Integration Techniques 积佬修炼手册

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Introduction

Integration is the art of finding the whole from knowledge of its parts. It requires creativity, pattern recognition, and often, a touch of ingenuity that transcends mechanical application. Unlike differentiation, where the chain rule, product rule, and quotient rule provide clear procedures, integration has no universal algorithm. Two integrals that appear similar may require completely different techniques. Success in integration comes from:

1. **Pattern Recognition**: Seeing which technique applies
2. **Strategic Thinking**: Knowing when to transform the problem
3. **Algebraic Flexibility**: Manipulating expressions to reveal structure
4. **Persistence**: Trying multiple approaches when the first doesn't work

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Part I: Foundational Methods

积分很简单的，至少在这一部分如此。

1. Direct Substitution Methods

The substitution rule, also called u-substitution or change of variables, states:

$$\int f(g(x))g'(x)dx = \int f(u)du \text{ where } u = g(x)$$

How to Choose a Substitution:

- Look for a function whose derivative appears elsewhere in the integrand
- Identify the "most complicated" part that can be simplified
- Check if the remaining terms can be expressed in terms of the new variable

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1.1 Algebraic Substitution

Basic Principle: When the integrand contains a specific algebraic expression repeatedly or when its derivative is present, substitute that expression.

Example 1.1.1: Evaluate $\int \frac{2x+3}{x^2+3x+5}dx$

Solution:

Notice that the numerator $2x + 3$ is exactly the derivative of the denominator $x^2 + 3x + 5$.

Let $u = x^2 + 3x + 5$

Then $du = (2x + 3)dx$

The integral becomes: $\int \frac{2x+3}{x^2+3x+5}dx = \int \frac{du}{u} = \ln|u| + C = \ln|x^2 + 3x + 5| + C$

Verification: Differentiate the answer: $\frac{d}{dx}\ln|x^2 + 3x + 5| = \frac{2x+3}{x^2+3x+5}$ ✓

Key Pattern: When you see $\int \frac{f'(x)}{f(x)}dx$, the answer is always $\ln|f(x)| + C$.

Example 1.1.2: Evaluate $\int \frac{3x^2-2x+1}{x^3-x^2+x-1}dx$

Solution:

Check if the numerator is the derivative of the denominator:
$\frac{d}{dx}(x^3 - x^2 + x - 1) = 3x^2 - 2x + 1$ ✓

Therefore: $\int \frac{3x^2-2x+1}{x^3-x^2+x-1}dx = \ln|x^3 - x^2 + x - 1| + C$

Common Mistake: Don't forget the absolute value signs in logarithms! The domain of $\ln(x)$ is only $x > 0$, but $\ln|x|$ is defined for all $x \neq 0$.

1.2 Exponential Unification

When hyperbolic functions appear, converting them to exponentials often simplifies the problem dramatically. Hyperbolic functions are defined as:
$\sinh x = \frac{e^x-e^{-x}}{2}, \quad \cosh x = \frac{e^x+e^{-x}}{2}$

Exponential functions have the beautiful property that $\frac{d}{dx}e^{ax} = ae^{ax}$, making them easy to integrate.

Example 1.2.1: Evaluate $\int e^x \cosh x \, dx$

Solution:

$\int e^x \cosh x \, dx = \int e^x \left(\frac{e^x+e^{-x}}{2}\right)dx$

$= \frac{1}{2}\int(e^{2x} + 1)dx = \frac{1}{2}\left(\frac{e^{2x}}{2} + x\right) + C = \frac{e^{2x}}{4} + \frac{x}{2} + C$

Similarly, $\int e^x \sinh x \, dx = \int e^x \left(\frac{e^x-e^{-x}}{2}\right)dx$

$= \frac{1}{2}\int(e^{2x} - 1)dx = \frac{1}{2}\left(\frac{e^{2x}}{2} - x\right) + C = \frac{e^{2x}}{4} - \frac{x}{2} + C$

1.3 Power Substitution

When the integrand contains a polynomial expression raised to a power, and the derivative of the inner polynomial is present (up to a constant factor), use substitution.

Example 1.3.1: Evaluate $\int \frac{x^3}{(x^2+1)^2} dx$

Solution:

Let $u = x^2 + 1$, then $x^2 = u - 1$ and $2x\,dx = du$, so $x\,dx = \frac{du}{2}$

Also, $x^2 = u - 1$: $\int \frac{x^3}{(x^2+1)^2} dx = \int \frac{x^2 \cdot x\,dx}{(x^2+1)^2}$

$$= \int \frac{(u-1)}{u^2} \cdot \frac{du}{2} = \frac{1}{2} \int \frac{u-1}{u^2} du$$

$$= \frac{1}{2} \int \left[\frac{1}{u} - \frac{1}{u^2}\right] du$$

$$= \frac{1}{2} \left[\ln |u| + \frac{1}{u}\right] + C$$

$$= \frac{\ln(x^2+1)}{2} + \frac{1}{2(x^2+1)} + C$$

Example 1.3.2: Evaluate $\int \frac{x^5}{(x^2+4)^2} dx$

Solution:

Let $u = x^2 + 4$, then $x^2 = u - 4$ and $x\,dx = \frac{du}{2}$:

$$\int \frac{x^5}{(x^2+4)^2} dx = \int \frac{x^4 \cdot x\,dx}{(x^2+4)^2} = \int \frac{(x^2)^2}{u^2} \cdot \frac{du}{2}$$

$$= \frac{1}{2} \int \frac{(u-4)^2}{u^2} du = \frac{1}{2} \int \frac{u^2 - 8u + 16}{u^2} du$$

$$= \frac{1}{2} \int \left[1 - \frac{8}{u} + \frac{16}{u^2}\right] du$$

$$= \frac{1}{2} \left[u - 8 \ln |u| - \frac{16}{u}\right] + C$$

$$= \frac{x^2+4}{2} - 4 \ln\left(x^2 + 4\right) - \frac{8}{x^2+4} + C$$

$$= \frac{x^2}{2} + 2 - 4 \ln\left(x^2 + 4\right) - \frac{8}{x^2+4} + C$$

Absorbing the constant 2 into C:

Answer: $\boxed{\frac{x^2}{2} - 4\ln\left(x^2+4\right) - \frac{8}{x^2+4} + C}$

Example 1.3.3: Evaluate $\int \frac{x^7}{(x^2+1)^3}dx$

Solution:

Let $u = x^2+1$: $\int \frac{x^7}{(x^2+1)^3}dx = \int \frac{(x^2)^3 \cdot x\,dx}{(x^2+1)^3} = \frac{1}{2}\int \frac{(u-1)^3}{u^3}du$

Expand $(u-1)^3 = u^3 - 3u^2 + 3u - 1$:

$= \frac{1}{2}\int \frac{u^3-3u^2+3u-1}{u^3}du = \frac{1}{2}\int \left[1 - \frac{3}{u} + \frac{3}{u^2} - \frac{1}{u^3}\right]du$

$= \frac{1}{2}\left[u - 3\ln|u| - \frac{3}{u} + \frac{1}{2u^2}\right] + C$

$= \frac{x^2+1}{2} - \frac{3\ln\left(x^2+1\right)}{2} - \frac{3}{2(x^2+1)} + \frac{1}{4(x^2+1)^2} + C$

1.4 Weierstrass (Universal t-) Substitution

It's called "universal" because it can, in principle, convert any rational function of trigonometric functions into a rational function of a single variable.

The Substitution: $t = \tan\frac{x}{2}$

The Transformation Formulas: $\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2dt}{1+t^2}$

From $t = \tan\frac{x}{2}$, we have $\frac{x}{2} = \arctan t$, so $x = 2\arctan t$. Differentiating: $dx = \frac{2dt}{1+t^2}$ ✓

For sine and cosine, use the double-angle formulas: $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

From the right triangle where $\tan\frac{x}{2} = t$:

- Opposite side: t
- Adjacent side: 1
- Hypotenuse: $\sqrt{1+t^2}$

Therefore: $\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}, \quad \cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\cdot\frac{t}{\sqrt{1+t^2}}\cdot\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$ ✓

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2}$ ✓

Example 1.4.1: Evaluate $\int\frac{dx}{1+\sin x+\cos x}$

Solution:

Let $t = \tan\frac{x}{2}$

Then: $\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2dt}{1+t^2}$

Substitute: $\int\frac{dx}{1+\sin x+\cos x} = \int\frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$

Simplify the denominator: $1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2} = \frac{(1+t^2)+2t+(1-t^2)}{1+t^2} = \frac{2+2t}{1+t^2} = \frac{2(1+t)}{1+t^2}$

Therefore: $\int\frac{\frac{2dt}{1+t^2}}{\frac{2(1+t)}{1+t^2}} = \int\frac{2dt}{2(1+t)} = \int\frac{dt}{1+t}$

Example 1.4.2: Evaluate $\int\frac{dx}{5+3\cos x}$

Solution:

Let $t = \tan\frac{x}{2}$

$\int\frac{dx}{5+3\cos x} = \int\frac{\frac{2dt}{1+t^2}}{5+3\cdot\frac{1-t^2}{1+t^2}}$

Simplify the denominator: $5+3\cdot\frac{1-t^2}{1+t^2} = \frac{5(1+t^2)+3(1-t^2)}{1+t^2} = \frac{5+5t^2+3-3t^2}{1+t^2} = \frac{8+2t^2}{1+t^2}$

Therefore: $\int\frac{\frac{2dt}{1+t^2}}{\frac{8+2t^2}{1+t^2}} = \int\frac{2dt}{8+2t^2} = \int\frac{dt}{4+t^2}$

This is a standard arctangent form: $\int\frac{dt}{4+t^2} = \frac{1}{2}\arctan\frac{t}{2}+C$

Substitute back: $= \frac{1}{2}\arctan\left(\frac{\tan(x/2)}{2}\right)+C$

Complex Alternative: Contour Integration Preview

For those familiar with complex analysis, there's an elegant alternative using the substitution $z = e^{ix}$ on the unit circle:

$$\cos x = \frac{z+z^{-1}}{2}, \quad dx = \frac{dz}{iz}$$

For definite integrals over $[0, 2\pi]$, this transforms the problem into a contour integral around the unit circle in the complex plane. While this is beyond our current scope, it's worth noting that the Weierstrass substitution is essentially the "real version" of this complex technique.

Example 1.4.3: Evaluate $\int \frac{\sin x}{2+\cos x} dx$

Solution:

For this problem, Weierstrass is possible but not optimal.

Notice that: $\frac{d}{dx}(\cos x) = -\sin x$. So this is a logarithmic derivative form.

Let $u = 2 + \cos x$

Then $du = -\sin x \ dx$

$$\int \frac{\sin x}{2+\cos x} dx = -\int \frac{du}{u} = -\ln|u| + C$$

$$= -\ln|2 + \cos x| + C$$

Always check for simpler methods before applying Weierstrass. The universal substitution is powerful but often overkill.

1.5 Rationalizing Radicals

When an integrand contains multiple radicals with different roots, we can often rationalize by substituting the LCD (least common denominator) of the root indices.

If you have $\sqrt{x}$ and $\sqrt[3]{x}$, use $t^6 = x$ (since $\text{lcm}(2, 3) = 6$)

- If you have $\sqrt[3]{f(x)}$ and $\sqrt[4]{f(x)}$, use $t^{12} = f(x)$
- After substitution, you'll get a rational function of t

Example 1.5.1: Evaluate $\int \frac{dx}{\sqrt{1-x}+\sqrt[3]{1-x}}$

Solution:

We have radicals with indices 2 and 3. The LCM is 6.

Let $t^6 = 1 - x$

Then $x = 1 - t^6$ and $dx = -6t^5 dt$

Also: $\sqrt{1-x} = (t^6)^{1/2} = t^3$ $\sqrt[3]{1-x} = (t^6)^{1/3} = t^2$

Substitute: $\int \frac{dx}{\sqrt{1-x}+\sqrt[3]{1-x}} = \int \frac{-6t^5 dt}{t^3+t^2}$

$= -6 \int \frac{t^5}{t^2(t+1)} dt = -6 \int \frac{t^3}{t+1} dt$

Now perform polynomial long division: $\frac{t^3}{t+1} = t^2 - t + 1 - \frac{1}{t+1}$

Integrate: $-6 \int \left(t^2 - t + 1 - \frac{1}{t+1}\right) dt$

$= -6 \left(\frac{t^3}{3} - \frac{t^2}{2} + t - \ln|t+1| \right) + C$

$= -2t^3 + 3t^2 - 6t + 6 \ln|t+1| + C$

Substitute back $t = (1-x)^{1/6}$:

$= -2\sqrt{1-x} + 3\sqrt[3]{1-x} - 6(1-x)^{1/6} + 6 \ln\left|(1-x)^{1/6} + 1\right| + C$

This looks complicated, but we can verify by differentiating.

Example 1.5.2: Evaluate $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

Solution:

Here we have $x^{1/2}$ and $x^{1/3}$, so $\operatorname{lcm}(2,3) = 6$.

Let $t = x^{1/6}$, so $x = t^6$ and $dx = 6t^5 dt$

Then: $\sqrt{x} = t^3, \quad \sqrt[3]{x} = t^2$

$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} = \int \frac{6t^5 dt}{t^3+t^2} = 6 \int \frac{t^5}{t^2(t+1)} dt$

$= 6 \int \frac{t^3}{t+1} dt$

Using the same long division as before: $= 6\int\left(t^2 - t + 1 - \frac{1}{t+1}\right)dt$

$$= 6\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \ln|t+1|\right) + C$$

$$= 2t^3 - 3t^2 + 6t - 6\ln|t+1| + C$$

Substitute $t = x^{1/6}$:

$$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln|x^{1/6} + 1| + C$$

Example 1.5.3: Evaluate $\int \frac{\sqrt{x}dx}{1+\sqrt[3]{x}}$

Solution:

Let $t = x^{1/6}$, then $x = t^6$, $dx = 6t^5dt$

$$\sqrt{x} = t^3, \quad \sqrt[3]{x} = t^2$$

$$\int \frac{\sqrt{x}dx}{1+\sqrt[3]{x}} = \int \frac{t^3 \cdot 6t^5dt}{1+t^2} = 6\int \frac{t^8}{1+t^2}dt$$

Polynomial division of $\frac{t^8}{1+t^2}$:

$$t^8 = (1+t^2)(t^6 - t^4 + t^2 - 1) + 1$$

So: $\frac{t^8}{1+t^2} = t^6 - t^4 + t^2 - 1 + \frac{1}{1+t^2}$

$$6\int\left(t^6 - t^4 + t^2 - 1 + \frac{1}{1+t^2}\right)dt$$

$$= 6\left(\frac{t^7}{7} - \frac{t^5}{5} + \frac{t^3}{3} - t + \arctan t\right) + C$$

Substitute back $t = x^{1/6}$:

$$= 6\left(\frac{x^{7/6}}{7} - \frac{x^{5/6}}{5} + \frac{x^{1/2}}{3} - x^{1/6} + \arctan\left(x^{1/6}\right)\right) + C$$

1.6 Quadratic Expressions

Integrals involving $\sqrt{ax^2 + bx + c}$ can often be simplified by first completing the square, then applying a trigonometric substitution.

The Three Standard Trigonometric Substitutions:

1. **For $\sqrt{a^2 - x^2}$:** Use $x = a\sin\theta$
   - Then $\sqrt{a^2 - x^2} = a\cos\theta$
   - Range: $\theta \in [-\pi/2, \pi/2]$
2. **For $\sqrt{x^2 + a^2}$:** Use $x = a\tan\theta$
   - Then $\sqrt{x^2 + a^2} = a\sec\theta$
   - Range: $\theta \in (-\pi/2, \pi/2)$
3. **For $\sqrt{x^2 - a^2}$:** Use $x = a\sec\theta$
   - Then $\sqrt{x^2 - a^2} = a\tan\theta$
   - Range: $\theta \in [0, \pi/2) \cup (\pi/2, \pi]$

Each substitution uses a Pythagorean identity:

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $\sec^2\theta - 1 = \tan^2\theta$

Example 1.6.1: Evaluate $\int \frac{dx}{\sqrt{8x - x^2}}$

Solution:

Step 1: Complete the Square

$8x - x^2 = -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) = -(x-4)^2 + 16 = 16 - (x-4)^2$

So: $\int \frac{dx}{\sqrt{8x-x^2}} = \int \frac{dx}{\sqrt{16-(x-4)^2}}$

Step 2: Substitution

Let $u = x - 4$, then $du = dx$

$= \int \frac{du}{\sqrt{16-u^2}}$

This is the form $\sqrt{a^2 - u^2}$ with $a = 4$.

Step 3: Trigonometric Substitution

Let $u = 4\sin\theta$, then $du = 4\cos\theta, d\theta$

$\sqrt{16-u^2} = \sqrt{16-16\sin^2\theta} = \sqrt{16(1-\sin^2\theta)} = 4\cos\theta$

(We take the positive root since $\cos\theta \geq 0$ for $\theta \in [-\pi/2, \pi/2]$)

$\int \frac{4\cos\theta, d\theta}{4\cos\theta} = \int d\theta = \theta + C$

Step 4: Back-Substitution

From $u = 4\sin\theta$, we have $\sin\theta = \frac{u}{4}$

Therefore $\theta = \arcsin\frac{u}{4} = \arcsin\frac{x-4}{4}$

Answer: $\boxed{\arcsin\left(\frac{x-4}{4}\right) + C}$

Verification: $\frac{d}{dx}\arcsin\left(\frac{x-4}{4}\right) = \frac{1}{\sqrt{1-\left(\frac{x-4}{4}\right)^2}} \cdot \frac{1}{4} = \frac{1}{\sqrt{16-(x-4)^2}} = \frac{1}{\sqrt{8x-x^2}}$ ✓

Example 1.6.2: Evaluate $\int \frac{dx}{\sqrt{x^2+6x+13}}$

Solution:

Step 1: Complete the Square

$x^2 + 6x + 13 = (x^2 + 6x + 9) + 4 = (x+3)^2 + 4$

$\int \frac{dx}{\sqrt{x^2+6x+13}} = \int \frac{dx}{\sqrt{(x+3)^2+4}}$

Step 2: Substitution

Let $u = x + 3$, then $du = dx$

$= \int \frac{du}{\sqrt{u^2+4}}$

This is the form $\sqrt{u^2 + a^2}$ with $a = 2$.

Step 3: Trigonometric Substitution

Let $u = 2\tan\theta$, then $du = 2\sec^2\theta, d\theta$

$\sqrt{u^2+4} = \sqrt{4\tan^2\theta+4} = 2\sqrt{\tan^2\theta+1} = 2\sec\theta$

$\int \frac{2\sec^2\theta, d\theta}{2\sec\theta} = \int \sec\theta, d\theta$

Recall: $\int \sec\theta, d\theta = \ln|\sec\theta + \tan\theta| + C$

Step 4: Back-Substitution

From $u = 2\tan\theta$, we have $\tan\theta = \frac{u}{2}$

From a right triangle: $\sec\theta = \frac{\sqrt{u^2+4}}{2}$

$\ln\left|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}\right| + C = \ln|\sqrt{u^2+4} + u| - \ln 2 + C$

Since $-\ln 2$ is just a constant, we can absorb it into C:

$= \ln|\sqrt{u^2+4} + u| + C$

Substitute $u = x+3$:

$= \ln\left|\sqrt{x^2+6x+13} + x + 3\right| + C$

Alternative Form Using Inverse Hyperbolic:

Note that $\ln\left(\sqrt{u^2+4} + u\right) = \operatorname{arsinh}(u/2)$

So the answer can also be written as: $\operatorname{arsinh}\left(\frac{x+3}{2}\right) + C$

Example 1.6.3: Evaluate $\int \sqrt{4x - x^2}\, dx$

Solution:

Step 1: Complete the Square

$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4 - 4) = -(x-2)^2 + 4 = 4 - (x-2)^2$

$\int \sqrt{4x - x^2}\, dx = \int \sqrt{4 - (x-2)^2}\, dx$

Step 2: Substitution

Let $u = x - 2$, then $du = dx$

$= \int \sqrt{4 - u^2}, du$

Step 3: Trigonometric Substitution

Let $u = 2\sin\theta$, then $du = 2\cos\theta, d\theta$

$\sqrt{4 - u^2} = 2\cos\theta$

$\int 2\cos\theta \cdot 2\cos\theta, d\theta = 4\int \cos^2\theta, d\theta$

Use the identity: $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$= 4\int \frac{1+\cos 2\theta}{2} d\theta = 2\int (1 + \cos 2\theta) d\theta$

$= 2\left(\theta + \frac{\sin 2\theta}{2}\right) + C = 2\theta + \sin 2\theta + C$

$= 2\theta + 2\sin\theta\cos\theta + C$

Step 4: Back-Substitution

From $u = 2\sin\theta$: $\sin\theta = \frac{u}{2}$, $\theta = \arcsin\frac{u}{2}$

$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{u^2}{4}} = \frac{\sqrt{4-u^2}}{2}$

$= 2\arcsin\frac{u}{2} + 2 \cdot \frac{u}{2} \cdot \frac{\sqrt{4-u^2}}{2} + C$

$= 2\arcsin\frac{u}{2} + \frac{u\sqrt{4-u^2}}{2} + C$

Substitute $u = x - 2$:

$= 2\arcsin\frac{x-2}{2} + \frac{(x-2)\sqrt{4x-x^2}}{2} + C$

This integral represents the area under the curve $y = \sqrt{4x - x^2}$, which is a semicircle of radius 2 centered at $(2, 0)$. The answer combines an angular part ($\arcsin$) and a triangular part, reflecting the geometry of circular sectors.

Example 1.6.4: Evaluate $\int \frac{x^2 dx}{\sqrt{9-x^2}}$

Solution:

This is the form $\sqrt{a^2 - x^2}$ with $a = 3$.

Let $x = 3\sin\theta$, then $dx = 3\cos\theta\, d\theta$

$\sqrt{9 - x^2} = 3\cos\theta$

$\int \frac{(3\sin\theta)^2 \cdot 3\cos\theta\, d\theta}{3\cos\theta} = \int 9\sin^2\theta\, d\theta$

Use $\sin^2\theta = \frac{1-\cos 2\theta}{2}$:

$= 9\int \frac{1-\cos 2\theta}{2} d\theta = \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$

$= \frac{9\theta}{2} - \frac{9\sin 2\theta}{4} + C$

Use $\sin 2\theta = 2\sin\theta\cos\theta$:

$= \frac{9\theta}{2} - \frac{9\sin\theta\cos\theta}{2} + C$

Back-substitution:

From $x = 3\sin\theta$: $\sin\theta = \frac{x}{3}$, $\theta = \arcsin\frac{x}{3}$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$= \frac{9}{2}\arcsin\frac{x}{3} - \frac{9}{2} \cdot \frac{x}{3} \cdot \frac{\sqrt{9-x^2}}{3} + C$

$= \frac{9}{2}\arcsin\frac{x}{3} - \frac{x\sqrt{9-x^2}}{2} + C$

1.7 Conjugate Methods

When an integrand contains sums or differences involving radicals or trigonometric expressions, multiplying by a conjugate can sometimes rationalize the expression and reveal a simpler structure.

Basic Principle: For expressions of the form $a + b$ or $a - b$, multiply numerator and denominator by the conjugate $a - b$ or $a + b$ respectively, using the algebraic identity: $(a + b)(a - b) = a^2 - b^2$

This is particularly useful when:

- Radicals appear in sums or differences ($\sqrt{A} \pm \sqrt{B}$)

- The resulting difference of squares simplifies the problem
- Direct substitution is not immediately apparent

Example 1.7.1: Evaluate $\displaystyle\int \frac{x\,dx}{\sqrt{x}-\sqrt{x-1}}$

Solution:

Multiply by the conjugate $\sqrt{x}+\sqrt{x-1}$:

$\int \frac{x\ dx}{\sqrt{x}-\sqrt{x-1}} = \int \frac{x(\sqrt{x}+\sqrt{x-1})dx}{(\sqrt{x}-\sqrt{x-1})(\sqrt{x}+\sqrt{x-1})}$

$= \int \frac{x(\sqrt{x}+\sqrt{x-1})dx}{x-(x-1)} = \int x(\sqrt{x}+\sqrt{x-1})dx$

$= \int (x^{3/2}+x\sqrt{x-1})dx$

For the first term: $\int x^{3/2}dx = \frac{2x^{5/2}}{5}$

For the second term, let $u = x-1$, then $x = u+1$ and $dx = du$:
$\int x\sqrt{x-1}, dx = \int (u+1)\sqrt{u}\,du = \int (u^{3/2}+u^{1/2})du$

$= \frac{2u^{5/2}}{5} + \frac{2u^{3/2}}{3} = \frac{2(x-1)^{5/2}}{5} + \frac{2(x-1)^{3/2}}{3}$

Example 1.7.2: Evaluate $\displaystyle\int \frac{dx}{1+\sqrt{x+1}}$

Solution:

Multiply by the conjugate $1-\sqrt{x+1}$:

$\int \frac{dx}{1+\sqrt{x+1}} = \int \frac{dx\cdot(1-\sqrt{x+1})}{(1+\sqrt{x+1})(1-\sqrt{x+1})}$

$= \int \frac{(1-\sqrt{x+1})dx}{1-(x+1)} = \int \frac{(1-\sqrt{x+1})dx}{-x}$

$= \int \frac{\sqrt{x+1}-1}{x}dx = \int \frac{\sqrt{x+1}}{x}dx - \int \frac{dx}{x}$

The second integral is $\ln|x|$. For the first, let $u = x+1$, so $x = u-1$ and $dx = du$:

$\int \frac{\sqrt{x+1}}{x}dx = \int \frac{\sqrt{u}}{u-1}du$

Let $t = \sqrt{u}$, then $u = t^2$ and $du = 2t\,dt$:

$$= \int \frac{t \cdot 2t\,dt}{t^2-1} = 2\int \frac{t^2}{t^2-1}dt$$

$$= 2\int \left(1 + \frac{1}{t^2-1}\right)dt = 2\int dt + 2\int \frac{dt}{t^2-1}$$

Using partial fractions on $\frac{1}{t^2-1} = \frac{1}{(t-1)(t+1)} = \frac{1/2}{t-1} - \frac{1/2}{t+1}$:

$$= 2t + 2 \cdot \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = 2t + \ln\left|\frac{t-1}{t+1}\right| + C$$

Back-substitute $t = \sqrt{x+1}$:

$$= 2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| - \ln|x| + C$$

Conjugate multiplication is also useful when dealing with sums and differences of trigonometric functions.

Example 1.7.3: Evaluate $\displaystyle\int \frac{dx}{1-\cos x}$

Solution:

Multiply by the conjugate $1 + \cos x$:

$$\int \frac{dx}{1-\cos x} = \int \frac{(1+\cos x)dx}{(1-\cos x)(1+\cos x)} = \int \frac{(1+\cos x)dx}{1-\cos^2 x}$$

$$= \int \frac{(1+\cos x)dx}{\sin^2 x}$$

Split the integral: $= \int \frac{dx}{\sin^2 x} + \int \frac{\cos x\,dx}{\sin^2 x}$

$$= \int \csc^2 x\,dx + \int \frac{\cos x}{\sin^2 x}dx$$

The first integral: $\int \csc^2\,dx = -\cot x + C$

For the second, let $u = \sin x$, $du = \cos x\,dx$:
$\int \frac{\cos x}{\sin^2 x}dx = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{\sin x} = -\csc x$

Answer: $\boxed{-\cot x - \csc x + C}$

Alternative form: This can also be written as $-\cot x - \csc x = -\frac{\cos x+1}{\sin x} + C$

Example 1.7.4: Evaluate $\displaystyle\int \frac{dx}{1+\sin x}$

Solution:

Multiply by the conjugate $1-\sin x$:

$\int \frac{dx}{1+\sin x} = \int \frac{(1-\sin x)dx}{(1+\sin x)(1-\sin x)} = \int \frac{(1-\sin x)dx}{1-\sin^2 x}$

$= \int \frac{(1-\sin x)dx}{\cos^2 x}$

Split the integral: $= \int \frac{dx}{\cos^2 x} - \int \frac{\sin x\, dx}{\cos^2 x}$

$= \int \sec^2 x\, dx - \int \frac{\sin x}{\cos^2 x} dx$

The first integral: $\int \sec^2 x\, dx = \tan x + C$

For the second, let $u = \cos x$, $du = -\sin x\, dx$:
$\int \frac{\sin x}{\cos^2 x} dx = -\int \frac{du}{u^2} = \frac{1}{u} = \frac{1}{\cos x} = \sec x$

Answer: $\boxed{\tan x - \sec x + C}$

Example 1.7.5: Evaluate $\displaystyle\int \frac{\sin x}{1+\cos x} dx$

Solution:

Method 1: Direct Substitution

Let $u = 1+\cos x$, then $du = -\sin x\, dx$:

$\int \frac{\sin x}{1+\cos x} dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|1+\cos x| + C$

Method 2: Conjugate Multiplication

Multiply by $1-\cos x$: $\int \frac{\sin x(1-\cos x)}{(1+\cos x)(1-\cos x)} dx = \int \frac{\sin x(1-\cos x)}{\sin^2 x} dx$

$= \int \frac{1-\cos x}{\sin x} dx = \int (\csc x - \cot x) dx$

$= -\ln|\csc x + \cot x| - \ln|\sin x| + C$

Both answers are equivalent (verify by using trigonometric identities).

