Functions Revisited in Complex

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> *"In the arithmetic of the universe, complex numbers are the natural language, and real numbers are but a shadow."*
>
> *— **Leonhard Euler***

Part I: Complex Numbers

1. Polar Form

Every complex number $z = x + iy$ can be written in **polar form**:

$z = r(\cos\theta + i\sin\theta)$

where:

- $r = |z| = \sqrt{x^2 + y^2}$ is the **modulus** (distance from origin)
- $\theta = \arg(z)$ is the **argument** (angle from positive real axis)

Example: Express $z = -\sqrt{3} + i$ in polar form.

Solution:

- $r = \sqrt{3+1} = 2$
- The point is in the second quadrant, so $\theta = \pi - \arctan\left(1/\sqrt{3}\right) = \pi - \pi/6 = 5\pi/6$
- Therefore: $-\sqrt{3} + i = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$

2. Fundamental Theorem

When multiplying complex numbers in polar form:

- **Moduli multiply**: $|z_1 z_2| = |z_1| \cdot |z_2|$
- **Arguments add**: $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

Proof:

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \end{aligned}$$

Recall the **angle addition formulas** from trigonometry:

- $\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$
- $\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$

Therefore:
$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

Multiplying by a complex number w transforms every point z by:

1. **Scaling** by a factor of $|w|$
2. **Rotating** counterclockwise by angle $\arg(w)$

Special Case: If $|w| = 1$, then w represents a **pure rotation**.

Similarly, for division:
$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

Moduli divide, arguments subtract.

Example: Compute $\frac{1+i}{-\sqrt{3}+i}$ using polar form.

Solution:
$\frac{1+i}{-\sqrt{3}+i} = \frac{\sqrt{2}}{2}[\cos(\pi/4 - 5\pi/6) + i\sin(\pi/4 - 5\pi/6)] = \frac{\sqrt{2}}{2}[\cos(-7\pi/12) + i\sin(-7\pi/12)]$

3. Power (De Moivre Theorem)

For any integer n:
$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

Proof by Induction:

Base case ($n = 1$): Trivially true.

Inductive step: Assume true for $n = k$. Then:

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k \cdot (\cos\theta + i\sin\theta)\\ &= [\cos(k\theta) + i\sin(k\theta)][\cos\theta + i\sin\theta]\\ &= \cos((k+1)\theta) + i\sin((k+1)\theta)\end{aligned}$$

where the last step uses the multiplication formula.

Remark: This also works for negative integers by noting that
$(\cos\theta + i\sin\theta)^{-1} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

Example: Calculate $(1+i)^{10}$.

Solution:

From polar form: $1 + i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

Therefore:

$$\begin{aligned}(1+i)^{10} &= (\sqrt{2})^{10}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})^{10}\\ &= 2^5[\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}]\\ &= 32[\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}]\\ &= 32[0 + i\cdot 1] = 32i\end{aligned}$$

Example: Use De Moivre's theorem to find a formula for $\cos(3\theta)$ in terms of $\cos\theta$.

Solution:

From De Moivre: $(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta)$

Expand the left side:
$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

Equating real parts:
$\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta) = 4\cos^3\theta - 3\cos\theta$

4. Inverse of Power: nth Roots

Theorem: The equation $z^n = w$ has exactly n solutions in $\mathbb{C}$.

If $w = re^{i\theta}$, then the solutions are:
$z_k = r^{1/n} e^{i(\theta+2\pi k)/n}, \quad k = 0, 1, 2, \ldots, n-1$

Why n roots? Each time we add 2π to the angle of w, we get the same complex number. But when we take the nth root, these different representations give different values:

- $w = re^{i\theta} = re^{i(\theta+2\pi)} = re^{i(\theta+4\pi)} = \cdots$
- $w^{1/n}$ has angles: $\frac{\theta}{n}, \frac{\theta+2\pi}{n}, \frac{\theta+4\pi}{n}, \ldots$

These repeat after n terms because $\frac{\theta+2\pi n}{n} = \frac{\theta}{n} + 2\pi$.

Geometric interpretation: The nth roots of w are evenly spaced around a circle of radius $r^{1/n}$, with angular spacing $\frac{2\pi}{n}$.

Example: Find all cube roots of $8i$.

