Note on cavity method for non-symmetric Wishart

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We try to use cavity method to calculate resolvent (the so called perturbative resolvent method [1]) and study the eigenvalue distribution of random matrices of the form AB'. The spectral density turns out to be quite complicated to calculate but we managed to obtain the boundary of eigenvalues. Curious about the relation among cavity calculation, the method in [2], diagrammatic method, and R-transform in free probability. There should be a way to simplify the susceptibilities in the note to obtain the spectral density.

I. NO EXTRA CORRELATION

A. Problem setup

We consider random matrix of the form AB', where both A and B are $N \times M$ matrices ($M \geq N$). To start, we assume

$$\langle A_{i\alpha} \rangle = 0,\ \langle B_{i\alpha} \rangle = 0, \tag{1}$$

and the second moments

$$\begin{aligned} \langle A_{i\alpha} A_{j\beta} \rangle &= \frac{\sigma_A^2}{M} \delta_{ij} \delta_{\alpha\beta}, \\ \langle B_{i\alpha} B_{j\beta} \rangle &= \frac{\sigma_B^2}{M} \delta_{ij} \delta_{\alpha\beta}, \\ \langle A_{i\alpha} B_{j\beta} \rangle &= \frac{\rho \sigma_A \sigma_B}{M} \delta_{ij} \delta_{\alpha\beta}. \end{aligned} \tag{2}$$

Later, we will take $M, N \to \infty$ but keep the ratio $r = N/M$ constant.

B. Resolvent equations

To obtain the correct distribution of eigenvalues on the complex plane, we construct the following matrix:

$$H(z) = \begin{pmatrix} O & AB' - zI \\ z^* I - BA' & O \end{pmatrix}. \tag{3}$$

With this matrix, we can construct the following linear system

$$\eta \begin{pmatrix} x \\ y \end{pmatrix} = H(z) \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}. \tag{4}$$

To perform cavity calculation, we further decompose the equations into

$$\begin{aligned} \eta x_i &= \sum_\alpha A_{i\alpha} u_\alpha - z y_i + a_i, \\ u_\alpha &= \sum_i B_{i\alpha} y_i + c_\alpha, \\ \eta y_i &= -\sum_\alpha B_{i\alpha} v_\alpha + z^* x_i + b_i, \\ v_\alpha &= \sum_i A_{i\alpha} x_i + d_\alpha. \end{aligned} \tag{5}$$

If we consider how x, y, u, and v are perturbed when a, b, c, and d change a little, we need to define the 16 susceptibilities (16 matrices):

$$\begin{aligned} &\frac{\partial x}{\partial a}, \frac{\partial x}{\partial b}, \frac{\partial x}{\partial c}, \frac{\partial x}{\partial d}, \\ &\frac{\partial y}{\partial a}, \frac{\partial y}{\partial b}, \frac{\partial y}{\partial c}, \frac{\partial y}{\partial d}, \\ &\frac{\partial u}{\partial a}, \frac{\partial u}{\partial b}, \frac{\partial u}{\partial c}, \frac{\partial u}{\partial d}, \\ &\frac{\partial v}{\partial a}, \frac{\partial v}{\partial b}, \frac{\partial v}{\partial c}, \frac{\partial v}{\partial d}. \end{aligned} \tag{6}$$

Differentiating Eq. (5), we can obtain

$$\begin{pmatrix} \eta I & zI - AB' \\ BA' - z^* I & \eta I \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{7}$$

from which, we know that only several susceptibilities will be relevant at the end, and the susceptibilities are constrained by Eq. (7):

$$\begin{aligned} \frac{\partial x}{\partial a} &= \frac{\partial y}{\partial b} \equiv \chi, \\ \frac{\partial y}{\partial a} &= -\big(\frac{\partial x}{\partial b}\big)' \equiv \nu. \end{aligned} \tag{8}$$

The susceptibility $\nu = \nu(\eta; z, z^*)$ will reduce to

$$\nu(0; z, z^*) = \frac{1}{zI - AB'}, \tag{9}$$

when $\eta = 0$. And the spectrum density can be then given by

$$f_{AB'}(x, y) = \frac{1}{\pi} \partial_{z^*} \left[\frac{1}{N} \mathrm{tr} \nu(0; z, z^*) \right]. \tag{10}$$

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Our task is therefore to find an expression of ν.