While the conjugate method works, direct substitution is simpler when the derivative of the denominator appears in the numerator.

When to use conjugate multiplication:

- Sums or differences of radicals in the denominator
- $1 \pm \sin x$ or $1 \pm \cos x$ in the denominator
- When simpler methods (like direct substitution) aren't obvious

When NOT to use it:

- If direct substitution is available (always check first)
- If it makes the problem more complicated
- If other methods (Weierstrass, completing the square) are more natural

Connection to rationalization:

- This is essentially the rationalization technique from algebra
- We're eliminating irrational expressions from denominators
- The goal is to transform to a more tractable form

The conjugate method is a powerful tool in the integration toolkit, but always look for the simplest approach first.

---

2. Integration by Parts

Integration by parts is the integral version of the product rule for differentiation. If $(uv)' = u'v + uv'$, then integrating both sides:

$\int (uv)' dx = \int u'v \, dx + \int uv' \, dx$

$uv = \int u'v \, dx + \int uv' \, dx$

Rearranging: $\boxed{\int u \, dv = uv - \int v \, du}$

The Art of Choosing u and dv:

The success of integration by parts depends critically on choosing which part is u and which is dv. A useful mnemonic is **LIATE**（对反幂三指）:

- **L**ogarithmic functions
- **I**nverse trigonometric functions
- **A**lgebraic (polynomial) functions
- **T**rigonometric functions
- **E**xponential functions

Choose u to be the function that appears earliest in this list. The remaining part becomes dv.

Why LIATE Works:

- Logarithmic and inverse trig functions become simpler when differentiated
- Polynomials reduce degree when differentiated
- Trig and exponential functions don't simplify much, but are easy to integrate

---

2.1 Inverse Functions: $\int f^{-1}(x)dx$

Theoretical Framework:

To integrate an inverse function, we transform it into a problem involving the original function.

The Method:

Let $y = f^{-1}(x)$. Then $x = f(y)$.

Differentiating both sides with respect to y: $\frac{dx}{dy} = f'(y)$

Therefore: $dx = f'(y)dy$

The integral becomes: $\int f^{-1}(x)dx = \int y \cdot f'(y)dy$

Now apply integration by parts with $u = y$ and $dv = f'(y)dy$:

- $du = dy$
- $v = f(y)$

$\int y \cdot f'(y)dy = y \cdot f(y) - \int f(y)dy$

The General Formula: $\int f^{-1}(x)dx = x \cdot f^{-1}(x) - \int f(f^{-1}(x))d(f^{-1}(x))$

Or more simply, after back-substitution: $\boxed{\int f^{-1}(x)dx = x \cdot f^{-1}(x) - F(f^{-1}(x)) + C}$

where F is an antiderivative of f.

Example 2.1.1: Evaluate $\int \arctan x \, dx$

Solution:

Here $f^{-1}(x) = \arctan x$, so $f(y) = \tan y$.

Step 1: Substitution

Let $y = \arctan x$. Then $x = \tan y$ and $dx = \sec^2 y, dy$.

$\int \arctan x \, dx = \int y \cdot \sec^2 y, dy$

Step 2: Integration by Parts

Let $u = y$ and $dv = \sec^2 y, dy$

Then $du = dy,\ v = \int \sec^2 y \, dy = \tan y$

$\int y \sec^2 y \, dy = y \tan y - \int \tan y \, dy$

Step 3: Integrate $\tan y$

$\int \tan y \, dy = \int \frac{\sin y}{\cos y} dy = \ln|\sec y| + C$

Step 4: Back-substitution

$\int y \sec^2 y \, dy = y \tan y - (-\ln|\cos y|) + C = y \tan y + \ln|\cos y| + C$

From $y = \arctan x$ and $x = \tan y$:

$= (\arctan x) \cdot x + \ln|\cos(\arctan x)| + C$

We need to express $\cos(\arctan x)$ in terms of x.

From a right triangle where $\tan y = x$: Opposite: x, Adjacent: 1, Hypotenuse: $\sqrt{1+x^2}$

Therefore: $\cos y = \frac{1}{\sqrt{1+x^2}}$

$= x \arctan x + \ln\left|\frac{1}{\sqrt{1+x^2}}\right| + C$

$= x \arctan x + \ln\left(1+x^2\right)^{-1/2} + C$

$\boxed{= x \arctan x - \frac{1}{2}\ln\left(1+x^2\right) + C}$

Verification: $\frac{d}{dx}\left[x \arctan x - \frac{1}{2}\ln\left(1+x^2\right)\right] = \arctan x + x \cdot \frac{1}{1+x^2} - \frac{1}{2} \cdot \frac{2x}{1+x^2}$

$= \arctan x + \frac{x}{1+x^2} - \frac{x}{1+x^2} = \arctan x \checkmark$

2.2 Complete Inverse Function Catalog

Inverse Trigonometric Functions

Example 2.2.1: $\int \arcsin x \, dx$

Let $y = \arcsin x$, so $x = \sin y$ and $dx = \cos y \, dy$

$\int \arcsin x \, dx = \int y \cos y, dy$

Integration by parts: $u = y$, $dv = \cos y \, dy$, $du = dy$, $v = \sin y$

$= y \sin y - \int \sin y \, dy = y \sin y + \cos y + C$

Back-substitute: $y = \arcsin x$, $\sin y = x$, $\cos y = \sqrt{1-x^2}$

Answer: $\boxed{x \arcsin x + \sqrt{1-x^2} + C}$

Example 2.2.2: $\int \arccos x \, dx$

Let $y = \arccos x$, so $x = \cos y$ and $dx = -\sin y \, dy$

$\int \arccos x \, dx = \int y \cdot (-\sin y) dy = -\int y \sin y \, dy$

Integration by parts: $u = y$, $dv = \sin y \, dy$, $du = dy$, $v = -\cos y$

$= -(-y \cos y - \int -\cos y \, dy) = -(-y \cos y + \sin y) + C$

$= y \cos y - \sin y + C$

Back-substitute: $y = \arccos x$, $\cos y = x$, $\sin y = \sqrt{1-x^2}$

Answer: $\boxed{x \arccos x - \sqrt{1-x^2} + C}$

Example 2.2.3: $\int \operatorname{arccot} x \, dx$

Let $y = \operatorname{arccot} x$, so $x = \cot y$ and $dx = -\csc^2 y \, dy$

$\int \operatorname{arccot} x \, dx = \int y \cdot (-\csc^2 y) dy = -\int y \csc^2 y \, dy$

Integration by parts: $u = y$, $dv = \csc^2 y, dy$, $du = dy$, $v = -\cot y$

$= -(-y \cot y - \int -\cot y, dy) = y \cot y - \int \cot y \, dy$

$= y \cot y - \ln|\sin y| + C$

Back-substitute: $y = \operatorname{arccot} x$, $\cot y = x$

For $\sin(\operatorname{arccot} x)$: from triangle with adjacent x, opposite 1, hypotenuse $\sqrt{1+x^2}$:

$\sin y = \frac{1}{\sqrt{1+x^2}}$

$= x \operatorname{arccot} x - \ln\left|\frac{1}{\sqrt{1+x^2}}\right| + C$

$= x \operatorname{arccot} x + \frac{1}{2} \ln\left(1+x^2\right) + C$

Answer: $\boxed{x \operatorname{arccot} x + \dfrac{1}{2} \ln\left(1+x^2\right) + C}$

Example 2.2.4: $\int \operatorname{arcsec} x \, dx$ (for $x \geq 1$)

Let $y = \operatorname{arcsec} x$, so $x = \sec y$ and $dx = \sec y \tan y \, dy$

$\int \operatorname{arcsec} x \, dx = \int y \sec y \tan y \, dy$

This requires integration by parts:

Let $u = y$, $dv = \sec y \tan y \, dy, \ du = dy, \ v = \sec y$

$= y \sec y - \int \sec y, dy$

$= y \sec y - \ln |\sec y + \tan y| + C$

Back-substitute: $y = \operatorname{arcsec} x$, $\sec y = x$

For $\tan y$: $\tan^2 y = \sec^2 y - 1 = x^2 - 1$, so $\tan y = \sqrt{x^2 - 1}$ (positive for $y \in [0, \pi/2)$)

$= x \operatorname{arcsec} x - \ln |x + \sqrt{x^2 - 1}| + C$

Answer: $\boxed{x \operatorname{arcsec} x - \ln |x + \sqrt{x^2 - 1}| + C}$

Inverse Hyperbolic Functions

Example 2.2.5: $\int \operatorname{arsinh} x \, dx$

Let $y = \operatorname{arsinh} x$, so $x = \sinh y$ and $dx = \cosh y \, dy$

$\int \operatorname{arsinh} x \, dx = \int y \cosh y \, dy$

Integration by parts: $u = y$, $dv = \cosh y \, dy, \ du = dy, \ v = \sinh y$

$= y \sinh y - \int \sinh y, dy = y \sinh y - \cosh y + C$

Back-substitute: $y = \operatorname{arsinh} x$, $\sinh y = x$

For $\cosh y$: $\cosh^2 y = 1 + \sinh^2 y = 1 + x^2$, so $\cosh y = \sqrt{1 + x^2}$

Answer: $\boxed{x \operatorname{arsinh} x - \sqrt{x^2 + 1} + C}$

Alternative form: Since $\operatorname{arsinh} x = \ln \left(x + \sqrt{x^2 + 1} \right)$:

$= x \ln \left(x + \sqrt{x^2 + 1} \right) - \sqrt{x^2 + 1} + C$

Example 2.2.6: $\int \operatorname{arcosh} x \, dx$ (for $x \geq 1$)

Let $y = \operatorname{arcosh} x$, so $x = \cosh y$ and $dx = \sinh y \, dy$

$\int \operatorname{arcosh} x \, dx = \int y \sinh y \, dy$

Integration by parts: $u = y$, $dv = \sinh y \, dy, \ du = dy, \ v = \cosh y$

$= y \cosh y - \int \cosh y, dy = y \cosh y - \sinh y + C$

Back-substitute: $y = \operatorname{arcosh} x$, $\cosh y = x$

For $\sinh y$: $\sinh^2 y = \cosh^2 y - 1 = x^2 - 1$, so $\sinh y = \sqrt{x^2 - 1}$ (positive for $y > 0$)

Answer: $\boxed{x \operatorname{arcosh} x - \sqrt{x^2 - 1} + C}$

Example 2.2.7: $\int \operatorname{artanh} x \, dx$ (for $|x| < 1$)

Let $y = \operatorname{artanh} x$, so $x = \tanh y$ and $dx = \operatorname{sech}^2 y \, dy$

$\int \operatorname{artanh} x \, dx = \int y \operatorname{sech}^2 y \, dy$

Integration by parts: $u = y, \, dv = \operatorname{sech}^2 y \, dy, \;\; du = dy, \;\; v = \tanh y$

$= y \tanh y - \int \tanh y, dy$

Recall: $\int \tanh y, dy = \ln(\cosh y) + C$

$= y \tanh y - \ln(\cosh y) + C$

Back-substitute: $y = \operatorname{artanh} x$, $\tanh y = x$

For $\cosh y$: $\operatorname{sech}^2 y = 1 - \tanh^2 y = 1 - x^2$, so $\cosh y = \frac{1}{\sqrt{1-x^2}}$

$= x \operatorname{artanh} x - \ln\left(\frac{1}{\sqrt{1-x^2}}\right) + C$

$= x \operatorname{artanh} x + \frac{1}{2} \ln\left(1 - x^2\right) + C$

Answer: $\boxed{x \operatorname{artanh} x + \dfrac{1}{2} \ln\left(1 - x^2\right) + C}$

Example 2.2.8: $\int \operatorname{arcoth} x \, dx$ (for $|x| > 1$)

Let $y = \operatorname{arcoth} x$, so $x = \coth y$ and $dx = -\operatorname{csch}^2 y \, dy$

$\int \operatorname{arcoth} x \, dx = \int y \cdot (-\operatorname{csch}^2 y) dy = -\int y \operatorname{csch}^2 y \, dy$

Integration by parts: $u = y, \, dv = \operatorname{csch}^2 y \, dy, \;\; du = dy, \;\; v = -\coth y$

$= -(-y \coth y + \int \coth y, dy) = y \coth y - \ln|\sinh y| + C$

Back-substitute: $y = \operatorname{arcoth} x$, $\coth y = x$

For $\sinh y$: $\operatorname{csch}^2 y = \coth^2 y - 1 = x^2 - 1$, so $|\sinh y| = \frac{1}{\sqrt{x^2-1}}$

$= x \operatorname{arcoth} x - \ln\left(\frac{1}{\sqrt{x^2-1}}\right) + C$

$= x \operatorname{arcoth} x + \frac{1}{2}\ln\left(x^2 - 1\right) + C$

Answer: $\boxed{x \operatorname{arcoth} x + \dfrac{1}{2}\ln\left(x^2 - 1\right) + C}$

2.3 Logarithmic Powers

Reduction Formula for $\int (\ln x)^n dx$

Derivation:

Use integration by parts with $u = (\ln x)^n$ and $dv = dx$:

- $du = n(\ln x)^{n-1} \cdot \frac{1}{x} dx$
- $v = x$

$I_n = \int (\ln x)^n dx = x(\ln x)^n - \int x \cdot n(\ln x)^{n-1} \cdot \frac{1}{x} dx$

$= x(\ln x)^n - n \int (\ln x)^{n-1} dx$

$= x(\ln x)^n - nI_{n-1}$

The Reduction Formula: $\boxed{I_n = x(\ln x)^n - nI_{n-1}}$

Base Case: $I_0 = \int 1\, dx = x + C$

Example 2.3.1: Evaluate $\int (\ln x)^3 dx$

Solution:

Using the reduction formula repeatedly:

$I_3 = x(\ln x)^3 - 3I_2$

$I_2 = x(\ln x)^2 - 2I_1$

$I_1 = x\ln x - I_0 = x\ln x - x$

Back-substitute:

$I_2 = x(\ln x)^2 - 2(x\ln x - x) = x(\ln x)^2 - 2x\ln x + 2x$

$I_3 = x(\ln x)^3 - 3[x(\ln x)^2 - 2x\ln x + 2x]$

$= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C$

Answer: $\boxed{x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C}$

Notice that the coefficients are related to $3!$ times alternating signs. In general:

$I_n = x\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}(\ln x)^k + C$

Example 2.3.2: Evaluate $\int x^2(\ln x)^2 dx$

Solution:

This combines polynomial and logarithmic factors. Use integration by parts with $u = (\ln x)^2$, $dv = x^2 dx$:

- $du = 2(\ln x)\cdot\frac{1}{x}dx$
- $v = \frac{x^3}{3}$

$\int x^2(\ln x)^2 dx = \frac{x^3}{3}(\ln x)^2 - \int\frac{x^3}{3}\cdot\frac{2\ln x}{x}dx$

$= \frac{x^3(\ln x)^2}{3} - \frac{2}{3}\int x^2\ln x\, dx$

Now integrate $\int x^2\ln x\, dx$ using IBP again:

Let $u = \ln x, dv = x^2 dx$:

- $du = \frac{1}{x}dx$
- $v = \frac{x^3}{3}$

$\int x^2\ln x\, dx = \frac{x^3\ln x}{3} - \int\frac{x^3}{3}\cdot\frac{1}{x}dx = \frac{x^3\ln x}{3} - \frac{1}{3}\int x^2 dx$

$= \frac{x^3 \ln x}{3} - \frac{x^3}{9}$

Substitute back:

$\int x^2 (\ln x)^2 dx = \frac{x^3 (\ln x)^2}{3} - \frac{2}{3}\left(\frac{x^3 \ln x}{3} - \frac{x^3}{9}\right) + C$

$= \frac{x^3 (\ln x)^2}{3} - \frac{2x^3 \ln x}{9} + \frac{2x^3}{27} + C$

Part II: Trigonometric and Hyperbolic Integrals

头痒痒的很正常，要开始长脑子了。

3. Trigonometric Power Reduction

Integrals of the form $\int \sin^m(x) \cos^n(x) dx$ are ubiquitous in applications ranging from Fourier analysis to quantum mechanics. The strategy depends critically on the parity of m and n.

3.1 Sine and Cosine Powers

The key insight is that $\sin^2(x) + \cos^2(x) = 1$ allows us to convert between sine and cosine.

Case 1: At least one exponent is odd

- Save one factor of the odd power for the differential
- Convert the remaining even power using $\sin^2 + \cos^2 = 1$
- Substitute

Case 2: Both exponents are even

- Use half-angle formulas:
  - $\sin^2(x) = \frac{1-\cos(2x)}{2}$
  - $\cos^2(x) = \frac{1+\cos(2x)}{2}$
- Or use complex exponentials (Euler's formula)

Method 1 - Traditional Substitution

Example 3.1.1: Evaluate $\int \cos^3(x)\ dx$

Solution:

The exponent 3 is odd, so save one cosine factor for du.

$\int \cos^3(x)\ dx = \int \cos^2(x) \cdot \cos(x)\ dx$

Use $\cos^2(x) = 1 - \sin^2(x)$:

$= \int (1 - \sin^2(x)) \cos(x)\ dx$

Let $u = \sin(x)$, then $du = \cos(x)dx$:

$= \int (1 - u^2)du = u - \frac{u^3}{3} + C$

$= \sin(x) - \frac{\sin^3(x)}{3} + C$

Method 2 - Complex Exponentials (Euler's Formula)

This method is more systematic for higher powers and reveals beautiful patterns.

Recall: $e^{ix} = \cos(x) + i\sin(x)$, therefore: $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}, \quad \sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$

Example 3.1.2: Evaluate $\int \cos^3(x)\ dx$ (using complex method)

Solution:

$\cos^3(x) = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^3 = \frac{1}{8}(e^{ix} + e^{-ix})^3$

Expand using the binomial theorem: $(e^{ix} + e^{-ix})^3 = e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix}$

Group terms: $= (e^{3ix} + e^{-3ix}) + 3(e^{ix} + e^{-ix})$

$= 2\cos(3x) + 6\cos(x)$

Therefore: $\cos^3(x) = \frac{1}{8}[2\cos(3x) + 6\cos(x)] = \frac{\cos(3x)}{4} + \frac{3\cos(x)}{4}$

Integrate: $\int \cos^3(x)\ dx = \int \left[\frac{\cos(3x)}{4} + \frac{3\cos(x)}{4}\right] dx$

$= \frac{\sin(3x)}{12} + \frac{3\sin(x)}{4} + C$

Verification that both answers are equivalent:

Using the triple-angle formula: $\sin(3x) = 3\sin(x) - 4\sin^3(x)$

$\frac{\sin(3x)}{12} + \frac{3\sin(x)}{4} = \frac{3\sin(x) - 4\sin^3(x)}{12} + \frac{3\sin(x)}{4}$

$= \frac{3\sin(x) - 4\sin^3(x) + 9\sin(x)}{12} = \frac{12\sin(x) - 4\sin^3(x)}{12}$

$= \sin(x) - \frac{\sin^3(x)}{3}$ ✓

General Reduction Formulas

For systematic computation of higher powers, we can derive reduction formulas using integration by parts.

For $\int \cos^n(x)dx$:

Write $\int \cos^n(x)dx = \int \cos^{n-1}(x) \cdot \cos(x)dx$

IBP: $u = \cos^{n-1}(x),\ dv = \cos(x)dx$

- $du = (n-1)\cos^{n-2}(x) \cdot (-\sin(x))dx$
- $v = \sin(x)$

$\int \cos^n(x)dx = \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\sin^2(x)dx$

Use $\sin^2(x) = 1 - \cos^2(x)$:

$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)[1 - \cos^2(x)]dx$

$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)dx - (n-1)\int \cos^n(x)dx$

Let $I_n = \int \cos^n(x)dx$:

$I_n = \cos^{n-1}(x)\sin(x) + (n-1)I_{n-2} - (n-1)I_n$

$I_n + (n-1)I_n = \cos^{n-1}(x)\sin(x) + (n-1)I_{n-2}$

$nI_n = \cos^{n-1}(x)\sin(x) + (n-1)I_{n-2}$

$$\boxed{\int \cos^n(x)dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)dx}$$

$$\boxed{\int \sin^n(x)dx = -\frac{\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)dx}$$

Example 3.1.3: Evaluate $\int \sin^4(x)\ dx$

Solution:

Method 1 (Half-angle formulas):

$\sin^2(x) = \frac{1-\cos(2x)}{2}$

$\sin^4(x) = \left(\frac{1-\cos(2x)}{2}\right)^2 = \frac{1-2\cos(2x)+\cos^2(2x)}{4}$

For $\cos^2(2x)$, use the half-angle formula again: $\cos^2(2x) = \frac{1+\cos(4x)}{2}$

Therefore: $\sin^4(x) = \frac{1-2\cos(2x)+\frac{1+\cos(4x)}{2}}{4}$

$= \frac{2-4\cos(2x)+1+\cos(4x)}{8} = \frac{3-4\cos(2x)+\cos(4x)}{8}$

Integrate: $\int \sin^4(x)dx = \int \frac{3-4\cos(2x)+\cos(4x)}{8}dx$

$= \frac{1}{8}\left[3x - 2\sin(2x) + \frac{\sin(4x)}{4}\right] + C$

$= \frac{3x}{8} - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C$

Example 3.1.4: Evaluate $\int \cos^6(x)\ dx$

Solution:

Method 1 (Repeated reduction formula):

$I_6 = \frac{\cos^5(x)\sin(x)}{6} + \frac{5}{6}I_4$

$I_4 = \frac{\cos^3(x)\sin(x)}{4} + \frac{3}{4}I_2$

$I_2 = \frac{\cos(x)\sin(x)}{2} + \frac{1}{2}I_0$

$I_0 = \int dx = x$

Back-substitute: $I_2 = \frac{\cos(x)\sin(x)}{2} + \frac{x}{2}$

$I_4 = \frac{\cos^3(x)\sin(x)}{4} + \frac{3}{4}\left[\frac{\cos(x)\sin(x)}{2} + \frac{x}{2}\right]$

$= \frac{\cos^3(x)\sin(x)}{4} + \frac{3\cos(x)\sin(x)}{8} + \frac{3x}{8}$

$I_6 = \frac{\cos^5(x)\sin(x)}{6} + \frac{5}{6}\left[\frac{\cos^3(x)\sin(x)}{4} + \frac{3\cos(x)\sin(x)}{8} + \frac{3x}{8}\right]$

$= \frac{\cos^5(x)\sin(x)}{6} + \frac{5\cos^3(x)\sin(x)}{24} + \frac{5\cos(x)\sin(x)}{16} + \frac{5x}{16}$

Method 2 (Complex exponentials):

$\cos^6(x) = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^6 = \frac{1}{64}(e^{ix} + e^{-ix})^6$

Using the binomial theorem: $(e^{ix} + e^{-ix})^6 = \sum_{k=0}^{6} \binom{6}{k} e^{i(6-2k)x}$

$= e^{6ix} + 6e^{4ix} + 15e^{2ix} + 20 + 15e^{-2ix} + 6e^{-4ix} + e^{-6ix}$

Group conjugate pairs: $= (e^{6ix} + e^{-6ix}) + 6(e^{4ix} + e^{-4ix}) + 15(e^{2ix} + e^{-2ix}) + 20$

$= 2\cos(6x) + 12\cos(4x) + 30\cos(2x) + 20$

Therefore: $\cos^6(x) = \frac{1}{64}[2\cos(6x) + 12\cos(4x) + 30\cos(2x) + 20]$

$= \frac{\cos(6x)}{32} + \frac{3\cos(4x)}{16} + \frac{15\cos(2x)}{32} + \frac{5}{16}$

Integrate: $\int \cos^6(x)dx = \frac{\sin(6x)}{192} + \frac{3\sin(4x)}{64} + \frac{15\sin(2x)}{64} + \frac{5x}{16} + C$

Example 3.1.5: Evaluate $\int \sin^5(x)\ dx$

Solution:

The exponent is odd, so use the substitution method.

$\int \sin^5(x)dx = \int \sin^4(x) \cdot \sin(x)dx$

$= \int (\sin^2(x))^2 \sin(x)dx = \int (1 - \cos^2(x))^2 \sin(x)dx$

Let $u = \cos(x)$, then $du = -\sin(x)dx$:

$$= -\int(1-u^2)^2 du = -\int(1-2u^2+u^4)du$$

$$= -\left(u - \frac{2u^3}{3} + \frac{u^5}{5}\right) + C$$

$$= -\cos(x) + \frac{2\cos^3(x)}{3} - \frac{\cos^5(x)}{5} + C$$

Example 3.1.6: Evaluate $\int \cos^7(x)\ dx$

Solution:

$$\int \cos^7(x)dx = \int \cos^6(x)\cos(x)dx = \int(\cos^2(x))^3\cos(x)dx$$

$$= \int(1-\sin^2(x))^3\cos(x)dx$$

Let $u = \sin(x)$, then $du = \cos(x)dx$:

$$= \int(1-u^2)^3 du$$

Expand $(1-u^2)^3$: $(1-u^2)^3 = 1 - 3u^2 + 3u^4 - u^6$

$$= \int(1-3u^2+3u^4-u^6)du$$

$$= u - u^3 + \frac{3u^5}{5} - \frac{u^7}{7} + C$$

$$= \sin(x) - \sin^3(x) + \frac{3\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C$$

3.2 Mixed Sine-Cosine Powers

For integrals of the form $\int \sin^m(x)\cos^n(x)dx$, the strategy depends on the parity of m and n.

Strategy Decision Tree:

1. **If m is odd:** Save one $\sin(x)$ for du, convert remaining $\sin^{m-1}(x)$ using $\sin^2 = 1 - \cos^2$, substitute $u = \cos(x)$
2. **If n is odd:** Save one $\cos(x)$ for du, convert remaining $\cos^{n-1}(x)$ using $\cos^2 = 1 - \sin^2$, substitute $u = \sin(x)$
3. **If both m and n are odd:** Choose either strategy (typically the one with smaller exponent)

4. **If both m and n are even:** Use half-angle formulas repeatedly or product-to-sum identities

---

Example 3.2.1: Evaluate $\int \sin^3(x)\cos^2(x)\ dx$

Solution:

Since the sine exponent (3) is odd, save one $\sin(x)$ and convert the rest.

$\int \sin^3(x)\cos^2(x)dx = \int \sin^2(x)\cos^2(x)\sin(x)dx$

$= \int (1-\cos^2(x))\cos^2(x)\sin(x)dx$

Let $u = \cos(x)$, then $du = -\sin(x)dx$:

$= -\int (1-u^2)u^2du = -\int (u^2-u^4)du$

$= -\left(\frac{u^3}{3}-\frac{u^5}{5}\right)+C$

$= -\frac{\cos^3(x)}{3}+\frac{\cos^5(x)}{5}+C$

Example 3.2.2: Evaluate $\int \sin^2(x)\cos^4(x)\ dx$

Solution:

Both exponents are even, so use half-angle formulas.

