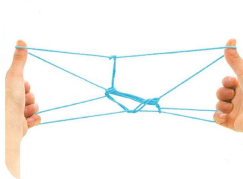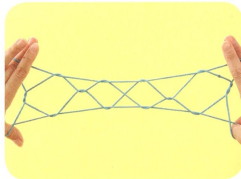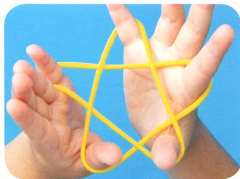# A Computational Approach to String Figures

Yulong Liu

2023-11-29

String Figures

- Designs formed from a loop of string
- Commonly known as a children's game

People have also been playing with the string throughout history.

- Entertainment during polar nights in the Arctic region
- Storytelling and illustrating scenes from myths and legends

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Noguchi, T. (2020). Ayatori Daizenshu. Shufunotomosha.

A Computational Approach

To make a string figure:
- ▶ Start with an initial position (opening)
- ▶ Apply a sequence of moves
- ▶ Each move transforms a string figure to another

String figures computations
- ▶ Represent string figures: simple, precise
- ▶ Apply moves directly to the representations

Motivation
- ▶ Precise language of describing string figures
- ▶ Computer simulations & animations

Representation: Diagrams

Fingers are named $L1, \cdots, L5$ and $R1, \cdots, R5$ from thumb to pinky

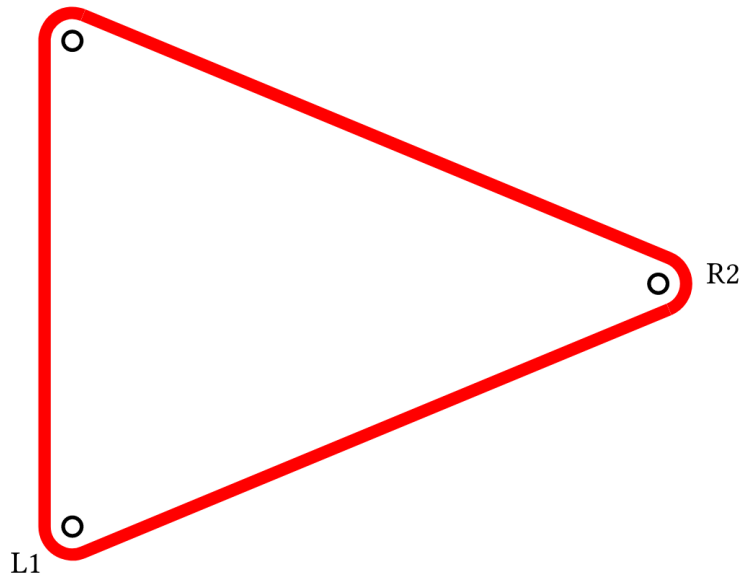- ▶ Ordered from nearest to furthest

$$L1 < L2 < L3 < L4 < L5$$
$$R1 < R2 < R3 < R4 < R5$$

String segments are named by finger $F \in \{L1, \cdots, L5, R1, \cdots, R5\}$

- ▶ *Fn* is the near string, *Ff* is the far string
- ▶ *Lp* and *Rp* are palmar strings

Representation: Diagrams

Representation: Linear Sequences
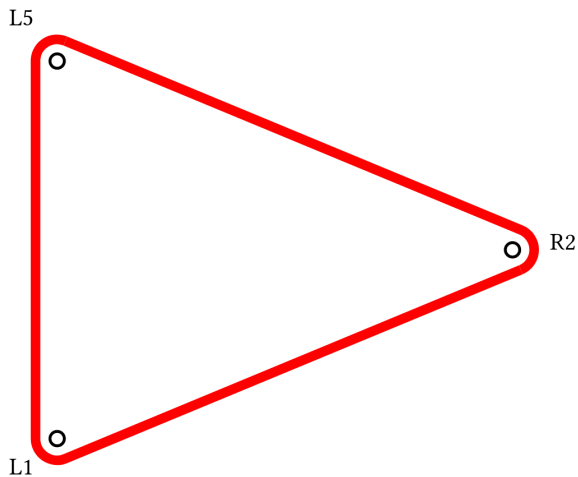
Two components
- Fingers that hold the string
- Crossings between two segments

Diagram → linear sequence
- Start with left nearest finger and travel clockwise
- Visit fingers and crossings

