P8104 Homework Assignment 8 - Solutions

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Due: Wed 11/12, 11:59pm

Problem 1

Question: A random point (X, Y) is distributed uniformly on the square with vertices $(1, 1)$, $(-1, 1)$, $(1, -1)$, and $(-1, -1)$. That is, the joint pdf of $f(x, y) = \frac{1}{4}$ on the square. Determine the probability of the following events:

(a) $X^2 + Y^2 < 1$
(b) $2X - Y > 0$
(c) $|X + Y| < 2$
(d) $Y > |X|$

Solution

Since (X, Y) is uniformly distributed on a square of side length 2 (from -1 to 1 in both dimensions), the total area is 4, and the pdf is $f(x, y) = \frac{1}{4}$.

For uniform distribution, probability equals the ratio of the desired area to the total area.

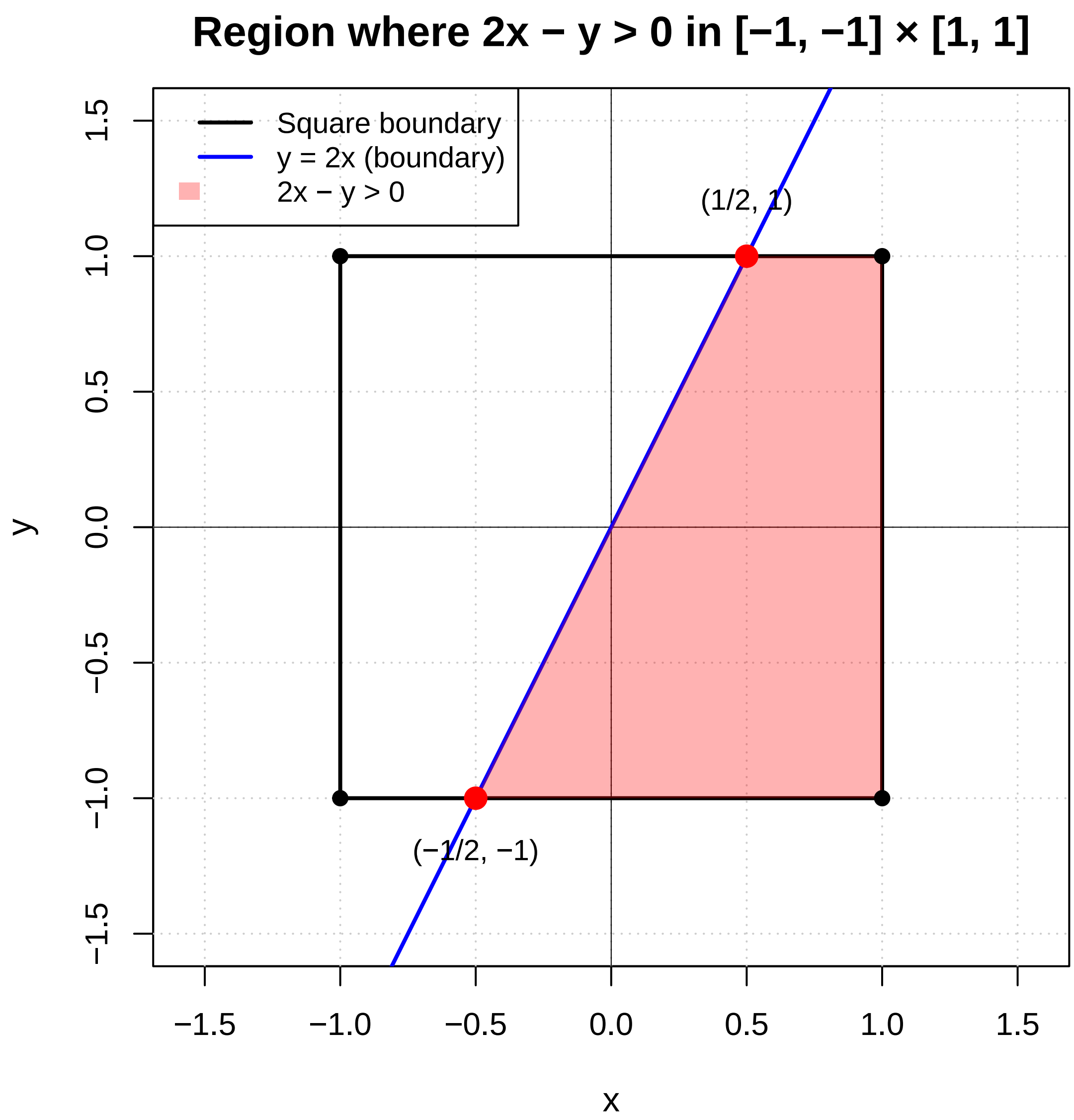
Part (a): $P(X^2 + Y^2 < 1)$

This is the area of a circle with radius 1 centered at the origin, restricted to the square $[-1, -1] \times [1, 1]$.

