

## P8104 Probability - Homework 2

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### Problem 1

Let  $X$  be the sum of the faces when rolling two fair six-sided dice.

(a) List all possible values of  $X$  and determine the probability mass function (pmf)  $p_X(x)$ .

$$X = \{2, 3, 4, \dots, 12\}$$
$$p_X(x) = \begin{cases} \frac{x-1}{36}, & 2 \leq x \leq 7 \\ \frac{13-x}{36}, & 8 \leq x \leq 12 \\ 0, & \text{otherwise} \end{cases}$$

(b) Compute:

$$\mathbb{P}(X \in \{7, 11\}) = \frac{6}{36} + \frac{2}{36} = \frac{2}{9}$$
$$\mathbb{P}(X \in \{2, 3, 12\}) = \frac{1}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{9}$$

(c) Determine the cumulative distribution function (cdf)  $F_X(x)$ .

$$F_X(x) = \begin{cases} 0, & x < 2, \\ \frac{1}{36}, & 2 \leq x < 3, \\ \frac{1+2}{36} = \frac{3}{36}, & 3 \leq x < 4, \\ \frac{1+2+3}{36} = \frac{6}{36}, & 4 \leq x < 5, \\ \frac{1+2+3+4}{36} = \frac{10}{36}, & 5 \leq x < 6, \\ \frac{1+2+3+4+5}{36} = \frac{15}{36}, & 6 \leq x < 7, \\ \frac{1+2+3+4+5+6}{36} = \frac{21}{36}, & 7 \leq x < 8, \\ \frac{1+2+3+4+5+6+7}{36} = \frac{26}{36}, & 8 \leq x < 9, \\ \frac{1+2+3+4+5+6+7+8}{36} = \frac{30}{36}, & 9 \leq x < 10, \\ \frac{1+2+3+4+5+6+7+8+9}{36} = \frac{33}{36}, & 10 \leq x < 11, \\ \frac{1+2+3+4+5+6+7+8+9+10}{36} = \frac{35}{36}, & 11 \leq x < 12, \\ 1, & x \geq 12. \end{cases}$$

## Problem 2

Let  $X$  be a discrete random variable with pmf

$$p_X(x) = \begin{cases} cx, & x = 1, 2, \dots, 10, \\ 0, & \text{otherwise.} \end{cases}$$

(a) What's the support of  $X$ ?

The support of  $X$  is  $\{x : p_X(x) > 0\} = \{1, 2, 3, \dots, 10\}$

(b) Find the constant  $c$ .

since  $\sum_{x \in X} p_X(x) = 1$ , we have

$$\sum_{x=1}^{10} cx = c \sum_{x=1}^{10} x = 55c = 1$$

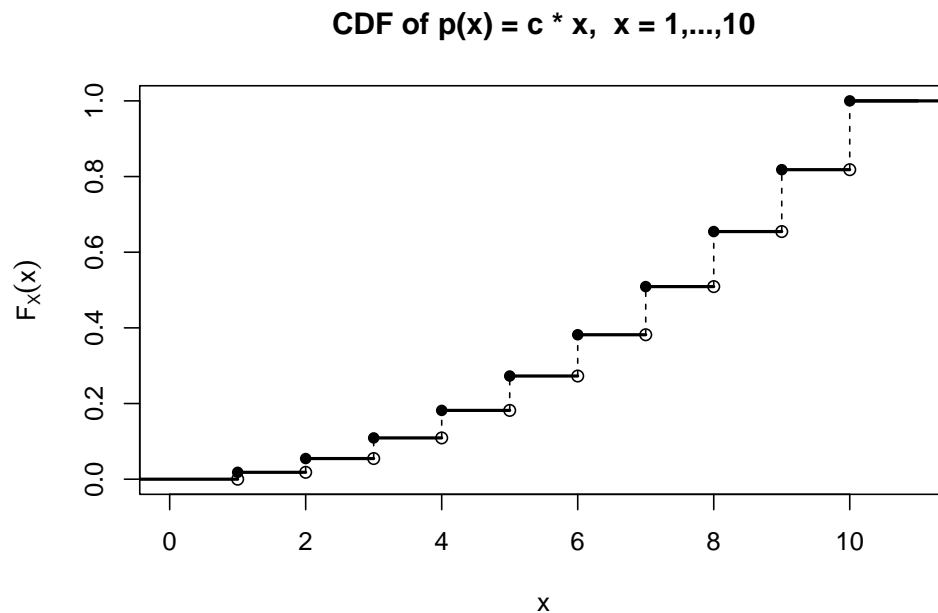
so,  $c = \frac{1}{55}$

(c) Compute  $\mathbb{P}(2 < X \leq 4)$ .

$$\mathbb{P}(2 < X \leq 4) = \mathbb{P}(3) + \mathbb{P}(4) = \frac{3}{55} + \frac{4}{55} = \frac{7}{55}$$

(d) Sketch the cdf  $F_X(x)$ . (You can also plot an R approximation.)

$$F_X(x) = \begin{cases} 0, & x < 1 \\ \sum_{i=1}^{\lfloor x \rfloor} \frac{i}{55}, & 1 \leq x \leq 10 \\ 1, & x > 10 \end{cases}$$



### Problem 3

Let  $X$  be the number of tails before the first head in independent flips of a coin with head probability  $p$ .