C. Cavity calculation

The susceptibilities can be calculated via cavity method. To do so, we introduce x_0, y_0, u_0, v_0, and consequently a new row and a new column to each of the matrices A and B, e.g., A gets A_{i0} and $A_{0\alpha}$. The original resolvent equations will be modified to

$$\begin{aligned}
\eta x_i &= \sum_\alpha A_{i\alpha} u_\alpha - z y_i + a_i + A_{i0} u_0, \\
u_\alpha &= \sum_i B_{i\alpha} y_i + c_\alpha + B_{0\alpha} y_0, \\
\eta y_i &= -\sum_\alpha B_{i\alpha} v_\alpha + z^* x_i + b_i - B_{i0} v_0, \\
v_\alpha &= \sum_i A_{i\alpha} x_i + d_\alpha + A_{0\alpha} x_0.
\end{aligned} \tag{11}$$

Since the newly added terms are small comparing to the original terms in the limit of large M and large N, we can use susceptibilities and original values of before introducing new variables, denoted by $x_{i\backslash 0}$...:

$$\begin{aligned}
x_i &= x_{i\backslash 0} + \sum_j \chi_{ij} A_{j0} u_0 + \sum_j \nu'_{ij} B_{j0} v_0 + \sum_\alpha \frac{\partial x_i}{\partial c_\alpha} B_{0\alpha} y_0 + \sum_\alpha \frac{\partial x_i}{\partial d_\alpha} A_{0\alpha} x_0, \\
y_i &= y_{i\backslash 0} + \sum_j \nu_{ij} A_{j0} u_0 - \sum_j \chi_{ij} B_{j0} v_0 + \sum_\alpha \frac{\partial y_i}{\partial c_\alpha} B_{0\alpha} y_0 + \sum_\alpha \frac{\partial y_i}{\partial d_\alpha} A_{0\alpha} x_0, \\
u_\alpha &= u_{\alpha\backslash 0} + \sum_j \frac{\partial u_\alpha}{\partial a_j} A_{j0} u_0 - \sum_j \frac{\partial u_\alpha}{\partial b_j} B_{j0} v_0 + \sum_\beta \frac{\partial u_\alpha}{\partial c_\beta} B_{0\beta} y_0 + \sum_\beta \frac{\partial u_\alpha}{\partial d_\beta} A_{0\beta} x_0, \\
v_\alpha &= v_{\alpha\backslash 0} + \sum_j \frac{\partial v_\alpha}{\partial a_j} A_{j0} u_0 - \sum_j \frac{\partial v_\alpha}{\partial b_j} B_{j0} v_0 + \sum_\beta \frac{\partial v_\alpha}{\partial c_\beta} B_{0\beta} y_0 + \sum_\beta \frac{\partial v_\alpha}{\partial d_\beta} A_{0\beta} x_0.
\end{aligned} \tag{12}$$

Next, we consider the equations the newly introduced variables should satisfy:

$$\begin{aligned}
\eta x_0 &= \sum_\alpha A_{0\alpha} u_\alpha - z y_0 + a_0 + A_{00} u_0, \\
u_0 &= \sum_i B_{i0} y_i + c_0 + B_{00} y_0, \\
\eta y_0 &= -\sum_\alpha B_{0\alpha} v_\alpha + z^* x_0 + b_0 - B_{00} v_0, \\
v_0 &= \sum_i A_{i0} x_i + d_0 + A_{00} x_0.
\end{aligned} \tag{13}$$

Substituting Eq. (12) into the above equations, we will have a lot of summations of many random variables. In the large N, M limit, by central limit theorem, only the mean values matter, we therefore have

$$\begin{aligned}
\eta x_0 &\approx \sum_\alpha A_{0\alpha} u_{\alpha\backslash 0} + \left\langle \frac{\partial u_\alpha}{\partial c_\alpha} \right\rangle \rho \sigma_A \sigma_B y_0 + \left\langle \frac{\partial u_\alpha}{\partial d_\alpha} \right\rangle \sigma_A^2 x_0 - z y_0 + a_0, \\
u_0 &\approx \sum_i B_{i0} y_{i\backslash 0} + r \langle \nu_{jj} \rangle \rho \sigma_A \sigma_B u_0 - r \langle \chi_{jj} \rangle \sigma_B^2 v_0 + c_0, \\
\eta y_0 &\approx -\sum_\alpha B_{0\alpha} v_{\alpha\backslash 0} - \left\langle \frac{\partial v_\alpha}{\partial c_\alpha} \right\rangle \sigma_B^2 y_0 - \left\langle \frac{\partial v_\alpha}{\partial d_\alpha} \right\rangle \rho \sigma_A \sigma_B x_0 + z^* x_0 + b_0, \\
v_0 &\approx \sum_i A_{i0} x_{i\backslash 0} + r \langle \chi_{jj} \rangle \sigma_A^2 u_0 + r \langle \nu^*_{jj} \rangle \rho \sigma_A \sigma_B v_0 + d_0.
\end{aligned} \tag{14}$$