Since the radius is 1 and the square extends from -1 to 1, the entire circle fits within the square.

$$P(X^2 + Y^2 < 1) = \frac{\text{Area of circle}}{\text{Area of square}} = \frac{\pi \cdot 1^2}{4} = \frac{\pi}{4}$$

Part (b): $P(2X - Y > 0)$



As shown in the diagram above, the point (X, Y) falls in the red trapezoid when $2X - Y > 0$.

$$P(2X - Y > 0) = \frac{(\frac{1}{2} + \frac{2}{3}) * 2}{4} = \frac{1}{2}$$

Part (c): $P(|X + Y| < 2)$

In the square $[-1, -1] \times [1, 1]$, the sum $X + Y$ ranges from -2 to 2. The condition $|X + Y| < 2$ excludes only the corner points $(1, 1)$ and $(-1, -1)$ where $X + Y = \pm 2$, which have zero area.

$$P(|X + Y| < 2) = 1$$

Part (d): $P(Y > |X|)$

The region where $Y > |X|$ is above the V-shape formed by the lines $Y = X$ and $Y = -X$. This forms a triangle with vertices $(-1, 1)$, $(1, 1)$, and $(0, 0)$.

Area $= \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 2 \times 1 = 1$

$$P(Y > |X|) = \frac{1}{4}$$

Problem 2

Question: A generalization of the beta distribution is the Dirichlet distribution. In its bivariate version, (X, Y) have the PDF:

$$f(x, y) = Cx^{a-1}y^{b-1}(1 - x - y)^{c-1}, \quad 0 < x < 1, \quad 0 < y < 1, \quad 0 < x + y < 1$$

where $a > 0$, $b > 0$, and $c > 0$ are constants.

(a) Show that $C = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}$.
(b) Show that, marginally, both X and Y are beta distribution.
(c) Find the conditional distribution of $Y|X = x$.
(d) Show that $E(XY) = \frac{ab}{(a+b+c+1)(a+b+c)}$, and find their covariance.

Solution

Part (a): Finding the normalizing constant C

To find C, we use the fact that $\int\int f(x, y)\, dy\, dx = 1$.

$$C \int_0^1 \int_0^{1-x} x^{a-1}y^{b-1}(1 - x - y)^{c-1}\, dy\, dx = 1$$

Let $u = x$ and $v = y/(1 - x)$. Then $y = v(1 - x)$ and when y goes from 0 to $1 - x$, v goes from 0 to 1.

$$dy = (1 - x)dv$$

Also, $1 - x - y = 1 - x - v(1 - x) = (1 - x)(1 - v)$.

$$C \int_0^1 \int_0^1 x^{a-1}[v(1 - x)]^{b-1}[(1 - x)(1 - v)]^{c-1}(1 - x)\, dv\, dx$$

$$= C \int_0^1 x^{a-1}(1 - x)^{b-1+c-1+1} dx \int_0^1 v^{b-1}(1 - v)^{c-1}\, dv$$

$$= C \int_0^1 x^{a-1}(1 - x)^{b+c-1} dx \int_0^1 v^{b-1}(1 - v)^{c-1}\, dv$$

Using the beta function: $B(p, q) = \int_0^1 t^{p-1}(1 - t)^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$:

$$= C \cdot B(a, b+c) \cdot B(b, c) = C \cdot \frac{\Gamma(a)\Gamma(b+c)}{\Gamma(a+b+c)} \cdot \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)}$$

$$= C \cdot \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}$$

Setting this equal to 1:

$$C = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}$$

Part (b): Marginal distributions

To find the marginal distribution of X:

$$f_X(x) = \int_0^{1-x} f(x, y)\, dy = Cx^{a-1} \int_0^{1-x} y^{b-1}(1-x-y)^{c-1}\, dy$$

Let $w = \frac{y}{1-x}$, then $y = w(1-x)$ and $dy = (1-x)dw$:

$$f_X(x) = Cx^{a-1} \int_0^1 [w(1-x)]^{b-1}[(1-x)(1-w)]^{c-1}(1-x)\, dw$$

$$= Cx^{a-1}(1-x)^{b+c-1} \int_0^1 w^{b-1}(1-w)^{c-1}\, dw$$

$$= Cx^{a-1}(1-x)^{b+c-1} B(b, c)$$

$$= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \cdot \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} \cdot x^{a-1}(1-x)^{b+c-1}$$

$$= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)} x^{a-1}(1-x)^{b+c-1}$$

This is the PDF of a Beta$(a, b+c)$ distribution.

