Homework 3 - P8104 Probability

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Problem 1

Let a point be chosen uniformly at random from the interior of a circular annulus with inner radius $r_0 = \dfrac{1}{2}$ and outer radius $r_1 = 1$. Let X be the distance from the origin to the selected point.

(a) What is the support of X?

$$\text{support}(X) = \{x|\frac{1}{2} < x < 1\}.$$

(b) Determine the cumulative distribution function (cdf) $F_X(x)$ of the distance X.

Let A_x be the area of the annulus with inner radius $r_0 = \dfrac{1}{2}$ and outer radius $r_x = x$.

Uniformity over area implies $P(X \leq x) = A_x/A_1$.

$$F_X(x) = \begin{cases} 0, & x < \frac{1}{2}, \\ \dfrac{\pi x^2 - \pi(\frac{1}{2})^2}{\pi - \pi(\frac{1}{2})^2} = \dfrac{4}{3}x^2 - \dfrac{1}{3}, & \frac{1}{2} \leq x \leq 1, \\ 1, & x > 1\ . \end{cases}$$

(c) Derive the probability density function (pdf) $f_X(x)$.

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{8}{3}x, \qquad \tfrac{1}{2} \leq x \leq 1.$$

(d) Compute the probability that the point lies between distances 0.6 and 0.8 from the origin.

$$F_X(0.8) - F_X(0.6) = \frac{4}{3}(0.8^2 - 0.6^2) = \frac{4 * 0.28}{3}$$

Problem 2

A median of a distribution of a random variable X is a value of x such that

$$P(X < x) \leq \tfrac{1}{2} \quad \text{and} \quad P(X \leq x) \geq \tfrac{1}{2}.$$

If there is only one such x, it is called the *median* of the distribution. For the following functions, first verify that it is a pmf or pdf, then find the median:

(a) Discrete case: $p_X(x) = \dfrac{4!}{x!(4-x)!}(0.8)^x(0.2)^{4-x},\ x = 0, 1, 2, 3, 4$

Hint: This is a Binomial distribution with $n = 4, p = 0.8$.

```
for (i in 0:4) {
  print(sprintf("p(%d) = %.3f", i, pbinom(i, 4, 0.8)))
}
```

```
## [1] "p(0) = 0.002"
## [1] "p(1) = 0.027"
## [1] "p(2) = 0.181"
## [1] "p(3) = 0.590"
## [1] "p(4) = 1.000"
```

From the result, we can see $p_X(2) < \frac{1}{2}$ and $p_X(3) > \frac{1}{2}$. So the median is 3.

(b) Continuous case: $f_X(x) = 4x^3,\ 0 < x < 1$

This is a valid cdf because $\int_0^1 4x^3\,dx = 1$.

The CDF is $F(x) = \int_0^x 4t^3\,dt = x^4$.

The median m solves $m^4 = \frac{1}{2}$, so $m = 2^{-1/4} \approx 0.84$.

Problem 3

Let $X \sim \text{Uniform}(0, 1)$, and define $Y = \sqrt{X}$.

(a) Find the cdf $F_Y(y)$.

$$F_Y(y) = P(\sqrt{X} \le y) = P(X \le y^2) = \begin{cases} 0, & y \le 0, \\ y^2, & 0 < y < 1, \\ 1, & y \ge 1\ . \end{cases}$$

(b) Find pdf $f_Y(y)$.

$$f_Y(y) = \frac{d}{dy}F_Y(y) = 2y, \qquad 0 < y < 1$$

(c) Verify that your pdf is valid.

$$\int_0^1 2y\,dy = F_Y(1) - F_Y(0) = 1$$

(d) Compute $E(Y)$.

$$E(Y) = \int_0^1 y * f_Y(y)\,dy = \int_0^1 2y^2\,dy = \frac{2}{3}$$

Problem 4

Find the pdf of Y in the following situations:

(a) $Y = \ln(X)$. The pdf for X is

$$f(x) = \frac{2}{\pi(1+x^2)}, \quad x \geq 0.$$

Use $x = e^y$ (so $y \in (-\infty, \infty)$) and $dx/dy = e^y$:

$$f_Y(y) = f_X(e^y)\,\frac{dx}{dy} = \frac{2}{\pi\left(1+e^{2y}\right)}\,e^y$$

(b) $Y = \sin(X)$. The pdf for X is

$$f(x) = \frac{c\,x}{\pi^2}, \quad x \in (0, \pi),$$

where c is a normalizing constant. Determine the value of c, and then find the pdf of Y.

Hint: The transformation $g(x) = \sin(x)$ is not one-to-one on $(0, \pi)$. For a given $y \in (0, 1)$, there are two values of x such that $\sin(x) = y$. You may need to consider both parts of the domain when deriving the distribution of Y.

Since

$$\int_0^{\pi} \frac{c\,x}{\pi^2}\,dx = \frac{c}{\pi^2} \cdot \frac{\pi^2}{2} = 1$$

We have $c = 2$.

