Solution 1

Using the Master Method, recurrence relations have the following form:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

the question in the homework has $a = 4$, $b = 2$, and $f(n)$ is $\Theta(1)$. There are three cases for the Master Method,

i) If $f(n) = O\left(n^{\log_b a - \varepsilon}\right)$, for some $\varepsilon > 0$, then $T(n) = \Theta\left(n^{\log_b a}\right)$.
ii) If $f(n) = O\left(n^{\log_b a}\right)$, then $T(n) = \Theta\left(n^{\log_b a} \log n\right)$.
iii) If $f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right)$ for some $\varepsilon > 0$, and $af\left(\frac{n}{b}\right) \le cf(n)$, for some $c < 1$ and for all n greater than some value n', then $T(n) = \Theta(f(n))$.

In order to match the requirement that $f(n) = \Theta(1)$ and if needed, $\varepsilon > 0$, $f(n)$ must be (i), while $\varepsilon = 2$. Therefore, $T(n) = \Theta\left(n^{\log_b a}\right) = \Theta\left(n^{\log_2 4}\right) = \Theta\left(n^2\right)$. Q.E.D.

Reference: http://goo.gl/lipThF

Solution 2

Since we know that $T(n) = 4T\left(\frac{n}{2}\right) + \Theta(1)$, so we know that $T(n) = 4T\left(\frac{n}{2}\right) + O(1)$ and so there is a $c > 0$, such that

$$T(n) \le 4T\left(\frac{n}{2}\right) + c$$

for all n sufficiently large. I hereby named my guess (not quite guess-y though) $T(n) = O\left(n^2\right)$, so that my guess will be that $T(n) \le kn^2$, since I know that there will be a constant left, I made a minor modification: $T(n) \le kn^2 + d$, d is an arbitrary number, for some $k > 0$ and all n sufficiently large.

During the try-n-error, for the k enlisted above, we have

$$\begin{aligned} T(n) &\le 4T\left(\frac{n}{2}\right) + c \\ &\le 4\left(k\left(\frac{n}{2}\right)^2 + d\right) + c = kn^2 + 4d + c \\ &\le kn^2 + d \end{aligned}$$

for $c \le -3d$, our case is valid. As for the $T(n) = 4T\left(\frac{n}{2}\right) + \Omega(1)$ (the Ω counterpart of the O), the entire proof is almost the same, but the equalities are reversed. Therefore, it's easy to see that $T(n) = \Theta\left(n^2\right)$. Q.E.D.

Reference: http://goo.gl/kt39kK and my roommate B03901057