Parametric Equations and Polar Coordinates

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Part I: Review

1. The Parametric Perspective

In elementary mathematics, we describe curves using **explicit functions** $y = f(x)$ or **implicit relations** $F(x, y) = 0$. But many natural curves resist these representations:

- The path of a projectile (both x and y depend on time)
- The motion of a planet (position changes continuously)
- A curve that loops back on itself (failing the vertical line test)

Parametric equations provide a more flexible framework. Instead of relating x and y directly, we express both as functions of a third variable t (the **parameter**):

$$x = f(t), \quad y = g(t)$$

Geometric interpretation: As t varies over an interval, the point $(x(t), y(t))$ traces out a curve in the plane. The parameter t often represents time, but it can be any convenient variable.

Example: The unit circle $x^2 + y^2 = 1$ can be parametrized as:
$x(t) = \cos t, \quad y(t) = \sin t, \quad t \in [0, 2\pi)$

As t increases from 0 to 2π, the point $(\cos t, \sin t)$ traces the circle counterclockwise, starting from $(1, 0)$.

Advantages of parametric form:

1. Can represent multi-valued functions (like circles)
2. Naturally encodes direction of motion
3. Simplifies computation of arc length and curvature
4. Separates geometric shape from its traversal speed

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2. Calculus with Parametric Curves

For a parametric curve $x = f(t), y = g(t)$, how do we find $\frac{dy}{dx}$?

Chain Rule approach: Both x and y are functions of t: $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

Solving for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}}$$

Requirement: $f'(t) \neq 0$ (otherwise the curve has a vertical tangent or cusp).

Example: For the unit circle $x = \cos t, y = \sin t$: $\frac{dy}{dx} = \frac{\sin'(t)}{\cos'(t)} = \frac{\cos t}{-\sin t} = -\cot t$

At $t = \pi/4$: $\frac{dy}{dx} = -\cot(\pi/4) = -1$, confirming the tangent line has slope -1 at $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

To find concavity, we need $\frac{d^2y}{dx^2}$. This requires careful application of the chain rule:

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$

Since $\frac{dt}{dx} = \frac{1}{dx/dt}$:

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}}$$

Common mistake: $\frac{d^2y}{dx^2} \neq \frac{d^2y/dt^2}{d^2x/dt^2}$

Example: For $x = t^2, y = t^3$:

- First derivative: $\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}$

- Second derivative: $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{3t}{2}\right)}{2t} = \frac{3/2}{2t} = \frac{3}{4t}$

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3. Arc Length

The arc length of a parametric curve from $t = a$ to $t = b$ is:

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Derivation: For a small increment Δt, the displacement is:
$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\,\Delta t$

Taking the limit as $\Delta t \to 0$ and integrating gives the formula.

Alternative notation: $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_a^b |r'(t)|\, dt$

where $r(t) = (x(t), y(t))$ is the position vector.

Example: For the unit circle $x = \cos t, y = \sin t, t \in [0, 2\pi]$:
$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \int_0^{2\pi} 1\, dt = 2\pi$

This confirms the circumference of the unit circle!

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4. Surface Area of Revolution

When a parametric curve is rotated about an axis, the surface area is:

About the x-axis: $\boxed{S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$

About the y-axis: $\boxed{S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$

Intuition: Each infinitesimal arc element ds sweeps out a thin band of circumference $2\pi r$ (where r is the distance from the axis).

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5. The Polar System

In **Cartesian coordinates**, we specify a point by its distances from two perpendicular axes: (x, y).

In **polar coordinates**, we specify a point by:

- r = distance from the origin (the **pole**)
- θ = angle from the positive x-axis (measured counterclockwise)

Conversion formulas: $\boxed{x = r\cos\theta, \quad y = r\sin\theta}$ $\boxed{r = \sqrt{x^2+y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)}$

Important notes:

1. Unlike Cartesian coordinates, polar coordinates are **not unique**: $(r, \theta) = (r, \theta + 2\pi k)$ for any integer k
2. Negative r values are allowed: $(r, \theta) = (-r, \theta + \pi)$
3. The pole (origin) has $r = 0$ and undefined θ

Example: The point $(1, 1)$ in Cartesian coordinates has polar coordinates: $r = \sqrt{1^2+1^2} = \sqrt{2}, \quad \theta = \arctan(1/1) = \pi/4$ So $(1, 1) = (\sqrt{2}, \pi/4)$ in polar form.

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6. Polar Curves

A **polar curve** is given by an equation $r = f(\theta)$. As θ varies, the distance r from the origin changes, tracing a curve.

Example: The circle $r = 2\cos\theta$

- At $\theta = 0$: $r = 2$, giving point $(2, 0)$
- At $\theta = \pi/2$: $r = 0$, giving the origin
- At $\theta = \pi$: $r = -2$, which is the same as $(2, 0)$

This traces a circle of radius 1 centered at $(1, 0)$.

To find $\frac{dy}{dx}$ for a polar curve $r = f(\theta)$:

Use the conversion $x = r\cos\theta$, $y = r\sin\theta$:
$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta, \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$

Therefore: $\boxed{\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r' \sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}}$

Special cases:

- **Horizontal tangent:** $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$
- **Vertical tangent:** $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$

Note: If both $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} = 0$ simultaneously, the curve may have a cusp or more complex behavior requiring further analysis using higher derivatives.

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7. Area in Polar Coordinates

The area enclosed by a polar curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is:

$$\boxed{A = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta}$$

Derivation: Divide the region into thin sectors. Each sector has approximate area $\frac{1}{2}r^2\Delta\theta$ (like a triangle with base $r\Delta\theta$ and height r).

Example: Area of the cardioid $r = 1 + \cos\theta$:

By symmetry, we can integrate from 0 to π and double:
$A = 2 \cdot \frac{1}{2}\int_0^\pi (1+\cos\theta)^2\, d\theta = \int_0^\pi (1 + 2\cos\theta + \cos^2\theta)\, d\theta$

Using $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$:
$A = \int_0^\pi \left(\frac{3}{2} + 2\cos\theta + \frac{\cos(2\theta)}{2}\right) d\theta = \left[\frac{3\theta}{2} + 2\sin\theta + \frac{\sin(2\theta)}{4}\right]_0^\pi = \frac{3\pi}{2}$

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8. Arc Length in Polar Coordinates

For a polar curve $r = f(\theta)$, the arc length from $\theta = \alpha$ to $\theta = \beta$ is:

$$\boxed{L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

Derivation: Using $x = r\cos\theta$, $y = r\sin\theta$: $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$

Example: Arc length of the spiral $r = \theta$ from $\theta = 0$ to $\theta = 2\pi$: $L = \int_0^{2\pi} \sqrt{\theta^2 + 1}\, d\theta$

This integral requires trigonometric substitution (left as exercise).

Part II: Advanced Topics in Plane Curves

1. Introduction

- General asymptote theories
- Conics sections revisited in polar
- Envelopes as boundaries of curve families
- Orthogonal trajectories

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2. Asymptotes

Asymptotes capture a curve's destiny at infinity. This calculus-driven approach demystifies limiting behaviors, essential for sketching and asymptotic analysis.

2.1 Cartesian Asymptotes

For $y = f(x)$, asymptotes emerge from limits.

1. **Vertical Asymptotes:** $\lim_{x\to a} |f(x)| = \infty$, finite a.
2. **Horizontal Asymptotes:** $\lim_{x\to\pm\infty} f(x) = L$, finite L.
3. **Slant (Oblique) Asymptotes:** $y = mx + b$ where $\lim_{x\to\pm\infty}[f(x) - (mx + b)] = 0$.
   - $m = \lim_{x\to\pm\infty} f(x)/x$.
   - $b = \lim_{x\to\pm\infty}[f(x) - mx]$.

Insight: Oblique asymptotes blend linear growth with subtler deviations, showing how a curve approaches a tilted line eternally.

Example: $f(x) = x + \frac{1}{x}$. $m = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty}\left(1 + \frac{1}{x^2}\right) = 1$. Then $b = \lim_{x\to\infty}\left(f(x) - x\right) = \lim_{x\to\infty} \frac{1}{x} = 0$, so the oblique asymptote is $y = x$.

2.2 Parametric Asymptotes

In $x = x(t), y = y(t)$, asymptotes arise as $t \to t_0$ (often $\pm\infty$).

1. **Vertical:** $\frac{dx}{dt} = 0, x(t) \to a$ finite, $|y(t)| \to \infty$.
2. **Horizontal:** $|x(t)| \to \infty, y(t) \to L$ finite.
3. **Oblique:** When both $x(t)$ and $y(t)$ approach infinity as $t \to t_0$, the asymptote has slope $m = \lim_{t\to t_0} \frac{y(t)}{x(t)}$ (finite) and intercept $b = \lim_{t\to t_0}[y(t) - mx(t)]$.

Comment: Parametrics shine in physics, like trajectories, where t is time—asymptotes reveal long-term trends.

Example: Hyperbola branch: $x = \cosh t, y = \sinh t$.

As $t \to \infty, m = \lim \frac{\cosh t}{\sinh t} = 1, b = \lim(\sinh t - \cosh t) = \lim -e^{-t} = 0$. Asymptote $y = x$.

As $t \to -\infty, m = -1, b = 0, y = -x$.

This mirrors the hyperbola's classic asymptotes, a beautiful unification.

Problem: Find oblique asymptotes for $x = t + 2/t, y = 3t + 4/t, t > 0$.

Solution: As $t \to \infty, m = \lim \frac{3-4/t^2}{1-2/t^2} = 3/1 = 3$.

$b = \lim[3t + 4/t - 3(t + 2/t)] = \lim(3t + 4/t - 3t - 6/t) = \lim(-2/t) = 0$.

Asymptote $y = 3x$. (Cartesian elimination yields $(3x - y)^2 = 20(y - 2x) + something$, but asymptote confirms linear trend.)

Insight: Such forms model rational functions parametrically, easing differentiation.

2.3 Polar Asymptotes

For $r = r(\theta)$, asymptotes occur when $r \to \infty$ at some finite $\theta = \theta_0$. The asymptote line can be found by converting to Cartesian coordinates and evaluating $\lim_{\theta\to\theta_0} y(\theta)$ or $\lim_{\theta\to\theta_0} x(\theta)$, or by finding the limiting line that the curve approaches.

Note: θ_0 aligns with the ray of approach, but for general lines, adjust to normal's angle if needed.

Insight: Polar asymptotes illuminate spirals and conics, where infinity dances with angles.

Example: Hyperbolic Spiral $r = a/\theta$.

As $\theta \to 0^+$, $r \to \infty$, $\theta_0 = 0$.

To find a horizontal asymptote, we examine $y = r \sin\theta \approx (a/\theta) \cdot \theta = a$ as $\theta \to 0$. This suggests the line $y = a$. The perpendicular distance from the pole to this line is $p = a$.

Thus, line $r \sin\theta = a$ ($y = a$), with effective $\theta_0 = \pi/2$, $\cos(\theta - \pi/2) = \sin\theta$, $p = a$.

This spiral coils infinitely, approaching a horizontal ledge—a poetic boundary.

Problem: For $r = \sec\theta + \csc\theta$, find asymptotes.

Solution: As $\theta \to 0^+$, $r \to \infty$, compute lim $r \sin\theta = \lim(\sec\theta + \csc\theta) \sin\theta = \tan\theta + 1 \to 1$.

Horizontal $y = 1$.

As $\theta \to \pi/2^-$, lim $r \cos\theta = 1 + \cot\theta \to 1$.

Vertical $x = 1$.

Cartesian: $y = x/(x - 1)$, confirms.

Insight: Polar reveals dual infinities, mirroring branch behaviors.

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3. Polar Conic Sections

Conics in polar form, anchored at a focus, bridge geometry to celestial mechanics.

3.1 Review: The Focus-Directrix Property

Conic: locus where $|PF|/|PL| = e$, F focus, L directrix.

$e < 1$ ellipse, $e = 1$ parabola, $e > 1$ hyperbola.

Comment: Eccentricity quantifies "deviance" from circularity—a metric of orbital wildness.

3.2 Derivation of the Polar Equation

Focus at pole, directrix $x = d > 0$: $r = e|d - r\cos\theta|$.

For different conic sections, we get different signs. For ellipses and parabolas, we typically use the positive case: $r = e(d - r\cos\theta)$.

Solving: $r = \frac{ed}{1+e\cos\theta}$.

Variants for directrix orientation.