Solution:

First, express in polar form: $8i = 8e^{i\pi/2}$

The cube roots are:
$z_k = 8^{1/3} e^{i(\pi/2+2\pi k)/3} = 2e^{i\pi(1+4k)/6}, \quad k = 0, 1, 2$

Computing each:

- $k = 0$: $z_0 = 2e^{i\pi/6} = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 2(\frac{\sqrt{3}}{2} + \frac{i}{2}) = \sqrt{3} + i$
- $k = 1$: $z_1 = 2e^{i5\pi/6} = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}) = 2(-\frac{\sqrt{3}}{2} + \frac{i}{2}) = -\sqrt{3} + i$
- $k = 2$: $z_2 = 2e^{i3\pi/2} = 2(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}) = 2(0 - i) = -2i$

Verification: We can check $(-2i)^3 = -8i^3 = -8(-i) = 8i$ ✓

Part II: The Complex Exponential Function

5. The Exponential Function

Recall from calculus that for real numbers x:
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

This **Taylor series** converges for all real x and has the fundamental property:
$e^{x_1+x_2} = e^{x_1} \cdot e^{x_2}$

Extending to Complex Numbers, for any complex number $z = x + iy$, we define:
$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Does this definition make sense? We need to verify:

1. Does the series converge? Yes, by ratio test (omitted for the moment)
2. Does it preserve the fundamental property $e^{z_1+z_2} = e^{z_1}e^{z_2}$?

$$e^{z_1} \cdot e^{z_2} = \left(\sum_{m=0}^{\infty} \frac{z_1^m}{m!}\right)\left(\sum_{n=0}^{\infty} \frac{z_2^n}{n!}\right)$$
$$= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} \frac{z_1^m}{m!}\frac{z_2^{k-m}}{(k-m)!}\right)$$

Factor out $\frac{1}{k!}$:
$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k} \frac{k!}{m!(k-m)!} z_1^m z_2^{k-m}\right)$

Recognize the binomial coefficient $\binom{k}{m} = \frac{k!}{m!(k-m)!}$:
$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k} \binom{k}{m} z_1^m z_2^{k-m}\right)$

By the binomial theorem:
$= \sum_{k=0}^{\infty} \frac{(z_1+z_2)^k}{k!} = e^{z_1+z_2}$

This proves the addition property! ✓

Corollary: For $z = x + iy$ where $x, y \in \mathbb{R}$:
$e^{x+iy} = e^x \cdot e^{iy}$

This separates the real and imaginary contributions!

Now we need to understand what e^{iy} means...

6. Euler's Formula

Let's substitute $z = i\theta$ (where θ is real) into the exponential series:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

Now use the powers of i:

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots$$

Separate real and imaginary parts, recall:

$$\cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

$$\sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

Theorem (Euler's Formula): $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

This is one of the most beautiful formulas in all of mathematics!

Special cases:

- $e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0i = -1$

  Therefore: $e^{i\pi} + 1 = 0$ (relating five fundamental constants!)
- $e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i = i$
- $e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1 + 0i = 1$

Part III: Trigonometric Functions

7. Definition

From Euler's formula:
$e^{iz} = \cos z + i\sin z$
$e^{-iz} = \cos(-z) + i\sin(-z) = \cos z - i\sin z$

(using the fact that cosine is even and sine is odd)

Add these equations:
$e^{iz} + e^{-iz} = 2\cos z$

Subtract them:
$e^{iz} - e^{-iz} = 2i\sin z$

Definition: For any complex number z:
$\cos z = \frac{e^{iz}+e^{-iz}}{2}$
$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$

These formulas extend sine and cosine to the entire complex plane.

Property 1: Periodicity

$\cos(z+2\pi) = \frac{e^{i(z+2\pi)}+e^{-i(z+2\pi)}}{2} = \frac{e^{iz}e^{2\pi i}+e^{-iz}e^{-2\pi i}}{2}$

Since $e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$:
$= \frac{e^{iz}+e^{-iz}}{2} = \cos z$

Similarly, $\sin(z+2\pi) = \sin z$.

Property 2: Even/Odd

$\cos(-z) = \frac{e^{-iz}+e^{iz}}{2} = \cos z \quad \text{(even)}$
$\sin(-z) = \frac{e^{-iz}-e^{iz}}{2i} = -\frac{e^{iz}-e^{-iz}}{2i} = -\sin z \quad \text{(odd)}$

Property 3: Values at Real Arguments

When $z = x$ is real, the definitions reduce to the familiar real trigonometric functions.

