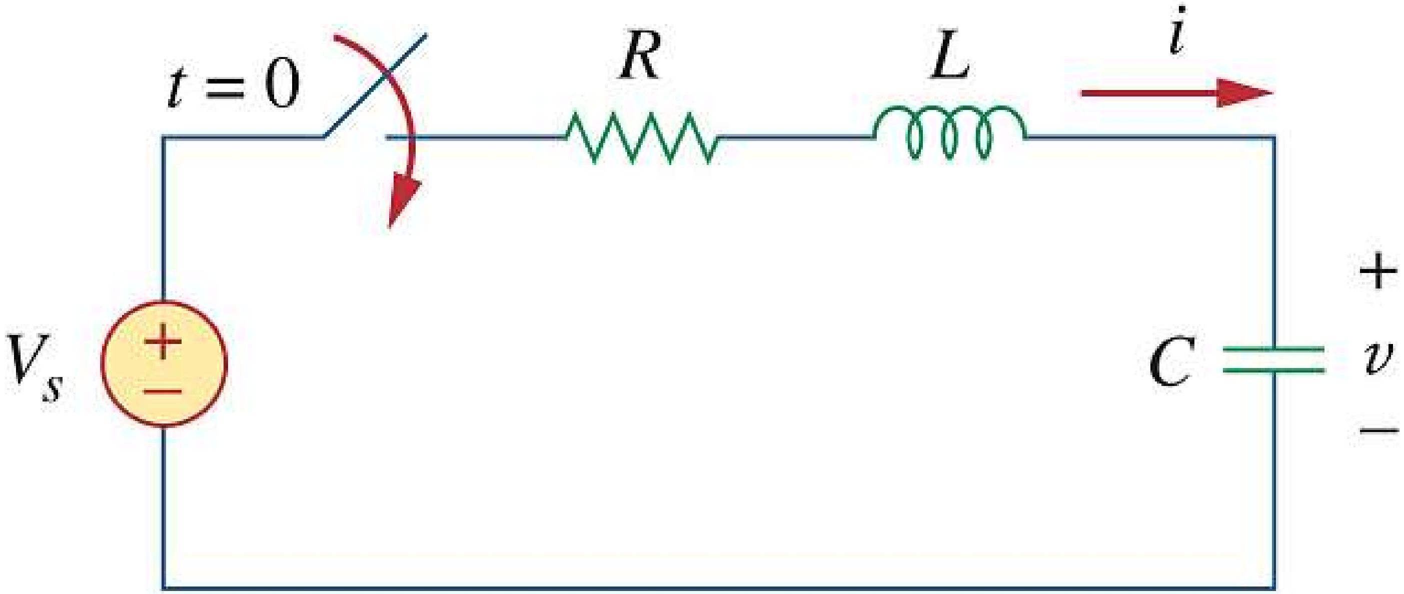


Figure 8.18 Step voltage applied to a series *RLC* circuit.

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}V_s$$

$\boxed{a_2y'' + a_1y' + a_0y = b_0x}$

V_s as the input

v as the output (response)

Being a complex quantity, $H(j\omega)$ has a magnitude and a phase; that is, $H(j\omega) = H\angle\phi$. The plot of H versus ω is called the *magnitude frequency response*. The plot of ϕ versus ω is called the *phase frequency response*.

$\mathbf{H}(j\omega) = H(\omega)\angle\phi(\omega)$
$H(\omega)$: magnitude frequency response
$\phi(\omega)$: phase frequency response

Example 14.1 For the *RC* circuit in Fig. 14.2(a), obtain the frequency response $\tilde{V}_o(\omega)/\tilde{V}_s(\omega)$. Let $v_s = V_m \cos \omega t$.

Solution :

voltage division

$$H(j\omega) = \frac{\tilde{V}_o(\omega)}{\tilde{V}_s(\omega)} = \frac{1/(j\omega C)}{R + 1/(j\omega C)}$$

$$= \frac{1}{1 + j\omega RC}$$

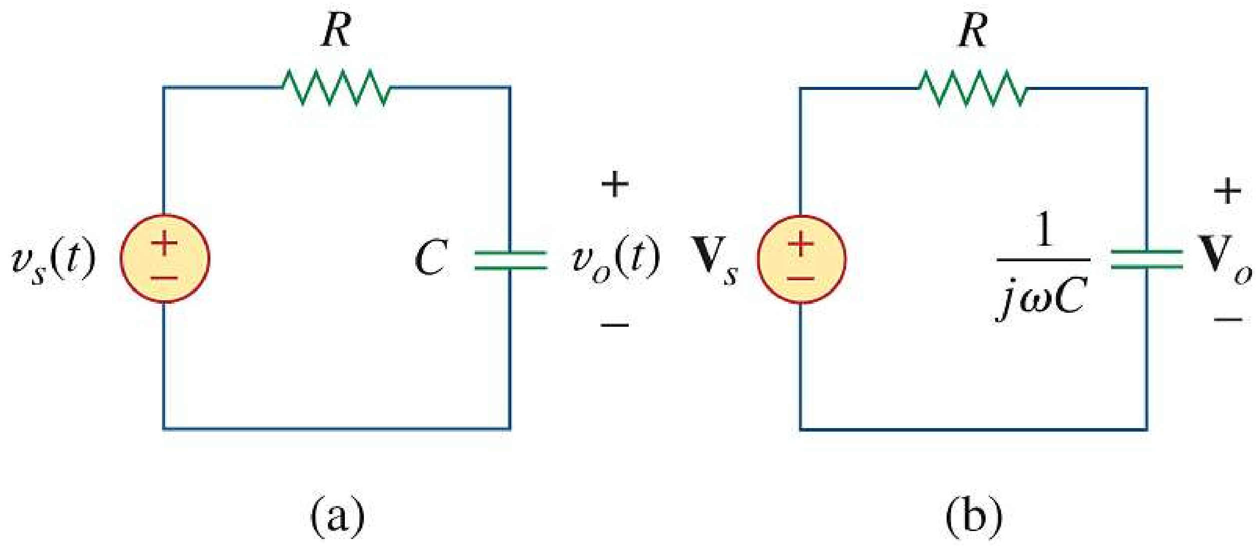


Figure 14.2

The magnitude and phase frequency responses are

$$H = \frac{1}{\sqrt{1+(\omega RC)^2}} = \frac{1}{\sqrt{1+(\omega/\omega_0)^2}}$$

$$\phi = -\tan^{-1}(\omega RC) = -\tan^{-1}(\omega/\omega_0)$$

where $\omega_o = \dfrac{1}{RC}$.

The plots of H an ϕ are shown in Fig. 14.3.

H = **N/D**
$\angle$**H** = $\angle$**N**−$\angle$**D**

H = Re + j×Im
$\angle$**H** = $\tan^{-1}$(Im/Re)

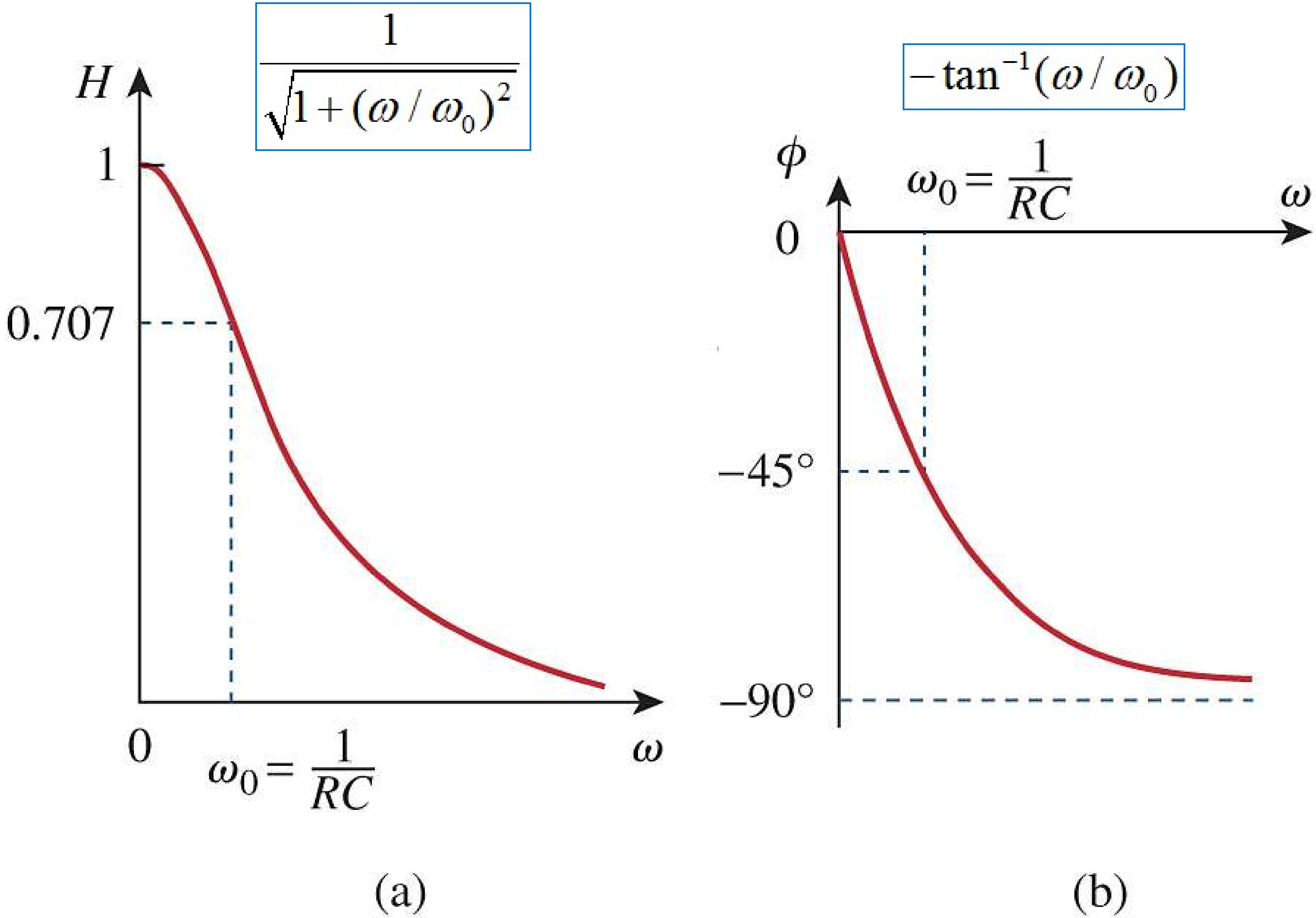


Figure 14.3 Frequency response of the RC circuit:
(a) amplitude response, (b) phase response.

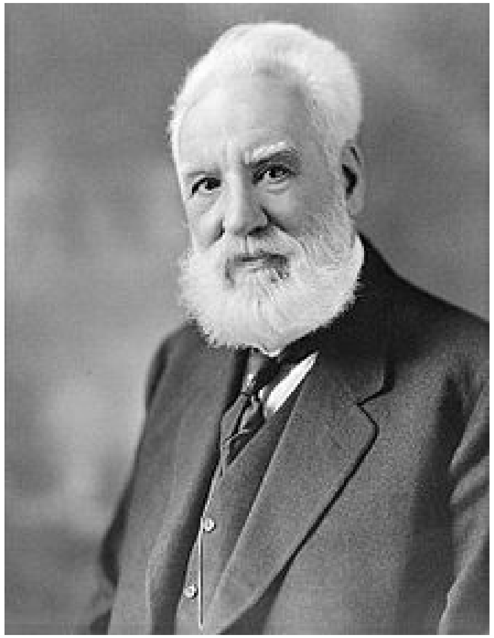
14.3 The Decibel Scale

In communication systems, gain is measured in bels. Historically, the *bel* is used to measure the ratio of two levels of power or power gain G; that is,

$$G = \text{Number of bels} = \log_{10} \frac{P_2}{P_1}$$

10^{G_bel} = linear scale
E.g.,
0 bel ⇔ 1 time
1 bel ⇔ 10 times
2 bels ⇔ 100 times
…

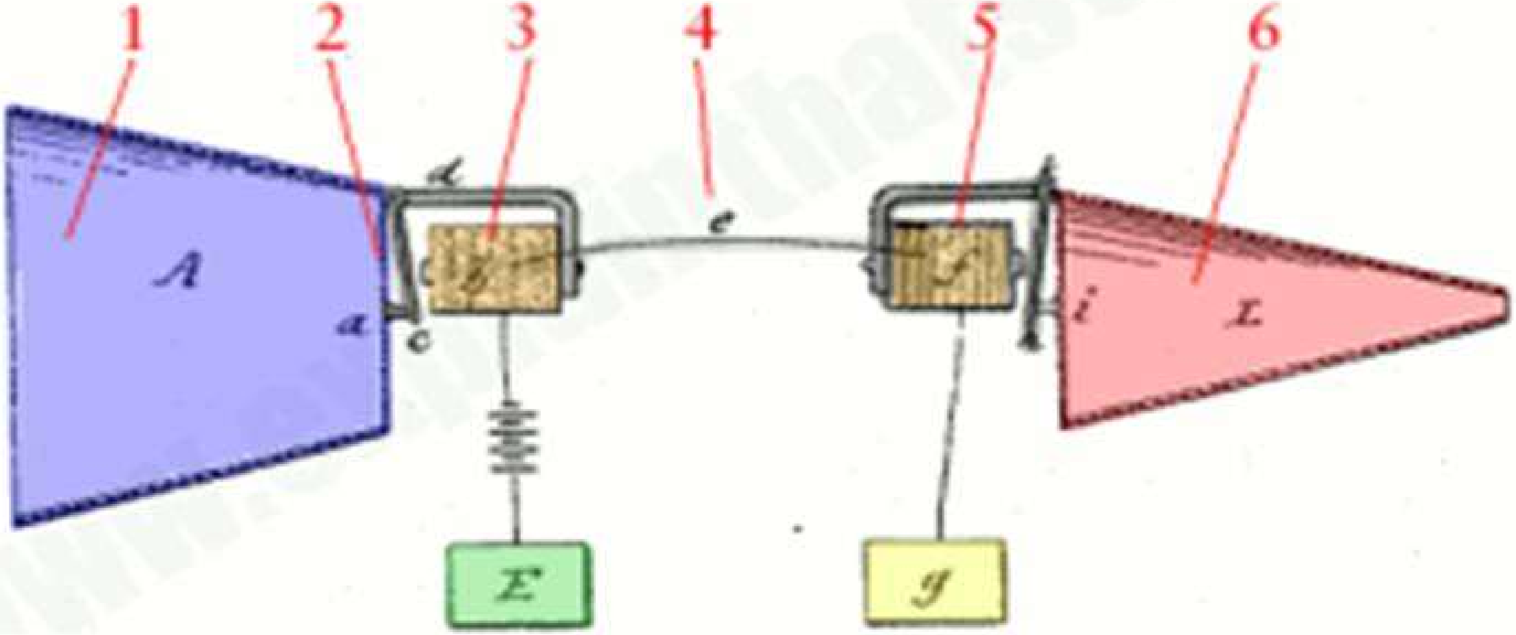
Alexander Graham Bell (March 3, 1847 – August 2, 1922) was an eminent scientist, inventor, engineer and innovator who is credited with inventing the first practical telephone.



A. G. BELL.

TELEGRAPHY.

No. 174,465. Patented March 7, 1876.



1. The speaker talks into a horn
2. Their sound makes a diaphragm vibrate
3. The vibrations move a coil near a magnet, converting mechanical sound energy into a fluctuating electric current
4. The electric current travels down a wire (in theory, can be any length)
5. At the receiving end, the process is reversed. The electric current flows into a coil placed near a magnet, making the coil move back and forth and pushing another diaphragm.
6. The diaphragm recreates the original sound. The shape amplifies the sound.