Insight: This compact form encodes all conics, a unified gem.

Example: Parabola $e = 1$, $r = \frac{d}{1+\cos\theta}$.

At $\theta = 0$, $r = d/2$, vertex.

3.3 Application: Kepler's First Law

Planets ellipse sun at focus.

Gravity yields $r = \frac{h^2/(GM)}{1+e\cos\theta}$, h angular momentum.

Geometric $ed = l = h^2/(GM)$, linking mass to shape.

Comment: Kepler's empiricism meets Newton's universality—conics as cosmic scripts.

Problem: Derive polar for hyperbola, $e = 2$, directrix $x = 1$.

Solution: $r = \frac{2\cdot 1}{1+2\cos\theta} = \frac{2}{1+2\cos\theta}$.

As $\theta \to \pm\cos^{-1}(-1/2) = \pm 2\pi/3$, denominator zero, r->infty, asymptote angles.

Insight: Hyperbolas model comet flybys, e>1 unbound.

Problem: For elliptical orbit $e = 0.5$, semi-latus $l = 2$, find perihelion distance.

Solution: $r = \frac{2}{1+0.5\cos\theta}$, min at $\theta = 0$, $r = 2/1.5 = 4/3$.

Standard $r_{\min} = l/(1+e) = 2/1.5 = 4/3$.

Comment: Perihelion whispers closest solar embrace.

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4. Envelopes of Families of Curves

Envelope tangents all, like the boundary of the shadow.

The standard method is:

1. $F(x, y, c) = 0$
2. $\frac{\partial F}{\partial c} = 0$
3. Eliminate c from these two equations

Example: Lines $y = cx + c^2$.

$F = y - cx - c^2 = 0$, $\partial F/\partial c = -x - 2c = 0$, $c = -x/2$.

Substitute: $y = (-x/2)x + (-x/2)^2 = -x^2/2 + x^2/4 = -x^2/4$.

Envelope $y = -x^2/4$, downward parabola.

Comment: This family envelopes a parabola.

Example: Tangents to $y = x^2$: $y = 2cx - c^2$, $F = y - 2cx + c^2 = 0$.

$\partial F = -2x + 2c = 0$, $c = x$, $y = 2x^2 - x^2 = x^2$. Recovers itself!

Insight: Self-envelope underscores tangents define the curve.

Problem: Family $y = cx + 1/c$.

Solution: $F = y - cx - 1/c = 0$.

$\partial F = -x + 1/c^2 = 0, c^2 = 1/x, c = \pm 1/\sqrt{x}(x > 0)$.

For +: $y = (1/\sqrt{x})x + \sqrt{x} = \sqrt{x} + \sqrt{x} = 2\sqrt{x}$.

For -: $y = -2\sqrt{x}$.

Envelopes $y = \pm 2\sqrt{x}$, branches of $y^2 = 4x$.

Comment: Lines envelope parabola, classic in optics.

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5. Orthogonal Trajectories

Orthogonal families intersect at 90°, modeling perpendicular fields. Trajectory crosses family orthogonally.

Insight: In electromagnetism, E-fields perpendicular to equipotentials—nature's right angles.

From family $f(x, y, c) = 0$, derive $dy/dx = g(x, y)$.

Orthogonal: $dy/dx = -1/g$.

Example: Concentric circles $x^2 + y^2 = c^2$.

$2x + 2yy' = 0, y' = -x/y$.

Orthogonal $y' = y/x$, $\frac{dy}{y} = \frac{dx}{x}$, $\ln y = \ln x + k$, $y = kx$. Radials.

Comment: Circles and rays: symmetry's perpendicular poetry.

Example: Hyperbolas $xy = c$.

$xy' + y = 0, y' = -y/x$.

Orthogonal $y' = x/y$, $ydy = xdx$, $y^2/2 = x^2/2 + k$, $x^2 - y^2 = K$. Rotated hyperbolas.

Problem: Orthogonal to parabolas $y = cx^2$.

Solution: $y' = 2cx$, $c = y/x^2$, but implicit: diff $y' = 2(y/x^2)x = 2y/x$.

Orthogonal $y' = -x/(2y)$.

$2ydy = -xdx$, $y^2 = -x^2/2 + k$, $x^2 + 2y^2 = K$. Ellipses.

Insight: Parabolas to ellipses—open to closed, infinite to bounded.

Part III: Ten Famous Curves

Parametric and polar coordinates reveal hidden symmetries and unexpected connections. The cycloid, discovered by studying the motion of wheels, turns out to solve the brachistochrone problem, showing that nature chooses the same elegant curves that mathematicians find beautiful. The lemniscate, first studied as a geometric curiosity, later became central to the theory of elliptic functions and modern cryptography.

We now explore ten remarkable curves from mathematical history. These curves reveal deep connections between geometry, analysis, and physics.

Curve 1: The Astroid—A Study in Envelopes and Hypocycloids

Parametric Equations:
$x(t) = a\cos^3 t, \quad y(t) = a\sin^3 t$

Cartesian Equation:
$x^{2/3} + y^{2/3} = a^{2/3}$

Construction Method 1: Hypocycloid Formation

The astroid can be formed by rolling a circle of radius $a/4$ on the inside of a circle of radius a. To see why, consider the general hypocycloid parametrization when a circle of radius b rolls inside a circle of radius a:

$$x = (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right), \quad y = (a-b)\sin t - b\sin\left(\frac{a-b}{b}t\right)$$

Setting $b = a/4$:
$x = \frac{3a}{4}\cos t + \frac{a}{4}\cos(3t), \quad y = \frac{3a}{4}\sin t - \frac{a}{4}\sin(3t)$

Using the triple angle formulas:
$\cos(3t) = 4\cos^3 t - 3\cos t, \quad \sin(3t) = 3\sin t - 4\sin^3 t$

Substituting:
$x = \frac{3a}{4} \cos t + \frac{a}{4} (4 \cos^3 t - 3 \cos t) = a \cos^3 t$
$y = \frac{3a}{4} \sin t - \frac{a}{4} (3 \sin t - 4 \sin^3 t) = a \sin^3 t$

This confirms the astroid parametrization!

Construction Method 2: Envelope of a Line Segment (Glissette)

Consider a line segment of length a moving with each endpoint constrained to one of the perpendicular coordinate axes. If the segment touches the x-axis at $(p, 0)$ and the y-axis at $(0, q)$ where $p^2 + q^2 = a^2$, we can parametrize: $p = a \cos t$, $q = a \sin t$.

The equation of this line is:
$\frac{x}{a \cos t} + \frac{y}{a \sin t} = 1$

Or: $x \sin t + y \cos t = a \sin t \cos t$

The **envelope** of this family of lines is found by eliminating t from:

1. $F(x, y, t) = x \sin t + y \cos t - a \sin t \cos t = 0$
2. $\frac{\partial F}{\partial t} = x \cos t - y \sin t - a(\cos^2 t - \sin^2 t) = 0$

From equation (2): $x \cos t - y \sin t = a \cos{(2t)}$

Squaring both equations and using $\sin^2 t + \cos^2 t = 1$:
$(x \sin t + y \cos t)^2 = a^2 \sin^2 t \cos^2 t$
$(x \cos t - y \sin t)^2 = a^2 \cos^2(2t)$

Adding: $x^2 + y^2 = a^2 \sin^2 t \cos^2 t + a^2 \cos^2(2t)$

After algebraic manipulation (using $\cos{(2t)} = \cos^2 t - \sin^2 t$), this yields the astroid equation $x^{2/3} + y^{2/3} = a^{2/3}$.

Deriving the Cartesian Equation from Parametric Form

From $x = a \cos^3 t$ and $y = a \sin^3 t$:
$\left(\frac{x}{a}\right)^{1/3} = \cos t, \quad \left(\frac{y}{a}\right)^{1/3} = \sin t$

Using $\cos^2 t + \sin^2 t = 1$:
$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1$

Therefore: $\boxed{x^{2/3} + y^{2/3} = a^{2/3}}$

Calculus Problem 1: Finding the Arc Length

Problem: Find the total length of the astroid.

Solution:

By the eight-fold symmetry of the astroid (symmetric with respect to both axes and both diagonals), we can compute the arc length in the first quadrant from $t = 0$ to $t = \pi/2$ and multiply by 4.

The arc length formula for parametric curves is:
$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

First, compute the derivatives:
$\frac{dx}{dt} = \frac{d}{dt}(a\cos^3 t) = a \cdot 3\cos^2 t \cdot (-\sin t) = -3a\cos^2 t \sin t$

$\frac{dy}{dt} = \frac{d}{dt}(a\sin^3 t) = a \cdot 3\sin^2 t \cdot \cos t = 3a\sin^2 t \cos t$

Now compute the sum of squares:
$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9a^2\cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t$

Factor out common terms:
$= 9a^2\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t) = 9a^2 \cos^2 t \sin^2 t$

Therefore:
$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3a|\cos t \sin t|$

For $t \in [0, \pi/2]$, both $\cos t$ and $\sin t$ are non-negative, so:
$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3a\cos t \sin t$

The arc length in the first quadrant is:
$L_1 = \int_0^{\pi/2} 3a\cos t \sin t\, dt$

Using the substitution $u = \sin t$, $du = \cos t\, dt$:
$L_1 = 3a\int_0^1 u\, du = 3a\left[\frac{u^2}{2}\right]_0^1 = \frac{3a}{2}$

Alternatively, using $\sin t \cos t = \frac{1}{2}\sin(2t)$:
$L_1 = \frac{3a}{2}\int_0^{\pi/2} \sin(2t)\,dt = \frac{3a}{2}\left[-\frac{\cos(2t)}{2}\right]_0^{\pi/2} = \frac{3a}{4}[-\cos\pi + \cos 0] = \frac{3a}{4}[1+1] = \frac{3a}{2}$

The total length is:

$$\boxed{L = 4L_1 = 4 \cdot \frac{3a}{2} = 6a}$$

Remarkable Result: The perimeter of the astroid is exactly **6 times the radius** of the generating circle!

Calculus Problem 2: Finding the Area Enclosed

Problem: Find the area enclosed by the astroid.

Solution:

By symmetry, we can compute the area in the first quadrant and multiply by 4. In the first quadrant, we integrate y with respect to x from $x = 0$ to $x = a$.

From the Cartesian equation: $y = a(1 - (x/a)^{2/3})^{3/2}$

This integral is complex. Instead, use parametric form with $t \in [0, \pi/2]$:

$A_1 = \int_0^a y\,dx$

Converting to parameter t (note: as t goes from $\pi/2$ to 0, x goes from 0 to a):
$A_1 = -\int_{\pi/2}^0 a\sin^3 t \cdot (-3a\cos^2 t \sin t)\,dt = 3a^2\int_0^{\pi/2} \sin^4 t \cos^2 t\,dt$

Using the reduction formula or $\sin^4 t = \left(\frac{1-\cos(2t)}{2}\right)^2$ and $\cos^2 t = \frac{1+\cos(2t)}{2}$:

After calculation: $A_1 = \frac{3\pi a^2}{32}$

Total area: $A = 4A_1 = \boxed{\frac{3\pi a^2}{8}}$

Calculus Problem 3: Tangent Line at Parameter $t = \theta_0$

Problem: Find the equation of the tangent line to the astroid at $t = \theta_0$.

Solution:

At $t = \theta_0$, the point on the astroid is:
$(x_0, y_0) = (a\cos^3\theta_0, a\sin^3\theta_0)$

The slope of the tangent line is:
$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a\sin^2\theta_0\cos\theta_0}{-3a\cos^2\theta_0\sin\theta_0} = -\frac{\sin\theta_0}{\cos\theta_0} = -\tan\theta_0$

The tangent line equation is:
$y - a\sin^3\theta_0 = -\tan\theta_0(x - a\cos^3\theta_0)$

Simplifying:
$y - a\sin^3\theta_0 = -\frac{\sin\theta_0}{\cos\theta_0}(x - a\cos^3\theta_0)$

Multiply through by $\cos\theta_0$:
$y\cos\theta_0 - a\sin^3\theta_0\cos\theta_0 = -x\sin\theta_0 + a\cos^3\theta_0\sin\theta_0$

$x\sin\theta_0 + y\cos\theta_0 = a\sin\theta_0\cos^3\theta_0 + a\cos\theta_0\sin^3\theta_0$

$= a\sin\theta_0\cos\theta_0(\cos^2\theta_0 + \sin^2\theta_0) = a\sin\theta_0\cos\theta_0$

Therefore, the tangent line can be written as:
$\boxed{\frac{x\sin\theta_0}{a^{1/3}} + \frac{y\cos\theta_0}{a^{1/3}} = a^{2/3}\sin\theta_0\cos\theta_0}$

Or more elegantly:
$\boxed{x^{1/3}\sin\theta_0 + y^{1/3}\cos\theta_0 = a^{2/3}}$

This form reveals that the tangent line touches the astroid at the point where the line segment from the origin makes angle θ_0 with the x-axis.

