Homework 5 - P8104 Probability

Yongyan Liu (yl6107)

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Problem 1

> Let the number of chocolate chips in a certain type of cookie have a Poisson distribution. We want the probability that a randomly chosen cookie has at least two chocolate chips to be greater than 0.99. Find the smallest value of the mean of the distribution that ensures this probability.

Let the number of chips be X ~ $Poisson(\lambda)$. We want P(X >= 2) > 0.99.

Since P(X >= 2) = 1 - P(X = 0) - P(X = 1), this becomes $1 - e^{-\lambda} - \lambda e^{-\lambda} > 0.99$, or equivalently $e^{-\lambda}(1 + \lambda) < 0.01$. We can use r to find the smallest λ as following.

```r
f <- function(lam) { exp(-lam) * (1 + lam) - 0.01 }

# find lambda
lambda = uniroot(f, c(0, 20))$root
lambda
```

```
## [1] 6.638354
```

```r
# Verify
1 - ppois(1, lambda)
```

```
## [1] 0.99
```

Problem 2

> A shipment of 1,000 items is delivered to a factory. Suppose 5% of the items are defective. The factory inspects a random sample of 10 items without replacement and records the number of defectives as X. If X >= 2, the factory decides to return the entire shipment. What is the probability that the factory will return a shipment?

Let X be the number of defectives in a simple random sample of size n = 10 from a population of size N = 1000 with K = 0.05 * 1000 = 50 defectives. Sampling is without replacement, so X ~ Hypergeometric(N = 1000, K = 50, n = 10).

The shipment is returned if X >= 2. Thus $P(X >= 2) = 1 - P(X <= 1) = 1 - P(X = 0) - P(X = 1)$.

Use r to compute:

```
N <- 1000
K <- 50
n <- 10

p_return <- 1 - dhyper(0, K, N-K, n) - dhyper(1, K, N-K, n)
p_return
```

```
## [1] 0.08530757
```

Problem 3

> Of all customers purchasing automatic garage-door openers, 75% purchase a chain-driven model, and the remaining 25% buy a shaft-driven model. Let X be the number among the next 15 buyers who select the chain-driven model. Assume that the preferences of individual customers are independent of each other.
>
> (a) Calculate the expectation and variance of X.

Let X ~ Binomial(n = 15, p = 0.75).

$E[X] = np = 15 * 0.75 = 11.25$

$Var(X) = np(1 - p) = 15 * 0.75 * 0.25 = 2.8125$

> (b) If the store currently has 12 chain-driven and 6 shaft-driven models, what is the probability that the requests of these 15 buyers can all be met from the existing stock?

For chain-driven models, we need X <= 12. And, for shaft-driven model, we need 15 - X <= 6.

So the probability is $P(9 <= X <= 12) = \sum_{k=9}^{12} P(X = k)$.

```
sum(dbinom(9:12, size = 15, prob = 0.75))
```

```
## [1] 0.7072919
```

Problem 4

You play a game where you toss a fair coin until the first head appears. If the first head shows up on the i-th toss, you receive 2i dollars. For example, if the first head is on toss 2, you win 22 = 4 dollars. Denote the reward you can get as X.

(a) Find the probability mass function of X.

Let T be the trial on which the first head appears with a fair coin. Then $P(T = i) = (1/2)^i$ for i = 1, 2, . . .

The reward is $X = 2^i$ when T = i. Therefore the pmf of X is

$$f_X(x) = \begin{cases} (1/2)^i & x = 2^i, i = 1, 2, ... \\ 0 & \text{other } x \end{cases}$$

(b) What is the expected value of the reward you can get?

$$E[X] = \sum_{i=1}^{\infty} 2^i * f_X(x) = \sum_{i=1}^{\infty} 2^i * (1/2)^i = \sum_{i=1}^{\infty} 1 = \infty$$

Thus the expected reward does not exist as a finite number.

Problem 5

Suppose X ~ Binomial(n, p), and its cdf is denoted as F_X. Suppose Y ~ Negative-Binomial(r, p), and its cdf is denoted as F_Y . In this problem, Y counts the number of failures before the r-th success (not the total number of trials). Show that $F_X(r - 1) = 1 - F_Y(n - r)$

$F_X(r - 1) = P(X \leq r - 1)$, which is the probability that there are at most r - 1 successes in n trials.

For Y, $1 - F_Y(n - r) = 1 - P(Y \leq n - r) = P(Y > n - r)$, which is the probability that, before the r-th success, there are more than n - r times failure and r - 1 times success. So, before the r-th success, there are more than (n - r) + (r - 1) = n - 1 trails, which is at least n trails.

P("at most r - 1 successes in n trials") and P("at least n trails before the r-th success") is the same thing. Therefore, proved.