A Computational Approach to String Figures

Yulong Liu

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String Figures

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- Commonly known as a children's game

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- Entertainment during polar nights in the Arctic region

1. the native inhabitants in the arctic region play string figures for entertainment

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- Entertainment during polar nights in the Arctic region
- Storytelling and illustrating scenes from myths and legends

1. the indigenous people in New Zealand play string figures for storytelling and illustrating scenes from myths and legends
2. and i play string figures when overleaf takes forever to compile my slides

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todo insert 3 images of strings figures from the encyclopedia: the star, something i can do, something i cant do

A Computational Approach

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- Each move transforms a string figure to another

1. (take steps) for example, to make the star, we start with this initial position, and then apply two moves of picking a segment of the string

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String figures computations

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- Represent string figures: simple, precise

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- Represent string figures: simple, precise
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Motivation

A Computational Approach

1. (mot) clear way of describing how to make string figures

String figures take a lot of steps to make

- Start with an initial position
- Apply a sequence of moves
- Each move transforms a string figure to another

String figures computations

- Represent string figures: simple, precise
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Motivation

- Do string figures on paper

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- Start with an initial position
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String figures computations

- Represent string figures: simple, precise
- Apply moves directly to the representations

Motivation

- Do string figures on paper
- Computer simulations & animations

1. (mot) teach computers how to play string figures
2. (record) store in computer, as database
3. simliar to music scores, rubiks cubes
4. one effective way of describing string figures is to draw them, as string diagrams

Representation: Diagrams

Fingers are named $L1, \cdots, L5$ and $R1, \cdots, R5$ from thumb to pinky

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- Fn is the near string, Ff is the far string
- Lp and Rp are palmar strings

1. (palmar) when we have a string between $L1$ and $L5$, we give a special name, called palmar string

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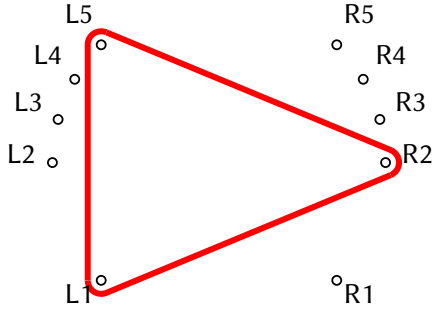

1. the diagrams are intuitive and very visual, but they are less computer friendly
2. what do computers like? they like array of symbols

Representation: Linear Sequences

Two key components

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- Fingers that hold the string

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- Start with left nearest finger and travel clockwise

1. travel along the string clockwise

Representation: Linear Sequences

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- Fingers that hold the string
- Crossings (with parity)

Diagram $\longrightarrow$ linear sequence

- Start with left nearest finger and travel clockwise
- Visit fingers and crossings (with parity)

1. note down the fingers and crossings with parity

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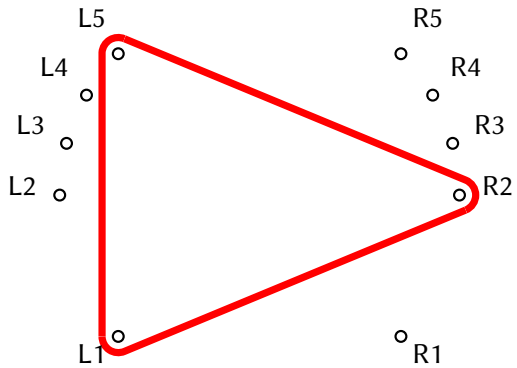

$\mapsto \quad L1 : L5 : R2$

Linear Sequences with Crossings

Diagram $\longrightarrow$ linear sequence

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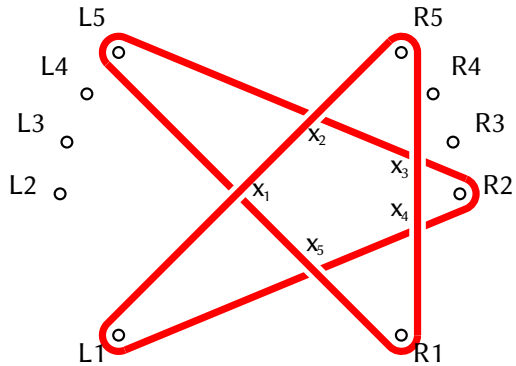

$L1{:}x_1(o){:}x_2(o){:}R5{:}x_3(o){:}x_4(o){:}R1{:}x_5(o){:}x_1(u){:}L5{:}x_2(u){:}x_3(u){:}R2{:}x_4(u){:}x_5(u)$

