Complex Numbers for Calculus

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1. Introduction

- Solve all polynomial equations
- Compute both rotation and scaling
- Unify trigonometric and exponential functions
- Simplify many calculus computations

2. Complex Solutions

Consider the quadratic equation: $x^2 + 1 = 0$. Complex numbers solve this by defining i as $\sqrt{-1}$.

For any quadratic $ax^2 + bx + c = 0$:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Case 1: $\Delta = b^2 - 4ac > 0 \rightarrow$ Two real solutions.
Case 2: $\Delta = b^2 - 4ac = 0 \rightarrow$ One real solution
Case 3: $\Delta = b^2 - 4ac < 0 \rightarrow$ Two complex solutions

Complex numbers ensure that **every** polynomial equation has solutions, completing algebra in a profound way.

3. Definition and Basic Operations

A complex number is written as $z = a + bi$ where:

- a is the **real part**: $\mathrm{Re}(z) = a$
- b is the **imaginary part**: $\mathrm{Im}(z) = b$

- i is the **imaginary unit**: $i^2 = -1$

$a + bi = c + di$ if and only if $a = c$ and $b = d$

$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$

$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$

$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$

4. Complex Conjugates and Modulus

For $z = a + bi$, the **complex conjugate** is $\overline{z} = a - bi$

1. **Conjugate of conjugate**: $\overline{\overline{z}} = z$
2. **Addition**: $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
3. **Multiplication**: $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
4. Real and imaginary parts:
    - $\text{Re}(z) = \frac{z+\overline{z}}{2}$
    - $\text{Im}(z) = \frac{z-\overline{z}}{2i}$
5. **Product with conjugate**: $z \cdot \overline{z} = a^2 + b^2$ (always real and non-negative)

The **modulus** of $z = a + bi$ is: $|z| = \sqrt{a^2 + b^2} = \sqrt{z \cdot \overline{z}}$

1. $|z| \geq 0$, with equality only when $z = 0$
2. $|\overline{z}| = |z|$
3. $|z_1 z_2| = |z_1||z_2|$
4. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ (when $z_2 \neq 0$)
5. **Triangle inequality**: $|z_1 + z_2| \leq |z_1| + |z_2|$

5. The Complex Plane and Polar Representation

We represent complex numbers geometrically:

- **Real axis** (horizontal): real part
- **Imaginary axis** (vertical): imaginary part
- Point (a, b) represents $z = a + bi$

Any complex number $z = a + bi$ can be written as: $z = r(\cos\theta + i\sin\theta)$

where:

- $r = |z| = \sqrt{a^2 + b^2}$ (modulus)
- $\theta = \arg(z)$ (argument/angle)

Rectangular to Polar:

- $r = \sqrt{a^2 + b^2}$
- $\theta = \arctan\left(\frac{b}{a}\right)$ (with quadrant adjustment)

Polar to Rectangular:

- $a = r\cos\theta$
- $b = r\sin\theta$

Example: Convert $z = -1 + i$ to polar form

$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\theta = \arctan\left(\frac{1}{-1}\right) = \arctan(-1)$

Since z is in the second quadrant: $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Therefore: $z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

6. Geometric Heart: Rotation

What is $\sqrt{-1}$?

On the real number line, -1 is the result of **rotating** 1 by 180° around the origin. Multiplying by -1 is equivalent to a 180° rotation. Then what operation, when applied twice, gives us this 180° rotation?

Answer: A 90° rotation!

Let i represent a 90° counterclockwise rotation. Then:

- $i \cdot i$ = (90° rotation) followed by (90° rotation) = 180° rotation = -1
- Therefore: $i^2 = -1$, so $i = \sqrt{-1}$

For any complex number $z = a + bi$, multiplication by i gives:
$iz = i(a + bi) = ai + bi^2 = ai - b = -b + ai$

Geometric effect: The point (a, b) becomes $(-b, a)$

—exactly what happens under 90° counterclockwise rotation!

General Pattern: i^n rotates by $90n$ degrees.

7. General Complex Multiplication

Complex Multiplication = Rotation + Scaling

Every complex number $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ can be thought of as encoding two geometric operations:

1. **Scaling** by factor $r = |z|$
2. **Rotation** by angle $\theta = \arg(z)$

Before we prove Euler's formula, let's understand what it should mean:

- $e^{i\theta}$ should represent pure rotation by angle θ
- $re^{i\theta}$ should represent scaling by r and rotating by θ

Consider multiplication: $(r_1e^{i\theta_1})(r_2e^{i\theta_2})$

If our interpretation is correct, this should:

1. Scale by $r_1 \times r_2$
2. Rotate by $\theta_1 + \theta_2$

Result: $r_1r_2e^{i(\theta_1+\theta_2)}$

Equivalently, w could also prove:

Multiplication: Multiply moduli, add arguments
$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

Division: Divide moduli, subtract arguments $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

Rotations should **add** when combined.

