Homework 7 - P8104 Probability

Yongyan Liu (yl6107)

Due Wed 11/5, 11:59pm

Problem 1

Suppose we toss a pair of fair, four-sided dice, in which one of the dice is RED and the other is BLACK. Let

- X: the outcome on the RED die $= \{1, 2, 3, 4\}$
- Y: the outcome on the BLACK die $= \{1, 2, 3, 4\}$

(a) What is the probability that X takes on a particular value x, and Y takes on a particular value y, i.e., what is $P(X = x, Y = y)$?

(b) What is the expectation of X?

Solution

Part (a)

By independence:

$$P(X = x, Y = y) = P(X = x) \cdot P(Y = y) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$$

Therefore, $P(X = x, Y = y) = \frac{1}{16}$ for any $(x, y) \in \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$.

Part (b)

The expectation of X is:

$$E(X) = \sum_{x=1}^{4} x \cdot P(X = x) = \sum_{x=1}^{4} x \cdot \frac{1}{4} = \frac{1}{4}(1 + 2 + 3 + 4) = \frac{10}{4} = \frac{5}{2}$$

Problem 2

A pdf is defined by

$$f(x) = \begin{cases} c(x + 2y), & 0 < x < 2 \text{ and } 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Calculate c.

(b) Find the marginal distribution of X.

(c) Find the joint cdf of X and Y.

Solution

Part (a)

For $f(x, y)$ to be a valid pdf, we need:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$$

$$\int_0^1 \int_0^2 c(x + 2y)\, dx\, dy = 1$$

First, integrate with respect to x:

$$\int_0^2 (x + 2y)\, dx = \left[\frac{x^2}{2} + 2xy\right]_0^2 = \frac{4}{2} + 4y = 2 + 4y$$

Then, integrate with respect to y:

$$\int_0^1 (2 + 4y)\, dy = \left[2y + 2y^2\right]_0^1 = 2 + 2 = 4$$

Therefore:

$$c \cdot 4 = 1 \implies c = \frac{1}{4}$$

Part (b)

The marginal distribution of X is:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_0^1 \frac{1}{4}(x + 2y)\, dy$$

for $0 < x < 2$:

$$f_X(x) = \frac{1}{4} \int_0^1 (x + 2y)\, dy = \frac{1}{4} \left[xy + y^2\right]_0^1 = \frac{1}{4}(x + 1)$$

Therefore:

$$f_X(x) = \begin{cases} \frac{1}{4}(x + 1), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Part (c)

The joint cdf is:

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y) = \int_0^y \int_0^x \frac{1}{4}(s + 2t)\, ds\, dt$$

for $0 < x < 2$ and $0 < y < 1$:

First, integrate with respect to s:

$$\int_0^x (s + 2t)\, ds = \left[\frac{s^2}{2} + 2st\right]_0^x = \frac{x^2}{2} + 2xt$$

Then, integrate with respect to t:

$$\int_0^y \left(\frac{x^2}{2} + 2xt\right) dt = \left[\frac{x^2 t}{2} + xt^2\right]_0^y = \frac{x^2 y}{2} + xy^2$$

Therefore:

$$F_{X,Y}(x,y) = \begin{cases} 0, & x \leq 0 \text{ or } y \leq 0 \\ \frac{1}{4}\left(\frac{x^2 y}{2} + xy^2\right), & 0 < x < 2,\, 0 < y < 1 \\ \frac{1}{4}\left(\frac{x^2}{2} + x\right), & 0 < x < 2,\, y \geq 1 \\ \frac{1}{4}\left(2y + 2y^2\right), & x \geq 2,\, 0 < y < 1 \\ 1, & x \geq 2,\, y \geq 1 \end{cases}$$

Problem 3

Let the joint pdf of X, Y be given by

$$f(x,y) = \begin{cases} \frac{2}{81}x^2 y, & 0 \leq x \leq c,\, 0 \leq y \leq c \\ 0, & \text{otherwise.} \end{cases}$$

(a) What value of c makes $f_{XY}(x,y)$ a valid joint pdf?

(b) Find $P(X > 3Y)$.

Solution

Part (a)

For a valid pdf:

$$\int_0^c \int_0^c \frac{2}{81} x^2 y \, dx \, dy = 1$$

First, integrate with respect to x:

$$\int_0^c x^2 \, dx = \left[\frac{x^3}{3}\right]_0^c = \frac{c^3}{3}$$

Then, integrate with respect to y:

$$\int_0^c y \, dy = \left[\frac{y^2}{2}\right]_0^c = \frac{c^2}{2}$$

Therefore:

$$\frac{2}{81} \cdot \frac{c^3}{3} \cdot \frac{c^2}{2} = (\frac{c}{3})^5 = 1 \implies c = 3$$

Part (b)

