

# Stability Analysis of Close-Loop Controller

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## I. PREFACE

This paper is a supplemented material serving for stability analysis of the close-loop controller comprising a PD controller and a neural augmented disturbance observer (NADOO), which is a innovatively designed observer introduced in manuscript submitted to IEEE/ASME Transactions on Mechatronics with paper ID TMECH-09-2025-22078. It can also provide reference for the stability analysis of other similar studies.

## II. PRELIMINARIES

The translational kinematic and dynamic of the drone is modeled as

$$\begin{cases} \dot{\mathbf{p}} = \mathbf{v}, \\ m\mathbf{a} = \mathbf{F} + \mathbf{G} + \mathbf{d}, \end{cases} \quad (1)$$

where  $\mathbf{p} \in \mathbb{R}^3$ ,  $\mathbf{v} = [v_x \ v_y \ v_z]^\top \in \mathbb{R}^3$ , and  $\mathbf{a} \in \mathbb{R}^3$  represent the position, velocity, acceleration of the drone in  $\mathcal{E}$ , respectively. The thrust of the drone in  $\mathcal{E}$  is defined as  $\mathbf{F} = -f\mathbf{b}_z$  where  $f$  is the total lift of all rotors. The gravity is defined as  $\mathbf{G} = mge_z$ , where  $m$  is mass of the drone and  $g$  is the gravitational acceleration.  $\mathbf{d}$  represents the disturbance encountered by the drone. According to Lemma 1 in the manuscript, the estimation deviation of the NN is assumed to be bounded, i.e.,  $\|\Delta\| \leq \bar{\Delta}$ , where  $\bar{\Delta}$  is a constant.

The objective of the close-loop controller is to follow a desired trajectory  $\mathbf{p}_d = [x_d \ y_d \ z_d]^\top$  and generate the desired control force  $\mathbf{F}_d$ . Tracking errors of position and velocity are represented as  $\mathbf{e}_p = \mathbf{p}_d - \mathbf{p}$  and  $\mathbf{e}_v = \mathbf{v}_d - \mathbf{v}$ , respectively. The baseline controller is designed as

$$\begin{cases} \mathbf{a}_d = \mathbf{K}_p \mathbf{e}_p + \mathbf{K}_v \mathbf{e}_v - ge_z + \ddot{\mathbf{p}}_d, \\ \mathbf{F}_d = m\mathbf{a}_d - \hat{\mathbf{d}}, \end{cases} \quad (2)$$

where  $\mathbf{a}_d$  is the desired translational acceleration,  $\mathbf{K}_p \in \mathbb{R}^{3 \times 3}$  and  $\mathbf{K}_v \in \mathbb{R}^{3 \times 3}$  are gains of the translational controller with positive diagonal structure,  $\ddot{\mathbf{p}}_d$  represents the desired acceleration

which is the feedforward term, define  $\hat{\mathbf{d}} = [\hat{d}_x \quad \hat{d}_y \quad \hat{d}_z]^\top \in \mathbb{R}^3$  as the disturbance estimation obtained from the NADO.

In the subsequent analysis process,  $\lambda_M(*)$  and  $\lambda_m(*)$  represent the maximum and minimum eigenvalues of a matrix, respectively.  $\hat{*}$ ,  $\dot{*}$  and  $\ddot{*}$  denote the estimation, the first-order time derivative and estimation error of  $*$ , respectively.

By combining the mathematical expressions of the baseline controller and model dynamics, the relationships between states in the translational loop can be formulated as a state-space equation in the following form, facilitating the evaluation of stability using theoretical analysis tools.

$$\underbrace{\begin{pmatrix} \dot{\mathbf{e}}_p \\ \dot{\mathbf{e}}_v \end{pmatrix}}_{\dot{\mathbf{e}}} = \underbrace{\begin{pmatrix} \mathbf{0}_{3 \times 3} & \mathbf{I}_{3 \times 3} \\ \frac{-\mathbf{K}_p}{m} & \frac{-\mathbf{K}_v}{m} \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} \mathbf{e}_p \\ \mathbf{e}_v \end{pmatrix}}_{\mathbf{e}} + \underbrace{\begin{pmatrix} \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \frac{\mathbf{I}_{3 \times 3}}{m} \end{pmatrix}}_{\mathbf{B}} \underbrace{\begin{pmatrix} \mathbf{0}_{3 \times 1} \\ \tilde{\mathbf{d}} \end{pmatrix}}_{\tilde{\mathbf{D}}}, \quad (3)$$

where  $\mathbf{e} = [\mathbf{e}_p^T, \mathbf{e}_v^T]^T$ ,  $\mathbf{e}_p$  and  $\mathbf{e}_v$  represent the tracking errors of position and velocity, respectively, the estimation derivation of the NADO is defined as  $\tilde{\mathbf{d}} = \hat{\mathbf{d}} - \mathbf{d}$ , which is derived from a convergence mechanism:

$$\dot{\tilde{\mathbf{d}}} = -\mathbf{L}\tilde{\mathbf{d}} + \Delta, \quad (4)$$

where  $\mathbf{L}$  is the NADO parameter to be defined and is needed to be definite.