Complex Numbers vs. Vectors

OPERATION	VECTORS	COMPLEX NUMBERS
Addition	Parallelogram law	Same as vectors
Scaling	Scalar multiplication	Multiplication by real numbers
Rotation	Matrix multiplication	Multiplication by $e^{i\theta}$

Key advantage: Complex multiplication naturally combines rotation and scaling in one operation.

8. Euler's Formula

- $e^{i\theta}$ should represent rotation by angle θ
- A point at distance 1 from origin, at angle θ, has coordinates $(\cos\theta, \sin\theta)$
- In complex notation: $\cos\theta + i\sin\theta$

Conjecture: $e^{i\theta} = \cos\theta + i\sin\theta$

Defining Complex Exponentials with Taylor Series

For real x:

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
- $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$
- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

Define e^{ix} using the exponential series:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

Using $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc. to separate the real and imaginary parts:

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

Recognizing the series: $e^{ix} = \cos x + i\sin x$

Beautiful consequence: $e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0i = -1$

Euler's Identity: $e^{i\pi} + 1 = 0$

Often called the most beautiful equation in mathematics, connecting e, i, π, 1, and 0 with $+$ and $=$.

An Alternative Illustration (Not Robust)

Consider the function $f(x) = e^{-ix}(\cos x + i \sin x)$

$f'(x) = \frac{d}{dx}[e^{-ix}(\cos x + i \sin x)]$

$= (-i)e^{-ix}(\cos x + i \sin x) + e^{-ix}(-\sin x + i \cos x)$

$= e^{-ix}[-i(\cos x + i \sin x) + (-\sin x + i \cos x)]$

$= e^{-ix}[-i \cos x + \sin x - \sin x + i \cos x] = e^{-ix} \cdot 0 = 0$

Since $f'(x) = 0$, $f(x)$ is constant.

At $x = 0$: $f(0) = e^0(\cos 0 + i \sin 0) = 1(1 + 0) = 1$

Therefore: $f(x) = 1$ for all x.

$\cos x + i \sin x = e^{ix}$

This illustrates Euler's formula through elementary calculus!

9. Exponential Form

Using Euler's formula: $z = re^{i\theta}$ where $r = |z|$ and $\theta = \arg(z)$

Multiplication: $z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

Division: $\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

Powers: $z^n = (re^{i\theta})^n = r^n e^{in\theta}$

Finding the nth Power (De Moivre's Theorem)

$(\cos\theta + i \sin\theta)^n = \cos(n\theta) + i \sin(n\theta)$, easier when expressed with the exponential form

Example: Find $(1 + i)^{10}$

Solution: $1+i=\sqrt{2}e^{i\pi/4},\ (1+i)^{10}=(\sqrt{2})^{10}e^{i10\pi/4}=2^5e^{i5\pi/2}=32e^{i\pi/2}=32i$

Finding nth Roots

The nth roots of $z=re^{i\theta}$ are: $z^{1/n}=r^{1/n}e^{i(\theta+2\pi k)/n},\quad k=0,1,2,\ldots,n-1$

Example: Find all cube roots of $8i$

Solution: $8i=8e^{i\pi/2},$ Cube roots: $2e^{i(\pi/2+2\pi k)/3}$ for $k=0,1,2$

- $k=0$: $2e^{i\pi/6}=2\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)=\sqrt{3}+i$
- $k=1$: $2e^{i5\pi/6}=2\left(-\frac{\sqrt{3}}{2}+\frac{i}{2}\right)=-\sqrt{3}+i$
- $k=2$: $2e^{i3\pi/2}=2(0-i)=-2i$

10. Trigonometric Functions

Euler's formulas:

- $e^{iz}=\cos z+i\sin z$
- $e^{-iz}=\cos z-i\sin z$

Inverse relationships:

- $\cos z=\frac{e^{iz}+e^{-iz}}{2}$
- $\sin z=\frac{e^{iz}-e^{-iz}}{2i}$

$$\frac{d}{dz}\cos z=\frac{d}{dz}\left(\frac{e^{iz}+e^{-iz}}{2}\right)=\frac{ie^{iz}-ie^{-iz}}{2}=-\frac{e^{iz}-e^{-iz}}{2i}=-\sin z$$

$$\frac{d}{dz}\sin z=\frac{d}{dz}\left(\frac{e^{iz}-e^{-iz}}{2i}\right)=\frac{ie^{iz}+ie^{-iz}}{2i}=\frac{e^{iz}+e^{-iz}}{2}=\cos z$$