Now with $c = 3$:

$$P(X > 3Y) = \int\int_{X>3Y} \frac{2}{81} x^2 y \, dx \, dy$$

The region where $X > 3Y$ with $0 \leq x \leq 3$ and $0 \leq y \leq 3$ is bounded by: - $x > 3y$ - $0 \leq y \leq 1$ (since when $y > 1$, $3y > 3$ which exceeds the domain)

$$P(X > 3Y) = \int_0^1 \int_{3y}^3 \frac{2}{81} x^2 y \, dx \, dy$$

First, integrate with respect to x:

$$\int_{3y}^3 x^2 \, dx = \left[\frac{x^3}{3} \right]_{3y}^3 = \frac{27}{3} - \frac{(3y)^3}{3} = 9 - 9y^3$$

Then, integrate with respect to y:

$$\frac{2}{81} \int_0^1 y(9 - 9y^3) \, dy = \frac{2}{81} \int_0^1 (9y - 9y^4) \, dy$$

$$= \frac{2}{81} \left[\frac{9y^2}{2} - \frac{9y^5}{5} \right]_0^1 = \frac{2}{81} \left(\frac{9}{2} - \frac{9}{5} \right) = \frac{1}{15}$$

Therefore, $P(X > 3Y) = \frac{1}{15}$.

Problem 4

Let the joint pdf of X, Y be given by

$$f(x, y) = \begin{cases} 6x_1x_2, & 0 \leq x_1 \leq x_2 \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Let $Y_1 = 2X_1^2X_2^3 + 3X_2$. Find $E(Y_1)$.

(b) Let $Y_2 = X_1/X_2^2$. Find $E(Y_2)$.

Solution

Part (a)

$$E(Y_1) = E(2X_1^2X_2^3 + 3X_2) = 2E(X_1^2X_2^3) + 3E(X_2)$$

For the region $0 \leq x_1 \leq x_2 \leq 1$:

$$E(X_1^2X_2^3) = \int_0^1 \int_0^{x_2} x_1^2 x_2^3 \cdot 6x_1x_2 \, dx_1 \, dx_2 = \int_0^1 \int_0^{x_2} 6x_1^3x_2^4 \, dx_1 \, dx_2$$

Integrate with respect to x_1:

$$\int_0^{x_2} 6x_1^3x_2^4 \, dx_1 = 6x_2^4 \left[\frac{x_1^4}{4} \right]_0^{x_2} = 6x_2^4 \cdot \frac{x_2^4}{4} = \frac{3x_2^8}{2}$$

Integrate with respect to x_2:

$$\int_0^1 \frac{3x_2^8}{2} \, dx_2 = \frac{3}{2} \left[\frac{x_2^9}{9} \right]_0^1 = \frac{3}{2} \cdot \frac{1}{9} = \frac{1}{6}$$

Now for $E(X_2)$:

$$E(X_2) = \int_0^1 \int_0^{x_2} x_2 \cdot 6x_1x_2 \, dx_1 \, dx_2 = \int_0^1 \int_0^{x_2} 6x_1x_2^2 \, dx_1 \, dx_2$$

Integrate with respect to x_1:

$$\int_0^{x_2} 6x_1x_2^2 \, dx_1 = 6x_2^2 \left[\frac{x_1^2}{2}\right]_0^{x_2} = 6x_2^2 \cdot \frac{x_2^2}{2} = 3x_2^4$$

Integrate with respect to x_2:

$$\int_0^1 3x_2^4 \, dx_2 = 3\left[\frac{x_2^5}{5}\right]_0^1 = \frac{3}{5}$$

Therefore:

$$E(Y_1) = 2 \cdot \frac{1}{6} + 3 \cdot \frac{3}{5} = \frac{1}{3} + \frac{9}{5} = \frac{5+27}{15} = \frac{32}{15}$$

Part (b)

$$E(Y_2) = E\left(\frac{X_1}{X_2^2}\right) = \int_0^1 \int_0^{x_2} \frac{x_1}{x_2^2} \cdot 6x_1x_2 \, dx_1 \, dx_2$$
$$= \int_0^1 \int_0^{x_2} 6\frac{x_1^2}{x_2} \, dx_1 \, dx_2$$

Integrate with respect to x_1:

$$\int_0^{x_2} 6\frac{x_1^2}{x_2} \, dx_1 = \frac{6}{x_2}\left[\frac{x_1^3}{3}\right]_0^{x_2} = \frac{6}{x_2} \cdot \frac{x_2^3}{3} = 2x_2^2$$

Integrate with respect to x_2:

$$\int_0^1 2x_2^2 \, dx_2 = 2\left[\frac{x_2^3}{3}\right]_0^1 = \frac{2}{3}$$

Therefore, $E(Y_2) = \frac{2}{3}$.

Problem 5

Let the joint pdf of X, Y be given by

$$f(x, y) = \begin{cases} cxy, & 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le x + y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Calculate c.

