Homework 4 - P8104 Probability

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Problem 1

Let $X \sim \text{Laplace}(0, 1)$ with pdf

$$f_X(x) = \tfrac{1}{2}e^{-|x|}, \qquad x \in \mathbb{R}.$$

(a) Validity of the pdf.

A function is a valid pdf if $f_X(x) \geq 0$ for all x, and $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.

Clearly $f_X(x) \geq 0$. And, We can split the pdf function at 0 and use that $|x| = -x$ for $x < 0$ and $|x| = x$ for $x \geq 0$:

$$\int_{-\infty}^{\infty} \frac{1}{2}e^{-|x|}dx = \frac{1}{2}\left(\int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{-x}\,dx\right) = \frac{1}{2}\,(1+1) = 1.$$

(b) The cdf $F_X(x)$.

For $x \leq 0$:

$$F_X(x) = \int_{-\infty}^{x} \tfrac{1}{2}e^{t}\,dt = \tfrac{1}{2}e^{x}$$

For $x > 0$:

$$F_X(x) = F_X(0) + \int_{0}^{x} \tfrac{1}{2}e^{-t}\,dt = \tfrac{1}{2} + \tfrac{1}{2}\,(1 - e^{-x}) = 1 - \tfrac{1}{2}e^{-x}$$

$$F_X(x) = \begin{cases} \tfrac{1}{2}e^{x}, & x \leq 0, \\ 1 - \tfrac{1}{2}e^{-x}, & x > 0. \end{cases}$$

(c) The mean $E[X]$ by symmetry.

The pdf is even: $f_X(x) = f_X(-x)$. Therefore the distribution is symmetric about 0 and $E[X] = 0$.

(d) The variance $\text{Var}(X)$.

Since $E[X] = 0$,

$$\text{Var}(X) = E[X^2] = 2\int_{0}^{\infty} x^2\tfrac{1}{2}e^{-x}dx = \int_{0}^{\infty} x^2 e^{-x}dx$$

```r
fx <- function(x) 0.5*exp(-abs(x))
integrate(function(x) x^2*fx(x), -Inf, Inf)
```

```
## 2 with absolute error < 7.1e-05
```

Thus $\text{Var}(X) = 2$.

Problem 2

Let X be a continuous random variable with pdf

$$f_X(x) = x^2\left(2x + \tfrac{3}{2}\right), \qquad 0 < x \le 1.$$

Let $Y = \dfrac{2}{X} + 3$.

(a) Express $\text{Var}(Y)$ in terms of the variance of a function of X.

$$\text{Var}(Y) = \text{Var}\left(\frac{2}{X} + 3\right) = 4\,\text{Var}\left(\frac{1}{X}\right).$$

(b) Compute $E[1/X]$ and $E[1/X^2]$.

$$E\left[\frac{1}{X}\right] = \int_0^1 \frac{1}{x}\,f_X(x)\,dx = \int_0^1 \left(2x^2 + \tfrac{3}{2}x\right)dx = \frac{2}{3} + \frac{3}{4} = \frac{17}{12}.$$

$$E\left[\frac{1}{X^2}\right] = \int_0^1 \frac{1}{x^2}\,f_X(x)\,dx = \int_0^1 \left(2x + \tfrac{3}{2}\right)dx = 1 + \frac{3}{2} = \frac{5}{2}.$$

(c) Compute $\text{Var}(Y)$.

$$\text{Var}(Y) = 4\left(E\left[\frac{1}{X^2}\right] - E\left[\frac{1}{X}\right]^2\right) = 4\left(\frac{5}{2} - \left(\frac{17}{12}\right)^2\right) = \frac{71}{36} \approx 1.9722.$$

Problem 3

Let X have

$$f_X(x) = \lambda^2\,x\,e^{-\lambda x}, \qquad x \ge 0 \quad \text{(and 0 otherwise)},$$

with $\lambda > 0$. (This is $\text{Gamma}(k = 2,\ \text{rate} = \lambda)$.)

(a) Mean $E[X]$.

Using gamma integrals,

$$E[X] = \int_0^\infty x\,\lambda^2 x e^{-\lambda x} dx = \lambda^2 \int_0^\infty x^2 e^{-\lambda x} dx = \lambda^2 \cdot \frac{2!}{\lambda^3} = \frac{2}{\lambda}.$$

(b) Variance $\text{Var}(X)$. First

$$E[X^2] = \lambda^2 \int_0^\infty x^3 e^{-\lambda x} dx = \lambda^2 \cdot \frac{3!}{\lambda^4} = \frac{6}{\lambda^2}$$

Hence

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = \frac{6}{\lambda^2} - \left(\frac{2}{\lambda}\right)^2 = \frac{2}{\lambda^2}.$$

Problem 4

Let X be the outcome of rolling a fair 4-sided die, so $P(X = k) = 1/4$ for $k = 1, 2, 3, 4$.

