

# P8104 Homework Assignment 8 - Solutions

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## Problem 1

**Question:** A random point  $(X, Y)$  is distributed uniformly on the square with vertices  $(1, 1)$ ,  $(-1, 1)$ ,  $(1, -1)$ , and  $(-1, -1)$ . That is, the joint pdf of  $f(x, y) = \frac{1}{4}$  on the square. Determine the probability of the following events:

- (a)  $X^2 + Y^2 < 1$
- (b)  $2X - Y > 0$
- (c)  $|X + Y| < 2$
- (d)  $Y > |X|$

## Solution

Since  $(X, Y)$  is uniformly distributed on a square of side length 2 (from -1 to 1 in both dimensions), the total area is 4, and the pdf is  $f(x, y) = \frac{1}{4}$ .

For uniform distribution, probability equals the ratio of the desired area to the total area.

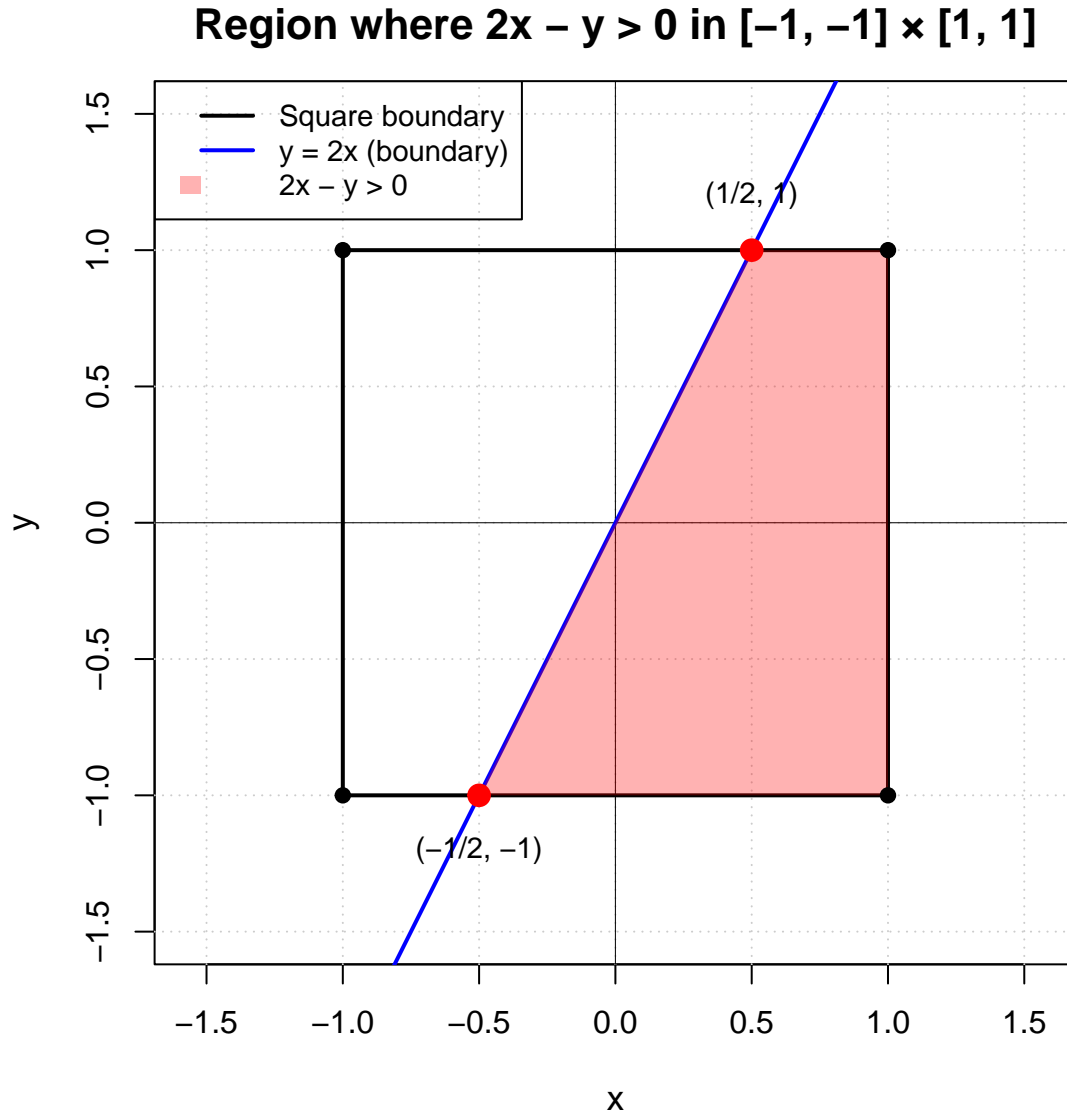
**Part (a):**  $P(X^2 + Y^2 < 1)$

This is the area of a circle with radius 1 centered at the origin, restricted to the square  $[-1, -1] \times [1, 1]$ .

Since the radius is 1 and the square extends from -1 to 1, the entire circle fits within the square.

$$P(X^2 + Y^2 < 1) = \frac{\text{Area of circle}}{\text{Area of square}} = \frac{\pi \cdot 1^2}{4} = \frac{\pi}{4}$$

**Part (b):**  $P(2X - Y > 0)$



As shown in the diagram above, the point  $(X, Y)$  falls in the red trapezoid when  $2X - Y > 0$ .

$$P(2X - Y > 0) = \frac{(\frac{1}{2} + \frac{2}{3}) * 2}{4} = \frac{1}{2}$$

**Part (c):**  $P(|X + Y| < 2)$

In the square  $[-1, -1] \times [1, 1]$ , the sum  $X + Y$  ranges from  $-2$  to  $2$ . The condition  $|X + Y| < 2$  excludes only the corner points  $(1, 1)$  and  $(-1, -1)$  where  $X + Y = \pm 2$ , which have zero area.