The *decibel* (dB) provides us with a unit of less magnitude. It is 1/10th of a bel and is given by

$$G_{dB} = 10\log_{10}\frac{P_2}{P_1}$$

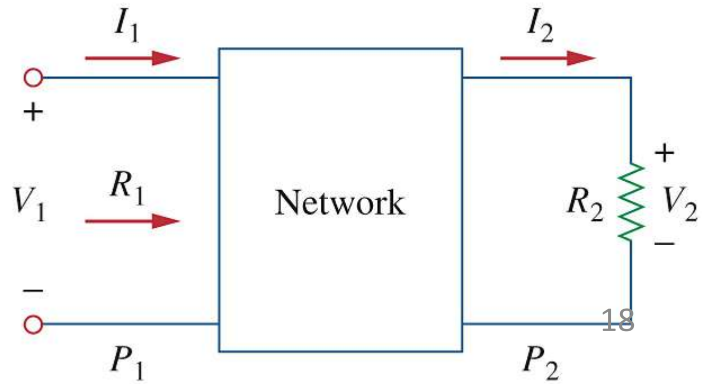
10^{G_bel} = linear scale
E.g.,
0 bel ⇔ 1 time
1 bels ⇔ 10 times
2 bels ⇔ 100 times
…

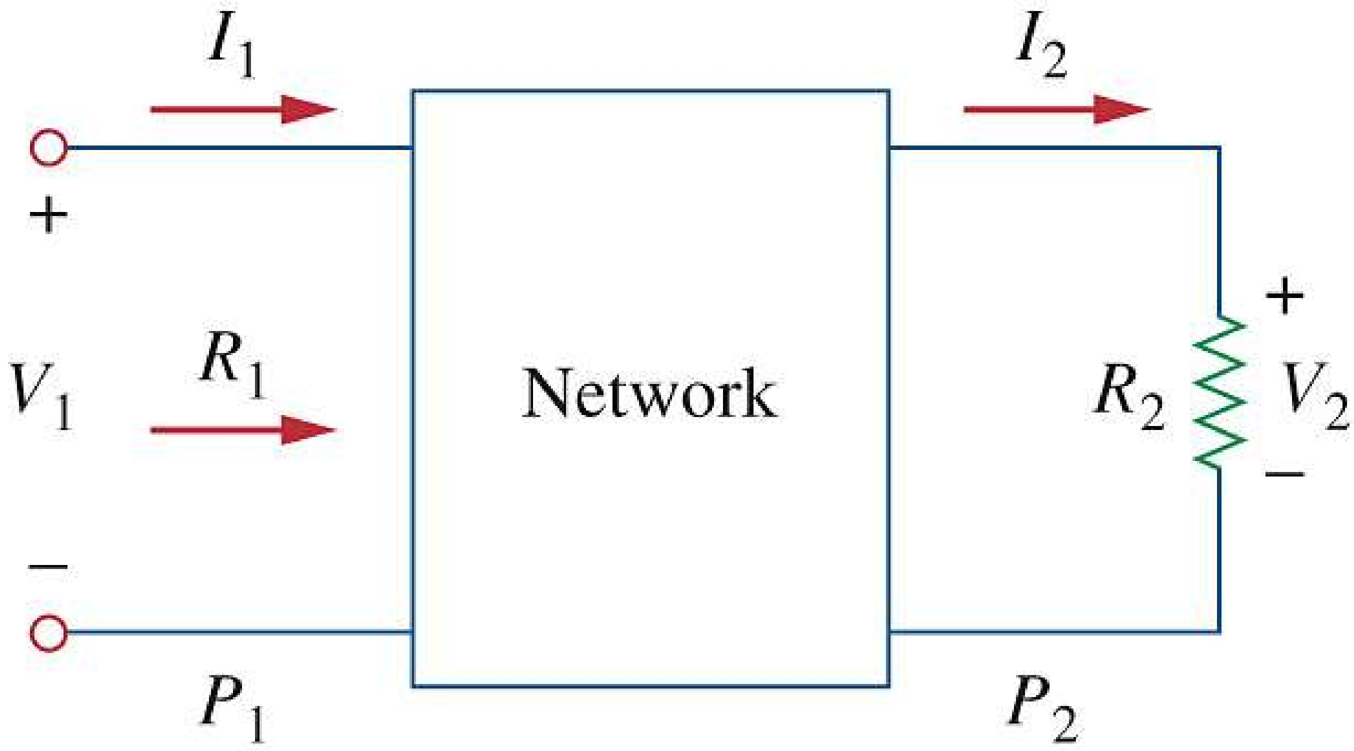
Use a smaller unit; make the number larger
E.g., from meter to centimeter

$10^{G\_dB/10}$ = linear scale
E.g.,
0 dB ⇔ 1 time
10 dB ⇔ 10 times
20 dB ⇔ 100 times
…

The gain G can be expressed in terms of voltage or current ratio. Consider the network shown in Fig. 14.8. If V_1 is the input voltage, V_2 is the output voltage, R_1 is the input resistance, and R_2 is the load resistance, then

$$G_{dB} = 10\log_{10}\frac{P_2}{P_1} = 10\log_{10}\left(\frac{V_2^2/R_2}{V_1^2/R_1}\right)$$





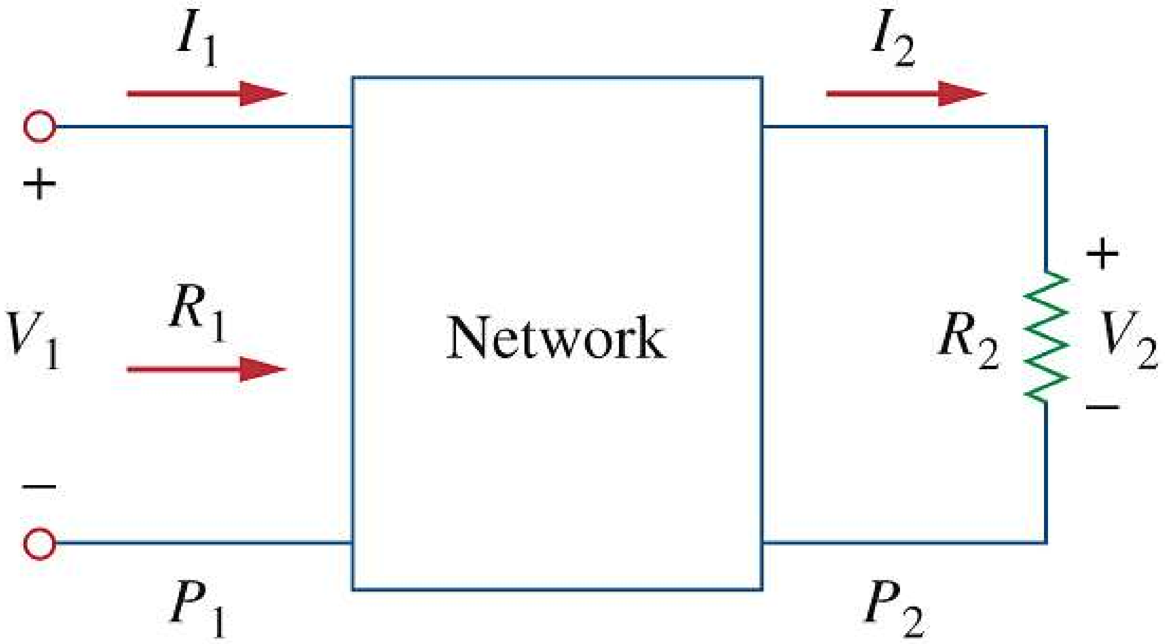
I_1
I_2
+
V_1
R_1
−
Network
R_2
+
V_2
−
P_1
P_2

For the case when $R_2 = R_1$, a condition that is often assumed when comparing voltage levels,

$$G_{dB} = 10\log_{10}\left(\frac{V_2}{V_1}\right)^2 = 20\log_{10}\frac{V_2}{V_1}$$

Similarly, for $R_1 = R_2$,

$$G_{dB} = 20\log_{10}\frac{I_2}{I_1}$$



For power

$10^{G\_dB/10}$ = linear scale

E.g.,

0 dB ⇔ P_2/P_1=1 time

10 dB ⇔ P_2/P_1=10 times

20 dB ⇔ P_2/P_1=100 times

…

Power ~ amplitude^2

Amplitude ~ power$^{0.5}$

For amplitude (voltage or current)

$10^{G\_dB/20}$ = linear scale

E.g.,

0 dB ⇔ P_2/P_1=1 time ⇔ V_2/V_1=1 time

10 dB ⇔ P_2/P_1=10 times ⇔ $V_2/V_1=(10)^{0.5}$ time= 3.16 times

20 dB ⇔ P_2/P_1=100 times ⇔ V_2/V_1=10 times

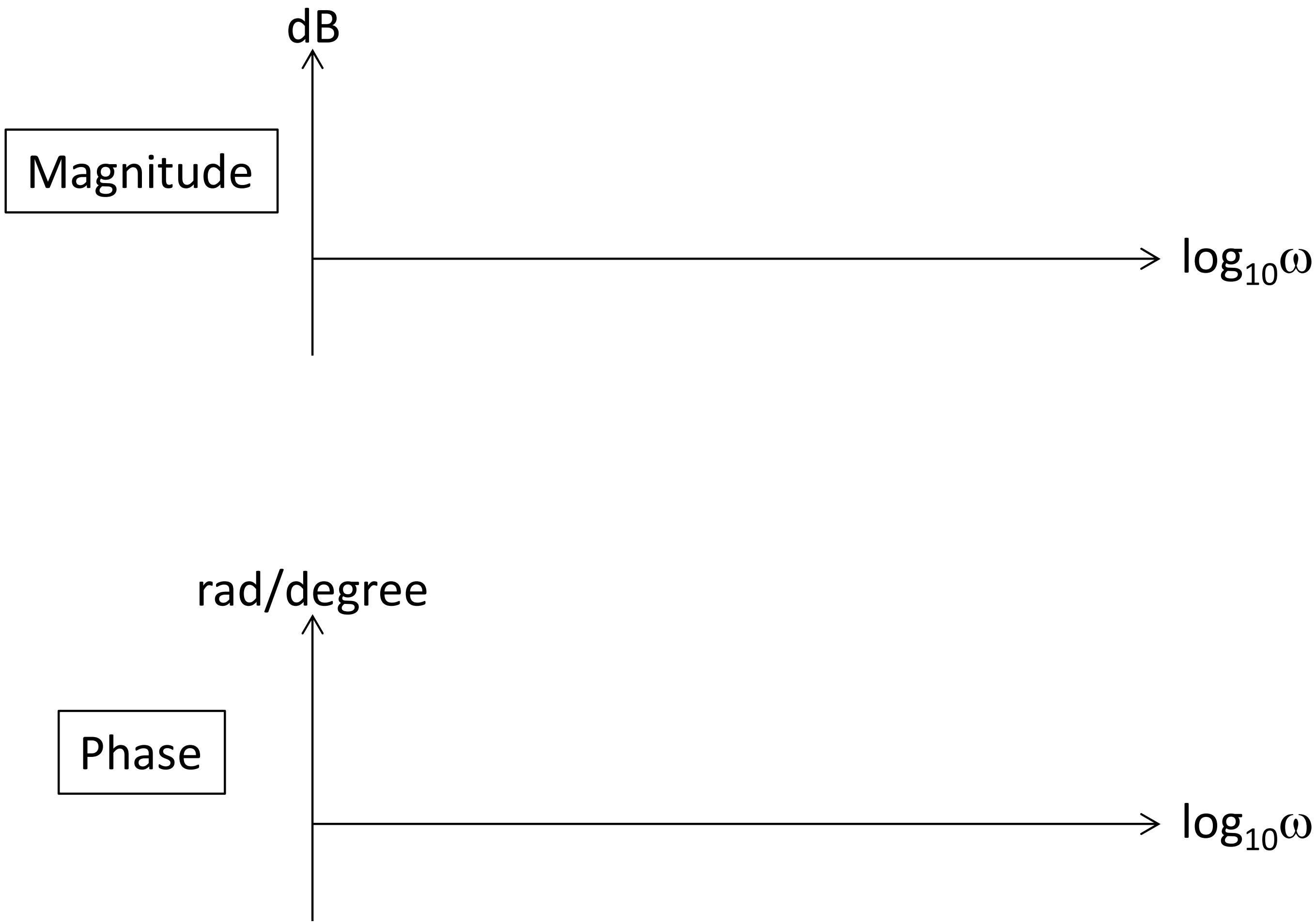
…

When we are talking about dB, we do not need to specify power or amplitude.
E.g., –3 dB means 0.5 times in power, and ~0.7 times in amplitude

14.4 Bode Plots

The frequency range required in plotting frequency response is often so wide that it is inconvenient to use a linear scale for the frequency axis. *Bode plots* are semilog plots in which the magnitude in decibels is plotted against the logarithm of the frequency, the phase in degrees is plotted against the logarithm of the frequency.

Bode plots



The frequency response may be written in factored form:

$$H(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{n=0}^{N} a_n (j\omega)^n}$$

$$= \frac{b_M (j\omega)^M + b_{M-1}(j\omega)^{M-1} + \quad \cdot + b_1 (j\omega) + b_0}{a_N (j\omega)^N + a_{N-1}(j\omega)^{N-1} + \quad \cdot + a_1 (j\omega) + a_0}$$

$$= \frac{b_M}{a_N} \frac{(j\omega)^M + \frac{b_{M-1}}{b_M}(j\omega)^{M-1} + \quad \cdot + \frac{b_0}{b_M}}{(j\omega)^N + \frac{a_{N-1}}{a_N}(j\omega)^{N-1} + \quad \cdot + \frac{a_0}{a_N}}$$

$$\Big\downarrow \boxed{\text{factorization}}$$

$$= \frac{b_M}{a_N} \frac{(j\omega + z_1)(j\omega + z_2) \quad \cdot (j\omega + z_M)}{(j\omega + p_1)(j\omega + p_2) \quad \cdot (j\omega + p_N)}$$

$$= \frac{\tilde{b} \prod_{m=1}^{M} (j\omega + z_m)}{\prod_{n=1}^{N} (j\omega + p_n)}$$

where $\tilde{b} = \frac{b_M}{a_N}$

All of the coefficients of $N(j\omega)$ are real, therefore the roots of $N(j\omega)=0$ must be

(1) Real (2) C.C.

either real or appear in complex conjugate pairs. That implies z_m, $m=1,\cdots,\rightleftharpoons,M$ are either real or appear in complex conjugate pairs. The same is true for p_n, $n=\cdots,2,\rightleftharpoons,N$.

E.g., Section 8.5 series RLC

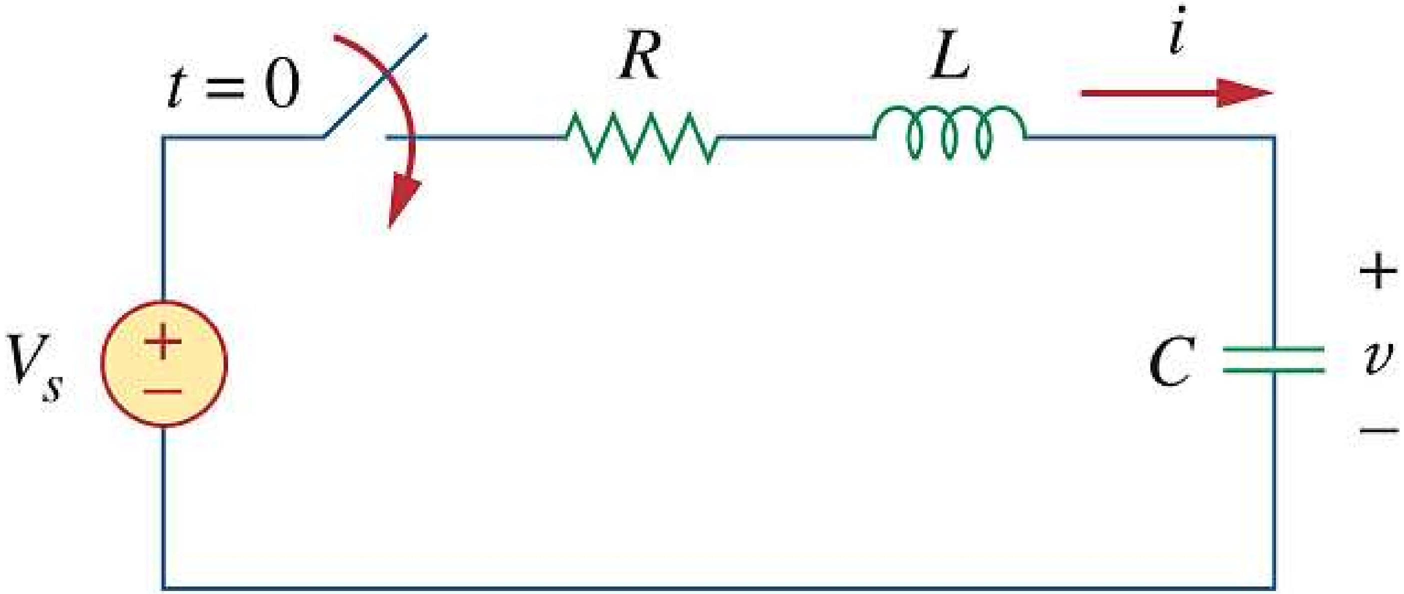


Figure 8.18 Step voltage applied to a series RLC circuit.

$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{LC}v=\frac{1}{LC}V_s$$

$\boxed{a_2y''+a_1y'+a_0y = b_0x}$

V_s as the input

v as the output (response)

$\boxed{\text{(1) Real}}$

If z_m is real, we write

$$j\omega + z_m = \begin{cases} z_m(1 + j\omega / z_m), & z_m \neq 0 \\ j\omega, & z_m = 0 \end{cases}$$

$\boxed{\text{(2) C.C.}}$

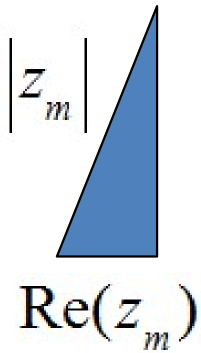
If z_m is complex, we group factors $(j\omega + z_m)$ and $(j\omega + z_m^*)$ into real-valued quadratic factor

$$(j\omega + z_m)(j\omega + z_m^*) = (j\omega)^2 + (z_m + z_m^*) j\omega + z_m z_m^*$$

$$= (j\omega)^2 + 2\,\mathrm{Re}(z_m)(j\omega) + \left|z_m\right|^2$$

$$= \left|z_m\right|^2 \left[1 + \frac{2\,\mathrm{Re}(z_m)(j\omega)}{\left|z_m\right|^2} + \frac{(j\omega)^2}{\left|z_m\right|^2} \right]$$

$$= \left|z_m\right|^2 \left[1 + \frac{2\,\mathrm{Re}(z_m)}{\left|z_m\right|} \left(\frac{j\omega}{\left|z_m\right|} \right) + \left(\frac{j\omega}{\left|z_m\right|} \right)^2 \right]$$



$$= \omega_k^2 \left[1 + 2\zeta_k (j\omega / \omega_k) + (j\omega / \omega_k)^2 \right]$$

where $\omega_k = \left|z_m\right|$ and $\zeta_k = \mathrm{Re}(z_m) / \left|z_m\right|$. It is evident that $0 \le \left|\zeta_k\right| \le 1$.

So, $H(j\omega)$ may be represented in the *standard form*:

1. The *b*

2. $z_m = 0$; $p_n = 0$

$$H(j\omega) = \underline{K}\,\underline{(j\omega)^{\pm 1}} \times$$

$$\frac{(1 + j\omega / z_1)[1 + j2\zeta_1(j\omega/\omega_k) + (j\omega/\omega_k)^2] \;\cdot}{\underline{(1 + j\omega / p_1)}\underline{[1 + j2\zeta_2(j\omega/\omega_n) + (j\omega/\omega_n)^2]} \;\cdot}$$

We will now plot <u>straight-line-segment</u> approximations (called *asymptotes*) associated with the factors in $H(j\omega)$.

4. z_m = c.c.; p_n = c.c.

3. z_m = real; p_n = real

1. The *b*

(1) For the gain K,

$$H_{dB} = 20\log_{10}|K|$$

Because of amplitude (voltage or current)

$$\phi = \begin{cases} 0^\circ, & K > 0 \\ 180^\circ, & K < 0 \end{cases}$$

as shown in Fig. 14.9.