- if we have n fingers holding the string figure that has m crossings, the sequence will have $n + 2m$ symbols
- in most string figures, having a lot of crossings is what makes them beautiful, and it would be tedious to write down the entire sequence by hand
- (you see that i cant even fit the linear sequence of this simple star with normal size font)
- so this gives us more motivation to develop a systematic way of computing the movements so we can let computer do the calculuation
- the most common movements involve a finger and sometimes a near/far string segment of some other finger, but the problem arises in identifying these segments from the sequence

Identifying String Segments from Linear Sequences

Consider a subsequence $\ldots : L_i : \ldots$ for any $L_i \in \{L1, \ldots, L5\}$

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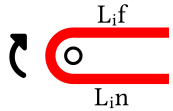
- Traverse clockwise 

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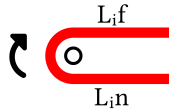
- Traverse clockwise 

$\implies \ldots : [L_i n] L_i [L_i f] : \ldots$

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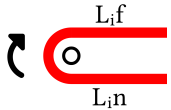
- Traverse clockwise 

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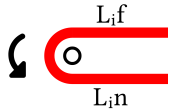
- Traverse counterclockwise 

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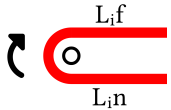
- Traverse clockwise 

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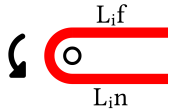
- Traverse counterclockwise 

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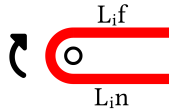
- Traverse clockwise 

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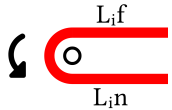
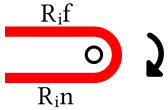
- Traverse counterclockwise 

$\implies \ldots : [L_i f] L_i [L_i n] : \ldots$

Similarly for finger R_i on the right hand

- 

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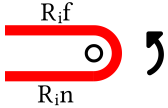
- 

$\implies \ldots : [R_i n] R_i [R_i f] : \ldots$

Identifying String Segments : Opposite Hand

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$$[L2n]L2[L2f] : [R2f]R2[R2n]$$

- Odd number of crossings between $L2$ and $R2$ $\implies$ rotation reverses

$$[L2n]L2[L2f] : x_1(u) : [R2n]R2[R2f] : x_1(o)$$

Identifying String Segments : Same Hand

Consider $\ldots : L2 : \ldots : L5 : \ldots$

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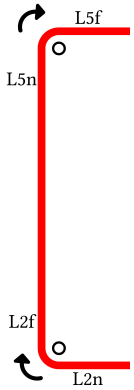

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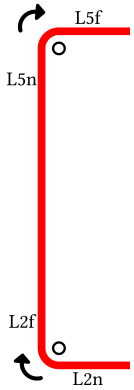

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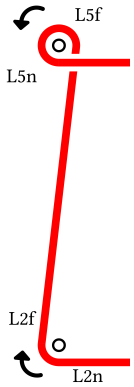

$\ldots : [L2n]L2[L2f] : x_1(u) : [L5f]L5[L5n] : x_1(o) : \ldots$

Identifying String Segments : Example

$$L1 : x_1(o) : R5 : x_2(o) : L5 : x_2(u) : x_1(u) : R2$$

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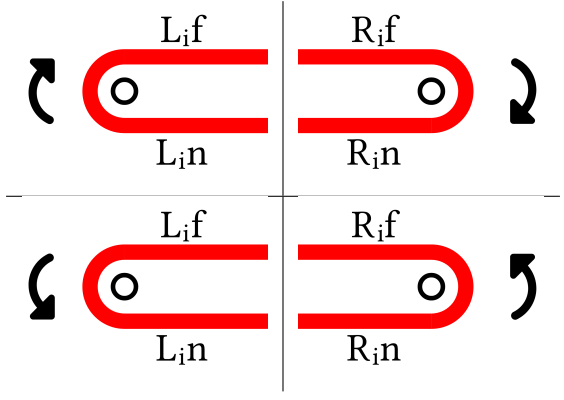