$\sin^2(x) = \frac{1-\cos(2x)}{2}, \quad \cos^2(x) = \frac{1+\cos(2x)}{2}$

$\int \sin^2(x)\cos^4(x)dx = \int \frac{1-\cos(2x)}{2}\cdot\left(\frac{1+\cos(2x)}{2}\right)^2 dx$

$= \frac{1}{8}\int (1-\cos(2x))(1+\cos(2x))^2dx$

Expand $(1+\cos(2x))^2 = 1+2\cos(2x)+\cos^2(2x)$:

$= \frac{1}{8}\int (1-\cos(2x))(1+2\cos(2x)+\cos^2(2x))dx$

$= \frac{1}{8}\int [1+2\cos(2x)+\cos^2(2x)-\cos(2x)-2\cos^2(2x)-\cos^3(2x)]dx$

$= \frac{1}{8}\int [1+\cos(2x)-\cos^2(2x)-\cos^3(2x)]dx$

For $\cos^2(2x)$, use $\cos^2(2x) = \frac{1+\cos(4x)}{2}$:

$$= \frac{1}{8}\int\left[1 + \cos(2x) - \frac{1+\cos(4x)}{2} - \cos^3(2x)\right]dx$$

$$= \frac{1}{8}\int\left[\frac{1}{2} + \cos(2x) - \frac{\cos(4x)}{2} - \cos^3(2x)\right]dx$$

For $\cos^3(2x)$, use $\cos^3(2x) = \frac{\cos(6x)}{4} + \frac{3\cos(2x)}{4}$ (from earlier):

$$= \frac{1}{8}\int\left[\frac{1}{2} + \cos(2x) - \frac{\cos(4x)}{2} - \frac{\cos(6x)}{4} - \frac{3\cos(2x)}{4}\right]dx$$

$$= \frac{1}{8}\int\left[\frac{1}{2} + \frac{\cos(2x)}{4} - \frac{\cos(4x)}{2} - \frac{\cos(6x)}{4}\right]dx$$

$$= \frac{1}{8}\left[\frac{x}{2} + \frac{\sin(2x)}{8} - \frac{\sin(4x)}{8} - \frac{\sin(6x)}{24}\right] + C$$

$$= \frac{x}{16} + \frac{\sin(2x)}{64} - \frac{\sin(4x)}{64} - \frac{\sin(6x)}{192} + C$$

Example 3.2.3: Evaluate $\int \sin^3(x)\cos^3(x)\ dx$

Solution:

Both exponents are odd. We can choose either; let's use $u = \sin(x)$ (saving one $\cos(x)$).

$$\int \sin^3(x)\cos^3(x)dx = \int \sin^3(x)\cos^2(x)\cos(x)dx$$

$$= \int \sin^3(x)(1-\sin^2(x))\cos(x)dx$$

Let $u = \sin(x)$, $du = \cos(x)dx$:

$$= \int u^3(1-u^2)du = \int(u^3-u^5)du$$

$$= \frac{u^4}{4} - \frac{u^6}{6} + C$$

$$= \frac{\sin^4(x)}{4} - \frac{\sin^6(x)}{6} + C$$

Alternative approach (using $u = \cos(x)$):

$$\int \sin^3(x)\cos^3(x)dx = \int \sin^2(x)\cos^3(x)\sin(x)dx$$

$$= \int(1-\cos^2(x))\cos^3(x)\sin(x)dx$$

Let $u = \cos(x)$, $du = -\sin(x)dx$:

$$= -\int(1-u^2)u^3du = -\int(u^3-u^5)du$$

$$= -\left(\frac{u^4}{4} - \frac{u^6}{6}\right) + C = -\frac{\cos^4(x)}{4} + \frac{\cos^6(x)}{6} + C$$

Both answers are correct and differ by a constant (which can be verified using the identity $\sin^2 + \cos^2 = 1$).

3.3 Hyperbolic Power Formulas

Hyperbolic functions follow similar patterns to trigonometric functions but with key sign differences in their identities.

Key Identities: $\cosh^2(x) - \sinh^2(x) = 1$ $\sinh(2x) = 2\sinh(x)\cosh(x)$ $\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 1 + 2\sinh^2(x)$

Half-argument formulas: $\sinh^2(x) = \frac{\cosh(2x)-1}{2}$ $\cosh^2(x) = \frac{\cosh(2x)+1}{2}$

General Reduction Formulas:

For $\int \sinh^n(x)dx$:

Write $\int \sinh^n(x)dx = \int \sinh^{n-1}(x)\sinh(x)dx$

IBP: $u = \sinh^{n-1}(x)$, $dv = \sinh(x)dx$

- $du = (n-1)\sinh^{n-2}(x)\cosh(x)dx$
- $v = \cosh(x)$

$$\int \sinh^n(x)dx = \sinh^{n-1}(x)\cosh(x) - (n-1)\int \sinh^{n-2}(x)\cosh^2(x)dx$$

Use $\cosh^2(x) = 1 + \sinh^2(x)$:

$$= \sinh^{n-1}(x)\cosh(x) - (n-1)\int \sinh^{n-2}(x)[1+\sinh^2(x)]dx$$

$$= \sinh^{n-1}(x)\cosh(x) - (n-1)\int \sinh^{n-2}(x)dx - (n-1)\int \sinh^n(x)dx$$

Let $I_n = \int \sinh^n(x)dx$:

$$I_n = \sinh^{n-1}(x)\cosh(x) - (n-1)I_{n-2} - (n-1)I_n$$

$$I_n + (n-1)I_n = \sinh^{n-1}(x)\cosh(x) - (n-1)I_{n-2}$$

$$nI_n = \sinh^{n-1}(x)\cosh(x) - (n-1)I_{n-2}$$

$$\boxed{\int \sinh^n(x)dx = \frac{\sinh^{n-1}(x)\cosh(x)}{n} - \frac{n-1}{n}\int \sinh^{n-2}(x)dx}$$

For $\int \cosh^n(x)dx$:

Similarly, using $\sinh^2(x) = \cosh^2(x) - 1$:

$$\boxed{\int \cosh^n(x)dx = \frac{\cosh^{n-1}(x)\sinh(x)}{n} + \frac{n-1}{n}\int \cosh^{n-2}(x)dx}$$

Note the sign difference compared to trig functions!

Example 3.3.1: Evaluate $\int \sinh^4(x)\ dx$

Solution:

Method 1 (Reduction formula):

$I_4 = \frac{\sinh^3(x)\cosh(x)}{4} - \frac{3}{4}I_2$

$I_2 = \frac{\sinh(x)\cosh(x)}{2} - \frac{1}{2}I_0$

$I_0 = \int dx = x$

Back-substitute: $I_2 = \frac{\sinh(x)\cosh(x)}{2} - \frac{x}{2}$

$I_4 = \frac{\sinh^3(x)\cosh(x)}{4} - \frac{3}{4}\left[\frac{\sinh(x)\cosh(x)}{2} - \frac{x}{2}\right]$

$= \frac{\sinh^3(x)\cosh(x)}{4} - \frac{3\sinh(x)\cosh(x)}{8} + \frac{3x}{8}$

Method 2 (Exponential):

$\sinh(x) = \frac{e^x - e^{-x}}{2}$

$\sinh^4(x) = \frac{(e^x - e^{-x})^4}{16}$

Expand using the binomial theorem: $(e^x - e^{-x})^4 = e^{4x} - 4e^{2x} + 6 - 4e^{-2x} + e^{-4x}$

Group terms: $= (e^{4x} + e^{-4x}) - 4(e^{2x} + e^{-2x}) + 6$

$= 2\cosh(4x) - 8\cosh(2x) + 6$

Therefore: $\sinh^4(x) = \frac{2\cosh(4x) - 8\cosh(2x) + 6}{16} = \frac{\cosh(4x)}{8} - \frac{\cosh(2x)}{2} + \frac{3}{8}$

Integrate: $\int \sinh^4(x)dx = \frac{\sinh(4x)}{32} - \frac{\sinh(2x)}{4} + \frac{3x}{8} + C$

Example 3.3.2: Evaluate $\int \cosh^3(x)\ dx$

Solution:

Method 1 (Direct substitution):

$\int \cosh^3(x)dx = \int \cosh^2(x)\cosh(x)dx$

Use $\cosh^2(x) = 1 + \sinh^2(x)$:

$= \int (1 + \sinh^2(x))\cosh(x)dx$

Let $u = \sinh(x)$, $du = \cosh(x)dx$:

$= \int (1 + u^2)du = u + \frac{u^3}{3} + C$

$= \sinh(x) + \frac{\sinh^3(x)}{3} + C$

Method 2 (Exponential):

$\cosh^3(x) = \left(\frac{e^x + e^{-x}}{2}\right)^3 = \frac{(e^x + e^{-x})^3}{8}$

$= \frac{e^{3x} + 3e^x + 3e^{-x} + e^{-3x}}{8}$

$= \frac{(e^{3x} + e^{-3x}) + 3(e^x + e^{-x})}{8} = \frac{2\cosh(3x) + 6\cosh(x)}{8}$

$= \frac{\cosh(3x)}{4} + \frac{3\cosh(x)}{4}$

Integrate: $\int \cosh^3(x)dx = \frac{\sinh(3x)}{12} + \frac{3\sinh(x)}{4} + C$

Using the triple-angle formula $\sinh(3x) = 3\sinh(x) + 4\sinh^3(x)$:

$\frac{\sinh(3x)}{12} + \frac{3\sinh(x)}{4} = \frac{3\sinh(x)+4\sinh^3(x)}{12} + \frac{3\sinh(x)}{4}$

$= \frac{3\sinh(x)+4\sinh^3(x)+9\sinh(x)}{12} = \sinh(x) + \frac{\sinh^3(x)}{3}$ ✓

Example 3.3.3: Evaluate $\int \sinh^5(x)\ dx$

Solution:

$\int \sinh^5(x)dx = \int \sinh^4(x)\sinh(x)dx$

$= \int (\sinh^2(x))^2 \sinh(x)dx$

Use $\sinh^2(x) = \cosh^2(x) - 1$:

$= \int (\cosh^2(x) - 1)^2 \sinh(x)dx$

Let $u = \cosh(x)$, $du = \sinh(x)dx$:

$= \int (u^2 - 1)^2 du = \int (u^4 - 2u^2 + 1)du$

$= \frac{u^5}{5} - \frac{2u^3}{3} + u + C$

$= \frac{\cosh^5(x)}{5} - \frac{2\cosh^3(x)}{3} + \cosh(x) + C$

Example 3.3.4: Evaluate $\int \cosh^4(x)\ dx$

Solution:

Using the exponential method:

$\cosh^4(x) = \left(\frac{e^x+e^{-x}}{2}\right)^4 = \frac{(e^x+e^{-x})^4}{16}$

$(e^x + e^{-x})^4 = e^{4x} + 4e^{2x} + 6 + 4e^{-2x} + e^{-4x}$

$= (e^{4x} + e^{-4x}) + 4(e^{2x} + e^{-2x}) + 6$

$= 2\cosh(4x) + 8\cosh(2x) + 6$

Therefore: $\cosh^4(x) = \frac{\cosh(4x)}{8} + \frac{\cosh(2x)}{2} + \frac{3}{8}$

Integrate: $\int \cosh^4(x)dx = \frac{\sinh(4x)}{32} + \frac{\sinh(2x)}{4} + \frac{3x}{8} + C$

Example 3.3.5: Evaluate $\int \sinh^2(x) \cosh^2(x) \ dx$

Solution:

Use the identity $\sinh(x) \cosh(x) = \frac{\sinh(2x)}{2}$:

$\sinh^2(x) \cosh^2(x) = [\sinh(x) \cosh(x)]^2 = \left[\frac{\sinh(2x)}{2}\right]^2 = \frac{\sinh^2(2x)}{4}$

Use $\sinh^2(2x) = \frac{\cosh(4x)-1}{2}$:

$= \frac{1}{4} \cdot \frac{\cosh(4x)-1}{2} = \frac{\cosh(4x)-1}{8}$

Integrate: $\int \sinh^2(x) \cosh^2(x) dx = \int \frac{\cosh(4x)-1}{8} dx$

$= \frac{1}{8}\left[\frac{\sinh(4x)}{4} - x\right] + C$

$= \frac{\sinh(4x)}{32} - \frac{x}{8} + C$

3.4 Hyperbolic Secant and Cosecant

These integrals require special techniques and often result in unexpected forms.

Example 3.4.1: Evaluate $\int \text{sech}(x) \ dx$

Solution:

Method 1 (Multiply by conjugate):

Multiply numerator and denominator by $\text{sech}(x) + \tanh(x)$:

$\int \text{sech}(x) dx = \int \frac{1}{\cosh(x)} dx = \int \frac{\text{sech}(x)+\tanh(x)}{\cosh(x)[\text{sech}(x)+\tanh(x)]} dx$

Wait, this doesn't simplify nicely. Let's try a different approach.

Method 2 (Convert to exponentials):

$\text{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x+e^{-x}}$

Multiply numerator and denominator by e^x:

$= \frac{2e^x}{e^{2x}+1}$

Let $u = e^x, du = e^x dx$, so $dx = \frac{du}{u}$:

$\int \frac{2e^x}{e^{2x}+1} dx = \int \frac{2}{u^2+1} du = 2\arctan(u) + C$

$= 2\arctan(e^x) + C$

Alternative form:

Using the identity $\arctan(e^x) = \frac{\pi}{2} - \text{arccot}(e^x)$ and $\text{arccot}(e^x) = \arctan(e^{-x})$:

We can also write this as: $\int \text{sech}(x) dx = \ln|\text{sech}(x) + \tanh(x)| + C$

To see this equivalence, note that: $\text{sech}(x) + \tanh(x) = \frac{1}{\cosh(x)} + \frac{\sinh(x)}{\cosh(x)} = \frac{1+\sinh(x)}{\cosh(x)}$

$= \frac{1+\frac{e^x-e^{-x}}{2}}{\frac{e^x+e^{-x}}{2}} = \frac{2+e^x-e^{-x}}{e^x+e^{-x}} = \frac{(e^x-e^{-x})+2}{e^x+e^{-x}}$

Multiplying by $\frac{e^x}{e^x}$:

$= \frac{e^{2x}-1+2e^x}{e^{2x}+1} = \frac{e^{2x}+2e^x-1}{e^{2x}+1}$

This can be shown to give $\ln(\text{sech}\, x + \tanh x) = 2\arctan(e^x) + C'$ for appropriate constant.

Answer: $\boxed{2\arctan(e^x) + C}$ or equivalently $\boxed{\ln|\text{sech}(x) + \tanh(x)| + C}$

Example 3.4.2: Evaluate $\int \text{csch}(x)\ dx$

Solution:

$\text{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x-e^{-x}}$

Multiply by $\frac{e^x}{e^x}$:

$= \frac{2e^x}{e^{2x}-1}$

Let $u = e^x, du = e^x dx$:

$\int \frac{2e^x}{e^{2x}-1} dx = \int \frac{2}{u^2-1} du$

Use partial fractions: $\frac{2}{u^2-1} = \frac{2}{(u-1)(u+1)} = \frac{1}{u-1} - \frac{1}{u+1}$:

$= \int \left[\frac{1}{u-1} - \frac{1}{u+1}\right] du = \ln|u-1| - \ln|u+1| + C$

$= \ln\left|\frac{u-1}{u+1}\right| + C = \ln\left|\frac{e^x-1}{e^x+1}\right| + C$

Alternative form:

$\frac{e^x-1}{e^x+1} = \frac{e^{x/2}(e^{x/2}-e^{-x/2})}{e^{x/2}(e^{x/2}+e^{-x/2})} = \frac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}} = \frac{2\sinh(x/2)}{2\cosh(x/2)} = \tanh(x/2)$

Answer: $\boxed{\ln|\tanh(x/2)| + C}$ or equivalently $\boxed{\ln\left|\frac{e^x-1}{e^x+1}\right| + C}$

Example 3.4.3: Evaluate $\int \operatorname{sech}^2(x)\tanh(x)\,dx$

Solution:

Notice that $\frac{d}{dx}[\operatorname{sech}(x)] = -\operatorname{sech}(x)\tanh(x)$.

So $\operatorname{sech}(x)\tanh(x)$ is almost the derivative of $\operatorname{sech}(x)$.

$\int \operatorname{sech}^2(x)\tanh(x)dx = \int \operatorname{sech}(x)\cdot[\operatorname{sech}(x)\tanh(x)]dx$

Let $u = \operatorname{sech}(x)$, then $du = -\operatorname{sech}(x)\tanh(x)dx$:

$= -\int u, du = -\frac{u^2}{2} + C$

$= -\frac{\operatorname{sech}^2(x)}{2} + C$

Answer: $\boxed{-\frac{\operatorname{sech}^2(x)}{2} + C}$

Example 3.4.4: Evaluate $\int \operatorname{csch}^3(x)\,dx$

Solution:

This requires integration by parts combined with reduction.

Write $\int \operatorname{csch}^3(x)dx = \int \operatorname{csch}^2(x)\cdot\operatorname{csch}(x)dx$

IBP: $u = \operatorname{csch}(x)$, $dv = \operatorname{csch}^2(x)dx$

- $du = -\operatorname{csch}(x)\coth(x)dx$
- $v = -\coth(x)$

$\int \operatorname{csch}^3(x)dx = -\operatorname{csch}(x)\coth(x) - \int \coth(x) \cdot \operatorname{csch}(x)\coth(x)dx$

$= -\operatorname{csch}(x)\coth(x) - \int \operatorname{csch}(x)\coth^2(x)dx$

Use $\coth^2(x) = 1 + \operatorname{csch}^2(x)$:

$= -\operatorname{csch}(x)\coth(x) - \int \operatorname{csch}(x)[1 + \operatorname{csch}^2(x)]dx$

$= -\operatorname{csch}(x)\coth(x) - \int \operatorname{csch}(x)dx - \int \operatorname{csch}^3(x)dx$

Let $I = \int \operatorname{csch}^3(x)dx$:

$I = -\operatorname{csch}(x)\coth(x) - \ln|\tanh(x/2)| - I$

$2I = -\operatorname{csch}(x)\coth(x) - \ln|\tanh(x/2)|$

$I = -\frac{\operatorname{csch}(x)\coth(x)}{2} - \frac{\ln|\tanh(x/2)|}{2} + C$

Answer: $\boxed{-\frac{\operatorname{csch}(x)\coth(x)}{2} - \frac{\ln|\tanh(x/2)|}{2} + C}$

Part IV: Rational and Algebraic Functions

有些助教感觉$\int \frac{dx}{x^5+1}$和$\int \frac{dx}{x+1}$味道应该差不多，你呢？

8. Partial Fractions

Partial fraction decomposition is the systematic method for integrating rational functions. It transforms a complex rational expression into a sum of simpler fractions that can be integrated term by term.

8.1 Standard Decomposition

Fundamental Principle:

Any rational function $\frac{P(x)}{Q(x)}$ where $\deg(P) < \deg(Q)$ can be decomposed into partial fractions based on the factorization of $Q(x)$.

(If $\deg(P) \geq \deg(Q)$, first perform polynomial long division.)

Decomposition Rules:

1. **Linear factor $(x-a)$:** Contributes $\frac{A}{x-a}$
2. **Repeated linear factor $(x-a)^n$:** Contributes $\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}$
3. **Irreducible quadratic x^2+px+q:** Contributes $\frac{Ax+B}{x^2+px+q}$
4. **Repeated irreducible quadratic $(x^2+px+q)^n$:** Contributes $\sum_{k=1}^{n} \frac{A_k x+B_k}{(x^2+px+q)^k}$

Example 8.1.1: $\int \frac{2x+1}{(x-1)(x+2)} dx$

Set up: $\frac{2x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$

Multiply through: $2x+1 = A(x+2) + B(x-1)$

- Set $x=1$: $3 = 3A \Rightarrow A = 1$
- Set $x=-2$: $-3 = -3B \Rightarrow B = 1$

Therefore: $\int \left[\frac{1}{x-1} + \frac{1}{x+2}\right] dx = \ln|x-1| + \ln|x+2| + C = \ln|(x-1)(x+2)| + C$

8.2 Irreducible Quadratic Denominators

The strategy involves completing the square, substitution, and splitting the numerator.

Strategy for $\int \frac{Ax+B}{(x^2+px+q)^n} dx$:

Step 1: Complete the square in the denominator $x^2 + px + q = \left(x + \frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right)$

Let $k^2 = q - \frac{p^2}{4}$ (assuming irreducible, so $k^2 > 0$)

Step 2: Substitute $u = x + \frac{p}{2}$, so $x = u - \frac{p}{2}$ and $dx = du$

The denominator becomes $(u^2 + k^2)^n$

Step 3: Express the numerator in terms of u:
$Ax + B = A\left(u - \frac{p}{2}\right) + B = Au + \left(B - \frac{Ap}{2}\right)$

Step 4: Split into twointegrals:
$\int \frac{Au+\left(B-\frac{Ap}{2}\right)}{(u^2+k^2)^n} du = A \int \frac{u}{(u^2+k^2)^n} du + \left(B - \frac{Ap}{2}\right) \int \frac{1}{(u^2+k^2)^n} du$

Step 5: Evaluate each part:

- **First integral:** Let $w = u^2 + k^2$, $dw = 2u, du$
  $\int \frac{u}{(u^2+k^2)^n} du = \frac{1}{2} \int \frac{dw}{w^n} = \frac{1}{2} \cdot \frac{w^{-n+1}}{-n+1} + C = -\frac{1}{2(n-1)(u^2+k^2)^{n-1}} + C$
- **Second integral:** Denote $I_n = \int \frac{du}{(u^2+k^2)^n}$
  - For $n = 1$: $I_1 = \frac{1}{k} \arctan \frac{u}{k} + C$
  - For $n \geq 2$: Use the **reduction formula** (derived below)

Reduction Formula for $I_n = \int \frac{du}{(u^2+k^2)^n}$:

Use integration by parts. Write: $I_n = \int \frac{1}{(u^2+k^2)^n} du$

IBP: $dv = du, u = \frac{1}{(u^2+k^2)^n}$ (confusing notation, let me rewrite)

Actually, write: $I_n = \int \frac{u^2+k^2-u^2}{(u^2+k^2)^n} du = \int \frac{u^2+k^2}{(u^2+k^2)^n} du - \int \frac{u^2}{(u^2+k^2)^n} du$

$= \int \frac{du}{(u^2+k^2)^{n-1}} - \int \frac{u^2}{(u^2+k^2)^n} du$

$= I_{n-1} - \int \frac{u \cdot u}{(u^2+k^2)^n} du$

For the second integral, use IBP: $v = u$, $dw = \frac{u,du}{(u^2+k^2)^n}$

- $dv = du$
- $w = -\frac{1}{2(n-1)(u^2+k^2)^{n-1}}$

$\int \frac{u^2}{(u^2+k^2)^n} du = -\frac{u}{2(n-1)(u^2+k^2)^{n-1}} + \frac{1}{2(n-1)} \int \frac{du}{(u^2+k^2)^{n-1}}$

$= -\frac{u}{2(n-1)(u^2+k^2)^{n-1}} + \frac{I_{n-1}}{2(n-1)}$

Substituting back: $I_n = I_{n-1} - \left[-\frac{u}{2(n-1)(u^2+k^2)^{n-1}} + \frac{I_{n-1}}{2(n-1)}\right]$

$= I_{n-1} + \frac{u}{2(n-1)(u^2+k^2)^{n-1}} - \frac{I_{n-1}}{2(n-1)}$

$= I_{n-1}\left(1 - \frac{1}{2(n-1)}\right) + \frac{u}{2(n-1)(u^2+k^2)^{n-1}}$

$= I_{n-1} \cdot \frac{2(n-1)-1}{2(n-1)} + \frac{u}{2(n-1)(u^2+k^2)^{n-1}}$

$= \frac{2n-3}{2(n-1)} I_{n-1} + \frac{u}{2(n-1)(u^2+k^2)^{n-1}}$

Solving for I_n:

Actually, let me reconsider. The cleaner form is:

Reduction Formula: $\boxed{I_n = \frac{u}{2k^2(n-1)(u^2+k^2)^{n-1}} + \frac{2n-3}{2k^2(n-1)} I_{n-1}}$

Example 8.2.1: Evaluate $\int \frac{x+1}{(x^2+4x+8)^2} dx$

Solution:

Step 1: Complete the square $x^2 + 4x + 8 = (x+2)^2 + 4$

So $k^2 = 4$, $k = 2$.

Step 2: Substitute $u = x + 2$, so $x = u - 2$, $dx = du$

$\int \frac{x+1}{(x^2+4x+8)^2} dx = \int \frac{(u-2)+1}{(u^2+4)^2} du = \int \frac{u-1}{(u^2+4)^2} du$

Step 3: Split $= \int \frac{u}{(u^2+4)^2} du - \int \frac{1}{(u^2+4)^2} du$

Step 4: First integral

Let $w = u^2 + 4, dw = 2u, du$: $\int \frac{u}{(u^2+4)^2} du = \frac{1}{2} \int \frac{dw}{w^2} = -\frac{1}{2w} + C = -\frac{1}{2(u^2+4)} + C$

Step 5: Second integral

This is I_2 with $k = 2$. Use the reduction formula: $I_2 = \frac{u}{2 \cdot 4 \cdot 1 \cdot (u^2+4)} + \frac{2(2)-3}{2 \cdot 4 \cdot 1} I_1$

$= \frac{u}{8(u^2+4)} + \frac{1}{8} I_1$

$I_1 = \frac{1}{2} \arctan \frac{u}{2}$

$I_2 = \frac{u}{8(u^2+4)} + \frac{1}{8} \cdot \frac{1}{2} \arctan \frac{u}{2}$

$= \frac{u}{8(u^2+4)} + \frac{\arctan(u/2)}{16}$

Step 6: Combine $= -\frac{1}{2(u^2+4)} - \left[\frac{u}{8(u^2+4)} + \frac{\arctan(u/2)}{16} \right] + C$

$= -\frac{1}{2(u^2+4)} - \frac{u}{8(u^2+4)} - \frac{\arctan(u/2)}{16} + C$

$= -\frac{4+u}{8(u^2+4)} - \frac{\arctan(u/2)}{16} + C$

Step 7: Back-substitute $u = x + 2$:

$= -\frac{x+6}{8(x^2+4x+8)} - \frac{\arctan((x+2)/2)}{16} + C$

Example 8.2.2: Evaluate $\int \frac{3x-2}{(x^2+2x+5)^2} dx$

Solution:

Step 1: Complete the square $x^2 + 2x + 5 = (x+1)^2 + 4$

Step 2: Substitute $u = x + 1$, so $x = u - 1$: $\int \frac{3(u-1)-2}{(u^2+4)^2} du = \int \frac{3u-5}{(u^2+4)^2} du$

Step 3: Split: $= 3 \int \frac{u}{(u^2+4)^2} du - 5 \int \frac{1}{(u^2+4)^2} du$

Step 4: First integral: $\int \frac{u}{(u^2+4)^2} du = -\frac{1}{2(u^2+4)}$

Step 5: Second integral (from previous example): $\int \frac{1}{(u^2+4)^2} du = \frac{u}{8(u^2+4)} + \frac{\arctan(u/2)}{16}$

Step 6: Combine: $= 3 \cdot \left(-\frac{1}{2(u^2+4)}\right) - 5\left[\frac{u}{8(u^2+4)} + \frac{\arctan(u/2)}{16}\right] + C$

$= -\frac{3}{2(u^2+4)} - \frac{5u}{8(u^2+4)} - \frac{5\arctan(u/2)}{16} + C$

$= -\frac{12+5u}{8(u^2+4)} - \frac{5\arctan(u/2)}{16} + C$

Step 7: Back-substitute $u = x + 1$:

$= -\frac{12+5(x+1)}{8((x+1)^2+4)} - \frac{5\arctan((x+1)/2)}{16} + C$

$= -\frac{5x+17}{8(x^2+2x+5)} - \frac{5\arctan((x+1)/2)}{16} + C$

Example 8.2.3: Evaluate $\int \frac{x^2+1}{x^2+4x+5} dx$

Solution:

The degree of numerator equals the degree of denominator, so perform polynomial division first.

Divide $x^2 + 1$ by $x^2 + 4x + 5$: $\frac{x^2+1}{x^2+4x+5} = 1 + \frac{-(4x+4)}{x^2+4x+5} = 1 - \frac{4x+4}{x^2+4x+5} = 1 - \frac{4(x+1)}{x^2+4x+5}$

Therefore: $\int \frac{x^2+1}{x^2+4x+5} dx = \int dx - 4\int \frac{x+1}{x^2+4x+5} dx$

$= x - 4\int \frac{x+1}{x^2+4x+5} dx$

For the second integral, complete the square: $x^2 + 4x + 5 = (x+2)^2 + 1$

Let $u = x + 2$, so $x = u - 2$ and $x + 1 = u - 1$: $\int \frac{x+1}{x^2+4x+5} dx = \int \frac{u-1}{u^2+1} du$

$= \int \frac{u}{u^2+1} du - \int \frac{1}{u^2+1} du$

$= \frac{\ln(u^2+1)}{2} - \arctan(u) + C$

$= \frac{\ln(x^2+4x+5)}{2} - \arctan(x+2) + C$

Therefore: $\int \frac{x^2+1}{x^2+4x+5} dx = x - 4\left[\frac{\ln(x^2+4x+5)}{2} - \arctan(x+2)\right] + C$

$= x - 2\ln\left(x^2 + 4x + 5\right) + 4\arctan(x+2) + C$

8.3 High-Degree Polynomial Factorization

When dealing with denominators like $x^n \pm 1$, we can use roots of unity to factor them into real quadratic factors (and possibly linear factors).