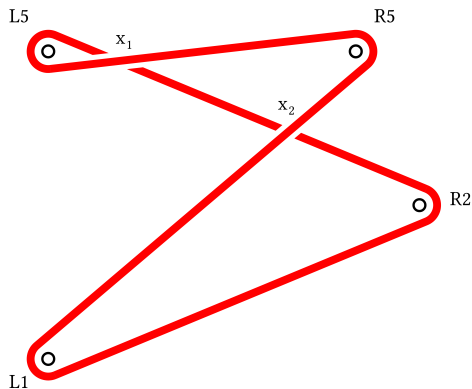# Representation: Linear Sequences

$L1 : L5 : R2$

Linear Sequences with Crossings

Diagram → linear sequence

- ▶ Name each crossing as x_i for some i
- ▶ Visit overcrossing $\implies$ write $x_i(o)$
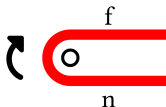- ▶ Visit undercrossing $\implies$ write $x_i(u)$

Linear Sequences with Crossings

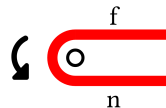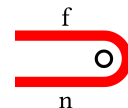$$L1 : x_2(o) : R5 : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

Identifying String Segments from Linear Sequences
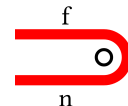
Consider a left-hand finger L_i in the sequence

- Traverse clockwise $\implies \ldots : [n]L_i[f] : \ldots$

- Traverse counterclockwise $\implies \ldots : [f]L_i[n] : \ldots$

Similarly for finger R_i on the right hand

- $\implies \ldots : [f]R_i[n] : \ldots$

- $\implies \ldots : [n]R_i[f] : \ldots$

Identifying String Segments : Opposite Hand

Consider $\ldots : L_i : \ldots : R_j : \ldots$

- Even number of crossings between L_i and R_j $\implies$ orientation persists

$$[n]L_i[f] : [f]R_j[n]$$

- Odd number of crossings between L_i and R_j $\implies$ orientation reverses

$$[n]L_i[f] : x_1(u) : [n]R_j[f] : x_1(o)$$

Identifying String Segments : Same Hand

Consider $\ldots : L_i : \ldots : L_j : \ldots$

Even $\implies$ orientation persists

...:[n]L_i[f]:[n]L_j[f]:...

Odd $\implies$ orientation reverses

...:[n]L_i[f]:$x_1(u)$:[f]L_j[n]:$x_1(o)$:...

Identifying String Segments : Example

$$L1 : x_2(o) : R5 : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

By convention, the first finger in the linear sequence is clockwise

$$L1: x_2(o) : R5 : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

Identifying String Segments : Example

$$L1 : x_2(o) : R5 : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

By convention, the first finger in the linear sequence is clockwise

$$\widehat{L1} : x_2(o) : R5 : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

Identifying String Segments : Example

$$L1 : x_2(o) : R5 : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

By convention, the first finger in the linear sequence is clockwise

$$[n] \; \widetilde{L1} \; [f] : x_2(o) : R5 : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

Identifying String Segments : Example

$$L1 : x_2(o) : R5 : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

By convention, the first finger in the linear sequence is clockwise

$$[n] \; \overset{\frown}{L1} \; [f] : x_2(o) : \overset{\frown}{R5} : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

Identifying String Segments : Example

$$L1 : x_2(o) : R5 : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

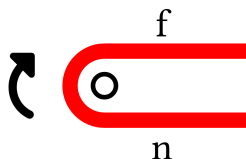By convention, the first finger in the linear sequence is clockwise

$$[n] \; \overset{\sim}{L1} \; [f] : x_2(o) : [n] \; \overset{\frown}{R5} \; [f] : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

Identifying String Segments : Example

$$L1 : x_2(o) : R5 : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

By convention, the first finger in the linear sequence is clockwise

$$[n]L1[f] : x_2(o) : [n] \overset{\frown}{R5} [f] : x_1(o) : \overset{\frown}{\widetilde{L5}} : x_1(u) : x_2(u) : R2$$
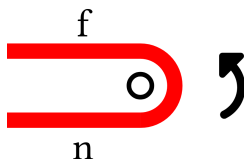
Identifying String Segments : Example

$$L1 : x_2(o) : R5 : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

By convention, the first finger in the linear sequence is clockwise

$$[n]L1[f] : x_2(o) : [n] \overset{\frown}{R5} [f] : x_1(o) : [n] \overset{\sim}{L5} [f] : x_1(u) : x_2(u) : R2$$