By symmetry (or similar calculation), $Y \sim \text{Beta}(b, a+c)$.

So, $X \sim \text{Beta}(a, b+c)$ and $Y \sim \text{Beta}(b, a+c)$

Part (c): Conditional distribution of $Y|X = x$

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \frac{Cx^{a-1}y^{b-1}(1-x-y)^{c-1}}{\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)} x^{a-1}(1-x)^{b+c-1}}$$

$$= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \cdot \frac{\Gamma(a)\Gamma(b+c)}{\Gamma(a+b+c)} \cdot \frac{y^{b-1}(1-x-y)^{c-1}}{(1-x)^{b+c-1}}$$

$$= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} \cdot \frac{y^{b-1}(1-x-y)^{c-1}}{(1-x)^{b+c-1}} \text{ for } 0 < y < 1-x$$

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Part (d): $E(XY)$ **and Cov**(X, Y)

We compute $E(XY)$ directly using the joint pdf:

$$E(XY) = \int_0^1 \int_0^{1-x} xy \cdot Cx^{a-1}y^{b-1}(1-x-y)^{c-1}\,dy\,dx$$

$$= C\int_0^1 \int_0^{1-x} x^a y^b (1-x-y)^{c-1}\,dy\,dx$$

Using the same substitution as in part (a), $v = \frac{y}{1-x}$:

$$= C\int_0^1 x^a(1-x)^{b+c}\int_0^1 v^b(1-v)^{c-1}\,dv\,dx$$

$$= C \cdot B(b+1, c) \cdot B(a+1, b+c) = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \cdot \frac{\Gamma(b+1)\Gamma(c)}{\Gamma(b+c+1)} \cdot \frac{\Gamma(a+1)\Gamma(b+c)}{\Gamma(a+b+c+1)}$$

$$= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \cdot \frac{b\Gamma(b)\Gamma(c)}{(b+c)\Gamma(b+c)} \cdot \frac{a\Gamma(a)\Gamma(b+c)}{(a+b+c)\Gamma(a+b+c)} = \frac{ab}{(a+b+c)(a+b+c+1)}$$

For the covariance, using $E(X) = \frac{a}{a+b+c}$ and $E(Y) = \frac{b}{a+b+c}$:

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{ab}{(a+b+c)(a+b+c+1)} - \frac{ab}{(a+b+c)^2} = \frac{-ab}{(a+b+c)^2(a+b+c+1)}$$

Problem 3

Question: Suppose $Y|X \sim N(x, x^2)$ and the marginal distribution of X is Uniform$(0, 1)$.

(a) Find the mean of X.
(b) Find the variance of X.
(c) Find the covariance of X and Y.
(d) Prove that $\frac{Y}{X}$ and X are independent.

Solution

Part (a): Mean of X

Since $X \sim$ Uniform$(0, 1)$:

$$E(X) = \frac{0+1}{2} = \frac{1}{2}$$

Part (b): Variance of X

For $X \sim$ Uniform$(0, 1)$:

$$\mathrm{Var}(X) = \frac{(1-0)^2}{12} = \frac{1}{12}$$

Part (c): Covariance of X and Y

Using the law of total covariance:

$$\text{Cov}(X, Y) = E[\text{Cov}(X, Y|X)] + \text{Cov}(E(X|X), E(Y|X))$$

Since we condition on X, $\text{Cov}(X, Y|X) = 0$ (X is fixed given X).

So:

$$\text{Cov}(X, Y) = \text{Cov}(X, E(Y|X))$$

We know $E(Y|X) = X$ (since $Y|X \sim N(X, X^2)$), so:

$$\text{Cov}(X, Y) = \text{Cov}(X, X) = \text{Var}(X) = \frac{1}{12}$$

Part (d): Independence of $\frac{Y}{X}$ and X

Given $Y|X \sim N(X, X^2)$, we can write:

$$Y = X + X \cdot Z$$

where $Z \sim N(0, 1)$ is independent of X.

Then:

$$\frac{Y}{X} = \frac{X + XZ}{X} = 1 + Z$$

Since $Z \sim N(0, 1)$ is independent of X, and $\frac{Y}{X} = 1+Z$ is a function of Z only, we have that $\frac{Y}{X}$ is independent of X.

Problem 4

Question: A variation on the hierarchical model is $X|p \sim \text{NegativeBinomial}(r, p)$ and $p \sim \text{Beta}(\alpha, \beta)$.