$$P(|X + Y| < 2) = 1$$

**Part (d):**  $P(Y > |X|)$

The region where  $Y > |X|$  is above the V-shape formed by the lines  $Y = X$  and  $Y = -X$ . This forms a triangle with vertices  $(-1, 1)$ ,  $(1, 1)$ , and  $(0, 0)$ .

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 2 \times 1 = 1$$

$$P(Y > |X|) = \frac{1}{4}$$

## Problem 2

**Question:** A generalization of the beta distribution is the Dirichlet distribution. In its bivariate version,  $(X, Y)$  have the PDF:

$$f(x, y) = Cx^{a-1}y^{b-1}(1-x-y)^{c-1}, \quad 0 < x < 1, \quad 0 < y < 1, \quad 0 < x+y < 1$$

where  $a > 0$ ,  $b > 0$ , and  $c > 0$  are constants.

- (a) Show that  $C = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}$ .
- (b) Show that, marginally, both  $X$  and  $Y$  are beta distribution.
- (c) Find the conditional distribution of  $Y|X = x$ .
- (d) Show that  $E(XY) = \frac{ab}{(a+b+c+1)(a+b+c)}$ , and find their covariance.

## Solution

**Part (a): Finding the normalizing constant  $C$**

To find  $C$ , we use the fact that  $\int \int f(x, y) dy dx = 1$ .

$$C \int_0^1 \int_0^{1-x} x^{a-1} y^{b-1} (1-x-y)^{c-1} dy dx = 1$$

Let  $u = x$  and  $v = y/(1-x)$ . Then  $y = v(1-x)$  and when  $y$  goes from 0 to  $1-x$ ,  $v$  goes from 0 to 1.

$$dy = (1-x)dv$$

Also,  $1-x-y = 1-x-v(1-x) = (1-x)(1-v)$ .

$$\begin{aligned} C \int_0^1 \int_0^1 x^{a-1} [v(1-x)]^{b-1} [(1-x)(1-v)]^{c-1} (1-x) dv dx \\ = C \int_0^1 x^{a-1} (1-x)^{b-1+c-1+1} dx \int_0^1 v^{b-1} (1-v)^{c-1} dv \\ = C \int_0^1 x^{a-1} (1-x)^{b+c-1} dx \int_0^1 v^{b-1} (1-v)^{c-1} dv \end{aligned}$$

