Stability Analysis of Observer-PD Controller

Kexin Guo, Yuhang Liu, Jindou Jia, Zihan Yang, Sicheng Zhou, Xiang Yu*, and Lei Guo

I. PREFACE

This paper is a supplemented material serving for stability analysis of the combination of PD controller and rain speed observer (RSO), which is a innovatively designed observer introduced in manuscript submitted to IEEE Transactions on Industrial Electronics with paper ID 24-2548. It can also provide reference for the stability analysis of other similar studies.

II. PRELIMINARIES

$\lambda_M(*)$ and $\lambda_m(*)$ represent the maximum and minimum eigenvalues of a matrix, respectively. $\hat{*}$, $\dot{*}$ and $\tilde{*}$ denote the estimation, the first-order time derivative and estimation error of $*$, respectively.

By combining the mathematical expressions of the baseline controller and model dynamics, the relationships between states in the translational loop can be formulated as a state-space equation in the following form, facilitating the evaluation of stability using theoretical analysis tools.

Kexin Guo is with the School of Aeronautic Science and Engineering, Beihang University, Beijing 100191, China, and also with the Hangzhou Innovation institute, Beihang University, Hangzhou 310051, China. E-mail: kxguo@buaa.edu.cn.

Yuhang Liu and Zihan Yang are with the School of Aeronautic Science and Engineering, Beihang University, Beijing 100191, China. E-mail: {lyhbuaa, snrt_zzhan }@buaa.edu.cn.

Jindou Jia and Sicheng Zhou are with the School of Automation Science and Electrical Engineering, Beihang University, Beijing 100191, China. E-mail: {jdjia, zb1903003}@buaa.edu.cn.

Xiang Yu and Lei Guo are with the School of Automation Science and Electrical Engineering, Beihang University, Beijing 100191, China, and also with the Hangzhou Innovation Institute, Beihang University, Hangzhou 310051, China. E-mail: {xiangyu_buaa, lguo}@buaa.edu.cn.

* Corresponding author.

$$\underbrace{\begin{pmatrix}\dot{\boldsymbol{e}}_p\\ \dot{\boldsymbol{e}}_v\end{pmatrix}}_{\dot{\boldsymbol{e}}}=\underbrace{\begin{pmatrix}\mathbf{0}_{3\times 3} & \boldsymbol{I}_{3\times 3}\\ \frac{-\boldsymbol{K}_p}{m} & \frac{-\boldsymbol{K}_v}{m}\end{pmatrix}}_{\boldsymbol{A}}\underbrace{\begin{pmatrix}\boldsymbol{e}_p\\ \boldsymbol{e}_v\end{pmatrix}}_{\boldsymbol{e}}+\underbrace{\begin{pmatrix}\mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3}\\ \mathbf{0}_{3\times 3} & \frac{\boldsymbol{I}_{3\times 3}}{m}\end{pmatrix}}_{\boldsymbol{B}}\underbrace{\begin{pmatrix}\mathbf{0}_{3\times 1}\\ \tilde{\boldsymbol{F}}_r^E\end{pmatrix}}_{\tilde{\boldsymbol{d}}}, \tag{1}$$

where $\boldsymbol{e}=\left[\boldsymbol{e}_p^T,\boldsymbol{e}_v^T\right]^T$, $\boldsymbol{e}_p$ and $\boldsymbol{e}_v$ represent the tracking errors of position and velocity, respectively, $\tilde{\boldsymbol{F}}_r^E=\hat{\boldsymbol{F}}_r^E-\boldsymbol{F}_r^E=-\boldsymbol{\vartheta}\tilde{\boldsymbol{v}}_r^E$ represents the estimation error of the rain disturbance $\boldsymbol{F}_r^E$ from RSO and the expression is derived from the designed model of disturbance.

