

Homework 4 - P8104 Probability

Yongyan Liu (yl6107)

December 16, 2025

Problem 1

Let $X \sim \text{Laplace}(0, 1)$ with pdf

$$f_X(x) = \frac{1}{2}e^{-|x|}, \quad x \in \mathbb{R}.$$

(a) Validity of the pdf.

A function is a valid pdf if $f_X(x) \geq 0$ for all x , and $\int_{-\infty}^{\infty} f_X(x) dx = 1$.

Clearly $f_X(x) \geq 0$. And, We can split the pdf function at 0 and use that $|x| = -x$ for $x < 0$ and $|x| = x$ for $x \geq 0$:

$$\int_{-\infty}^{\infty} \frac{1}{2}e^{-|x|} dx = \frac{1}{2} \left(\int_{-\infty}^0 e^x dx + \int_0^{\infty} e^{-x} dx \right) = \frac{1}{2} (1 + 1) = 1.$$

(b) The cdf $F_X(x)$.

For $x \leq 0$:

$$F_X(x) = \int_{-\infty}^x \frac{1}{2}e^t dt = \frac{1}{2}e^x$$

For $x > 0$:

$$F_X(x) = F_X(0) + \int_0^x \frac{1}{2}e^{-t} dt = \frac{1}{2} + \frac{1}{2} (1 - e^{-x}) = 1 - \frac{1}{2}e^{-x}$$

$$F_X(x) = \begin{cases} \frac{1}{2}e^x, & x \leq 0, \\ 1 - \frac{1}{2}e^{-x}, & x > 0. \end{cases}$$

(c) The mean $E[X]$ by symmetry.

The pdf is even: $f_X(x) = f_X(-x)$. Therefore the distribution is symmetric about 0 and $E[X] = 0$.

(d) The variance $\text{Var}(X)$.

Since $E[X] = 0$,

$$\text{Var}(X) = E[X^2] = 2 \int_0^{\infty} x^2 \frac{1}{2}e^{-x} dx = \int_0^{\infty} x^2 e^{-x} dx$$

```
fx <- function(x) 0.5*exp(-abs(x))
integrate(function(x) x^2*fx(x), -Inf, Inf)
```

```
## 2 with absolute error < 7.1e-05
```

Thus $\text{Var}(X) = 2$.

Problem 2

Let X be a continuous random variable with pdf

$$f_X(x) = x^2 \left(2x + \frac{3}{2}\right), \quad 0 < x \leq 1.$$

Let $Y = \frac{2}{X} + 3$.

(a) **Express** $\text{Var}(Y)$ in terms of the variance of a function of X .

$$\text{Var}(Y) = \text{Var}\left(\frac{2}{X} + 3\right) = 4 \text{Var}\left(\frac{1}{X}\right).$$

(b) **Compute** $E[1/X]$ and $E[1/X^2]$.

$$E\left[\frac{1}{X}\right] = \int_0^1 \frac{1}{x} f_X(x) dx = \int_0^1 (2x^2 + \frac{3}{2}x) dx = \frac{2}{3} + \frac{3}{4} = \frac{17}{12}.$$

$$E\left[\frac{1}{X^2}\right] = \int_0^1 \frac{1}{x^2} f_X(x) dx = \int_0^1 (2x + \frac{3}{2}) dx = 1 + \frac{3}{2} = \frac{5}{2}.$$

(c) **Compute** $\text{Var}(Y)$.

$$\text{Var}(Y) = 4 \left(E\left[\frac{1}{X^2}\right] - E\left[\frac{1}{X}\right]^2 \right) = 4 \left(\frac{5}{2} - \left(\frac{17}{12}\right)^2 \right) = \frac{71}{36} \approx 1.9722.$$

Problem 3

Let X have

$$f_X(x) = \lambda^2 x e^{-\lambda x}, \quad x \geq 0 \quad (\text{and } 0 \text{ otherwise}),$$

with $\lambda > 0$. (This is Gamma($k = 2$, rate = λ).)

(a) **Mean** $E[X]$.

Using gamma integrals,

$$E[X] = \int_0^\infty x \lambda^2 x e^{-\lambda x} dx = \lambda^2 \int_0^\infty x^2 e^{-\lambda x} dx = \lambda^2 \cdot \frac{2!}{\lambda^3} = \frac{2}{\lambda}.$$

(b) **Variance** $\text{Var}(X)$. First

$$E[X^2] = \lambda^2 \int_0^\infty x^3 e^{-\lambda x} dx = \lambda^2 \cdot \frac{3!}{\lambda^4} = \frac{6}{\lambda^2}$$

Hence

$$\text{Var}(X) = E[X^2] - E[X]^2 = \frac{6}{\lambda^2} - \left(\frac{2}{\lambda}\right)^2 = \frac{2}{\lambda^2}.$$

Problem 4

Let X be the outcome of rolling a fair 4-sided die, so $P(X = k) = 1/4$ for $k = 1, 2, 3, 4$.

