P8104 Probability - Homework 2

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Problem 1

Let X be the sum of the faces when rolling two fair six-sided dice.

(a) List all possible values of X and determine the probability mass function (pmf) $p_X(x)$.

$$X = \{2, 3, 4, ..., 12\}$$
$$p_X(x) = \begin{cases} \dfrac{x-1}{36}, & 2 \le x \le 7 \\ \dfrac{13-x}{36}, & 8 \le x \le 12 \\ 0, & otherwise \end{cases}$$

(b) Compute:

$$\mathbb{P}(X \in \{7, 11\}) = \frac{6}{36} + \frac{2}{36} = \frac{2}{9}$$
$$\mathbb{P}(X \in \{2, 3, 12\}) = \frac{1}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{9}$$

(c) Determine the cumulative distribution function (cdf) $F_X(x)$.

$$F_X(x) = \begin{cases} 0, & x < 2, \\ \frac{1}{36}, & 2 \le x < 3, \\ \frac{1+2}{36} = \frac{3}{36}, & 3 \le x < 4, \\ \frac{1+2+3}{36} = \frac{6}{36}, & 4 \le x < 5, \\ \frac{1+2+3+4}{36} = \frac{10}{36}, & 5 \le x < 6, \\ \frac{1+2+3+4+5}{36} = \frac{15}{36}, & 6 \le x < 7, \\ \frac{1+2+3+4+5+6}{36} = \frac{21}{36}, & 7 \le x < 8, \\ \frac{1+2+3+4+5+6+5}{36} = \frac{26}{36}, & 8 \le x < 9, \\ \frac{1+2+3+4+5+6+5+4}{36} = \frac{30}{36}, & 9 \le x < 10, \\ \frac{1+2+3+4+5+6+5+4+3}{36} = \frac{33}{36}, & 10 \le x < 11, \\ \frac{1+2+3+4+5+6+5+4+3+2}{36} = \frac{35}{36}, & 11 \le x < 12, \\ 1, & x \ge 12. \end{cases}$$

Problem 2

Let X be a discrete random variable with pmf

$$p_X(x) = \begin{cases} cx, & x = 1, 2, \ldots, 10, \\ 0, & \text{otherwise.} \end{cases}$$

(a) What's the support of X?

The support of X is $\{x : p_X(x) > 0\} = \{1, 2, 3, ..., 10\}$

(b) Find the constant c.

since $\sum_{x \in X} p_X(x) = 1$, we have

$$\sum_{x=1}^{10} cx = c \sum_{x=1}^{10} x = 55c = 1$$

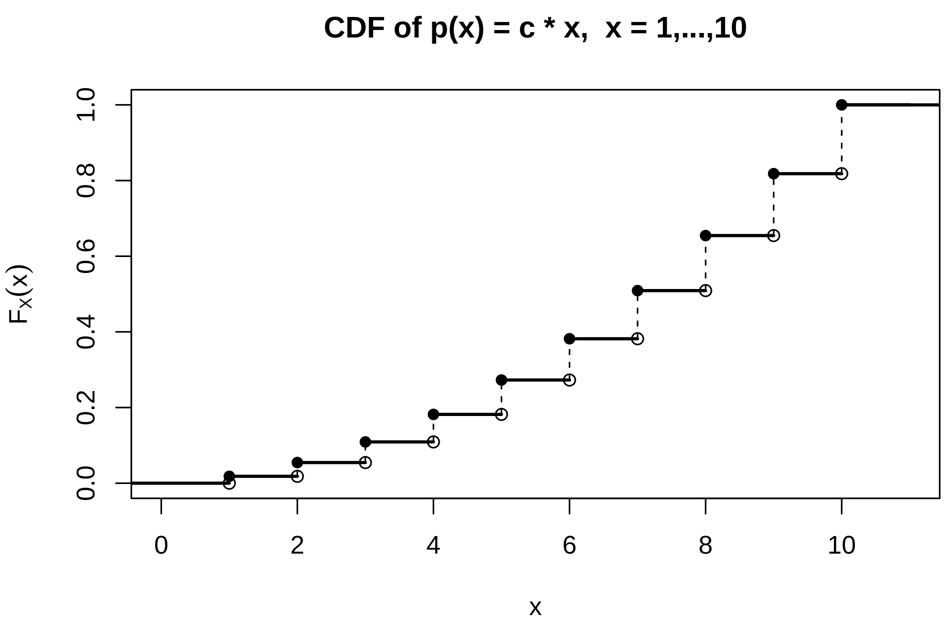
so, $c = \dfrac{1}{55}$

(c) Compute $\mathbb{P}(2 < X \leq 4)$.

$$\mathbb{P}(2 < X \leq 4) = \mathbb{P}(3) + \mathbb{P}(4) = \frac{3}{55} + \frac{4}{55} = \frac{7}{55}$$

(d) Sketch the cdf $F_X(x)$. (You can also plot an R approximation.)

$$F_X(x) = \begin{cases} 0, & x < 1 \\ \sum_{i=1}^{\lfloor x \rfloor} \frac{i}{55}, & 1 \leq x \leq 10 \\ 1, & x > 10 \end{cases}$$



Problem 3

Let X be the number of tails before the first head in independent flips of a coin with head probability p.