Calculus Problem 4: Distance from X-intercept to Y-intercept

Problem: A tangent line to the astroid intersects the x-axis at point X and the y-axis at point Y. Find the distance from X to Y.

Solution:

From the tangent line equation $x \sin\theta_0 + y \cos\theta_0 = a \sin\theta_0 \cos\theta_0$:

Finding X-intercept (set $y = 0$):
$x \sin\theta_0 = a \sin\theta_0 \cos\theta_0 \implies x = a\cos\theta_0$

So $X = (a\cos\theta_0, 0)$.

Finding Y-intercept (set $x = 0$):
$y \cos\theta_0 = a \sin\theta_0 \cos\theta_0 \implies y = a\sin\theta_0$

So $Y = (0, a\sin\theta_0)$.

Distance from X to Y:
$|XY| = \sqrt{(a\cos\theta_0)^2 + (a\sin\theta_0)^2} = a\sqrt{\cos^2\theta_0 + \sin^2\theta_0} = \boxed{a}$

Remarkable Property: The tangent line segment between the coordinate axes has **constant length** a regardless of where it touches the astroid! This is the converse of the envelope construction—the astroid is the envelope of all line segments of length a with endpoints on the axes.

This property makes the astroid the solution to the **brachistochrone problem in a uniform gravitational field with friction**.

Curve 2: The Cardioid—Heart of Mathematics

Parametric Equations:
$x(t) = a(2\cos t - \cos 2t), \quad y(t) = a(2\sin t - \sin 2t)$

Polar Equation:
$r = 2a(1 + \cos\theta)$

Cartesian Equation:
$(x^2 + y^2 - 2ax)^2 = 4a^2(x^2 + y^2)$

Geometric Construction and Properties

Construction Method 1: Epicycloid Formation

The cardioid is a special case of an epicycloid where the rolling circle has the same radius as the fixed circle. When a circle of radius a rolls externally on another circle of radius a, the path traced by a point on the rolling circle's circumference forms a cardioid.

From the general epicycloid equations with $a = b$:
$x = 2a\cos t - a\cos 2t, \quad y = 2a\sin t - a\sin 2t$

Using trigonometric identities:
$\cos 2t = 2\cos^2 t - 1, \quad \sin 2t = 2\sin t\cos t$

We get the standard parametric form:
$x = a(2\cos t - 2\cos^2 t + 1) = a(1 + 2\cos t - 2\cos^2 t)$
$y = a(2\sin t - 2\sin t\cos t) = 2a\sin t(1 - \cos t)$

Construction Method 2: Envelope of Circles

The cardioid is also the envelope of all circles passing through a fixed point on a given circle and having their centers on that circle.

Deriving the Polar Equation

Consider a circle of radius a with center at $(a, 0)$. In polar coordinates (r, θ):

$(x^2 + y^2 - 2ax)^2 = 4a^2(x^2 + y^2)$

Substitute $x = r\cos\theta, y = r\sin\theta, x^2 + y^2 = r^2$:

$(r^2 - 2ar\cos\theta)^2 = 4a^2r^2$

Taking square roots:
$r^2 - 2ar\cos\theta = \pm 2ar$

The positive case: $r^2 - 2ar\cos\theta = 2ar \Rightarrow r^2 - 2ar(\cos\theta + 1) = 0$

The negative case: $r^2 - 2ar\cos\theta = -2ar \Rightarrow r^2 - 2ar(\cos\theta - 1) = 0$

These combine to give the standard polar form:
$\boxed{r = 2a(1 + \cos\theta)}$

Calculus Problem 1: Arc Length of the Cardioid

Problem: Find the total length of the cardioid.

Solution:

Using the polar arc length formula:
$L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

For $r = 2a(1 + \cos\theta)$:
$\frac{dr}{d\theta} = -2a\sin\theta$

Compute the integrand:
$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 4a^2(1 + \cos\theta)^2 + 4a^2\sin^2\theta$
$= 4a^2(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta)$
$= 4a^2(2 + 2\cos\theta) = 8a^2(1 + \cos\theta)$

Using the identity $1 + \cos\theta = 2\cos^2(\theta/2)$:
$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{8a^2 \cdot 2\cos^2(\theta/2)} = 4a|\cos(\theta/2)|$

Due to symmetry, integrate from 0 to 2π:
$L = \int_0^{2\pi} 4a|\cos(\theta/2)|\, d\theta$

Since $\cos(\theta/2)$ changes sign at π, split the integral:
$L = 4a\left[\int_0^{\pi} \cos(\theta/2)\, d\theta - \int_{\pi}^{2\pi} \cos(\theta/2)\, d\theta\right]$

Compute each integral:
$\int \cos(\theta/2)\, d\theta = 2\sin(\theta/2)$

Therefore:
$L = 4a\left[(2\sin(\theta/2)\big|_0^{\pi}) - (2\sin(\theta/2)\big|_{\pi}^{2\pi})\right]$
$= 4a\left[(2\sin(\pi/2) - 2\sin 0) - (2\sin\pi - 2\sin(\pi/2))\right]$
$= 4a\left[(2 - 0) - (0 - 2)\right] = 4a(2 + 2) = 16a$

$\boxed{L = 16a}$

Remarkable Result: The perimeter of the cardioid is exactly **16 times the radius** of the generating circle!

Calculus Problem 2: Area Enclosed by the Cardioid

Problem: Find the area enclosed by the cardioid.

Solution:

Using the polar area formula:
$A = \frac{1}{2} \int_0^{2\pi} r^2 \, d\theta$

Substitute $r = 2a(1 + \cos\theta)$:
$A = \frac{1}{2} \int_0^{2\pi} 4a^2(1 + \cos\theta)^2 \, d\theta$
$= 2a^2 \int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta) \, d\theta$

Using $\cos^2\theta = \frac{1+\cos 2\theta}{2}$:
$A = 2a^2 \int_0^{2\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$
$= 2a^2 \int_0^{2\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$

Now integrate term by term:
$\int_0^{2\pi} \frac{3}{2} \, d\theta = 3\pi$
$\int_0^{2\pi} 2\cos\theta \, d\theta = 0$
$\int_0^{2\pi} \frac{1}{2}\cos 2\theta \, d\theta = 0$

Therefore:
$A = 2a^2 \cdot 3\pi = \boxed{6\pi a^2}$

Calculus Problem 3: Tangent Lines at Special Points

Problem: Find points where the tangent line is vertical or horizontal.

Solution:

Using parametric form $x(t) = a(2\cos t - \cos 2t)$, $y(t) = a(2\sin t - \sin 2t)$:

Compute derivatives:
$\frac{dx}{dt} = -2a\sin t + 2a\sin 2t = 2a(\sin 2t - \sin t)$
$\frac{dy}{dt} = 2a\cos t - 2a\cos 2t = 2a(\cos t - \cos 2t)$

- **Horizontal tangents** occur when $\frac{dy}{dt} = 0$:
  $\cos t - \cos 2t = 0$

  Using $\cos 2t = 2\cos^2 t - 1$:
  $\cos t - (2\cos^2 t - 1) = 0 \Rightarrow 2\cos^2 t - \cos t - 1 = 0$

Solving the quadratic: $(2\cos t + 1)(\cos t - 1) = 0$

Thus $\cos t = 1$ or $\cos t = -\frac{1}{2}$

    - $\cos t = 1 \Rightarrow t = 0, 2\pi$: point $(a, 0)$
    - $\cos t = -\frac{1}{2} \Rightarrow t = \frac{2\pi}{3}, \frac{4\pi}{3}$: points $\left(-\frac{a}{2}, \pm\frac{3\sqrt{3}a}{2}\right)$

- **Vertical tangents** occur when $\frac{dx}{dt} = 0$:
  $\sin 2t - \sin t = 0$

  Using $\sin 2t = 2\sin t \cos t$:
  $2\sin t\cos t - \sin t = \sin t(2\cos t - 1) = 0$

  Thus $\sin t = 0$ or $\cos t = \frac{1}{2}$

    - $\sin t = 0 \Rightarrow t = 0, \pi, 2\pi$: points $(a, 0)$ and $(-3a, 0)$
    - $\cos t = \frac{1}{2} \Rightarrow t = \frac{\pi}{3}, \frac{5\pi}{3}$: points $\left(\frac{3a}{2}, \pm\frac{\sqrt{3}a}{2}\right)$

Note that $(a, 0)$ has both horizontal and vertical tangents—this is the cusp point.

Advanced Property: Parallel Tangents

The cardioid has the remarkable property that from any external point, there are exactly three tangents to the curve. This makes it useful in singularity theory.

Connection to Complex Dynamics: The cardioid appears as the main cardioid of the Mandelbrot set, where c values inside the cardioid give connected Julia sets. The boundary equation is given by:
$c = \frac{e^{i\theta}}{2} - \frac{e^{2i\theta}}{4}$

This is exactly the cardioid in the complex plane, revealing a deep connection between classical differential geometry and modern complex dynamics.

Curve 3: The Cissoid of Diocles

Cartesian Equation:
$y^2 = \frac{x^3}{2a-x}$

Polar Equation:
$r = 2a\tan\theta\sin\theta$

Parametric Equations:
$x(t) = \frac{2at^2}{1+t^2}, \quad y(t) = \frac{2at^3}{1+t^2}$

Construction Method: Circle and Tangent Line

Given a circle with diameter OA along the x-axis, where O is at the origin and A is at $(2a, 0)$, and a vertical tangent line at A.

For any line through O that intersects:

- The circle at point Q
- The tangent line at point R

The cissoid is the locus of points P such that OP = QR.

Algebraic Derivation of the Cartesian Equation

Let the circle have equation: $(x - a)^2 + y^2 = a^2$

The vertical tangent at A: $x = 2a$

A line through O with slope m: $y = mx$

This line intersects:

- The circle: substitute $y = mx$ into circle equation
  $(x - a)^2 + m^2x^2 = a^2$
  $x^2 - 2ax + a^2 + m^2x^2 = a^2$
  $x^2(1 + m^2) - 2ax = 0$
  $x[(1 + m^2)x - 2a] = 0$

So intersection Q has coordinates: $x_Q = \frac{2a}{1+m^2}$, $y_Q = \frac{2am}{1+m^2}$

- The tangent line: $x = 2a$, so $y_R = 2am$

Now, QR = $y_R - y_Q = 2am - \frac{2am}{1+m^2} = \frac{2am^3}{1+m^2}$

Since OP = QR and P lies on the line $y = mx$, we have:
$OP = \sqrt{x_P^2 + y_P^2} = \sqrt{x_P^2 + m^2x_P^2} = x_P\sqrt{1 + m^2}$

Set equal to QR: $x_P\sqrt{1 + m^2} = \frac{2am^3}{1+m^2}$

Thus: $x_P = \frac{2am^3}{(1+m^2)^{3/2}}$

And: $y_P = mx_P = \frac{2am^4}{(1+m^2)^{3/2}}$

Let $t = m$, we get the parametric form. To eliminate the parameter, note that:

From $x = \frac{2at^2}{1+t^2}$ and $y = \frac{2at^3}{1+t^2}$, we have:

$\frac{y}{x} = t$

Substitute into x-equation:
$x = \frac{2a(y/x)^2}{1+(y/x)^2} = \frac{2ay^2/x^2}{(x^2+y^2)/x^2} = \frac{2ay^2}{x^2+y^2}$

Thus: $x(x^2 + y^2) = 2ay^2$

Or: $y^2 = \frac{x^3}{2a-x}$

$$\boxed{y^2 = \frac{x^3}{2a - x}}$$

Calculus Problem 1: Area Between Curve and Asymptote

Problem: Find the area bounded by the cissoid and its asymptote $x = 2a$.

Solution:

The cissoid has a vertical asymptote at $x = 2a$. Due to symmetry about the x-axis, we compute the area in the first quadrant and double it.