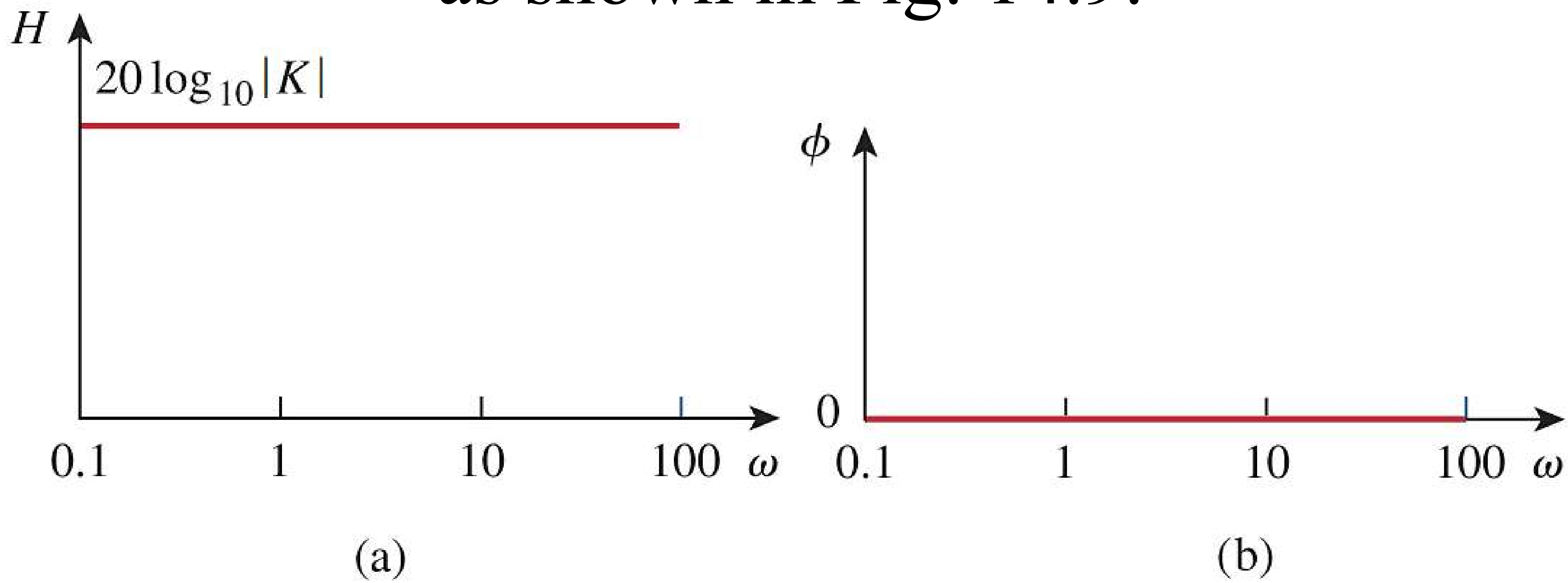


Figure 14.9 Bode plots for gain K: (a) magnitude plot, (b) phase plot.

2. $z_m = 0$; $p_n = 0$

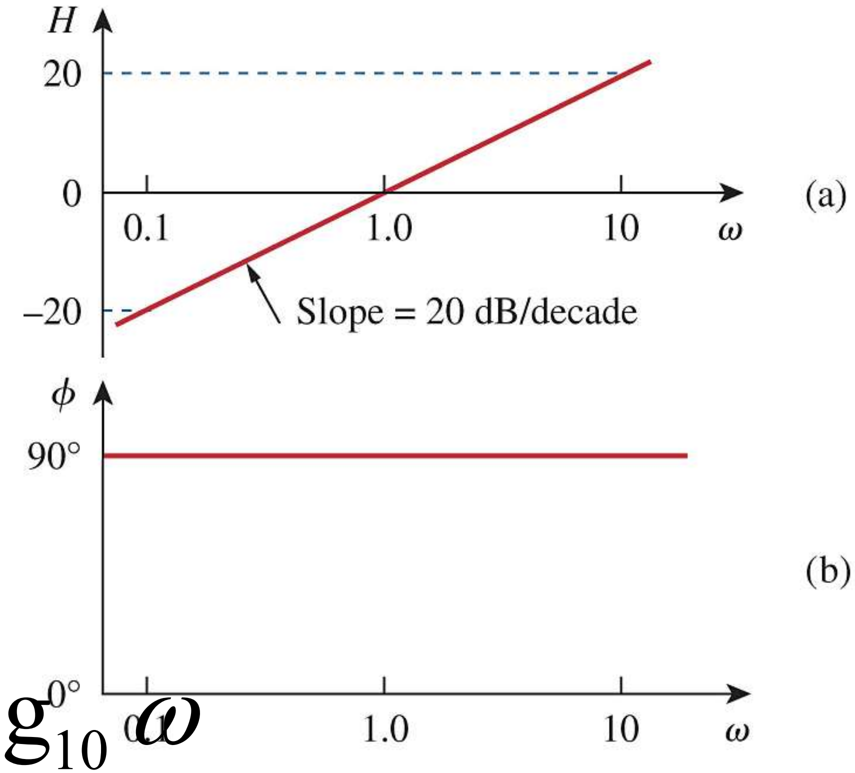


Figure 14.10 Bode plots for $j\omega$.

(2) For $j\omega$,

$$\underbrace{H_{dB}}_{y} = 20\log_{10}|j\omega| = \underbrace{\boxed{20}}_{\text{slope}}\underbrace{\log_{10}\omega}_{x}$$

$$\phi = 90°$$

These are shown in Fig. 14.10, where we notice that the slope of the magnitude plot is 20 dB/decade, where the word *decade* means a group or series of ten.

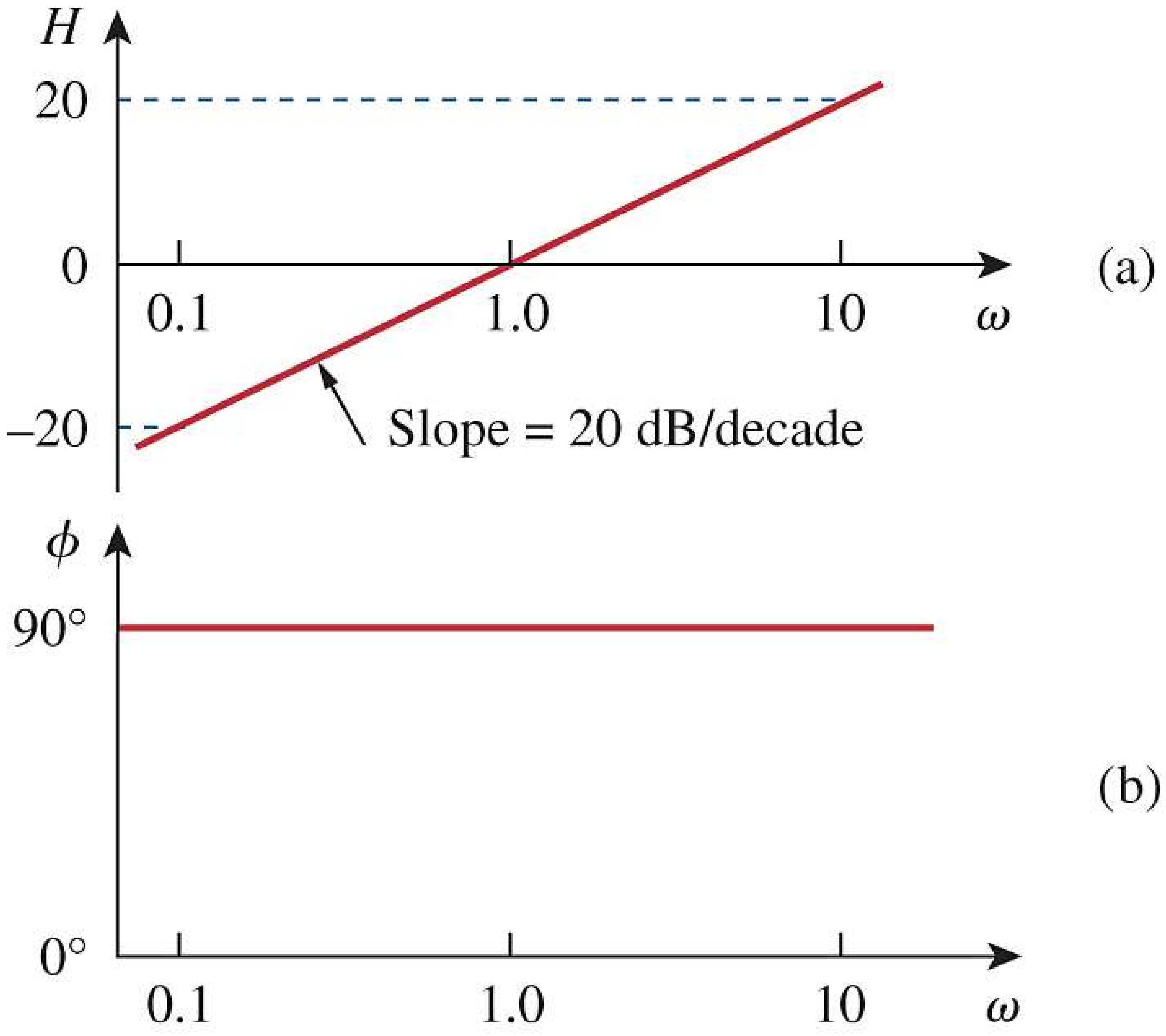


Figure 14.10 Bode plots for $j\omega$.

The Bode plots for $(j\omega)^{-1}$ are similar except that the slope of the magnitude plot is -20 dB/decade while the phase is $-90°$. In general, for $(j\omega)^{N}$, where N is an interger, the magnitude plot will have a slope of $20N$ dB/decade, while the phase is $90N°$.

$H_{dB} = 20\log_{10}|(j\omega)^{-1}| = -20\log_{10}\omega$
$\angle(j\omega)^{-1} = \angle 1 - \angle(j\omega) = 0° - 90° = -90°$

$H_{dB} = 20\log_{10}|(j\omega)^{N}| = 20N\log_{10}\omega$
$\angle(j\omega)^{N} = \angle(j\omega) + \angle(j\omega) + \ldots = 90N°$

3. z_m = real; p_n = real

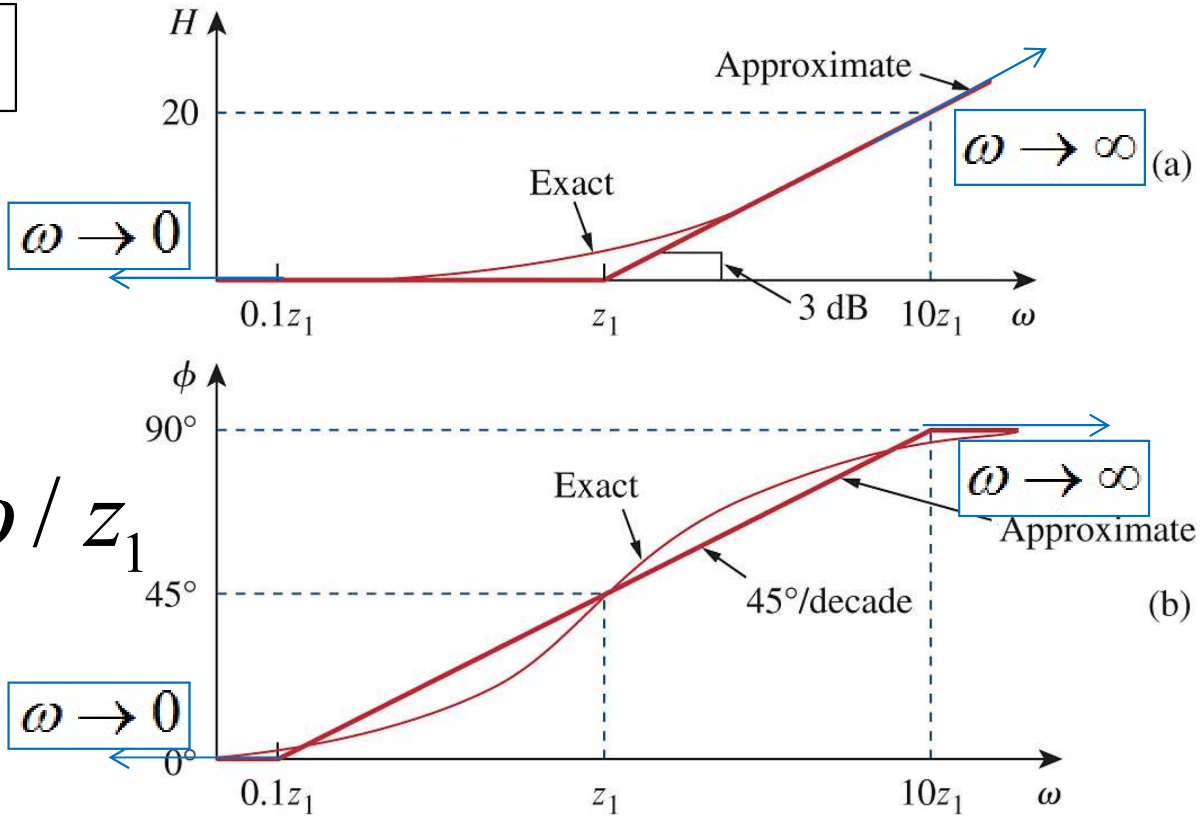


Figure 14.11 Bode plots for $1+j\omega/z_1$: (a) magnitude plot, (b) phase plot.

(3) For $(1+j\omega/z_1)$,

$$H_{dB} = 20\log_{10}|1+j\omega/z_1|$$

$$\phi = \tan^{-1}(\omega/z_1)$$

We notice that

$$\begin{cases} H_{dB} = 20\log_{10} 1 = 0 \\ \phi = 0^\circ \end{cases}, \quad \omega \to 0$$

$\tan^{-1}(0)=0°$

$$\begin{cases} H_{dB} = 20\log_{10}(\omega/z_1) \\ \phi = 90^\circ \end{cases}, \quad \omega \to \infty$$

$\tan^{-1}(\infty)=90°$

0 dB at $\omega = z_1$
Slope = 20dB/decade

Compare the approximate and exact curves

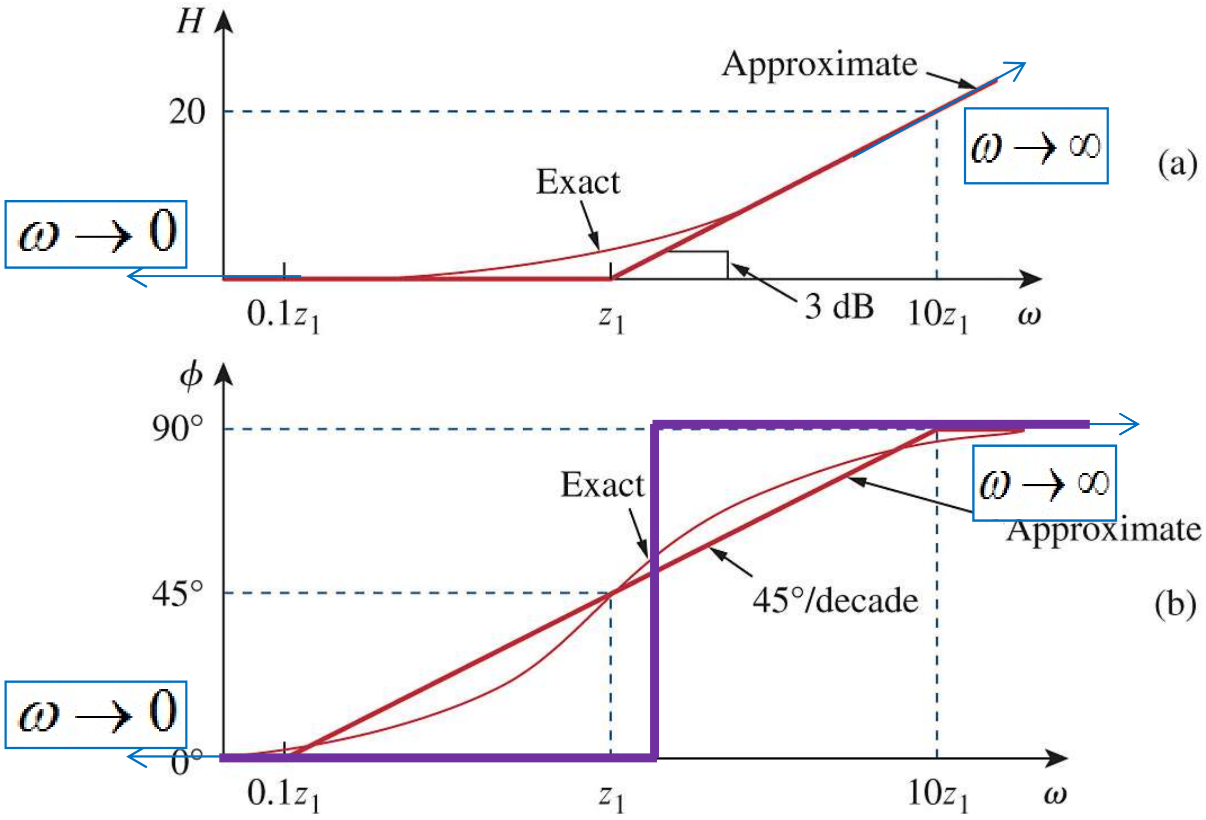


Figure 14.11 Bode plots for $1+j\omega/z_1$: (a) magnitude plot, (b) phase plot.

2 sections are enough

2 sections are not enough; 3 sections needed

As a straight-line approximation, we let

$$H_{dB} = \begin{cases} 0, & \omega \le z_1 \\ 20\log_{10}(\omega / z_1), & \omega \ge z_1 \end{cases}$$

At $\omega = z_1$, $\tan^{-1}(1)$=45°
Use a straight line to connect the three points $\omega = 0.1z_1$, z_1, $10z_1$

$$\phi = \begin{cases} 0°, \quad \boxed{\tan^{-1}(0)=0°} & \omega \le 0.1z_1 \\ 45° + 45°\log_{10}(\omega / z_1), & 0.1z_1 \le \omega \le 10z_1 \\ 90°, \quad \boxed{\tan^{-1}(\infty)=90°} & \omega \ge 10z_1 \end{cases}$$

as shown in Fig. 14.11. The frequency $\omega =$ z_1 is called the *corner frequency* or *break frequency*.