(a) mgf $M_X(t)$.

$$M_X(t) = E[e^{tX}] = \frac{1}{4}\big(e^t + e^{2t} + e^{3t} + e^{4t}\big).$$

(b) Use $M_X(t)$ to compute $E[X]$ and $\mathrm{Var}(X)$.

From $M_X(t)$, we have

$$M_X'(t) = \frac{1}{4}\big(e^t + 2e^{2t} + 3e^{3t} + 4e^{4t}\big)$$

$$M_X''(t) = \frac{1}{4}\big(e^t + 4e^{2t} + 9e^{3t} + 16e^{4t}\big)$$

Hence $E[X] = M_X'(0) = \frac{1+2+3+4}{4} = 2.5$ and $E[X^2] = M_X''(0) = \frac{1+4+9+16}{4} = 7.5$.

So

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = 7.5 - 2.5^2 = 1.25$$

(c) Direct calculation from the pmf.
Directly from the pmf

$$E[X] = \sum_{k=1}^{4} k \cdot \tfrac{1}{4} = 2.5$$

$$E[X^2] = \sum_{k=1}^{4} k^2 \cdot \tfrac{1}{4} = 7.5$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = 1.25$$

So the results are the same.

Problem 5

Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathrm{Exp}(\lambda)$ with rate $\lambda > 0$.

(a) Show that if X and Y are independent, then $M_{X+Y}(t) = M_X(t)M_Y(t)$.

$$M_{X+Y}(t) = E\big[e^{t(X+Y)}\big] = E\big[e^{tX}e^{tY}\big]$$

Since X and Y are independent random variables, $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$. Thus

$$M_{X+Y}(t) = E\left[e^{tX}e^{tY}\right] = E[e^{tX}]\,E[e^{tY}] \quad = M_X(t)M_Y(t).$$

(b) For $S_n = \sum_{j=1}^{n} X_j$, find $M_{S_n}(t)$.

The mgf of $X_i \sim Exp(\lambda)$ is

$$\begin{aligned} M_{X_i}(t) = E[e^{tx}] &= \int_0^\infty e^{tx} * \lambda e^{-\lambda x} dx \\ &= \lambda \int_0^\infty e^{(t-\lambda)x} dx \end{aligned}$$

Assuming $t < \lambda$, we have

$$M_{X_i}(t) = \frac{\lambda}{\lambda - t}$$

From 5(a), independence gives

$$M_{S_n}(t) = \prod_{j=1}^{n} M_{X_j}(t) = \left(\frac{\lambda}{\lambda - t}\right)^n, \quad t < \lambda$$

(c) If $Y = aX + b$, then $M_Y(t) = e^{bt}M_X(at)$.

$$M_Y(t) = E\left[e^{t(aX+b)}\right] = e^{bt}\,E\left[e^{(at)X}\right] = e^{bt}\,M_X(at).$$

(d) Let Y_n be the standardized version of S_n, and find $M_{Y_n}(t)$.

$$\begin{aligned} M'_{S_n}(t) &= \lambda^n \cdot n(\lambda - t)^{-n-1} \\ M''_{S_n}(t) &= \lambda^n \cdot n(n+1)(\lambda - t)^{-n-2} \end{aligned}$$

Thus $E[S_n] = M'_{S_n}(0) = \lambda^n \cdot n\lambda^{-n-1} = n/\lambda$ and $\mathrm{Var}(S_n) = E[S_n^2] - (E[S_n])^2 = n/\lambda^2$.

Define

$$Y_n = \frac{S_n - E[S_n]}{\sqrt{\mathrm{Var}(S_n)}} = \frac{\lambda}{\sqrt{n}}\,S_n - \sqrt{n}.$$

Using part (c) with $a = \lambda/\sqrt{n}$ and $b = -\sqrt{n}$:

$$M_{Y_n}(t) = e^{-\sqrt{n}\,t}\,M_{S_n}\left(\frac{\lambda t}{\sqrt{n}}\right) = e^{-\sqrt{n}\,t}\left(\frac{\lambda}{\lambda - \frac{\lambda t}{\sqrt{n}}}\right)^n = e^{-\sqrt{n}\,t}\left(\frac{1}{1 - \frac{t}{\sqrt{n}}}\right)^n.$$