In the above equations, indices in the terms like $\frac{\partial u_\alpha}{\partial c_\alpha}$ do not mean any specific value, as $\frac{\partial u_\alpha}{\partial c_\alpha}$ have the same distribution across α. This is a key to the following, as we will write down self-consistent equations making use of $\left\langle \frac{\partial u_\alpha}{\partial c_\alpha} \right\rangle = \left\langle \frac{\partial u_0}{\partial c_0} \right\rangle$, etc. For simplicity, we further define $\overline{\chi} \equiv \langle \chi_{jj} \rangle$ and $\overline{\nu} \equiv \langle \nu_{jj} \rangle$.

We first solve for u_0 and v_0, which are

$$\begin{aligned} u_0 &= \frac{(1 - r\overline{\nu}^* \rho\sigma_A\sigma_B)(\sum_i B_{i0} y_{i\backslash 0} + c_0) - r\overline{\chi}\sigma_B^2(\sum_i A_{i0} x_{i\backslash 0} + d_0)}{r^2\overline{\chi}^2\sigma_A^2\sigma_B^2 + |1 - r\overline{\nu}\rho\sigma_A\sigma_B|^2}, \\ v_0 &= \frac{r\overline{\chi}\sigma_A^2(\sum_i B_{i0} y_{i\backslash 0} + c_0) + (1 - r\overline{\nu}\rho\sigma_A\sigma_B)(\sum_i A_{i0} x_{i\backslash 0} + d_0)}{r^2\overline{\chi}^2\sigma_A^2\sigma_B^2 + |1 - r\overline{\nu}\rho\sigma_A\sigma_B|^2}. \end{aligned} \tag{15}$$

We therefore can obtain

$$\left\langle \frac{\partial u_\alpha}{\partial c_\alpha} \right\rangle = \frac{1 - r\overline{\nu}^*\rho\sigma_A\sigma_B}{r^2\overline{\chi}^2\sigma_A^2\sigma_B^2 + |1 - r\overline{\nu}\rho\sigma_A\sigma_B|^2}, \ \left\langle \frac{\partial u_\alpha}{\partial d_\alpha} \right\rangle = \frac{-r\overline{\chi}\sigma_B^2}{r^2\overline{\chi}^2\sigma_A^2\sigma_B^2 + |1 - r\overline{\nu}\rho\sigma_A\sigma_B|^2}, \tag{16}$$

and

$$\left\langle \frac{\partial v_\alpha}{\partial c_\alpha} \right\rangle = \frac{r\overline{\chi}\sigma_A^2}{r^2\overline{\chi}^2\sigma_A^2\sigma_B^2 + |1 - r\overline{\nu}\rho\sigma_A\sigma_B|^2}, \ \left\langle \frac{\partial v_\alpha}{\partial d_\alpha} \right\rangle = \frac{1 - r\overline{\nu}\rho\sigma_A\sigma_B}{r^2\overline{\chi}^2\sigma_A^2\sigma_B^2 + |1 - r\overline{\nu}\rho\sigma_A\sigma_B|^2}. \tag{17}$$

Next, we evaluate x_0 and y_0 when $\eta = 0$, which are

$$\begin{aligned} x_0 &= \frac{\left\langle \frac{\partial v_\alpha}{\partial c_\alpha} \right\rangle \sigma_B^2 (a_0 + \sum_\alpha A_{0\alpha} u_{\alpha\backslash 0}) + (\left\langle \frac{\partial u_\alpha}{\partial c_\alpha} \right\rangle \rho\sigma_A\sigma_B - z)(b_0 - \sum_\alpha B_{0\alpha} v_{\alpha\backslash 0})}{-\left\langle \frac{\partial u_\alpha}{\partial d_\alpha} \right\rangle \left\langle \frac{\partial v_\alpha}{\partial c_\alpha} \right\rangle \sigma_A^2\sigma_B^2 + \left| \left\langle \frac{\partial u_\alpha}{\partial c_\alpha} \right\rangle \rho\sigma_A\sigma_B - z \right|^2}, \\ y_0 &= \frac{\left\langle \frac{\partial u_\alpha}{\partial d_\alpha} \right\rangle \sigma_A^2 (b_0 - \sum_\alpha B_{0\alpha} v_{\alpha\backslash 0}) - (z^* - \left\langle \frac{\partial v_\alpha}{\partial d_\alpha} \right\rangle \rho\sigma_A\sigma_B)(a_0 + \sum_\alpha A_{0\alpha} u_{\alpha\backslash 0})}{\left\langle \frac{\partial u_\alpha}{\partial d_\alpha} \right\rangle \left\langle \frac{\partial v_\alpha}{\partial c_\alpha} \right\rangle \sigma_A^2\sigma_B^2 - \left| \left\langle \frac{\partial u_\alpha}{\partial c_\alpha} \right\rangle \rho\sigma_A\sigma_B - z \right|^2}. \end{aligned} \tag{18}$$

