P8104 Homework Assignment 8

Due Wed 11/12. 11:59pm

Problem 1

A random point (X, Y) is distributed uniformly on the square with vertices $(1,1)$, $(-1,1)$, $(1,-1)$, and $(-1,-1)$. That is, the joint pdf of $f(x,y) = \frac{1}{4}$ on the square. Determine the probability of the following events:

(a) $X^2 + Y^2 < 1$;

(b) $2X - Y > 0$;

(c) $|X + Y| < 2$;

(d) $Y > |X|$.

Problem 2

A generalization of the beta distribution is the *Dirichlet distribution*. In its bivariate version, (X, Y) have the PDF:

$$f(x,y) = Cx^{a-1}y^{b-1}(1-x-y)^{c-1}, \quad 0 < x < 1, \quad 0 < y < 1, \quad 0 < x + y < 1$$

where $a > 0$, $b > 0$, and $c > 0$ are constants.

(a) Show that $C = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}$.

(b) Show that, marginally, both X and Y are beta distribution.

(c) Find the conditional distribution of $Y|X = x$.

(d) Show that $E(XY) = \frac{ab}{(a+b+c+1)(a+b+c)}$, and find their covariance.

Problem 3

Suppose $Y|X \sim N(x, x^2)$ and the marginal distribution of X is $Uniform(0, 1)$.

(a) Find the mean of X.

(b) Find the variance of X.

(c) Find the covariance of X and Y.

(d) Prove that $\frac{Y}{X}$ and X are independent.

Problem 4

A variation on the hierarchical model is $X|p \sim NegativeBinomial(r, p)$ and $p \sim Beta(\alpha, \beta)$.

(a) Find the marginal pmf of X.

(b) Find the mean of X.

(c) Find the variance of X.

Problem 5

For any two random variables X and Y with finite variances, prove that

(a) $Cov(X, Y) = Cov(X, E(Y|X))$

(b) X and $Y - E(Y|X)$ are uncorrelated, i.e., $Cov(X, Y - E(Y|X)) = 0$.

(c) $Var[Y - E(Y|X)] = E[Var(Y|X)]$.