(b) Obtain the conditional pdf of X given $Y = y$.

(c) Calculate $E(XY)$.

(d) Find the joint cdf $F_{X,Y}(x, y)$.

Solution

Part (a)

For a given $y \in [0,1]$, we have $x \in [0, 1-y]$.

$$\int_0^1 \int_0^{1-y} cxy \, dx \, dy = 1$$

Integrate with respect to x:

$$\int_0^{1-y} xy \, dx = y \left[\frac{x^2}{2} \right]_0^{1-y} = y \cdot \frac{(1-y)^2}{2} = \frac{y(1-y)^2}{2}$$

Integrate with respect to y:

$$c \int_0^1 \frac{y(1-y)^2}{2} \, dy = c \int_0^1 \frac{y(1-2y+y^2)}{2} \, dy$$
$$= \frac{c}{2} \int_0^1 (y - 2y^2 + y^3) \, dy = \frac{c}{2} \left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4} \right]_0^1$$
$$= \frac{c}{2} \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{c}{24}$$

Therefore: $\frac{c}{24} = 1 \implies c = 24$

Part (b)

First, find the marginal pdf of Y:

$$f_Y(y) = \int_0^{1-y} 24xy \, dx = 24y \left[\frac{x^2}{2} \right]_0^{1-y} = 12y(1-y)^2$$

for $0 \le y \le 1$.

The conditional pdf is:

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{24xy}{12y(1-y)^2} = \frac{2x}{(1-y)^2}$$

for $0 \le x \le 1-y$.

Therefore:

$$f_{X|Y}(x|y) = \begin{cases} \frac{2x}{(1-y)^2}, & 0 \le x \le 1-y \\ 0, & \text{otherwise} \end{cases}$$

Part (c)

$$E(XY) = \int_0^1 \int_0^{1-y} xy \cdot 24xy \, dx \, dy = 24 \int_0^1 \int_0^{1-y} x^2 y^2 \, dx \, dy$$

Integrate with respect to x:

$$\int_0^{1-y} x^2 \, dx = \left[\frac{x^3}{3} \right]_0^{1-y} = \frac{(1-y)^3}{3}$$

Integrate with respect to y:

$$24 \int_0^1 y^2 \cdot \frac{(1-y)^3}{3}\, dy = 8 \int_0^1 y^2 (1-y)^3\, dy$$

Expanding $(1-y)^3 = 1 - 3y + 3y^2 - y^3$:

$$8 \int_0^1 y^2 (1 - 3y + 3y^2 - y^3)\, dy = 8 \int_0^1 (y^2 - 3y^3 + 3y^4 - y^5)\, dy$$

$$= 8 \left[\frac{y^3}{3} - \frac{3y^4}{4} + \frac{3y^5}{5} - \frac{y^6}{6} \right]_0^1$$

$$= 8 \left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6} \right) = \frac{2}{15}$$

Therefore, $E(XY) = \frac{2}{15}$.

Part (d)

The joint cdf is:

$$F_{X,Y}(x, y) = P(X \le x, Y \le y) = \int_0^y \int_0^{\min(x, 1-t)} 24st\, ds\, dt$$

This requires considering different regions:

Case 1: $x + y \le 1$ (inside the triangular region)

$$F_{X,Y}(x, y) = \int_0^y \int_0^x 24st\, ds\, dt = \int_0^y 24t \left[\frac{s^2}{2} \right]_0^x dt$$

$$= \int_0^y 12tx^2\, dt = 12x^2 \left[\frac{t^2}{2} \right]_0^y = 6x^2 y^2$$

Case 2: $x + y > 1$, $x < 1$, $y < 1$

$$F_{X,Y}(x, y) = \int_0^{1-x} \int_0^x 24st\, ds\, dt + \int_{1-x}^y \int_0^{1-t} 24st\, ds\, dt$$

The first integral: $6x^2(1-x)^2$

The second integral:

$$\int_{1-x}^y 24t \cdot \frac{(1-t)^2}{2}\, dt = 12 \int_{1-x}^y t(1-t)^2\, dt$$

$$= 12 \int_{1-x}^y (t - 2t^2 + t^3)\, dt = 12 \left[\frac{t^2}{2} - \frac{2t^3}{3} + \frac{t^4}{4} \right]_{1-x}^y$$

Therefore:

$$F_{X,Y}(x, y) = \begin{cases} 0, & x \le 0 \text{ or } y \le 0 \\ 6x^2 y^2, & 0 < x, y,\ x + y \le 1 \\ 6x^2(1-x)^2 + 12 \left[\frac{t^2}{2} - \frac{2t^3}{3} + \frac{t^4}{4} \right]_{1-x}^y, & 0 < x < 1,\ 0 < y < 1,\ x + y > 1 \\ 1, & x \ge 1,\ y \ge 1 \end{cases}$$