Generating Functions and Infinite Series

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Exercises

1. Evaluate $S = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$
2. Evaluate $\sum_{n=1}^{\infty} \frac{n+1}{3^n}$.
3. Evaluate $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$.
4. Evaluate $\sum_{k=0}^{n} \frac{\binom{n}{k}}{k+2}$.
5. Evaluate $\sum_{k=0}^{n} k^2 \binom{n}{k}$.
6. Evaluate $\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$.
7. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} x^n$ and find its value at $x = 1$.

Solution 1: Alternating Series with Odd Terms

Define: $G(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

We want $S = G(1)$.

Differentiate: $G'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot (2n+1) x^{2n} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$= 1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1+x^2}$

Integrate: $G(x) = \int_0^x \frac{1}{1+t^2} , dt = \arctan(t) \Big|_0^x = \arctan(x)$

Therefore: $S = G(1) = \arctan(1) = \frac{\pi}{4}$

Solution 2: Geometric-Weighted Series with Shift

$\sum_{n=1}^{\infty} \frac{n+1}{3^n} = \sum_{n=1}^{\infty} \frac{n}{3^n} + \sum_{n=1}^{\infty} \frac{1}{3^n}$

From the standard formula: $\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1/3}{(1-1/3)^2} = \frac{1/3}{4/9} = \frac{3}{4}$

And: $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1/3}{1-1/3} = \frac{1/3}{2/3} = \frac{1}{2}$

Therefore: $\sum_{n=1}^{\infty} \frac{n+1}{3^n} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

Solution 3: Cubic Numerator

Start with $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

First derivative: $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$, so $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$

Second derivative: $\sum_{n=1}^{\infty} n^2x^{n-1} = \frac{1+x}{(1-x)^3}$, so $\sum_{n=1}^{\infty} n^2x^n = \frac{x(1+x)}{(1-x)^3}$

Third derivative: $\frac{d}{dx}\left[\frac{x(1+x)}{(1-x)^3}\right] = \frac{(1+2x)(1-x)^3+3x(1+x)(1-x)^2}{(1-x)^6}$

$= \frac{(1-x)^2[(1+2x)(1-x)+3x(1+x)]}{(1-x)^6} = \frac{(1+2x-x-2x^2)+3x+3x^2}{(1-x)^4}$

$= \frac{1+4x+x^2}{(1-x)^4}$

Therefore: $\sum_{n=1}^{\infty} n^3x^n = \frac{x(1+4x+x^2)}{(1-x)^4}$

At $x = 1/2$: $\sum_{n=1}^{\infty} \frac{n^3}{2^n} = \frac{(1/2)(1+2+1/4)}{(1/2)^4} = \frac{(1/2)(13/4)}{1/16} = \frac{13/8}{1/16} = 26$

Solution 4: Binomial with Linear Denominator

$\sum_{k=0}^{n} \frac{\binom{n}{k}}{k+2} = \sum_{k=0}^{n} \binom{n}{k} \int_0^1 x^{k+1}, dx = \int_0^1 x \sum_{k=0}^{n} \binom{n}{k} x^k, dx$

$= \int_0^1 x(1+x)^n, dx$

Let $u = 1 + x$, then $du = dx$, and when $x = 0$, $u = 1$; when $x = 1$, $u = 2$:

$= \int_1^2 (u-1)u^n, du = \int_1^2 (u^{n+1} - u^n), du$

$= \left[\frac{u^{n+2}}{n+2} - \frac{u^{n+1}}{n+1} \right]_1^2$

$= \frac{2^{n+2}}{n+2} - \frac{2^{n+1}}{n+1} - \frac{1}{n+2} + \frac{1}{n+1}$

$= \frac{2^{n+2}(n+1) - 2^{n+1}(n+2)}{(n+1)(n+2)} - \frac{(n+1)-(n+2)}{(n+1)(n+2)}$

$= \frac{2^{n+1}[2(n+1)-(n+2)]+1}{(n+1)(n+2)} = \frac{2^{n+1} \cdot n+1}{(n+1)(n+2)}$

$= \frac{n \cdot 2^{n+1}+1}{(n+1)(n+2)}$

Solution 5: Squared Binomial Sum

Consider $G(x) = \sum_{k=0}^{n} k^2 \binom{n}{k} x^k$.

We know that $\frac{d}{dx}[(1+x)^n] = n(1+x)^{n-1}$, so: $\sum_{k=0}^{n} k \binom{n}{k} x^{k-1} = n(1+x)^{n-1}$

Multiply by x: $\sum_{k=0}^{n} k \binom{n}{k} x^k = nx(1+x)^{n-1}$

Differentiate again: $\sum_{k=0}^{n} k^2 \binom{n}{k} x^{k-1} = \frac{d}{dx}[nx(1+x)^{n-1}]$

$= n(1+x)^{n-1} + nx(n-1)(1+x)^{n-2} = n(1+x)^{n-2}[(1+x) + x(n-1)]$

$= n(1+x)^{n-2}[1+nx]$

Multiply by x: $\sum_{k=0}^{n} k^2 \binom{n}{k} x^k = nx(1+x)^{n-2}[1+nx]$

At $x = 1$: $\sum_{k=0}^{n} k^2 \binom{n}{k} = n \cdot 2^{n-2} \cdot (1+n) = n(n+1)2^{n-2}$

Solution 6: Composite Substitution

Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Consider: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

Similarly: $e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

Adding: $e^x + e^{-x} = 2 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

Subtracting: $e^x - e^{-x} = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

Therefore: $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \frac{e^x - e^{-x}}{2} = \sinh(x)$

At $x = 1$: $\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} = \sinh(1) = \frac{e - e^{-1}}{2} = \frac{e^2 - 1}{2e}$

Solution 7: Harmonic-Like Series

Note that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ (partial fractions).

$G(x) = \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{x^n}{n} - \frac{x^n}{n+1}\right)$

$= \sum_{n=1}^{\infty} \frac{x^n}{n} - \sum_{n=1}^{\infty} \frac{x^n}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n} - \sum_{m=2}^{\infty} \frac{x^{m-1}}{m}$

$= \sum_{n=1}^{\infty} \frac{x^n}{n} - \frac{1}{x} \sum_{m=2}^{\infty} \frac{x^m}{m}$

$= \sum_{n=1}^{\infty} \frac{x^n}{n} - \frac{1}{x} \left(\sum_{m=1}^{\infty} \frac{x^m}{m} - x\right)$

$= \sum_{n=1}^{\infty} \frac{x^n}{n} - \frac{1}{x} \sum_{m=1}^{\infty} \frac{x^m}{m} + 1$

Since $\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$:

$G(x) = -\ln(1-x) + \frac{\ln(1-x)}{x} + 1 = 1 + \frac{\ln(1-x)}{x}(1-x)$

At $x = 1$ (using L'Hôpital's rule): $G(1) = 1 + \lim_{x \to 1} \frac{\ln(1-x)(1-x)}{x} = 1 + 0 = 1$

Therefore: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$