### III. STABILITY ANALYSIS OF THE NADO

According to (4), rigorous analysis of the boundedness of  $\tilde{\mathbf{d}}$  can be referenced by the properties of solutions to first-order linear differential equations, and it can be derived as

$$\tilde{\mathbf{d}}(t) = e^{-\mathbf{L}t}\tilde{\mathbf{d}}(0) + \int_0^t e^{-\mathbf{L}(t-\tau)}\Delta(\tau)d\tau, \quad (5)$$

where  $\tilde{\mathbf{d}}(0)$  represents the initial value of the estimation derivation  $\tilde{\mathbf{d}}$ .

Based on the properties of matrix exponents, for positive definite matrix  $\mathbf{L}$ , there exists  $\|e^{-\mathbf{L}t}\| \leq e^{-\lambda_m(\mathbf{L})t}$  and  $\|\Delta\| \leq \bar{\Delta}$  is satisfied. Consequently, it can be obtained that

$$\begin{aligned} \|\tilde{\mathbf{d}}(t)\| &\leq e^{-\lambda_m(\mathbf{L})t}\|\tilde{\mathbf{d}}(0)\| + \bar{\Delta} \int_0^t e^{-\lambda_m(\mathbf{L})(t-\tau)}d\tau \\ &\leq e^{-\lambda_m(\mathbf{L})t}\|\tilde{\mathbf{d}}(0)\| + \frac{\bar{\Delta}}{\lambda_m(\mathbf{L})}(1 - e^{-\lambda_m(\mathbf{L})t}). \end{aligned} \quad (6)$$

It can be checked that  $e^{-\mathbf{L}t}$  is an exponentially decaying function, since  $\mathbf{L}$  is positive definite. Moreover,  $e^{-\mathbf{L}t}$  converges to  $\mathbf{0}$  as time  $t \rightarrow +\infty$ , indicating that  $\tilde{\mathbf{d}}$  is bounded with a upper bound of stable state, i.e.,  $\frac{\bar{\Delta}}{\lambda_m(\mathbf{L})}$ . While  $\mathbf{L}$  can be adjusted to possess a large minimum eigenvalue to effectively mitigate the impact of the NN estimation derivation  $\Delta$  and improve estimation accuracy of the NADO, the gain must not be excessive, otherwise inherently amplifying measurement noise presented in input signals of the NADO.

#### IV. STABILITY ANALYSIS OF THE NADO BASED CLOSE-LOOP CONTROLLER

It can be checked that  $\mathbf{A}$  in (3) is a Hurwitz matrix if  $\mathbf{K}_p$  and  $\mathbf{K}_v$  are positive definite. As a consequence, there must exist a positive definite symmetric matrix  $\mathbf{P}$  that meets the equation  $\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A} = -\mathbf{I}$ . Subsequently, a Lyapunov function is designed as follows,

$$V = \underbrace{\mathbf{e}^T \mathbf{P} \mathbf{e}}_{V_e} + \sigma \underbrace{\tilde{\mathbf{d}}^T \tilde{\mathbf{d}}}_{V_d}, \quad (7)$$

where  $\sigma > 0$  is a weight coefficient for balancing the convergence rate of the two subsystems, i.e.,  $V_e$  and  $V_d$ . According to (3) and (4), derivative of  $V$  can be obtained as

$$\begin{aligned} \dot{V} &= (\dot{\mathbf{e}}^T \mathbf{P} \mathbf{e} + \mathbf{e}^T \mathbf{P} \dot{\mathbf{e}}) + \sigma (\dot{\tilde{\mathbf{d}}}^T \tilde{\mathbf{d}} + \tilde{\mathbf{d}}^T \dot{\tilde{\mathbf{d}}}) \\ &= \mathbf{e}^T (\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A}) \mathbf{e} + 2\mathbf{e}^T \mathbf{P} \mathbf{B} \tilde{\mathbf{D}} + \sigma \left[ -\tilde{\mathbf{d}}^T (\mathbf{L} + \mathbf{L}^T) \tilde{\mathbf{d}} + 2\tilde{\mathbf{d}}^T \Delta \right] \\ &= -\|\mathbf{e}\|^2 + 2\mathbf{e}^T \mathbf{P} \mathbf{B} \tilde{\mathbf{D}} + \sigma \left[ -\tilde{\mathbf{d}}^T (\mathbf{L} + \mathbf{L}^T) \tilde{\mathbf{d}} + 2\tilde{\mathbf{d}}^T \Delta \right] \end{aligned} \quad (8)$$

Furthermore, by resorting to the Young's inequality [1], it can be verified that

$$2\mathbf{e}^T \mathbf{P} \mathbf{B} \tilde{\mathbf{D}} \leq \varepsilon \|\mathbf{e}\|^2 + \frac{\|\mathbf{P}\|^2}{m^2 \varepsilon} \|\tilde{\mathbf{d}}\|^2, \quad (9)$$

where  $\varepsilon$  is an arbitrary positive constant.

