Stability Analysis of Close-Loop Controller

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I. PREFACE

This paper is a supplemented material serving for stability analysis of the close-loop controller comprising a PD controller and a neural augmented disturbance observer (NADOO), which is a innovatively designed observer introduced in manuscript submitted to IEEE/ASME Transactions on Mechatronics with paper ID TMECH-09-2025-22078. It can also provide reference for the stability analysis of other similar studies.

II. PRELIMINARIES

The objective of the close-loop controller is to follow a desired trajectory $\boldsymbol{p}_d = [\begin{array}{ccc} x_d & y_d & z_d \end{array}]^\top$ and generate the desired control force $\boldsymbol{F}_d$. Tracking errors of position and velocity are represented as $\boldsymbol{e}_p = \boldsymbol{p}_d - \boldsymbol{p}$ and $\boldsymbol{e}_v = \boldsymbol{v}_d^E - \boldsymbol{v}$, respectively. The baseline controller is designed as

$$\begin{cases} \boldsymbol{a}_d = \boldsymbol{K}_p \boldsymbol{e}_p + \boldsymbol{K}_v \boldsymbol{e}_v - g\boldsymbol{e}_z + \ddot{\boldsymbol{p}}_d, \\ \boldsymbol{F}_d = m\boldsymbol{a}_d - \hat{\boldsymbol{d}}, \end{cases} \tag{1}$$

where $\boldsymbol{a}_d$ is the desired translational acceleration, $\boldsymbol{K}_p \in \mathbb{R}^{3\times3}$ and $\boldsymbol{K}_v \in \mathbb{R}^{3\times3}$ are gains of the translational controller with positive diagonal structure, $\ddot{\boldsymbol{p}}_d$ represents the desired acceleration which is the feedforward term, define $\hat{\boldsymbol{d}} = [\begin{array}{ccc} \hat{d}_x & \hat{d}_y & \hat{d}_z \end{array}]^\top \in \mathbb{R}^3$ as the disturbance estimation obtained from the NADO.

$\lambda_M(*)$ and $\lambda_m(*)$ represent the maximum and minimum eigenvalues of a matrix, respectively. $\hat{*}$, $\dot{*}$ and $\tilde{*}$ denote the estimation, the first-order time derivative and estimation error of $*$, respectively.

By combining the mathematical expressions of the baseline controller and model dynamics, the relationships between states in the translational loop can be formulated as a state-space equation in the following form, facilitating the evaluation of stability using theoretical analysis tools.

$$\underbrace{\begin{pmatrix} \dot{\boldsymbol{e}}_p \\ \dot{\boldsymbol{e}}_v \end{pmatrix}}_{\dot{\boldsymbol{e}}} = \underbrace{\begin{pmatrix} \boldsymbol{0}_{3\times3} & \boldsymbol{I}_{3\times3} \\ \frac{-\boldsymbol{K}_p}{m} & \frac{-\boldsymbol{K}_v}{m} \end{pmatrix}}_{\boldsymbol{A}} \underbrace{\begin{pmatrix} \boldsymbol{e}_p \\ \boldsymbol{e}_v \end{pmatrix}}_{\boldsymbol{e}} + \underbrace{\begin{pmatrix} \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \frac{\boldsymbol{I}_{3\times3}}{m} \end{pmatrix}}_{\boldsymbol{B}} \underbrace{\begin{pmatrix} \boldsymbol{0}_{3\times1} \\ \tilde{\boldsymbol{d}} \end{pmatrix}}_{\tilde{\boldsymbol{D}}}, \tag{2}$$

where $\boldsymbol{e} = \left[\boldsymbol{e}_p^T, \boldsymbol{e}_v^T\right]^T$, $\boldsymbol{e}_p$ and $\boldsymbol{e}_v$ represent the tracking errors of position and velocity, respectively, the estimation derivation of the NADO is defined as $\tilde{\boldsymbol{d}} = \hat{\boldsymbol{d}} - \boldsymbol{d}$, which is derived from a convergence mechanism:

$$\dot{\tilde{\boldsymbol{d}}} = -\boldsymbol{L}\tilde{\boldsymbol{d}} + \boldsymbol{\Delta}, \tag{3}$$

where $\boldsymbol{L}$ is the NADO parameter to be defined and is needed to be definite. Estimation derivation of the NN is represented as $\boldsymbol{\Delta}$, and $\|\boldsymbol{\Delta}\| \leq \bar{\Delta}$ is satisfied.