$\cos x = \frac{e^{ix}+e^{-ix}}{2} = \frac{(\cos x+i\sin x)+(\cos x-i\sin x)}{2} = \cos x \checkmark$

Property 4: $\cos^2 z + \sin^2 z = 1$ **for all complex** z

$$\begin{aligned}\cos^2 z + \sin^2 z &= \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 \\ &= \frac{e^{2iz}+2+e^{-2iz}}{4} + \frac{e^{2iz}-2+e^{-2iz}}{-4} = 1\end{aligned}$$

8. Trigonometric Identities

$\sin(A+B) = \sin A\cos B + \cos A\sin B$
$\cos(A+B) = \cos A\cos B - \sin A\sin B$

Proof of cosine formula:

$$\begin{aligned}\cos(A+B) &= \frac{e^{i(A+B)} + e^{-i(A+B)}}{2} \\ &= \frac{e^{iA}e^{iB} + e^{-iA}e^{-iB}}{2} \\ &= \frac{(\cos A + i\sin A)(\cos B + i\sin B) + (\cos A - i\sin A)(\cos B - i\sin B)}{2} \\ &= \cos A\cos B - \sin A\sin B\end{aligned}$$

Setting $A = B = z$ in the addition formulas:

$\sin(2z) = 2\sin z\cos z$
$\cos(2z) = \cos^2 z - \sin^2 z = 1 - 2\sin^2 z$

$\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
$\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

Proof of first formula:

$$\begin{aligned}\cos A\cos B &= \frac{1}{4}(e^{iA} + e^{-iA})(e^{iB} + e^{-iB}) \\ &= \frac{1}{4}(e^{i(A+B)} + e^{i(A-B)} + e^{-i(A-B)} + e^{-i(A+B)}) \\ &= \frac{1}{4}(e^{i(A+B)} + e^{-i(A+B)} + e^{i(A-B)} + e^{-i(A-B)}) \\ &= \frac{1}{2}\left[\frac{e^{i(A+B)} + e^{-i(A+B)}}{2} + \frac{e^{i(A-B)} + e^{-i(A-B)}}{2}\right] \\ &= \frac{1}{2}[\cos(A+B) + \cos(A-B)]\end{aligned}$$

Unlike real sine and cosine (which are bounded by 1), complex trigonometric functions can take arbitrarily large values.

$\sin(iy) = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y} - e^{y}}{2i} = \frac{-(e^{y} - e^{-y})}{2i} = i\frac{e^{y} - e^{-y}}{2} = i\sinh y$

As $y \to \infty$, $\sinh y \to \infty$, so $|\sin(iy)| \to \infty$.

$\sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$

$|\sin{(x+iy)}|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$

Using $\cosh^2 y - \sinh^2 y = 1$:
$= \sin^2 x(1+\sinh^2 y) + \cos^2 x \sinh^2 y = \sin^2 x + \sinh^2 y$

So: $|\sin{(x+iy)}|^2 = \sin^2 x + \sinh^2 y$

Zeros:
$\sin z = 0$ if and only if $z = n\pi$ for some integer n.
$\cos z = 0$ if and only if $z = \frac{(2n+1)\pi}{2}$ for some integer n.

$\sin z = 0 \Leftrightarrow \frac{e^{iz}-e^{-iz}}{2i} = 0 \Leftrightarrow e^{iz} = e^{-iz}$
$\Leftrightarrow e^{2iz} = 1 = e^{2\pi in}$
$\Leftrightarrow 2iz = 2\pi in \Leftrightarrow z = n\pi$

$\cos z = 0 \Leftrightarrow \frac{e^{iz}+e^{-iz}}{2} = 0 \Leftrightarrow e^{iz} = -e^{-iz}$
$\Leftrightarrow e^{2iz} = -1 = e^{i\pi+2\pi in} = e^{i(2n+1)\pi}$
$\Leftrightarrow 2iz = i(2n+1)\pi \Leftrightarrow z = \frac{(2n+1)\pi}{2}$

The zeros of complex sine and cosine are exactly the same as for real sine and cosine —they're all on the real axis!

9. Hyperbolic Functions

Definition:
$\cosh z = \frac{e^z+e^{-z}}{2}$
$\sinh z = \frac{e^z-e^{-z}}{2}$

Why "hyperbolic"? Just as $(\cos t, \sin t)$ parameterizes the unit circle $x^2+y^2=1$, the pair $(\cosh t, \sinh t)$ parameterizes the unit hyperbola $x^2-y^2=1$.