III. STABILITY ANALYSIS

It can be checked that $\boldsymbol{A}$ in (1) has negative definite structure if $\boldsymbol{K}_p$ and $\boldsymbol{K}_v$ are positive definite. As a consequence, there must exist a positive definite symmetric matrix $\boldsymbol{M}$ that meets the equation $\boldsymbol{A}^T\boldsymbol{M}+\boldsymbol{M}\boldsymbol{A}=-\boldsymbol{I}$. Subsequently, a Lyapunov function is designed as follows,

$$V=\boldsymbol{e}^T\boldsymbol{M}\boldsymbol{e}+\frac{1}{2}(\tilde{\boldsymbol{v}}_r^E)^T\tilde{\boldsymbol{v}}_r^E. \tag{2}$$

Differentiate V, it can be implied that

$$\begin{aligned}\dot{V}=&\dot{\boldsymbol{e}}^T\boldsymbol{M}\boldsymbol{e}+\boldsymbol{e}^T\boldsymbol{M}\dot{\boldsymbol{e}}+(\tilde{\boldsymbol{v}}_r^E)^T\dot{\tilde{\boldsymbol{v}}}_r^E\\ =&\boldsymbol{e}^T\boldsymbol{A}^T\boldsymbol{M}\boldsymbol{e}+\tilde{\boldsymbol{d}}^T\boldsymbol{B}^T\boldsymbol{M}\boldsymbol{e}+\boldsymbol{e}^T\boldsymbol{M}\boldsymbol{A}\boldsymbol{e}\\ &+\boldsymbol{e}^T\boldsymbol{M}\boldsymbol{B}\tilde{\boldsymbol{d}}+(\tilde{\boldsymbol{v}}_r^E)^T\dot{\tilde{\boldsymbol{v}}}_r^E\\ =&\boldsymbol{e}^T\left(\boldsymbol{A}^T\boldsymbol{M}+\boldsymbol{M}\boldsymbol{A}\right)\boldsymbol{e}+2\boldsymbol{e}^T\boldsymbol{M}\boldsymbol{B}\tilde{\boldsymbol{d}}+(\tilde{\boldsymbol{v}}_r^E)^T\dot{\tilde{\boldsymbol{v}}}_r^E\\ =&-\boldsymbol{e}^T\boldsymbol{e}+2\boldsymbol{e}^T\boldsymbol{M}\boldsymbol{B}\tilde{\boldsymbol{d}}+(\tilde{\boldsymbol{v}}_r^E)^T\dot{\tilde{\boldsymbol{v}}}_r^E.\end{aligned} \tag{3}$$

Furthermore, by resorting to Young's inequality [1] , it can be verified that

$$-\boldsymbol{e}^\top\boldsymbol{e}\le-\frac{\boldsymbol{e}^\top\boldsymbol{M}\boldsymbol{e}}{\lambda_M\left(\boldsymbol{M}\right)}, \tag{4a}$$

$$-(\tilde{\boldsymbol{v}}_r^E)^\top\dot{\boldsymbol{v}}_r^E\le\frac{1}{4}(\tilde{\boldsymbol{v}}_r^E)^\top\tilde{\boldsymbol{v}}_r^E+\delta, \tag{4b}$$

$$\boldsymbol{e}^\top\boldsymbol{M}\boldsymbol{B}\tilde{\boldsymbol{d}}\le\frac{1}{2\varepsilon}\lambda_M^2\left(\boldsymbol{M}\boldsymbol{B}\right)\boldsymbol{e}^\top\boldsymbol{e}+\frac{\varepsilon}{2}\tilde{\boldsymbol{d}}^\top\tilde{\boldsymbol{d}}, \tag{4c}$$

where ε is an arbitrary positive constant, and $\delta\le(\dot{\boldsymbol{v}}_r^E)^T\dot{\boldsymbol{v}}_r^E$ is a constant value with an upper bound concerned with $\dot{\boldsymbol{v}}_r^E$.

From the stability analysis of the RSO, i.e., $\dot{\tilde{\boldsymbol{v}}}_r^E = -\boldsymbol{L}\boldsymbol{\vartheta}\tilde{\boldsymbol{v}}_r^E - \dot{\boldsymbol{v}}_r^E$, where $\boldsymbol{L}\boldsymbol{\vartheta}$ is a diagonal matrix, it can be obtained that