Roots of Unity

The n-th roots of unity are solutions to $z^n = 1$:
$z_k = e^{2\pi ik/n} = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}, \quad k = 0, 1, \ldots, n-1$

Similarly, for $z^n = -1$: $z_k = e^{i\pi(2k+1)/n}, \quad k = 0, 1, \ldots, n-1$

Method 1: Complex roots come in conjugate pairs, and each pair gives a real quadratic factor:

If $z = e^{i\theta}$ is a root, then so is $\bar{z} = e^{-i\theta}$, and:
$(x - e^{i\theta})(x - e^{-i\theta}) = x^2 - (e^{i\theta} + e^{-i\theta})x + e^{i\theta}e^{-i\theta} = x^2 - 2\cos\theta \cdot x + 1$

Fundamental Theorem of Algebra: Every polynomial of degree n has exactly n complex roots (counting multiplicity).

Method 2 (See 8.4): Complex Decomposition

1. Factor $Q(x)$ into complex linear factors
2. Write partial fractions with complex coefficients
3. Integrate to obtain complex logarithms
4. Combine conjugate terms to recover real-valued expressions

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Example 8.3.1: Evaluate $\int \frac{dx}{x^4+1}$

Step 1: Factorization: $x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$

Step 2: Partial fraction decomposition

$\frac{1}{x^4+1} = \frac{Ax+B}{x^2-\sqrt{2}x+1} + \frac{Cx+D}{x^2+\sqrt{2}x+1}$

$1 = (Ax + B)(x^2 + \sqrt{2}x + 1) + (Cx + D)(x^2 - \sqrt{2}x + 1)$

Comparing coefficients:

- Coefficient of x^3: $0 = A + C$, so $C = -A$
- Coefficient of x^2: $0 = \sqrt{2}A + B - \sqrt{2}C + D = 2\sqrt{2}A + B + D$
- Coefficient of x^1: $0 = A + \sqrt{2}B - C - \sqrt{2}D = 2A + \sqrt{2}(B - D)$
- Coefficient of x^0: $1 = B + D$

Solving the system gives:
$A = -\frac{\sqrt{2}}{4}, B = \frac{1}{2}, C = \frac{\sqrt{2}}{4}, D = \frac{1}{2}$

$\frac{1}{x^4+1} = \frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2-\sqrt{2}x+1} + \frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}$

Step 3: Integrate

For the first integral: $x^2 - \sqrt{2}x + 1 = \left(x - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}$

Let $u = x - \frac{\sqrt{2}}{2}$: $\int \frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2-\sqrt{2}x+1} dx = \int \frac{-\frac{\sqrt{2}}{4}(u+\frac{\sqrt{2}}{2})+\frac{1}{2}}{u^2+\frac{1}{2}} du$

$= \int \frac{-\frac{\sqrt{2}}{4}u-\frac{1}{4}+\frac{1}{2}}{u^2+\frac{1}{2}} du = \int \frac{-\frac{\sqrt{2}}{4}u+\frac{1}{4}}{u^2+\frac{1}{2}} du$

$= -\frac{\sqrt{2}}{4} \int \frac{u}{u^2+\frac{1}{2}} du + \frac{1}{4} \int \frac{1}{u^2+\frac{1}{2}} du$

$= -\frac{\sqrt{2}}{8} \ln\left(u^2 + \frac{1}{2}\right) + \frac{1}{4} \cdot \sqrt{2} \arctan\left(\sqrt{2}u\right) + C$

$= -\frac{\sqrt{2}}{8} \ln\left(x^2 - \sqrt{2}x + 1\right) + \frac{\sqrt{2}}{4} \arctan\left(\sqrt{2}x - 1\right) + C$

By similar analysis for the second integral:
$\int \frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+\sqrt{2}x+1} dx = \frac{\sqrt{2}}{8} \ln\left(x^2 + \sqrt{2}x + 1\right) + \frac{\sqrt{2}}{4} \arctan\left(\sqrt{2}x + 1\right) + C$

Answer:

$$\boxed{\frac{\sqrt{2}}{8} \ln\left|\frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1}\right| + \frac{\sqrt{2}}{4}\left[\arctan\left(\sqrt{2}x + 1\right) + \arctan\left(\sqrt{2}x - 1\right)\right] + C}$$

Example 8.3.2: Evaluate $\int \frac{dx}{x^5+1}$

Step 1: Factor $x^5 + 1$

The roots of $x^5 = -1$ are $z_k = e^{i\pi(2k+1)/5}$ for $k = 0, 1, 2, 3, 4$. The angles are $\pi/5, 3\pi/5, \pi, 7\pi/5, 9\pi/5$.

The real factor is $(x+1)$ from $z_2 = e^{i\pi} = -1$.

Coupling symmetric angles:

$$x^5+1 = (x+1)\underbrace{(x^2-2x\cos(\pi/5)+1)}_{Q_1(x)}\underbrace{(x^2-2x\cos(3\pi/5)+1)}_{Q_2(x)}$$

where $\cos(\pi/5) = \frac{1+\sqrt{5}}{4}$ and $\cos(3\pi/5) = \frac{1-\sqrt{5}}{4}$.

Step 2: Partial Fraction Decomposition (PFD)

The decomposition is $\frac{1}{x^5+1} = \frac{A}{x+1} + \frac{B_1x+C_1}{Q_1(x)} + \frac{B_2x+C_2}{Q_2(x)}$.

A. Linear Term Coefficient:

Coefficients comparison is applicable, but here we will adopt a method that can be derived from Taylor expansion: the residue method.

$$A = \frac{1}{(x^5+1)'}\Big|_{x=-1} = \frac{1}{5x^4}\Big|_{x=-1} = \frac{1}{5}$$

B. Quadratic Term Coefficients (based on the general solution form):

Let $\alpha = \pi/5$ and $\beta = 3\pi/5$. Coefficients comparison gives:

$$\begin{cases} B_1 = \frac{2}{5}\cos\alpha - \frac{1}{5}, & C_1 = \frac{2}{5} - \frac{1}{5}\cos\alpha \\ B_2 = \frac{2}{5}\cos\beta - \frac{1}{5}, & C_2 = \frac{2}{5} - \frac{1}{5}\cos\beta \end{cases}$$

Step 3: Integration and Final Answer

The general solution structure for $I = \int \frac{dx}{x^5+1}$ is:

$$\int \frac{B_kx+C_k}{x^2-2x\cos\theta+1}dx = \frac{B_k}{2}\ln\left(x^2-2x\cos\theta+1\right) + \frac{B_k\cos\theta+C_k}{\sin\theta}\arctan\left(\frac{x-\cos\theta}{\sin\theta}\right)$$

$$\boxed{\int \frac{dx}{x^5+1} = \frac{1}{5}\ln|x+1|}$$
$$\boxed{+\sum_{k=1}^{2}\left[\frac{B_k}{2}\ln\left(x^2-2x\cos\theta_k+1\right) + \frac{B_k\cos\theta_k+C_k}{\sin\theta_k}\arctan\left(\frac{x-\cos\theta_k}{\sin\theta_k}\right)\right]+C}$$

where $\theta_1 = \pi/5, \theta_2 = 3\pi/5$, and B_k, C_k are the PFD coefficients given above.

Example 8.3.3: Evaluate $\int \frac{dx}{x^8+1}$

Step 1: Factor $x^8 + 1$

The roots of $x^8 = -1$ are $z_k = e^{i\pi(2k+1)/8}$ for $k = 0, 1, \dots, 7$.

Since $n = 8$ is even and the power is $x^n + 1$, there are no real roots. The roots form **four conjugate pairs** with angles $\theta_k = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$.

$x^8 + 1 = \prod_{k=0}^{3} \left(x^2 - 2x\cos\left(\frac{(2k+1)\pi}{8}\right) + 1\right)$
$= (x^2 - 2x\cos(\pi/8) + 1)(x^2 - 2x\cos(3\pi/8) + 1)$
$\cdot(x^2 - 2x\cos(5\pi/8) + 1)(x^2 - 2x\cos(7\pi/8) + 1)$

Step 2: Partial Fraction Decomposition and Integration

The PFD is $\frac{1}{x^8+1} = \sum_{k=1}^{4} \frac{A_k x + B_k}{Q_k(x)}$. The final integral is a sum of four complex quadratic integrals.

Final Answer:

$$\boxed{\int \frac{dx}{x^8+1} = \sum_{k=1}^{4}\left[\frac{A_k}{2}\ln(Q_k(x)) + \frac{A_k\cos\theta_k + B_k}{\sin\theta_k}\arctan\left(\frac{x - \cos\theta_k}{\sin\theta_k}\right)\right] + C}$$

where $Q_k(x) = x^2 - 2x\cos\theta_k + 1$, $\theta_k \in \{\pi/8, 3\pi/8, 5\pi/8, 7\pi/8\}$, and A_k, B_k are the PFD coefficients determined by routine algebra.

Example 8.3.4: Evaluate $\int \frac{dx}{x^6-1}$ (has linear factors from real roots)

Step 1: Factor $x^6 - 1$

$x^6 - 1 = (x^3 - 1)(x^3 + 1) = (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)$

Step 2: Partial Fraction Decomposition (PFD)

$\frac{1}{x^6-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2-x+1}$

A. Linear Term Coefficients:

- Set $x = 1$: $A(2)(3)(1) = 1 \Rightarrow A = \frac{1}{6}$
- Set $x = -1$: $B(-2)(1)(3) = 1 \Rightarrow B = -\frac{1}{6}$

B. Quadratic Term Coefficients (Full PFD):

$\frac{1}{x^6-1} = \frac{1/6}{x-1} - \frac{1/6}{x+1} - \frac{(x+2)/6}{x^2+x+1} + \frac{(x-2)/6}{x^2-x+1}$

Step 3: Integration

$I = \frac{1}{6}\int\left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx - \frac{1}{6}\int\frac{x+2}{x^2+x+1}dx + \frac{1}{6}\int\frac{x-2}{x^2-x+1}dx$

For $\int\frac{x+2}{x^2+x+1}dx$: Complete the square: $x^2+x+1 = (x+\frac{1}{2})^2 + \frac{3}{4}$

Split: $\frac{1}{2}\int\frac{2x+1}{x^2+x+1}dx + \frac{3}{2}\int\frac{1}{x^2+x+1}dx$

$= \frac{1}{2}\ln\left(x^2+x+1\right) + \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$

$= \frac{1}{2}\ln\left(x^2+x+1\right) + \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$

For $\int\frac{x-2}{x^2-x+1}dx$: Complete the square: $x^2-x+1 = (x-\frac{1}{2})^2 + \frac{3}{4}$

Split: $\frac{1}{2}\int\frac{2x-1}{x^2-x+1}dx - \frac{3}{2}\int\frac{1}{x^2-x+1}dx$

$= \frac{1}{2}\ln\left(x^2-x+1\right) - \sqrt{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$

Final Answer:

$$\boxed{\int\frac{dx}{x^6-1} = \frac{1}{6}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{12}\ln\left(x^2+x+1\right) + \frac{1}{12}\ln\left(x^2-x+1\right)}$$
$$\boxed{-\frac{\sqrt{3}}{6}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{\sqrt{3}}{6}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C}$$

Example 8.3.5: Evaluate $\int\frac{x^2dx}{x^4+1}$

Solution:

$x^4+1 = (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$

Partial fractions: $\frac{x^2}{x^4+1} = \frac{Ax+B}{x^2-\sqrt{2}x+1} + \frac{Cx+D}{x^2+\sqrt{2}x+1}$

Multiply through: $x^2 = (Ax+B)(x^2+\sqrt{2}x+1) + (Cx+D)(x^2-\sqrt{2}x+1)$

By symmetry and coefficient matching:

- x^3: $0 = A + C$

- x^2: $1 = \sqrt{2}A + B - \sqrt{2}C + D$
- x^1: $0 = A + \sqrt{2}B - C - \sqrt{2}D$
- x^0: $0 = B + D$

Solving gives: $A = \frac{\sqrt{2}}{4}, B = 0, C = -\frac{\sqrt{2}}{4}, D = 0$

$\frac{x^2}{x^4+1} = \frac{\frac{\sqrt{2}}{4}x}{x^2-\sqrt{2}x+1} + \frac{-\frac{\sqrt{2}}{4}x}{x^2+\sqrt{2}x+1}$

Integrate each term:

For $\int \frac{\frac{\sqrt{2}}{4}x}{x^2-\sqrt{2}x+1}dx$:

Complete the square: $x^2 - \sqrt{2}x + 1 = (x - \frac{\sqrt{2}}{2})^2 + \frac{1}{2}$

$= \frac{\sqrt{2}}{4} \int \frac{x}{x^2-\sqrt{2}x+1}dx$

Let $u = x - \frac{\sqrt{2}}{2}$: $= \frac{\sqrt{2}}{4} \int \frac{u+\frac{\sqrt{2}}{2}}{u^2+\frac{1}{2}}du$

$= \frac{\sqrt{2}}{4} \int \frac{u}{u^2+\frac{1}{2}}du + \frac{\sqrt{2}}{4} \cdot \frac{\sqrt{2}}{2} \int \frac{1}{u^2+\frac{1}{2}}du$

$= \frac{\sqrt{2}}{8}\ln\left(u^2 + \frac{1}{2}\right) + \frac{1}{4} \cdot \sqrt{2}\arctan\left(\sqrt{2}u\right) + C$

$= \frac{\sqrt{2}}{8}\ln\left(x^2 - \sqrt{2}x + 1\right) + \frac{\sqrt{2}}{4}\arctan\left(\sqrt{2}x - 1\right) + C$

Similarly for $\int \frac{-\frac{\sqrt{2}}{4}x}{x^2+\sqrt{2}x+1}dx$: $= -\frac{\sqrt{2}}{8}\ln\left(x^2 + \sqrt{2}x + 1\right) + \frac{\sqrt{2}}{4}\arctan\left(\sqrt{2}x + 1\right) + C$

Combine:
$\int \frac{x^2dx}{x^4+1} = \frac{\sqrt{2}}{8}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + \frac{\sqrt{2}}{4}[\arctan\left(\sqrt{2}x - 1\right) + \arctan\left(\sqrt{2}x + 1\right)] + C$

Answer:

$$\boxed{\frac{\sqrt{2}}{8}\ln\left|\frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1}\right| + \frac{\sqrt{2}}{4}[\arctan\left(\sqrt{2}x - 1\right) + \arctan\left(\sqrt{2}x + 1\right)] + C}$$

Example 8.3.6: Evaluate $\int \sqrt{\tan x}\, dx$

(MATH1560J 2021 Second Midterm Examination, Exercise 2)

Step 1: Substitution to Rationalize.
Let $t = \sqrt{\tan x}$.

Then $t^2 = \tan x$, and $2t\,dt = \sec^2 x \; dx$.

Since $\sec^2 x = 1 + \tan^2 x = 1 + t^4$, we have $dx = \frac{2t}{1+t^4}\,dt$.

The integral becomes a rational function in t:

$$\int \sqrt{\tan x} \; dx = \int t \cdot \frac{2t}{1+t^4}\,dt = \int \mathbf{\frac{2t^2}{t^4+1}\,dt}$$

Step 2: Integration of $\frac{2t^2}{t^4+1}$

Using the algebraic splitting $\frac{2t^2}{t^4+1} = \frac{t^2+1}{t^4+1} + \frac{t^2-1}{t^4+1}$, integrating, and combining the resulting logarithmic and arctangent terms:

$$\int \frac{2t^2}{t^4+1} dt = \frac{1}{2\sqrt{2}} \ln\left(\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right) + \frac{1}{\sqrt{2}} \arctan\left(\frac{t^2-1}{\sqrt{2}t}\right) + C$$

Step 3: Back Substitution ($t = \sqrt{\tan x}$)

Final Answer:

$$\boxed{\int \sqrt{\tan x} \; dx = \frac{1}{2\sqrt{2}} \ln\left(\frac{\tan x - \sqrt{2\tan x} + 1}{\tan x + \sqrt{2\tan x} + 1}\right) + \frac{1}{\sqrt{2}} \arctan\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + C}$$

Note that 8.3.5 and 8.3.6 have the same algebraic expression, but the answer differs in the tangent argument. This is because the substitution $t = \sqrt{\tan x}$ only maps $x \in (0, \frac{\pi}{2})$ to $t \in (0, +\infty)$, and the arctangent function has a period of π. Therefore, the two answers differ by a constant.

Differing by a constant is a common occurrence when dealing with inverse trigonometric functions, as is also seen in the integration of $\sec x$ and $\csc x$, $arcsinx$ and $arccosx$, etc. Besides, trigonometric identities, such as double angle formulas or $\tan\left(x + \frac{\pi}{4}\right) = \frac{\tan x + 1}{1 - \tan x}$, can also lead to different forms. In the examinations, TAs will try as many forms as possible to check for equivalence. You will get full marks as long as you can justify the equivalence.

8.4 Complex Partial Fraction Decomposition

We can extend partial fraction decomposition to the complex domain by factoring polynomials into linear factors over $\mathbb{C}$. This approach often simplifies the integration process for high-degree polynomials.

Complex Analysis and Residue Calculus will be covered in MATH2560J, so don't be worried if you struggle to understand this method.

Fundamental Theorem of Algebra: Every polynomial of degree n has exactly n complex roots (counting multiplicity).

Theorem (Residue Formula for Partial Fractions):
Let $Q(x)$ be a polynomial of degree n with distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then for any polynomial $P(x)$ with $\deg(P) < n$, we have:

$\frac{P(x)}{Q(x)} = \sum_{k=1}^{n} \frac{A_k}{x-\alpha_k}$

where the coefficients are given by the residue formula:

$$\boxed{A_k = \frac{P(\alpha_k)}{Q'(\alpha_k)}}$$

Proof:

Let $Q(x) = c(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ where c is the leading coefficient.

Step 1: Consider the function
$R(x) = \frac{P(x)}{Q(x)} - \sum_{k=1}^{n} \frac{A_k}{x-\alpha_k}$

We want to choose A_k such that $R(x) \equiv 0$.

Step 2: Multiply both sides by $Q(x)$:
$P(x) - \sum_{k=1}^{n} A_k \cdot \frac{Q(x)}{x-\alpha_k} = Q(x)R(x)$

Step 3: Evaluate at $x = \alpha_j$:
For $k \neq j$:
$\lim_{x\to\alpha_j} \frac{Q(x)}{x-\alpha_k} = \frac{Q(\alpha_j)}{\alpha_j-\alpha_k} = 0$

For $k = j$, we have the indeterminate form $\frac{0}{0}$. Using L'Hôpital's rule:
$\lim_{x\to\alpha_j} \frac{Q(x)}{x-\alpha_j} = Q'(\alpha_j)$

Therefore, evaluating at $x = \alpha_j$ gives:
$P(\alpha_j) - A_j Q'(\alpha_j) = Q(\alpha_j)R(\alpha_j) = 0$

Step 4: Solve for A_j:
$A_j = \frac{P(\alpha_j)}{Q'(\alpha_j)}$

Step 5: Verify completeness:
Both sides of the original equation are rational functions with the same poles and the same principal parts at each pole. Therefore, their difference is an entire rational function. Since $\deg(P) < n$ and the partial fraction sum has degree < 0, the difference is a polynomial of negative degree, hence identically zero.

Corollary: For the special case $P(x) = 1$, we have:
$A_k = \frac{1}{Q'(\alpha_k)}$

Example Application:
For $Q(x) = x^4 + 1$ with roots $\alpha_k = e^{i\pi(2k+1)/4}$, we have:
$Q'(x) = 4x^3 \Rightarrow A_k = \frac{1}{4\alpha_k^3}$

This is exactly the formula used in our complex partial fraction decompositions.

Key Insight 1: Logarithm Sum $\Rightarrow$ Real Logarithm
For a complex conjugate pair α and $\bar{\alpha}$, and real x:

$$\ln(x - \alpha) + \ln(x - \bar{\alpha}) = \ln(x - \alpha)(x - \bar{\alpha})$$

Let $\alpha = a + bi$, then:
$(x - \alpha)(x - \bar{\alpha}) = (x - a - bi)(x - a + bi) = (x - a)^2 + b^2$

Therefore:
$\ln(x - \alpha) + \ln(x - \bar{\alpha}) = \ln(x - a)^2 + b^2$

This is the mechanism by which complex logarithms combine to yield real logarithms of quadratic factors.

Key Insight 2: Logarithm Difference $\Rightarrow$ Arctangent
For the same conjugate pair:

$$\ln(x - \alpha) - \ln(x - \bar{\alpha}) = \ln\left(\frac{x-\alpha}{x-\bar{\alpha}}\right)$$

Let's analyze the argument:
$\frac{x-\alpha}{x-\bar{\alpha}} = \frac{x-a-bi}{x-a+bi} = \frac{(x-a-bi)^2}{(x-a)^2+b^2}$

The magnitude is 1, so we're dealing purely with phase. Writing in polar form:
$\frac{x-\alpha}{x-\bar{\alpha}} = e^{2i\theta} \quad \text{where} \quad \theta = -\arg(x-\alpha)$

But more usefully, note that:
$\frac{x-\alpha}{x-\bar{\alpha}} = \frac{(x-a)-bi}{(x-a)+bi}$

This is a complex number on the unit circle. Its argument is:
$\arg\left(\frac{x-\alpha}{x-\bar{\alpha}}\right) = -2\arctan\left(\frac{b}{x-a}\right)$

Therefore:
$\frac{1}{2i}[\ln(x-\alpha) - \ln(x-\bar{\alpha})] = \frac{1}{2i}\ln\left(\frac{x-\alpha}{x-\bar{\alpha}}\right) = \arctan\left(\frac{b}{x-a}\right)$

This is the mechanism by which differences of complex logarithms yield arctangent terms.

General Combination Formula:
For complex coefficients A and $\bar{A}$, and conjugate roots α and $\bar{\alpha}$:

$$A\ln(x-\alpha) + \bar{A}\ln(x-\bar{\alpha}) = 2\text{Re}(A)\ln|x-\alpha| + 2\text{Im}(A)\arg(x-\alpha)$$

Since $|x-\alpha|^2 = (x-a)^2 + b^2$ and $\arg(x-\alpha) = -\arctan\left(\frac{b}{x-a}\right)$, this becomes:

$$= \text{Re}(A)\ln^{(x-a)^2+b^2} - 2\text{Im}(A)\arctan\left(\frac{b}{x-a}\right)$$

This is the fundamental identity that allows us to recover real-valued expressions from complex partial fraction decompositions.

Complex Decomposition Strategy:

1. Factor $Q(x)$ into complex linear factors
2. Write partial fractions with complex coefficients using the residue theorem
3. Integrate to obtain complex logarithms
4. Combine conjugate terms using the above identities to recover real-valued expressions

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Example 8.4.1: Evaluate $\int \frac{dx}{x^4+1}$ using complex methods

Step 1: Complex Factorization

Find all complex roots of $x^4 = -1$:
$x^4 = -1 = e^{i(\pi+2\pi k)} \Rightarrow x_k = e^{i\pi(2k+1)/4}, \quad k = 0, 1, 2, 3$

- $x_0 = e^{i\pi/4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$
- $x_1 = e^{i3\pi/4} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$
- $x_2 = e^{i5\pi/4} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$
- $x_3 = e^{i7\pi/4} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$x^4 + 1 = (x - x_0)(x - x_1)(x - x_2)(x - x_3)$

Step 2: Complex Partial Fractions

$\frac{1}{x^4+1} = \frac{A}{x-x_0} + \frac{B}{x-x_1} + \frac{C}{x-x_2} + \frac{D}{x-x_3}$

Using the residue formula: $A = \frac{1}{Q'(x_0)} = \frac{1}{4x_0^3}$

Calculate each coefficient:

- $A = \frac{1}{4e^{i3\pi/4}} = \frac{1}{4}e^{-i3\pi/4} = -\frac{\sqrt{2}}{8} - i\frac{\sqrt{2}}{8}$
- $B = \frac{1}{4e^{i9\pi/4}} = \frac{1}{4}e^{-i9\pi/4} = \frac{1}{4}e^{-i\pi/4} = \frac{\sqrt{2}}{8} - i\frac{\sqrt{2}}{8}$
- $C = \bar{B} = \frac{\sqrt{2}}{8} + i\frac{\sqrt{2}}{8}$
- $D = \bar{A} = -\frac{\sqrt{2}}{8} + i\frac{\sqrt{2}}{8}$

Step 3: Integration

$\int \frac{dx}{x^4+1} = A\ln(x - x_0) + B\ln(x - x_1) + C\ln(x - x_2) + D\ln(x - x_3) + K$

Step 4: Combine Conjugate Terms

Group conjugate pairs: (A, D) and (B, C)

First pair (A and D):
$A\ln(x - x_0) + D\ln(x - x_3) = A\ln(x - x_0) + \bar{A}\ln(x - \bar{x_0})$

Using the identity:
$A\ln(x - x_0) + \bar{A}\ln(x - \bar{x_0}) = 2\text{Re}(A)\ln|x - x_0| + 2\text{Im}(A)\arg(x - x_0)$

Since $\text{Re}(A) = -\frac{\sqrt{2}}{8}$ and $\text{Im}(A) = -\frac{\sqrt{2}}{8}$:

$= -\frac{\sqrt{2}}{4}\ln|x - x_0| - \frac{\sqrt{2}}{4}\arg(x - x_0)$

But $|x - x_0|^2 = x^2 - \sqrt{2}x + 1$ and $\arg(x - x_0) = -\arctan\left(\frac{\sqrt{2}/2}{x - \sqrt{2}/2}\right)$

Second pair (B and C):
$B\ln(x - x_1) + C\ln(x - x_2) = B\ln(x - x_1) + \bar{B}\ln(x - \bar{x_1})$

With $\mathrm{Re}(B) = \frac{\sqrt{2}}{8}$ and $\mathrm{Im}(B) = -\frac{\sqrt{2}}{8}$:

$= \frac{\sqrt{2}}{4}\ln|x - x_1| - \frac{\sqrt{2}}{4}\arg(x - x_1)$

Where $|x - x_1|^2 = x^2 + \sqrt{2}x + 1$ and $\arg(x - x_1) = -\arctan\left(\frac{\sqrt{2}/2}{x + \sqrt{2}/2}\right)$

Step 5: Final Combination

Combining all terms and simplifying the arctangent expressions:

$$\int \frac{dx}{x^4+1} = \frac{\sqrt{2}}{8}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right| + \frac{\sqrt{2}}{4}[\arctan\left(\sqrt{2}x + 1\right) + \arctan\left(\sqrt{2}x - 1\right)] + C$$

Example 8.4.2: Evaluate $\int \frac{dx}{x^5+1}$ using complex methods

Step 1: Complex Factorization

The roots of $x^5 = -1$ are:
$x_k = e^{i\pi(2k+1)/5}, \quad k = 0, 1, 2, 3, 4$

Explicit angles: $\frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$

$x^5 + 1 = (x+1)(x - e^{i\pi/5})(x - e^{-i\pi/5})(x - e^{i3\pi/5})(x - e^{-i3\pi/5})$

Step 2: Complex Partial Fractions

$$\frac{1}{x^5+1} = \frac{A}{x+1} + \frac{B}{x - e^{i\pi/5}} + \frac{C}{x - e^{-i\pi/5}} + \frac{D}{x - e^{i3\pi/5}} + \frac{E}{x - e^{-i3\pi/5}}$$

Using residue formula:

- $A = \frac{1}{5(-1)^4} = \frac{1}{5}$
- $B = \frac{1}{5e^{i4\pi/5}} = \frac{1}{5}e^{-i4\pi/5}$
- $C = \bar{B} = \frac{1}{5}e^{i4\pi/5}$
- $D = \frac{1}{5e^{i12\pi/5}} = \frac{1}{5}e^{-i12\pi/5} = \frac{1}{5}e^{-i2\pi/5}$
- $E = \bar{D} = \frac{1}{5}e^{i2\pi/5}$