(a) **mgf** $M_X(t)$.

$$M_X(t) = E[e^{tX}] = \frac{1}{4}(e^t + e^{2t} + e^{3t} + e^{4t}).$$

(b) **Use** $M_X(t)$ to compute $E[X]$ and $\text{Var}(X)$.

From $M_X(t)$, we have

$$M'_X(t) = \frac{1}{4}(e^t + 2e^{2t} + 3e^{3t} + 4e^{4t})$$

$$M''_X(t) = \frac{1}{4}(e^t + 4e^{2t} + 9e^{3t} + 16e^{4t})$$

Hence $E[X] = M'_X(0) = \frac{1+2+3+4}{4} = 2.5$ and $E[X^2] = M''_X(0) = \frac{1+4+9+16}{4} = 7.5$.

So

$$\text{Var}(X) = E[X^2] - E[X]^2 = 7.5 - 2.5^2 = 1.25$$

(c) **Direct calculation from the pmf.**

Directly from the pmf

$$E[X] = \sum_{k=1}^4 k \cdot \frac{1}{4} = 2.5$$

$$E[X^2] = \sum_{k=1}^4 k^2 \cdot \frac{1}{4} = 7.5$$

$$\text{Var}(X) = E[X^2] - E[X]^2 = 1.25$$

So the results are the same.

Problem 5

Let $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} \text{Exp}(\lambda)$ with rate $\lambda > 0$.

(a) Show that if X and Y are independent, then $M_{X+Y}(t) = M_X(t)M_Y(t)$.

$$M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX}e^{tY}]$$

Since X and Y are independent random variables, $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$. Thus

$$M_{X+Y}(t) = E[e^{tX}e^{tY}] = E[e^{tX}]E[e^{tY}] = M_X(t)M_Y(t).$$

(b) For $S_n = \sum_{j=1}^n X_j$, find $M_{S_n}(t)$.

The mgf of $X_i \sim \text{Exp}(\lambda)$ is

$$\begin{aligned} M_{X_i}(t) &= E[e^{tx}] = \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x} dx \\ &= \lambda \int_0^\infty e^{(t-\lambda)x} dx \end{aligned}$$

Assuming $t < \lambda$, we have

$$M_{X_i}(t) = \frac{\lambda}{\lambda - t}$$

From 5(a), independence gives

$$M_{S_n}(t) = \prod_{j=1}^n M_{X_j}(t) = \left(\frac{\lambda}{\lambda - t} \right)^n, \quad t < \lambda$$

(c) If $Y = aX + b$, then $M_Y(t) = e^{bt}M_X(at)$.

$$M_Y(t) = E[e^{t(aX+b)}] = e^{bt} E[e^{(at)X}] = e^{bt} M_X(at).$$

(d) Let Y_n be the standardized version of S_n , and find $M_{Y_n}(t)$.

$$M'_{S_n}(t) = \lambda^n \cdot n(\lambda - t)^{-n-1}$$

$$M''_{S_n}(t) = \lambda^n \cdot n(n+1)(\lambda - t)^{-n-2}$$

Thus $E[S_n] = M'_{S_n}(0) = \lambda^n \cdot n\lambda^{-n-1} = n/\lambda$ and $\text{Var}(S_n) = E[S_n^2] - (E[S_n])^2 = n/\lambda^2$.

Define

$$Y_n = \frac{S_n - E[S_n]}{\sqrt{\text{Var}(S_n)}} = \frac{\lambda}{\sqrt{n}} S_n - \sqrt{n}.$$

Using part (c) with $a = \lambda/\sqrt{n}$ and $b = -\sqrt{n}$:

$$M_{Y_n}(t) = e^{-\sqrt{n}t} M_{S_n}\left(\frac{\lambda t}{\sqrt{n}}\right) = e^{-\sqrt{n}t} \left(\frac{\lambda}{\lambda - \frac{\lambda t}{\sqrt{n}}} \right)^n = e^{-\sqrt{n}t} \left(\frac{1}{1 - \frac{t}{\sqrt{n}}} \right)^n.$$