(a) Write the pmf of X and justify it.

Let $f_X(x)$ be the pmf of X.

$$f_X(x) = \begin{cases} 0, & x < 0 \\ (1-p)^x p, & x \geq 0 \end{cases}$$

The probability of flipping the coin with a head is p, so the probability of flipping with a tail is 1-p. So, f_X(x) is the probability that the first X flips all result in tails and followed by one flip results in a head, which is $(1-p)^x * p$.

(b) Compute $\mathbb{P}(X \text{ is even})$.

$$\begin{aligned} \mathbb{P}(\text{X is even}) &= \sum_{x=0,2,4,\ldots}^{\infty} (1-p)^x p \\ &= p \sum_{i=0,1,2,\ldots}^{\infty} ((1-p)^2)^i \\ &= \frac{p}{(1-(1-p)^2)} \\ &= \frac{p}{2p-p^2} \\ &= \frac{1}{2-p} \end{aligned}$$

(c) What is the support of X?

The support of X is $\{0, 1, 2, 3, \ldots\}$.

Problem 4

Let $X \sim \text{Uniform}(0,1)$, i.e., a real number chosen at random between 0 and 1.

(a) Write the cdf $F_X(x)$ and pdf $f_X(x)$.

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & otherwise \end{cases}$$

(b) Verify that $f_X(x) = \frac{d}{dx} F_X(x)$.

When x < 0 or x > 1, $F_X(x)$ is constant. So $F'_X(x) = 0$. As $f_X(x) = 0$ by definition, the equality holds.

For $0 \le x \le 1$, $\frac{d}{dx}F_X(x) = \frac{dx}{dx} = 1 = f_X(x)$, thus the equality also holds.

Note $F_X(x)$ is not differentiable at $x = 0$ and $x = 1$, but the equation holds everywhere else.

(c) Let $A = (0.3, 0.7]$. Compute $\mathbb{P}(X \in A)$ using both the cdf and the pdf.

$$\mathbb{P}(X \in A) = F_X(0.7) - F_X(0.3) = 0.4$$
$$\mathbb{P}(X \in (0.3, 0.7]) = \int_{0.3}^{0.7} dx = 0.4.$$

(d) State whether X is discrete, continuous, or neither, and explain why.

X is continuous, since the probabilities are given by integrals over intervals; for any single point a, $\mathbb{P}(X = a) = 0$.

Problem 5

Let $X \sim \text{Binomial}(n = 4, p)$, i.e., the number of heads in 4 independent coin flips where the probability of heads is p. Define a new random variable $Y = |X - 2|$.

(a) List all possible values of X and write out its pmf $p_X(x)$.

$$X = \{0, 1, 2, 3, 4\}$$

$$p_X(x) = \binom{4}{x} p^x (1-p)^{4-x}$$

(b) List all possible values of Y and determine its support.

$$Y = \{0, 1, 2\}$$

For $0 < p < 1$, all are in the support. If $p = 0$ or $p = 1$, the support reduces to $\{2\}$.

(c) Express each y value in terms of the corresponding x values from the binomial distribution.

$$y = \begin{cases} 0, & x = 2 \\ 1, & x = 1 \; or \; 3 \\ 2, & x = 0 \; or \; 4 \end{cases}$$

(d) Is the transformation one-to-one? Derive the pmf of Y.

No, as shown above.

Since $y = |x - 2|$, we have $y = x - 2$ or $y = -(x - 2)$. Thus $x = 2 + y$ or $x = 2 - y$.

$$p_Y(y) = \begin{cases} p_X(y+2), & y = 0 \\ p_X(2+y) + p_X(2-y), & otherwise \end{cases}$$

(e) Verify that $\sum_y p_Y(y) = 1$.

$$\sum_y p_Y(y) = p_Y(0) + p_Y(1) + p_Y(2) = p_X(2) + (p_X(3) + p_X(1)) + (p_X(4) + p_X(0)) = \sum_x p_X(x) = 1$$

So the equality holds.