Stability Analysis of Observer-PD Controller

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I. PREFACE

This paper is a supplemented material serving for stability analysis of the combination of PD controller and rain speed observer (RSO), which is a innovatively designed observer introduced in manuscript submitted to IEEE Transactions on Industrial Electronics with paper ID 24-2548. It can also provide reference for the stability analysis of other similar studies.

II. PRELIMINARIES

$\lambda_M(*)$ and $\lambda_m(*)$ represent the maximum and minimum eigenvalues of a matrix, respectively. $\hat{*}$, $\dot{*}$ and $\tilde{*}$ denote the estimation, the first-order time derivative and estimation error of $*$, respectively.

By combining the mathematical expressions of the baseline controller and model dynamics, the relationships between states in the translational loop can be formulated as a state-space equation in the following form, facilitating the evaluation of stability using theoretical analysis tools.

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$$\underbrace{\begin{pmatrix} \dot{\boldsymbol{e}}_p \\ \dot{\boldsymbol{e}}_v \end{pmatrix}}_{\dot{\boldsymbol{e}}} = \underbrace{\begin{pmatrix} \mathbf{0}_{3\times3} & \boldsymbol{I}_{3\times3} \\ \frac{-\boldsymbol{K}_p}{m} & \frac{-\boldsymbol{K}_v}{m} \end{pmatrix}}_{\boldsymbol{A}} \underbrace{\begin{pmatrix} \boldsymbol{e}_p \\ \boldsymbol{e}_v \end{pmatrix}}_{\boldsymbol{e}} + \underbrace{\begin{pmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \frac{\boldsymbol{I}_{3\times3}}{m} \end{pmatrix}}_{\boldsymbol{B}} \underbrace{\begin{pmatrix} \mathbf{0}_{3\times1} \\ \tilde{\boldsymbol{F}}_r^E \end{pmatrix}}_{\tilde{\boldsymbol{d}}}, \tag{1}$$

where $\boldsymbol{e} = \left[\boldsymbol{e}_p^T, \boldsymbol{e}_v^T\right]^T$, $\boldsymbol{e}_p$ and $\boldsymbol{e}_v$ represent the tracking errors of position and velocity, respectively, $\tilde{\boldsymbol{F}}_r^E = \hat{\boldsymbol{F}}_r^E - \boldsymbol{F}_r^E = -\boldsymbol{\vartheta}\tilde{\boldsymbol{v}}_r^E$ represents the estimation error of the rain disturbance $\boldsymbol{F}_r^E$ from RSO and the expression is derived from the designed model of disturbance.

III. STABILITY ANALYSIS

It can be checked that $\boldsymbol{A}$ in (1) has negative definite structure if $\boldsymbol{K}_p$ and $\boldsymbol{K}_v$ are positive definite. As a consequence, there must exist a positive definite symmetric matrix $\boldsymbol{M}$ that meets the equation $\boldsymbol{A}^T\boldsymbol{M} + \boldsymbol{M}\boldsymbol{A} = -\boldsymbol{I}$. Subsequently, a Lyapunov function is designed as follows,

$$V = \boldsymbol{e}^T\boldsymbol{M}\boldsymbol{e} + \frac{1}{2}(\tilde{\boldsymbol{v}}_r^E)^T\tilde{\boldsymbol{v}}_r^E. \tag{2}$$

Differentiate V, it can be implied that

$$\begin{aligned} \dot{V} =& \dot{\boldsymbol{e}}^T\boldsymbol{M}\boldsymbol{e} + \boldsymbol{e}^T\boldsymbol{M}\dot{\boldsymbol{e}} + (\tilde{\boldsymbol{v}}_r^E)^T\dot{\tilde{\boldsymbol{v}}}_r^E \\ =& \boldsymbol{e}^T\boldsymbol{A}^T\boldsymbol{M}\boldsymbol{e} + \tilde{\boldsymbol{d}}^T\boldsymbol{B}^T\boldsymbol{M}\boldsymbol{e} + \boldsymbol{e}^T\boldsymbol{M}\boldsymbol{A}\boldsymbol{e} \\ &+ \boldsymbol{e}^T\boldsymbol{M}\boldsymbol{B}\tilde{\boldsymbol{d}} + (\tilde{\boldsymbol{v}}_r^E)^T\dot{\tilde{\boldsymbol{v}}}_r^E \\ =& \boldsymbol{e}^T\left(\boldsymbol{A}^T\boldsymbol{M} + \boldsymbol{M}\boldsymbol{A}\right)\boldsymbol{e} + 2\boldsymbol{e}^T\boldsymbol{M}\boldsymbol{B}\tilde{\boldsymbol{d}} + (\tilde{\boldsymbol{v}}_r^E)^T\dot{\tilde{\boldsymbol{v}}}_r^E \\ =& -\boldsymbol{e}^T\boldsymbol{e} + 2\boldsymbol{e}^T\boldsymbol{M}\boldsymbol{B}\tilde{\boldsymbol{d}} + (\tilde{\boldsymbol{v}}_r^E)^T\dot{\tilde{\boldsymbol{v}}}_r^E. \end{aligned} \tag{3}$$

Furthermore, by resorting to Young's inequality [1] , it can be verified that

$$-\boldsymbol{e}^\top\boldsymbol{e} \leq -\frac{\boldsymbol{e}^\top\boldsymbol{M}\boldsymbol{e}}{\lambda_M(\boldsymbol{M})}, \tag{4a}$$

$$-(\tilde{\boldsymbol{v}}_r^E)^\top\dot{\boldsymbol{v}}_r^E \leq \frac{1}{4}(\tilde{\boldsymbol{v}}_r^E)^\top\tilde{\boldsymbol{v}}_r^E + \delta, \tag{4b}$$

$$\boldsymbol{e}^\top\boldsymbol{M}\boldsymbol{B}\tilde{\boldsymbol{d}} \leq \frac{1}{2\varepsilon}\lambda_M^2(\boldsymbol{M}\boldsymbol{B})\,\boldsymbol{e}^\top\boldsymbol{e} + \frac{\varepsilon}{2}\tilde{\boldsymbol{d}}^\top\tilde{\boldsymbol{d}}, \tag{4c}$$

where ε is an arbitrary positive constant, and $\delta \leq (\dot{\boldsymbol{v}}_r^E)^T\dot{\boldsymbol{v}}_r^E$ is a constant value with an upper bound concerned with $\dot{\boldsymbol{v}}_r^E$.