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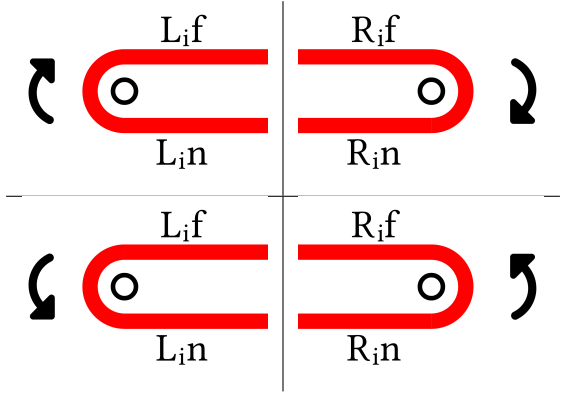

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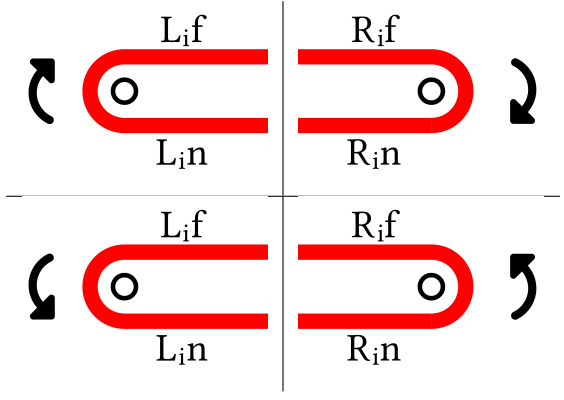

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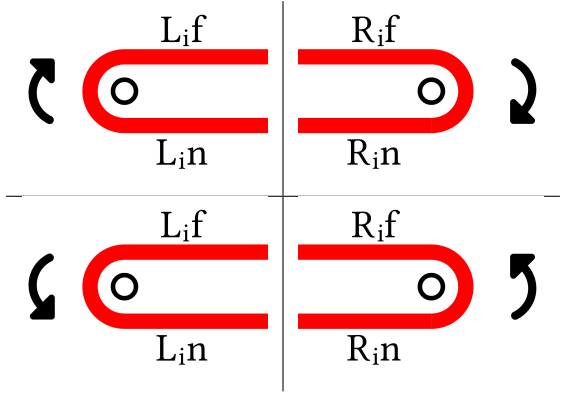

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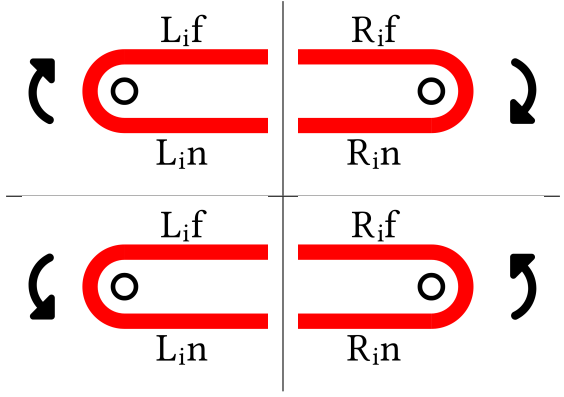

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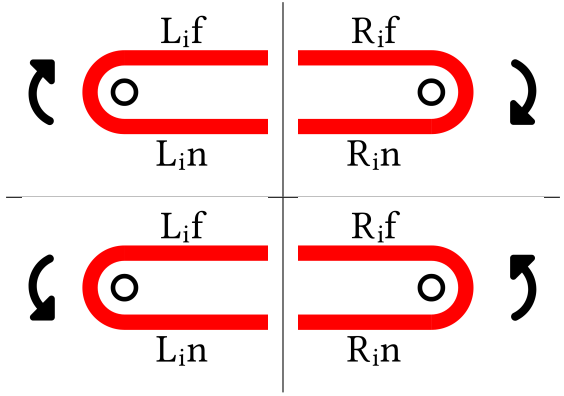

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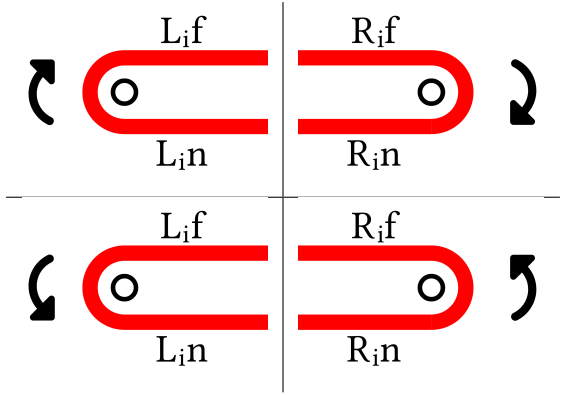