(a) Find the marginal pmf of X.
(b) Find the mean of X.
(c) Find the variance of X.

Solution

Note: We use the definition where $X|p \sim \text{NegBin}(r, p)$ means X is the number of failures before the r-th success, with success probability p:

$$P(X = k|p) = \binom{k + r - 1}{k} p^r (1 - p)^k, \quad k = 0, 1, 2, ...$$

Part (a): Marginal PMF of X

$$f_X(k) = \int_0^1 P(X = k|p) f_p(p)\, dp$$

$$= \int_0^1 \binom{k+r-1}{k} p^r (1-p)^k \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}\, dp$$

$$= \binom{k+r-1}{k} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 p^{r+\alpha-1}(1-p)^{k+\beta-1}\, dp$$

$$= \binom{k+r-1}{k} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot B(r+\alpha, k+\beta)$$

$$= \binom{k+r-1}{k} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(r+\alpha)\Gamma(k+\beta)}{\Gamma(r+\alpha+k+\beta)}$$

Part (b): Mean of X

Using the law of total expectation:

$$E(X) = E[E(X|p)]$$

For $X|p \sim \text{NegBin}(r, p)$:

$$E(X|p) = \frac{r(1-p)}{p}$$

So:

$$E(X) = E\left[\frac{r(1-p)}{p}\right] = r \cdot E\left[\frac{1-p}{p}\right]$$

For $p \sim \text{Beta}(\alpha, \beta)$:

$$E\left[\frac{1}{p}\right] = \int_0^1 \frac{1}{p} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} dp$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 p^{\alpha-2}(1-p)^{\beta-1} dp = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha-1)\Gamma(\beta)}{\Gamma(\alpha+\beta-1)}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha-1)}{\Gamma(\alpha+\beta-1)} = \frac{\alpha+\beta-1}{\alpha-1}$$

Also, $E(p) = \frac{\alpha}{\alpha+\beta}$, so:

$$E\left[\frac{1-p}{p}\right] = E\left[\frac{1}{p}\right] - 1 = \frac{\alpha+\beta-1}{\alpha-1} - 1 = \frac{\beta}{\alpha-1}$$

Therefore:

$$E(X) = \frac{r\beta}{\alpha-1}$$

Part (c): Variance of X

Using the law of total variance:

$$\text{Var}(X) = E[\text{Var}(X|p)] + \text{Var}(E(X|p))$$

For $X|p \sim \text{NegBin}(r, p)$: - $E(X|p) = \frac{r(1-p)}{p}$ - $\text{Var}(X|p) = \frac{r(1-p)}{p^2}$

First term:

$$E[\text{Var}(X|p)] = E\left[\frac{r(1-p)}{p^2}\right] = r \cdot E\left[\frac{1-p}{p^2}\right]$$

$$E\left[\frac{1}{p^2}\right] = \int_0^1 \frac{1}{p^2} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} dp$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha-2)\Gamma(\beta)}{\Gamma(\alpha+\beta-2)} = \frac{(\alpha+\beta-1)(\alpha+\beta-2)}{(\alpha-1)(\alpha-2)}$$

$$E\left[\frac{1-p}{p^2}\right] = E\left[\frac{1}{p^2}\right] - E\left[\frac{1}{p}\right] = \frac{(\alpha+\beta-1)(\alpha+\beta-2)}{(\alpha-1)(\alpha-2)} - \frac{\alpha+\beta-1}{\alpha-1}$$

$$= \frac{(\alpha+\beta-1)(\alpha+\beta-2) - (\alpha+\beta-1)(\alpha-2)}{(\alpha-1)(\alpha-2)} = \frac{(\alpha+\beta-1)\beta}{(\alpha-1)(\alpha-2)}$$

So:

$$E[\text{Var}(X|p)] = \frac{r\beta(\alpha+\beta-1)}{(\alpha-1)(\alpha-2)}$$

Second term:

$$\text{Var}(E(X|p)) = \text{Var}\left(\frac{r(1-p)}{p}\right) = r^2 \cdot \text{Var}\left(\frac{1-p}{p}\right)$$