III. STABILITY ANALYSIS OF THE NADO

According to (3), rigorous analysis of the boundedness of $\tilde{\boldsymbol{d}}$ can be referenced by the properties of solutions to first-order linear differential equations, and it can be derived as

$$\tilde{\boldsymbol{d}}(t) = e^{-\boldsymbol{L}t}\tilde{\boldsymbol{d}}(0) + \int_0^t e^{-\boldsymbol{L}(t-\tau)}\boldsymbol{\Delta}(\tau)d\tau, \tag{4}$$

where $\tilde{\boldsymbol{d}}(0)$ represents the initial value of the estimation derivation $\tilde{\boldsymbol{d}}$.

Based on the properties of matrix exponents, for positive definite matrix $\boldsymbol{L}$, there exists $\|e^{-\boldsymbol{L}t}\| \leq e^{-\lambda_m(\boldsymbol{L})t}$ and $\|\boldsymbol{\Delta}\| \leq \bar{\Delta}$ is satisfied. Consequently, it can be obtained that

$$\begin{aligned} \|\tilde{\boldsymbol{d}}(t)\| &\leq e^{-\lambda_m(\boldsymbol{L})t}\|\tilde{\boldsymbol{d}}(0)\| + \bar{\Delta}\int_0^t e^{-\lambda_m(\boldsymbol{L})(t-\tau)}d\tau \\ &\leq e^{-\lambda_m(\boldsymbol{L})t}\|\tilde{\boldsymbol{d}}(0)\| + \frac{\bar{\Delta}}{\lambda_m(\boldsymbol{L})}(1 - e^{-\lambda_m(\boldsymbol{L})t}) \end{aligned} \tag{5}$$

It can be checked that $e^{-\boldsymbol{L}t}$ is an exponentially decaying function, since $\boldsymbol{L}$ is positive definite. Moreover, $e^{-\boldsymbol{L}t}$ converges to $\boldsymbol{0}$ as time $t \to +\infty$, indicating that $\tilde{\boldsymbol{d}}$ is bounded with a upper bound of stable state, i.e., $\frac{\bar{\Delta}}{\lambda_m(\boldsymbol{L})}$. While $\boldsymbol{L}$ can be adjusted to possess a large minimum eigenvalue to effectively mitigate the impact of the NN estimation derivation $\boldsymbol{\Delta}$ and improve estimation accuracy of the NADO, the gain must not be excessive, otherwise inherently amplifying measurement noise presented in the NADO input signals.

IV. STABILITY ANALYSIS OF THE CLOSE-LOOP CONTROLLER

It can be checked that $\boldsymbol{A}$ in (2) is a Hurwitz matrix if $\boldsymbol{K}_p$ and $\boldsymbol{K}_v$ are positive definite. As a consequence, there must exist a positive definite symmetric matrix $\boldsymbol{P}$ that meets the equation $\boldsymbol{A}^T\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A}=-\boldsymbol{I}$. Subsequently, a Lyapunov function is designed as follows,

$$V=\underbrace{\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{e}}_{V_e}+\sigma\underbrace{\tilde{\boldsymbol{d}}^T\tilde{\boldsymbol{d}}}_{V_d}, \tag{6}$$

where $\sigma>0$ is a weight coefficient for balancing the convergence rate of the two subsystems, i.e., V_e and V_d. According to (2) and (3), derivative of V can be obtained as

$$\begin{aligned}\dot{V}&=\left(\dot{\boldsymbol{e}}^T\boldsymbol{P}\boldsymbol{e}+\boldsymbol{e}^T\boldsymbol{P}\dot{\boldsymbol{e}}\right)+\sigma\left(\dot{\tilde{\boldsymbol{d}}}^T\tilde{\boldsymbol{d}}+\tilde{\boldsymbol{d}}^T\dot{\tilde{\boldsymbol{d}}}\right)\\&=\boldsymbol{e}^T(\boldsymbol{A}^T\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A})\boldsymbol{e}+2\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{B}\tilde{\boldsymbol{D}}+\sigma\left[-\tilde{\boldsymbol{d}}^T(\boldsymbol{L}+\boldsymbol{L}^T)\tilde{\boldsymbol{d}}+2\tilde{\boldsymbol{d}}^T\boldsymbol{\Delta}\right]\\&=-\|\boldsymbol{e}\|^2+2\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{B}\tilde{\boldsymbol{D}}++\sigma\left[-\tilde{\boldsymbol{d}}^T(\boldsymbol{L}+\boldsymbol{L}^T)\tilde{\boldsymbol{d}}+2\tilde{\boldsymbol{d}}^T\boldsymbol{\Delta}\right]\end{aligned} \tag{7}$$

Furthermore, by resorting to the Young's inequality [1], it can be verified that

$$2\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{B}\tilde{\boldsymbol{D}}\leq\varepsilon\|\boldsymbol{e}\|^2+\frac{\|\boldsymbol{P}\|^2}{m^2\varepsilon}\|\tilde{\boldsymbol{d}}\|^2, \tag{8}$$

where ε is an arbitrary positive constant.

