Stability Analysis of NADO Based Closed-Loop Controller

Yuhang Liu, Jindou Jia, Zihan Yang, Qianyuan Liu, Kexin Guo*, and Xiang Yu

I. PREFACE

This paper is a supplemented material serving for stability analysis of the close-loop controller comprising a PD controller and a neural augmented disturbance observer (NADOO), which is a innovatively designed observer introduced in manuscript submitted to IEEE/ASME Transactions on Mechatronics with paper ID TMECH-09-2025-22078. It can also provide reference for the stability analysis of other similar studies.

II. PRELIMINARIES

The translational kinematic and dynamic of the drone is modeled as

$$\begin{cases} \dot{\boldsymbol{p}} = \boldsymbol{v}, \\ m\boldsymbol{a} = \boldsymbol{F} + \boldsymbol{G} + \boldsymbol{d}, \end{cases} \tag{1}$$

where $\boldsymbol{p} \in \mathbb{R}^3$, $\boldsymbol{v} = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}^\top \in \mathbb{R}^3$, and $\boldsymbol{a} \in \mathbb{R}^3$ represent the position, velocity, acceleration of the drone in $\mathcal{E}$, respectively. The thrust of the drone in $\mathcal{E}$ is defined as $\boldsymbol{F} = -f\boldsymbol{b}_z$ where f is the total lift of all rotors. The gravity is defined as $\boldsymbol{G} = mg\boldsymbol{e}_z$, where m is mass of the drone and g is the gravitational acceleration. $\boldsymbol{d}$ represents the disturbance encountered by the drone. According to Lemma 1 in the manuscript, the estimation deviation of the NN is assumed to be bounded, i.e., $\|\boldsymbol{\Delta}\| \leq \bar{\Delta}$, where $\bar{\Delta}$ is a constant.

The objective of the close-loop controller is to follow a desired trajectory $\boldsymbol{p}_d = \begin{bmatrix} x_d & y_d & z_d \end{bmatrix}^\top$ and generate the desired control force $\boldsymbol{F}_d$. Tracking errors of position and velocity are represented as $\boldsymbol{e}_p = \boldsymbol{p}_d - \boldsymbol{p}$ and $\boldsymbol{e}_v = \boldsymbol{v}_d - \boldsymbol{v}$, respectively. The baseline controller is designed as

$$\begin{cases} \boldsymbol{a}_d = \boldsymbol{K}_p\boldsymbol{e}_p + \boldsymbol{K}_v\boldsymbol{e}_v - g\boldsymbol{e}_z + \ddot{\boldsymbol{p}}_d, \\ \boldsymbol{F}_d = m\boldsymbol{a}_d - \hat{\boldsymbol{d}}, \end{cases} \tag{2}$$

where $\boldsymbol{a}_d$ is the desired translational acceleration, $\boldsymbol{K}_p \in \mathbb{R}^{3\times3}$ and $\boldsymbol{K}_v \in \mathbb{R}^{3\times3}$ are gains of the translational controller with positive diagonal structure, $\ddot{\boldsymbol{p}}_d$ represents the desired acceleration which is the feedforward term, define $\hat{\boldsymbol{d}} = [\ \hat{d}_x \quad \hat{d}_y \quad \hat{d}_z\ ]^\top \in \mathbb{R}^3$ as the disturbance estimation obtained from the NADO.

In the subsequent analysis process, $\lambda_M(*)$ and $\lambda_m(*)$ represent the maximum and minimum eigenvalues of a matrix, respectively. $\hat{*}$, $\dot{*}$ and $\tilde{*}$ denote the estimation, the first-order time derivative and estimation error of $*$, respectively.