Trigonometric identities:

$$\cos(A+B)=\mathrm{Re}(e^{i(A+B)})=\mathrm{Re}(e^{iA}e^{iB})=\cos A\cos B-\sin A\sin B$$

$$\cos A\cos B=\frac{1}{4}[(e^{iA}+e^{-iA})(e^{iB}+e^{-iB})]=\frac{1}{2}[\cos(A+B)+\cos(A-B)]$$

11. Hyperbolic Functions

$\cosh z = \frac{e^z+e^{-z}}{2}, \quad \sinh z = \frac{e^z-e^{-z}}{2}$

Compare with trigonometric functions: $\cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$

The deep connection becomes clear when we substitute $z \leftrightarrow iz$:

From trigonometric to hyperbolic:

- $\cos(iz) = \frac{e^{i(iz)}+e^{-i(iz)}}{2} = \frac{e^{-z}+e^{z}}{2} = \cosh z$
- $\sin(iz) = \frac{e^{i(iz)}-e^{-i(iz)}}{2i} = \frac{e^{-z}-e^{z}}{2i} = \frac{-(e^z-e^{-z})}{2i} = \frac{i(e^z-e^{-z})}{2} = i\sinh z$

From hyperbolic to trigonometric:

- $\cosh(iz) = \frac{e^{iz}+e^{-iz}}{2} = \cos z$
- $\sinh(iz) = \frac{e^{iz}-e^{-iz}}{2} = \frac{2i\sin z}{2i} = i\sin z$

Zeros of Trigonometric Functions

- $\cos z = 0 \Rightarrow \frac{e^{iz}+e^{-iz}}{2} = 0 \Rightarrow e^{2iz} = -1 \Rightarrow 2iz = (2n+1)\pi i$
- Therefore: $z = \frac{(2n+1)\pi}{2}$ for integer n
- $\sin z = 0 \Rightarrow \frac{e^{iz}-e^{-iz}}{2i} = 0 \Rightarrow e^{2iz} = 1 \Rightarrow 2iz = 2n\pi i$
- Therefore: $z = n\pi$ for integer n

Zeros of Hyperbolic Functions

- $\cosh z = 0 \Rightarrow \cos(iz) = 0 \Rightarrow iz = \frac{(2n+1)\pi}{2}$
- Therefore: $z = \frac{(2n+1)\pi i}{2}$ for integer n
- $\sinh z = 0 \Rightarrow i\sin(iz) = 0 \Rightarrow \sin(iz) = 0 \Rightarrow iz = n\pi$
- Therefore: $z = n\pi i$ for integer n

1. Summary of Course Related Insights

FUNCTION	REAL ZEROS	COMPLEX ZEROS
$\cos z$	$z = \frac{(2n+1)\pi}{2}$	Same (all real)
$\sin z$	$z = n\pi$	Same (all real)
$\cosh z$	None	$z = \frac{(2n+1)\pi i}{2}$ (purely imaginary)
$\sinh z$	$z = 0$ only	$z = n\pi i$ (purely imaginary)

The transformation $z \mapsto iz$ represents a 90° rotation in the complex plane. This reveals why:

1. **Trigonometric functions** (oscillatory on real axis) become **hyperbolic functions** (exponential growth/decay) when rotated
2. **Real zeros** of trigonometric functions become **imaginary zeros** of hyperbolic functions
3. The **periodic behavior** of trigonometric functions becomes **exponential behavior** of hyperbolic functions

On the real axis:

- $\cos x$ and $\sin x$ oscillate between -1 and 1
- $\cosh x \geq 1$ for all real x (never zero!)
- $\sinh x$ is strictly increasing (zero only at $x = 0$)

In the complex plane:

- All four functions have infinite zeros
- Trigonometric functions have real zeros, hyperbolic functions have imaginary zeros
- The zeros are related by the transformation $z \mapsto iz$

Appendix A: Linear Independence (Not Required)

Two functions $f(x)$ and $g(x)$ are **linearly independent** if the only way to make $af(x) + bg(x) = 0$ for all x is to have $a = 0$ and $b = 0$.

Intuitive meaning: Neither function can be written as a constant multiple of the other.

The Fundamental Theorem for Complex Functions

If $z(x) = u(x) + iv(x)$ where $u(x)$ and $v(x)$ are real-valued functions, then $u(x)$ and $v(x)$ are automatically linearly independent (unless one of them is identically zero).

Proof: Suppose $au(x) + bv(x) = 0$ for all x, where a, b are real constants.