The area between the curve and asymptote from $x = 0$ to $x = 2a$ is:

$A = 2\int_0^{2a} y\,dx = 2\int_0^{2a} \sqrt{\frac{x^3}{2a-x}}\,dx$

Use substitution: $x = 2a\sin^2\theta$, $dx = 4a\sin\theta\cos\theta\,d\theta$

When $x = 0, \theta = 0$; when $x = 2a, \theta = \pi/2$

Now compute the integrand:
$y = \sqrt{\frac{(2a\sin^2\theta)^3}{2a-2a\sin^2\theta}} = \sqrt{\frac{8a^3\sin^6\theta}{2a\cos^2\theta}} = \sqrt{4a^2\sin^6\theta/\cos^2\theta} = 2a\frac{\sin^3\theta}{\cos\theta}$

Thus:
$A = 2\int_0^{\pi/2}\left(2a\frac{\sin^3\theta}{\cos\theta}\right)(4a\sin\theta\cos\theta)\,d\theta$
$= 16a^2\int_0^{\pi/2}\sin^4\theta\,d\theta$

Using the reduction formula: $\int_0^{\pi/2} \sin^4\theta\, d\theta = \frac{3\pi}{16}$

Therefore:
$A = 16a^2 \cdot \frac{3\pi}{16} = \boxed{3\pi a^2}$

Remarkable Result: The area between the cissoid and its asymptote is finite and equals $3\pi a^2$, despite the curve approaching the asymptote infinitely closely.

Calculus Problem 2: Tangent Lines and Their Properties

Problem: Find the equation of the tangent line at any point on the cissoid.

Solution:

Using implicit differentiation on $y^2 = \frac{x^3}{2a-x}$:

$$2y\frac{dy}{dx} = \frac{(2a-x)(3x^2)-x^3(-1)}{(2a-x)^2} = \frac{6ax^2-3x^3+x^3}{(2a-x)^2} = \frac{6ax^2-2x^3}{(2a-x)^2}$$

Thus:
$\frac{dy}{dx} = \frac{3ax-x^2}{y(2a-x)^2}$

Using the curve equation $y^2(2a-x) = x^3$, we can simplify:

$$\frac{dy}{dx} = \frac{x(3a-x)}{y(2a-x)^2}$$

At point (x_0, y_0) on the curve, the tangent line is:

$$y - y_0 = \frac{x_0(3a-x_0)}{y_0(2a-x_0)^2}(x - x_0)$$

Special Property: From any point in the plane, there are either one or three tangents to the cissoid. This makes it an example of a curve of class 3.

Advanced Applications

Rolling Parabolas and the Cissoid

If a parabola rolls over another equal parabola, the vertex traces a cissoid. This connection between mechanical motion and algebraic curves was important in the development of kinematics.

Inverse Curve of a Parabola

The cissoid is the inverse of a parabola with respect to its vertex. This property connects it to projective geometry and was used by Newton in his classification of cubic curves.

Connection to the Folium of Descartes

The cissoid can be transformed into the folium of Descartes through a rational transformation, revealing deep connections between different cubic curves in the projective plane.

The cissoid represents an important milestone in the history of mathematics—bridging Greek geometry with the emerging algebraic methods that would later develop into calculus.

Curve 4: The Lemniscate of Bernoulli

Cartesian Equation:
$(x^2 + y^2)^2 = a^2(x^2 - y^2)$

Polar Equation:
$r^2 = a^2 \cos(2\theta)$

Parametric Equations:
$x(t) = \frac{a \cos t}{1+\sin^2 t}, \quad y(t) = \frac{a \sin t \cos t}{1+\sin^2 t}$

Special Case of Cassinian Ovals

The lemniscate is a special case of Cassinian ovals where the product of distances from two fixed points (foci) is constant and equal to $a^2/2$. When $c = a$ in the Cassinian oval equation:
$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) - a^4 + c^4 = 0$
we get the lemniscate equation.

Derivation of the Polar Equation

Starting from the Cartesian equation:
$(x^2 + y^2)^2 = a^2(x^2 - y^2)$

Substitute polar coordinates $x = r\cos\theta, y = r\sin\theta$:
$(r^2)^2 = a^2(r^2\cos^2\theta - r^2\sin^2\theta)$
$r^4 = a^2r^2(\cos^2\theta - \sin^2\theta)$

Using the double-angle identity $\cos^2\theta - \sin^2\theta = \cos(2\theta)$:
$r^4 = a^2r^2\cos(2\theta)$

Dividing by r^2 (for $r \neq 0$):
$\boxed{r^2 = a^2\cos(2\theta)}$

Domain Restrictions: For real values of r, we require $\cos(2\theta) \geq 0$, which occurs when:
$-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ and $\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$

This explains why the lemniscate consists of two symmetric loops.

Calculus Problem 1: Area of One Loop

Problem: Find the area enclosed by one loop of the lemniscate.

Solution:

Using the polar area formula and symmetry:
$A_{\text{loop}} = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta$

For one loop, we integrate from $-\pi/4$ to $\pi/4$:
$A_{\text{loop}} = \frac{1}{2}\int_{-\pi/4}^{\pi/4} a^2\cos(2\theta)\,d\theta$

Due to even symmetry:
$A_{\text{loop}} = a^2\int_0^{\pi/4}\cos(2\theta)\,d\theta$

$$= a^2\left[\frac{\sin(2\theta)}{2}\right]_0^{\pi/4} = a^2\left(\frac{\sin(\pi/2)}{2} - 0\right) = \frac{a^2}{2}$$

$$\boxed{A_{\text{loop}} = \frac{a^2}{2}}$$

Total Area: Since there are two identical loops, the total area enclosed by the lemniscate is:
$A_{\text{total}} = 2\cdot\frac{a^2}{2} = a^2$

Calculus Problem 2: Arc Length of the Entire Curve

Problem: Find the total length of the lemniscate.

Solution:

Using the polar arc length formula:
$L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

From $r^2 = a^2 \cos(2\theta)$, differentiate implicitly:
$2r\frac{dr}{d\theta} = -2a^2 \sin(2\theta) \Rightarrow \frac{dr}{d\theta} = -\frac{a^2 \sin(2\theta)}{r}$

Thus:
$\left(\frac{dr}{d\theta}\right)^2 = \frac{a^4 \sin^2(2\theta)}{r^2} = \frac{a^4 \sin^2(2\theta)}{a^2 \cos(2\theta)} = a^2 \tan^2(2\theta)$

Now compute the integrand:
$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2 \cos(2\theta) + a^2 \tan^2(2\theta) \cos^2(2\theta)$
$= a^2 \cos(2\theta) + a^2 \sin^2(2\theta)$
$= a^2(\cos(2\theta) + \sin^2(2\theta))$

This is not simplifying nicely. The length of the lemniscate is actually given by a special constant:

$$L = 4a \int_0^{\pi/4} \frac{d\theta}{\sqrt{\cos(2\theta)}}$$

This integral cannot be expressed in elementary functions—it's related to the lemniscate constant:
$L = 2\pi a \cdot \frac{\varpi}{\pi} \approx 2\pi a \cdot 0.8346$

Where ϖ is the lemniscate constant:
$\varpi = 2\int_0^1 \frac{dt}{\sqrt{1-t^4}} \approx 2.622$

Thus:
$\boxed{L = 2a\varpi \approx 5.244a}$

Calculus Problem 3: Tangent Line Slope and Special Points

Problem: Find the slope of the tangent line at any point on the lemniscate.

Solution:

Using implicit differentiation on the Cartesian equation:
$(x^2 + y^2)^2 = a^2(x^2 - y^2)$

Differentiate both sides:
$2(x^2 + y^2)(2x + 2y\frac{dy}{dx}) = a^2(2x - 2y\frac{dy}{dx})$

Divide by 2:
$(x^2 + y^2)(2x + 2y\frac{dy}{dx}) = a^2(x - y\frac{dy}{dx})$

Expand:
$2x(x^2 + y^2) + 2y(x^2 + y^2)\frac{dy}{dx} = a^2x - a^2y\frac{dy}{dx}$

Collect $\frac{dy}{dx}$ terms:
$[2y(x^2 + y^2) + a^2y]\frac{dy}{dx} = a^2x - 2x(x^2 + y^2)$

Factor:
$y[2(x^2 + y^2) + a^2]\frac{dy}{dx} = x[a^2 - 2(x^2 + y^2)]$

Therefore:
$\frac{dy}{dx} = \frac{x[a^2 - 2(x^2+y^2)]}{y[2(x^2+y^2)+a^2]}$

Special Case at the Origin: At $(0, 0)$, both numerator and denominator are zero, indicating the curve crosses itself. The two tangent lines at the origin can be found by analyzing the lowest-order terms in the equation.

Horizontal Tangents: Occur when $\frac{dy}{dx} = 0$, which requires:
$x[a^2 - 2(x^2 + y^2)] = 0$

- $x = 0$: Substitute into original equation: $(y^2)^2 = a^2(-y^2) \Rightarrow y^4 = -a^2y^2$
  This gives $y = 0$ or $y^2 = -a^2$ (no real solutions). So only the origin.
- $a^2 - 2(x^2 + y^2) = 0 \Rightarrow x^2 + y^2 = \frac{a^2}{2}$
  Substitute into original equation: $\left(\frac{a^2}{2}\right)^2 = a^2(x^2 - y^2)$
  $\frac{a^4}{4} = a^2(x^2 - y^2) \Rightarrow x^2 - y^2 = \frac{a^2}{4}$
  Solve the system:
  $x^2 + y^2 = \frac{a^2}{2}$
  $x^2 - y^2 = \frac{a^2}{4}$
  Adding: $2x^2 = \frac{3a^2}{4} \Rightarrow x = \pm\frac{a\sqrt{3}}{2\sqrt{2}}$
  Subtracting: $2y^2 = \frac{a^2}{4} \Rightarrow y = \pm\frac{a}{2\sqrt{2}}$

  These give four points with horizontal tangents.

Advanced Properties and Connections

Lemniscate Constant and Elliptic Integrals

The arc length of the lemniscate involves the lemniscate constant ϖ, which is analogous to π for the circle. This constant appears in the theory of elliptic functions and was studied by Gauss, who discovered that the lemniscate can be divided into 5 equal parts with straightedge and compass—a remarkable fact given the impossibility of angle trisection.

Connection to Hyperbolas

The lemniscate is the inverse of a rectangular hyperbola with respect to its center. This transformation property connects it to projective geometry and was important in the development of complex analysis.

Applications in Physics

The lemniscate appears as the focal set in certain optical systems and as equipotential lines in dipole fields. Its unique shape makes it useful in antenna design and other electromagnetic applications.

The lemniscate represents a beautiful intersection of geometry, analysis, and number theory, showcasing how a simple algebraic curve can encode deep mathematical relationships.

Curve 5: The Catenary

Cartesian Equation:
$y = a\cosh\left(\frac{x}{a}\right) = \frac{a}{2}(e^{x/a} + e^{-x/a})$

Alternative Form:
$y = a\cosh\left(\frac{x}{a}\right)$

Historical Context and Physical Significance

The Hanging Chain Problem

The catenary derives its name from the Latin "catena" meaning "chain." It describes the curve formed by a flexible, uniform chain hanging under its own weight. The problem of determining this shape was posed by Galileo Galilei, who incorrectly believed it to be a parabola.

Jacob Bernoulli's Challenge

In 1690, Jacob Bernoulli challenged mathematicians to find the equation of the hanging chain. The solution was independently discovered by Gottfried Leibniz, Christiaan Huygens, and Johann Bernoulli in 1691.

Derivation from Physical Principles

Consider a chain of uniform density ρ hanging between two points. The forces acting on a small segment are:

- Weight: $\rho g\, ds$ downward
- Tension forces at each end

By resolving forces horizontally and vertically and taking limits, we obtain the differential equation:
$\frac{d^2y}{dx^2} = \frac{1}{a}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

Where $a = \frac{H}{\rho g}$ and H is the horizontal component of tension.

The solution to this equation is:
$y = a \cosh\left(\frac{x}{a}\right)$

Calculus Problem 1: Arc Length Between Two Points

Problem: Find the length of the catenary between points $x = x_1$ and $x = x_2$.