Step 3: Integration

$$\int \frac{dx}{x^5+1} = \frac{1}{5}\ln|x+1| + B\ln\left(x - e^{i\pi/5}\right) + C\ln\left(x - e^{-i\pi/5}\right)$$
$$+D\ln\left(x - e^{i3\pi/5}\right) + E\ln\left(x - e^{-i3\pi/5}\right) + K$$

Step 4: Combine Conjugate Terms

First conjugate pair (B and C):
Let $\theta_1 = \pi/5$, then:
$B\ln\left(x - e^{i\theta_1}\right) + C\ln\left(x - e^{-i\theta_1}\right) = 2\mathrm{Re}(B)\ln|x - e^{i\theta_1}| + 2\mathrm{Im}(B)\arg(x - e^{i\theta_1})$

Calculate real and imaginary parts:
$\mathrm{Re}(B) = \frac{1}{5}\cos\left(-\frac{4\pi}{5}\right) = \frac{1}{5}\cos\left(\frac{4\pi}{5}\right) = -\frac{1}{5}\cos\left(\frac{\pi}{5}\right)$
$\mathrm{Im}(B) = \frac{1}{5}\sin\left(-\frac{4\pi}{5}\right) = -\frac{1}{5}\sin\left(\frac{4\pi}{5}\right) = -\frac{1}{5}\sin\left(\frac{\pi}{5}\right)$

After simplification, this gives:
$= -\frac{2}{5}\cos(\pi/5)\ln\sqrt{x^2 - 2x\cos(\pi/5) + 1} - \frac{2}{5}\sin(\pi/5)\arctan\left(\frac{\sin(\pi/5)}{x - \cos(\pi/5)}\right)$

Second conjugate pair (D and E):
Let $\theta_2 = 3\pi/5$, similar calculation gives:
$= -\frac{2}{5}\cos(3\pi/5)\ln\sqrt{x^2 - 2x\cos(3\pi/5) + 1} - \frac{2}{5}\sin(3\pi/5)\arctan\left(\frac{\sin(3\pi/5)}{x - \cos(3\pi/5)}\right)$

Step 5: Final Answer

Combining all terms and simplifying coefficients:

$$\int \frac{dx}{x^5+1} = \frac{1}{5}\ln|x+1|$$
$$+\sum_{k=1}^{2}\left[\frac{B_k}{2}\ln\left(x^2 - 2x\cos\theta_k + 1\right) + \frac{B_k\cos\theta_k + C_k}{\sin\theta_k}\arctan\left(\frac{x - \cos\theta_k}{\sin\theta_k}\right)\right] + C$$

where $\theta_1 = \pi/5, \theta_2 = 3\pi/5$, and:

- $B_1 = -\frac{2}{5}\cos(\pi/5), C_1 = -\frac{2}{5}\sin(\pi/5)$
- $B_2 = -\frac{2}{5}\cos(3\pi/5), C_2 = -\frac{2}{5}\sin(3\pi/5)$

Example 8.4.3: Evaluate $\int \frac{dx}{x^6-1}$ using complex methods

Step 1: Complex Factorization

The roots of $x^6 = 1$ are the 6th roots of unity:
$x_k = e^{2\pi ik/6} = e^{\pi ik/3}, \quad k = 0, 1, 2, 3, 4, 5$

- $x_0 = 1$
- $x_1 = e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$
- $x_2 = e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$
- $x_3 = -1$
- $x_4 = e^{i4\pi/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$
- $x_5 = e^{i5\pi/3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

$$x^6 - 1 = (x-1)(x+1)(x-e^{i\pi/3})(x-e^{-i\pi/3})(x-e^{i2\pi/3})(x-e^{-i2\pi/3})$$

Step 2: Complex Partial Fractions

$$\frac{1}{x^6-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-e^{i\pi/3}} + \frac{D}{x-e^{-i\pi/3}} + \frac{E}{x-e^{i2\pi/3}} + \frac{F}{x-e^{-i2\pi/3}}$$

Using residue formula:

- $A = \frac{1}{6(1)^5} = \frac{1}{6}$
- $B = \frac{1}{6(-1)^5} = -\frac{1}{6}$
- $C = \frac{1}{6e^{i5\pi/3}} = \frac{1}{6}e^{-i5\pi/3} = \frac{1}{6}e^{i\pi/3}$
- $D = \bar{C} = \frac{1}{6}e^{-i\pi/3}$
- $E = \frac{1}{6e^{i10\pi/3}} = \frac{1}{6}e^{-i10\pi/3} = \frac{1}{6}e^{i2\pi/3}$
- $F = \bar{E} = \frac{1}{6}e^{-i2\pi/3}$

Step 3: Integration

$$\int \frac{dx}{x^6-1} = \frac{1}{6}\ln|x-1| - \frac{1}{6}\ln|x+1| + C\ln\left(x-e^{i\pi/3}\right)$$
$$+D\ln\left(x-e^{-i\pi/3}\right) + E\ln\left(x-e^{i2\pi/3}\right) + F\ln\left(x-e^{-i2\pi/3}\right) + K$$

Step 4: Combine Conjugate Terms

First conjugate pair (C and D):
Let $\theta = \pi/3$, then:
$$C\ln\left(x-e^{i\theta}\right) + D\ln\left(x-e^{-i\theta}\right) = 2\mathrm{Re}(C)\ln|x-e^{i\theta}| + 2\mathrm{Im}(C)\arg(x-e^{i\theta})$$

Calculate:
$\mathrm{Re}(C) = \frac{1}{6}\cos(\pi/3) = \frac{1}{12}$
$\mathrm{Im}(C) = \frac{1}{6}\sin(\pi/3) = \frac{\sqrt{3}}{12}$

This gives:
$$= \frac{1}{6}\ln\sqrt{x^2-x+1} + \frac{\sqrt{3}}{6}\arctan\left(\frac{\sqrt{3}/2}{x-1/2}\right)$$

Second conjugate pair (E and F):
Let $\phi = 2\pi/3$, then:
$E \ln\left(x - e^{i\phi}\right) + F \ln\left(x - e^{-i\phi}\right) = 2\mathrm{Re}(E) \ln|x - e^{i\phi}| + 2\mathrm{Im}(E) \arg(x - e^{i\phi})$

Calculate:
$\mathrm{Re}(E) = \frac{1}{6}\cos\left(2\pi/3\right) = -\frac{1}{12}$
$\mathrm{Im}(E) = \frac{1}{6}\sin\left(2\pi/3\right) = \frac{\sqrt{3}}{12}$

This gives:
$= -\frac{1}{6}\ln\sqrt{x^2+x+1} + \frac{\sqrt{3}}{6}\arctan\left(\frac{\sqrt{3}/2}{x+1/2}\right)$

Step 5: Final Combination

Combining all terms and simplifying:

$\int \frac{dx}{x^6-1} = \frac{1}{6}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{12}\ln\left(x^2+x+1\right) + \frac{1}{12}\ln\left(x^2-x+1\right)$
$-\frac{\sqrt{3}}{6}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{\sqrt{3}}{6}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$

Summary: Complex partial fraction decomposition provides a systematic approach to integrating rational functions. The key insight is that complex logarithms from conjugate pairs combine to yield real logarithms and arctangent functions in the final answer.

Part VI: Definite Integral

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9. Common Methods for Definite Integral

Some integrals, despite their simple appearance, cannot be expressed in terms of elementary functions (polynomials, exponentials, logarithms, trigonometric functions, and their inverses). However, we can still evaluate definite integrals involving these forms using clever techniques, and in the process, encounter important special functions that appear throughout mathematics and physics.

9.1 King Property (Substitution $x \to a+b-x$)

Theorem: For any integrable function f: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Proof: Let $u = a+b-x$, then $du = -dx$. When $x = a$, $u = b$; when $x = b$, $u = a$: $\int_a^b f(a+b-x)dx = -\int_b^a f(u)du = \int_a^b f(u)du = \int_a^b f(x)dx$

Application Strategy:

1. Compute $I = \int_a^b f(x)dx$
2. Also write $I = \int_a^b f(a+b-x)dx$
3. Add both expressions: $2I = \int_a^b [f(x) + f(a+b-x)]dx$
4. The sum often simplifies dramatically

Example 9.1.1: Evaluate $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$

Solution:

Let $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$

Apply King property with $a = 0, b = \pi/2$, so $a+b-x = \pi/2 - x$: $I = \int_0^{\pi/2} \frac{\sin(\pi/2-x)}{\sin(\pi/2-x)+\cos(\pi/2-x)} dx$

Using $\sin(\pi/2 - x) = \cos x$ and $\cos(\pi/2 - x) = \sin x$: $I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx$

Add both expressions: $2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}$

Therefore: $I = \frac{\pi}{4}$

Example 9.1.2: Evaluate $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

Solution:

Let $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

Apply King property with $a+b-x = \pi - x$: $I = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)} dx$

Using $\sin(\pi - x) = \sin x$ and $\cos(\pi - x) = -\cos x$: $I = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x} dx$

Add both: $2I = \int_0^{\pi} \frac{x \sin x + (\pi - x)\sin x}{1+\cos^2 x} dx = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x} dx$

$2I = \pi \int_0^\pi \frac{\sin x}{1+\cos^2 x} dx$

Let $u = \cos x$, $du = -\sin x\, dx$. When $x = 0, u = 1$; when $x = \pi, u = -1$:
$2I = \pi \int_1^{-1} \frac{-du}{1+u^2} = \pi \int_{-1}^{1} \frac{du}{1+u^2}$

$= \pi [\arctan{(u)}]_{-1}^{1} = \pi \left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi \cdot \frac{\pi}{2} = \frac{\pi^2}{2}$

Therefore: $I = \frac{\pi^2}{4}$

Example 9.1.3: Show that $\int_0^\pi x f(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx$

Solution:

Let $I = \int_0^\pi x f(\sin x) dx$

Apply King property with $\pi - x$:
$I = \int_0^\pi (\pi - x) f(\sin{(\pi - x)}) dx = \int_0^\pi (\pi - x) f(\sin x) dx$

(since $\sin{(\pi - x)} = \sin x$)

Add: $2I = \int_0^\pi [x + (\pi - x)] f(\sin x) dx = \pi \int_0^\pi f(\sin x) dx$

Therefore: $\boxed{\int_0^\pi x f(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx}$

Application: This is a powerful reduction formula for integrals of the form $\int_0^\pi x g(\sin x) dx$.

Example 9.1.4: Evaluate $\int_0^{2\pi} \frac{x \sin^2 x}{1+\cos x} dx$

Solution:

Use King property on $[0, 2\pi]$ with $2\pi - x$:

Let $I = \int_0^{2\pi} \frac{x \sin^2 x}{1+\cos x} dx$

$I = \int_0^{2\pi} \frac{(2\pi - x)\sin^2(2\pi - x)}{1+\cos(2\pi - x)} dx = \int_0^{2\pi} \frac{(2\pi - x)\sin^2 x}{1+\cos x} dx$

Adding: $2I = 2\pi \int_0^{2\pi} \frac{\sin^2 x}{1+\cos x} dx$

Now evaluate $J = \int_0^{2\pi} \frac{\sin^2 x}{1+\cos x} dx = \int_0^{2\pi} \frac{1-\cos^2 x}{1+\cos x} dx$

$= \int_0^{2\pi} \frac{(1-\cos x)(1+\cos x)}{1+\cos x} dx = \int_0^{2\pi} (1-\cos x) dx = [x - \sin x]_0^{2\pi} = 2\pi$

Therefore: $2I = 2\pi \cdot 2\pi = 4\pi^2$, so $I = 2\pi^2$

9.2 Even-Odd Function Properties

Theorem:

1. If $f(-x) = f(x)$ (even): $\int_{-a}^{a} f(x) dx = 2 \int_0^a f(x) dx$
2. If $f(-x) = -f(x)$ (odd): $\int_{-a}^{a} f(x) dx = 0$

Example 9.2.1: Evaluate $\int_{-\pi}^{\pi} \frac{x^2 \sin x}{1+e^x} dx$

Solution:

Key observation: For $g(x) = \frac{1}{1+e^x}$, we have: $g(-x) = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x}$

So: $g(x) + g(-x) = \frac{1}{1+e^x} + \frac{e^x}{1+e^x} = 1$

Let $I = \int_{-\pi}^{\pi} \frac{x^2 \sin x}{1+e^x} dx$

Split: $I = \int_{-\pi}^{0} \frac{x^2 \sin x}{1+e^x} dx + \int_0^{\pi} \frac{x^2 \sin x}{1+e^x} dx$

For the first integral, substitute $u = -x$:

$\int_{-\pi}^{0} \frac{x^2 \sin x}{1+e^x} dx = \int_{\pi}^{0} \frac{u^2 \sin(-u)}{1+e^{-u}} (-du) = \int_0^{\pi} \frac{u^2 \sin u \cdot e^u}{1+e^u} du$

Therefore: $I = \int_0^{\pi} \frac{x^2 \sin x \cdot e^x}{1+e^x} dx + \int_0^{\pi} \frac{x^2 \sin x}{1+e^x} dx = \int_0^{\pi} x^2 \sin x \, dx$

Now use integration by parts twice. Let $u = x^2$, $dv = \sin x \, dx$, so $v = -\cos x$:

$\int_0^{\pi} x^2 \sin x \, dx = [-x^2 \cos x]_0^{\pi} + 2 \int_0^{\pi} x \cos x \, dx = \pi^2 + 2 \int_0^{\pi} x \cos x \, dx$

For $\int_0^{\pi} x \cos x \, dx$, let $u = x$, $dv = \cos x \, dx$, so $v = \sin x$:

$= [x \sin x]_0^{\pi} - \int_0^{\pi} \sin x \, dx = 0 - [-\cos x]_0^{\pi} = -[1 - (-1)] = -2$

Therefore: $I = \pi^2 + 2(-2) = \pi^2 - 4$

9.3 Periodic Function Integrals

Properties:

1. If $f(x+T) = f(x)$, then $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$
2. **Shift property:** $\int_a^{a+T} f(x)dx = \int_0^T f(x)dx$

Example 9.3.1: Evaluate $\int_0^{3\pi} |\sin x|dx$

Solution:

The function $|\sin x|$ has period π (not 2π).

Therefore: $\int_0^{3\pi} |\sin x|dx = 3\int_0^{\pi} |\sin x|dx$

On $[0, \pi]$, $\sin x \geq 0$, so $|\sin x| = \sin x$:

$= 3\int_0^{\pi} \sin x\, dx = 3[-\cos x]_0^{\pi} = 3[-(-1) - (-1)] = 3 \cdot 2 = 6$

Example 9.3.2: Evaluate $\int_5^{11} \cos^2(\pi x)dx$

Solution:

The function $\cos^2(\pi x)$ has period $T = 1$

(since $\cos^2(\pi(x+1)) = \cos^2(\pi x + \pi) = \cos^2(\pi x)$).

The interval $[5, 11]$ has length $6 = 6T$, so:

$\int_5^{11} \cos^2(\pi x)dx = \int_0^6 \cos^2(\pi x)dx = 6\int_0^1 \cos^2(\pi x)dx$

Now: $\cos^2(\pi x) = \frac{1+\cos(2\pi x)}{2}$

$\int_0^1 \cos^2(\pi x)dx = \int_0^1 \frac{1+\cos(2\pi x)}{2}dx = \frac{1}{2}\left[x + \frac{\sin(2\pi x)}{2\pi}\right]_0^1 = \frac{1}{2}[1 + 0 - 0 - 0] = \frac{1}{2}$

Therefore: $\int_5^{11} \cos^2(\pi x)dx = 6 \cdot \frac{1}{2} = 3$

9.4 Wallis's Formula

For integrals of the form $I_n = \int_0^{\pi/2} \sin^n x\, dx$ or $\int_0^{\pi/2} \cos^n x\, dx$:

Key observations:

1. By substitution $u = \pi/2 - x$: $\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx$
2. Both satisfy the reduction formula: $I_n = \frac{n-1}{n} I_{n-2}$

Wallis's Formula: $\int_0^{\pi/2} \sin^{2n} x\, dx = \int_0^{\pi/2} \cos^{2n} x\, dx = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2}$

$\int_0^{\pi/2} \sin^{2n+1} x\, dx = \int_0^{\pi/2} \cos^{2n+1} x\, dx = \frac{(2n)!!}{(2n+1)!!}$

where $n!! = n(n-2)(n-4)\cdots$ (double factorial).

Connection: The ratio $\frac{I_{2n}}{I_{2n+1}} \to 1$ as $n \to \infty$, which leads to Wallis's product:

$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)(2n)}{(2n-1)(2n+1)} = \frac{2\cdot 2}{1\cdot 3} \cdot \frac{4\cdot 4}{3\cdot 5} \cdot \frac{6\cdot 6}{5\cdot 7} \cdots$

9.5 Feynman's Trick for Evaluation

Feynman's trick involves introducing a parameter to make an integral easier, differentiating with respect to that parameter, solving the simpler problem, then integrating back.

Nevertheless, chances are that the extra parameter to introduce has multiple choices but only one (or even zero) agreeable solution. Therefore, this trick involves deeper considerations beyond the scope of elementary calculus, and the following examples are just for illustration.

Example 9.5.1: Evaluate $\int_0^1 \frac{x^2-1}{\ln x} dx$

Solution:

Step 1: Introduce parameter

Consider: $I(a) = \int_0^1 \frac{x^a-1}{\ln x} dx$. We want $I(2)$.

Step 2: Differentiate

$I'(a) = \frac{d}{da} \int_0^1 \frac{x^a - 1}{\ln x} dx = \int_0^1 \frac{\partial}{\partial a} \left(\frac{x^a - 1}{\ln x} \right) dx$

$= \int_0^1 \frac{x^a \ln x}{\ln x} dx = \int_0^1 x^a \, dx = \left[\frac{x^{a+1}}{a+1} \right]_0^1 = \frac{1}{a+1}$

Step 3: Integrate back

$I(a) = \int \frac{1}{a+1} da = \ln(a+1) + C$

Step 4: Find constant

At $a = 0$: $I(0) = \int_0^1 \frac{1-1}{\ln x} dx = 0$

So: $0 = \ln(1) + C$, giving $C = 0$

Therefore: $I(a) = \ln(a+1)$. $I(2) = \ln(3)$

Example 9.5.2: Evaluate $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx$

Solution:

Step 1: Introduce parameter

Consider: $I(a) = \int_0^\infty \frac{e^{-ax}}{x} dx$

We want: $I(a) - I(b) = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx$

Step 2: Differentiate

$I'(a) = \frac{d}{da} \int_0^\infty \frac{e^{-ax}}{x} dx = \int_0^\infty \frac{\partial}{\partial a} \left(\frac{e^{-ax}}{x} \right) dx$

$= \int_0^\infty \frac{-xe^{-ax}}{x} dx = -\int_0^\infty e^{-ax} dx$

$= -\left[-\frac{e^{-ax}}{a} \right]_0^\infty = -\frac{1}{a}$

Step 3: Integrate back

$I(a) = -\int \frac{da}{a} = -\ln|a| + C$

Step 4: Apply to difference

$I(a) - I(b) = -\ln|a| + \ln|b| = \ln\left|\frac{b}{a}\right|$

For $a, b > 0$:

Answer: $\boxed{\ln\left(\frac{b}{a}\right)}$

9.6 Riemann Sum Limits

Recognition: Limits of the form: $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(k/n) = \int_0^1 f(x)dx$

General form: $\lim_{n\to\infty} \frac{b-a}{n}\sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right) = \int_a^b f(x)dx$

Example 9.6.1: Evaluate $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \sqrt{\frac{k}{n}}$

Solution: Recognize as: $\int_0^1 \sqrt{x}\,dx = \int_0^1 x^{1/2}dx = \left[\frac{x^{3/2}}{3/2}\right]_0^1 = \frac{2}{3}$

Example 9.6.2: Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \ln(1 + k/n) \cdot (1/n)$

Solution:

Recognize as a Riemann sum for $f(x) = \ln(1+x)$ on $[0, 1]$:

$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \ln(1 + k/n) = \int_0^1 \ln(1+x)dx$

Evaluate the integral using integration by parts:

Let $u = \ln(1+x)$, $dv = dx$, so $du = \frac{dx}{1+x}$, $v = x$:

$\int_0^1 \ln(1+x)dx = [x\ln(1+x)]_0^1 - \int_0^1 \frac{x}{1+x}dx$

$= 1 \cdot \ln(2) - 0 - \int_0^1 \frac{x}{1+x}dx$

For the remaining integral, write: $\frac{x}{1+x} = \frac{(1+x)-1}{1+x} = 1 - \frac{1}{1+x}$

$\int_0^1 \frac{x}{1+x}dx = \int_0^1 \left(1 - \frac{1}{1+x}\right)dx = [x - \ln(1+x)]_0^1$

$= 1 - \ln(2) - 0 = 1 - \ln(2)$

Therefore: $= \ln(2) - (1 - \ln(2)) = 2\ln(2) - 1$

Example 9.6.3: Evaluate $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{n}{n^2+k^2}$

Solution:

Rewrite: $\frac{n}{n^2+k^2} = \frac{1}{n} \cdot \frac{1}{1+(k/n)^2}$

Therefore: $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{n}{n^2+k^2} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+(k/n)^2} \cdot \frac{1}{n}$

$= \int_0^1 \frac{dx}{1+x^2} = [\arctan x]_0^1 = \frac{\pi}{4}$

Example 9.6.4: Evaluate $\lim_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n} \sin\left(\frac{k\pi}{n}\right)$

Solution:

This is a Riemann sum with $\Delta x = \frac{\pi}{n}$ on $[0, \pi]$:

$\lim_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n} \sin\left(\frac{k\pi}{n}\right) = \int_0^{\pi} \sin x \, dx$

$= [-\cos x]_0^{\pi} = -(-1) - (-1) = 2$

Example 9.6.5: Evaluate $\lim_{n\to\infty} \left(\frac{n!}{n^n}\right)^{1/n}$

(MATH2560J 2021 Second Midterm Examination, Exercise 4)

Solution:

Let L be the limit, then:
$\ln L = \lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{n!}{n^n}\right) = \lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{1}{n} \cdot \frac{2}{n} \cdots \frac{n}{n}\right)$
$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n}\right)$

Recognize as a Riemann sum:
$= \int_0^1 \ln x dx = [x \ln x - x]_0^1 = -1$

Therefore: $L = e^{-1}$

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4. Secant-Tangent Power Reduction

Integrals involving powers of secant and tangent require careful case analysis based on parity. The fundamental relationship $\tan^2(x) = \sec^2(x) - 1$ is the key to most transformations.

4.1 Classification by Parity

To integrate $\int \sec^m(x) \tan^n(x)\, dx$:

- If m is odd: Save $\sec(x) \tan(x)$, let $u = \sec(x)$.
- If m is even:
    - If n is even: Extract $\sec^2(x)$, use $\tan^2(x) = \sec^2(x) - 1$, let $u = \tan(x)$.
    - If n is odd: Use a reduction formula or special method.

Key Differentials:

- $d(\sec x) = \sec x \tan x\, dx$
- $d(\tan x) = \sec^2 x\, dx$

4.2 Secant Powers

Secant Reduction Formula Derivation:

For $I_n = \int \sec^n(x) dx$, write: $I_n = \int \sec^{n-2}(x) \sec^2(x) dx$

IBP: $u = \sec^{n-2}(x)$, $dv = \sec^2(x) dx$

- $du = (n-2) \sec^{n-3}(x) \cdot \sec(x) \tan(x) dx = (n-2) \sec^{n-2}(x) \tan(x) dx$
- $v = \tan(x)$

$I_n = \sec^{n-2}(x) \tan(x) - (n-2) \int \sec^{n-2}(x) \tan^2(x) dx$

Use $\tan^2(x) = \sec^2(x) - 1$:

$= \sec^{n-2}(x) \tan(x) - (n-2) \int \sec^{n-2}(x) [\sec^2(x) - 1] dx$

$= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^n(x)dx + (n-2)\int \sec^{n-2}(x)dx$

$= \sec^{n-2}(x)\tan(x) - (n-2)I_n + (n-2)I_{n-2}$

$I_n + (n-2)I_n = \sec^{n-2}(x)\tan(x) + (n-2)I_{n-2}$

$(n-1)I_n = \sec^{n-2}(x)\tan(x) + (n-2)I_{n-2}$

$$\boxed{\int \sec^n(x)dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)dx}$$

Base Cases:

$I_0 = \int dx = x + C$

$I_1 = \int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$

(Derivation of I_1: Multiply by $\frac{\sec x + \tan x}{\sec x + \tan x}$:

$\int \sec(x)dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx$

Let $u = \sec x + \tan x$, then $du = (\sec x \tan x + \sec^2 x)dx$:

$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$)

Example 4.2.1: Evaluate $\int \sec^3(x)\,dx$

Solution:

Using the reduction formula with $n = 3$:

$I_3 = \frac{\sec(x)\tan(x)}{2} + \frac{1}{2}I_1$

$I_1 = \ln|\sec(x) + \tan(x)|$

Therefore: $I_3 = \frac{\sec(x)\tan(x)}{2} + \frac{\ln|\sec(x)+\tan(x)|}{2} + C$

This is a famous integral that appears frequently in applications (arc length of parabolas, catenary curves, etc.).

Example 4.2.2: Evaluate $\int \sec^5(x)\,dx$

Solution:

Using the reduction formula:

$$I_5 = \frac{\sec^3(x)\tan(x)}{4} + \frac{3}{4} I_3$$

We already know $I_3 = \frac{\sec(x)\tan(x)}{2} + \frac{\ln|\sec(x)+\tan(x)|}{2}$:

$$I_5 = \frac{\sec^3(x)\tan(x)}{4} + \frac{3}{4}\left[\frac{\sec(x)\tan(x)}{2} + \frac{\ln|\sec(x)+\tan(x)|}{2}\right]$$

$$= \frac{\sec^3(x)\tan(x)}{4} + \frac{3\sec(x)\tan(x)}{8} + \frac{3\ln|\sec(x)+\tan(x)|}{8} + C$$

Example 4.2.3: Evaluate $\int \sec^4(x)\,dx$

Solution:

The exponent is even, so we can use a different approach. Write:

$$\sec^4(x) = \sec^2(x) \cdot \sec^2(x) = (1+\tan^2(x))\sec^2(x)$$

Let $u = \tan(x)$, $du = \sec^2(x)dx$:

$$\int \sec^4(x)dx = \int (1+u^2)du = u + \frac{u^3}{3} + C$$

$$= \tan(x) + \frac{\tan^3(x)}{3} + C$$

Alternative form: Factor out $\tan(x)$:

$$= \tan(x)\left(1 + \frac{\tan^2(x)}{3}\right) + C = \tan(x) \cdot \frac{3+\tan^2(x)}{3} + C$$

$$= \frac{\tan(x)(3+\tan^2(x))}{3} + C = \frac{\tan(x)(\sec^2(x)+2)}{3} + C$$

4.3 Tangent Powers

Tangent Reduction Strategy:

For $\int \tan^n(x)dx$, use the identity $\tan^2(x) = \sec^2(x) - 1$ repeatedly.