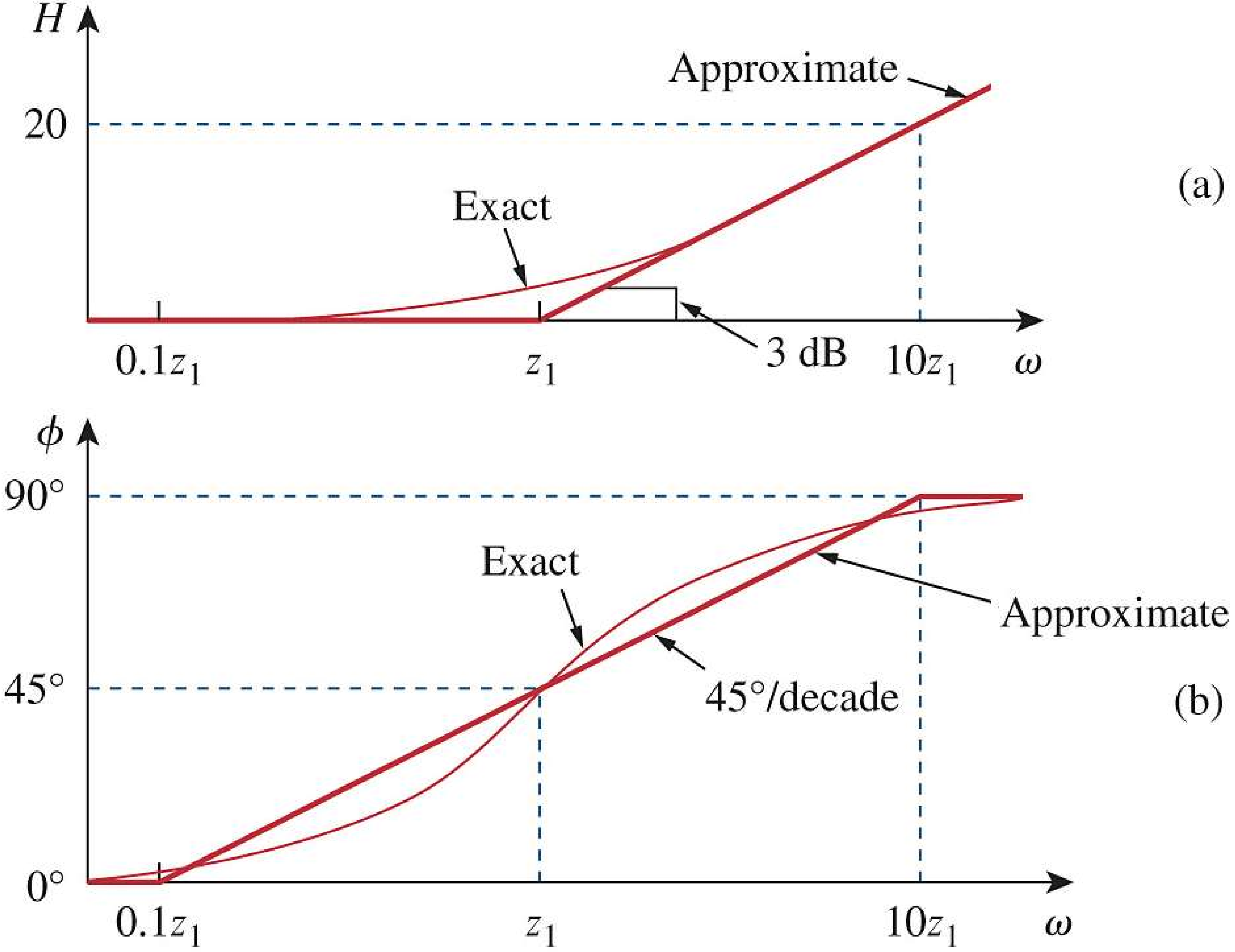


Figure 14.11 Bode plots for $1+j\omega/z_1$: (a) magnitude plot, (b) phase plot.

The Bode plots for $1/(1+j\omega/p_1)$ are similar to those in Fig. 14.11 except that the corner frequency is at $\omega = p_1$, the magnitude has a slope of -20 dB/decade, and the phase has a slope of $-45°$ per decade.

In general, for $(1+j\omega/z_1)^N$, where N is an interger,

$$H_{dB} = \begin{cases} 0, & \omega \le z_1 \\ 20N\log_{10}(\omega/z_1), & \omega \ge z_1 \end{cases}$$

$$\phi = \begin{cases} 0^\circ, & \omega \le 0.1z_1 \\ 45N^\circ + 45N^\circ\log_{10}(\omega/z_1), & 0.1z_1 \le \omega \le 10z_1 \\ 90N^\circ, & \omega \ge 10z_1 \end{cases}$$

4. z_m = c.c.; p_n = c.c.

(4) For $1/[1+2\zeta_2(j\omega/\omega_n)+(j\omega/\omega_n)^2]$,

$$H_{dB} = -20\log_{10}\left|1+2\zeta_2(j\omega/\omega_n)+(j\omega/\omega_n)^2\right|$$

$$\phi = -\tan^{-1}\left(\frac{2\zeta_2\omega/\omega_n}{1-\omega^2/\omega_n^2}\right)$$

We notice that

$$\begin{cases} H_{dB} = -20\log_{10}1 = 0 \\ \phi = 0^\circ \end{cases}, \quad \omega \to 0$$

tan⁻¹(0) = 0°

$$\begin{cases} H_{dB} = -40\log_{10}(\omega/\omega_n) \\ \phi = -180^\circ \end{cases}, \quad \omega \to \infty$$

–tan⁻¹(0) = 0° or –180°?

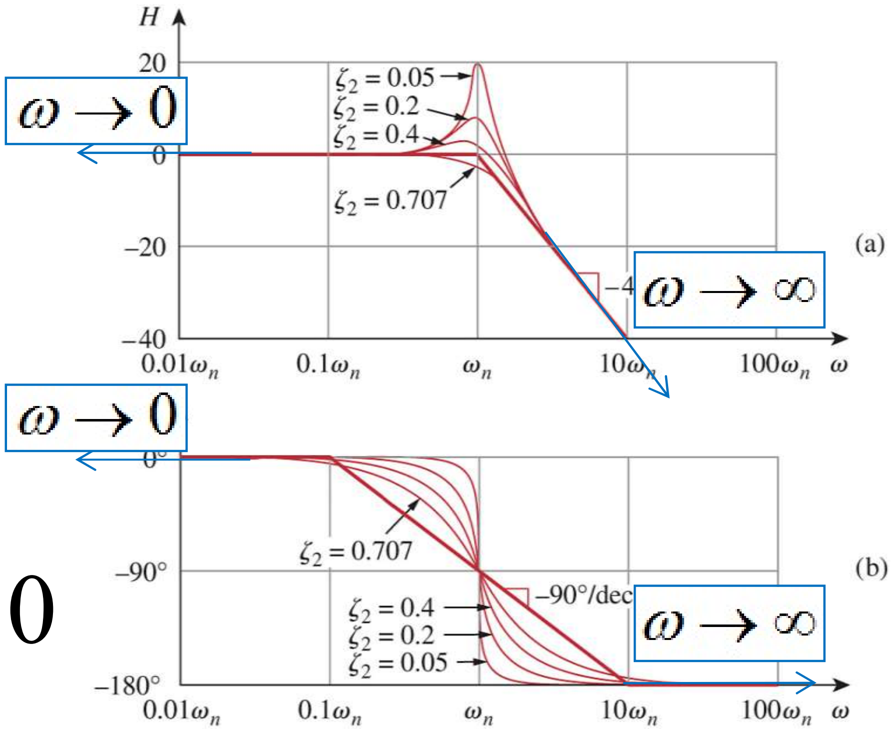


Figure 14.12 Bode plots of $1/[1+2\zeta_2(j\omega/\omega_n)+(j\omega/\omega_n)^2]$: (a) magnitude plot, (b) phase plot.

As a straight-line approximation, we let

$$H_{dB} = \begin{cases} 0, & \omega \le \omega_n \\ -40\log_{10}(\omega/\omega_n), & \omega \ge \omega_n \end{cases}$$

At $\omega = \omega_n$, $\phi = -\tan^{-1}(\infty)$ = $-90°$
Use a straight line to connect the three points $\omega = 0.1\omega_n$, ω_n, $10\omega_n$

$$\phi = \begin{cases} 0°, & \omega \le 0.1\omega_n \\ -90° - 90°\log_{10}(\omega/\omega_n), & 0.1\omega_n \le \omega \le 10\omega_n \\ -180°, & \omega \ge 10\omega_n \end{cases}$$

$\tan^{-1}(0)$ = 0°

$-\tan^{-1}(0)$=0° or –180°? ➔ –180°

as shown in Fig. 14.12. The frequency $\omega = \omega_n$ is called the *corner frequency* or *break frequency*.

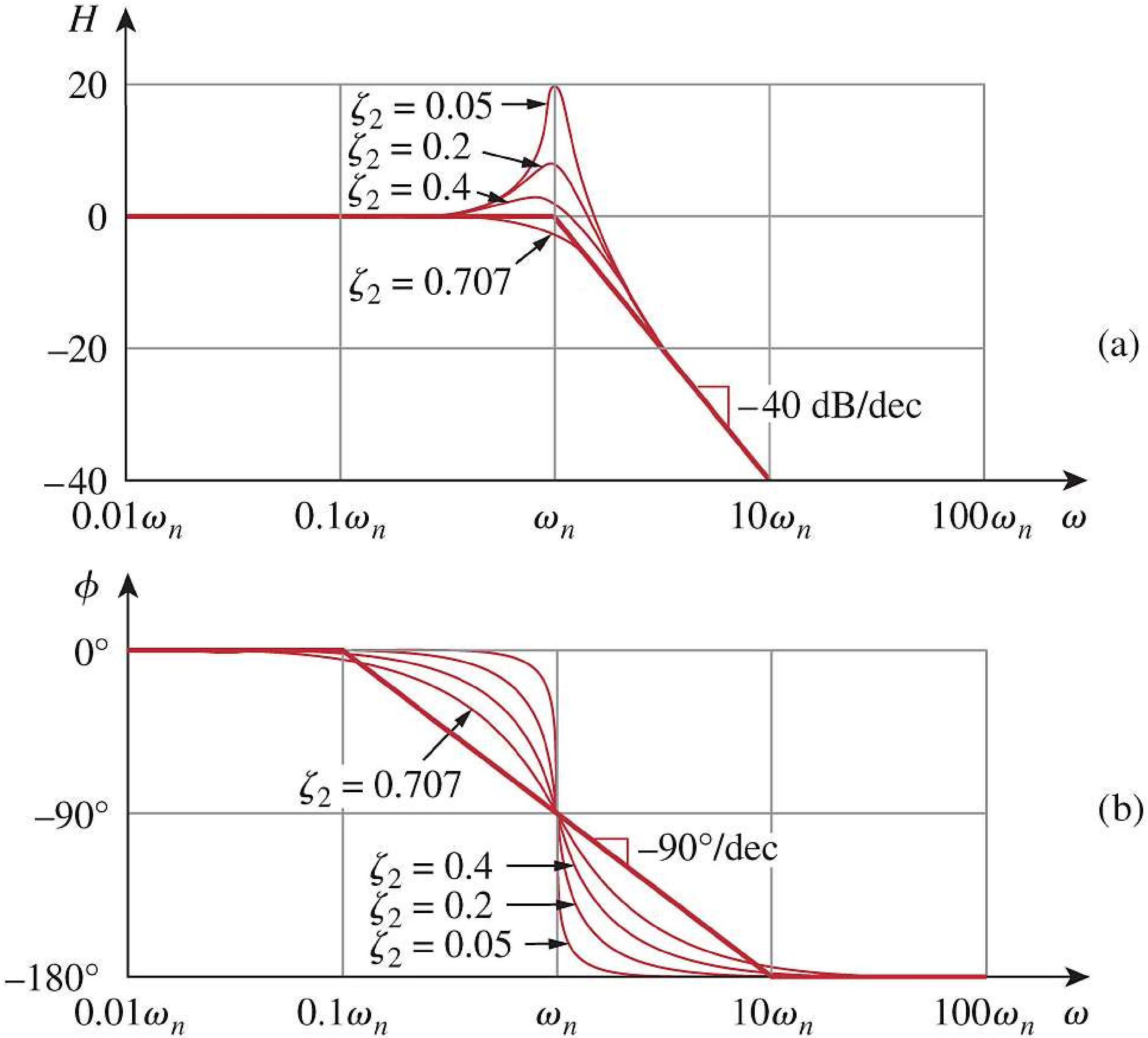
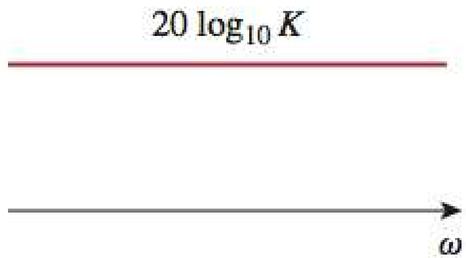
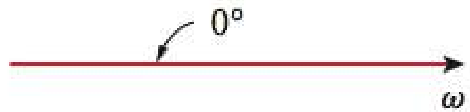
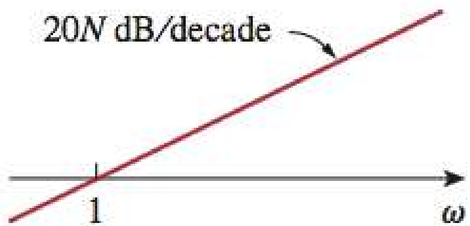
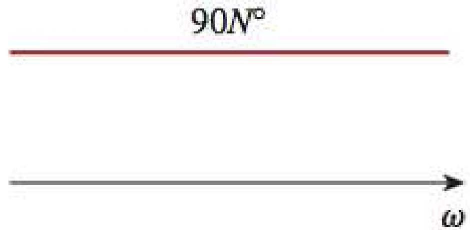
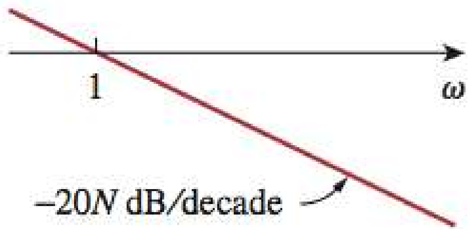
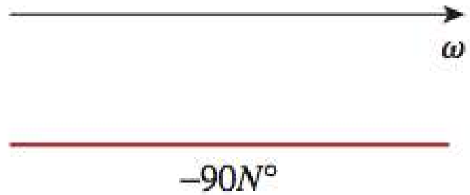
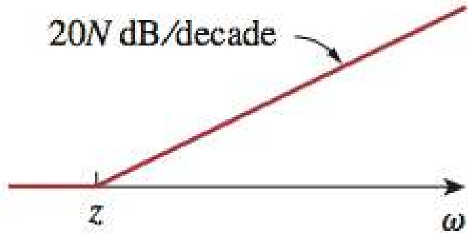
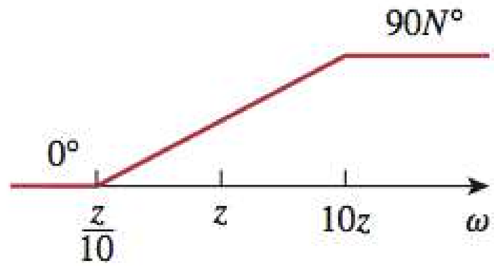
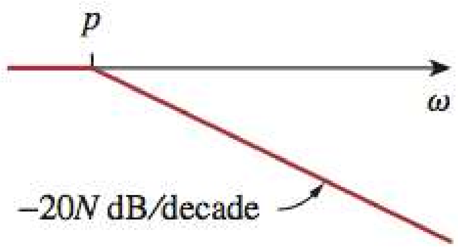


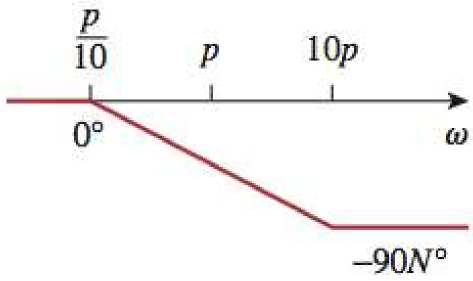
Figure 14.12 Bode plots of $1/[1+2\zeta_2(j\omega/\omega_n)+(j\omega/\omega_n)^2]$: (a) magnitude plot, (b) phase plot.

The Bode plots for $[1+2\zeta_1(j\omega/\omega_k)+(j\omega/\omega_k)^2]$ are similar to those in Fig. 14.12 except that the corner frequency is at $\omega = \omega_k$, the magnitude has a slope of +40 dB/decade, and the phase has a slope of +90° per decade.

Factor	Magnitude	Phase
1. The *b* or *K* K		
2. $z_m = 0$; $p_n = 0$ $(j\omega)^N$		
$\dfrac{1}{(j\omega)^N}$		
3. z_m = real; p_n = real $\left(1 + \dfrac{j\omega}{z}\right)^N$		

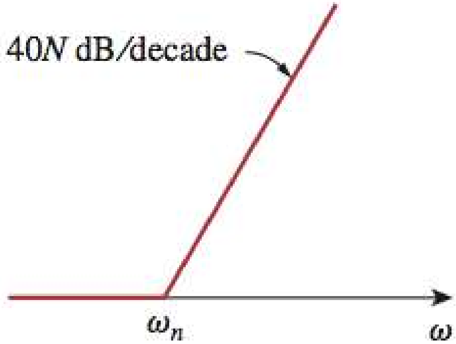
$$\frac{1}{(1 + j\omega/p)^N}$$

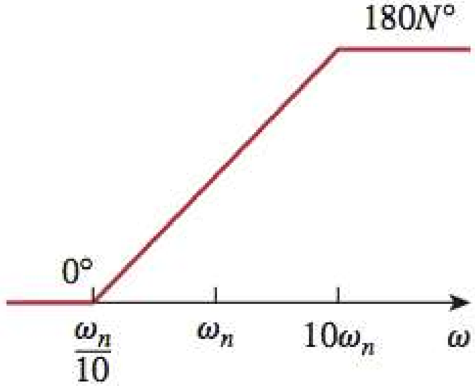




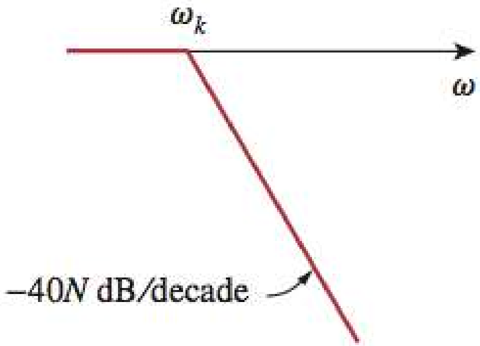
4. z_m = c.c.; p_n = c.c.