Solution:

Using the arc length formula:
$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

First, compute the derivative:
$\frac{dy}{dx} = \frac{d}{dx}\left[a \cosh\left(\frac{x}{a}\right)\right] = \sinh\left(\frac{x}{a}\right)$

Now compute the integrand:
$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2\left(\frac{x}{a}\right) = \cosh^2\left(\frac{x}{a}\right)$

Therefore:
$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \cosh\left(\frac{x}{a}\right)$

The arc length becomes:
$L = \int_{x_1}^{x_2} \cosh\left(\frac{x}{a}\right) dx$

$= a\left[\sinh\left(\frac{x}{a}\right)\right]_{x_1}^{x_2} = a\left[\sinh\left(\frac{x_2}{a}\right) - \sinh\left(\frac{x_1}{a}\right)\right]$

$$\boxed{L = a\left[\sinh\left(\frac{x_2}{a}\right) - \sinh\left(\frac{x_1}{a}\right)\right]}$$

Special Case: From the lowest point $(0, a)$ to a general point (x, y):
$L = a \sinh\left(\frac{x}{a}\right)$

Calculus Problem 2: Area Under the Catenary

Problem: Find the area bounded above by the catenary and below by the x-axis between $x = x_1$ and $x = x_2$.

Solution:

The area is given by:
$A = \int_{x_1}^{x_2} y\, dx = \int_{x_1}^{x_2} a \cosh\left(\frac{x}{a}\right) dx$

$= a^2\left[\sinh\left(\frac{x}{a}\right)\right]_{x_1}^{x_2} = a^2\left[\sinh\left(\frac{x_2}{a}\right) - \sinh\left(\frac{x_1}{a}\right)\right]$

$$\boxed{A = a^2\left[\sinh\left(\frac{x_2}{a}\right) - \sinh\left(\frac{x_1}{a}\right)\right]}$$

Geometric Interpretation: The area under the catenary from $-x$ to x equals the area of a rectangle with width x and height equal to the arc length from the vertex to (x, y).

Calculus Problem 3: Tangent Line and Its Properties

Problem: Find the equation of the tangent line at point $(c, a\cosh(c/a))$.

Solution:

The slope at point $x = c$ is:
$m = \left.\frac{dy}{dx}\right|_{x=c} = \sinh\left(\frac{c}{a}\right)$

Using the point-slope form:
$y - a\cosh\left(\frac{c}{a}\right) = \sinh\left(\frac{c}{a}\right)(x - c)$

Important Property: The length of the tangent from the point of tangency to the x-axis is constant.

Let's find the x-intercept by setting $y = 0$:
$0 - a\cosh\left(\frac{c}{a}\right) = \sinh\left(\frac{c}{a}\right)(x_{\text{int}} - c)$

$x_{\text{int}} = c - a\frac{\cosh(c/a)}{\sinh(c/a)} = c - a\coth\left(\frac{c}{a}\right)$

The distance along the tangent from $(c, a\cosh(c/a))$ to the x-axis can be shown to be constant, related to the parameter a.

Calculus Problem 4: Volume of Revolution

Problem: Let R be the region bounded above by $y = 2$ and below by $y = \cosh x$ (with $a = 1$). Find the volume when R is rotated about $y = -1$.

Solution:

First, find the intersection points of $y = \cosh x$ and $y = 2$:
$\cosh x = 2 \Rightarrow \frac{e^x + e^{-x}}{2} = 2$
$e^x + e^{-x} = 4$
Multiply by e^x: $e^{2x} + 1 = 4e^x$
$e^{2x} - 4e^x + 1 = 0$

Let $u = e^x$: $u^2 - 4u + 1 = 0$
$u = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$

So $x = \ln\left(2 \pm \sqrt{3}\right) = \pm\ln\left(2 + \sqrt{3}\right)$ (since $\ln\left(2 - \sqrt{3}\right) = -\ln\left(2 + \sqrt{3}\right)$)

Using the method of washers, the volume is:
$$V = \pi \int_{-\ln\left(2+\sqrt{3}\right)}^{\ln\left(2+\sqrt{3}\right)} [R_{\text{outer}}^2 - R_{\text{inner}}^2]\, dx$$

Where:

- Outer radius: distance from $y = -1$ to $y = 2$: $2 - (-1) = 3$
- Inner radius: distance from $y = -1$ to $y = \cosh x$: $\cosh x - (-1) = \cosh x + 1$

Thus:
$$V = \pi \int_{-\ln\left(2+\sqrt{3}\right)}^{\ln\left(2+\sqrt{3}\right)} [3^2 - (\cosh x + 1)^2]\, dx$$

$$= \pi \int_{-\ln\left(2+\sqrt{3}\right)}^{\ln\left(2+\sqrt{3}\right)} [9 - (\cosh^2 x + 2\cosh x + 1)]\, dx$$

$$= \pi \int_{-\ln\left(2+\sqrt{3}\right)}^{\ln\left(2+\sqrt{3}\right)} [8 - \cosh^2 x - 2\cosh x]\, dx$$

Using $\cosh^2 x = \frac{\cosh 2x+1}{2}$:
$$V = \pi \int_{-\ln\left(2+\sqrt{3}\right)}^{\ln\left(2+\sqrt{3}\right)} \left[8 - \frac{\cosh 2x+1}{2} - 2\cosh x\right] dx$$

$$= \pi \int_{-\ln\left(2+\sqrt{3}\right)}^{\ln\left(2+\sqrt{3}\right)} \left[\tfrac{15}{2} - \tfrac{1}{2}\cosh 2x - 2\cosh x\right] dx$$

Due to even symmetry, we can compute from 0 to $\ln\left(2+\sqrt{3}\right)$ and double:

$$V = 2\pi\left[\tfrac{15}{2}x - \tfrac{1}{4}\sinh 2x - 2\sinh x\right]_0^{\ln\left(2+\sqrt{3}\right)}$$

After computation (using hyperbolic identities), we get:
$$\boxed{V = \pi\left[15\ln\left(2+\sqrt{3}\right) - 6\sqrt{3}\right]}$$

Advanced Applications and Properties

The Catenoid—Minimal Surface of Revolution

When a catenary is rotated about the x-axis, it generates a catenoid—a minimal surface. This is the only minimal surface of revolution and has the smallest area for given boundary conditions.

Architecture and Engineering

The catenary shape is used in architecture for arches and domes, as it experiences only compressive forces under uniform loading. Gaudí famously used catenary models to design the Sagrada Família.

Inverse Catenary Property

If a catenary is inverted, it becomes the ideal shape for an arch supporting its own weight, with only compressive forces and no bending moments.

Connection to the Tractrix

The catenary is the evolute of the tractrix, meaning it's the envelope of normals to the tractrix. This relationship connects two important curves in differential geometry.

The catenary exemplifies the beautiful interplay between mathematics and physics—a simple physical problem leading to profound mathematical insights with far-reaching applications.

Curve 6: The Epicycloid—Rolling Circles and Beautiful Patterns

Parametric Equations:
$x(t) = (a+b)\cos t - b\cos\left(\frac{a+b}{b}t\right)$
$y(t) = (a+b)\sin t - b\sin\left(\frac{a+b}{b}t\right)$

Geometric Construction and Family Relations

The Rolling Circle Mechanism

An epicycloid is generated by tracing a point on a circle of radius b that rolls externally on a fixed circle of radius a. The parametric equations describe the path of point P on the rolling circle.

Mathematical Derivation of Parametric Equations

Consider:

- Fixed circle: center at origin, radius a
- Rolling circle: radius b, initially touching fixed circle at point $(a, 0)$

- Point P starts at $(a+b, 0)$

As the rolling circle rotates through angle ϕ, it travels along the fixed circle. The arc lengths must be equal:
$a\phi = b\theta \Rightarrow \theta = \frac{a}{b}\phi$

The center of the rolling circle moves to:
$C = ((a+b)\cos\phi, (a+b)\sin\phi)$

Relative to its center, point P is at:
$P_{\text{relative}} = (-b\cos(\phi+\theta), -b\sin(\phi+\theta))$

Substituting $\theta = \frac{a}{b}\phi$:
$P_{\text{relative}} = \left(-b\cos\left(\phi + \frac{a}{b}\phi\right), -b\sin\left(\phi + \frac{a}{b}\phi\right)\right)$
$= \left(-b\cos\left(\frac{a+b}{b}\phi\right), -b\sin\left(\frac{a+b}{b}\phi\right)\right)$

Adding the center position:
$x = (a+b)\cos\phi - b\cos\left(\frac{a+b}{b}\phi\right)$
$y = (a+b)\sin\phi - b\sin\left(\frac{a+b}{b}\phi\right)$

Letting $t = \phi$, we obtain the standard form.

Special Cases:

- **Cardioid:** When $a = b$ $(m = 2)$
- **Nephroid:** When $a = 2b$ $(m = 3)$
- **Astroid:** Actually a hypocycloid, but related

Calculus Problem 1: Arc Length for Integer Ratio

Problem: Find the length of the epicycloid when $a = (m-1)b$ where m is an integer.

Solution:

When $a = (m-1)b$, the parametric equations become:
$x(t) = mb\cos t - b\cos(mt)$
$y(t) = mb\sin t - b\sin(mt)$

Compute derivatives:
$\frac{dx}{dt} = -mb\sin t + mb\sin(mt)$
$\frac{dy}{dt} = mb\cos t - mb\cos(mt)$

Now compute the arc length integrand:
$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=m^2b^2[\sin^2t-2\sin t\sin(mt)+\sin^2(mt)]$
$+m^2b^2[\cos^2t-2\cos t\cos(mt)+\cos^2(mt)]$

Combine terms:
$=m^2b^2[(\sin^2t+\cos^2t)+(\sin^2(mt)+\cos^2(mt))-2(\sin t\sin(mt)+\cos t\cos(mt))]$

Using trigonometric identities:
$\sin^2t+\cos^2t=1$
$\sin^2(mt)+\cos^2(mt)=1$
$\sin t\sin(mt)+\cos t\cos(mt)=\cos(mt-t)=\cos((m-1)t)$

Therefore:
$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=m^2b^2[2-2\cos((m-1)t)]$
$=2m^2b^2[1-\cos((m-1)t)]$

Using $1-\cos\theta=2\sin^2(\theta/2)$:
$=4m^2b^2\sin^2\left(\frac{m-1}{2}t\right)$

Thus:
$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=2mb\left|\sin\left(\frac{m-1}{2}t\right)\right|$

The complete curve is traced as t goes from 0 to 2π. Due to symmetry, we can compute from 0 to 2π and account for the m lobes:

$L=\int_0^{2\pi}2mb\left|\sin\left(\frac{m-1}{2}t\right)\right|dt$

Since $\sin\left(\frac{m-1}{2}t\right)$ changes sign, and there are $2(m-1)$ sign changes in $[0,2\pi]$, we can compute:

$L=2mb\cdot2(m-1)\int_0^{\pi/(m-1)}\sin\left(\frac{m-1}{2}t\right)dt$

$=4mb(m-1)\left[-\frac{2}{m-1}\cos\left(\frac{m-1}{2}t\right)\right]_0^{\pi/(m-1)}$

$=8mb\left[-\cos\left(\frac{m-1}{2}\cdot\frac{\pi}{m-1}\right)+\cos 0\right]$

$=8mb\left[-\cos\left(\frac{\pi}{2}\right)+1\right]=8mb(0+1)=8mb$

Since $a=(m-1)b$, we have $b=\frac{a}{m-1}$, so:
$L=8m\cdot\frac{a}{m-1}=\frac{8ma}{m-1}$

$$\boxed{L = \frac{8ma}{m-1}}$$

Special Cases:

- Cardioid $(m = 2)$: $L = \frac{16a}{1} = 16a$
- Nephroid $(m = 3)$: $L = \frac{24a}{2} = 12a$

Calculus Problem 2: Area Enclosed by Epicycloid

Problem: Find the area enclosed by the epicycloid when $a = (m-1)b$.

Solution:

Using the parametric area formula:
$A = \frac{1}{2} \oint (x\,dy - y\,dx)$

From our derivatives:
$x\,dy - y\,dx = [mb\cos t - b\cos(mt)][mb\cos t - mb\cos(mt)]dt$
$-[mb\sin t - b\sin(mt)][-mb\sin t + mb\sin(mt)]dt$

This simplifies to:
$x\,dy - y\,dx = m^2b^2[1 - \cos((m-1)t)]dt$

Therefore:
$A = \frac{1}{2} \int_0^{2\pi} mb^2(m+1)[1 - \cos((m-1)t)]dt$

The integral of $\cos((m-1)t)$ over $[0, 2\pi]$ is zero, so:
$A = \frac{1}{2} mb^2(m+1) \cdot 2\pi = \pi mb^2(m+1)$

Since $b = \frac{a}{m-1}$:
$A = \pi m(m+1)\left(\frac{a}{m-1}\right)^2 = \frac{\pi m(m+1)a^2}{(m-1)^2}$

$$\boxed{A = \frac{\pi m(m+1)a^2}{(m-1)^2}}$$

Special Cases:

- Cardioid $(m = 2)$: $A = \frac{\pi \cdot 2 \cdot 3a^2}{1} = 6\pi a^2$ ✓
- Nephroid $(m = 3)$: $A = \frac{\pi \cdot 3 \cdot 4a^2}{4} = 3\pi a^2$

Calculus Problem 3: Tangent Line Slope

Problem: Find the slope of the tangent line to the epicycloid at any point.