Compare the definitions:

Trigonometric:
$\cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$

Hyperbolic:
$\cosh z = \frac{e^z+e^{-z}}{2}, \quad \sinh z = \frac{e^z-e^{-z}}{2}$

Key observation: If we replace z with iz in the trigonometric functions:

$\cos(iz) = \frac{e^{i(iz)}+e^{-i(iz)}}{2} = \frac{e^{-z}+e^{z}}{2} = \cosh z$

$\sin(iz) = \frac{e^{i(iz)}-e^{-i(iz)}}{2i} = \frac{e^{-z}-e^{z}}{2i} = \frac{-(e^{z}-e^{-z})}{2i} = i\frac{e^{z}-e^{-z}}{2} = i\sinh z$

Property 1: Even/Odd

$\cosh(-z) = \frac{e^{-z}+e^{z}}{2} = \cosh z \quad \text{(even)}$

$\sinh(-z) = \frac{e^{-z}-e^{z}}{2} = -\sinh z \quad \text{(odd)}$

Property 2: Hyperbolic Pythagorean Identity

$$\cosh^2 z - \sinh^2 z = 1$$

$$\begin{aligned}\cosh^2 z - \sinh^2 z &= \left(\frac{e^z+e^{-z}}{2}\right)^2 - \left(\frac{e^z-e^{-z}}{2}\right)^2 \\ &= \frac{(e^z+e^{-z})^2-(e^z-e^{-z})^2}{4} \\ &= \frac{e^{2z}+2+e^{-2z}-e^{2z}+2-e^{-2z}}{4} = 1\end{aligned}$$

Property 3: Addition Formulas

$$\sinh(A+B) = \sinh A\cosh B + \cosh A\sinh B$$
$$\cosh(A+B) = \cosh A\cosh B + \sinh A\sinh B$$

Note: The sign in the cosine formula is **plus**, not minus!

$$\begin{aligned}\sinh(A+B) &= \frac{e^{A+B}-e^{-(A+B)}}{2} \\ &= \frac{e^{A}e^{B}-e^{-A}e^{-B}}{2} \\ &= \frac{1}{4}[(e^A+e^{-A})(e^B-e^{-B})+(e^A-e^{-A})(e^B+e^{-B})] \\ &= \sinh A\cosh B + \cosh A\sinh B\end{aligned}$$

Zeros:

$\sinh z = 0$ if and only if $z = n\pi i$ for some integer n.
$\cosh z = 0$ if and only if $z = \frac{(2n+1)\pi i}{2}$ for some integer n.

Since $\sin(iz) = i\sinh z$:
$\sinh z = 0 \Leftrightarrow \sin(iz) = 0 \Leftrightarrow iz = n\pi \Leftrightarrow z = n\pi i$

Since $\cos(iz) = \cosh z$:
$\cosh z = 0 \Leftrightarrow \cos(iz) = 0 \Leftrightarrow iz = \frac{(2n+1)\pi}{2} \Leftrightarrow z = \frac{(2n+1)\pi i}{2}$

Summary Table:

FUNCTION	REAL ZEROS	COMPLEX ZEROS
$\sin z$	$z = n\pi$	Same (all real)
$\cos z$	$z = \frac{(2n+1)\pi}{2}$	Same (all real)
$\sinh z$	$z = 0$ only	$z = n\pi i$ (purely imaginary)
$\cosh z$	None	$z = \frac{(2n+1)\pi i}{2}$ (purely imaginary)

Geometric insight: The transformation $z \mapsto iz$ is a 90° rotation in the complex plane. This rotation transforms:

- Real zeros → Imaginary zeros
- Oscillatory behavior → Exponential behavior
- Bounded functions → Unbounded functions

Part IV: Complex Methods

10. Sine and Cosine in Pairs

Example: Compute $S = \sum_{k=0}^{n} \cos(kx)$.

Solution:

Use $\cos(kx) = \mathrm{Re}(e^{ikx})$: $S = \mathrm{Re}\left(\sum_{k=0}^{n} e^{ikx}\right)$

The sum is a geometric series: $\sum_{k=0}^{n} e^{ikx} = \frac{1-e^{i(n+1)x}}{1-e^{ix}}$
(provided $e^{ix} \neq 1$, i.e., $x \neq 2\pi m$)

To simplify, multiply numerator and denominator by $e^{-ix/2}$:
$= \frac{e^{-ix/2}-e^{i(n+1/2)x}}{e^{-ix/2}-e^{ix/2}} = \frac{e^{inx/2}(e^{-i(n+1)x/2}-e^{i(n+1)x/2})}{e^{-ix/2}-e^{ix/2}}$