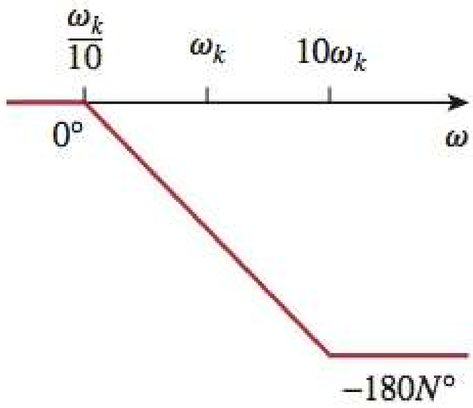
$$\left[1 + \frac{2j\omega\zeta}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2\right]^N$$





$$\frac{1}{[1 + 2j\omega\zeta/\omega_k + (j\omega/\omega_k)^2]^N}$$





Zeros: upward turn

Poles: downward turn

Practice Problem 14.3 Draw the Bode plots for

$$H(j\omega) = \frac{5(j\omega + 2)}{j\omega(j\omega + 10)}$$

Solution :

$$H(j\omega) = \frac{\underbrace{(1 + j\omega/2)}_{\text{Type 3, zero}}}{\underbrace{j\omega}_{\text{Type 2, pole}}\underbrace{(1 + j\omega/10)}_{\text{Type 3, pole}}}$$

$$H_{dB} = 20\log_{10}\left|\frac{(1 + j\omega/2)}{j\omega(1 + j\omega/10)}\right|$$

$$= 20\log_{10}\left|1 + j\omega/2\right| + 20\log_{10}\frac{1}{\left|j\omega\right|}$$

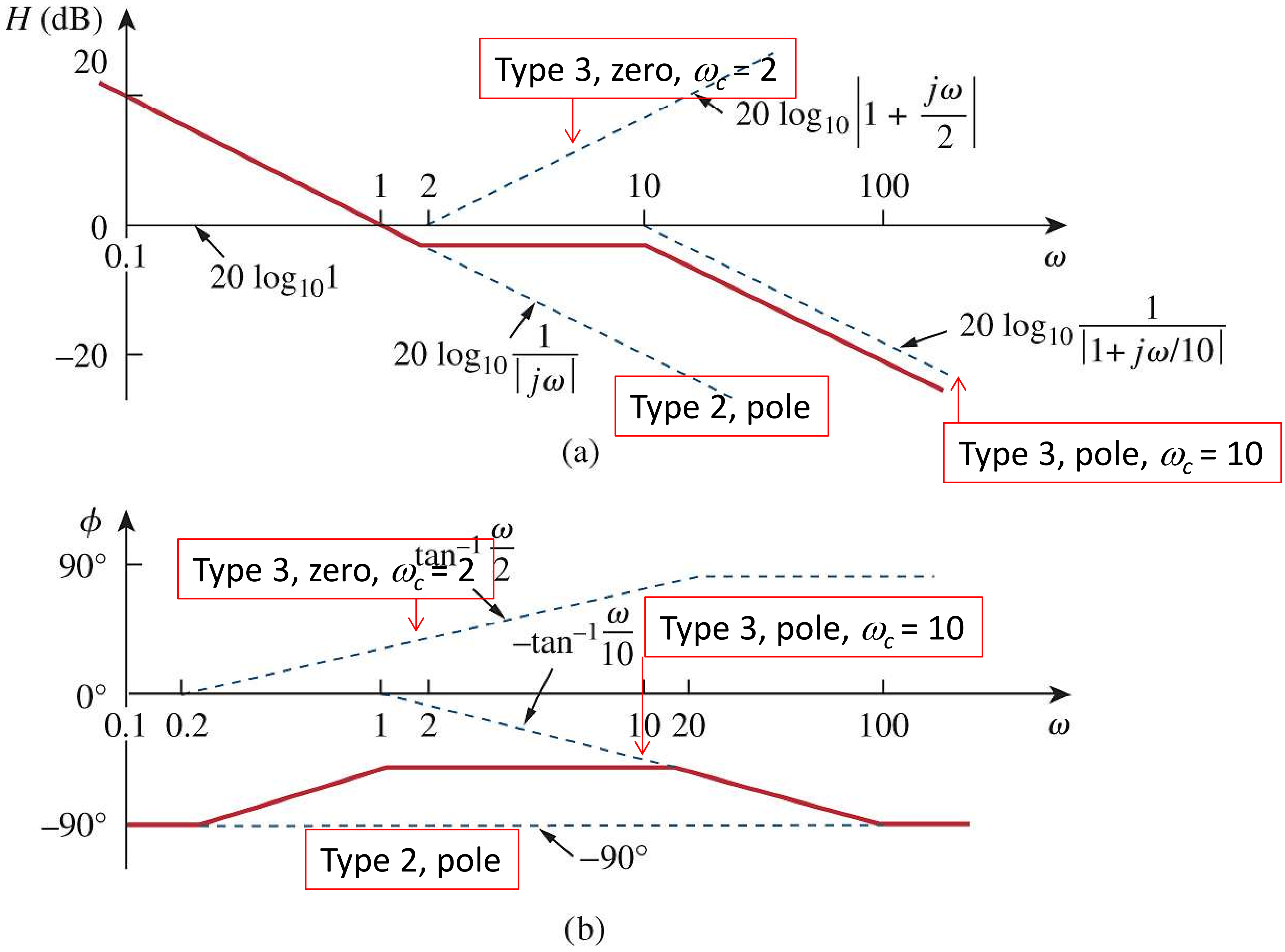
$$+20\log_{10}\frac{1}{\left|1 + j\omega/10\right|}$$

$$= 20\log_{10}\left|1 + j\omega/2\right| - 20\log_{10}\left|j\omega\right|$$

$$-20\log_{10}\left|1 + j\omega/10\right|$$

$$\phi = \tan^{-1}\left(\omega/2\right) - 90^\circ - 2\tan^{-1}(\omega/10)$$

The Bode plots are in Fig. 14.14.



H (dB)
20
0
–20
0.1
1
2
10
100
ω
Type 3, zero, $\omega_c = 2$
$20 \log_{10}\left|1 + \frac{j\omega}{2}\right|$
$20 \log_{10} 1$
$20 \log_{10} \frac{1}{|j\omega|}$
Type 2, pole
$20 \log_{10} \frac{1}{|1 + j\omega/10|}$
Type 3, pole, $\omega_c = 10$
(a)
ϕ
90°
0°
–90°
0.1
0.2
1
2
10
20
100
ω
Type 3, zero, $\omega_c = 2$
$\tan^{-1} \frac{\omega}{2}$
$-\tan^{-1} \frac{\omega}{10}$
Type 3, pole, $\omega_c = 10$
Type 2, pole
–90°
(b)

Figure 14.14

Example 14.4 Obtain the Bode plots for

$$H(j\omega) = \frac{j\omega + 10}{j\omega(j\omega + 5)^2}$$

Solution :

$$H(j\omega) = \frac{\overbrace{0.4}^{\text{Type 1}}\overbrace{(1 + j\omega/10)}^{\text{Type 3, zero}}}{\underbrace{j\omega}_{\text{Type 2, pole}}\underbrace{(1 + j\omega/5)^2}_{\text{Type 3, } N = 2\text{, pole}}}$$

$$H_{dB} = 20\log_{10}\left|\frac{0.4(1 + j\omega/10)}{j\omega(1 + j\omega/5)^2}\right|$$

$$= 20\log_{10} 0.4 + 20\log_{10} \left|1 + j\omega/10\right| +$$

$$-20\log_{10} \left|j\omega\right| - 40\log_{10} \left|1 + j\omega/5\right|$$

$$\phi = 0° + \tan^{-1}\left(\omega/10\right) - 90° - 2\tan^{-1}(\omega/5)$$

The Bode plots are in Fig. 14.15.

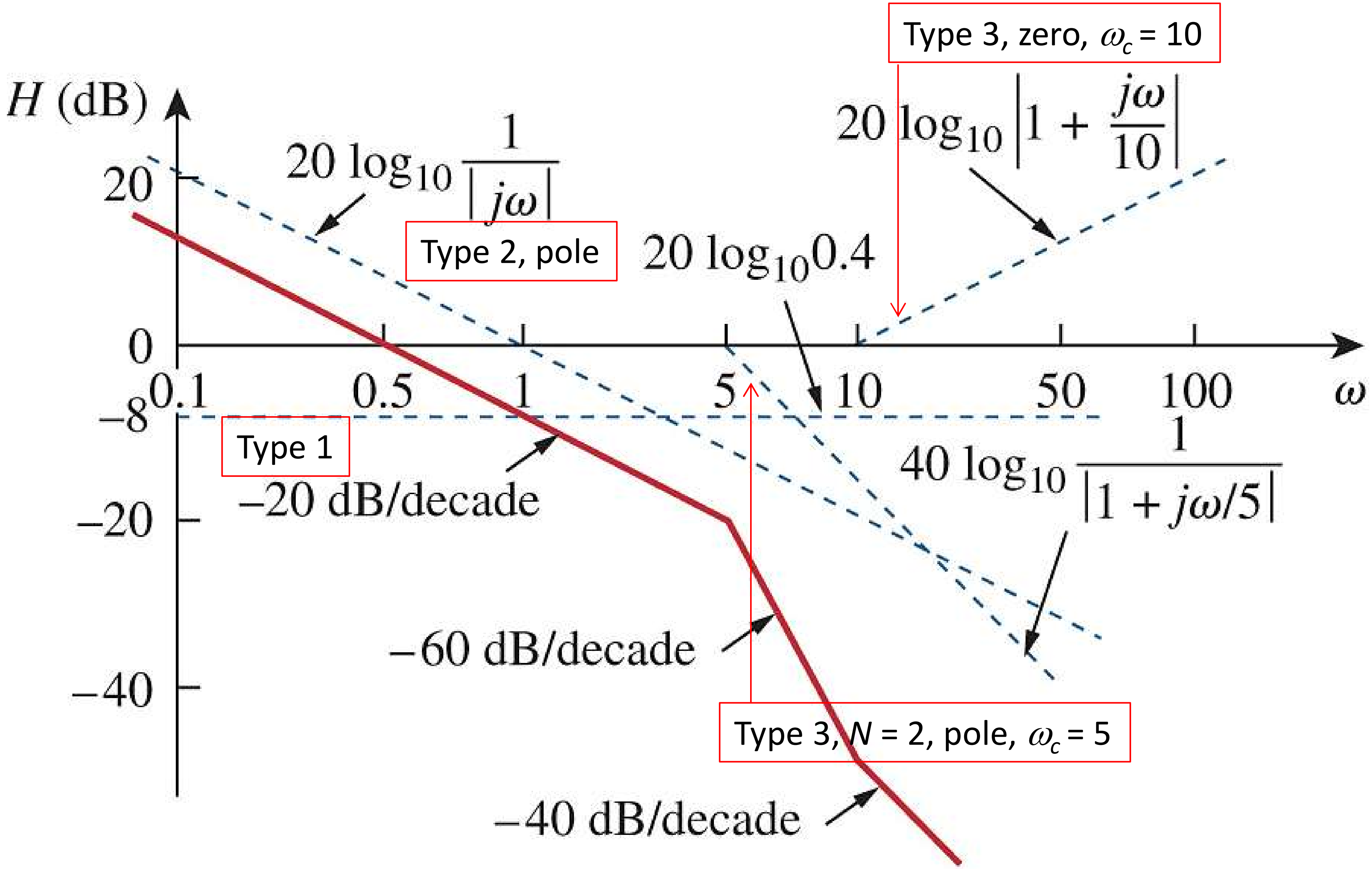


Figure 14.15(a)

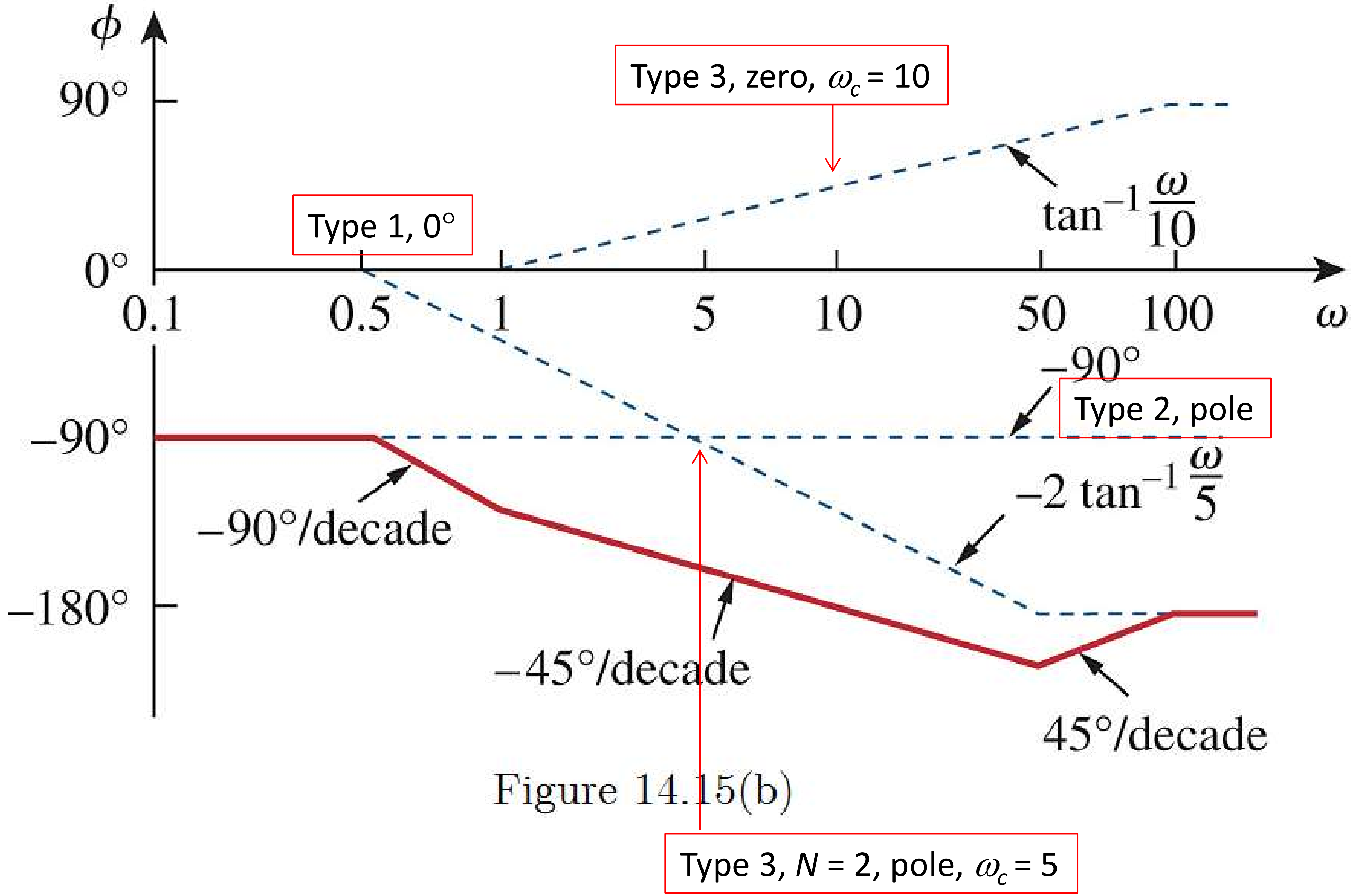


Figure 14.15(b)

Example 14.5 Draw the Bode plots for

$$H(j\omega)=\frac{j\omega+1}{(j\omega)^2+12(j\omega)+100}$$

Solution :

$$H(j\omega)=\frac{0.01(1+j\omega/1)}{1+0.12(j\omega)+(j\omega/10)^2}$$

$$=\frac{\underbrace{0.01}_{\text{Type 1}}\underbrace{(1+j\omega/1)}_{\text{Type 3, zero}}}{\underbrace{1+2\times0.6(j\omega/10)+(j\omega/10)^2}_{\text{Type 4, pole}}}$$

$$H_{dB} = 20\log_{10}\left|\frac{0.01(1+j\omega/1)}{1+2\times 0.6(j\omega/10)+(j\omega/10)^2}\right|$$

$$= 20\log_{10} 0.01 + 20\log_{10}\left|1+j\omega/1\right|$$

$$-20\log_{10}\left|1+2\times 0.6(j\omega/10)+(j\omega/10)^2\right|$$

$$\phi = 0° + \tan^{-1}\left(\omega/1\right) - \tan^{-1}\left[\frac{2\times 0.6(\omega/10)}{1-(\omega/10)^2}\right]$$

The Bode plots are in Fig. 14.17.