(a) Write the pmf of  $X$  and justify it.

Let  $f_X(x)$  be the pmf of  $X$ .

$$f_X(x) = \begin{cases} 0, & x < 0 \\ (1-p)^x p, & x \geq 0 \end{cases}$$

The probability of flipping the coin with a head is  $p$ , so the probability of flipping with a tail is  $1-p$ . So,  $f_X(x)$  is the probability that the first  $x$  flips all result in tails and followed by one flip results in a head, which is  $(1-p)^x * p$ .

(b) Compute  $\mathbb{P}(X \text{ is even})$ .

$$\begin{aligned} \mathbb{P}(X \text{ is even}) &= \sum_{x=0,2,4,\dots}^{\infty} (1-p)^x p \\ &= p \sum_{i=0,1,2,\dots}^{\infty} ((1-p)^2)^i \\ &= \frac{p}{(1 - (1-p)^2)} \\ &= \frac{p}{2p - p^2} \\ &= \frac{1}{2 - p} \end{aligned}$$

(c) What is the support of  $X$ ?

The support of  $X$  is  $\{0, 1, 2, 3, \dots\}$ .

### Problem 4

Let  $X \sim \text{Uniform}(0, 1)$ , i.e., a real number chosen at random between 0 and 1.

(a) Write the cdf  $F_X(x)$  and pdf  $f_X(x)$ .

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$
$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) Verify that  $f_X(x) = \frac{d}{dx} F_X(x)$ .

When  $x < 0$  or  $x > 1$ ,  $F_X(x)$  is constant. So  $F'_X(x) = 0$ . As  $f_X(x) = 0$  by definition, the equality holds.

For  $0 \leq x \leq 1$ ,  $\frac{d}{dx}F_X(x) = \frac{dx}{dx} = 1 = f_X(x)$ , thus the equality also holds.

Note  $F_X(x)$  is not differentiable at  $x = 0$  and  $x = 1$ , but the equation holds everywhere else.

(c) Let  $A = (0.3, 0.7]$ . Compute  $\mathbb{P}(X \in A)$  using both the cdf and the pdf.

$$\mathbb{P}(X \in A) = F_X(0.7) - F_X(0.3) = 0.4$$

$$\mathbb{P}(X \in (0.3, 0.7]) = \int_{0.3}^{0.7} dx = 0.4.$$

(d) State whether  $X$  is discrete, continuous, or neither, and explain why.

$X$  is continuous, since the probabilities are given by integrals over intervals; for any single point  $a$ ,  $\mathbb{P}(X = a) = 0$ .

## Problem 5

Let  $X \sim \text{Binomial}(n = 4, p)$ , i.e., the number of heads in 4 independent coin flips where the probability of heads is  $p$ . Define a new random variable  $Y = |X - 2|$ .

(a) List all possible values of  $X$  and write out its pmf  $p_X(x)$ .

$$X = \{0, 1, 2, 3, 4\}$$

$$p_X(x) = \binom{4}{x} p^x (1-p)^{4-x}$$

(b) List all possible values of  $Y$  and determine its support.

$$Y = \{0, 1, 2\}$$

For  $0 < p < 1$ , all are in the support. If  $p = 0$  or  $p = 1$ , the support reduces to  $\{2\}$ .

(c) Express each  $y$  value in terms of the corresponding  $x$  values from the binomial distribution.

$$y = \begin{cases} 0, & x = 2 \\ 1, & x = 1 \text{ or } 3 \\ 2, & x = 0 \text{ or } 4 \end{cases}$$

(d) Is the transformation one-to-one? Derive the pmf of  $Y$ .

No, as shown above.

Since  $y = |x - 2|$ , we have  $y = x - 2$  or  $y = -(x - 2)$ . Thus  $x = 2 + y$  or  $x = 2 - y$ .

$$p_Y(y) = \begin{cases} p_X(y+2), & y = 0 \\ p_X(2+y) + p_X(2-y), & \textit{otherwise} \end{cases}$$

(e) Verify that  $\sum_y p_Y(y) = 1$ .

$$\sum_y p_Y(y) = p_Y(0) + p_Y(1) + p_Y(2) = p_X(2) + (p_X(3) + p_X(1)) + (p_X(4) + p_X(0)) = \sum_x p_X(x) = 1$$

So the equality holds.