By combining the mathematical expressions of the baseline controller and model dynamics, the relationships between states in the translational loop can be formulated as a state-space equation in the following form, facilitating the evaluation of stability using theoretical analysis tools.

$$\underbrace{\begin{pmatrix} \dot{\boldsymbol{e}}_p \\ \dot{\boldsymbol{e}}_v \end{pmatrix}}_{\dot{\boldsymbol{e}}} = \underbrace{\begin{pmatrix} \boldsymbol{0}_{3\times3} & \boldsymbol{I}_{3\times3} \\ \frac{-\boldsymbol{K}_p}{m} & \frac{-\boldsymbol{K}_v}{m} \end{pmatrix}}_{\boldsymbol{A}} \underbrace{\begin{pmatrix} \boldsymbol{e}_p \\ \boldsymbol{e}_v \end{pmatrix}}_{\boldsymbol{e}} + \underbrace{\begin{pmatrix} \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \frac{\boldsymbol{I}_{3\times3}}{m} \end{pmatrix}}_{\boldsymbol{B}} \underbrace{\begin{pmatrix} \boldsymbol{0}_{3\times1} \\ \tilde{\boldsymbol{d}} \end{pmatrix}}_{\tilde{\boldsymbol{D}}}, \tag{3}$$

where $\boldsymbol{e} = \left[\boldsymbol{e}_p^T, \boldsymbol{e}_v^T\right]^T$, $\boldsymbol{e}_p$ and $\boldsymbol{e}_v$ represent the tracking errors of position and velocity, respectively, the estimation deviation of the NADO is defined as $\tilde{\boldsymbol{d}} = \hat{\boldsymbol{d}} - \boldsymbol{d}$, which is derived from a convergence mechanism:

$$\dot{\tilde{\boldsymbol{d}}} = -\boldsymbol{L}\tilde{\boldsymbol{d}} + \boldsymbol{\Delta}, \tag{4}$$

where $\boldsymbol{L}$ is the NADO parameter to be defined and is needed to be definite.

III. STABILITY ANALYSIS OF THE NADO

According to (4), rigorous analysis of the boundedness of $\tilde{\boldsymbol{d}}$ can be referenced by the properties of solutions to first-order linear differential equations, and it can be derived as

$$\tilde{\boldsymbol{d}}(t) = e^{-\boldsymbol{L}t}\tilde{\boldsymbol{d}}(0) + \int_0^t e^{-\boldsymbol{L}(t-\tau)}\boldsymbol{\Delta}(\tau)d\tau, \tag{5}$$

where $\tilde{\boldsymbol{d}}(0)$ represents the initial value of the estimation deviation $\tilde{\boldsymbol{d}}$.

Based on the properties of matrix exponents, for positive definite matrix $\boldsymbol{L}$, there exists $\|e^{-\boldsymbol{L}t}\| \le e^{-\lambda_m(\boldsymbol{L})t}$ and $\|\boldsymbol{\Delta}\| \le \bar{\Delta}$ is satisfied. Consequently, it can be obtained that

$$\begin{aligned} \|\tilde{\boldsymbol{d}}(t)\| &\le e^{-\lambda_m(\boldsymbol{L})t}\|\tilde{\boldsymbol{d}}(0)\| + \bar{\Delta}\int_0^t e^{-\lambda_m(\boldsymbol{L})(t-\tau)}d\tau \\ &\le e^{-\lambda_m(\boldsymbol{L})t}\|\tilde{\boldsymbol{d}}(0)\| + \frac{\bar{\Delta}}{\lambda_m(\boldsymbol{L})}(1 - e^{-\lambda_m(\boldsymbol{L})t}). \end{aligned} \tag{6}$$

It can be checked that $e^{-\boldsymbol{L}t}$ is an exponentially decaying function, since $\boldsymbol{L}$ is positive definite. Moreover, $e^{-\boldsymbol{L}t}$ converges to $\mathbf{0}$ as time $t \to +\infty$, indicating that $\tilde{\boldsymbol{d}}$ is bounded with a upper bound of stable state, i.e., $\frac{\bar{\Delta}}{\lambda_m(\boldsymbol{L})}$. While $\boldsymbol{L}$ can be adjusted to possess a large minimum eigenvalue to effectively mitigate the impact of the NN estimation deviation $\boldsymbol{\Delta}$ and improve estimation accuracy of the NADO, the gain must not be excessive, otherwise inherently amplifying measurement noise presented in input signals of the NADO.