Solution:

The slope is given by:
$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{mb\cos t - mb\cos(mt)}{-mb\sin t + mb\sin(mt)}$

Simplify:
$\frac{dy}{dx} = \frac{\cos t - \cos(mt)}{-\sin t + \sin(mt)}$

Using sum-to-product identities:
$\cos t - \cos(mt) = -2\sin\left(\frac{1+m}{2}t\right)\sin\left(\frac{1-m}{2}t\right)$
$\sin(mt) - \sin t = 2\cos\left(\frac{1+m}{2}t\right)\sin\left(\frac{m-1}{2}t\right)$

Therefore:
$\frac{dy}{dx} = \frac{-2\sin\left(\frac{1+m}{2}t\right)\sin\left(\frac{1-m}{2}t\right)}{-2\cos\left(\frac{1+m}{2}t\right)\sin\left(\frac{m-1}{2}t\right)}$

Since $\sin\left(\frac{1-m}{2}t\right) = -\sin\left(\frac{m-1}{2}t\right)$:
$\frac{dy}{dx} = \frac{-2\sin\left(\frac{1+m}{2}t\right)\left(-\sin\left(\frac{m-1}{2}t\right)\right)}{-2\cos\left(\frac{1+m}{2}t\right)\sin\left(\frac{m-1}{2}t\right)}$
$= \frac{\sin\left(\frac{1+m}{2}t\right)}{\cos\left(\frac{1+m}{2}t\right)} = \tan\left(\frac{1+m}{2}t\right)$

$$\boxed{\frac{dy}{dx} = \tan\left(\frac{1+m}{2}t\right)}$$

Geometric Interpretation: The tangent makes an angle of $\frac{1+m}{2}t$ with the x-axis, showing a beautiful linear relationship between the parameter and the tangent direction.

Advanced Properties and Applications

Gear Design and Engineering

Epicycloids are used in gear design, particularly for cycloidal gears which have the advantage of lower friction and smoother operation compared to involute gears.

Astronomical Connections

The epicycloid model was historically used in Ptolemaic astronomy to describe planetary orbits as epicycles upon deferents, though this was later superseded by Kepler's elliptical orbits.

Complex Representation

Epicycloids can be represented in complex form:
$z(t) = (a+b)e^{it} - be^{i\frac{a+b}{b}t}$

This compact representation reveals the underlying harmonic nature of these curves.

Relation to Fourier Series

The parametric equations of epicycloids are essentially finite Fourier series, connecting them to harmonic analysis and signal processing.

The epicycloid demonstrates how simple mechanical motion—one circle rolling on another—can generate curves of remarkable mathematical beauty and practical utility.

Curve 7: The Witch of Agnesi—Mistaken Identity with Profound Implications

Cartesian Equation:
$y(x^2 + a^2) = a^3$

Parametric Equations:
$x(t) = at$
$y(t) = \frac{a}{1+t^2}$

Alternative Forms:
$y = \frac{a^3}{x^2+a^2}$

Historical Context and Naming

Maria Gaetana Agnesi and Her "Witch"

The curve was studied by Maria Gaetana Agnesi in 1748 in her influential calculus textbook. The name "witch" comes from a mistranslation of the Italian word "versiera" (curve), which was confused with "avversiera" (witch or devil's wife). Agnesi herself called it the "cubical parabola."

Earlier Discoveries

The curve was actually first studied by Pierre de Fermat in 1630, and later by Luigi Guido Grandi in 1703, who gave it the name "versoria."

Geometric Construction

Circle and Tangent Construction

1. Start with a circle of diameter a centered at $(0, \frac{a}{2})$
2. Draw a horizontal line through the top of the circle
3. For any line through the origin with slope t, it intersects:
    - The circle at point Q
    - The horizontal line at point R
4. The Witch is the locus of points P such that the y-coordinate of P equals that of Q, and the x-coordinate equals that of R

Algebraic Derivation

The circle has equation: $x^2 + (y - \frac{a}{2})^2 = (\frac{a}{2})^2$

A line through origin: $y = tx$

Intersection with circle:
$x^2 + (tx - \frac{a}{2})^2 = \frac{a^2}{4}$

Expanding: $x^2 + t^2x^2 - atx + \frac{a^2}{4} = \frac{a^2}{4}$

Thus: $x^2(1 + t^2) = atx$

So $x = \frac{at}{1+t^2}$ (excluding $x = 0$)

Then $y = tx = \frac{at^2}{1+t^2}$

This gives the point Q on the circle.

The horizontal line through top of circle: $y = a$

Intersection with line $y = tx$: $x = \frac{a}{t}$, giving point $R = (\frac{a}{t}, a)$

The Witch point P has coordinates $(\frac{a}{t}, \frac{at^2}{1+t^2})$

Letting t be the parameter, we get the parametric form.

To eliminate parameter, let $x = \frac{a}{t} \Rightarrow t = \frac{a}{x}$

Then $y = \frac{a(\frac{a}{x})^2}{1+(\frac{a}{x})^2} = \frac{a^3/x^2}{(x^2+a^2)/x^2} = \frac{a^3}{x^2+a^2}$

Thus: $\boxed{y(x^2 + a^2) = a^3}$

Calculus Problem 1: Area Under the Curve

Problem: Find the area under the Witch of Agnesi.

Solution:

The curve is symmetric about the y-axis, so we compute from $x = 0$ to $x = \infty$ and double:

$A = 2\int_0^\infty \frac{a^3}{x^2+a^2}\,dx$

Use substitution: $x = a\tan\theta$, $dx = a\sec^2\theta\,d\theta$

When $x = 0, \theta = 0$; when $x \to \infty, \theta \to \frac{\pi}{2}$

Then:
$A = 2\int_0^{\pi/2} \frac{a^3}{a^2\tan^2\theta+a^2} \cdot a\sec^2\theta\,d\theta$
$= 2\int_0^{\pi/2} \frac{a^4}{a^2(\tan^2\theta+1)} \cdot \sec^2\theta\,d\theta$
$= 2\int_0^{\pi/2} \frac{a^2}{\sec^2\theta} \cdot \sec^2\theta\,d\theta$
$= 2a^2\int_0^{\pi/2} d\theta = 2a^2 \cdot \frac{\pi}{2} = \pi a^2$

$\boxed{A = \pi a^2}$

Remarkable Result: The area under the entire Witch equals the area of a circle of radius a!

Calculus Problem 2: Inflection Points

Problem: Find the inflection points of the Witch of Agnesi.

Solution:

We have $y = \frac{a^3}{x^2+a^2}$

First derivative:
$\frac{dy}{dx} = a^3 \cdot \frac{-2x}{(x^2+a^2)^2} = \frac{-2a^3x}{(x^2+a^2)^2}$

Second derivative:
$\frac{d^2y}{dx^2} = \frac{-2a^3(x^2+a^2)^2+2a^3x\cdot2(x^2+a^2)\cdot2x}{(x^2+a^2)^4}$
$= \frac{-2a^3(x^2+a^2)^2+8a^3x^2(x^2+a^2)}{(x^2+a^2)^4}$

Factor numerator:
$= \frac{2a^3(x^2+a^2)[-(x^2+a^2)+4x^2]}{(x^2+a^2)^4}$
$= \frac{2a^3(3x^2-a^2)}{(x^2+a^2)^3}$

Set $\frac{d^2y}{dx^2} = 0$:
$3x^2 - a^2 = 0 \Rightarrow x = \pm\frac{a}{\sqrt{3}}$

The corresponding y-values:
$y = \frac{a^3}{\frac{a^2}{3}+a^2} = \frac{a^3}{\frac{4a^2}{3}} = \frac{3a}{4}$

So the inflection points are:
$$\boxed{\left(\pm\frac{a}{\sqrt{3}}, \frac{3a}{4}\right)}$$

Calculus Problem 3: Tangent Line Equation

Problem: Find the equation of the tangent line at any point P on the Witch.

Solution:

At point (x_0, y_0) on the curve, the slope is:
$m = \frac{-2a^3x_0}{(x_0^2+a^2)^2}$

Using the point-slope form:
$y - y_0 = \frac{-2a^3x_0}{(x_0^2+a^2)^2}(x - x_0)$

Since $y_0 = \frac{a^3}{x_0^2+a^2}$, we can write:
$y - \frac{a^3}{x_0^2+a^2} = \frac{-2a^3x_0}{(x_0^2+a^2)^2}(x - x_0)$

Multiply through by $(x_0^2 + a^2)^2$:
$y(x_0^2 + a^2)^2 - a^3(x_0^2 + a^2) = -2a^3x_0(x - x_0)$

This is the general tangent line equation.

Special Case: At the vertex $(0, a)$, the slope is 0, so the tangent is horizontal:
$y = a$

Calculus Problem 4: Volume of Revolution

Problem: Find the volume when the region under the Witch is rotated about the x-axis.

Solution:

Using the disk method:
$V = \pi \int_{-\infty}^{\infty} [y(x)]^2 \, dx = \pi \int_{-\infty}^{\infty} \left(\frac{a^3}{x^2+a^2}\right)^2 dx$

Due to symmetry:
$V = 2\pi \int_0^{\infty} \frac{a^6}{(x^2+a^2)^2} \, dx$

Use substitution $x = a \tan\theta$, $dx = a \sec^2\theta \, d\theta$:

$V = 2\pi \int_0^{\pi/2} \frac{a^6}{a^4 \sec^4\theta} \cdot a \sec^2\theta \, d\theta$
$= 2\pi \int_0^{\pi/2} \frac{a^3}{\sec^2\theta} \, d\theta$
$= 2\pi a^3 \int_0^{\pi/2} \cos^2\theta \, d\theta$

Using $\int_0^{\pi/2} \cos^2\theta \, d\theta = \frac{\pi}{4}$:

$V = 2\pi a^3 \cdot \frac{\pi}{4} = \frac{\pi^2 a^3}{2}$

$$\boxed{V = \frac{\pi^2 a^3}{2}}$$

Advanced Properties and Applications

Probability and Statistics

The Witch of Agnesi is essentially the probability density function of the Cauchy distribution:
$f(x) = \frac{1}{\pi(1+x^2)}$ (when $a = 1$)

This distribution has the remarkable property that its mean does not exist, illustrating the difference between the Cauchy distribution and more familiar distributions like the normal distribution.

Optics and Wave Phenomena

The curve appears in optics as the intensity distribution of certain diffraction patterns and in the study of wave propagation.

Complex Analysis

In complex analysis, the Witch appears in the study of conformal mappings and as an example of a function with interesting analytic properties.

Connection to the Lorentzian Function

The Witch is essentially a scaled version of the Lorentzian function, which appears in physics describing spectral line shapes and resonance phenomena.

The Witch of Agnesi represents a beautiful example of how a simple geometric construction can lead to profound mathematical insights with applications across multiple disciplines, from probability theory to physics.

Curve 8: The Tractrix—Path of the Dragged Object

Parametric Equations:
$x(t) = \frac{1}{\cosh t}$
$y(t) = t - \tanh t$

Alternative Forms:
$x = a \operatorname{sech}\left(\frac{y}{a}\right)$
$y = a \ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right) - \sqrt{a^2-x^2}$

Historical Context and Physical Interpretation

Leibniz's Drag Problem

The tractrix (from Latin "trahere" meaning "to pull") was first studied by Claude Perrault in 1670 and later by Leibniz and Huygens. Leibniz posed the problem: "What is the path of an object dragged along a horizontal plane by a string of constant length when the free end moves along a straight line?"

The Towel and Pocket Watch

A simple physical model: imagine pulling a towel across a table by one corner, or dragging a pocket watch by its chain along a straight line.

Mathematical Derivation

Differential Equation Approach

Let the object start at $(a, 0)$ and be pulled along the x-axis. At any point (x, y) on the curve, the string of length a is tangent to the curve.

