P8104 Homework Assignment 1

Due Wed 9/17. 11:59pm

Problem 1

Let the sample space be defined as $C = \{1, 2, 3, 4, 5, 6\}$. Define the events: $A = \{2, 4, 6\}, \quad B = \{1, 2, 3\}$.

(a) Compute the following set operations: $A \cup B$, $A \cap B$, A^c, and $(A \cup B)^c$.

(b) Assume each outcome in C is equally likely. Compute $P(A)$, $P(B)$, $P(A \cap B)$, and $P(A \cup B)$. Then, verify $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

(c) Define a set function Q that maps a subset $A \subseteq C$ to the number of even elements in A. That is, $Q(A) =$ number of even numbers in A. Compute $Q(A)$, $Q(B)$, and $Q(A \cup B)$.

Problem 2

(a) An experiment consists of tossing a fair coin three times. Define the events A: the first toss is heads; and B: at least two heads occur in the three tosses.

- List the full sample space of the experiment
- Identify the outcomes that make up events A and B
- Compute $P(B)$, $P(A \cap B)$, and $P(B \mid A)$.

(b) A hand of 5 cards is drawn at random (without replacement) from a standard 52-card deck. Compute the probability that it contains all spades, given that it contains at least 4 spades.

Problem 3

Suppose two events A and B satisfy $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.2$.

(a) Are A and B independent? Justify using at least two equivalent definitions.

(b) Are A and B mutually exclusive? Why or why not?

(c) Suppose C is independent of A, and $P(C) = 0.6$. What is $P(A \cap C)$?

Problem 4

A rare disease affects 1 in 1,000 people. A diagnostic test for the disease has 98% sensitivity (true positive rate) and 97% specificity (true negative rate). That is,

- P(Test is positive | disease) = 0.98
- P(Test is negative | no disease) = 0.97

Let D be the event that a person has the disease, and T^+ be the event that the test result is positive.

(a) Compute the probability that a randomly selected individual tests positive.

(b) Compute the probability that a person actually has the disease given a positive test result, i.e., $P(D \mid T^+)$.

(c) Interpret the result from (b). What does it suggest about the usefulness of the test?

Problem 5

Let E_1, E_2, and E_3 be three events with $P(E_1) = 0.3, \quad P(E_2) = 0.4, \quad P(E_3) = 0.5$.

(a) Use Boole's inequality to give an upper bound for $P(E_1 \cup E_2 \cup E_3)$.

(b) Suppose that $P(E_1 \cap E_2) = 0.12$, $P(E_1 \cap E_3) = 0.10$, and $P(E_2 \cap E_3) = 0.08$. Use the following second-order Bonferroni's inequality

$$P(\bigcup_{i=1}^{n} A_i) \quad \geq \quad \sum_{i=1}^{n} P(A_i) \quad - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j)$$

to give a lower bound for the same union $P(E_1 \cup E_2 \cup E_3)$.

(c) Briefly interpret your bounds from (a)–(b).
What extra information would tighten them further?