Tangent Reduction Formula Derivation:

Write $\tan^n(x) = \tan^{n-2}(x) \cdot \tan^2(x) = \tan^{n-2}(x)[\sec^2(x) - 1]$:

$\int \tan^n(x)dx = \int \tan^{n-2}(x)\sec^2(x)dx - \int \tan^{n-2}(x)dx$

For the first integral, let $u = \tan(x)$, $du = \sec^2(x)dx$:

$= \int u^{n-2}du - \int \tan^{n-2}(x)dx = \frac{u^{n-1}}{n-1} - \int \tan^{n-2}(x)dx$

$$\boxed{\int \tan^n(x)dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)dx}$$

Base Cases:

$\int \tan^0(x)dx = \int dx = x + C$

$\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}dx = -\ln|\cos(x)| + C = \ln|\sec(x)| + C$

$\int \tan^2(x)dx = \int(\sec^2(x) - 1)dx = \tan(x) - x + C$

Example 4.3.1: Evaluate $\int \tan^4(x)\,dx$

Solution:

Using the reduction formula:

$\int \tan^4(x)dx = \frac{\tan^3(x)}{3} - \int \tan^2(x)dx$

$= \frac{\tan^3(x)}{3} - [\tan(x) - x] + C$

$= \frac{\tan^3(x)}{3} - \tan(x) + x + C$

Example 4.3.2: Evaluate $\int \tan^6(x)\,dx$

Solution:

$\int \tan^6(x)dx = \frac{\tan^5(x)}{5} - \int \tan^4(x)dx$

$= \frac{\tan^5(x)}{5} - \left[\frac{\tan^3(x)}{3} - \tan(x) + x\right] + C$

$= \frac{\tan^5(x)}{5} - \frac{\tan^3(x)}{3} + \tan(x) - x + C$

Example 4.3.3: Evaluate $\int \tan^3(x)\, dx$

Solution:

$\int \tan^3(x)dx = \int \tan(x) \cdot \tan^2(x)dx = \int \tan(x)[\sec^2(x) - 1]dx$

$= \int \tan(x)\sec^2(x)dx - \int \tan(x)dx$

For the first integral, let $u = \tan(x)$, $du = \sec^2(x)dx$:

$= \int u, du - \int \tan(x)dx = \frac{u^2}{2} - \ln|\sec(x)| + C$

$= \frac{\tan^2(x)}{2} - \ln|\sec(x)| + C$

$= \frac{\tan^2(x)}{2} + \ln|\cos(x)| + C$

Example 4.3.4: Evaluate $\int \tan^5(x)\, dx$

Solution:

Using the reduction formula:

$\int \tan^5(x)dx = \frac{\tan^4(x)}{4} - \int \tan^3(x)dx$

$= \frac{\tan^4(x)}{4} - \frac{\tan^2(x)}{2} - \ln|\cos(x)| + C$

Example 4.3.5: Evaluate $\int \tan^7(x)\, dx$ (verify reduction pattern)

Solution:

$\int \tan^7(x)dx = \frac{\tan^6(x)}{6} - \int \tan^5(x)dx$

$= \frac{\tan^6(x)}{6} - \left[\frac{\tan^4(x)}{4} - \frac{\tan^2(x)}{2} + \ln|\sec(x)|\right] + C$

$= \frac{\tan^6(x)}{6} - \frac{\tan^4(x)}{4} + \frac{\tan^2(x)}{2} - \ln|\sec(x)| + C$

Pattern Observation:

For odd powers of tangent:
$\int \tan^{2k+1}(x)dx = \sum_{j=1}^{k} \frac{(-1)^{k-j}\tan^{2j}(x)}{2j} + (-1)^k \ln|\sec(x)| + C$

For even powers: $\int \tan^{2k}(x)dx = \sum_{j=1}^{k} \frac{(-1)^{k-j}\tan^{2j-1}(x)}{2j-1} + (-1)^{k+1}x + C$

4.4 Mixed Secant-Tangent

When both secant and tangent appear with various powers, the strategy depends on identifying which differential form is present (or can be created).

Strategy Summary:

1. **If tangent power is odd AND secant power is at least 1:** Extract $\sec(x)\tan(x)dx = d(\sec x)$, convert remaining tangent powers using $\tan^2 = \sec^2 - 1$
2. **If tangent power is even:** Extract $\sec^2(x)dx = d(\tan x)$, convert remaining secant powers using $\sec^2 = 1 + \tan^2$
3. **Otherwise:** Use reduction formulas or convert to sines and cosines

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Example 4.4.1: Evaluate $\int \tan^3(x)\sec^3(x)\,dx$

Solution:

Tangent power is odd, so extract $\sec(x)\tan(x)$:

$\int \tan^3(x)\sec^3(x)dx = \int \tan^2(x)\sec^2(x)\cdot[\sec(x)\tan(x)]dx$

Use $\tan^2(x) = \sec^2(x) - 1$:

$= \int[\sec^2(x) - 1]\sec^2(x)\cdot\sec(x)\tan(x)\,dx$

$= \int[\sec^4(x) - \sec^2(x)]\sec(x)\tan(x)dx$

Let $u = \sec(x)$, $du = \sec(x)\tan(x)dx$:

$= \int[u^4 - u^2]du = \frac{u^5}{5} - \frac{u^3}{3} + C$

$= \frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3} + C$

Example 4.4.2: Evaluate $\int \tan^2(x)\sec^4(x)\,dx$

Solution:

Tangent power is even, so extract $\sec^2(x)$:

$$\int \tan^2(x)\sec^4(x)dx = \int \tan^2(x)\sec^2(x)\cdot\sec^2(x)dx$$

Use $\sec^2(x) = 1 + \tan^2(x)$:

$$= \int \tan^2(x)[1 + \tan^2(x)]\sec^2(x)dx$$

$$= \int [\tan^2(x) + \tan^4(x)]\sec^2(x)dx$$

Let $u = \tan(x)$, $du = \sec^2(x)dx$:

$$= \int [u^2 + u^4]du = \frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= \frac{\tan^3(x)}{3} + \frac{\tan^5(x)}{5} + C$$

Example 4.4.3: Evaluate $\int \tan^4(x)\sec(x)\,dx$

Solution:

Tangent power is even, but we only have $\sec(x)$ (not $\sec^2$). We need a different approach.

Convert everything to sines and cosines:

$$\int \tan^4(x)\sec(x)dx = \int \frac{\sin^4(x)}{\cos^4(x)}\cdot\frac{1}{\cos(x)}dx = \int \frac{\sin^4(x)}{\cos^5(x)}dx$$

This doesn't simplify nicely either. Let's try a substitution approach.

Write $\tan^4(x) = (\sec^2(x) - 1)^2 = \sec^4(x) - 2\sec^2(x) + 1$:

$$\int \tan^4(x)\sec(x)dx = \int [\sec^4(x) - 2\sec^2(x) + 1]\sec(x)dx$$

$$= \int \sec^5(x)dx - 2\int \sec^3(x)dx + \int \sec(x)dx$$

We've computed these before:

- $\int \sec^5(x)dx = \frac{\sec^3(x)\tan(x)}{4} + \frac{3\sec(x)\tan(x)}{8} + \frac{3\ln|\sec(x)+\tan(x)|}{8}$
- $\int \sec^3(x)dx = \frac{\sec(x)\tan(x)}{2} + \frac{\ln|\sec(x)+\tan(x)|}{2}$
- $\int \sec(x)dx = \ln|\sec(x) + \tan(x)|$

Therefore: $= \frac{\sec^3(x)\tan(x)}{4} + \frac{3\sec(x)\tan(x)}{8} + \frac{3\ln|\sec(x)+\tan(x)|}{8}$
$-2\left[\frac{\sec(x)\tan(x)}{2} + \frac{\ln|\sec(x)+\tan(x)|}{2}\right] \quad + \ln|\sec(x) + \tan(x)| + C$

$$= \frac{\sec^3(x)\tan(x)}{4} + \frac{3\sec(x)\tan(x)}{8} - \sec(x)\tan(x) + \frac{3\ln|\sec(x)+\tan(x)|}{8} - \ln|\sec(x) + \tan(x)| + \ln|\sec(x) + \tan(x)| + C$$

$$= \frac{\sec^3(x)\tan(x)}{4} + \left[\frac{3}{8} - 1\right]\sec(x)\tan(x) + \frac{3\ln|\sec(x)+\tan(x)|}{8} + C$$

$$= \frac{\sec^3(x)\tan(x)}{4} - \frac{5\sec(x)\tan(x)}{8} + \frac{3\ln|\sec(x)+\tan(x)|}{8} + C$$

Example 4.4.4: Evaluate $\int \tan^5(x)\sec^3(x)\,dx$

Solution:

Tangent power is odd, extract $\sec(x)\tan(x)$:

$$\int \tan^5(x)\sec^3(x)dx = \int \tan^4(x)\sec^2(x) \cdot [\sec(x)\tan(x)]dx$$

Use $\tan^2(x) = \sec^2(x) - 1$, so $\tan^4(x) = (\sec^2(x) - 1)^2$:

$$= \int (\sec^2(x) - 1)^2 \sec^2(x) \cdot \sec(x)\tan(x)dx$$

$$= \int [\sec^4(x) - 2\sec^2(x) + 1]\sec^2(x) \cdot \sec(x)\tan(x)dx$$

$$= \int [\sec^6(x) - 2\sec^4(x) + \sec^2(x)]\sec(x)\tan(x)dx$$

Let $u = \sec(x)$, $du = \sec(x)\tan(x)dx$:

$$= \int [u^6 - 2u^4 + u^2]du = \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + C$$

$$= \frac{\sec^7(x)}{7} - \frac{2\sec^5(x)}{5} + \frac{\sec^3(x)}{3} + C$$

Example 4.4.5: Evaluate $\int \tan(x)\sec^5(x)\,dx$

Solution:

This is straightforward since we have $\sec(x)\tan(x)$:

Let $u = \sec(x)$, $du = \sec(x)\tan(x)dx$:

$$\int \tan(x)\sec^5(x)dx = \int \sec^4(x) \cdot [\sec(x)\tan(x)]dx$$

$= \int u^4 du = \frac{u^5}{5} + C$

$= \frac{\sec^5(x)}{5} + C$

Part III: Multiplication of Polynominal, Exponential, and Trigonometric

我见分部积分多妩媚，料分部积分见我应如是。

5. Polynomial × Exponential

5.1 The Reduction Formula

For $I_n = \int x^n e^{ax} dx$, using integration by parts with $u = x^n$, $dv = e^{ax} dx$:

$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$

Base case: $I_0 = \int e^{ax} dx = \frac{e^{ax}}{a} + C$

Tabular Method (Efficient for multiple reductions):

For integrals requiring multiple IBP applications, the tabular method is systematic:

DIFFERENTIATE	SIGN	INTEGRATE
x^n	$+$	e^{ax}
nx^{n-1}	$-$	$\frac{e^{ax}}{a}$
$n(n-1)x^{n-2}$	$+$	$\frac{e^{ax}}{a^2}$
$\vdots$	$\vdots$	$\vdots$
$n!$	$(-1)^n$	$\frac{e^{ax}}{a^n}$
0		$\frac{e^{ax}}{a^{n+1}}$

The answer is the sum of products along diagonals with alternating signs.

Example 5.1.1: Evaluate $\int x^3 e^{2x} dx$

Solution (Reduction Formula):

$$I_3 = \frac{x^3e^{2x}}{2} - \frac{3}{2}I_2$$

$$I_2 = \frac{x^2e^{2x}}{2} - \frac{2}{2}I_1 = \frac{x^2e^{2x}}{2} - I_1$$

$$I_1 = \frac{xe^{2x}}{2} - \frac{1}{2}I_0 = \frac{xe^{2x}}{2} - \frac{1}{2} \cdot \frac{e^{2x}}{2} = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4}$$

Back-substitute:

$$I_2 = \frac{x^2e^{2x}}{2} - \left(\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right) = \frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{e^{2x}}{4}$$

$$I_3 = \frac{x^3e^{2x}}{2} - \frac{3}{2}\left(\frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{e^{2x}}{4}\right)$$

$$= \frac{x^3e^{2x}}{2} - \frac{3x^2e^{2x}}{4} + \frac{3xe^{2x}}{4} - \frac{3e^{2x}}{8} + C$$

Factor out e^{2x}:

$$= e^{2x}\left(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8}\right) + C$$

Solution (Tabular Method):

DIFFERENTIATE	SIGN	INTEGRATE
x^3	$+$	e^{2x}
$3x^2$	$-$	$\frac{e^{2x}}{2}$
$6x$	$+$	$\frac{e^{2x}}{4}$
6	$-$	$\frac{e^{2x}}{8}$
0		

Result: $x^3 \cdot \frac{e^{2x}}{2} - 3x^2 \cdot \frac{e^{2x}}{4} + 6x \cdot \frac{e^{2x}}{8} - 6 \cdot \frac{e^{2x}}{16}$

$$= \frac{x^3e^{2x}}{2} - \frac{3x^2e^{2x}}{4} + \frac{3xe^{2x}}{4} - \frac{3e^{2x}}{8} + C \checkmark$$

Example 5.1.2: Evaluate $\int x^4e^{-x}dx$

Solution (Tabular Method):

DIFFERENTIATE	SIGN	INTEGRATE
x^4	$+$	e^{-x}
$4x^3$	$-$	$-e^{-x}$
$12x^2$	$+$	e^{-x}
$24x$	$-$	$-e^{-x}$
24	$+$	e^{-x}
0		

Result: $-x^4e^{-x} - 4x^3e^{-x} - 12x^2e^{-x} - 24xe^{-x} - 24e^{-x} + C$

$= -e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + C$

5.2 Gaussian-Type Integrals

Strategy for $\int x^{2n+1}e^{-x^2}dx$:

1. Substitute $u = -x^2$, so $du = -2x, dx$
2. This transforms to $-\frac{1}{2}\int u^n e^u du$
3. Apply polynomial-exponential reduction

General Pattern: The result is always $-\frac{e^{-x^2}}{2}P_n(x^2)$ where P_n is a polynomial of degree n.

Example 5.2.1: Evaluate $\int x^3e^{-x^2}dx$

Solution:

Let $u = -x^2$, then $du = -2x, dx$, so $x, dx = -\frac{1}{2}du$

Also, $x^2 = -u$

$\int x^3e^{-x^2}dx = \int x^2 \cdot e^{-x^2} \cdot x, dx = \int(-u) \cdot e^u \cdot \left(-\frac{1}{2}du\right)$

$= \frac{1}{2}\int ue^u du$

Integration by parts: u (the variable), $dv = e^u du$: $du = du, v = e^u$

$= \frac{1}{2}(ue^u - \int e^u du) = \frac{1}{2}(ue^u - e^u) + C$

$= \frac{e^u(u-1)}{2} + C$

Substitute back $u = -x^2$:

$= \frac{e^{-x^2}(-x^2-1)}{2} + C = -\frac{e^{-x^2}(x^2+1)}{2} + C$

Example 5.2.2: Evaluate $\int x^5 e^{-x^2} dx$

Solution:

Let $u = -x^2$, then $x^4 = u^2$ and $x, dx = -\frac{1}{2} du$

$\int x^5 e^{-x^2} dx = \int x^4 \cdot e^{-x^2} \cdot x, dx = \int u^2 \cdot e^u \cdot \left(-\frac{1}{2} du\right)$

$= -\frac{1}{2} \int u^2 e^u du$

Use tabular method:

DIFFERENTIATE	SIGN	INTEGRATE
u^2	$+$	e^u
$2u$	$-$	e^u
2	$+$	e^u
0		

$\int u^2 e^u du = u^2 e^u - 2ue^u + 2e^u + C = e^u(u^2 - 2u + 2) + C$

Therefore:

$-\frac{1}{2} \int u^2 e^u du = -\frac{e^u(u^2-2u+2)}{2} + C$

Substitute back $u = -x^2$:

$= -\frac{e^{-x^2}((-x^2)^2-2(-x^2)+2)}{2} + C$

$= -\frac{e^{-x^2}(x^4+2x^2+2)}{2} + C$

General Reduction Pattern:

For $\int x^{2n+1} e^{-x^2} dx$, let $u = -x^2$:

$\int x^{2n+1}e^{-x^2}dx = -\frac{1}{2}\int u^n e^u du$

Using the reduction formula for $\int u^n e^u du = e^u(u^n - nu^{n-1} + n(n-1)u^{n-2} - \cdots + (-1)^n n!)$:

$= -\frac{e^{-x^2}}{2}\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}(-x^2)^k + C$

$= -\frac{e^{-x^2}}{2}\sum_{k=0}^{n}(-1)^{n-k+k}\frac{n!}{k!}x^{2k} + C$

$= -\frac{e^{-x^2}}{2}\sum_{k=0}^{n}(-1)^{n}\frac{n!}{k!}x^{2k} + C$

5.3 Fractional Exponent Exponentials

General Strategy for $\int e^{x^{p/q}}dx$:

Step 1: Rationalize the exponent

- Let $t = x^{1/q}$ (where q is the denominator)
- Then $x = t^q$ and $dx = qt^{q-1}dt$
- So $x^{p/q} = (t^q)^{p/q} = t^p$

Step 2: Transform the integral $\int e^{x^{p/q}}dx = \int e^{t^p} \cdot qt^{q-1}dt$

Step 3: Evaluate based on p:

- If $p = 1$: Direct exponential-polynomial integration
- If $p > 1$: Generally non-elementary (elliptic-type)

Example 5.3.1: Evaluate $\int e^{x^{1/3}}dx$

Solution:

Here $p/q = 1/3$, so $p = 1$, $q = 3$.

Let $t = x^{1/3}$, so $x = t^3$ and $dx = 3t^2dt$

$\int e^{x^{1/3}}dx = \int e^t \cdot 3t^2dt = 3\int t^2e^tdt$

Use tabular method:

DIFFERENTIATE	SIGN	INTEGRATE
t^2	$+$	e^t
$2t$	$-$	e^t
2	$+$	e^t
0		

$\int t^2 e^t dt = t^2 e^t - 2te^t + 2e^t + C = e^t(t^2 - 2t + 2) + C$

Therefore:

$3 \int t^2 e^t dt = 3e^t(t^2 - 2t + 2) + C$

Substitute back $t = x^{1/3}$:

$= 3e^{x^{1/3}}((x^{1/3})^2 - 2x^{1/3} + 2) + C$

$= 3e^{x^{1/3}}(x^{2/3} - 2x^{1/3} + 2) + C$

Example 5.3.2: Evaluate $\int e^{\sqrt{x}} dx$

Solution:

Here we have $x^{1/2}$, so let $t = \sqrt{x}$.

Then $x = t^2$ and $dx = 2t, dt$

$\int e^{\sqrt{x}} dx = \int e^t \cdot 2t\, dt = 2 \int te^t dt$

Integration by parts: $u = t, dv = e^t dt, du = dt, v = e^t$

$= 2(te^t - \int e^t dt) = 2(te^t - e^t) + C$

$= 2e^t(t - 1) + C$

Substitute back $t = \sqrt{x}$:

$= 2e^{\sqrt{x}}(\sqrt{x} - 1) + C$

---

6. Exponential $\times$ Trigonometric

6.1 Basic Forms: $\int e^{Ax} \sin{(Bx)} dx$ and $\int e^{Ax} \cos{(Bx)} dx$

These integrals arise frequently in applications (differential equations, Fourier analysis, signal processing). They require either:

1. **Two applications of integration by parts** (traditional method)
2. **Complex exponentials** (Euler's formula method)

Method 1: Double Integration by Parts

Derivation for $I = \int e^{Ax} \sin{(Bx)} dx$:

First IBP: Let $u = \sin{(Bx)}$, $dv = e^{Ax} dx$

- $du = B \cos{(Bx)} dx$
- $v = \frac{e^{Ax}}{A}$

$$I = \frac{e^{Ax} \sin(Bx)}{A} - \frac{B}{A} \int e^{Ax} \cos{(Bx)} dx$$

Let $J = \int e^{Ax} \cos{(Bx)} dx$. Then:

$$I = \frac{e^{Ax} \sin(Bx)}{A} - \frac{B}{A} J \quad \cdots (1)$$

Second IBP on J: Let $u = \cos{(Bx)}$, $dv = e^{Ax} dx$

- $du = -B \sin{(Bx)} dx$
- $v = \frac{e^{Ax}}{A}$

$$J = \frac{e^{Ax} \cos(Bx)}{A} + \frac{B}{A} \int e^{Ax} \sin{(Bx)} dx$$

$$J = \frac{e^{Ax} \cos(Bx)}{A} + \frac{B}{A} I \quad \cdots (2)$$

Now solve the system of equations (1) and (2):

From (1): $AI = e^{Ax} \sin{(Bx)} - BJ$

From (2): $AJ = e^{Ax} \cos{(Bx)} + BI$

Multiply (2) by B: $ABJ = Be^{Ax} \cos{(Bx)} + B^2 I$

Substitute into the first equation:

$$AI = e^{Ax}\sin(Bx) - Be^{Ax}\cos(Bx) - B^2 I$$

$$AI + B^2 I = e^{Ax}(A\sin(Bx) - B\cos(Bx))$$

$$(A^2 + B^2)I = Ae^{Ax}\sin(Bx) - Be^{Ax}\cos(Bx)$$

$$\boxed{\int e^{Ax}\sin(Bx)dx = \frac{e^{Ax}(A\sin(Bx) - B\cos(Bx))}{A^2 + B^2} + C}$$

$$\boxed{\int e^{Ax}\cos(Bx)dx = \frac{e^{Ax}(A\cos(Bx) + B\sin(Bx))}{A^2 + B^2} + C}$$

Method 2: Complex Exponentials (Euler's Formula)

This method is more systematic and less prone to algebraic errors. It leverages the beautiful relationship: $e^{i\theta} = \cos\theta + i\sin\theta$

Consider the complex integral:

$$K = \int e^{Ax}\cdot e^{iBx}dx = \int e^{(A+iB)x}dx$$

This is straightforward:

$$K = \frac{e^{(A+iB)x}}{A+iB} + C_{\text{complex}}$$

Now rationalize the denominator by multiplying by the conjugate:

$$\frac{1}{A+iB} = \frac{A-iB}{(A+iB)(A-iB)} = \frac{A-iB}{A^2+B^2}$$

Therefore:

$$K = \frac{(A-iB)e^{(A+iB)x}}{A^2+B^2} + C_{\text{complex}}$$

$$= \frac{(A-iB)e^{Ax}e^{iBx}}{A^2+B^2} + C_{\text{complex}}$$

$$= \frac{(A-iB)e^{Ax}(\cos(Bx)+i\sin(Bx))}{A^2+B^2} + C_{\text{complex}}$$

Expand:

$= \frac{e^{Ax}}{A^2+B^2}[(A\cos{(Bx)} + B\sin{(Bx)}) + i(A\sin{(Bx)} - B\cos{(Bx)})] + C_{\text{complex}}$

But also:

$K = \int e^{Ax}(\cos{(Bx)} + i\sin{(Bx)})dx = \int e^{Ax}\cos{(Bx)}dx + i\int e^{Ax}\sin{(Bx)}dx$

$= J + iI$

Comparing real and imaginary parts:

$I = \int e^{Ax}\sin{(Bx)}dx = \frac{e^{Ax}(A\sin(Bx) - B\cos(Bx))}{A^2+B^2} + C$

$J = \int e^{Ax}\cos{(Bx)}dx = \frac{e^{Ax}(A\cos(Bx) + B\sin(Bx))}{A^2+B^2} + C$

The complex method is more systematic and generalizes beautifully. Once you're comfortable with complex numbers, it's often the preferred approach.

Example 6.1.1: Evaluate $\int e^{3x}\sin{(2x)}dx$

Solution (Using Formula):

Here $A = 3$, $B = 2$, so $A^2 + B^2 = 9 + 4 = 13$

$\int e^{3x}\sin{(2x)}dx = \frac{e^{3x}(3\sin(2x) - 2\cos(2x))}{13} + C$

Example 6.1.2: Evaluate $\int e^{-x}\cos{(3x)}dx$

Solution:

Here $A = -1$, $B = 3$, so $A^2 + B^2 = 1 + 9 = 10$

$\int e^{-x}\cos{(3x)}dx = \frac{e^{-x}(-\cos(3x) + 3\sin(3x))}{10} + C$

$= \frac{e^{-x}(3\sin(3x) - \cos(3x))}{10} + C$

6.2 Product-to-Sum Reduction

Strategy: When you have products like $\sin(Ax)\cos(Bx)$, use product-to-sum identities:

$\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\cos(A)\cos(B) = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

After applying these identities, integrate each term using the standard formulas.

Example 6.2.1: Evaluate $\int e^{2x}\sin(3x)\cos(x)dx$

Solution:

First, use the product-to-sum identity for $\sin(3x)\cos(x)$:

$\sin(3x)\cos(x) = \frac{1}{2}[\sin(3x+x) + \sin(3x-x)] = \frac{1}{2}[\sin(4x) + \sin(2x)]$

Therefore:

$\int e^{2x}\sin(3x)\cos(x)dx = \frac{1}{2}\int e^{2x}\sin(4x)dx + \frac{1}{2}\int e^{2x}\sin(2x)dx$

Apply the standard formula to each integral:

First integral: $A = 2$, $B = 4$, $A^2 + B^2 = 4 + 16 = 20$

$\int e^{2x}\sin(4x)dx = \frac{e^{2x}(2\sin(4x) - 4\cos(4x))}{20} = \frac{e^{2x}(\sin(4x) - 2\cos(4x))}{10}$

Second integral: $A = 2$, $B = 2$, $A^2 + B^2 = 8$

$\int e^{2x}\sin(2x)dx = \frac{e^{2x}(2\sin(2x) - 2\cos(2x))}{8} = \frac{e^{2x}(\sin(2x) - \cos(2x))}{4}$

Combine:

$= \frac{1}{2} \cdot \frac{e^{2x}(\sin(4x) - 2\cos(4x))}{10} + \frac{1}{2} \cdot \frac{e^{2x}(\sin(2x) - \cos(2x))}{4} + C$

$= \frac{e^{2x}(\sin(4x) - 2\cos(4x))}{20} + \frac{e^{2x}(\sin(2x) - \cos(2x))}{8} + C$

Example 6.2.2: Evaluate $\int e^{x}\sin(x)\sin(2x)dx$

Solution:

Use the product-to-sum identity for $\sin(x)\sin(2x)$:

$\sin(x)\sin(2x) = \frac{1}{2}[\cos(x-2x) - \cos(x+2x)] = \frac{1}{2}[\cos(-x) - \cos(3x)]$

$= \frac{1}{2}[\cos(x) - \cos(3x)]$

Therefore:

$\int e^x \sin(x)\sin(2x)dx = \frac{1}{2}\int e^x \cos(x)dx - \frac{1}{2}\int e^x \cos(3x)dx$

First integral: $A = 1, B = 1, A^2 + B^2 = 2$

$\int e^x \cos(x)dx = \frac{e^x(\cos(x)+\sin(x))}{2}$

Second integral: $A = 1, B = 3, A^2 + B^2 = 10$

$\int e^x \cos(3x)dx = \frac{e^x(\cos(3x)+3\sin(3x))}{10}$

Combine:

$= \frac{1}{2} \cdot \frac{e^x(\cos(x)+\sin(x))}{2} - \frac{1}{2} \cdot \frac{e^x(\cos(3x)+3\sin(3x))}{10} + C$

$= \frac{e^x(\cos(x)+\sin(x))}{4} - \frac{e^x(\cos(3x)+3\sin(3x))}{20} + C$

Example 6.2.3: Evaluate $\int e^{-x}\cos(2x)\cos(x)dx$

Solution:

Use $\cos(2x)\cos(x) = \frac{1}{2}[\cos(3x) + \cos(x)]$

$\int e^{-x}\cos(2x)\cos(x)dx = \frac{1}{2}\int e^{-x}\cos(3x)dx + \frac{1}{2}\int e^{-x}\cos(x)dx$

First integral: $A = -1, B = 3, A^2 + B^2 = 10$

$\int e^{-x}\cos(3x)dx = \frac{e^{-x}(-\cos(3x)+3\sin(3x))}{10}$

Second integral: $A = -1, B = 1, A^2 + B^2 = 2$

$\int e^{-x}\cos(x)dx = \frac{e^{-x}(-\cos(x)+\sin(x))}{2}$

Combine:

$$= \frac{1}{2} \cdot \frac{e^{-x}(3\sin(3x) - \cos(3x))}{10} + \frac{1}{2} \cdot \frac{e^{-x}(\sin(x) - \cos(x))}{2} + C$$

$$= \frac{e^{-x}(3\sin(3x) - \cos(3x))}{20} + \frac{e^{-x}(\sin(x) - \cos(x))}{4} + C$$

6.3 Mixed Trigonometric-Hyperbolic

When mixing trigonometric and hyperbolic functions, convert everything to exponentials. Recall:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

After conversion, the problem reduces to exponential-trigonometric integrals.

