

# Homework Assignment 3

Due Wed 10/1. 11:59pm

## Problem 1

Let a point be chosen uniformly at random from the interior of a circular annulus with inner radius  $r_0 = \frac{1}{2}$  and outer radius  $r_1 = 1$ . Let  $X$  be the distance from the origin to the selected point.

- (a) What is the support of  $X$ ?
- (b) Determine the cumulative distribution function (cdf)  $F_X(x)$  of the distance  $X$ .
- (c) Derive the probability density function (pdf)  $f_X(x)$ .
- (d) Compute the probability that the point lies between distances 0.6 and 0.8 from the origin.

## Problem 2

A median of a distribution of a random variable  $X$  is a value of  $x$  such that

$$P(X < x) \leq \frac{1}{2} \quad \text{and} \quad P(X \leq x) \geq \frac{1}{2}.$$

If there is only one such  $x$ , it is called the median of the distribution. For the following functions, first verify that it is a pmf or pdf, then find the median:

- (a) Discrete case:  $p_X(x) = \frac{4!}{x!(4-x)!}(0.8)^x(0.2)^{4-x}, \quad x = 0, 1, 2, 3, 4$

*Hint: This is a Binomial distribution with  $n = 4$ ,  $p = 0.8$ .*

- (b) Continuous case:  $f_X(x) = 4x^3, \quad 0 < x < 1$

## Problem 3

Let  $X \sim \text{Uniform}(0, 1)$ , and define  $Y = \sqrt{X}$ .

- (a) Find the cdf  $F_Y(y)$ .
- (b) Find pdf  $f_Y(y)$ .
- (c) Verify that your pdf is valid.
- (d) Compute  $E(Y)$ .

## Problem 4

Find the pdf of  $Y$  in the following situations:

- (a)  $Y = \ln(X)$ . The pdf for  $X$  is

$$f(x) = \frac{2}{\pi(1+x^2)}, \quad x \geq 0.$$

- (b)  $Y = \sin(X)$ . The pdf for  $X$  is

$$f(x) = \frac{cx}{\pi^2}, \quad \text{where } x \in (0, \pi),$$

where  $c$  is a normalizing constant.

Determine the value of  $c$ , and then find the pdf of  $Y$ .

**Hint:** The transformation  $g(x) = \sin(x)$  is not one-to-one on  $(0, \pi)$ . For a given  $y \in (0, 1)$ , there are two values of  $x$  such that  $\sin(x) = y$ . You may need to consider both parts of the domain when deriving the distribution of  $Y$ .

## Problem 5

Let  $X$  be a continuous random variable with pdf:

$$f_X(x) = \begin{cases} \frac{3}{2}x^2, & 0 < x < 1 \\ \frac{3}{2}(1-x)^2, & 1 \leq x < 2 \\ 0, & \text{otherwise} \end{cases}$$

- (a) Verify that it is a valid pdf.
- (b) Sketch the pdf.
- (c) Compute  $E(X)$ .
- (d) Compute  $E(X^2)$ , and then compute  $\text{Var}(X)$ .