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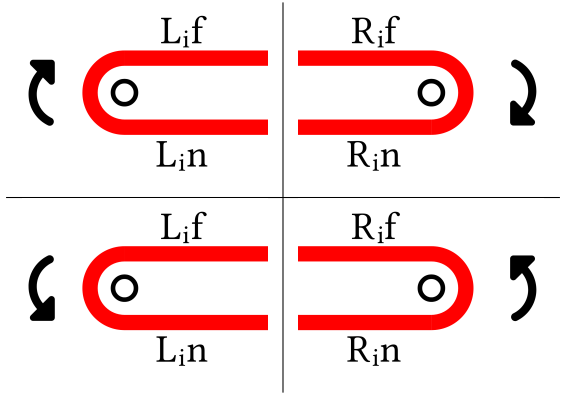

- now that we can recover the string segments from the linear sequence
- when we want to pick a specific string segment, we know where it is in the linear sequence
- "use a finger to pick some segment" is essentially insert the finger to where the segment is in the linear sequence along with some crossings if needed

Moves: Twist

Moves: Twist

Two variations: twist towards and twist away

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Two variations: twist towards and twist away

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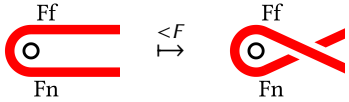
- 

$$\ldots : [Fn]F[Ff] : \ldots \quad \overset{<F}{\mapsto} \quad \ldots : x(u) : F : x(o) : \ldots$$

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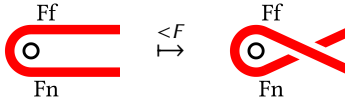
- 

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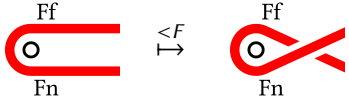
- 

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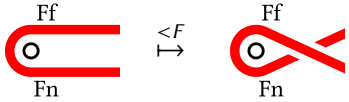
- 

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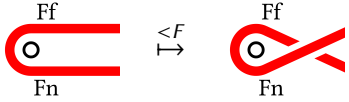
- 

  $$\ldots : [Fn]F[Ff] : \ldots \overset{>F}{\mapsto} \ldots : x(o) : F : x(u) : \ldots$$

$\ldots : [Ff]F[Fn] : \ldots \implies$ crossing parities are reversed

- works on right hand too
- Storer p382

Moves: Pick

Finger F picks a string segment s

- Written as $F(s)$

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Examples

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Finger F picks a string segment s

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Examples

- "$R5$ passes *over* all intermediate segments and picks Lp from *above*" is denoted as $\overleftarrow{R5}(\overline{Lp})$

Moves: Pick

1. the first 2 moves makes the star

Finger F picks a string segment s

- Written as $F(s)$

Two types of variations:

- F passes *over/under* all intermediate segments
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Examples

- "$R5$ passes *over* all intermediate segments and picks Lp from *above*" is denoted as $\overleftarrow{R5}(\overline{Lp})$
- "$R1$ passes *over* all intermediate segments and picks $R5n$ from *below*" is denoted as $\overrightarrow{R1}(\underline{R5n})$

Moves: Pick

Finger F picks a string segment s

- Written as $F(s)$

Two types of variations:

- F passes *over/under* all intermediate segments
- F picks s from *above/below*

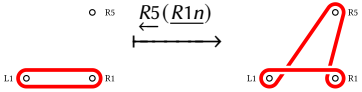
Examples

- "$R5$ passes *over* all intermediate segments and picks Lp from *above*" is denoted as $\overleftarrow{R5}(\overline{Lp})$
- "$R1$ passes *over* all intermediate segments and picks $R5n$ from *below*" is denoted as $\overrightarrow{R1}(\underline{R5n})$
- "$R4$ passes *below* all intermediate segments and picks $L1n$ from *below*" is denoted as $\overleftarrow{R4}(\underline{L1n})$

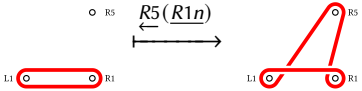
1. pick moves can be really hard to perform in reality, e.g. apply 3rd one after 1 and 2