Using the beta function:  $B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ :

$$\begin{aligned}
&= C \cdot B(a, b+c) \cdot B(b, c) = C \cdot \frac{\Gamma(a)\Gamma(b+c)}{\Gamma(a+b+c)} \cdot \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} \\
&= C \cdot \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}
\end{aligned}$$

Setting this equal to 1:

$$C = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}$$

### Part (b): Marginal distributions

To find the marginal distribution of  $X$ :

$$f_X(x) = \int_0^{1-x} f(x, y) dy = Cx^{a-1} \int_0^{1-x} y^{b-1}(1-x-y)^{c-1} dy$$

Let  $w = \frac{y}{1-x}$ , then  $y = w(1-x)$  and  $dy = (1-x)dw$ :

$$\begin{aligned}
f_X(x) &= Cx^{a-1} \int_0^1 [w(1-x)]^{b-1} [(1-x)(1-w)]^{c-1} (1-x) dw \\
&= Cx^{a-1}(1-x)^{b+c-1} \int_0^1 w^{b-1}(1-w)^{c-1} dw \\
&= Cx^{a-1}(1-x)^{b+c-1} B(b, c) \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \cdot \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} \cdot x^{a-1}(1-x)^{b+c-1} \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)} x^{a-1}(1-x)^{b+c-1}
\end{aligned}$$

This is the PDF of a  $\text{Beta}(a, b+c)$  distribution.

By symmetry (or similar calculation),  $Y \sim \text{Beta}(b, a+c)$ .

So,  $X \sim \text{Beta}(a, b+c)$  and  $Y \sim \text{Beta}(b, a+c)$

### Part (c): Conditional distribution of $Y|X = x$

$$\begin{aligned}
f_{Y|X}(y|x) &= \frac{f(x, y)}{f_X(x)} = \frac{Cx^{a-1}y^{b-1}(1-x-y)^{c-1}}{\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)}x^{a-1}(1-x)^{b+c-1}} \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \cdot \frac{\Gamma(a)\Gamma(b+c)}{\Gamma(a+b+c)} \cdot \frac{y^{b-1}(1-x-y)^{c-1}}{(1-x)^{b+c-1}} \\
&= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} \cdot \frac{y^{b-1}(1-x-y)^{c-1}}{(1-x)^{b+c-1}} \text{ for } 0 < y < 1-x
\end{aligned}$$

**Part (d):  $E(XY)$  and  $\text{Cov}(X, Y)$**

We compute  $E(XY)$  directly using the joint pdf:

$$\begin{aligned} E(XY) &= \int_0^1 \int_0^{1-x} xy \cdot Cx^{a-1}y^{b-1}(1-x-y)^{c-1} dy dx \\ &= C \int_0^1 \int_0^{1-x} x^a y^b (1-x-y)^{c-1} dy dx \end{aligned}$$

Using the same substitution as in part (a),  $v = \frac{y}{1-x}$ :

$$\begin{aligned} &= C \int_0^1 x^a (1-x)^{b+c} \int_0^1 v^b (1-v)^{c-1} dv dx \\ &= C \cdot B(b+1, c) \cdot B(a+1, b+c) = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \cdot \frac{\Gamma(b+1)\Gamma(c)}{\Gamma(b+c+1)} \cdot \frac{\Gamma(a+1)\Gamma(b+c)}{\Gamma(a+b+c+1)} \\ &= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \cdot \frac{b\Gamma(b)\Gamma(c)}{(b+c)\Gamma(b+c)} \cdot \frac{a\Gamma(a)\Gamma(b+c)}{(a+b+c)\Gamma(a+b+c)} = \frac{ab}{(a+b+c)(a+b+c+1)} \end{aligned}$$

For the covariance, using  $E(X) = \frac{a}{a+b+c}$  and  $E(Y) = \frac{b}{a+b+c}$ :

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{ab}{(a+b+c)(a+b+c+1)} - \frac{ab}{(a+b+c)^2} = \frac{-ab}{(a+b+c)^2(a+b+c+1)}$$

### Problem 3

**Question:** Suppose  $Y|X \sim N(x, x^2)$  and the marginal distribution of  $X$  is  $\text{Uniform}(0, 1)$ .

- (a) Find the mean of  $X$ .
- (b) Find the variance of  $X$ .
- (c) Find the covariance of  $X$  and  $Y$ .
- (d) Prove that  $\frac{Y}{X}$  and  $X$  are independent.