Identifying String Segments : Example

$$L1 : x_2(o) : R5 : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

By convention, the first finger in the linear sequence is clockwise

$$[n]L1[f] : x_2(o) : [n]R5[f] : x_1(o) : [n] \widetilde{L5} [f] : x_1(u) : x_2(u) : \widetilde{R2}$$

Identifying String Segments : Example

$$L1 : x_2(o) : R5 : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

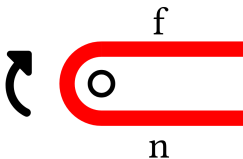By convention, the first finger in the linear sequence is clockwise
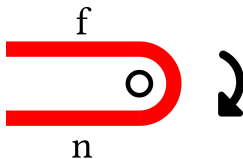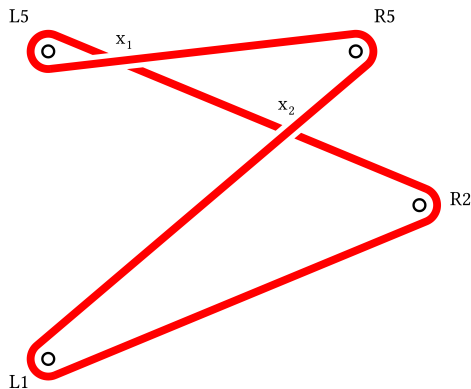
$$[n]L1[f] : x_2(o) : [n]R5[f] : x_1(o) : [n] \widetilde{L5} [f] : x_1(u) : x_2(u) : [f] \widetilde{R2} [n]$$

Identifying String Segments : Example

$$[n]L1[f] : x_2(o) : [n]R5[f] : x_1(o) : [n]L5[f] : x_1(u) : x_2(u) : [f]R2[n]$$

Two variations: twist towards and twist away

▶ Twist the loop on finger F *towards* player: $< F$
▶ Twist the loop on finger F *away* from player: $> F$

Consider $\ldots : [n]F[f] : \ldots$

▶
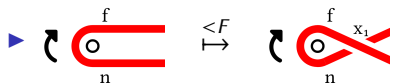
$$\ldots : [n]F[f] : \ldots \quad \overset{<F}{\mapsto} \quad \ldots : x_1(u) : F : x_1(o) : \ldots$$

▶
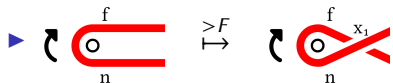
$$\ldots : [n]F[f] : \ldots \quad \overset{>F}{\mapsto} \quad \ldots : x_1(o) : F : x_1(u) : \ldots$$

▶ $\ldots : [f]F[n] : \ldots \overset{<F}{\mapsto} \ldots : x(o) : F : x(u) : \ldots$
▶ $\ldots : [f]F[n] : \ldots \overset{>F}{\mapsto} \ldots : x(u) : F : x(o) : \ldots$

Moves: Pick

Finger *F* picks a string segment *s*

- ▶ Written as $F(s)$

Four variations:

- ▶ *F* passes *over/under* all intermediate segments
- ▶ *F* picks *s* from *above/below*

Examples

- ▶ "*R*5 passes *over* all intermediate segments and picks *Lp* from *above*" is denoted as $\overleftarrow{R5}(\overline{Lp})$
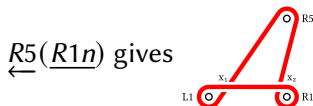- ▶ "*R*1 passes *over* all intermediate segments and picks *R*5*n* from *below*" is denoted as $\overrightarrow{R1}(\underline{R5n})$
- ▶ "*R*4 passes *below* all intermediate segments and picks *L*1*n* from *below*" is denoted as $\underleftarrow{R4}(\underline{L1n})$

Pick: Examples

Starting with $_{L1}\!\bigcirc\!\!\!\!\!\longrightarrow\!\!\!\!\!\bigcirc_{R1}$ $[n]L1[f] : [f]R1[n]$

$\underleftarrow{R5}(\underline{R1n})$ gives

$L1 : x_1(o) : x_2(o) : R1 : x_2(u) : R5 :$
$x_1(u)$

$\underleftarrow{\overleftarrow{R5}}(\underline{R1n})$ gives
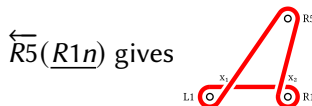
$L1 : x_1(u) : x_2(u) : R1 : x_2(o) : R5 :$
$x_1(o)$

$\underleftarrow{R5}(\overline{R1n})$ gives
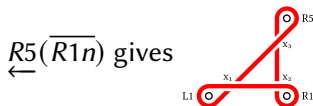
$L1 : x_1(o) : x_2(o) : R1 : x_2(u) : x_3(u) :$
$R5 : x_3(o) : x_1(u)$