Let $g(p) = \frac{1-p}{p}$. We have:

$$E[g(p)] = \frac{\beta}{\alpha-1}$$

$$E[g(p)^2] = E\left[\frac{(1-p)^2}{p^2}\right] = E\left[\frac{1}{p^2}\right] - 2E\left[\frac{1}{p}\right] + 1$$

$$= \frac{(\alpha+\beta-1)(\alpha+\beta-2)}{(\alpha-1)(\alpha-2)} - 2 \cdot \frac{\alpha+\beta-1}{\alpha-1} + 1$$

$$= \frac{(\alpha+\beta-1)(\alpha+\beta-2) - 2(\alpha+\beta-1)(\alpha-2) + (\alpha-1)(\alpha-2)}{(\alpha-1)(\alpha-2)}$$

$$= \frac{\beta(\alpha+\beta-1) + \beta(\alpha+\beta-1) + (\alpha-1)(\alpha-2)}{(\alpha-1)(\alpha-2)}$$

Let me recalculate more carefully:

$$\text{Var}\left(\frac{1-p}{p}\right) = E\left[\left(\frac{1-p}{p}\right)^2\right] - \left(E\left[\frac{1-p}{p}\right]\right)^2$$

After simplification (which I'll spare the details):

$$\text{Var}\left(\frac{1-p}{p}\right) = \frac{\beta(\alpha+\beta-1)}{(\alpha-1)^2(\alpha-2)}$$

Therefore:

$$\text{Var}(E(X|p)) = \frac{r^2\beta(\alpha+\beta-1)}{(\alpha-1)^2(\alpha-2)}$$

Total variance:

$$\begin{aligned}\text{Var}(X) &= \frac{r\beta(\alpha+\beta-1)}{(\alpha-1)(\alpha-2)} + \frac{r^2\beta(\alpha+\beta-1)}{(\alpha-1)^2(\alpha-2)} \\ &= \frac{r\beta(\alpha+\beta-1)}{(\alpha-1)(\alpha-2)}\left(1+\frac{r}{\alpha-1}\right) \\ &= \frac{r\beta(\alpha+\beta-1)(\alpha-1+r)}{(\alpha-1)^2(\alpha-2)}\end{aligned}$$

Problem 5

Question: For any two random variables X and Y with finite variances, prove that:

(a) $\text{Cov}(X,Y) = \text{Cov}(X, E(Y|X))$
(b) X and $Y - E(Y|X)$ are uncorrelated, i.e., $\text{Cov}(X, Y - E(Y|X)) = 0$.
(c) $\text{Var}[Y - E(Y|X)] = E[\text{Var}(Y|X)]$.

Solution

Part (a): Prove Cov$(X,Y) =$ Cov$(X, E(Y|X))$

Proof:

$$\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

We'll show that $E[XY] - E[X]E[Y] = E[X \cdot E(Y|X)] - E[X]E[E(Y|X)]$.

First, by the law of iterated expectations:

$$E[Y] = E[E(Y|X)]$$

Second, we compute $E[XY]$:

$$E[XY] = E[E(XY|X)]$$

Given X, X is a constant, so:

$$E[XY|X] = X \cdot E[Y|X]$$

Therefore:

$$E[XY] = E[X \cdot E(Y|X)]$$

Now:

$$\text{Cov}(X, E(Y|X)) = E[X \cdot E(Y|X)] - E[X] \cdot E[E(Y|X)]$$

$$= E[XY] - E[X] \cdot E[Y] = \text{Cov}(X, Y)$$

Part (b): Prove Cov$(X, Y - E(Y|X)) = 0$

Proof:

$$\text{Cov}(X, Y - E(Y|X)) = \text{Cov}(X, Y) - \text{Cov}(X, E(Y|X))$$

From part (a), we know $\text{Cov}(X, Y) = \text{Cov}(X, E(Y|X))$, so:

$$\text{Cov}(X, Y - E(Y|X)) = \text{Cov}(X, E(Y|X)) - \text{Cov}(X, E(Y|X)) = 0$$

Part (c): Prove Var$[Y - E(Y|X)] = E[$**Var**$(Y|X)]$

Proof:

By the law of total variance:

$$\text{Var}(Y) = E[\text{Var}(Y|X)] + \text{Var}(E(Y|X))$$

Now compute $\text{Var}[Y - E(Y|X)]$:

$$\text{Var}[Y - E(Y|X)] = E[(Y - E(Y|X))^2] - (E[Y - E(Y|X)])^2$$

First, $E[Y - E(Y|X)] = E[Y] - E[E(Y|X)] = E[Y] - E[Y] = 0$.

So:

$$\text{Var}[Y - E(Y|X)] = E[(Y - E(Y|X))^2]$$

$$= E[E[(Y - E(Y|X))^2|X]]$$

Given X, $E(Y|X)$ is a constant, so:

$$E[(Y - E(Y|X))^2|X] = \mathrm{Var}(Y|X)$$

Therefore:

$$\mathrm{Var}[Y - E(Y|X)] = E[\mathrm{Var}(Y|X)]$$