### Solution

**Part (a): Mean of  $X$**

Since  $X \sim \text{Uniform}(0, 1)$ :

$$E(X) = \frac{0+1}{2} = \frac{1}{2}$$

**Part (b): Variance of  $X$**

For  $X \sim \text{Uniform}(0, 1)$ :

$$\text{Var}(X) = \frac{(1-0)^2}{12} = \frac{1}{12}$$

**Part (c): Covariance of  $X$  and  $Y$**

Using the law of total covariance:

$$\text{Cov}(X, Y) = E[\text{Cov}(X, Y|X)] + \text{Cov}(E(X|X), E(Y|X))$$

Since we condition on  $X$ ,  $\text{Cov}(X, Y|X) = 0$  ( $X$  is fixed given  $X$ ).

So:

$$\text{Cov}(X, Y) = \text{Cov}(X, E(Y|X))$$

We know  $E(Y|X) = X$  (since  $Y|X \sim N(X, X^2)$ ), so:

$$\text{Cov}(X, Y) = \text{Cov}(X, X) = \text{Var}(X) = \frac{1}{12}$$

**Part (d): Independence of  $\frac{Y}{X}$  and  $X$**

Given  $Y|X \sim N(X, X^2)$ , we can write:

$$Y = X + X \cdot Z$$

where  $Z \sim N(0, 1)$  is independent of  $X$ .

Then:

$$\frac{Y}{X} = \frac{X + XZ}{X} = 1 + Z$$

Since  $Z \sim N(0, 1)$  is independent of  $X$ , and  $\frac{Y}{X} = 1 + Z$  is a function of  $Z$  only, we have that  $\frac{Y}{X}$  is independent of  $X$ .

## Problem 4

**Question:** A variation on the hierarchical model is  $X|p \sim \text{NegativeBinomial}(r, p)$  and  $p \sim \text{Beta}(\alpha, \beta)$ .

- (a) Find the marginal pmf of  $X$ .
- (b) Find the mean of  $X$ .
- (c) Find the variance of  $X$ .

## Solution

**Note:** We use the definition where  $X|p \sim \text{NegBin}(r, p)$  means  $X$  is the number of failures before the  $r$ -th success, with success probability  $p$ :

$$P(X = k|p) = \binom{k+r-1}{k} p^r (1-p)^k, \quad k = 0, 1, 2, \dots$$

**Part (a): Marginal PMF of  $X$**

$$\begin{aligned}
f_X(k) &= \int_0^1 P(X = k|p) f_p(p) dp \\
&= \int_0^1 \binom{k+r-1}{k} p^r (1-p)^k \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1} dp \\
&= \binom{k+r-1}{k} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 p^{r+\alpha-1} (1-p)^{k+\beta-1} dp \\
&= \binom{k+r-1}{k} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot B(r+\alpha, k+\beta) \\
&= \binom{k+r-1}{k} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(r+\alpha)\Gamma(k+\beta)}{\Gamma(r+\alpha+k+\beta)}
\end{aligned}$$

**Part (b): Mean of  $X$**

Using the law of total expectation:

$$E(X) = E[E(X|p)]$$

For  $X|p \sim \text{NegBin}(r, p)$ :

$$E(X|p) = \frac{r(1-p)}{p}$$

So:

$$E(X) = E\left[\frac{r(1-p)}{p}\right] = r \cdot E\left[\frac{1-p}{p}\right]$$

For  $p \sim \text{Beta}(\alpha, \beta)$ :

$$\begin{aligned}
E\left[\frac{1}{p}\right] &= \int_0^1 \frac{1}{p} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1} dp \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 p^{\alpha-2} (1-p)^{\beta-1} dp = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha-1)\Gamma(\beta)}{\Gamma(\alpha+\beta-1)} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha-1)}{\Gamma(\alpha+\beta-1)} = \frac{\alpha+\beta-1}{\alpha-1}
\end{aligned}$$