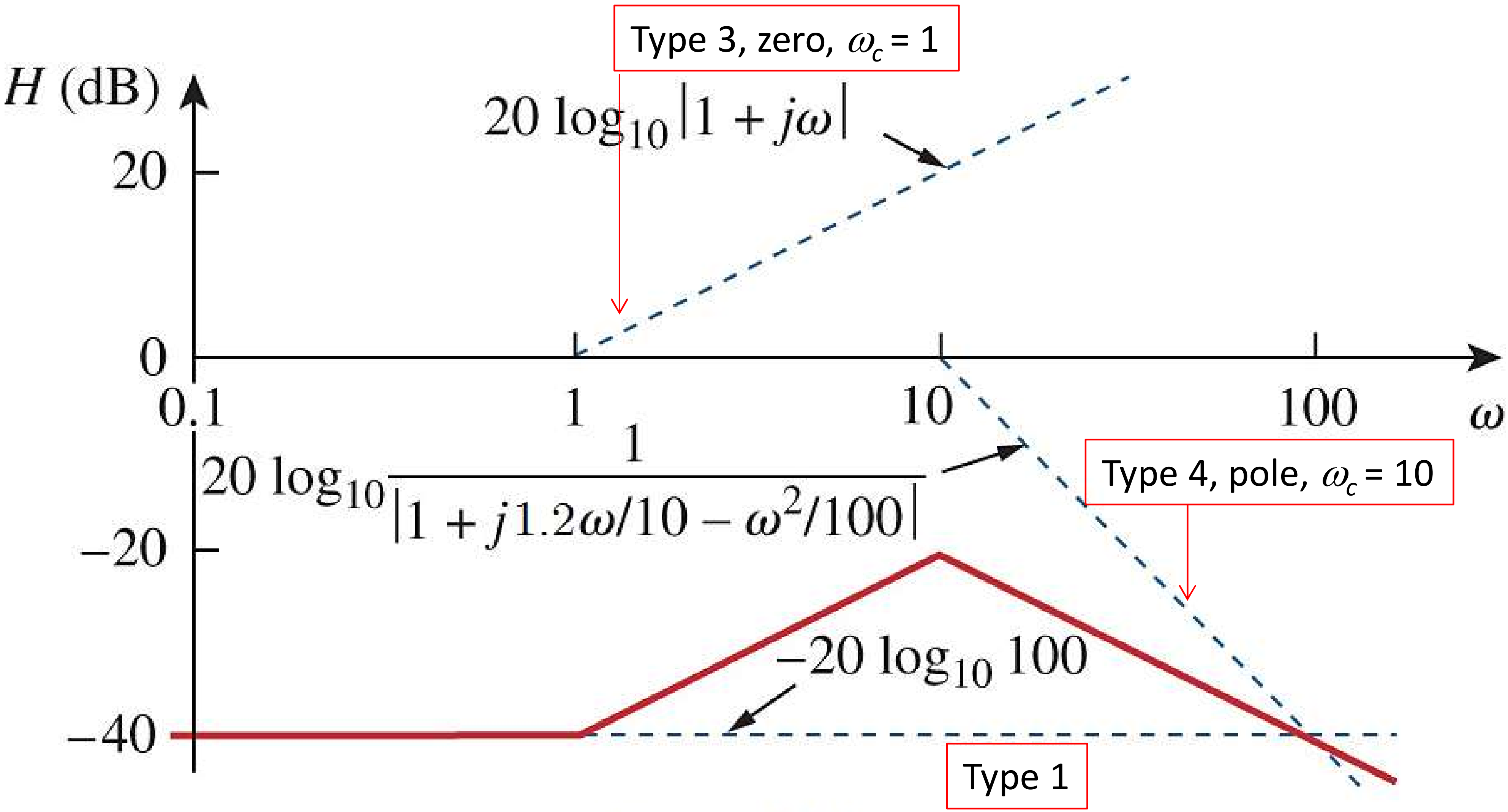


Figure 14.17(a)

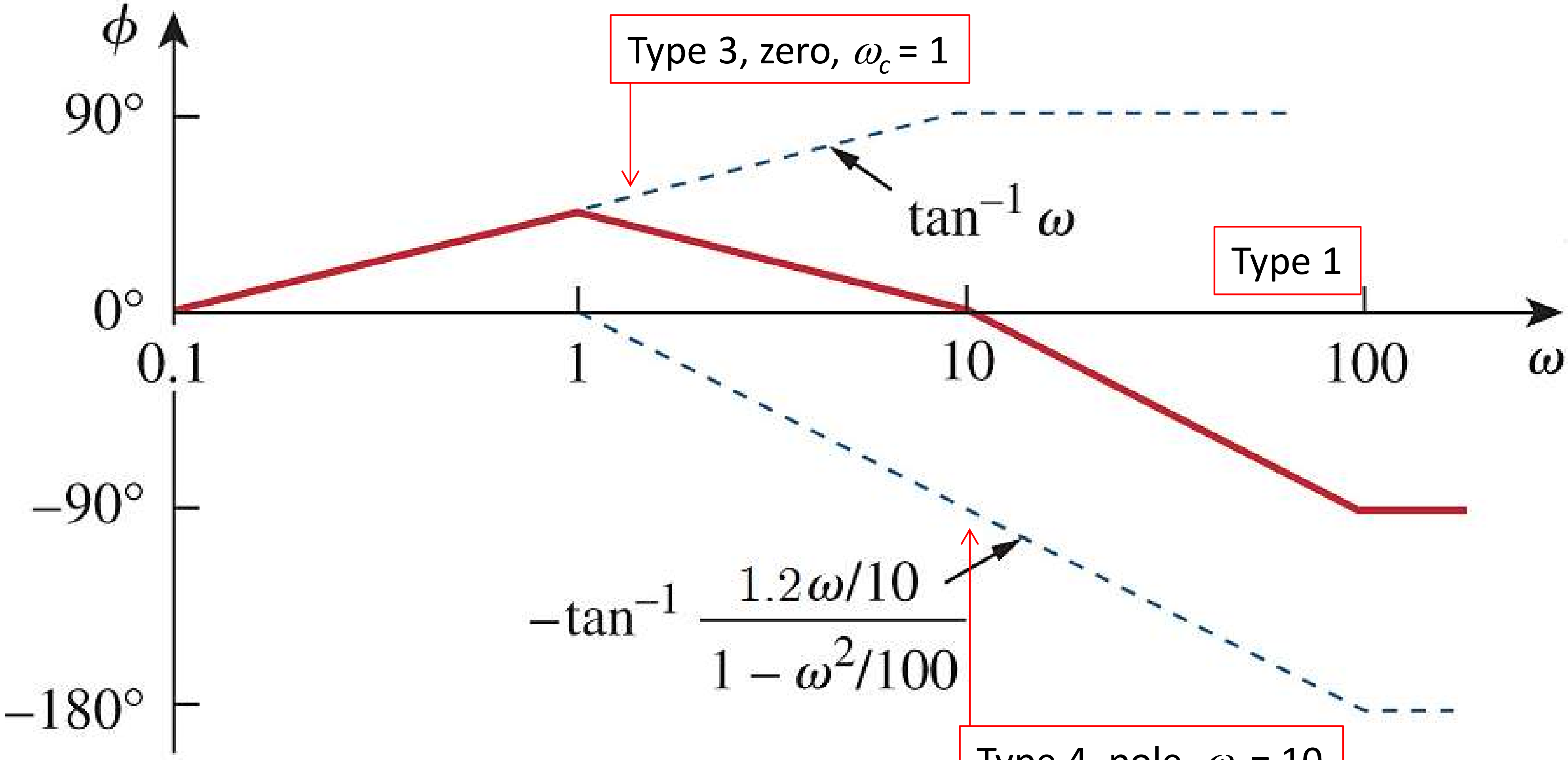


Figure 14.17(b)

14.5 Series Resonance

Resonance occurs in any circuits that has at least one inductor and one capacitor. Consider the series *RLC* circuit shown in Fig. 14.21. The input impedance is

$$Z = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

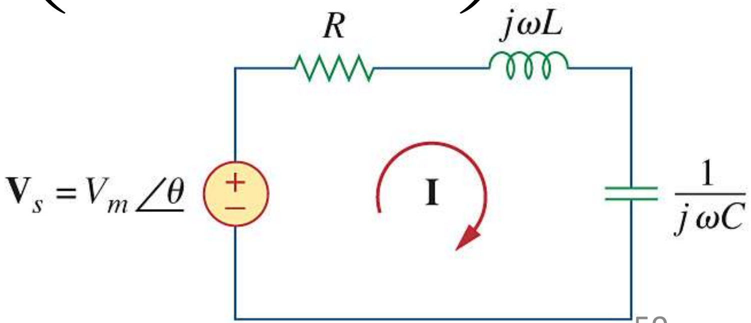


Figure 14.21 The series resonant circuit.

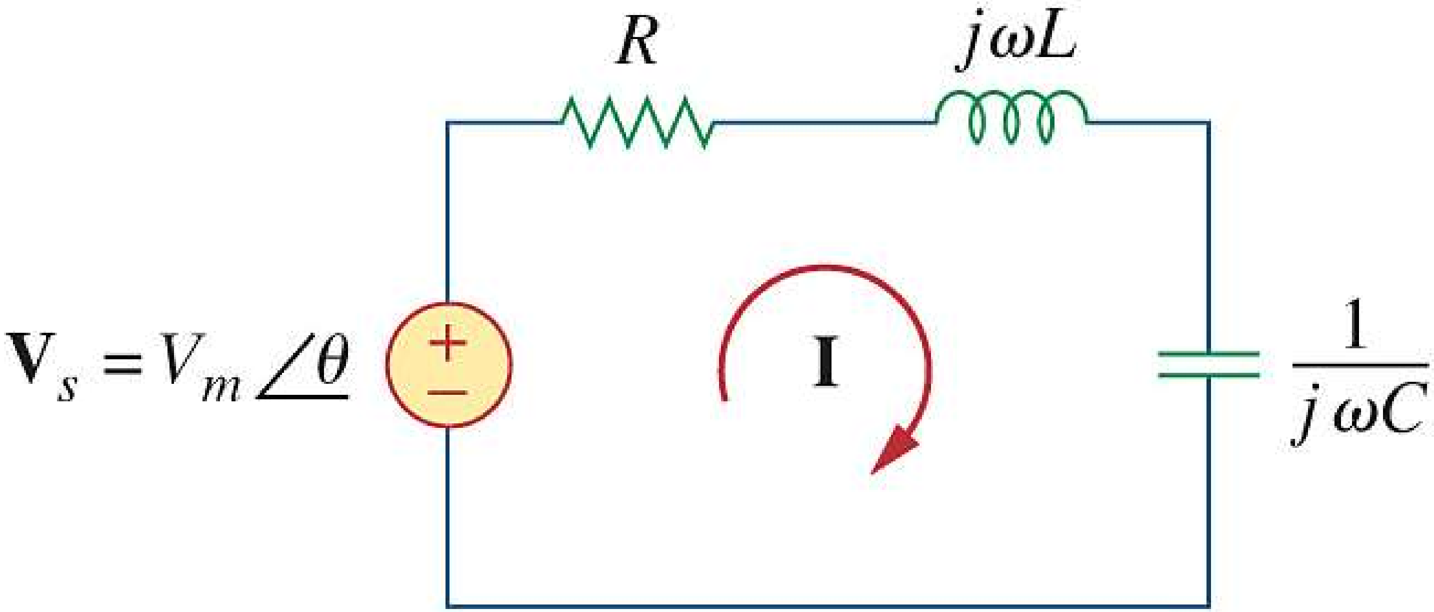


Figure 14.21 The series resonant circuit.

Resonance results when $\text{Im}(Z) = 0$, that is, the capacitive and inductive reactances are equal in magnitude. The value of ω that satisfies this condition is called the *resonant frequency* ω_0. Thus, the resonance condition is

$$\omega_0 L = \frac{1}{\omega_0 C}$$

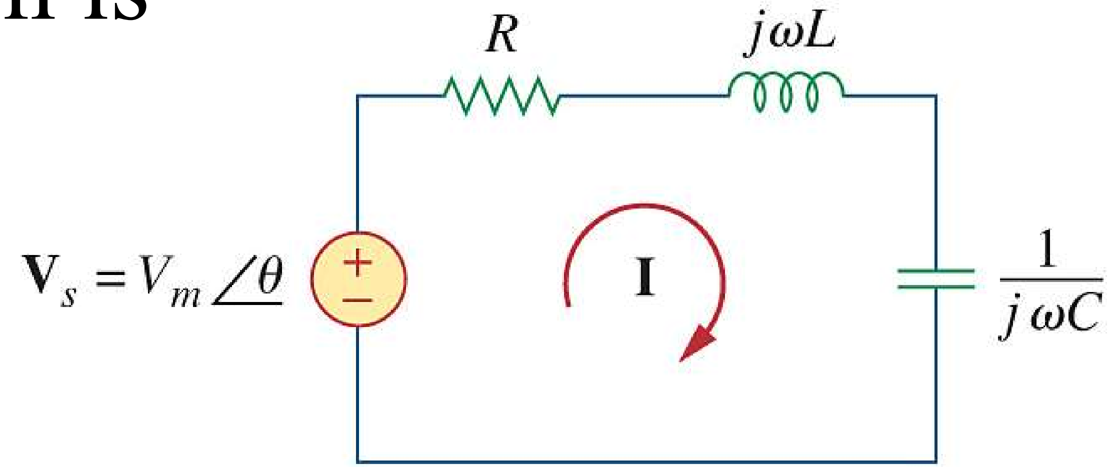


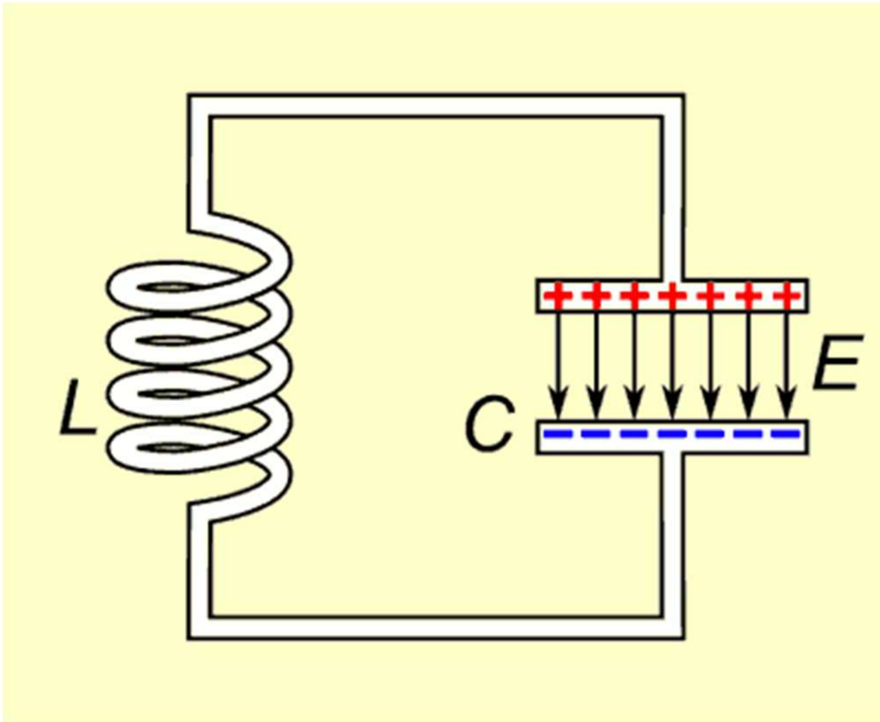
Figure 14.21 The series resonant circuit.

Resonance

$$\mathrm{Im}(Z) = \omega L - \frac{1}{\omega C} = 0 \qquad (14.24)$$

$|Q_L| = |\mathbf{I}^2 X| = \mathbf{I}^2 \omega L$
$|Q_C| = |\mathbf{I}^2 X| = \mathbf{I}^2/\omega C$
$|Q_L| = |Q_C|$

At resonance, the energies from inductive and capacitive circuits are equal.



Ease of excitation at resonance

It is easy to get an object to vibrate at its resonant frequencies, hard at other frequencies. A child's playground swing is an example of a pendulum, a resonant system with only one resonant frequency. With a tiny push on the swing each time it comes back to you, you can continue to build up the **amplitude** of swing.
If you try to force it to swing a twice that frequency, you will find it very difficult, and might even lose teeth in the process!

Swinging a child in a playground swing is an easy job because you are helped by its natural frequency.

But can you swing it at some other frequency?

In circuit, if the driving frequency is the resonant frequency ω_0, the amplitude (voltage or current) can be built up.

$$\omega_0 = 2\pi f_0 = \frac{1}{\sqrt{LC}}$$

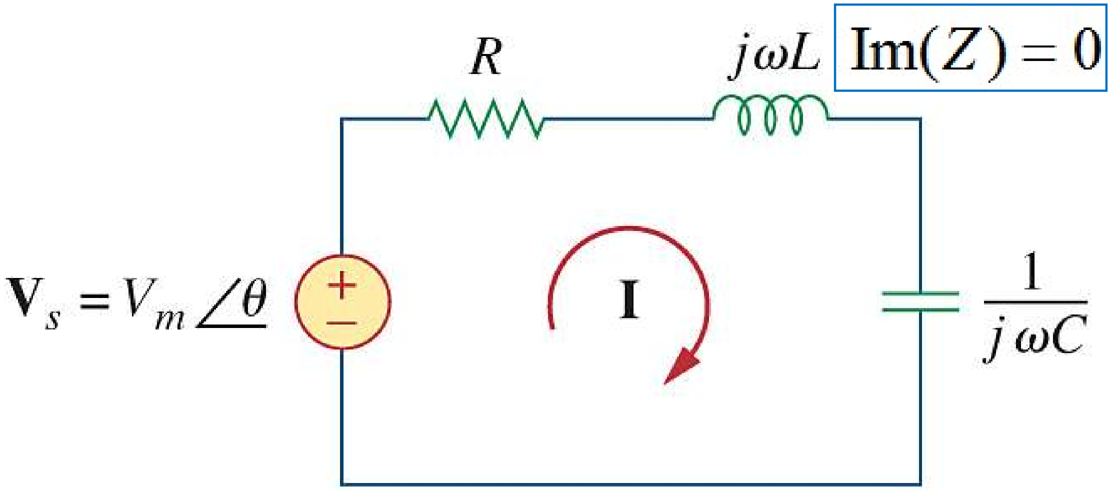


Figure 14.21 The series resonant circuit.

Note that at resonance:

1. The impedance is purely resistive, thus, $Z = R$. In other words, the LC series combination acts like a short circuit, and the entire voltage is across R.