Example 6.3.1: Evaluate $\int \sin(x)\sinh(x)dx$

Solution:

Convert to exponentials:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\int \sin(x)\sinh(x)dx = \int \sin(x) \cdot \frac{e^x - e^{-x}}{2} dx$$

$$= \frac{1}{2}\int e^x \sin(x)dx - \frac{1}{2}\int e^{-x}\sin(x)dx$$

First integral: $A = 1$, $B = 1$, $A^2 + B^2 = 2$

$$\int e^x \sin(x)dx = \frac{e^x(\sin(x) - \cos(x))}{2}$$

Second integral: $A = -1$, $B = 1$, $A^2 + B^2 = 2$

$$\int e^{-x}\sin(x)dx = \frac{e^{-x}(-\sin(x) - \cos(x))}{2} = -\frac{e^{-x}(\sin(x) + \cos(x))}{2}$$

Combine:

$$= \frac{1}{2} \cdot \frac{e^x(\sin(x) - \cos(x))}{2} - \frac{1}{2} \cdot \left(-\frac{e^{-x}(\sin(x) + \cos(x))}{2}\right) + C$$

$= \frac{e^x(\sin(x)-\cos(x))}{4} + \frac{e^{-x}(\sin(x)+\cos(x))}{4} + C$

$= \frac{1}{4}[e^x(\sin(x) - \cos(x)) + e^{-x}(\sin(x) + \cos(x))] + C$

Factor by terms:

$= \frac{1}{4}[\sin(x)(e^x + e^{-x}) - \cos(x)(e^x - e^{-x})] + C$

$= \frac{1}{2}[\sin(x)\cosh(x) - \cos(x)\sinh(x)] + C$

Example 6.3.2: Evaluate $\int \cos(x)\cosh(x)dx$

Solution:

$\int \cos(x)\cosh(x)dx = \int \cos(x) \cdot \frac{e^x+e^{-x}}{2} dx$

$= \frac{1}{2}\int e^x \cos(x)dx + \frac{1}{2}\int e^{-x}\cos(x)dx$

First integral: $A = 1, B = 1$

$\int e^x \cos(x)dx = \frac{e^x(\cos(x)+\sin(x))}{2}$

Second integral: $A = -1, B = 1$

$\int e^{-x}\cos(x)dx = \frac{e^{-x}(-\cos(x)+\sin(x))}{2} = \frac{e^{-x}(\sin(x)-\cos(x))}{2}$

Combine:

$= \frac{1}{2} \cdot \frac{e^x(\cos(x)+\sin(x))}{2} + \frac{1}{2} \cdot \frac{e^{-x}(\sin(x)-\cos(x))}{2} + C$

$= \frac{e^x(\cos(x)+\sin(x))}{4} + \frac{e^{-x}(\sin(x)-\cos(x))}{4} + C$

$= \frac{1}{4}[e^x(\cos(x) + \sin(x)) + e^{-x}(\sin(x) - \cos(x))] + C$

Factor:

$= \frac{1}{4}[\sin(x)(e^x + e^{-x}) + \cos(x)(e^x - e^{-x})] + C$

$= \frac{1}{2}[\sin(x)\cosh(x) + \cos(x)\sinh(x)] + C$

Example 6.3.3: Evaluate $\int \sin(x)\cosh(2x)dx$

Solution:

$\int \sin(x)\cosh(2x)dx = \int \sin(x) \cdot \frac{e^{2x}+e^{-2x}}{2}dx$

$= \frac{1}{2}\int e^{2x}\sin(x)dx + \frac{1}{2}\int e^{-2x}\sin(x)dx$

First integral: $A = 2$, $B = 1$, $A^2 + B^2 = 5$

$\int e^{2x}\sin(x)dx = \frac{e^{2x}(2\sin(x)-\cos(x))}{5}$

Second integral: $A = -2$, $B = 1$, $A^2 + B^2 = 5$

$\int e^{-2x}\sin(x)dx = \frac{e^{-2x}(-2\sin(x)-\cos(x))}{5}$

Combine:

$= \frac{1}{2} \cdot \frac{e^{2x}(2\sin(x)-\cos(x))}{5} + \frac{1}{2} \cdot \frac{e^{-2x}(-2\sin(x)-\cos(x))}{5} + C$

$= \frac{e^{2x}(2\sin(x)-\cos(x))}{10} - \frac{e^{-2x}(2\sin(x)+\cos(x))}{10} + C$

Example 6.3.4: Evaluate $\int \cos(3x)\sinh(x)dx$

Solution:

$\int \cos(3x)\sinh(x)dx = \int \cos(3x) \cdot \frac{e^{x}-e^{-x}}{2}dx$

$= \frac{1}{2}\int e^{x}\cos(3x)dx - \frac{1}{2}\int e^{-x}\cos(3x)dx$

First integral: $A = 1$, $B = 3$, $A^2 + B^2 = 10$

$\int e^{x}\cos(3x)dx = \frac{e^{x}(\cos(3x)+3\sin(3x))}{10}$

Second integral: $A = -1$, $B = 3$, $A^2 + B^2 = 10$

$\int e^{-x}\cos(3x)dx = \frac{e^{-x}(-\cos(3x)+3\sin(3x))}{10}$

Combine:

$= \frac{1}{2} \cdot \frac{e^{x}(\cos(3x)+3\sin(3x))}{10} - \frac{1}{2} \cdot \frac{e^{-x}(3\sin(3x)-\cos(3x))}{10} + C$

$= \frac{e^{x}(\cos(3x)+3\sin(3x))}{20} - \frac{e^{-x}(3\sin(3x)-\cos(3x))}{20} + C$

$= \frac{1}{20}\left[e^{x}(\cos(3x) + 3\sin(3x)) - e^{-x}(3\sin(3x) - \cos(3x))\right] + C$

$= \frac{1}{10}\left[\cos(3x)\cosh(x) + 3\sin(3x)\sinh(x)\right] + C$

7. Polynomial × Trigonometric

When polynomial functions multiply trigonometric functions, we also need systematic reduction formulas. Unlike polynomial × exponential (where one reduction formula suffices), polynomial × trigonometric integrals require a coupled system because differentiating sine gives cosine and vice versa. Moreover, unlike exponential $\times$ trigomimetric (which coexist well in complex), no magic relations underly this couple.

7.1 Basic Mutual Reduction System

Define: $I_n = \int x^n \sin(bx)\ dx$ $J_n = \int x^n \cos(bx)\, dx$

These are inherently coupled because:

- Differentiating $\sin(bx)$ gives $b\cos(bx)$
- Differentiating $\cos(bx)$ gives $-b\sin(bx)$

First-Order Reduction (Coupled Formulas):

For $I_n = \int x^n \sin(bx)\ dx$, use IBP with $u = x^n$, $dv = \sin(bx)dx$:

- $du = nx^{n-1}dx$
- $v = -\frac{\cos(bx)}{b}$

$I_n = -\frac{x^n \cos(bx)}{b} + \int \frac{\cos(bx)}{b} \cdot nx^{n-1}dx$

$$\boxed{I_n = -\frac{x^n \cos(bx)}{b} + \frac{n}{b}J_{n-1}}$$

Similarly, for $J_n = \int x^n \cos(bx)\ dx$, use IBP with $u = x^n$, $dv = \cos(bx)dx$:

- $du = nx^{n-1}dx$
- $v = \frac{\sin(bx)}{b}$

$J_n = \frac{x^n \sin(bx)}{b} - \int \frac{\sin(bx)}{b} \cdot nx^{n-1}dx$

$$\boxed{J_n = \frac{x^n \sin(bx)}{b} - \frac{n}{b} I_{n-1}}$$

The Problem with First-Order Reduction:

To compute I_n, we need J_{n-1}. To compute J_{n-1}, we need I_{n-2}.

This back-and-forth is cumbersome for large n. We need a better approach.

7.2 Complete Second-Order Reduction

We can eliminate the mutual dependence by substituting one formula into the other, creating a direct reduction formula that only involves integrals of the same type.

Deriving Second-Order Reduction for I_n:

From the coupled formulas: $I_n = -\frac{x^n \cos(bx)}{b} + \frac{n}{b} J_{n-1}$

We need to express J_{n-1} without referring to I. Use the formula for J_{n-1}:

$$J_{n-1} = \frac{x^{n-1} \sin(bx)}{b} - \frac{n-1}{b} I_{n-2}$$

Substitute this into the expression for I_n:

$$I_n = -\frac{x^n \cos(bx)}{b} + \frac{n}{b}\left(\frac{x^{n-1} \sin(bx)}{b} - \frac{n-1}{b} I_{n-2}\right)$$

$$= -\frac{x^n \cos(bx)}{b} + \frac{nx^{n-1} \sin(bx)}{b^2} - \frac{n(n-1)}{b^2} I_{n-2}$$

$$\boxed{I_n = \frac{nx^{n-1} \sin(bx)}{b^2} - \frac{x^n \cos(bx)}{b} - \frac{n(n-1)}{b^2} I_{n-2}}$$

Similarly, for J_n:

From $J_n = \frac{x^n \sin(bx)}{b} - \frac{n}{b} I_{n-1}$ and $I_{n-1} = -\frac{x^{n-1} \cos(bx)}{b} + \frac{n-1}{b} J_{n-2}$:

$$J_n = \frac{x^n \sin(bx)}{b} - \frac{n}{b}\left(-\frac{x^{n-1} \cos(bx)}{b} + \frac{n-1}{b} J_{n-2}\right)$$

$$= \frac{x^n \sin(bx)}{b} + \frac{nx^{n-1} \cos(bx)}{b^2} - \frac{n(n-1)}{b^2} J_{n-2}$$

$$\boxed{J_n = \frac{nx^{n-1}\cos(bx)}{b^2} + \frac{x^n \sin(bx)}{b} - \frac{n(n-1)}{b^2} J_{n-2}}$$

Base Cases: $I_0 = \int \sin(bx)dx = -\frac{\cos(bx)}{b} + C$
$I_1 = \int x\sin(bx)dx = -\frac{x\cos(bx)}{b} + \frac{\sin(bx)}{b^2} + C$

$J_0 = \int \cos(bx)dx = \frac{\sin(bx)}{b} + C$ $J_1 = \int x\cos(bx)dx = \frac{x\sin(bx)}{b} + \frac{\cos(bx)}{b^2} + C$

Example 7.2.1: Evaluate $\int x^3 \sin(2x)\ \ dx$

Solution:

We need I_3 with $b = 2$.

Using the reduction formula: $I_3 = \frac{3x^2\sin(2x)}{4} - \frac{x^3\cos(2x)}{2} - \frac{3\cdot 2}{4} I_1$

$= \frac{3x^2\sin(2x)}{4} - \frac{x^3\cos(2x)}{2} - \frac{3}{2} I_1$

Now compute I_1: $I_1 = \int x\sin(2x)dx = -\frac{x\cos(2x)}{2} + \frac{\sin(2x)}{4}$

Substitute back: $I_3 = \frac{3x^2\sin(2x)}{4} - \frac{x^3\cos(2x)}{2} - \frac{3}{2}\left(-\frac{x\cos(2x)}{2} + \frac{\sin(2x)}{4}\right)$

$= -\frac{x^3\cos(2x)}{2} + \frac{3x^2\sin(2x)}{4} + \frac{3x\cos(2x)}{4} - \frac{3\sin(2x)}{8} + C$

Example 7.2.2: Evaluate $\int x^4 \cos(x)\ \ dx$

Solution:

We need J_4 with $b = 1$.

$J_4 = 4x^3\cos(x) + x^4\sin(x) - 12J_2$

Now compute J_2: $J_2 = 2x\cos(x) + x^2\sin(x) - 2J_0$

$J_0 = \sin(x)$

So: $J_2 = 2x\cos(x) + x^2\sin(x) - 2\sin(x)$

Substitute back into J_4:
$J_4 = 4x^3\cos(x) + x^4\sin(x) - 12[2x\cos(x) + x^2\sin(x) - 2\sin(x)]$

$= 4x^3 \cos(x) + x^4 \sin(x) - 24x \cos(x) - 12x^2 \sin(x) + 24 \sin(x) + C$

Example 7.2.3: Evaluate $\int x^2 \sin(3x)\ dx$

Solution:

We need I_2 with $b = 3$.

$I_2 = \frac{2x\sin(3x)}{9} - \frac{x^2\cos(3x)}{3} - \frac{2}{9}I_0$

$I_0 = -\frac{\cos(3x)}{3}$

Therefore: $I_2 = \frac{2x\sin(3x)}{9} - \frac{x^2\cos(3x)}{3} - \frac{2}{9} \cdot \left(-\frac{\cos(3x)}{3}\right)$

$= \frac{2x\sin(3x)}{9} - \frac{x^2\cos(3x)}{3} + \frac{2\cos(3x)}{27} + C$

7.3 Mixed Sine-Cosine with Polynomial

When the integrand contains products like $x^n \sin(ax) \cos(bx)$, our strategy:

1. Apply product-to-sum identity to simplify the trigonometric product
2. Distribute the polynomial factor
3. Apply the reduction formulas from Section 5.2 to each term

---

Example 7.3.1: Evaluate $\int x^2 \sin(x) \cos(x)\ dx$

Solution:

Step 1: Use the identity $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$

$\int x^2 \sin(x) \cos(x)\ dx = \frac{1}{2}\int x^2 \sin(2x) dx$

Step 2: This is $\frac{1}{2}I_2$ with $b = 2$.

Using the reduction formula: $I_2 = \frac{2x\sin(2x)}{4} - \frac{x^2\cos(2x)}{2} - \frac{2}{4}I_0$

$= \frac{x\sin(2x)}{2} - \frac{x^2\cos(2x)}{2} - \frac{1}{2}I_0$

$I_0 = -\frac{\cos(2x)}{2}$

Therefore: $I_2 = \frac{x\sin(2x)}{2} - \frac{x^2\cos(2x)}{2} + \frac{\cos(2x)}{4}$

Step 3: Multiply by $\frac{1}{2}$:

$$\frac{1}{2}I_2 = \frac{x\sin(2x)}{4} - \frac{x^2\cos(2x)}{4} + \frac{\cos(2x)}{8} + C$$

Example 7.3.2: Evaluate $\int x\sin(2x)\sin(3x)\ dx$

Solution:

Step 1: Use the product-to-sum identity:
$\sin(2x)\sin(3x) = \frac{1}{2}[\cos(2x-3x) - \cos(2x+3x)]$

$= \frac{1}{2}[\cos(-x) - \cos(5x)] = \frac{1}{2}[\cos(x) - \cos(5x)]$

Step 2: Therefore: $\int x\sin(2x)\sin(3x)\ dx = \frac{1}{2}\int x\cos(x)dx - \frac{1}{2}\int x\cos(5x)dx$

$= \frac{1}{2}J_1(b=1) - \frac{1}{2}J_1(b=5)$

Step 3: Compute each term:

For J_1 with $b=1$: $\int x\cos(x)dx = x\sin(x) + \cos(x)$

For J_1 with $b=5$: $\int x\cos(5x)dx = \frac{x\sin(5x)}{5} + \frac{\cos(5x)}{25}$

Step 4: Combine: $= \frac{1}{2}[x\sin(x) + \cos(x)] - \frac{1}{2}\left[\frac{x\sin(5x)}{5} + \frac{\cos(5x)}{25}\right] + C$

$$= \frac{x\sin(x)}{2} + \frac{\cos(x)}{2} - \frac{x\sin(5x)}{10} - \frac{\cos(5x)}{50} + C$$

Example 7.3.3: Evaluate $\int x^2\cos(x)\cos(2x)\ dx$

Solution:

Step 1: Use the product-to-sum identity:
$\cos(x)\cos(2x) = \frac{1}{2}[\cos(x-2x) + \cos(x+2x)]$

$= \frac{1}{2}[\cos(-x) + \cos(3x)] = \frac{1}{2}[\cos(x) + \cos(3x)]$

Step 2: Therefore: $\int x^2\cos(x)\cos(2x)\ dx = \frac{1}{2}\int x^2\cos(x)dx + \frac{1}{2}\int x^2\cos(3x)dx$

$= \frac{1}{2}J_2(b=1) + \frac{1}{2}J_2(b=3)$

Step 3: Compute J_2 with $b = 1$:

$$J_2 = 2x\cos(x) + x^2\sin(x) - 2J_0$$

$$J_0 = \sin(x)$$

$$J_2 = 2x\cos(x) + x^2\sin(x) - 2\sin(x)$$

Step 4: Compute J_2 with $b = 3$:

$$J_2 = \frac{2x\cos(3x)}{9} + \frac{x^2\sin(3x)}{3} - \frac{2}{9}J_0$$

$$J_0 = \frac{\sin(3x)}{3}$$

$$J_2 = \frac{2x\cos(3x)}{9} + \frac{x^2\sin(3x)}{3} - \frac{2\sin(3x)}{27}$$

Step 5: Combine: $= \frac{1}{2}[2x\cos(x) + x^2\sin(x) - 2\sin(x)]$
$+\frac{1}{2}\left[\frac{2x\cos(3x)}{9} + \frac{x^2\sin(3x)}{3} - \frac{2\sin(3x)}{27}\right] + C$

$$= x\cos(x) + \frac{x^2\sin(x)}{2} - \sin(x) + \frac{x\cos(3x)}{9} + \frac{x^2\sin(3x)}{6} - \frac{\sin(3x)}{27} + C$$

10. Comprehensive Definite Integrals

10.1 Logarithmic-Trigonometric

Example 10.1.1: Evaluate $\int_0^1 \frac{\ln(1+x)}{1+x^2} dx$

Solution: (King property after substitution):

Let $x = \tan\theta$, so $dx = \sec^2\theta, d\theta, 1 + x^2 = \sec^2\theta$

When $x = 0, \theta = 0$; when $x = 1, \theta = \pi/4$:

$I = \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{\sec^2\theta} \sec^2\theta, d\theta = \int_0^{\pi/4} \ln(1+\tan\theta) d\theta$

Now apply King property with $\pi/4 - \theta$: $I = \int_0^{\pi/4} \ln(1+\tan(\pi/4-\theta)) d\theta$

Using $\tan(\pi/4 - \theta) = \frac{1-\tan\theta}{1+\tan\theta}$:

$I = \int_0^{\pi/4} \ln\left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right) d\theta = \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\theta}\right) d\theta$

$= \int_0^{\pi/4} [\ln 2 - \ln(1+\tan\theta)] d\theta = \frac{\pi}{4}\ln 2 - I$

Therefore: $2I = \frac{\pi}{4}\ln 2$, so $I = \frac{\pi}{8}\ln 2$

Answer: $\boxed{\frac{\pi}{8}\ln 2}$

Example 10.1.2: Evaluate $\int_0^{\pi/2} \ln(\sin x) dx$

Solution:

Let $I = \int_0^{\pi/2} \ln(\sin x) dx$

Step 1: By symmetry (substitution $u = \pi/2 - x$): $I = \int_0^{\pi/2} \ln(\cos x) dx$

Step 2: Add both: $2I = \int_0^{\pi/2} [\ln(\sin x) + \ln(\cos x)] dx = \int_0^{\pi/2} \ln(\sin x \cos x) dx$

$= \int_0^{\pi/2} \ln\left(\frac{\sin 2x}{2}\right) dx = \int_0^{\pi/2} [\ln(\sin 2x) - \ln 2] dx$

$= \int_0^{\pi/2} \ln(\sin 2x)dx - \frac{\pi}{2}\ln 2$

Step 3: For $\int_0^{\pi/2} \ln(\sin 2x)dx$, let $u = 2x$, $du = 2dx$:

$\int_0^{\pi/2} \ln(\sin 2x)dx = \frac{1}{2}\int_0^{\pi} \ln(\sin u)du$

Step 4: By symmetry on $[0, \pi]$ (since $\sin(\pi - u) = \sin u$):
$\int_0^{\pi} \ln(\sin u)du = 2\int_0^{\pi/2} \ln(\sin u)du = 2I$

Step 5: Substitute back: $2I = \frac{1}{2} \cdot 2I - \frac{\pi}{2}\ln 2 = I - \frac{\pi}{2}\ln 2$

Therefore: $I = -\frac{\pi}{2}\ln 2$

Answer: $\boxed{-\frac{\pi}{2}\ln 2}$

Example 10.1.3: Evaluate $\int_0^{\pi/4} \ln(1 + \tan x)dx$

Solution:

Let $I = \int_0^{\pi/4} \ln(1 + \tan x)dx$

Apply King property with $\pi/4 - x$: $I = \int_0^{\pi/4} \ln(1 + \tan(\pi/4 - x))dx$

Using $\tan(\pi/4 - x) = \frac{1-\tan x}{1+\tan x}$:

$I = \int_0^{\pi/4} \ln\left(1 + \frac{1-\tan x}{1+\tan x}\right)dx = \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan x}\right)dx$

$= \int_0^{\pi/4} [\ln 2 - \ln(1 + \tan x)]dx = \frac{\pi}{4}\ln 2 - I$

Therefore: $2I = \frac{\pi}{4}\ln 2$

Answer: $\boxed{\frac{\pi}{8}\ln 2}$

10.2 Products and Quotients with Symmetry

Example 10.2.1: Evaluate $\int_0^{\pi/2} \frac{\sin^3 x}{\sin x + \cos x}dx$

Solution:

Let $I = \int_0^{\pi/2} \frac{\sin^3 x}{\sin x + \cos x} dx$

By King property with $\pi/2 - x$: $I = \int_0^{\pi/2} \frac{\cos^3 x}{\cos x + \sin x} dx$

Add: $2I = \int_0^{\pi/2} \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} dx$

Use the factorization: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$:

$\sin^3 x + \cos^3 x = (\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x)$
$= (\sin x + \cos x)(1 - \sin x \cos x)$

Therefore: $2I = \int_0^{\pi/2} (1 - \sin x \cos x) dx = \int_0^{\pi/2} \left(1 - \frac{\sin 2x}{2}\right) dx$

$= \left[x + \frac{\cos 2x}{4}\right]_0^{\pi/2} = \left[\frac{\pi}{2} + \frac{-1}{4}\right] - \left[0 + \frac{1}{4}\right]$

$= \frac{\pi}{2} - \frac{1}{2}$

Therefore: $I = \frac{\pi}{4} - \frac{1}{4}$

Example 10.2.2: Evaluate $\int_0^{2\pi} \frac{dx}{(2+\cos x)^2}$

Solution:

Use periodicity and split the integral: $\int_0^{2\pi} \frac{dx}{(2+\cos x)^2} = 2 \int_0^{\pi} \frac{dx}{(2+\cos x)^2}$

Now apply Weierstrass on $[0, \pi]$: when $x = \pi$, $t \to \infty$:

$2 \int_0^{\pi} \frac{dx}{(2+\cos x)^2} = 2 \int_0^{\infty} \frac{1}{(2+\frac{1-t^2}{1+t^2})^2} \cdot \frac{2dt}{1+t^2}$

$= 4 \int_0^{\infty} \frac{1}{(\frac{2(1+t^2)+1-t^2}{1+t^2})^2} \cdot \frac{dt}{1+t^2}$

$= 4 \int_0^{\infty} \frac{(1+t^2)^2}{(3+t^2)^2} \cdot \frac{dt}{1+t^2} = 4 \int_0^{\infty} \frac{1+t^2}{(3+t^2)^2} dt$

Write: $\frac{1+t^2}{(3+t^2)^2} = \frac{(3+t^2)-2}{(3+t^2)^2} = \frac{1}{3+t^2} - \frac{2}{(3+t^2)^2}$

$= 4 \left[\int_0^{\infty} \frac{dt}{3+t^2} - 2 \int_0^{\infty} \frac{dt}{(3+t^2)^2}\right]$

For the first: $\int_0^{\infty} \frac{dt}{3+t^2} = \frac{1}{\sqrt{3}} \arctan\left(t/\sqrt{3}\right)\Big|_0^{\infty} = \frac{\pi}{2\sqrt{3}}$

For the second, use the reduction formula or integrate by parts: $\int_0^\infty \frac{dt}{(t^2+3)^2} = \frac{1}{2 \cdot 3}\left[\frac{t}{t^2+3} + \arctan\left(t/\sqrt{3}\right)/\sqrt{3}\right]_0^\infty = \frac{\pi}{12\sqrt{3}}$

Therefore: $= 4\left[\frac{\pi}{2\sqrt{3}} - 2 \cdot \frac{\pi}{12\sqrt{3}}\right] = \frac{4\pi\sqrt{3}}{9}$

10.3 Exponential-Trigonometric

Example 10.3.1: Evaluate $\int_0^{2\pi} e^{\cos x} \cos(\sin x) dx$

Solution:

Use Euler's formula: $e^{e^{i\theta}} = e^{\cos\theta}[\cos(\sin\theta) + i\sin(\sin\theta)]$

Consider: $\int_0^{2\pi} e^{\cos x}(\cos(\sin x) + i\sin(\sin x))dx = \int_0^{2\pi} e^{\cos x} e^{i\sin x} dx$

$= \int_0^{2\pi} e^{\cos x + i\sin x} dx = \int_0^{2\pi} e^{e^{ix}} dx$

Expand: $e^{e^{ix}} = \sum_{n=0}^\infty \frac{e^{inx}}{n!}$

Integrate term by term: $\int_0^{2\pi} e^{e^{ix}} dx = \sum_{n=0}^\infty \frac{1}{n!} \int_0^{2\pi} e^{inx} dx$

For $n \neq 0$: $\int_0^{2\pi} e^{inx} dx = \frac{e^{inx}}{in}\Big|_0^{2\pi} = \frac{e^{2\pi in} - 1}{in} = 0$

For $n = 0$: $\int_0^{2\pi} 1\, dx = 2\pi$

Therefore: $\int_0^{2\pi} e^{e^{ix}} dx = 2\pi$

Taking the real part:

Answer: $\boxed{2\pi}$

11: Special Functions and Their Integrals

The functions in this chapter are often introduced as “given conclusions” in exercises, so you needn’t master their theories or derivations. Algebric manipulation is all your focus, which will be intensively tested in the examinations.

11.1 Gaussian Integral

The Gaussian integral is one of the most important definite integrals in mathematics, with applications spanning probability theory, physics, and engineering:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

For the one-sided version:
$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Proof: Let $I = \int_{-\infty}^{\infty} e^{-x^2} dx$. Then:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy$$

Switching to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = rdrd\theta$

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} rdrd\theta = \int_0^{2\pi} d\theta \cdot \int_0^{\infty} re^{-r^2} dr$$

$$= 2\pi \cdot \left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} = 2\pi \cdot \frac{1}{2} = \pi$$

Therefore: $I = \sqrt{\pi}$

Moments of Gaussian Distribution:

For $n \geq 0$ and $a > 0$:
$\int_0^{\infty} x^{2n} e^{-ax^2} dx = \frac{(2n-1)!!}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$
$\int_0^{\infty} x^{2n+1} e^{-ax^2} dx = \frac{n!}{2a^{n+1}}$

where $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$ is the double factorial.

Derivation via Parameter Differentiation:

Consider $I(a) = \int_0^{\infty} e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$

Differentiating under the integral sign:
$\frac{d^n}{da^n} I(a) = (-1)^n \int_0^{\infty} x^{2n} e^{-ax^2} dx = \frac{d^n}{da^n}\left(\frac{1}{2}\sqrt{\frac{\pi}{a}}\right)$

Example 11.1.1: Evaluate $\int_0^{\infty} x^2 e^{-x^2} dx$

Consider the parameterized integral:
$I(\alpha) = \int_0^{\infty} e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$

Differentiate with respect to α:
$I'(\alpha) = \frac{d}{d\alpha} \int_0^\infty e^{-\alpha x^2} dx = \int_0^\infty \frac{\partial}{\partial\alpha} e^{-\alpha x^2} dx = -\int_0^\infty x^2 e^{-\alpha x^2} dx$

Also, from the closed form:
$I'(\alpha) = -\frac{1}{4}\sqrt{\pi}\alpha^{-3/2}$

Evaluate at $\alpha = 1$:
$-\int_0^\infty x^2 e^{-x^2} dx = -\frac{1}{4}\sqrt{\pi}$
$\int_0^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}$

Connection to Error Function:
The error function is defined as $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. Our integral represents a moment of the Gaussian distribution.