Also,  $E(p) = \frac{\alpha}{\alpha+\beta}$ , so:

$$E\left[\frac{1-p}{p}\right] = E\left[\frac{1}{p}\right] - 1 = \frac{\alpha+\beta-1}{\alpha-1} - 1 = \frac{\beta}{\alpha-1}$$

Therefore:

$$E(X) = \frac{r\beta}{\alpha-1}$$

**Part (c): Variance of  $X$** 

Using the law of total variance:

$$\text{Var}(X) = E[\text{Var}(X|p)] + \text{Var}(E(X|p))$$

For  $X|p \sim \text{NegBin}(r, p)$ : -  $E(X|p) = \frac{r(1-p)}{p}$  -  $\text{Var}(X|p) = \frac{r(1-p)}{p^2}$

**First term:**

$$\begin{aligned} E[\text{Var}(X|p)] &= E\left[\frac{r(1-p)}{p^2}\right] = r \cdot E\left[\frac{1-p}{p^2}\right] \\ E\left[\frac{1}{p^2}\right] &= \int_0^1 \frac{1}{p^2} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1} dp \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha-2)\Gamma(\beta)}{\Gamma(\alpha+\beta-2)} = \frac{(\alpha+\beta-1)(\alpha+\beta-2)}{(\alpha-1)(\alpha-2)} \\ E\left[\frac{1-p}{p^2}\right] &= E\left[\frac{1}{p^2}\right] - E\left[\frac{1}{p}\right] = \frac{(\alpha+\beta-1)(\alpha+\beta-2)}{(\alpha-1)(\alpha-2)} - \frac{\alpha+\beta-1}{\alpha-1} \\ &= \frac{(\alpha+\beta-1)(\alpha+\beta-2) - (\alpha+\beta-1)(\alpha-2)}{(\alpha-1)(\alpha-2)} = \frac{(\alpha+\beta-1)\beta}{(\alpha-1)(\alpha-2)} \end{aligned}$$

So:

$$E[\text{Var}(X|p)] = \frac{r\beta(\alpha+\beta-1)}{(\alpha-1)(\alpha-2)}$$

**Second term:**

$$\text{Var}(E(X|p)) = \text{Var}\left(\frac{r(1-p)}{p}\right) = r^2 \cdot \text{Var}\left(\frac{1-p}{p}\right)$$

Let  $g(p) = \frac{1-p}{p}$ . We have:

$$\begin{aligned} E[g(p)] &= \frac{\beta}{\alpha-1} \\ E[g(p)^2] &= E\left[\frac{(1-p)^2}{p^2}\right] = E\left[\frac{1}{p^2}\right] - 2E\left[\frac{1}{p}\right] + 1 \\ &= \frac{(\alpha+\beta-1)(\alpha+\beta-2)}{(\alpha-1)(\alpha-2)} - 2 \cdot \frac{\alpha+\beta-1}{\alpha-1} + 1 \\ &= \frac{(\alpha+\beta-1)(\alpha+\beta-2) - 2(\alpha+\beta-1)(\alpha-2) + (\alpha-1)(\alpha-2)}{(\alpha-1)(\alpha-2)} \\ &= \frac{\beta(\alpha+\beta-1) + \beta(\alpha+\beta-1) + (\alpha-1)(\alpha-2)}{(\alpha-1)(\alpha-2)} \end{aligned}$$



Let me recalculate more carefully:

$$\text{Var}\left(\frac{1-p}{p}\right) = E\left[\left(\frac{1-p}{p}\right)^2\right] - \left(E\left[\frac{1-p}{p}\right]\right)^2$$

