P8104 Homework Assignment 10

Due Wed 11/26. 11:59pm

Problem 1

Let X and Y have a bivariate normal distribution with parameters $\mu_1 = 5, \mu_2 = 10, \sigma_1^2 = 1, \sigma_2^2 = 25$, and $\rho > 0$. If $P(4 < Y < 16|X = 5) = 0.954$, determine ρ.

Problem 2

Let $X \sim N(0, 1)$, $Y = X^2$

(a) What is the distribution of Y?

(b) Calculate the mean and variance of Y.
Hint: (Stein's Lemma) When $X \sim N(\mu, \sigma^2)$, $\mathbb{E}[g(X)(X - \mu)] = \sigma^2 \mathbb{E}[g'(X)]$.

(c) If $W_i \sim \chi^2_{(1)}$, with all W_i independent. Define $U = \sum_{i=1}^{r} W_i$. What is the distribution of U?

(d) Calculate the mean and variance of U.

Problem 3

Let $U \sim \chi^2_p$ and $V \sim \chi^2_q$, where U and V are independent of each other. Define

$$X = \frac{U/p}{V/q}.$$

(a) Show that $X \sim F_{p,q}$.

(b) Derive the mean and variance of X.

(c) Show that $1/X \sim F_{q,p}$.

(d) Show that the *median* of an $F_{p,p}$ random variable equals 1 for any p.

(e) Show that

$$\frac{(p/q)X}{1 + (p/q)X} \sim \text{Beta}\left(\frac{p}{2}, \frac{q}{2}\right).$$

Problem 4

Let $X_1, ..., X_n$ be a random sample from a population with pdf $f_X(x) = \begin{cases} \frac{1}{\theta}, & 0 < x < \theta \\ 0, & \text{otherwise} \end{cases}$

Let $X_{(1)}, ..., X_{(n)}$ be the order statistics. Show that $\frac{X_{(1)}}{X_{(n)}}$ and $X_{(n)}$ are independent.

Problem 5

Let $X_1, ..., X_n$ be i.i.d *lognormal*(μ, σ^2) random variables. Let $G_n = \prod_{i=1}^n X_i^{1/n}$ be the sample Geometric mean.

(a) What is the distribution of G_n?

(b) Show that $G_n \xrightarrow{P} e^\mu$.