2. The voltage $\tilde{V}_s$ and the current $\tilde{I}$ are in phase.

3. The magnitude of the current is maximum.

Because $|\boldsymbol{Z}|$ achieves minimum

Proof :

The circuit's cuurent magnitude

$$I = \left|\frac{\tilde{V}_s}{Z}\right| = \left|\frac{V_m\angle\theta}{R + j(\omega L - \dfrac{1}{\omega C})}\right|$$

$$= \frac{V_m}{\sqrt{R^2 + (\omega L - \dfrac{1}{\omega C})^2}} \le \frac{V_m}{R} = I_{\max}$$

$\mathrm{Im}(Z) = 0$

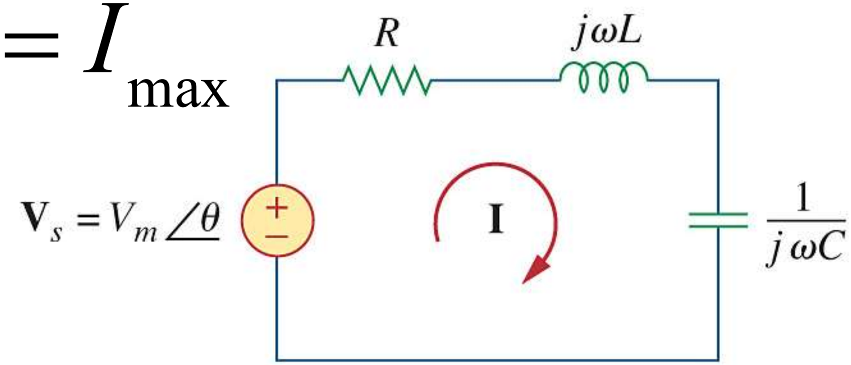


Figure 14.21 The series resonant circuit.

$$I = \frac{V_m}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

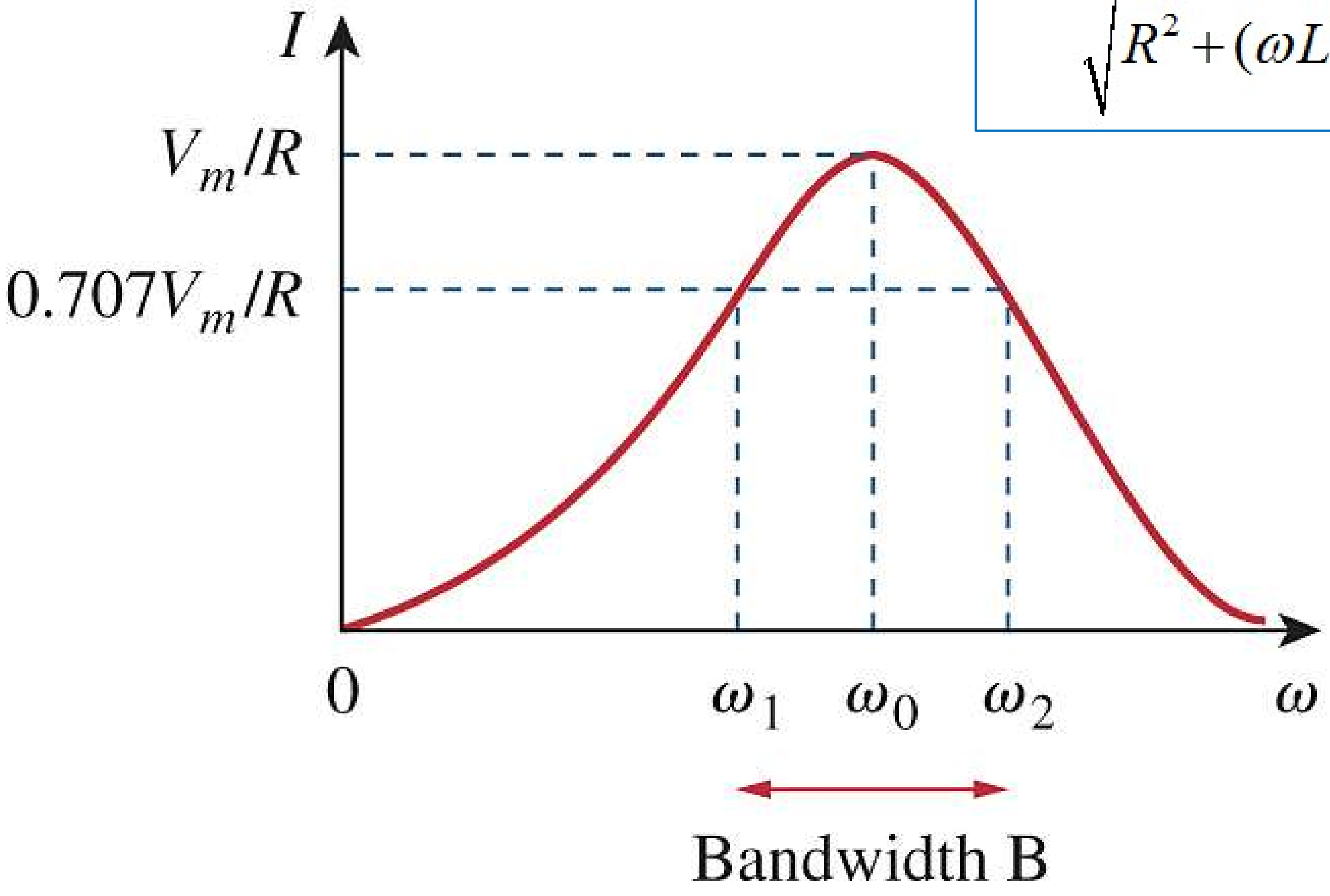


Figure 14.22 The current amplitude versus frequency for the series resonant circuit of Fig. 14.21.

The average power dissipated by the *RLC* circuit is

$$P(\omega) = \frac{1}{2} I^2 R = \frac{1}{2} \frac{V_m^2}{R^2 + \left(\omega L - 1/(\omega C)\right)^2} R$$

The highest power dissipated occurs at resonance,

$$P(\omega_0) = \frac{1}{2} I_{\max}^2 R = \frac{1}{2} \frac{V_m^2}{R}$$

At certain frequencies $\omega = \omega_1, \omega_2$, the dissipated power is half the maximum value; that is,

$$P(\omega_1) = P(\omega_2) = \frac{1}{2} P(\omega_0) = \frac{V_m^2}{4R}$$

$$\boxed{P(\omega_0) = \frac{1}{2} I_{\max}^2 R}$$

Hence, ω_1 and ω_2 are called the *half - power frequencies*. $I(\omega_1) = I(\omega_2) = \dfrac{I_{\max}}{\sqrt{2}}$.

The half-power frequencies are obtained by solving the equation

$$P(\omega) = \frac{1}{2}\frac{V_m^2}{R^2 + \left(\omega L - 1/(\omega C)\right)^2} R = \frac{V_m^2}{4R} = \frac{1}{2}P(\omega_0)$$

$$\left(\omega L - 1/(\omega C)\right)^2 = R^2$$

$$\omega L - 1/(\omega C) = \pm R$$

$$LC\omega^2 \mp RC\omega - 1 = 0$$

$$\omega = \frac{\pm RC \pm \sqrt{(RC)^2 + 4LC}}{2LC}$$

Neglect "–" solution

$$= \pm \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

ω_1 and ω_2 are positive,

$$\omega = \pm \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

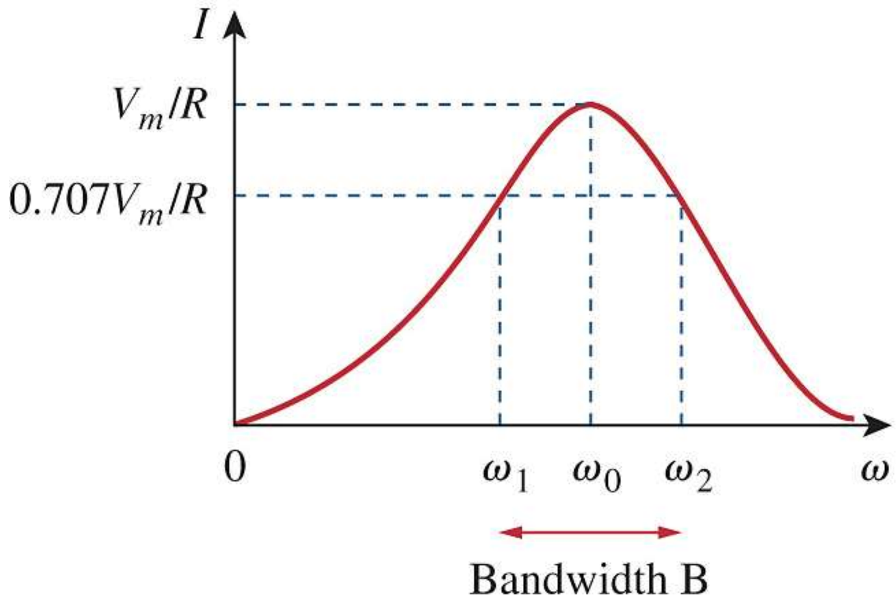


Figure 14.22 The current amplitude versus frequency for the series resonant circuit of Fig. 14.21.

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

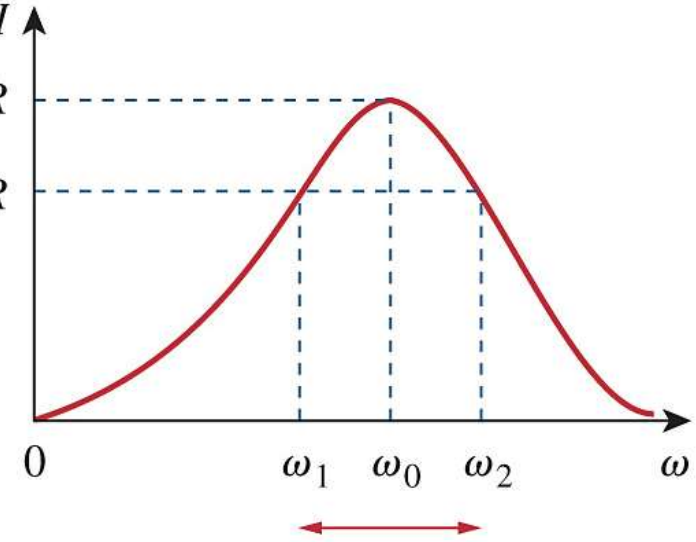


Figure 14.22 The current amplitude versus frequency for the series resonant circuit of Fig. 14.21.

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

We can relate the half-power frequencies and the resonant frequency,

$$\omega_1\omega_2 = \frac{1}{LC} = \omega_0^2 \Rightarrow \omega_0 = \sqrt{\omega_1\omega_2}$$

showing that the resonant frequency is the geometric mean of the half-power frequencies. Notice that ω_1 and ω_2 are in general not symmetrical around ω_0 because the frequency response is not generally symmetrical. However, if the frequency axis is a logarithm, we have $\log_{10}\omega_0 = (\log_{10}\omega_1 + \log_{10}\omega_2)/2.$

The *half - power bandwidth* is defined as the difference between the two half-power frequencies,

$$BW = \omega_2 - \omega_1 = \frac{R}{L}$$

The "sharpness" of the resonance is measured quantitatively by the *quality factor Q*.

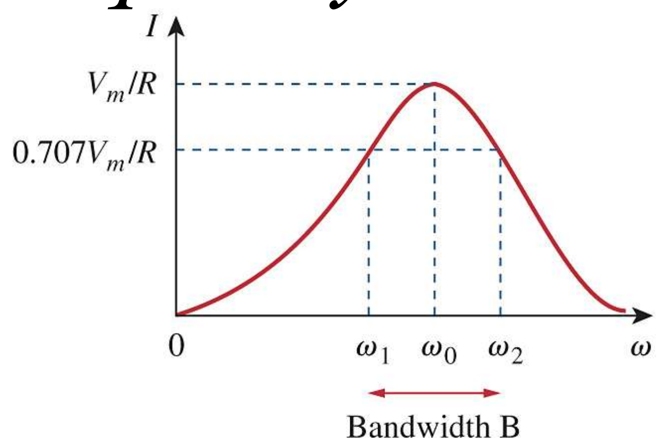


Figure 14.22 The current amplitude versus frequency for the series resonant circuit of Fig. 14.21.

The quality factor Q can be defined by

$$Q = 2\pi \frac{E_s}{E_d}$$

where E_s is the peak energy stored in the circuit and E_d is the energy dissipated in one period at resonance.

$L = 1/(\omega_0{}^2 C)$

$$Q = 2\pi \frac{\frac{1}{2} L I_{\max}^2}{\frac{1}{2} I_{\max}^2 R (1/f_0)} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$$

$E = P_{av} \times T$

Higher *Q*: more stored energy and/or less loss

The relationship between B and Q is

$$BW = \frac{R}{L} = \frac{\omega_0}{Q} \Rightarrow Q = \frac{\omega_0}{BW}$$

Thus, the quality factor of an RLC circuit can be defined as the ratio of its resonant frequency to its bandwidth.

For same ω_0,

$Q \uparrow \Leftrightarrow BW \downarrow \Leftrightarrow$ sharpness $\uparrow$

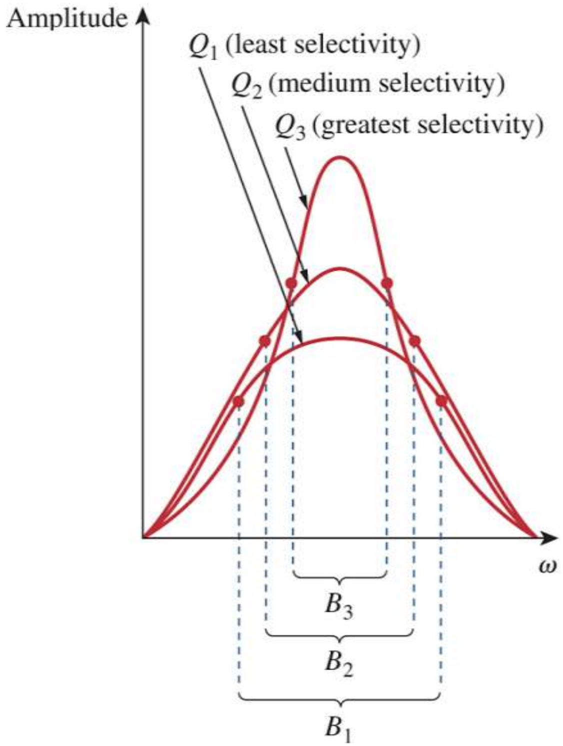


Figure 14.23 The higher the circuit Q, the smaller the bandwidth.

As illustrated in Fig. 14.23, the higher the value of Q, the more selective the circuit is. The *selectivity* of an *RLC* circuit is the ability of the circuit to respond to a certain frequency and discriminate against all other frequencies.

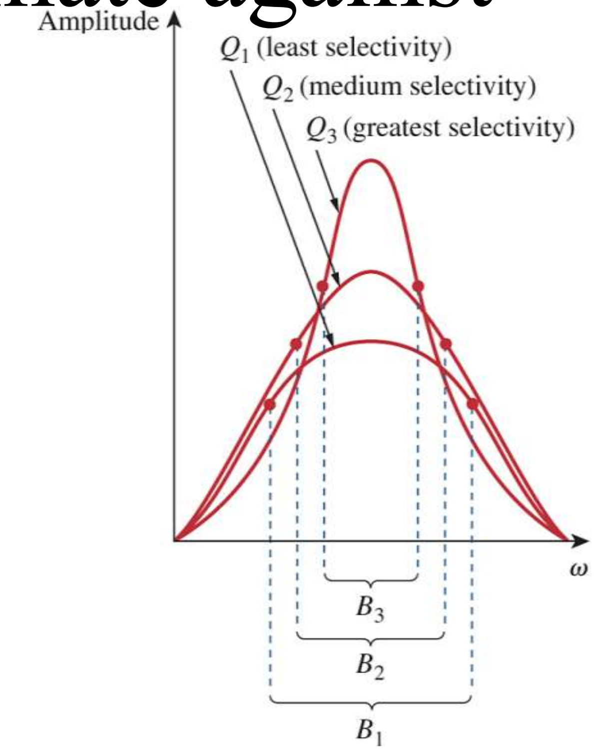


Figure 14.23 The higher the circuit Q, the smaller the bandwidth.

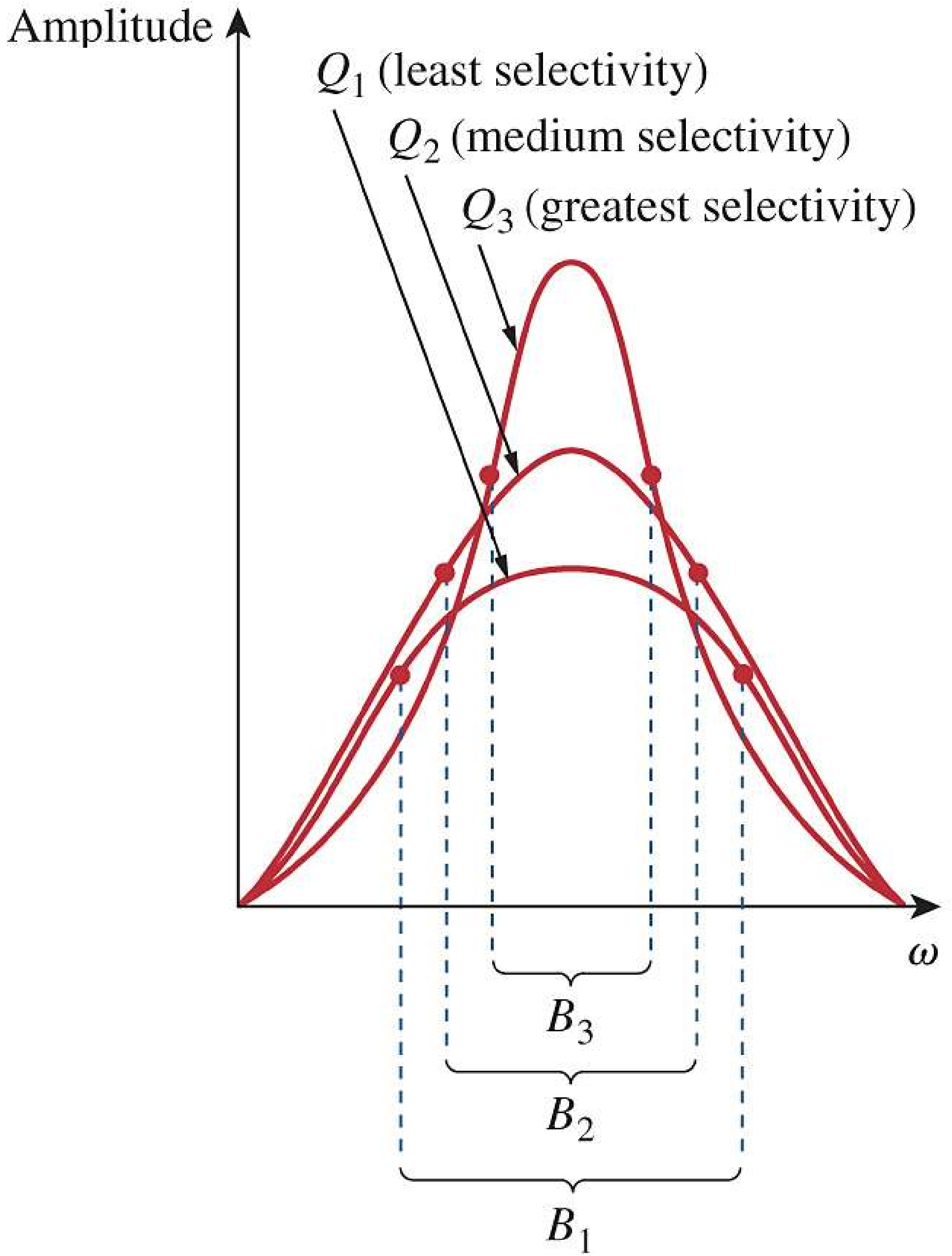


Figure 14.23 The higher the circuit Q, the smaller the bandwidth.

A resonant circuit is designed to operate at or near its resonant frequency. It is said to be a *high - Q circuit* when $Q \geq 10$. For high-Q circuits,

1/(2Q) << 1

$$\omega_{1,2} = \mp\frac{\omega_0}{2Q} + \omega_0\sqrt{\left(\frac{1}{2Q}\right)^2 + 1} \approx \mp\frac{\omega_0}{2Q} + \omega_0$$

$$= \mp\frac{BW}{2} + \omega_0$$

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \qquad Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \qquad BW = \frac{R}{L} = \frac{\omega_0}{Q}$$

The inductor and capacitor voltages can be much more than the source voltage at resonance.

$$V_L = \frac{V_m}{R}\omega_0 L = \frac{V_m}{R}\frac{1}{\omega_0 C} = V_C$$

$|\boldsymbol{V_C}| = |\boldsymbol{IZ_C}| = |\boldsymbol{I}| \times |\boldsymbol{Z_C}|$

For high-Q circuits, $V_L = V_C = QV_m >> V_m$.