Scaling based on inequalities, it can be obtained that

$$-\|\boldsymbol{e}\|^2\leq-\frac{\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{e}}{\lambda_M(\boldsymbol{P})}, \tag{9a}$$

$$-\tilde{\boldsymbol{d}}^T(\boldsymbol{L}+\boldsymbol{L}^T)\tilde{\boldsymbol{d}}\leq-2\lambda_m(\boldsymbol{L})\|\tilde{\boldsymbol{d}}\|^2, \tag{9b}$$

$$\tilde{\boldsymbol{d}}^T\boldsymbol{\Delta}\leq\|\tilde{\boldsymbol{d}}\|\bar{\Delta}, \tag{9c}$$

where $\|\tilde{\boldsymbol{d}}\|\bar{\Delta}$ can be further treated according to the Young's inequality [1].

$$\|\tilde{\boldsymbol{d}}\|\bar{\Delta}\leq\frac{\lambda_m(\boldsymbol{L})}{2}\|\tilde{\boldsymbol{d}}\|^2+\frac{\bar{\Delta}^2}{2\lambda_m(\boldsymbol{L})}, \tag{10}$$

where $\lambda_m(\boldsymbol{L})$ is chosen as the constant factor for ease of the latter expression simplification, specially.

Based on (8), (9) and (10), the $\dot{V}$ in (7) can be derived as

$$\begin{aligned}\dot{V}&=-\|\boldsymbol{e}\|^2+2\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{B}\tilde{\boldsymbol{D}}+\sigma\left[-\tilde{\boldsymbol{d}}^T(\boldsymbol{L}+\boldsymbol{L}^T)\tilde{\boldsymbol{d}}+2\tilde{\boldsymbol{d}}^T\boldsymbol{\Delta}\right]\\&\leq\left[-(1-\varepsilon)\|\boldsymbol{e}\|^2+\frac{\|\boldsymbol{P}\|^2}{m^2\varepsilon}\|\tilde{\boldsymbol{d}}\|^2\right]+\left[-\sigma\lambda_m(\boldsymbol{L})\|\tilde{\boldsymbol{d}}\|^2+\frac{\sigma\bar{\Delta}^2}{\lambda_m(\boldsymbol{L})}\right]\\&\leq-(1-\varepsilon)\frac{\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{e}}{\lambda_M(\boldsymbol{P})}-\left(\sigma\lambda_m(\boldsymbol{L})-\frac{\|\boldsymbol{P}\|^2}{m^2\varepsilon}\right)\|\tilde{\boldsymbol{d}}\|^2+\frac{\sigma\bar{\Delta}^2}{\lambda_m(\boldsymbol{L})}\\&\leq-\delta_1\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{e}-\delta_2\sigma\|\tilde{\boldsymbol{d}}\|^2+\delta_3\end{aligned} \tag{11}$$

where $\delta_1 = \frac{1-\varepsilon}{\lambda_M(\boldsymbol{P})}$, $\delta_2 = \left(\lambda_m(\boldsymbol{L}) - \frac{\|\boldsymbol{P}\|^2}{m^2\sigma\varepsilon}\right)$ and $\delta_3 = \frac{\sigma\bar{\Delta}^2}{\lambda_m(\boldsymbol{L})}$.

According to (11), the stability condition of the close-loop system is $\delta_1, \delta_2, \delta_3 > 0$. The first coefficient δ_1 can be ensured to be positive with $\varepsilon < 1$. Coefficient δ_2 can be guaranteed to be positive by adjusting $\boldsymbol{L}$ properly. Coefficient δ_3 is positive since the observer gain $\boldsymbol{L}$ is positive and definite.

Further, overall convergence rate of the close-loop system can be obtained that $\mu = \min(\delta_1, \delta_2)$, and it can be obtained that $\dot{V} \leq -\mu V + \delta_3$, which implies the time domain range of V is

$$0 \leq V(t) \leq e^{-\mu t}V(0) - \frac{\delta_3}{\mu}(1 - e^{-\mu t}). \tag{12}$$

According to (12), the upper bound of V (i.e., $\bar{V}$) as $t \to \infty$ can be obtained as $\bar{V} = \frac{\delta_3}{\mu}$.

V. Conclusion

It can be concluded from (12) that all signals of the close-loop controller are globally uniformly bounded. Besides, it can be found that the estimation derivation of NADO can converge to a small residual set related to δ_3.

References

[1] W. H. Young, "On classes of summable functions and their Fourier series," *Proc. Roy. Soc. Lond. Ser. A Math. Phys. Sci.*, vol. 87, no. 594, pp. 225–229, 1912, doi: 10.1098/rspa.1912.0076.