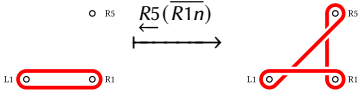
Pick: Examples

Pick: Examples



Pick: Examples





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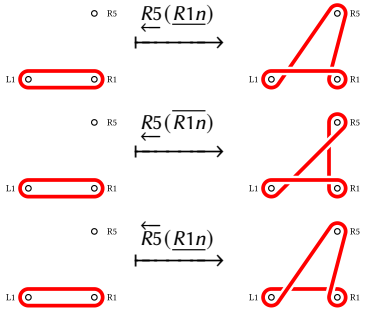

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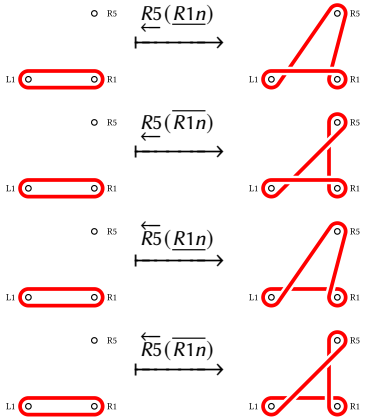
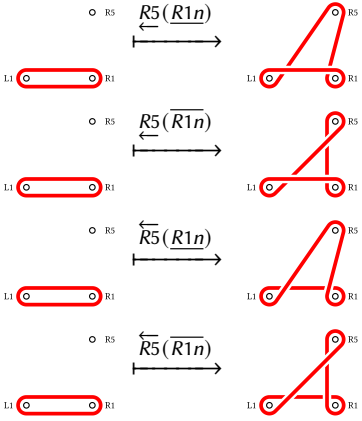
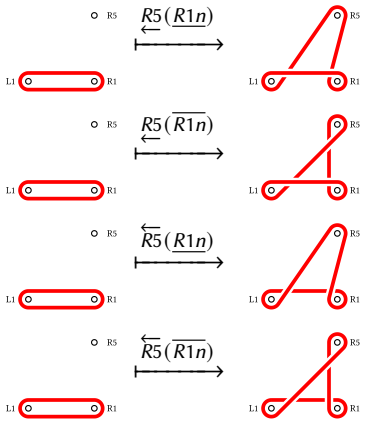


Pick: Examples



Observations

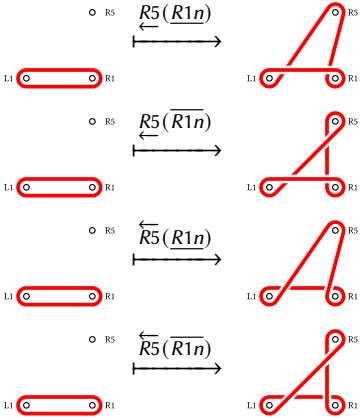
Pick: Examples



Observations

- A pair of crossings for each intermediate string

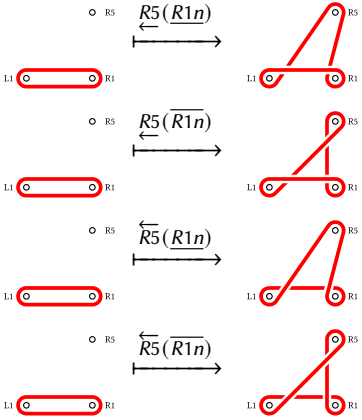
Pick: Examples



Observations

- A pair of crossings for each intermediate string
- $F(\overline{s})$ and $F(\underline{s})$ differ by a twist

Pick: Examples



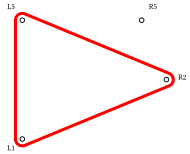
Observations

- A pair of crossings for each intermediate string
- $F(\overline{s})$ and $F(\underline{s})$ differ by a twist
- $\overleftarrow{F}(s)$ and $\underleftarrow{F}(s)$ differ by crossing parity

- when F moves towards, the new twist is a twist towards
- when F moves away, the new twist is away, e.g. $L5 : R5$ and pick with $R1$

Pick: Construction

General steps for $F(s)$

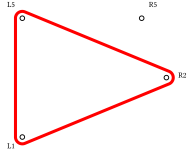


1. note that we can check if a move is valid since we know how to identify all the segments in a seq

Pick: Construction

General steps for $F(s)$

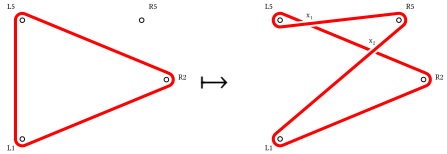
- Identify intermediate segments ($Fn < F < Ff$)