From the stability analysis of the RSO, i.e., $\dot{\tilde{\boldsymbol{v}}}_r^E = -\boldsymbol{L}\boldsymbol{\vartheta}\tilde{\boldsymbol{v}}_r^E - \dot{\boldsymbol{v}}_r^E$, where $\boldsymbol{L}\boldsymbol{\vartheta}$ is a diagonal matrix, it can be obtained that

$$\begin{aligned}(\tilde{\boldsymbol{v}}_r^E)^\top \dot{\tilde{\boldsymbol{v}}}_r^E &= -\boldsymbol{L}\boldsymbol{\vartheta}(\tilde{\boldsymbol{v}}_r^E)^\top \tilde{\boldsymbol{v}}_r^E - (\tilde{\boldsymbol{v}}_r^E)^\top \dot{\boldsymbol{v}}_r^E \\ &\leq -\lambda_m(\boldsymbol{L}\boldsymbol{\vartheta})(\tilde{\boldsymbol{v}}_r^E)^\top \tilde{\boldsymbol{v}}_r^E + \frac{1}{4}(\tilde{\boldsymbol{v}}_r^E)^\top \tilde{\boldsymbol{v}}_r^E + \delta, \end{aligned} \tag{5a}$$

$$\tilde{\boldsymbol{d}}^\top \tilde{\boldsymbol{d}} = (\tilde{\boldsymbol{v}}_r^E)^\top \boldsymbol{\vartheta}^\top \boldsymbol{\vartheta} \tilde{\boldsymbol{v}}_r^E \leq \lambda_M(\boldsymbol{\vartheta}^\top \boldsymbol{\vartheta})(\tilde{\boldsymbol{v}}_r^E)^\top \tilde{\boldsymbol{v}}_r^E. \tag{5b}$$

Combining (3) - (5), (3) can be adjusted that

$$\begin{aligned}\dot{V} \leq &- \boldsymbol{e}^\top \boldsymbol{e} + \frac{1}{\varepsilon}\lambda_M^2\left(\boldsymbol{M}\boldsymbol{B}\right)\boldsymbol{e}^\top \boldsymbol{e} + \varepsilon \tilde{\boldsymbol{d}}^\top \tilde{\boldsymbol{d}} + (\tilde{\boldsymbol{v}}_r^E)^\top \dot{\tilde{\boldsymbol{v}}}_r^E \\ \leq &- \frac{\varepsilon - \lambda_M^2\left(\boldsymbol{M}\boldsymbol{B}\right)}{\varepsilon \lambda_M\left(\boldsymbol{M}\right)}\boldsymbol{e}^\top \boldsymbol{M}\boldsymbol{e} + \varepsilon\lambda_M(\boldsymbol{\vartheta}^\top \boldsymbol{\vartheta})(\tilde{\boldsymbol{v}}_r^E)^\top \tilde{\boldsymbol{v}}_r^E \\ &- \lambda_m(\boldsymbol{L}\boldsymbol{\vartheta})(\tilde{\boldsymbol{v}}_r^E)^\top \tilde{\boldsymbol{v}}_r^E + \frac{1}{4}(\tilde{\boldsymbol{v}}_r^E)^\top \tilde{\boldsymbol{v}}_r^E + \delta \\ \leq &- \sigma_1 \boldsymbol{e}^\top \boldsymbol{M}\boldsymbol{e} - \sigma_2(\tilde{\boldsymbol{v}}_r^E)^\top \tilde{\boldsymbol{v}}_r^E + \delta, \end{aligned} \tag{6}$$

where $\sigma_1 = \frac{\varepsilon - \lambda_M^2(\boldsymbol{M}\boldsymbol{B})}{\varepsilon\lambda_M(\boldsymbol{M})}$ and $\sigma_2 = \lambda_m(\boldsymbol{L}\boldsymbol{\vartheta}) - \varepsilon\lambda_M(\boldsymbol{\vartheta}^T\boldsymbol{\vartheta}) - \frac{1}{4}$. σ_1 can be ensured to be positive with $\varepsilon - \lambda_M^2\left(\boldsymbol{M}\boldsymbol{B}\right) > 0$. σ_2 can be guaranteed to be positive by adjusting $\boldsymbol{L}$ properly. We can obtain that $\dot{V} \leq -\gamma V + \delta$ where $\gamma = \min\{\sigma_1, 2\sigma_2\}$, which implies that

$$0 \leq V(t) \leq e^{-\gamma t}\left[V(0) - \frac{\delta}{\gamma}\right] + \frac{\delta}{\gamma}. \tag{7}$$

IV. CONCLUSION

It can be concluded from (7) that all signals of the translational loop are globally uniformly bounded. According to (5b) and the boundedness of $V(t)$, we can find that the estimation error of RSO can converge to a small residual set related to δ.

REFERENCES

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