IV. STABILITY ANALYSIS OF THE NADO BASED CLOSE-LOOP CONTROLLER

It can be checked that $\boldsymbol{A}$ in (3) is a Hurwitz matrix if $\boldsymbol{K}_p$ and $\boldsymbol{K}_v$ are positive definite. As a consequence, there must exist a positive definite symmetric matrix $\boldsymbol{P}$ that meets the equation $\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{I}$. Subsequently, a Lyapunov function is designed as follows,

$$V = \underbrace{\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{e}}_{V_e} + \sigma \underbrace{\tilde{\boldsymbol{d}}^T\tilde{\boldsymbol{d}}}_{V_d}, \tag{7}$$

where $\sigma > 0$ is a weight coefficient for balancing the convergence rate of the two subsystems, i.e., V_e and V_d. According to (3) and (4), derivative of V can be obtained as

$$\begin{aligned}\dot{V} &= \left(\dot{\boldsymbol{e}}^T\boldsymbol{P}\boldsymbol{e} + \boldsymbol{e}^T\boldsymbol{P}\dot{\boldsymbol{e}}\right) + \sigma\left(\dot{\tilde{\boldsymbol{d}}}^T\tilde{\boldsymbol{d}} + \tilde{\boldsymbol{d}}^T\dot{\tilde{\boldsymbol{d}}}\right) \\ &= \boldsymbol{e}^T(\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A})\boldsymbol{e} + 2\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{B}\tilde{\boldsymbol{D}} + \sigma\left[-\tilde{\boldsymbol{d}}^T(\boldsymbol{L} + \boldsymbol{L}^T)\tilde{\boldsymbol{d}} + 2\tilde{\boldsymbol{d}}^T\boldsymbol{\Delta}\right] \\ &= -\|\boldsymbol{e}\|^2 + 2\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{B}\tilde{\boldsymbol{D}} + \sigma\left[-\tilde{\boldsymbol{d}}^T(\boldsymbol{L} + \boldsymbol{L}^T)\tilde{\boldsymbol{d}} + 2\tilde{\boldsymbol{d}}^T\boldsymbol{\Delta}\right]\end{aligned} \tag{8}$$

Furthermore, by resorting to the Young's inequality [1], it can be verified that

$$2\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{B}\tilde{\boldsymbol{D}} \leq \varepsilon\|\boldsymbol{e}\|^2 + \frac{\|\boldsymbol{P}\|^2}{m^2\varepsilon}\|\tilde{\boldsymbol{d}}\|^2, \tag{9}$$

where ε is an arbitrary positive constant.

Scaling based on inequalities, it can be obtained that

$$-\|\boldsymbol{e}\|^2 \leq -\frac{\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{e}}{\lambda_M(\boldsymbol{P})}, \tag{10a}$$

$$-\tilde{\boldsymbol{d}}^T(\boldsymbol{L} + \boldsymbol{L}^T)\tilde{\boldsymbol{d}} \leq -2\lambda_m(\boldsymbol{L})\|\tilde{\boldsymbol{d}}\|^2, \tag{10b}$$

$$\tilde{\boldsymbol{d}}^T\boldsymbol{\Delta} \leq \|\tilde{\boldsymbol{d}}\|\bar{\Delta}, \tag{10c}$$

where $\|\tilde{\boldsymbol{d}}\|\bar{\Delta}$ can be further treated according to the Young's inequality [1].

$$\|\tilde{\boldsymbol{d}}\|\bar{\Delta} \leq \frac{\lambda_m(\boldsymbol{L})}{2}\|\tilde{\boldsymbol{d}}\|^2 + \frac{\bar{\Delta}^2}{2\lambda_m(\boldsymbol{L})}, \tag{11}$$

where $\lambda_m(\boldsymbol{L})$ is chosen as the constant factor for ease of the latter expression simplification, specially.