$= \frac{e^{inx/2}\cdot(-2i\sin((n+1)x/2))}{-2i\sin(x/2)} = e^{inx/2}\frac{\sin((n+1)x/2)}{\sin(x/2)}$

Taking the real part:
$\sum_{k=0}^{n} \cos(kx) = \cos(nx/2)\frac{\sin((n+1)x/2)}{\sin(x/2)}$

Taking the imaginary part of the same calculation:
$\sum_{k=0}^{n} \sin(kx) = \sin(nx/2)\frac{\sin((n+1)x/2)}{\sin(x/2)}$

11. Real and Imaginary Separate

Problem: Evaluate $I = \int_0^{\pi/2} \cos^6 x dx$.

Traditional approach: Use reduction formulas, which are messy and time-consuming.

Complex solution:

Use the identity $\cos x = \frac{e^{ix}+e^{-ix}}{2}$:
$\cos^6 x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^6 = \frac{1}{64}(e^{6ix} + 6e^{4ix} + 15e^{2ix} + 20 + 15e^{-2ix} + 6e^{-4ix} + e^{-6ix})$

Since the integral is real, we only need the constant term:
$I = \int_0^{\pi/2} \cos^6 x dx = \frac{1}{64} \int_0^{\pi/2} 20 dx = \frac{20}{64} \cdot \frac{\pi}{2} = \frac{5\pi}{32}$

Problem: Prove that $\sum_{k=0}^{n} \binom{n}{k} \cos(kx) = 2^n \cos^n\left(\frac{x}{2}\right) \cos\left(\frac{nx}{2}\right)$.

Traditional approach: Extremely difficult with combinatorial identities.

Complex solution:

Use the binomial theorem with complex numbers:
$\sum_{k=0}^{n} \binom{n}{k} e^{ikx} = (1 + e^{ix})^n$

But $1 + e^{ix} = e^{ix/2}(e^{-ix/2} + e^{ix/2}) = 2\cos(x/2)e^{ix/2}$

So $(1 + e^{ix})^n = 2^n \cos^n(x/2) e^{inx/2}$

Taking real parts:
$\sum_{k=0}^{n} \binom{n}{k} \cos(kx) = 2^n \cos^n(x/2) \cos(nx/2)$

12. Unity Roots

Problem: Prove that $\cos\frac{\pi}{7} \cos\frac{2\pi}{7} \cos\frac{3\pi}{7} = \frac{1}{8}$.

Traditional approach: Nearly impossible with standard trig identities.

Complex solution:

Let $z = e^{i\pi/7}$, then $z^7 = -1$.

The product becomes:
$P = \left(\frac{z+z^{-1}}{2}\right)\left(\frac{z^2+z^{-2}}{2}\right)\left(\frac{z^3+z^{-3}}{2}\right) = \frac{1}{8}(z+z^{-1})(z^2+z^{-2})(z^3+z^{-3})$

$= z^6 + 1 + z^4 + z^2 + z^{-2} + z^{-4} + 1 + z^{-6}$

But $z^7 = -1 \Rightarrow z^{-k} = -z^{7-k}$, so:
$= z^6 + z^4 + z^2 + 2 - z^5 - z^3 - z$

Since $1 + z + z^2 + \cdots + z^6 = 0$, this sum equals 1.

Thus $P = \frac{1}{8}$.

Problem: Factor $x^8 + x^4 + 1$ over the reals.

Traditional approach: Difficult to see the pattern.

Complex solution:

Recognize this as related to the 12th roots of unity:
$x^{12} - 1 = (x^4 - 1)(x^8 + x^4 + 1)$

So $x^8 + x^4 + 1 = \frac{x^{12}-1}{x^4-1}$

The roots are the primitive 12th roots of unity: $e^{\pm i\pi/6}, e^{\pm i\pi/2}, e^{\pm i5\pi/6}$

Group conjugate pairs:
$(x - e^{i\pi/6})(x - e^{-i\pi/6}) = x^2 - 2\cos(\pi/6)x + 1 = x^2 - \sqrt{3}x + 1$

$(x - e^{i\pi/2})(x - e^{-i\pi/2}) = x^2 + 1$

$(x - e^{i5\pi/6})(x - e^{-i5\pi/6}) = x^2 - 2\cos(5\pi/6)x + 1 = x^2 + \sqrt{3}x + 1$