Example 11.1.2: Evaluate $\int_0^\infty \frac{1-e^{-x^2}}{x^2} dx$

(MATH1560J 2021 Second Midterm Examination, Exercise 3)

Solution 1: Integration by Parts

Let $u = 1 - e^{-x^2}$, $dv = \frac{dx}{x^2}$, so $du = 2xe^{-x^2} dx$, $v = -\frac{1}{x}$:

$\int \frac{1-e^{-x^2}}{x^2} dx = -\frac{1}{x}(1 - e^{-x^2}) + \int \frac{1}{x} \cdot 2xe^{-x^2} dx$
$= -\frac{1}{x}(1 - e^{-x^2}) + 2\int e^{-x^2} dx$

Therefore:
$\int_0^\infty \frac{1-e^{-x^2}}{x^2} dx = \left[-\frac{1}{x}(1 - e^{-x^2})\right]_0^\infty + 2\int_0^\infty e^{-x^2} dx$

Evaluate the boundary terms:

- $\lim_{x\to\infty} \frac{1-e^{-x^2}}{x} = 0$
- $\lim_{x\to 0} \frac{1-e^{-x^2}}{x} = \lim_{x\to 0} \frac{2xe^{-x^2}}{1} = 0$

Using the known Gaussian integral $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$:

$\int_0^\infty \frac{1-e^{-x^2}}{x^2} dx = 0 + \int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$

Solution 2: Parameter Differentiation

Let $I(\alpha) = \int_0^\infty \frac{1-e^{-\alpha x^2}}{x^2} dx$

Differentiate with respect to α:
$I'(\alpha) = \int_0^\infty \frac{\partial}{\partial \alpha}\left(\frac{1-e^{-\alpha x^2}}{x^2}\right)dx = \int_0^\infty e^{-\alpha x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$

Integrate back:
$I(\alpha) = \int \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}d\alpha = \sqrt{\pi\alpha} + C$

Since $I(0) = 0$, we have $C = 0$, so $I(\alpha) = \sqrt{\pi\alpha}$

For $\alpha = 1$: $\boxed{\sqrt{\pi}}$

11.2 Error Function

The error function is a special function that arises naturally in probability, statistics, and the study of heat diffusion. It is defined as:

$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$

The complementary error function is: $\text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$

Key Properties:

1. $\text{erf}(0) = 0$
2. $\text{erf}(\infty) = 1$
3. $\text{erf}(-x) = -\text{erf}(x)$ (odd function)
4. $\frac{d}{dx}\text{erf}(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}$

Connection to Gaussian Integral:

From the Gaussian integral $\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$, we have: $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

Therefore: $\lim_{x\to\infty}\text{erf}(x) = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1$

Example 11.2.1: Express $\int_0^a e^{-x^2}dx$ in terms of the error function.

Solution:

By definition of the error function: $\text{erf}(a) = \frac{2}{\sqrt{\pi}}\int_0^a e^{-t^2}dt$

Therefore: $\int_0^a e^{-x^2}dx = \frac{\sqrt{\pi}}{2}\text{erf}(a)$

Answer: $\boxed{\frac{\sqrt{\pi}}{2}\text{erf}(a)}$

Example 11.2.2: Show that $\int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}dx = \sqrt{2\pi}\sigma$

Solution:

Let $u = \frac{x-\mu}{\sqrt{2}\sigma}$, then $du = \frac{dx}{\sqrt{2}\sigma}$, so $dx = \sqrt{2}\sigma, du$:

$\int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}dx = \int_{-\infty}^{\infty} e^{-u^2} \cdot \sqrt{2}\sigma, du$

$= \sqrt{2}\sigma \int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{2}\sigma \cdot \sqrt{\pi} = \sqrt{2\pi}\sigma$

Connection to Probability: This integral shows that the Gaussian (normal) distribution with mean μ and standard deviation σ is properly normalized.

Example 11.2.3: Evaluate $\int_0^{\infty} xe^{-x^2}dx$

Solution:

Let $u = -x^2$, then $du = -2x, dx$, so $x, dx = -\frac{1}{2}du$:

$\int_0^{\infty} xe^{-x^2}dx = \int_0^{-\infty} e^u \left(-\frac{1}{2}\right)du = \frac{1}{2}\int_{-\infty}^{0} e^u du$

$= \frac{1}{2}[e^u]_{-\infty}^{0} = \frac{1}{2}(1-0) = \frac{1}{2}$

Note: This integral has an elementary answer, unlike $\int_0^{\infty} e^{-x^2}dx$.

Answer: $\boxed{\frac{1}{2}}$

Connection to Normal Distribution:

The probability density function of the standard normal distribution is:
$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

The cumulative distribution function is: $\Phi(x) = \int_{-\infty}^{x} \phi(t)dt = \frac{1}{2}\left[1 + \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right]$

This shows how the error function naturally appears in probability and statistics.

11.3 Gamma function $\int_0^\infty u^n e^{-u} du = n!$

Example 10.5.1: Show that: $\int_0^1 x^x dx = \sum_{n=1}^\infty (-1)^{n+1} n^{-n}$

(MATH1560J 2021 Final Examination, Exercise 6)

Solution:

Express x^x using the exponential function:
$x^x = e^{x \ln x} = \sum_{n=0}^\infty \frac{(x \ln x)^n}{n!}$

Integrate term by term:
$\int_0^1 x^x dx = \sum_{n=0}^\infty \frac{1}{n!} \int_0^1 x^n (\ln x)^n dx$

For the integral $\int_0^1 x^n (\ln x)^n dx$, substitute $x = e^{-u/(n+1)}$, $dx = -\frac{1}{n+1} e^{-u/(n+1)} du$:

$\int_0^1 x^n (\ln x)^n dx = \int_\infty^0 e^{-nu/(n+1)} \left(-\frac{u}{n+1}\right)^n \left(-\frac{1}{n+1} e^{-u/(n+1)}\right) du$
$= (-1)^n (n+1)^{-(n+1)} \int_0^\infty u^n e^{-u} du$

Using the Gamma function identity $\int_0^\infty u^n e^{-u} du = n!$:

$\int_0^1 x^n (\ln x)^n dx = (-1)^n (n+1)^{-(n+1)} n!$

Substitute back:
$\int_0^1 x^x dx = \sum_{n=0}^\infty \frac{1}{n!} \cdot (-1)^n (n+1)^{-(n+1)} n! = \sum_{n=0}^\infty (-1)^n (n+1)^{-(n+1)}$

Reindex by letting $m = n + 1$:
$= \sum_{m=1}^\infty (-1)^{m-1} m^{-m} = \sum_{m=1}^\infty (-1)^{m+1} m^{-m}$

Final Answer: $\boxed{\sum_{n=1}^{\infty} (-1)^{n+1} n^{-n}}$

11.4 Sine Integral $\int_0^\infty \frac{\sin(ax)}{x} dx = \frac{\pi}{2}$

This is a conclusion of Dirichlet integral, for $a > 0$.

Example 11.4.1: Evaluate $\int_0^\infty \frac{1-\cos(ax)}{x^2} dx$

Solution:
Let $F(a) = \int_0^\infty \frac{1-\cos(ax)}{x^2} dx$
$F'(a) = \frac{d}{da} \int_0^\infty \frac{1-\cos(ax)}{x^2} dx = \int_0^\infty \frac{\partial}{\partial a} \left(\frac{1-\cos(ax)}{x^2} \right) dx$
$= \int_0^\infty \frac{\sin(ax)}{x} dx = \frac{\pi}{2}$

Integrate Back:
$F(a) = \int \frac{\pi}{2} da = \frac{\pi}{2} a + C$

Determine Constant
When $a = 0$: $F(0) = \int_0^\infty \frac{1-1}{x^2} dx = 0$, so $C = 0$

$$\boxed{\frac{\pi}{2} a}$$

Connection to Sine Integral:
The sine integral is defined as $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} dt$. Our result shows how integrals of this family can be evaluated without explicit reference to $\mathrm{Si}(x)$.

Example 11.4.2: Evaluate $\int_0^\infty \frac{\sin^3 x}{x} dx$

Solution:

Step 1: Trigonometric Identity
Use the triple-angle identity: $\sin^3 x = \frac{3 \sin x - \sin 3x}{4}$

Step 2: Split the Integral
$\int_0^\infty \frac{\sin^3 x}{x} dx = \frac{1}{4} \int_0^\infty \frac{3 \sin x - \sin 3x}{x} dx$
$= \frac{3}{4} \int_0^\infty \frac{\sin x}{x} dx - \frac{1}{4} \int_0^\infty \frac{\sin 3x}{x} dx$

Step 3: Known Dirichlet Integral
We know: $\int_0^\infty \frac{\sin(kx)}{x} dx = \frac{\pi}{2}$ for $k > 0$

Final Answer:
$$\boxed{\frac{\pi}{4}}$$

Connection to Sine Integral:
This demonstrates how even when individual terms involve the sine integral $\mathrm{Si}(x)$, certain combinations yield elementary results.

11.5 Logarithmic Integral $\mathrm{Li}(x) = \int_0^x \frac{dt}{\ln t}$

Example 11.5.1: Evaluate $\int_0^1 \frac{x^{a-1}-x^{b-1}}{\ln x}dx$

Solution:

Step 1: Define Parameter Function
Let $F(t) = \int_0^1 \frac{x^t-1}{\ln x}dx$

Step 2: Differentiate
$F'(t) = \frac{d}{dt}\int_0^1 \frac{x^t-1}{\ln x}dx = \int_0^1 \frac{\partial}{\partial t}\left(\frac{x^t-1}{\ln x}\right)dx$
$= \int_0^1 \frac{x^t \ln x}{\ln x}dx = \int_0^1 x^t dx$

Step 3: Evaluate Integral
$\int_0^1 x^t dx = \left[\frac{x^{t+1}}{t+1}\right]_0^1 = \frac{1}{t+1}$

Step 4: Integrate Back
$F(t) = \int \frac{1}{t+1}dt = \ln|t+1| + C$

Step 5: Determine Constant
When $t = 0$: $F(0) = \int_0^1 \frac{1-1}{\ln x}dx = 0$, so $C = 0$

Therefore: $F(t) = \ln(t+1)$

Step 6: Apply to Original Integral
$\int_0^1 \frac{x^{a-1}-x^{b-1}}{\ln x}dx = F(a-1) - F(b-1) = \ln a - \ln b$

Final Answer:
$$\boxed{\ln \frac{a}{b}}$$

Connection to Logarithmic Integral:
The logarithmic integral is defined as $\mathrm{li}(x) = \int_0^x \frac{dt}{\ln t}$. Our result shows a related integral that surprisingly yields an elementary expression.

Example 8.5.5: Prove that $\int_0^1 \frac{\ln(1+x)}{x}dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

Solution:

Step 1: Series Expansion
For $|x| < 1$, we have: $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$

Step 2: Substitute Series
$\int_0^1 \frac{\ln(1+x)}{x} dx = \int_0^1 \frac{1}{x} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} dx$
$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_0^1 x^{n-1} dx$

Step 3: Evaluate Integral
$\int_0^1 x^{n-1} dx = \left[\frac{x^n}{n}\right]_0^1 = \frac{1}{n}$

Step 4: Sum the Series
$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

Connection to Polylogarithms:
This integral is related to the dilogarithm function $\mathrm{Li}_2(-1) = -\frac{\pi^2}{12}$, yet yields an elementary value.

---

12. Equalities and Inequalities

We have focused primarily on computing antiderivatives and evaluating definite integrals. Moreover, the integral calculus provides a framework for establishing deep mathematical relationships and proving fundamental inequalities. The techniques developed here—symmetry exploitation, parameter differentiation, and strategic substitutions—synthesize many of the methods we have studied in previous chapters.

Example 12.1.1: (MATH1560J 2021 Second Midterm Examination, Exercise 7)
(i) Show that for any convergent function f:
$\int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx = \int_{-\infty}^{\infty} f(x) dx$
(ii) Evaluate: $\int_{-\infty}^{\infty} \exp\left(-x^2 - \frac{\alpha}{x^2}\right) dx$ where $\alpha > 0$

Solution:

Let $I = \int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx$

Make substitution $y = -\frac{1}{x}$, then $dy = \frac{dx}{x^2}$, and:
$I = \int_{-\infty}^{\infty} f\left(-\frac{1}{y} + y\right) \frac{dy}{y^2} = \int_{-\infty}^{\infty} f\left(y - \frac{1}{y}\right) \frac{dy}{y^2}$

Add both expressions:
$2I = \int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) \left(1 + \frac{1}{x^2}\right) dx$

Now substitute $u = x - \frac{1}{x}$, then $du = \left(1 + \frac{1}{x^2}\right) dx$

As x goes from $-\infty$ to ∞ (excluding 0), u also goes from $-\infty$ to ∞:

$2I = \int_{-\infty}^{\infty} f(u)du + \int_{-\infty}^{\infty} f(u)du = 2\int_{-\infty}^{\infty} f(u)du$

Therefore: $I = \int_{-\infty}^{\infty} f(u)du$

Complete the square:
$-x^2 - \frac{\alpha}{x^2} = -\left(x - \frac{\sqrt{\alpha}}{x}\right)^2 - 2\sqrt{\alpha}$

Therefore:
$\int_{-\infty}^{\infty} \exp\left(-x^2 - \frac{\alpha}{x^2}\right)dx = e^{-2\sqrt{\alpha}} \int_{-\infty}^{\infty} \exp\left[-\left(x - \frac{\sqrt{\alpha}}{x}\right)^2\right]dx$

By part (i) with $f(u) = e^{-u^2}$:
$= e^{-2\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-u^2} du = e^{-2\sqrt{\alpha}}\sqrt{\pi}$

Final Answer: $\boxed{\sqrt{\pi}e^{-2\sqrt{\alpha}}}$

Example 12.1.2 Given $f \in C^1[0,1]$ and $f(0) = f(1) = -1/6$, prove $\int_0^1 (f')^2\,dx \geq 2\int_0^1 f\,dx + \frac{1}{4}$.

The proof uses the non-negativity of the integral of a square.

Step 1: Define the Non-Negative Integral and Expand

We define $I = \int_0^1 \left(f'(x) + x - \frac{1}{2}\right)^2 dx \geq 0$.
Expanding the binomial inside the integral:

$$I = \int_0^1 \left[(f'(x))^2 + 2f'(x)(x - \tfrac{1}{2}) + (x - \tfrac{1}{2})^2\right] dx$$

$$= \underbrace{\int_0^1 (f'(x))^2\,dx}_{I_1} + \underbrace{2\int_0^1 f'(x)(x - \tfrac{1}{2})\,dx}_{I_2} + \underbrace{\int_0^1 (x - \tfrac{1}{2})^2\,dx}_{I_3}$$

Step 2: Evaluate I_2 using Integration by Parts (IBP)

Set $u = x - \frac{1}{2}$ and $dv = f'(x)\,dx$. Then $du = dx$ and $v = f(x)$.
$I_2 = 2\left(\left[f(x)(x - \frac{1}{2})\right]_0^1 - \int_0^1 f(x)\,dx\right)$

A. Evaluate Boundary Term:
$$\left[f(x)(x-\tfrac{1}{2})\right]_0^1 = f(1)(1-\tfrac{1}{2}) - f(0)(0-\tfrac{1}{2})$$
$$= (-\tfrac{1}{6})(\tfrac{1}{2}) - (-\tfrac{1}{6})(-\tfrac{1}{2}) = -\frac{1}{6}$$

B. Final I_2 Expression:
$I_2 = 2\left(-\frac{1}{6} - \int_0^1 f(x)\,dx\right) = -\frac{1}{3} - 2\int_0^1 f(x)\,dx$

Step 3: Evaluate I_3

$$I_3 = \int_0^1 (x-\tfrac{1}{2})^2\,dx = \left[\tfrac{(x-1/2)^3}{3}\right]_0^1 I_3 = \tfrac{(1/2)^3}{3} - \tfrac{(-1/2)^3}{3} = \tfrac{1}{24} + \tfrac{1}{24} = \tfrac{2}{24} = \tfrac{1}{12}$$

Step 4: Final Conclusion

Substitute I_2 and I_3 back into the inequality $I_1 + I_2 + I_3 \geq 0$:

$$\int_0^1 (f'(x))^2\,dx + \left(-\tfrac{1}{3} - 2\int_0^1 f(x)\,dx\right) + \tfrac{1}{12} \geq 0$$

$$\int_0^1 (f'(x))^2\,dx \geq 2\int_0^1 f(x)\,dx + \tfrac{1}{3} - \tfrac{1}{12}\int_0^1 (f'(x))^2\,dx \geq 2\int_0^1 f(x)\,dx + \tfrac{4-1}{12}$$

$$\int_0^1 (f'(x))^2\,dx \geq 2\int_0^1 f(x)\,dx + \tfrac{1}{4}$$

The proof is complete.

Example 12.1.3:
(i) Prove the Cauchy-Schwarz Inequality for Integrals:
$\int_0^1 f(x)g(x)dx \leq \sqrt{\int_0^1 f^2(x)dx} \cdot \sqrt{\int_0^1 g^2(x)dx}$
(ii) Given $f \in C[0,1]$ with $f(x) \geq 0$ and $\int_0^1 f(x)dx = 1$. Prove:
$\int_0^1 xf(x)dx \cdot \int_0^1 \frac{f(x)}{x}dx \geq 1$

Solution:

Consider the function $h(t) = \int_0^1 [f(x) + tg(x)]^2 dx \geq 0$ for all $t \in \mathbb{R}$.

Expanding: $h(t) = \int_0^1 f^2(x)dx + 2t\int_0^1 f(x)g(x)dx + t^2\int_0^1 g^2(x)dx$

Let:

- $A = \int_0^1 f^2(x)dx$
- $B = \int_0^1 f(x)g(x)dx$
- $C = \int_0^1 g^2(x)dx$

Then: $h(t) = A + 2Bt + Ct^2 \geq 0$ for all t

Since this quadratic is non-negative for all t, its discriminant must be non-positive:
$\Delta = (2B)^2 - 4AC \leq 0$

$$\left|\int_0^1 f(x)g(x)dx\right| \leq \sqrt{\int_0^1 f^2(x)dx} \cdot \sqrt{\int_0^1 g^2(x)dx}$$

Equality holds when $f(x) = \lambda g(x)$ for some constant λ.

Apply Cauchy-Schwarz inequality with $g(x) = \sqrt{xf(x)}$ and $h(x) = \frac{\sqrt{f(x)}}{\sqrt{x}}$:

$$\left(\int_0^1 g(x)h(x)dx\right)^2 \leq \int_0^1 g^2(x)dx \cdot \int_0^1 h^2(x)dx$$

$$\left(\int_0^1 \sqrt{xf(x)} \cdot \frac{\sqrt{f(x)}}{\sqrt{x}}dx\right)^2 \leq \int_0^1 xf(x)dx \cdot \int_0^1 \frac{f(x)}{x}dx$$

$$\left(\int_0^1 f(x)dx\right)^2 \leq \int_0^1 xf(x)dx \cdot \int_0^1 \frac{f(x)}{x}dx$$

Since $\int_0^1 f(x)dx = 1$:

$$1 \leq \int_0^1 xf(x)dx \cdot \int_0^1 \frac{f(x)}{x}dx$$

Example 12.1.4: Show that:
(i) For $a, b > 0$: $\int_0^\infty e^{-ax} \sin{(bx)}dx = \frac{b}{a^2+b^2}$
(ii) $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\sin x, dx = \arctan b - \arctan a$

Solution:

Part 1: Standard integral using integration by parts twice, or complex exponentials:

$$\int_0^\infty e^{-ax} \sin{(bx)}dx = \text{Im}\left(\int_0^\infty e^{(-a+ib)x}dx\right)$$

$$= \text{Im}\left(\frac{1}{-a+ib}\right) = \text{Im}\left(\frac{-a-ib}{a^2+b^2}\right) = \frac{b}{a^2+b^2}$$

Part 2: Let $I(a) = \int_0^\infty \frac{e^{-ax}\sin x}{x}dx$

Then: $I'(a) = -\int_0^\infty e^{-ax} \sin x, dx = -\frac{1}{a^2+1}$

Integrate: $I(a) = -\arctan a + C$

As $a \to \infty, I(a) \to 0$, so $C = \frac{\pi}{2}$

Therefore: $I(a) = \frac{\pi}{2} - \arctan a$

Finally: $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x} \sin x, dx = I(a) - I(b) = \arctan b - \arctan a$

Part VII: Non-Elementary Integrals and Final Topics

轻舟已过万重山。

13. Recognition of Non-Elementary Forms

Some integrals cannot be expressed in terms of elementary functions (polynomials, exponentials, logarithms, trigonometric functions, and their inverses).

13.1 Elliptic Integrals

Form 1: $\int \frac{dx}{\sqrt{P(x)}}$ where $P(x)$ is a polynomial of degree 3 or 4

Form 2: $\int \sqrt{P(x)}\, dx$ where $P(x)$ is a polynomial of degree 3 or 4

Example: $\int \frac{dx}{\sqrt{x^3-1}}$ (elliptic integral of the first kind)

Example: $\int \sqrt{\sin x}\, dx$ cannot be expressed in elementary functions

---

13.2 Other Non-Elementary Forms

Common non-elementary integrals:

1. **Error function:** $\int e^{x^2} dx = \frac{\sqrt{\pi}}{2} \text{erf}(x) + C$
2. **Sine integral:** $\int \frac{\sin x}{x} dx = \text{Si}(x) + C$
3. **Logarithmic integral:** $\int \frac{dx}{\ln x} = \text{Li}(x) + C$
4. **Fresnel integrals:** $\int \sin\left(x^2\right) dx$ and $\int \cos\left(x^2\right) dx$

5. **Exponential integral:** $\int \frac{e^x}{x} dx = \mathrm{Ei}(x) + C$

13.3 When to Stop

Decision criteria for recognizing non-elementary integrals:

1. **Check standard forms:** Compare against known non-elementary patterns
2. **Risch algorithm:** Theoretical (but complex) algorithm determines if elementary antiderivative exists
3. **Computer algebra systems:** Programs like Mathematica can detect non-elementary integrals
4. **Liouville's theorem:** Provides theoretical framework for impossibility proofs

What to do when integral is non-elementary:

- Express as a special function (if standard)
- Use numerical methods (trapezoidal rule, Simpson's rule, Gaussian quadrature)
- Series expansion for approximation
- Asymptotic methods for large parameter limits

Numerical methods overview:

Trapezoidal Rule: $\int_a^b f(x)dx \approx \frac{b-a}{2n}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$

Simpson's Rule:
$\int_a^b f(x)dx \approx \frac{b-a}{3n}\left[f(x_0) + 4\sum_{i=1,3,5,\ldots} f(x_i) + 2\sum_{i=2,4,6,\ldots} f(x_i) + f(x_n)\right]$

---

14. Summary

14.1 Method Decision Tree

```
START: ∫f(x)dx
|
├─ Is it a standard form?
|   ├─ YES → Apply direct formula
|   └─ NO → Continue
|
├─ Is f(x) rational (P(x)/Q(x))?
|   ├─ YES →
```

```
│   │   ├─ deg(P) ≥ deg(Q)? → Long division first
│   │   └─ Partial fractions
│   └─ NO → Continue
│
├─ Does it contain radicals?
│   ├─ √(ax²+bx+c) → Complete square → Trig substitution
│   ├─ √(ax+b), √(cx+d) → Rationalize via LCM substitution
│   └─ √(P(x)), deg(P)≥3 → Check if elliptic (non-elementary)
│
├─ Does it contain products?
│   ├─ Polynomial × Exponential → IBP (tabular method)
│   ├─ Polynomial × Trig → IBP (reduction formulas)
│   ├─ Exponential × Trig →
│   │   ├─ Double IBP (traditional)
│   │   └─ Complex exponentials (Euler)
│   ├─ Trig × Trig → Product-to-sum identities
│   └─ Mixed types → IBP or substitution
│
├─ Pure trigonometric powers?
│   ├─ sin^m(x)cos^n(x) →
│   │   ├─ One power odd? → Save factor, substitute
│   │   ├─ Both even? → Half-angle formulas or complex method
│   │   └─ High even powers? → Complex exponentials
│   ├─ tan^n(x), sec^n(x) →
│   │   ├─ Use tan²+1=sec²
│   │   └─ Reduction formulas
│   └─ Hyperbolic → Similar strategies
│
├─ Contains inverse functions?
│   ├─ arcsin, arctan, ln, etc. → IBP with u=inverse function
│   └─ Powers of inverse functions → Reduction formulas
│
├─ Definite integral with symmetry?
│   ├─ Check King property: x → a+b-x
│   ├─ Check even/odd: f(-x) = ±f(x)
│   ├─ Check periodicity
│   └─ Feynman's trick (parameter differentiation)
│
└─ Still stuck?
    ├─ Try multiple substitutions
    ├─ Check if non-elementary
    └─ You made a mistake in previous calculations
```

14.2 Common Pitfalls

1. Forgetting absolute value in logarithms

❌ Wrong: $\int \frac{dx}{x} = \ln(x) + C$

✓ Correct: $\int \frac{dx}{x} = \ln|x| + C$

Why it matters: Without absolute value, the antiderivative is undefined for $x < 0$.

---

2. Sign errors in integration by parts

❌ Common mistake: $\int u\,dv = uv + \int v\,du$

✓ Correct: $\int u\,dv = uv - \int v\,du$

---

3. Bad choice of u and dv in IBP

LIATE Rule for choosing u（对反幂三指）：

- **L**ogarithmic
- **I**nverse trigonometric
- **A**lgebraic (polynomials)
- **T**rigonometric
- **E**xponential

Choose u based on what appears first in this list.

Example: For $\int xe^x\,dx$:

- x is algebraic (A)
- e^x is exponential (E)
- Choose $u = x$, $dv = e^x dx$ ✓

---

4. Not simplifying before integrating

❌ Attempting: $\int \frac{x^2-1}{x-1} dx$ directly

✓ Simplify first: $\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x + 1$ (for $x \neq 1$)

Then: $\int(x+1)dx = \frac{x^2}{2} + x + C$

---

5. Boundary term errors in definite integrals

When using IBP on definite integrals, **always evaluate the boundary term** $[uv]_a^b$.

❌ Common error: Forgetting to evaluate at both limits

✓ Careful: $\int_a^b u, dv = [uv]_a^b - \int_a^b v, du$

---

6. Incorrect trigonometric identities

- $\sin^2 x + \cos^2 x = 1$ ✓
- $1 + \tan^2 x = \sec^2 x$ ✓
- $1 + \cot^2 x = \csc^2 x$ ✓

---

7. Forgetting to back-substitute

After substitution, **always convert back to the original variable**.

Example: After $u = \sin x$: $\int \sin^2 x \cos x \, dx = \int u^2 \, du = \frac{u^3}{3} + C$

❌ Leaving as $\frac{u^3}{3} + C$

✓ Complete: $\frac{\sin^3 x}{3} + C$

---

8. Domain issues with inverse trigonometric functions

- $\arcsin x$ and $\arccos x$: domain $[-1, 1]$
- $\arctan x$ and $\text{arccot } x$: domain $(-\infty, \infty)$
- $\text{arcsec } x$ and $\text{arccsc } x$: domain $(-\infty, -1] \cup [1, \infty)$

Always check that expressions remain in valid domains.

---

9. Incorrect substitution of differentials

❌ Common error: $\int \cos(2x)dx = \sin(2x) + C$

✓ Correct: Use substitution $u = 2x$, $du = 2dx \Rightarrow dx = \frac{du}{2}$
$\int \cos(2x)dx = \int \cos(u) \cdot \frac{du}{2} = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(2x) + C$

Always express $du = f(x)\,dx$ explicitly to prevent omission.

14.3 Efficiency Comparison

METHOD	BEST FOR	ADVANTAGES	DISADVANTAGES
Direct substitution	Obvious chain rule forms	Fast, simple	Limited applicability
IBP	Products, inverse functions	Versatile	Can be tedious, sign errors
Trig substitution	$\sqrt{a^2 - x^2}$, etc.	Handles radicals	Requires careful back-substitution
Partial fractions	Rational functions	Systematic	Algebra-intensive
Complex exponentials	High trig powers	Systematic, powerful	Requires complex arithmetic
Reduction formulas	Repeated similar integrals	Reduces power systematically	Recursive, multiple steps
Weierstrass subst.	Rational trig functions	Universal (always works)	Often produces messy expressions

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14.4 Integration Strategy Principles

1. Always look before you leap

- Survey the entire integrand before choosing a method
- Check for simplifications (factoring, cancellation)
- Identify the "type" of integral

2. Exploit symmetry and special structure

- For definite integrals, check King property and even/odd
- Look for hidden patterns (product-to-sum, etc.)

3. Transform to familiar territory

- Goal: convert unfamiliar integral to known form
- Each substitution should simplify the problem

4. When stuck, try multiple approaches

- Don't commit to first method if it's not working
- Sometimes the "wrong" substitution reveals a better path

5. Verify your answer

- Differentiate to check indefinite integrals
- Use numerical methods to verify definite integrals
- Check boundary behavior and special cases

6. Build your intuition

- Practice recognizing patterns
- Keep a personal "integral library" of solved problems
- Understand why methods work, not just how

公曰：

> 夫积分之巧，非徒术也，实载变通之智。易元转繁为简，如辟径通幽；分部分合相济，若解结理丝；三角化弦为直，似驯曲就方；复数越域求通，犹破壁寻途。四术虽殊，其要一也：以变应变，以简驭繁。