P8104 Homework Assignment 11

Due Thursday 12/4. 11:59pm

Problem 1

Let X_1 and X_2 be two continuous i.i.d. random variables, and let m denote the median of their common distribution.

(a) What is the probability that the larger of X_1 and X_2, i.e., $\max(X_1, X_2) = X_{(2)}$, will exceed the population median?

(b) Generalize the result in (a) to samples of size n, i.e., for n continuous i.i.d. random variables, compute the probability of $\max(X_1, ..., X_n) = X_{(n)} > m$.

Hint: The following facts about order statistics may be helpful

$$P(X_{(n)} < m) = P(X_1 < m, \ldots, X_n < m); P(X_{(1)} > m) = P(X_1 > m, \ldots, X_n > m).$$

Problem 2

Suppose $X_1, \ldots, X_n$ are independent random variables with $X_i \sim N(\mu_i, \sigma_i^2)$ for all $i \in \{1, \ldots, n\}$, and let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be fixed constants. Use moment-generating functions to prove that

$$Z = \sum_{i=1}^{n}(a_i X_i + b_i) \ \sim \ N\left(\sum_{i=1}^{n}(a_i \mu_i + b_i),\ \sum_{i=1}^{n} a_i^2 \sigma_i^2\right).$$

Problem 3

Consider a random sample $X_1, \cdots, X_n$ of size n from a Poisson distribution with mean λ. Let $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ be the sample mean and define the statistic $Y_n = \exp(-\overline{X})$.

(a) Find a constant c (if it exists) such that $Y_n \xrightarrow{P} c$.

(b) Find the asymptotic normal distribution for (suitably scaled and centered) Y_n, i.e., find sequences of constants a_n, b_n such that

$$a_n(Y_n - b_n) \xrightarrow{d} N(0, 1).$$

Hint: You may use the Delta Method: Let $\{X_n\}$ be a sequence of random variables and the function g is differentiable at θ and $g'(\theta) \neq 0$. If $\sqrt{n}\,(X_n - \theta) \xrightarrow{d} N(0, \sigma^2)$, Then $\sqrt{n}\,\big(g(X_n) - g(\theta)\big) \xrightarrow{d} N\Big(0,\ \sigma^2\big(g'(\theta)\big)^2\Big)$.

Problem 4

A manufacturer of booklets packages them in boxes of 100. It is known that on average, the booklets weigh 1 ounce, with a standard deviation of 0.05 ounce. The manufacturer is interested in calculating the following probability

$$P(100 \text{ booklets weigh more than } 100.4 \text{ ounces}),$$

a number that would help detect whether too many booklets are being put in a box. Explain how you would calculate the (approximate) value of this probability. Mention any relevant theorems or assumptions needed.