For $y \in (0, 1)$, there are two x mapping to each y, $x_1 = \arcsin y$ and $x_2 = \pi - \arcsin y$.

$$\begin{aligned} f_Y(y) &= \left[f_X(x_1) + f_X(x_2)\right]\left|\frac{dx}{dy}\right| \\ &= \left[\frac{2}{\pi^2}\big(\arcsin y + \pi - \arcsin y\big)\right]\frac{1}{\sqrt{1-y^2}} \\ &= \frac{2}{\pi}\,\frac{1}{\sqrt{1-y^2}}, \quad 0 < y < 1, \end{aligned}$$

and $f_Y(y) = 0$ otherwise.

Problem 5

Let X be a continuous random variable with pdf:

$$f_X(x) = \begin{cases} \frac{3}{2}x^2, & 0 < x < 1, \\ \frac{3}{2}(1-x)^2, & 1 \leq x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Verify that it is a valid pdf.

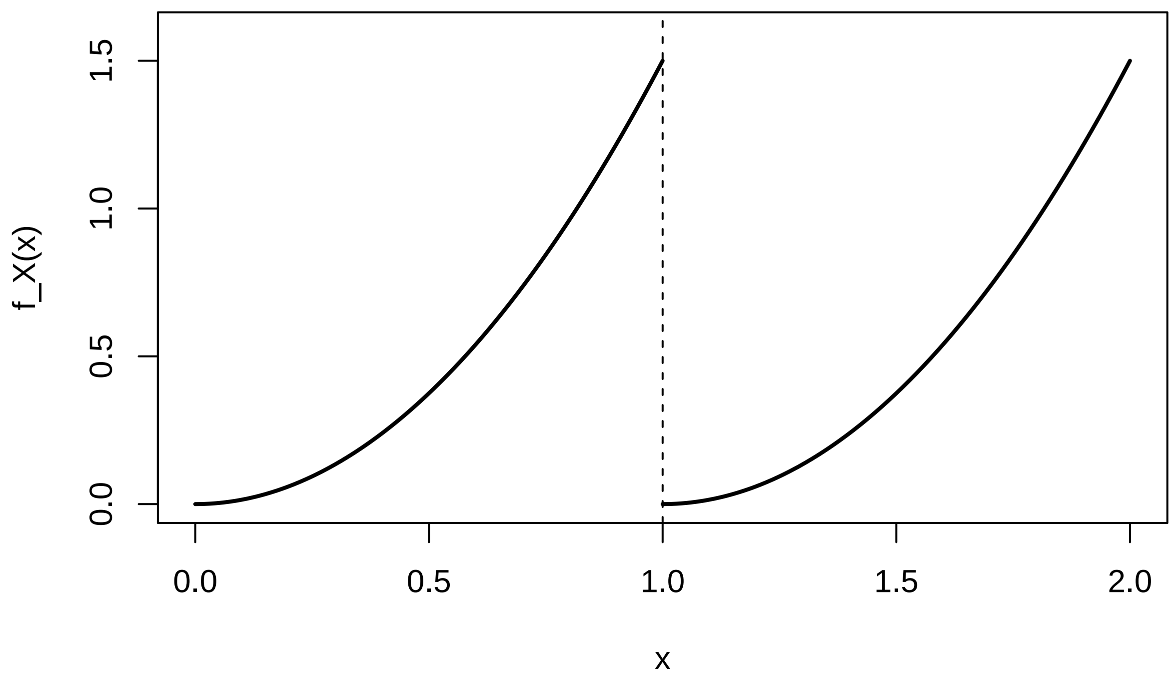
Since $\int_0^1 \frac{3}{2}x^2\,dx = \frac{1}{2}$ and $\int_1^2 \frac{3}{2}(1-x)^2\,dx = \frac{1}{2}$, verified the total is 1 and \$f_X(x) is a valid pdf.

(b) Sketch the pdf.

Please see the plot below.

```r
xs1 <- seq(0, 1, length.out = 201)
xs2 <- seq(1, 2, length.out = 201)
fx1 <- 1.5 * xs1^2
fx2 <- 1.5 * (1 - xs2)^2

plot(xs1, fx1, type = "l", lwd = 2, xlab = "x", ylab = "f_X(x)", xlim = c(0, 2), ylim = c(0, 1
lines(xs2, fx2, lwd = 2)
abline(v = 1, lty = 2)
```



(c) Compute $E(X)$.

$$E(X) = \int_0^1 x \cdot \tfrac{3}{2}x^2 \, dx + \int_1^2 x \cdot \tfrac{3}{2}(1-x)^2 \, dx = \tfrac{3}{2}\left(\tfrac{1}{4}\right) + \tfrac{7}{8} = \tfrac{5}{4}.$$

(d) Compute $E(X^2)$, and then compute $\mathrm{Var}(X)$.

$$E(X^2) = \int_0^1 x^2 \cdot \tfrac{3}{2}x^2 \, dx + \int_1^2 x^2 \cdot \tfrac{3}{2}(1-x)^2 \, dx = \frac{37}{20}.$$

$$\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{37}{20} - \left(\frac{5}{4}\right)^2 = \frac{23}{80}.$$