Scaling based on inequalities, it can be obtained that

$$-\|\mathbf{e}\|^2 \leq -\frac{\mathbf{e}^T \mathbf{P} \mathbf{e}}{\lambda_M(\mathbf{P})}, \quad (10a)$$

$$-\tilde{\mathbf{d}}^T (\mathbf{L} + \mathbf{L}^T) \tilde{\mathbf{d}} \leq -2\lambda_m(\mathbf{L}) \|\tilde{\mathbf{d}}\|^2, \quad (10b)$$

$$\tilde{\mathbf{d}}^T \Delta \leq \|\tilde{\mathbf{d}}\| \bar{\Delta}, \quad (10c)$$

where  $\|\tilde{\mathbf{d}}\| \bar{\Delta}$  can be further treated according to the Young's inequality [1].

$$\|\tilde{\mathbf{d}}\| \bar{\Delta} \leq \frac{\lambda_m(\mathbf{L})}{2} \|\tilde{\mathbf{d}}\|^2 + \frac{\bar{\Delta}^2}{2\lambda_m(\mathbf{L})}, \quad (11)$$

where  $\lambda_m(\mathbf{L})$  is chosen as the constant factor for ease of the latter expression simplification, specially.

Based on (9), (10) and (11), the  $\dot{V}$  in (8) can be derived as

$$\begin{aligned}
\dot{V} &= -\|\mathbf{e}\|^2 + 2\mathbf{e}^T \mathbf{P} \mathbf{B} \tilde{\mathbf{d}} + \sigma \left[ -\tilde{\mathbf{d}}^T (\mathbf{L} + \mathbf{L}^T) \tilde{\mathbf{d}} + 2\tilde{\mathbf{d}}^T \Delta \right] \\
&\leq \left[ -(1-\varepsilon)\|\mathbf{e}\|^2 + \frac{\|\mathbf{P}\|^2}{m^2\varepsilon} \|\tilde{\mathbf{d}}\|^2 \right] + \left[ -\sigma\lambda_m(\mathbf{L})\|\tilde{\mathbf{d}}\|^2 + \frac{\sigma\bar{\Delta}^2}{\lambda_m(\mathbf{L})} \right] \\
&\leq -(1-\varepsilon)\frac{\mathbf{e}^T \mathbf{P} \mathbf{e}}{\lambda_M(\mathbf{P})} - \left( \sigma\lambda_m(\mathbf{L}) - \frac{\|\mathbf{P}\|^2}{m^2\varepsilon} \right) \|\tilde{\mathbf{d}}\|^2 + \frac{\sigma\bar{\Delta}^2}{\lambda_m(\mathbf{L})} \\
&\leq -\delta_1 \mathbf{e}^T \mathbf{P} \mathbf{e} - \delta_2 \sigma \|\tilde{\mathbf{d}}\|^2 + \delta_3
\end{aligned} \tag{12}$$

where  $\delta_1 = \frac{1-\varepsilon}{\lambda_M(\mathbf{P})}$ ,  $\delta_2 = \left( \lambda_m(\mathbf{L}) - \frac{\|\mathbf{P}\|^2}{m^2\varepsilon} \right)$  and  $\delta_3 = \frac{\sigma\bar{\Delta}^2}{\lambda_m(\mathbf{L})}$ .

According to (12), the stability condition of the close-loop system is  $\delta_1, \delta_2, \delta_3 > 0$ . The first coefficient  $\delta_1$  can be ensured to be positive with  $\varepsilon < 1$ . Coefficient  $\delta_2$  can be guaranteed to be positive by adjusting  $\mathbf{L}$  properly. Coefficient  $\delta_3$  is obviously positive since the observer gain  $\mathbf{L}$  is positive and definite.

Further, overall convergence rate of the close-loop system can be obtained that  $\mu = \min(\delta_1, \delta_2)$ , and it can be obtained that  $\dot{V} \leq -\mu V + \delta_3$ , which implies the time domain range of  $V$  is

$$0 \leq V(t) \leq e^{-\mu t} V(0) + \frac{\delta_3}{\mu} (1 - e^{-\mu t}). \tag{13}$$

According to (13), the upper bound of  $V$  (i.e.,  $\bar{V}$ ) as  $t \rightarrow +\infty$  can be obtained as  $\bar{V} = \frac{\delta_3}{\mu}$ .

## V. CONCLUSION

It can be concluded from (13) that signals of the close-loop controller are globally uniformly bounded. Besides, it can be found that the estimation derivation of NADO can converge to a small residual set related to  $\delta_3$  and the impact of NN estimation derivation can be mitigated by adjusting  $\mathbf{L}$  properly.

## REFERENCES

- [1] W. H. Young, “On classes of summable functions and their Fourier series,” *Proc. Roy. Soc. Lond. Ser. A Math. Phys. Sci.*, vol. 87, no. 594, pp. 225–229, 1912, doi: 10.1098/rspa.1912.0076.