Pick: Construction

General steps for $F(s)$

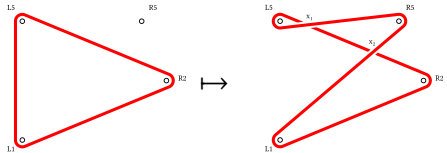
- Identify intermediate segments ($Fn < F < Ff$)
- Insert a pair of crossings for each intermediate segment



Pick: Construction

General steps for $F(s)$

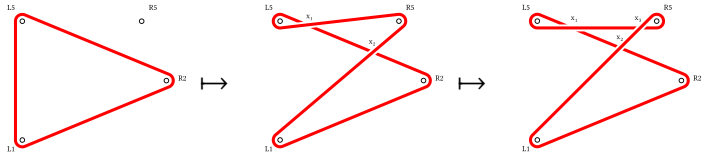
- Identify intermediate segments ($Fn < F < Ff$)
- Insert a pair of crossings for each intermediate segment
- Insert F at s with crossings (aka Spike)



Pick: Construction

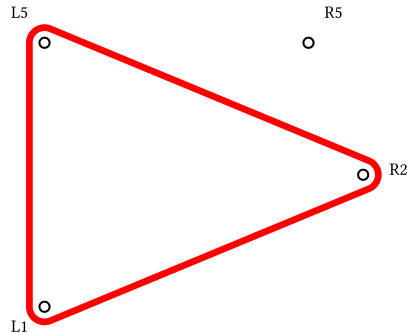
General steps for $F(s)$

- Identify intermediate segments ($Fn < F < Ff$)
- Insert a pair of crossings for each intermediate segment
- Insert F at s with crossings (aka Spike)
- Add twist if pick from above



Pick: Construction Example

$$L1 : L5 : R2 \xmapsto{\overleftarrow{R5}(\overline{Lp})} ???$$



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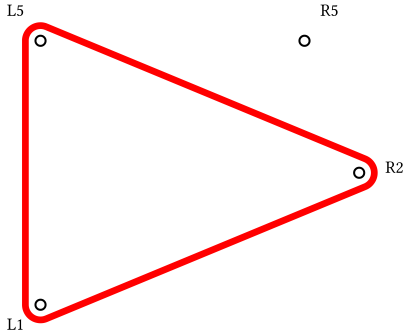
- $[L1n]L1[Lp]L5[L5f] : [R2f]R2[R2n]$



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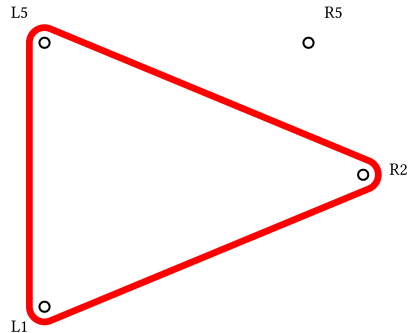
$L1 : L5 : R2 \xmapsto{\overleftarrow{R5}(\overline{Lp})} ???$

- $[L1n]L1[Lp]L5[L5f] : [R2f]R2[R2n]$
- Only the segment $[L5f] : [R2f]$ is intermediate

$$Lp = L5n < L5f = R2f < R2 < R5$$



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$$L1 : L5[L5f] : x_1(u) : x_2(u) : [R2f]R2[R2n]$$

1. under because our finger is going above

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Make spike for $R5$ and insert at Lp

- Spike: $x_2(o) : R5 : x_1(o)$

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$$\overset{\curvearrowright}{L1}: x_2(o) : \overset{\curvearrowleft}{R5}: x_1(o) : \overset{\curvearrowright}{L5}: x_1(u) : x_2(u) : R2$$

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$$L1 : x_2(o) : R5 : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

Make twist on $R5$ (towards)

$$L1 : x_2(o) : R5 : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

Pick: Construction Example

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Found $[L5f] : [R2f]$ as an intermediate segment

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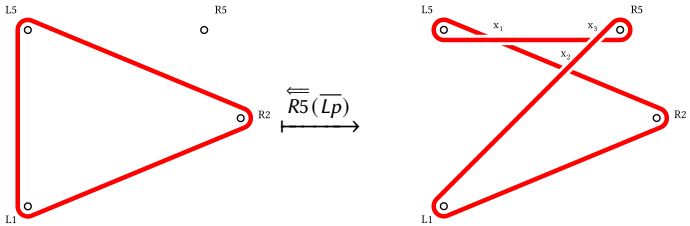

Extension Cancellation