The condition for the tangent: $\frac{dy}{dx} = -\frac{\sqrt{a^2-x^2}}{x}$

This leads to the differential equation:
$\frac{dy}{dx} = -\frac{y}{\sqrt{a^2-y^2}}$

Solving this separable equation gives the Cartesian form.

Parametric Derivation

A more elegant approach uses the fact that the distance from the point of tangency to the x-axis along the tangent equals a.

From the geometry:
$\frac{dy}{dx} = -\frac{y}{\sqrt{a^2-y^2}}$

Let $y = a\sin\theta$, then:
$\frac{dy}{dx} = -\frac{a\sin\theta}{a\cos\theta} = -\tan\theta$

But $\frac{dy}{dx} = a\cos\theta\frac{d\theta}{dx}$, so:
$a\cos\theta\frac{d\theta}{dx} = -\tan\theta$

This leads to the parametric equations.

Calculus Problem 1: Arc Length to Asymptote

Problem: Find the length of the tractrix from the x-axis to a point on the curve.

Solution:

Using the parametric form $x(t) = \frac{1}{\cosh t}$, $y(t) = t - \tanh t$ (with $a = 1$ for simplicity):

Compute derivatives:
$\frac{dx}{dt} = -\frac{\sinh t}{\cosh^2 t}$
$\frac{dy}{dt} = 1 - \operatorname{sech}^2 t = \frac{\sinh^2 t}{\cosh^2 t}$

Arc length element:
$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$
$= \sqrt{\frac{\sinh^2 t}{\cosh^4 t} + \frac{\sinh^4 t}{\cosh^4 t}} dt$
$= \frac{|\sinh t|}{\cosh^2 t} \sqrt{1 + \sinh^2 t} dt$
$= \frac{\sinh t}{\cosh^2 t} \cosh t dt = \frac{\sinh t}{\cosh t} dt = \tanh t dt$

Thus the length from $t = t_0$ to $t = t_1$ is:
$L = \int_{t_0}^{t_1} \tanh t dt = [\ln(\cosh t)]_{t_0}^{t_1}$

From the starting point $(1, 0)$ where $t = 0$, to a general point with parameter t:
$L = \ln(\cosh t) - \ln(\cosh 0) = \ln(\cosh t)$

As $t \to \infty$, $\cosh t \sim \frac{e^t}{2}$, so the total length to the asymptote is infinite.

However, the length from a point to the asymptote along a tangent is constant and equals a.

Calculus Problem 2: Area Between Curve and Asymptote

Problem: Find the area between the tractrix and its asymptote.

Solution:

The tractrix approaches the x-axis as an asymptote. The area between the curve and the x-axis from $x = 0$ to $x = a$ is:

$A = \int_0^a y dx$

Using the derivative relation $\frac{dy}{dx} = -\frac{y}{\sqrt{a^2-y^2}}$, we can change variables:

$$dx = -\frac{\sqrt{a^2-y^2}}{y}dy$$

So:
$$A = \int_a^0 y\left(-\frac{\sqrt{a^2-y^2}}{y}\right)dy = \int_0^a \sqrt{a^2-y^2}dy$$

This is a standard integral:
$$A = \left[\frac{y}{2}\sqrt{a^2-y^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right)\right]_0^a$$
$$= \left(0 + \frac{a^2}{2}\cdot\frac{\pi}{2}\right) - 0 = \frac{\pi a^2}{4}$$

$$\boxed{A = \frac{\pi a^2}{4}}$$

Remarkable Result: The area between the tractrix and its asymptote equals one-quarter the area of a circle of radius a!

Calculus Problem 3: Tangent Line Properties

Problem: Show that the length of the tangent from the point of tangency to the asymptote is constant.

Solution:

At point (x, y) on the tractrix, the tangent line has slope:
$$\frac{dy}{dx} = -\frac{y}{\sqrt{a^2-y^2}}$$

The equation of the tangent line:
$$Y - y = -\frac{y}{\sqrt{a^2-y^2}}(X - x)$$

Find the x-intercept by setting $Y = 0$:
$$-y = -\frac{y}{\sqrt{a^2-y^2}}(X_{\text{int}} - x)$$

Assuming $y \neq 0$:
$$1 = \frac{1}{\sqrt{a^2-y^2}}(X_{\text{int}} - x)$$

So: $X_{\text{int}} = x + \sqrt{a^2-y^2}$

The distance along the tangent from (x, y) to $(X_{\text{int}}, 0)$ is:
$d = \sqrt{(X_{\text{int}} - x)^2 + y^2} = \sqrt{(a^2 - y^2) + y^2} = a$

$\boxed{d = a}$

This confirms the defining property: the string length remains constant.

Calculus Problem 4: Surface and Volume of Revolution

Problem: Find the surface area and volume when the tractrix is rotated about its asymptote.

Solution:

Volume of Revolution (about x-axis):

Using the disk method:
$V = \pi \int_0^a y^2 dx$

Change variables using $dx = -\frac{\sqrt{a^2-y^2}}{y} dy$:

$V = \pi \int_a^0 y^2 \left(-\frac{\sqrt{a^2-y^2}}{y}\right) dy = \pi \int_0^a y\sqrt{a^2 - y^2} dy$

Use substitution $u = a^2 - y^2$, $du = -2ydy$:

$V = \pi \int_{a^2}^0 \sqrt{u} \left(-\frac{du}{2}\right) = \frac{\pi}{2} \int_0^{a^2} u^{1/2} du$
$= \frac{\pi}{2} \left[\frac{2}{3} u^{3/2}\right]_0^{a^2} = \frac{\pi}{3} a^3$

$$\boxed{V = \frac{\pi a^3}{3}}$$

Surface Area of Revolution:

$S = 2\pi \int y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

We have $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{y^2}{a^2-y^2}} = \frac{a}{\sqrt{a^2-y^2}}$

So:
$S = 2\pi \int y \cdot \frac{a}{\sqrt{a^2-y^2}} dx$

Using $dx = -\frac{\sqrt{a^2-y^2}}{y}dy$:

$$S = 2\pi a \int_a^0 y \cdot \frac{1}{\sqrt{a^2-y^2}} \cdot \left(-\frac{\sqrt{a^2-y^2}}{y}\right)dy$$
$$= 2\pi a \int_0^a dy = 2\pi a^2$$

$$\boxed{S = 2\pi a^2}$$

Remarkable Fact: The surface generated by rotating the tractrix about its asymptote is a pseudosphere—a surface of constant negative curvature.

Advanced Properties and Applications

The Pseudosphere

When the tractrix is rotated about its asymptote, it generates a pseudosphere—a surface of constant negative Gaussian curvature $-1/a^2$. This was the first discovered surface with constant negative curvature.

Connection to Hyperbolic Geometry

The pseudosphere provides a model for hyperbolic geometry (Lobachevsky geometry), where geodesics play the role of "straight lines." This was crucial in the development of non-Euclidean geometry.

Catenary-Tractrix Duality

The tractrix is the evolute of the catenary, meaning it's the envelope of normals to the catenary. This reciprocal relationship connects two fundamental curves.

The tractrix beautifully illustrates how a simple physical problem—dragging an object—leads to profound mathematical concepts including non-Euclidean geometry and surfaces of constant curvature.

Curve 9: The Conchoid of Nicomedes

Cartesian Equation:
$(x-b)^2(x^2+y^2) - a^2x^2 = 0$

Polar Equation:
$r = a + b\sec\theta$

Parametric Equations:
$x(t) = a + \cos t$
$y(t) = a\tan t + \sin t$

Historical Context and Geometric Construction

Nicomedes and the Angle Trisection Problem

The conchoid (from Greek "konche" meaning "shell") was discovered by Nicomedes around 200 BCE to solve the classical problems of angle trisection and cube duplication. The curve's shell-like shape gives it its name.

The Pole and Ruler Construction

Given a fixed line (the "ruler"), a fixed point O (the "pole") at distance b from the line, and a fixed length a:

1. Draw a line m through O intersecting the ruler at point Q
2. Mark points P and P' on m such that QP = QP' = a
3. The locus of P and P' forms the two branches of the conchoid

Mathematical Derivation

Deriving the Cartesian Equation

Let the ruler be the y-axis (x = 0) and the pole be at (b, 0). A line through (b, 0) with slope t has equation:
$y = t(x - b)$

This line intersects the ruler (x = 0) at:
$Q = (0, -tb)$

We want points P such that QP = a. The distance from Q to P is:
$\sqrt{(x-0)^2 + (y+tb)^2} = a$

But from the line equation, $y = t(x - b)$, so:
$\sqrt{x^2 + [t(x-b)+tb]^2} = \sqrt{x^2 + (tx)^2} = |x|\sqrt{1+t^2} = a$

Also, from the line equation: $t = \frac{y}{x-b}$

Substituting:
$|x|\sqrt{1 + \left(\frac{y}{x-b}\right)^2} = a$

Squaring:
$x^2 \left(1 + \frac{y^2}{(x-b)^2}\right) = a^2$
$x^2 \frac{(x-b)^2 + y^2}{(x-b)^2} = a^2$
$x^2[(x-b)^2 + y^2] = a^2(x-b)^2$

Thus:
$\boxed{(x-b)^2(x^2+y^2) - a^2x^2 = 0}$

Deriving the Polar Equation

Let the pole be at the origin and the ruler be the line x = b. In polar coordinates:
$x = r\cos\theta, y = r\sin\theta$

The distance from a point (r, θ) to the ruler x = b is:
$|r\cos\theta - b|$

The distance from the point to the pole is r.

By the conchoid definition: $r = a + |r\cos\theta - b|$

Assuming $r\cos\theta \geq b$ (for the right branch):
$r = a + r\cos\theta - b$
$r - r\cos\theta = a - b$
$r(1-\cos\theta) = a - b$

This gives $r = \frac{a-b}{1-\cos\theta}$, which is not our desired form.

Let's reconsider: The distance from Q to P should be a, where Q is the intersection of line OP with the ruler.

The line OP makes angle θ with the x-axis. The ruler x = b intersects this line at:
$Q = (b, b\tan\theta)$

The distance OQ = $\frac{b}{\cos\theta}$

We want points P such that QP = a. Since P lies on the same line:
$r = OQ \pm a = \frac{b}{\cos\theta} \pm a$

Thus:
$\boxed{r = b\sec\theta \pm a}$

Calculus Problem 1: Area Between Branch and Asymptote

Problem: Find the area between either branch and the asymptote.

Solution:

The conchoid has a vertical asymptote at x = b. Consider the right branch (r = b\secθ + a) in the first quadrant.

The area between the curve and asymptote from y = -c to y = c is:

Using the formula for area in polar coordinates, but it's easier to use Cartesian coordinates.

From the polar equation r = b\secθ + a, we have:
$x = r\cos\theta = b + a\cos\theta$
$y = r\sin\theta = (b + a\cos\theta)\sin\theta$

The area between the curve and the asymptote x = b is:
$A = \int_{y_1}^{y_2} (x - b) dy$

But $x - b = a\cos\theta$ and $dy = \frac{dy}{d\theta} d\theta$

Compute $\frac{dy}{d\theta} = -a\sin^2\theta + (b + a\cos\theta)\cos\theta$
$= -a\sin^2\theta + b\cos\theta + a\cos^2\theta$
$= b\cos\theta + a(\cos^2\theta - \sin^2\theta)$
$= b\cos\theta + a\cos(2\theta)$

Thus:
$A = \int_{\theta_1}^{\theta_2} (a\cos\theta)[b\cos\theta + a\cos(2\theta)] d\theta$

This integral can be evaluated, but the result depends on the specific branch and limits.

For the loop area (when it exists), we can use the polar area formula.

Calculus Problem 2: Area of the Loop

Problem: Find the area of the loop when -1 < a < 0.

Solution:

When -1 < a < 0, the conchoid has a loop. Using the polar area formula for the loop:

$A = \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 d\theta$

For the loop, we use r = b\secθ + a (taking a negative since a < 0), and find where r = 0:
$b \sec\theta + a = 0 \Rightarrow \sec\theta = -\frac{a}{b} \Rightarrow \cos\theta = -\frac{b}{a}$

Since |a| < b in this case, there are two angles θ_1 and θ_2 where cosθ = -b/a.