$\overleftarrow{R5}(\overline{R1n})$ gives
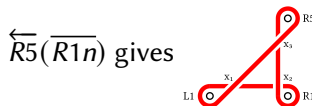
$L1 : x_1(u) : x_2(u) : R1 : x_2(o) : x_3(u) :$
$R5 : x_3(o) : x_1(o)$

Observations

- A pair of crossings for each intermediate string
- $\overleftarrow{F}(s)$ and $\underleftarrow{F}(s)$ differ by crossing parity
- $F(\bar{s})$ and $F(\underline{s})$ differ by a twist

Pick: Construction

General steps for applying $F(s)$ to a string figure
- Identify intermediate segments
- Insert a pair of crossings for each intermediate segment
- Insert F at s with crossings
- Add twist if pick from above
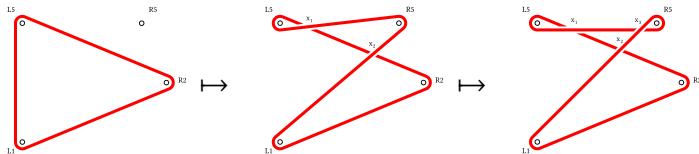
Pick: Construction Example

$L1 : L5 : R2 \xrightarrow{\overleftarrow{R5(\overline{Lp})}}$???

- Identify Lp

$$[n]L1[f] : [n]L5[f] : [f]R2[n]$$

Pick: Construction Example

$L1 : L5 : R2 \xrightarrow{\overleftarrow{R5(\overline{Lp})}}$???

- Identify Lp

$$[n]L1[Lp]L5[f] : [f]R2[n]$$

Pick: Construction Example

$L1 : L5 : R2 \xrightarrow{\overleftarrow{R5(\overline{Lp})}}$???

- Identify Lp

$$[n]L1[Lp]L5[f] : [f]R2[n]$$

Pick: Construction Example

$L1 : L5 : R2 \xmapsto{\overleftarrow{R5(\overline{Lp})}}$???

▶ Identify Lp

$$[n]L1[Lp]L5[f] : [f]R2[n]$$

▶ Only the segment between $L5$ and $R2$ is intermediate

$$\underline{R2n = L1n} < L1 < Lp = L5n < \underline{L5f = R2f} < R5$$

Pick: Construction Example

$$L1 : L5 : R2 \xmapsto{\overleftarrow{\overline{R5(\overline{Lp})}}} \text{ ???}$$

Found $L5f = R2f$ as an intermediate segment

- ▶ Insert crossings x_1 and x_2 at intermediate segment

Pick: Construction Example

$L1 : L5 : R2 \xrightarrow{\overleftarrow{\overline{R5(\overline{Lp})}}}$???

Found $L5f = R2f$ as an intermediate segment

- ▶ Insert crossings x_1 and x_2 at intermediate segment

$$L1 : L5 : [f] : [f] : R2$$

Pick: Construction Example

$L1 : L5 : R2 \xrightarrow{\overleftarrow{\overline{R5}(\overline{Lp})}}$???

Found $L5f = R2f$ as an intermediate segment

▶ Insert crossings x_1 and x_2 at intermediate segment

$$L1 : L5 : x_1(u) : x_2(u) : R2$$

Pick: Construction Example

$$L1 : L5 : R2 \xrightarrow{\overleftleftarrow{R5(\overline{Lp})}} \text{ ???}$$

Found $L5f = R2f$ as an intermediate segment

- ▶ Insert crossings x_1 and x_2 at intermediate segment

$$L1 : L5 : x_1(u) : x_2(u) : R2$$

- ▶ Insert $R5$ at Lp with x_1 and x_2

Pick: Construction Example

$L1 : L5 : R2 \xrightarrow{\overleftarrow{\overline{R5}(\overline{Lp})}}$???

Found $L5f = R2f$ as an intermediate segment

- Insert crossings x_1 and x_2 at intermediate segment

$$L1 : L5 : x_1(u) : x_2(u) : R2$$

- Insert $R5$ at Lp with x_1 and x_2

$$\widetilde{L1} \; [Lp] \; \widetilde{L5} : x_1(u) : x_2(u) : \widetilde{R2}$$

Pick: Construction Example

$L1 : L5 : R2 \xrightarrow{\overleftarrow{\overline{R5}}(\overline{Lp})}$???