$$\begin{aligned}(\tilde{\boldsymbol{v}}_r^E)^\top \dot{\tilde{\boldsymbol{v}}}_r^E &= -\boldsymbol{L}\boldsymbol{\vartheta}(\tilde{\boldsymbol{v}}_r^E)^\top \tilde{\boldsymbol{v}}_r^E - (\tilde{\boldsymbol{v}}_r^E)^\top \dot{\boldsymbol{v}}_r^E \\ &\leq -\lambda_m(\boldsymbol{L}\boldsymbol{\vartheta})(\tilde{\boldsymbol{v}}_r^E)^\top \tilde{\boldsymbol{v}}_r^E + \frac{1}{4}(\tilde{\boldsymbol{v}}_r^E)^\top \tilde{\boldsymbol{v}}_r^E + \delta, \end{aligned} \tag{5a}$$

$$\tilde{\boldsymbol{d}}^\top \tilde{\boldsymbol{d}} = (\tilde{\boldsymbol{v}}_r^E)^\top \boldsymbol{\vartheta}^\top \boldsymbol{\vartheta} \tilde{\boldsymbol{v}}_r^E \leq \lambda_M(\boldsymbol{\vartheta}^\top \boldsymbol{\vartheta})(\tilde{\boldsymbol{v}}_r^E)^\top \tilde{\boldsymbol{v}}_r^E. \tag{5b}$$

Combining (3) - (5), (3) can be adjusted that

$$\begin{aligned}\dot{V} \leq &- \boldsymbol{e}^\top \boldsymbol{e} + \frac{1}{\varepsilon}\lambda_M^2\left(\boldsymbol{M}\boldsymbol{B}\right)\boldsymbol{e}^\top \boldsymbol{e} + \varepsilon \tilde{\boldsymbol{d}}^\top \tilde{\boldsymbol{d}} + (\tilde{\boldsymbol{v}}_r^E)^\top \dot{\tilde{\boldsymbol{v}}}_r^E \\ \leq &- \frac{\varepsilon - \lambda_M^2\left(\boldsymbol{M}\boldsymbol{B}\right)}{\varepsilon \lambda_M\left(\boldsymbol{M}\right)}\boldsymbol{e}^\top \boldsymbol{M}\boldsymbol{e} + \varepsilon\lambda_M(\boldsymbol{\vartheta}^\top \boldsymbol{\vartheta})(\tilde{\boldsymbol{v}}_r^E)^\top \tilde{\boldsymbol{v}}_r^E \\ &- \lambda_m(\boldsymbol{L}\boldsymbol{\vartheta})(\tilde{\boldsymbol{v}}_r^E)^\top \tilde{\boldsymbol{v}}_r^E + \frac{1}{4}(\tilde{\boldsymbol{v}}_r^E)^\top \tilde{\boldsymbol{v}}_r^E + \delta \\ \leq &- \sigma_1 \boldsymbol{e}^\top \boldsymbol{M}\boldsymbol{e} - \sigma_2(\tilde{\boldsymbol{v}}_r^E)^\top \tilde{\boldsymbol{v}}_r^E + \delta, \end{aligned} \tag{6}$$

where $\sigma_1 = \frac{\varepsilon - \lambda_m^2(\boldsymbol{M}\boldsymbol{B})}{\varepsilon\lambda_m(\boldsymbol{M})}$ and $\sigma_2 = \lambda_m(\boldsymbol{L}\boldsymbol{\vartheta}) - \varepsilon\lambda_M(\boldsymbol{\vartheta}^T\boldsymbol{\vartheta}) - \frac{1}{4}$. σ_1 can be ensured to be positive with $\varepsilon - \lambda_m^2\left(\boldsymbol{M}\boldsymbol{B}\right) > 0$. σ_2 can be guaranteed to be positive by adjusting $\boldsymbol{L}$ properly. We can obtain that $\dot{V} \leq -\sigma_1 V + \delta$ where $\gamma = \min\{\sigma_1, 2\sigma_2\}$, which implies that

$$0 \leq V(t) \leq e^{-\gamma t}\left[V(0) - \frac{\delta}{\gamma}\right] + \frac{\delta}{\gamma}. \tag{7}$$

IV. Conclusion

It can be concluded from (7) that all signals of the translational loop are globally uniformly bounded. According to (5b) and the boundedness of $V(t)$, we can find that the estimation error of RSO can converge to a small residual set related to δ.

References

[1] W. H. Young, "On classes of summable functions and their Fourier series," *Proc. Roy. Soc. Lond. Ser. A Math. Phys. Sci.*, vol. 87, no. 594, pp. 225–229, 1912, doi: 10.1098/rspa.1912.0076.