Based on (9), (10) and (11), the $\dot{V}$ in (8) can be derived as

$$\begin{aligned}
\dot{V} &= -\|\boldsymbol{e}\|^2 + 2\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{B}\tilde{\boldsymbol{D}} + \sigma\left[-\tilde{\boldsymbol{d}}^T(\boldsymbol{L}+\boldsymbol{L}^T)\tilde{\boldsymbol{d}} + 2\tilde{\boldsymbol{d}}^T\boldsymbol{\Delta}\right] \\
&\leq \left[-(1-\varepsilon)\|\boldsymbol{e}\|^2 + \frac{\|\boldsymbol{P}\|^2}{m^2\varepsilon}\|\tilde{\boldsymbol{d}}\|^2\right] + \left[-\sigma\lambda_m(\boldsymbol{L})\|\tilde{\boldsymbol{d}}\|^2 + \frac{\sigma\bar{\Delta}^2}{\lambda_m(\boldsymbol{L})}\right] \\
&\leq -(1-\varepsilon)\frac{\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{e}}{\lambda_M(\boldsymbol{P})} - \left(\sigma\lambda_m(\boldsymbol{L}) - \frac{\|\boldsymbol{P}\|^2}{m^2\varepsilon}\right)\|\tilde{\boldsymbol{d}}\|^2 + \frac{\sigma\bar{\Delta}^2}{\lambda_m(\boldsymbol{L})} \\
&\leq -\delta_1\boldsymbol{e}^T\boldsymbol{P}\boldsymbol{e} - \delta_2\sigma\|\tilde{\boldsymbol{d}}\|^2 + \delta_3
\end{aligned} \tag{12}$$

where $\delta_1 = \frac{1-\varepsilon}{\lambda_M(\boldsymbol{P})}$, $\delta_2 = \left(\lambda_m(\boldsymbol{L}) - \frac{\|\boldsymbol{P}\|^2}{m^2\sigma\varepsilon}\right)$ and $\delta_3 = \frac{\sigma\bar{\Delta}^2}{\lambda_m(\boldsymbol{L})}$.

According to (12), the stability condition of the close-loop system is $\delta_1, \delta_2, \delta_3 > 0$. The first coefficient δ_1 can be ensured to be positive with $\varepsilon < 1$. Coefficient δ_2 can be guaranteed to be positive by adjusting $\boldsymbol{L}$ properly. Coefficient δ_3 is obviously positive since the observer gain $\boldsymbol{L}$ is positive and definite.

Further, overall convergence rate of the close-loop system can be obtained that $\mu = \min(\delta_1, \delta_2)$, and it can be obtained that $\dot{V} \leq -\mu V + \delta_3$, which implies the time domain range of V is

$$0 \leq V(t) \leq e^{-\mu t}V(0) + \frac{\delta_3}{\mu}(1 - e^{-\mu t}). \tag{13}$$

According to (13), the upper bound of V (i.e., $\bar{V}$) as $t \to +\infty$ can be obtained as $\bar{V} = \frac{\delta_3}{\mu}$.

V. Conclusion

It can be concluded from (13) that signals of the close-loop controller are globally uniformly bounded. Besides, it can be found that the estimation deviation of NADO can converge to a small residual set related to δ_3 and the impact of NN estimation deviation can be mitigated by adjusting $\boldsymbol{L}$ properly.

References

[1] W. H. Young, "On classes of summable functions and their Fourier series," *Proc. Roy. Soc. Lond. Ser. A Math. Phys. Sci.*, vol. 87, no. 594, pp. 225–229, 1912, doi: 10.1098/rspa.1912.0076.