Found $L5f = R2f$ as an intermediate segment

▶ Insert crossings x_1 and x_2 at intermediate segment

$$L1 : L5 : x_1(u) : x_2(u) : R2$$

▶ Insert $R5$ at Lp with x_1 and x_2

$$\widetilde{L1}: x_2(o) : \widehat{R5}: x_1(o) : \widetilde{L5}: x_1(u) : x_2(u) : \widetilde{R2}$$

Pick: Construction Example

$L1 : L5 : R2 \xmapsto{\overleftarrow{\overline{R5}(\overline{Lp})}}$???

Found $L5f = R2f$ as an intermediate segment

▶ Insert crossings x_1 and x_2 at intermediate segment

$$L1 : L5 : x_1(u) : x_2(u) : R2$$

▶ Insert $R5$ at Lp with x_1 and x_2

$$\widehat{L}1: x_2(o) : \widehat{R5}: x_1(o) : \widetilde{L5}: x_1(u) : x_2(u) : \widetilde{R}2$$

Pick: Construction Example

$L1 : L5 : R2 \xlongleftarrow{\overleftarrow{R5(\overline{Lp})}}$???

Found $L5f = R2f$ as an intermediate segment

- Insert crossings x_1 and x_2 at intermediate segment

$$L1 : L5 : x_1(u) : x_2(u) : R2$$

- Insert $R5$ at Lp with x_1 and x_2

$$\overset{\frown}{L1}: x_2(o) : \overset{\frown}{R5}: x_1(o) : \overset{\frown}{L5}: x_1(u) : x_2(u) : \overset{\frown}{R2}$$

$$L1 : x_2(o) : R5 : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

Pick: Construction Example

$L1 : L5 : R2 \xmapsto{\overleftarrow{\overline{R5}(\overline{Lp})}}$???

Found $L5f = R2f$ as an intermediate segment

▶ Insert crossings x_1 and x_2 at intermediate segment

$$L1 : L5 : x_1(u) : x_2(u) : R2$$

▶ Insert $R5$ at Lp with x_1 and x_2

$$\overset{\frown}{\widetilde{L1}}\colon x_2(o) : \overset{\frown}{R5}\colon x_1(o) :\overset{\frown}{\widetilde{L5}}\colon x_1(u) : x_2(u) :\overset{\frown}{\widetilde{R2}}$$

$$L1 : x_2(o) : x_3(o) : R5 : x_3(u) : x_1(o) : L5 : x_1(u) : x_2(u) : R2$$

Pick: Construction Example

$L1 : L5 : R2 \xmapsto{\overleftarrow{\overline{R5(\overline{Lp})}}}$???

Found $L5f = R2f$ as an intermediate segment

- ▶ Insert crossings x_1 and x_2 at intermediate segment

$$L1 : L5 : x_1(u) : x_2(u) : R2$$
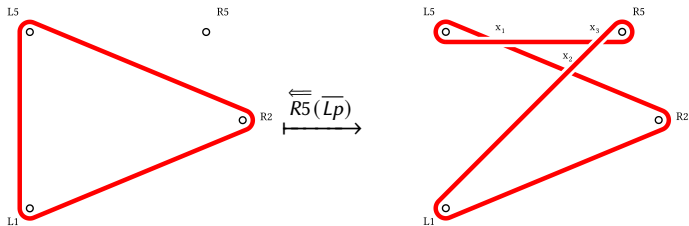
- ▶ Insert $R5$ at Lp with x_1 and x_2

$$\overset{\frown}{L}1 : x_2(o) : \overset{\frown}{R}5 : x_1(o) : \tilde{L}5 : x_1(u) : x_2(u) : \tilde{R}2$$

$$\overset{\frown}{L}1 : x_2(o) : x_3(o) : \tilde{R}5 : x_3(u) : x_1(o) : \overset{\frown}{L}5 : x_1(u) : x_2(u) : \tilde{R}2$$

Pick: Construction Example

$$L1{:}L5{:}R2 \xmapsto{\overleftarrow{R5}\,(\overline{Lp})} \widetilde{L}1{:}x_2(o){:}x_3(o){:}\widetilde{R}5{:}x_3(u){:}x_1(o){:}\widetilde{L}5{:}x_1(u){:}x_2(u){:}\widetilde{R}2$$

Summary

What we covered
- ▶ Representing string figures as linear sequences
- ▶ Applying twist and pick to linear sequences

Going deeper
- ▶ More moves
- ▶ Drawing diagrams from linear sequences

Thank you!