Thus:
$A = \frac{1}{2} \int_{\theta_1}^{\theta_2} (b \sec\theta + a)^2 d\theta$

Expand:
$A = \frac{1}{2} \int_{\theta_1}^{\theta_2} (b^2 \sec^2\theta + 2ab \sec\theta + a^2) d\theta$

Compute term by term:

- $\int \sec^2\theta d\theta = \tan\theta$
- $\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta|$
- $\int a^2 d\theta = a^2\theta$

After evaluation between θ_1 and θ_2, we get:
$\boxed{A = b^2\sqrt{\frac{1}{\cos^2\theta_1} - 1} + 2ab \ln\left|\frac{1 + \sin\theta_1}{\cos\theta_1}\right| + a^2\theta_1}$ (with appropriate sign adjustments)

Where $\cos\theta_1 = -\frac{b}{a}$

Calculus Problem 3: Tangent Line Slope

Problem: Find the slope of the tangent line at any point on the conchoid.

Solution:

Using implicit differentiation on the Cartesian equation:
$(x-b)^2(x^2+y^2)-a^2x^2=0$

Differentiate:
$2(x-b)(x^2+y^2)+(x-b)^2(2x+2y\frac{dy}{dx})-2a^2x=0$

Solve for $\frac{dy}{dx}$:
$2(x-b)^2y\frac{dy}{dx}=2a^2x-2(x-b)(x^2+y^2)-2x(x-b)^2$

$\frac{dy}{dx}=\frac{a^2x-(x-b)(x^2+y^2)-x(x-b)^2}{y(x-b)^2}$

Using the original equation $(x-b)^2(x^2+y^2)=a^2x^2$, we can simplify:

$(x-b)(x^2+y^2)=\frac{a^2x^2}{x-b}$

Substitute:
$\frac{dy}{dx}=\frac{a^2x-\frac{a^2x^2}{x-b}-x(x-b)^2}{y(x-b)^2}$
$=\frac{\frac{a^2x(x-b)-a^2x^2}{x-b}-x(x-b)^2}{y(x-b)^2}$
$=\frac{-\frac{a^2bx}{x-b}-x(x-b)^2}{y(x-b)^2}$
$=-\frac{x}{y(x-b)^2}\left(\frac{a^2b}{x-b}+(x-b)^2\right)$

$$\boxed{\frac{dy}{dx}=-\frac{x}{y}\left(\frac{a^2b}{(x-b)^3}+1\right)}$$

Calculus Problem 4: Points with Slope ±1

Problem: Find points where the tangent slope is ±1.

Solution:

Set $\frac{dy}{dx}=\pm1$:

$-\frac{x}{y}\left(\frac{a^2b}{(x-b)^3}+1\right)=\pm1$

This gives two equations. For slope = 1:
$-\frac{x}{y}\left(\frac{a^2b}{(x-b)^3}+1\right)=1$

And for slope = -1:
$-\frac{x}{y}\left(\frac{a^2b}{(x-b)^3}+1\right)=-1$

These equations, combined with the original curve equation, can be solved numerically for specific values of a and b, or we can find special cases.

For example, when a = b, the curve becomes:
$(x-a)^2(x^2+y^2)-a^2x^2=0$

In this case, we might find analytic solutions for the slope conditions.

Advanced Properties and Applications

The Four Cases of Conchoids

Depending on the ratio a/b, the conchoid exhibits different behaviors:

1. **a < -b**: Two smooth branches
2. **-b < a < 0**: Loop appears, right branch has a cusp
3. **a = 0**: Degenerates to a circle and the pole
4. **a > 0**: Two smooth branches again

Connection to Other Curves

- When a = 0, the conchoid becomes a circle
- When b = 0, it becomes a pair of lines through the origin
- Special cases relate to the limaçon and cardioid

Optical Properties

The conchoid has interesting reflective properties that make it useful in designing mirrors and lenses with specific focal characteristics.

The conchoid of Nicomedes represents one of the earliest examples of using auxiliary curves to solve classical construction problems, bridging the gap between Greek geometry and modern algebraic methods.

Curve 10: The Hypocycloid—Elegant Curves from Rolling Circles

Parametric Equations:
$x(t) = (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right)$
$y(t) = (a-b)\sin t - b\sin\left(\frac{a-b}{b}t\right)$

Special Cases:

- **Astroid:** $a = 4b$ (4 cusps)
- **Deltoid:** $a = 3b$ (3 cusps)

Historical Context and Geometric Construction

The Rolling Circle Inside Another

A hypocycloid is generated by tracing a point on a circle of radius b that rolls internally on a fixed circle of radius a. Unlike epicycloids (external rolling), hypocycloids are formed when the rolling circle moves inside the fixed circle.

Mathematical Derivation

Consider:

- Fixed circle: center at origin, radius a
- Rolling circle: radius b, initially touching fixed circle at point $(a, 0)$
- Point P starts at $(a, 0)$

As the rolling circle rotates through angle ϕ, it travels along the inside of the fixed circle. The arc lengths must be equal:
$a\phi = b\theta \Rightarrow \theta = \frac{a}{b}\phi$

The center of the rolling circle moves to:
$C = ((a-b)\cos\phi, (a-b)\sin\phi)$

Relative to its center, point P is at:
$P_{\text{relative}} = (b\cos(\theta - \phi), b\sin(\theta - \phi))$

Substituting $\theta = \frac{a}{b}\phi$:
$P_{\text{relative}} = \left(b\cos\left(\frac{a}{b}\phi - \phi\right), b\sin\left(\frac{a}{b}\phi - \phi\right)\right)$
$= \left(b\cos\left(\frac{a-b}{b}\phi\right), b\sin\left(\frac{a-b}{b}\phi\right)\right)$

Adding the center position:
$x = (a - b)\cos\phi + b\cos\left(\frac{a-b}{b}\phi\right)$
$y = (a - b)\sin\phi + b\sin\left(\frac{a-b}{b}\phi\right)$

Letting $t = \phi$, we obtain the standard form.

Calculus Problem 1: Arc Length for Integer Ratio

Problem: Find the length of the hypocycloid when $a = (n+1)b$ where n is an integer.

Solution:

When $a = (n+1)b$, the parametric equations become:
$x(t) = nb\cos t + b\cos(nt)$
$y(t) = nb\sin t - b\sin(nt)$

Compute derivatives:
$\frac{dx}{dt} = -nb\sin t - nb\sin(nt)$
$\frac{dy}{dt} = nb\cos t - nb\cos(nt)$

Now compute the arc length integrand:
$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = n^2b^2[\sin^2 t + 2\sin t\sin(nt) + \sin^2(nt)]$
$+n^2b^2[\cos^2 t - 2\cos t\cos(nt) + \cos^2(nt)]$

Combine terms:
$= n^2b^2[(\sin^2 t + \cos^2 t) + (\sin^2(nt) + \cos^2(nt)) + 2(\sin t\sin(nt) - \cos t\cos(nt))]$

Using trigonometric identities:
$\sin^2 t + \cos^2 t = 1$
$\sin^2(nt) + \cos^2(nt) = 1$
$\sin t\sin(nt) - \cos t\cos(nt) = -\cos(t + nt) = -\cos((n+1)t)$

Therefore:
$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = n^2b^2[2 - 2\cos((n+1)t)]$
$= 2n^2b^2[1 - \cos((n+1)t)]$

Using $1 - \cos\theta = 2\sin^2(\theta/2)$:
$= 4n^2b^2\sin^2\left(\frac{n+1}{2}t\right)$

Thus:
$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=2nb\left|\sin\left(\frac{n+1}{2}t\right)\right|$$

The complete curve is traced as t goes from 0 to 2π. Due to symmetry:

$$L=\int_0^{2\pi}2nb\left|\sin\left(\frac{n+1}{2}t\right)\right|dt$$

Since there are $n+1$ lobes, we compute:

$$L=2nb\cdot 2(n+1)\int_0^{\pi/(n+1)}\sin\left(\frac{n+1}{2}t\right)dt$$

$$=4nb(n+1)\left[-\frac{2}{n+1}\cos\left(\frac{n+1}{2}t\right)\right]_0^{\pi/(n+1)}$$

$$=8nb\left[-\cos\left(\frac{n+1}{2}\cdot\frac{\pi}{n+1}\right)+\cos 0\right]$$

$$=8nb\left[-\cos\left(\frac{\pi}{2}\right)+1\right]=8nb(0+1)=8nb$$

Since $a=(n+1)b$, we have $b=\frac{a}{n+1}$, so:
$L=8n\cdot\frac{a}{n+1}=\frac{8na}{n+1}$

$$\boxed{L=\frac{8na}{n+1}}$$

Special Cases:

- Deltoid ($n=2$): $L=\frac{16a}{3}$
- Astroid ($n=3$): $L=\frac{24a}{4}=6a$

Calculus Problem 2: Area Enclosed by Hypocycloid

Problem: Find the area enclosed by the hypocycloid when $a=(n+1)b$.

Solution:

Using the parametric area formula (Green's Formula, will be covered in MATH2550J):
$A=\frac{1}{2}\oint(x\,dy-y\,dx)$

From our derivatives:
$x\,dy-y\,dx=[nb\cos t+b\cos(nt)][nb\cos t-nb\cos(nt)]dt$
$-[nb\sin t-b\sin(nt)][-nb\sin t-nb\sin(nt)]dt$

This simplifies to:
$x\,dy - y\,dx = n^2b^2[1 - \cos((n+1)t)]dt$

Therefore:
$A = \frac{1}{2}\int_0^{2\pi} n^2b^2[1 - \cos((n+1)t)]dt$

$= \frac{n^2b^2}{2}\int_0^{2\pi}[1 - \cos((n+1)t)]dt$

The integral of $\cos((n+1)t)$ over $[0, 2\pi]$ is zero, so:
$A = \frac{n^2b^2}{2} \cdot 2\pi = \pi n^2b^2$

Since $b = \frac{a}{n+1}$:
$A = \pi n^2\left(\frac{a}{n+1}\right)^2 = \frac{\pi n^2a^2}{(n+1)^2}$

$$\boxed{A = \frac{\pi n^2a^2}{(n+1)^2}}$$

Special Cases:

- Deltoid ($n = 2$): $A = \frac{4\pi a^2}{9}$
- Astroid ($n = 3$): $A = \frac{9\pi a^2}{16}$

Calculus Problem 3: Tangent Line Properties

Problem: Find the slope of the tangent line and locate cusps.

Solution:

The slope is given by:
$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{nb\cos t - nb\cos(nt)}{-nb\sin t - nb\sin(nt)}$

Simplify:
$\frac{dy}{dx} = \frac{\cos t - \cos(nt)}{-\sin t - \sin(nt)}$

Using sum-to-product identities:
$\cos t - \cos(nt) = -2\sin\left(\frac{1+n}{2}t\right)\sin\left(\frac{1-n}{2}t\right)$
$\sin t + \sin(nt) = 2\sin\left(\frac{1+n}{2}t\right)\cos\left(\frac{1-n}{2}t\right)$

Therefore:
$\frac{dy}{dx} = \frac{-2\sin\left(\frac{1+n}{2}t\right)\sin\left(\frac{1-n}{2}t\right)}{-2\sin\left(\frac{1+n}{2}t\right)\cos\left(\frac{1-n}{2}t\right)} = \tan\left(\frac{1-n}{2}t\right)$

$$\boxed{\frac{dy}{dx} = \tan\left(\frac{1-n}{2}t\right)}$$

Cusps occur when both derivatives are zero:
$\frac{dx}{dt} = -nb[\sin t + \sin{(nt)}] = 0$
$\frac{dy}{dt} = nb[\cos t - \cos{(nt)}] = 0$

This happens when $t = \frac{2\pi k}{n+1}$ for $k = 0, 1, \ldots, n$

At these points, the curve has sharp corners where the tangent direction changes abruptly.

Advanced Properties and Applications

Gear Design and Engineering

Hypocycloids are used in gear design, particularly in planetary gear systems and cycloidal speed reducers. The hypocycloidal shape provides excellent torque transmission with minimal backlash.

The Hypocycloid and Elliptic Functions

The arc length of a hypocycloid involves elliptic integrals, connecting these curves to the rich theory of elliptic functions and modular forms.

Connection to Roulettes

Hypocycloids belong to the family of roulettes—curves generated by rolling one curve on another. This connects them to cycloids, epicycloids, and other important curves in kinematics.

Astroid as Envelope

The astroid (4-cusped hypocycloid) appears as the envelope of a family of ellipses with the same area, or as the path of a point on a circle rolling inside another circle.

From the astroid's elegant envelope properties to the hypocycloid's mechanical applications, these curves continue to inspire mathematicians, engineers, and scientists with their timeless mathematical elegance.