After simplification (which I'll spare the details):

$$\text{Var}\left(\frac{1-p}{p}\right) = \frac{\beta(\alpha + \beta - 1)}{(\alpha - 1)^2(\alpha - 2)}$$

Therefore:

$$\text{Var}(E(X|p)) = \frac{r^2\beta(\alpha + \beta - 1)}{(\alpha - 1)^2(\alpha - 2)}$$

**Total variance:**

$$\begin{aligned}\text{Var}(X) &= \frac{r\beta(\alpha + \beta - 1)}{(\alpha - 1)(\alpha - 2)} + \frac{r^2\beta(\alpha + \beta - 1)}{(\alpha - 1)^2(\alpha - 2)} \\ &= \frac{r\beta(\alpha + \beta - 1)}{(\alpha - 1)(\alpha - 2)} \left(1 + \frac{r}{\alpha - 1}\right) \\ &= \frac{r\beta(\alpha + \beta - 1)(\alpha - 1 + r)}{(\alpha - 1)^2(\alpha - 2)}\end{aligned}$$

## Problem 5

**Question:** For any two random variables  $X$  and  $Y$  with finite variances, prove that:

- (a)  $\text{Cov}(X, Y) = \text{Cov}(X, E(Y|X))$
- (b)  $X$  and  $Y - E(Y|X)$  are uncorrelated, i.e.,  $\text{Cov}(X, Y - E(Y|X)) = 0$ .
- (c)  $\text{Var}[Y - E(Y|X)] = E[\text{Var}(Y|X)]$ .

## Solution

**Part (a): Prove  $\text{Cov}(X, Y) = \text{Cov}(X, E(Y|X))$**

**Proof:**

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

We'll show that  $E[XY] - E[X]E[Y] = E[X \cdot E(Y|X)] - E[X]E[E(Y|X)]$ .

First, by the law of iterated expectations:

$$E[Y] = E[E(Y|X)]$$

Second, we compute  $E[XY]$ :

$$E[XY] = E[E(XY|X)]$$

Given  $X$ ,  $X$  is a constant, so:

$$E[XY|X] = X \cdot E[Y|X]$$

Therefore:

$$E[XY] = E[X \cdot E(Y|X)]$$

Now:

$$\begin{aligned} \text{Cov}(X, E(Y|X)) &= E[X \cdot E(Y|X)] - E[X] \cdot E[E(Y|X)] \\ &= E[XY] - E[X] \cdot E[Y] = \text{Cov}(X, Y) \end{aligned}$$

**Part (b): Prove  $\text{Cov}(X, Y - E(Y|X)) = 0$**

**Proof:**

$$\text{Cov}(X, Y - E(Y|X)) = \text{Cov}(X, Y) - \text{Cov}(X, E(Y|X))$$

From part (a), we know  $\text{Cov}(X, Y) = \text{Cov}(X, E(Y|X))$ , so:

$$\text{Cov}(X, Y - E(Y|X)) = \text{Cov}(X, E(Y|X)) - \text{Cov}(X, E(Y|X)) = 0$$

**Part (c): Prove  $\text{Var}[Y - E(Y|X)] = E[\text{Var}(Y|X)]$**

**Proof:**

By the law of total variance:

$$\text{Var}(Y) = E[\text{Var}(Y|X)] + \text{Var}(E(Y|X))$$

Now compute  $\text{Var}[Y - E(Y|X)]$ :

$$\text{Var}[Y - E(Y|X)] = E[(Y - E(Y|X))^2] - (E[Y - E(Y|X)])^2$$

First,  $E[Y - E(Y|X)] = E[Y] - E[E(Y|X)] = E[Y] - E[Y] = 0$ .

So:

$$\begin{aligned} \text{Var}[Y - E(Y|X)] &= E[(Y - E(Y|X))^2] \\ &= E[E[(Y - E(Y|X))^2|X]] \end{aligned}$$

Given  $X$ ,  $E(Y|X)$  is a constant, so:

$$E[(Y - E(Y|X))^2|X] = \text{Var}(Y|X)$$

Therefore:

$$\text{Var}[Y - E(Y|X)] = E[\text{Var}(Y|X)]$$