$|\boldsymbol{V_L}| = |\boldsymbol{IZ_L}| = |\boldsymbol{I}| \times |\boldsymbol{Z_L}|$

Note that at resonance, the LC series combination acts like a short circuit, but each of L and C is not a short circuit.

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$$

Practice Problem 14.7 A series-connected circuit has $R = 4\ \Omega$ and $L = 25$ mH. (a) Calculate the value of C that will produce a quality factor of 50. (b) Find ω_1, ω_2, and BW. (c) Determine the average power dissipated at $\omega = \omega_0$, ω_1, ω_2. Take $V_m = 100$ V.

Solution :

(a) $\boxed{Q = \dfrac{\omega_0 L}{R}}$

$$\omega_0 = \frac{QR}{L} = \frac{50 \times 4}{25 \times 10^{-3}} = 8000 \text{ (rad/s)}$$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

$$C = \frac{1}{\omega_0^2 L} = \frac{1}{8000^2 \times 25 \times 10^{-3}}$$

$$= 6.25 \times 10^{-7} \text{ (F)} = 0.625\ \mu\text{F}$$

(b)

$$\frac{R}{2L} = \frac{4}{2 \times 25 \times 10^{-3}} = 80 \text{ (rad)}$$

$$\sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$= \sqrt{80^2 + 8000^2}$$

$$\approx 8000.40 \text{ (rad)}$$

$$\boxed{\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}}$$

$$= -80 + 8000.40 = 7920.40 \text{ (rad/s)}$$

$$\boxed{\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}}$$

$$= 80 + 8000.40 = 8080.40 \text{ (rad/s)}$$

$$\boxed{BW = \omega_2 - \omega_1} = 8080.40 - 7920.40$$

$$= 160 \text{ (rad/s)}$$

(c)

$$\boxed{P(\omega_0) = \frac{1}{2}\frac{V_m^2}{R}} = \frac{1}{2} \times \frac{100^2}{4} = 1250 \text{ (W)}$$

$$P(\omega_1) = P(\omega_2) = \frac{1}{2}P(\omega_0)$$

$$= \frac{1}{2} \times 1250$$

$$= 625 \text{ (W)}$$

14.6 Parallel Resonance

The parallel *RLC* circuit in Fig. 14.25 is the dual of the series *RLC* circuit. So we will avoid needless repetition.

$$Y = \frac{1}{R} + j\omega C + \frac{1}{j\omega L} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

Resonance occurs when $\mathrm{Im}(Y) = 0$,

$$\omega_0 C - \frac{1}{\omega_0 L} = 0 \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

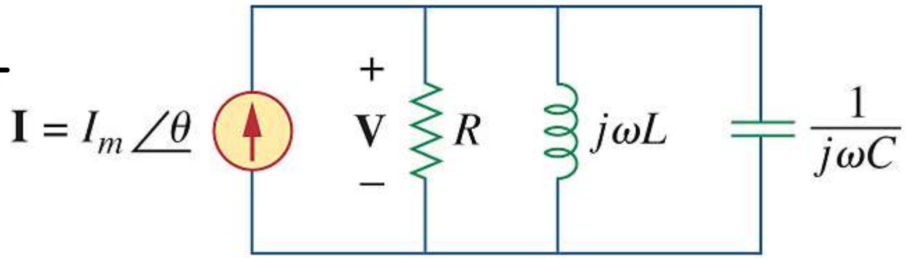


Figure 14.25 The parallel resonant circuit.

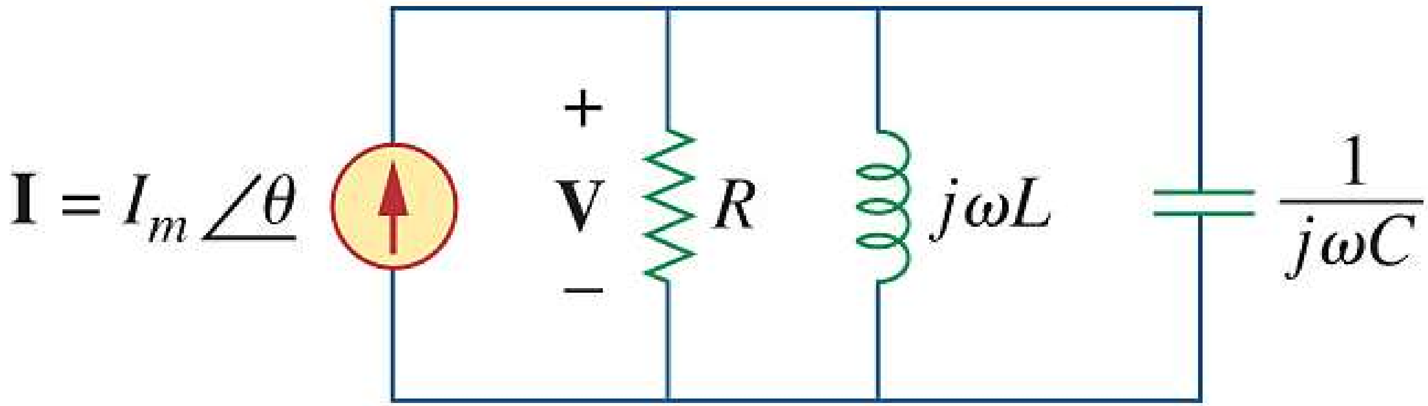


Figure 14.25 The parallel resonant circuit.

The magnitude of voltage $\tilde{V}$ is sketched in Fig. 14.26 as a function of frequency. Notice that at resonance, the parallel LC combination acts like an open circuit, so $\boxed{\mathrm{Im}(Y) = 0}$ that the entire current flows through R. Also, the inductor and capacitor currents can be much more than the source current at resonance.

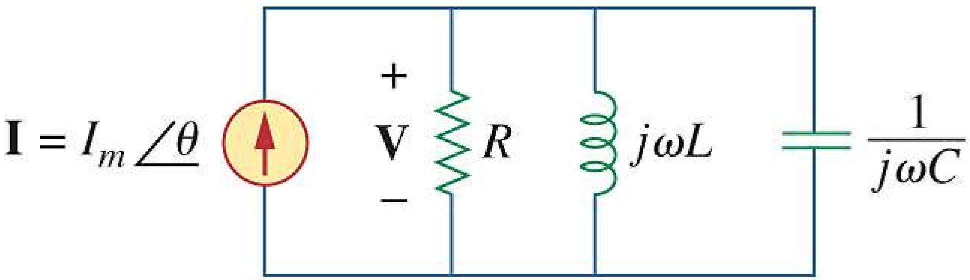


Figure 14.25 The parallel resonant circuit.

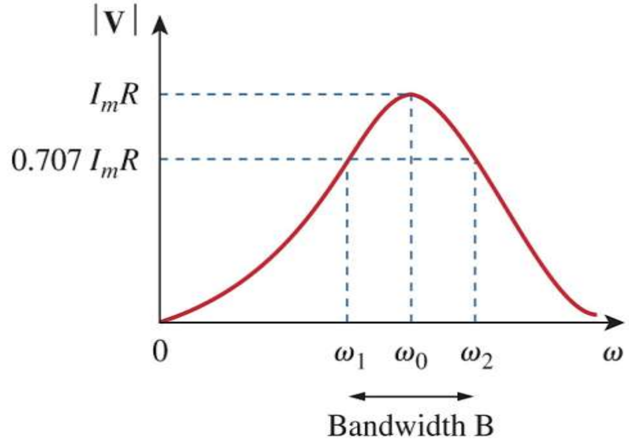


Figure 14.26 The voltage amplitude versus frequency for the parallel resonant circuit of Fig. 14.25.

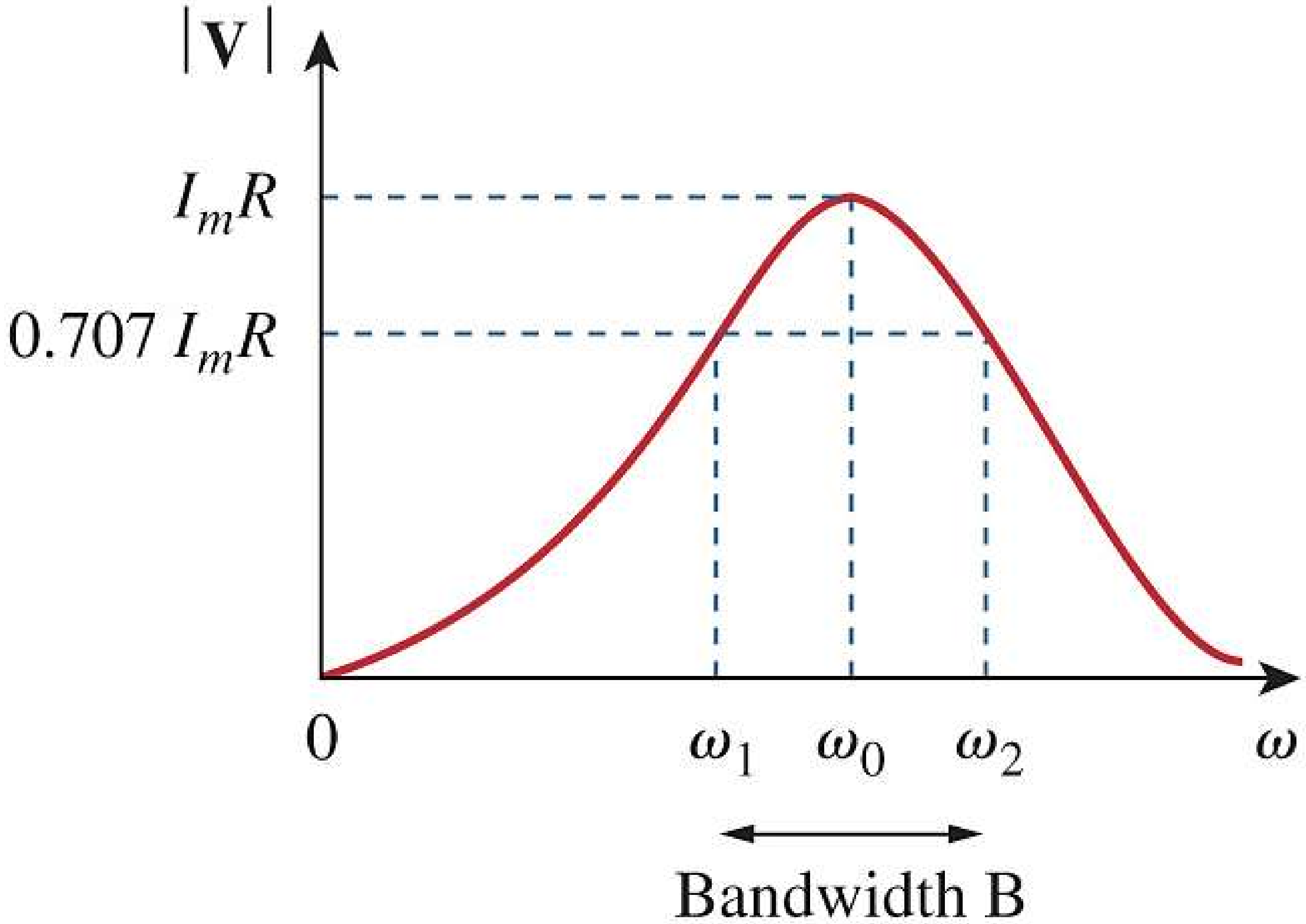


Figure 14.26 The voltage amplitude versus frequency for the parallel resonant circuit of Fig. 14.25.

Recall Chapter 8

TABLE 8.1

Dual pairs.

Resistance R	Conductance G
Inductance L	Capacitance C
Voltage v	Current i
Voltage source	Current source
Node	Mesh
Series path	Parallel path
Open circuit	Short circuit
KVL	KCL
Thevenin	Norton

Series RLC

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$BW = \frac{R}{L} = \frac{\omega_0}{Q}$$

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$$

Parallel RLC

By exploiting duality, we have

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$BW = \omega_2 - \omega_1 = \frac{1}{RC}$$

$$Q = \frac{\omega_0}{BW} = \omega_0 RC = \frac{R}{\omega_0 L}$$

Practice Problem 14.8 A parallel resonant circuit has $R = 100$ kΩ and $L = 20$ mH, and $C = 5$ nF. Calculate ω_0, ω_1, ω_2, Q, and BW.

Solution :

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}} = \frac{1}{\sqrt{20\times10^{-3}\times5\times10^{-9}}}$$

$$= 10^5 \text{ (rad/s)}$$

$$\frac{1}{2RC} = \frac{1}{2\times100\times10^3\times5\times10^{-9}} = 1000 \text{ (rad/s)}$$

$$\sqrt{\left(\frac{1}{2RC}\right)^2+\frac{1}{LC}}=\sqrt{1000^2+(10^5)^2}$$

$$\approx 100{,}005.00 \text{ (rad/s)}$$

$$\boxed{\omega_1=-\frac{1}{2RC}+\sqrt{\left(\frac{1}{2RC}\right)^2+\frac{1}{LC}}}$$

$$=-1000+100{,}005.00=99{,}005.00 \text{ (rad/s)}$$

$$\boxed{\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}}$$

$$= 1000 + 100{,}005.00 = 101{,}005.00 \text{ (rad/s)}$$

$$\boxed{BW = \omega_2 - \omega_1} = 101{,}005.00 - 99{,}005.00$$

$$= 2000 \text{ (rad/s)}$$

$$\boxed{Q = \frac{\omega_0}{B}} = \frac{10^5}{2000} = 50$$

Practice Problem 14.9 Calculate the resonant frequency of the circuit in Fig. 14.29. $\boxed{\text{A more general case}}$

Solution :

$$Z = j\omega L + R \,||\, \frac{1}{j\omega C} = j\omega L + \frac{R}{1 + j\omega RC}$$

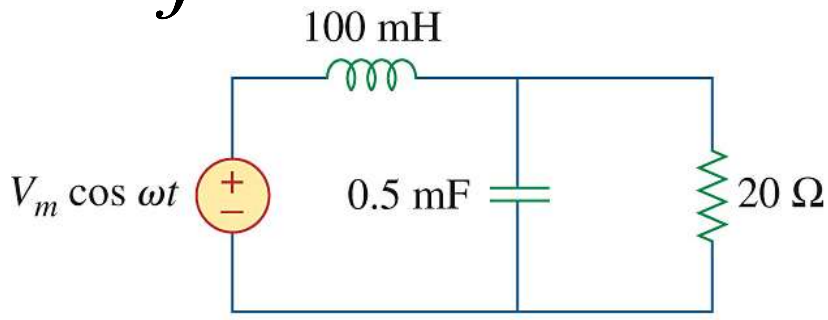


Figure 14.29

$$= j\omega L + \frac{R - j\omega R^2 C}{1 + (\omega RC)^2}$$

$$= \frac{R}{1 + (\omega RC)^2} + j\left(\omega L - \frac{\omega R^2 C}{1 + (\omega RC)^2}\right)$$

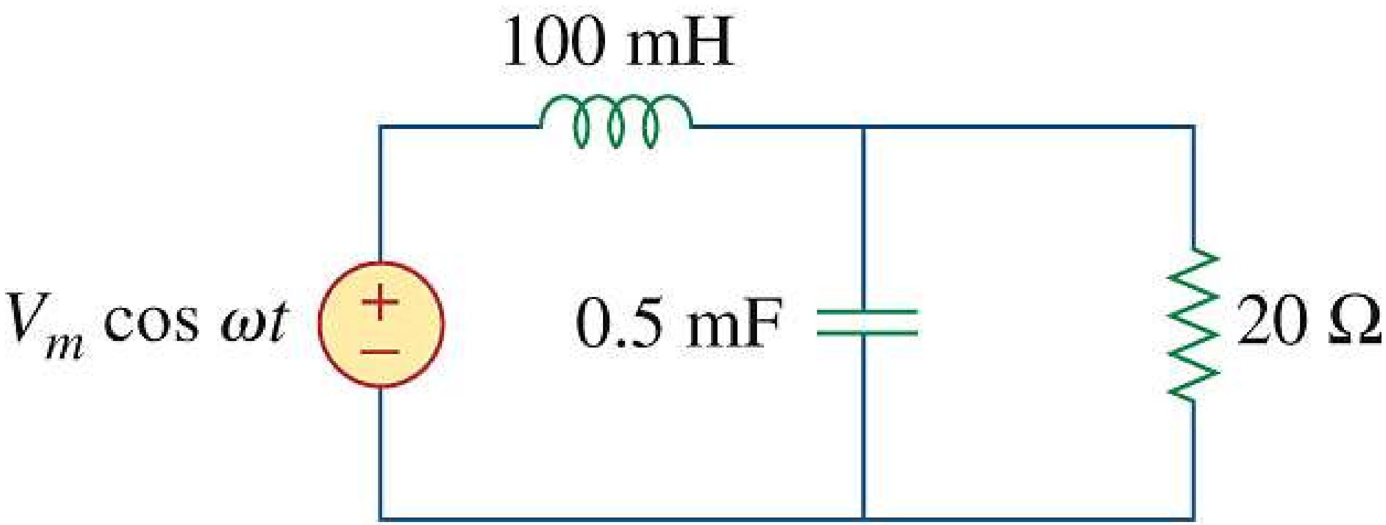


Figure 14.29

When $\omega = \omega_0, \quad \mathrm{Im}(Z) = 0,$

$$\omega_0 L - \frac{\omega_0 R^2 C}{1 + (\omega_0 RC)^2} = 0$$

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{1}{(RC)^2}}$$

$$= \sqrt{\frac{1}{100 \times 10^{-3} \times 0.5 \times 10^{-3}} - \frac{1}{(20 \times 0.5 \times 10^{-3})^2}}$$

$$= 100 \text{ (rad/s)}$$

The same to use $\mathrm{Im}(Y) = 0$ and $\mathrm{Im}(Z) = 0$

- Proof:
  $Z = R+jX$ ➔ $\mathrm{Im}(Z) = 0$ results in $X = 0$

  $Y = 1/Z = (R-jX)/(R^2+X^2)$ ➔ $\mathrm{Im}(Y) = 0$ results in $X = 0$