We know $u(x) = \frac{z(x)+\overline{z(x)}}{2}$ and $v(x) = \frac{z(x)-\overline{z(x)}}{2i}$

Substituting: $a \cdot \frac{z(x)+\overline{z(x)}}{2} + b \cdot \frac{z(x)-\overline{z(x)}}{2i} = 0$

Since $\frac{b}{i} = -ib$, $\frac{a-ib}{2} z(x) + \frac{a+ib}{2} \overline{z(x)} = 0$

For this to hold for all x (assuming $z(x)$ is not trivial), we need: $a - ib = 0$ and $a + ib = 0$

This gives us $a = 0$ and $b = 0$. ✓

Example 1: The Harmonic Oscillator $y'' + \omega^2 y = 0$

Step 1: Try complex solution $y = e^{i\omega x}$

- $y' = i\omega e^{i\omega x}$
- $y'' = (i\omega)^2 e^{i\omega x} = -\omega^2 e^{i\omega x}$
- Check: $y'' + \omega^2 y = -\omega^2 e^{i\omega x} + \omega^2 e^{i\omega x} = 0$ ✓

Step 2: Extract real and imaginary parts
$e^{i\omega x} = \cos(\omega x) + i \sin(\omega x)$

Step 3: Since both real and imaginary parts satisfy the equation:

- $y_1 = \cos(\omega x)$ is a solution
- $y_2 = \sin(\omega x)$ is a solution
- These are linearly independent (neither is a constant multiple of the other)

Step 4: General real solution
$y = A\cos(\omega x) + B\sin(\omega x)$

Why this works: The complex exponential "packages" both independent solutions together!

Example 2: Harmonic Functions

Definition: A function $u(x, y)$ is harmonic if $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Key Property: If $f(z) = u(x, y) + iv(x, y)$ is analytic (complex differentiable), then both u and v are harmonic.

Example: Let $f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + i(2xy)$

- $u(x, y) = x^2 - y^2$
- $v(x, y) = 2xy$

Verification:

- $\frac{\partial^2 u}{\partial x^2} = 2$, $\frac{\partial^2 u}{\partial y^2} = -2$, so $\nabla^2 u = 0$ ✓
- $\frac{\partial^2 v}{\partial x^2} = 0$, $\frac{\partial^2 v}{\partial y^2} = 0$, so $\nabla^2 v = 0$ ✓

Power: Starting with one analytic function, we get **two** linearly independent harmonic functions.

The Big Picture

Linear independence of real and imaginary parts explains why complex methods are so powerful:

1. **Efficiency**: One complex calculation gives two real results
2. **Completeness**: The two parts span the solution space for second-order linear equations
3. **Natural structure**: Complex functions respect the underlying symmetries of the problems

Complex numbers reveal the **structural relationships** between seemingly different real-valued solutions.

Appendix B: Complex Methods (Not Required)

Many real calculus problems become easier when extended to complex numbers, then we extract the real or imaginary part of the solution.

Example 1: Integrating $\int e^{ax} \cos{(bx)} dx$

Complex approach: Consider $\int e^{ax} e^{ibx} dx = \int e^{(a+ib)x} dx = \frac{e^{(a+ib)x}}{a+ib}$

Rationalizing: $\frac{e^{(a+ib)x}}{a+ib} \cdot \frac{a-ib}{a-ib} = \frac{(a-ib)e^{ax}e^{ibx}}{a^2+b^2}$

$= \frac{(a-ib)e^{ax}(\cos bx + i \sin bx)}{a^2+b^2}$

Taking the real part: $\int e^{ax} \cos{(bx)} dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2+b^2} + C$

Example 2: Finding Particular Solutions to $y'' + \omega^2 y = e^{ax}\cos(bx)$

Step 1: Replace the forcing function with $e^{ax}e^{ibx} = e^{(a+ib)x}$

Step 2: Try particular solution $y_p = Ae^{(a+ib)x}$

Step 3: Substitute and solve for A: $A[(a+ib)^2 + \omega^2]e^{(a+ib)x} = e^{(a+ib)x}$

$A = \frac{1}{(a+ib)^2+\omega^2} = \frac{1}{a^2-b^2+2abi+\omega^2}$

Step 4: Rationalize and take real part to get solution to original problem.

Example 3: Summing Trigonometric Series

Problem: Find $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$

Solution: Consider the complex series $\sum_{n=1}^{\infty} \frac{e^{inx}}{n^2}$

This relates to the polylogarithm function. The real part gives our desired sum.

Using $e^{inx} = \cos(nx) + i\sin(nx)$: $\sum_{n=1}^{\infty} \frac{e^{inx}}{n^2} = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} + i\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$

The complex approach often provides more powerful techniques for evaluating such series.

When to use complex methods:

1. Trigonometric functions appear in the problem
2. Exponential and trigonometric functions are combined
3. Oscillatory behavior needs to be analyzed
4. Linear differential equations with constant coefficients

The Big Picture

Complex exponentials $e^{(a+ib)x}$ are easier to differentiate and integrate than separate trigonometric functions, and they automatically encode both the exponential growth/decay and the oscillatory behavior.