Thus:
$x^8 + x^4 + 1 = (x^2 - \sqrt{3}x + 1)(x^2 + 1)(x^2 + \sqrt{3}x + 1)$

Part V: Exercises

Problem Set A: Polar Form and De Moivre

A1. Convert to polar form:
(a) $3 + 3i$
(b) $-1 + \sqrt{3}i$
(c) $-4i$
(d) $-2 - 2i$

A2. Convert to rectangular form:
(a) $2e^{i\pi/6}$
(b) $5e^{3i\pi/4}$
(c) $e^{-i\pi/3}$

A3. Compute using De Moivre's theorem:
(a) $(1 + i\sqrt{3})^6$
(b) $(\frac{1}{2} + \frac{\sqrt{3}}{2}i)^{12}$
(c) $(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8})^{16}$

A4. Find all solutions:
(a) $z^4 = 16$
(b) $z^3 = -8$
(c) $z^6 = -1$

A5. Use De Moivre to prove:
(a) $\cos(4\theta) = 8\cos^4\theta - 8\cos^2\theta + 1$
(b) $\sin(5\theta)$ in terms of $\sin\theta$

Problem Set B: Complex Exponentials

B1. Compute exactly:
(a) $|e^{2+3i}|$
(b) $e^{i\pi/3} \cdot e^{i\pi/6}$
(c) $(e^{i\pi/4})^8$

B2. Solve for all complex z:
(a) $e^z = 1$
(b) $e^z = -1$
(c) $e^z = i$
(d) $e^z = 2i$

B3. Prove that $|e^{iz}| = e^{-\text{Im}(z)}$ for all complex z.

B4. Show that $e^{z+2\pi i} = e^z$ for all z.

B5. Find all z such that e^z is:
(a) Real and positive
(b) Real and negative
(c) Purely imaginary

Problem Set C: Trigonometric Functions

C1. Compute exactly (in $a + bi$ form):
(a) $\sin(i)$
(b) $\cos(i)$
(c) $\sin(1+i)$
(d) $\cos(2i)$

C2. Solve for all complex z:
(a) $\sin z = 0$
(b) $\cos z = 0$
(c) $\sin z = 2$
(d) $\cos z = i$

C3. Prove the identities:
(a) $\sin(z+\pi) = -\sin z$
(b) $\cos(z+\pi) = -\cos z$
(c) $\sin(\pi/2 - z) = \cos z$
(d) $|\sin z|^2 + |\cos z|^2 =?$ (Is it always 1?)

C4. Find all z such that $\sin z = \cos z$.

Problem Set D: Hyperbolic Functions

D1. Compute:
(a) $\sinh(i\pi/2)$
(b) $\cosh(i\pi)$
(c) $\sinh(\ln 2)$
(d) $\cosh(\ln 3)$

D2. Prove:
(a) $\sinh(2z) = 2\sinh z\cosh z$
(b) $\cosh(2z) = \cosh^2 z + \sinh^2 z$
(c) $\cosh z + \sinh z = e^z$
(d) $\cosh z - \sinh z = e^{-z}$

D3. Solve for all complex z:
(a) $\sinh z = 0$
(b) $\cosh z = 2$
(c) $\sinh z = i$

D4. Express in terms of trig functions:
(a) $\sinh(ix)$ where x is real
(b) $\cosh(ix)$ where x is real

Problem Set E: Applications and Synthesis

E1. Chebyshev Polynomials: Define $T_n(\cos\theta) = \cos(n\theta)$.
(a) Show that $T_n(x)$ is indeed a polynomial in x.
(b) Find explicit formulas for $T_2(x), T_3(x), T_4(x)$.
(c) Prove that $T_n(x)$ satisfies: $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

E2. Lagrange's trigonometric identities:
Prove: $\sum_{k=0}^{n} \cos(kx) = \frac{\sin((n+1)x/2)\cos(nx/2)}{\sin(x/2)}$

E3. Evaluate:
(a) $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)d\theta$
(b) $\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n}$
(c) $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ for $0 < x < 2\pi$

E4. Lucas's theorem on cube roots:
Let $\omega = e^{2\pi i/3}$. Show that for any integers a, b, c:
$(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) = a^2 + b^2 + c^2 - ab - bc - ca$

E5. Prove that $\prod_{k=1}^{n-1} \sin\frac{k\pi}{n} = \frac{n}{2^{n-1}}$.

E6. Partial fractions in the complex domain:
Show that $\frac{1}{1-z^n} = \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{1-\omega^k z}$ where $\omega = e^{2\pi i/n}$.