# P8104 Homework Assignment 6

Due Wed 10/29. 11:59pm

## Problem 1

Verify that the following probability density functions (pdfs) have the indicated hazard function. The hazard function of a random variable $T$ is defined as

$$h_T(t) = \lim_{\delta \to 0} \frac{P(t \leq T < t + \delta \mid T \geq t)}{\delta} = \frac{f_T(t)}{1 - F_T(t)}$$

Compute $h_T(t)$ in each case and verify that it matches the stated form:

(a) $T \sim \text{Exponential}(\beta)$: $h_T(t) = \dfrac{1}{\beta}$.

(b) $T \sim \text{Weibull}(r, \beta)$: $h_T(t) = \dfrac{r}{\beta} t^{r-1}$.

(c) $T \sim \text{Logistic}(\mu, \beta)$, where $F_T(t) = \dfrac{1}{1 + e^{-(t-\mu)/\beta}}$: $h_T(t) = \dfrac{1}{\beta} F_T(t)$.

## Problem 2

Let $X \sim \text{DoubleExponential}(\mu, \sigma)$ with pdf

$$f(x|\mu, \sigma) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma}, \quad -\infty < x < \infty$$

Compute

(a) $E(X)$ using its definition.

(b) $Var(X)$ using its definition.

(c) The moment generating function (MGF) using its definition.

Please show all intermediate steps to get full credits.

## Problem 3

The pdf of $X$, representing the lifetime of a device (in years) is

$$f(x) = \begin{cases} \frac{1}{4} x e^{-x/2}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

(a) What is the probability that the device survives at least 5 years?

(b) Calculate the expected lifetime of the device.

(c) In a batch of 100 such devices, what is the (approximate) probability that 25 or more will survive at least 5 years?

## Problem 4

Simple transformations of many distributions conform to a known distribution. In this exercise, we provide a set of random variables $X$ paired with a transformation $Y = g(X)$. In each case, (i) derive and identify the distribution of $Y$ and specify its domain, and (ii) calculate $E(Y)$ and $\text{Var}(Y)$.

(a) $X \sim N(\mu, \sigma^2), \quad Y = e^X.$

(b) $X \sim \text{Gamma}(n, \beta),\ n$ integer, $\quad Y = \dfrac{2X}{\beta}.$

(c) $X \sim U(0,1), \quad Y = \max(X, 1 - X).$

(d) $X \sim \text{Cauchy}(0), \quad Y = \dfrac{1}{X}.$

(e) $X \sim \text{Exp}(\beta), \quad Y =$ smallest integer $\geq X$. For example, if $X = 5.13$, then $Y = 6$.

## Problem 5

There is an interesting relationship between the negative binomial and the gamma distribution, which can sometimes provide a useful approximation. Let $Y \sim \text{NegativeBinomial}(r, p)$, where $f(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}$. Show that the mgf of $Y$ converges to Gamma($r$,1) when $p$ goes to 0.
*Hint: You may use L′Hôpital′s rule:* $\lim_{x \to c} \frac{g(x)}{f(x)} = \lim_{x \to c} \frac{g'(x)}{f'(x)}$.