Since we know

$$\overline{\chi} \approx \frac{\partial x_0}{\partial a_0} = \frac{\partial y_0}{\partial b_0}, \ \overline{\nu} \approx \frac{\partial y_0}{\partial a_0} = -\left(\frac{\partial x_0}{\partial b_0} \right)^*, \tag{19}$$

we can solve $\overline{\chi}$ and $\overline{\nu}$ as

$$\overline{\chi} = \frac{\left\langle \frac{\partial v_\alpha}{\partial c_\alpha} \right\rangle \sigma_B^2}{-\left\langle \frac{\partial u_\alpha}{\partial d_\alpha} \right\rangle \left\langle \frac{\partial v_\alpha}{\partial c_\alpha} \right\rangle \sigma_A^2\sigma_B^2 + \left| \left\langle \frac{\partial u_\alpha}{\partial c_\alpha} \right\rangle \rho\sigma_A\sigma_B - z \right|^2}, \ \overline{\nu} = \frac{-(z^* - \left\langle \frac{\partial v_\alpha}{\partial d_\alpha} \right\rangle \rho\sigma_A\sigma_B)}{\left\langle \frac{\partial u_\alpha}{\partial d_\alpha} \right\rangle \left\langle \frac{\partial v_\alpha}{\partial c_\alpha} \right\rangle \sigma_A^2\sigma_B^2 - \left| \left\langle \frac{\partial u_\alpha}{\partial c_\alpha} \right\rangle \rho\sigma_A\sigma_B - z \right|^2}. \tag{20}$$

Substituting the terms and only leaving $\overline{\chi}$ and $\overline{\nu}$, we have

$$\begin{aligned} \overline{\chi} &= \frac{r\overline{\chi}\sigma_A^2\sigma_B^2(r^2\overline{\chi}^2\sigma_A^2\sigma_B^2 + |1 - r\overline{\nu}\rho\sigma_A\sigma_B|^2)}{r^2\overline{\chi}^2\sigma_A^4\sigma_B^4 + |z^*(r^2\overline{\chi}^2\sigma_A^2\sigma_B^2 + |1 - r\overline{\nu}\rho\sigma_A\sigma_B|^2) - \rho\sigma_A\sigma_B(1 - r\overline{\nu}\rho\sigma_A\sigma_B)|^2}, \\ \overline{\nu} &= \frac{(r^2\overline{\chi}^2\sigma_A^2\sigma_B^2 + |1 - r\overline{\nu}\rho\sigma_A\sigma_B|^2)(z^*(r^2\overline{\chi}^2\sigma_A^2\sigma_B^2 + |1 - r\overline{\nu}\rho\sigma_A\sigma_B|^2) - \rho\sigma_A\sigma_B(1 - r\overline{\nu}\rho\sigma_A\sigma_B))}{r^2\overline{\chi}^2\sigma_A^4\sigma_B^4 + |z^*(r^2\overline{\chi}^2\sigma_A^2\sigma_B^2 + |1 - r\overline{\nu}\rho\sigma_A\sigma_B|^2) - \rho\sigma_A\sigma_B(1 - r\overline{\nu}\rho\sigma_A\sigma_B)|^2}. \end{aligned} \tag{21}$$

Without loss of generality, we can always rescale the system as

$$\sigma_A\sigma_B\overline{\nu} \to \overline{\nu}, \ \sigma_A\sigma_B\overline{\chi} \to \overline{\chi}, \ \frac{z}{\sigma_A\sigma_B} \to z. \tag{22}$$

This scaling is equivalent to set $\sigma_A = \sigma_B = 1$. We can rescale the system back adding σ_A and σ_B back after finding the results.

We now try to solve for $\overline{\nu}$ for the rescaled system. First, we notice that $\overline{\chi}$ has two solutions, $\overline{\chi} = 0$ or not. If $\overline{\chi} = 0$, we have

$$\overline{\nu}^2 - \left(\frac{1}{r\rho} - \frac{1}{rz} + \frac{1}{z} \right) \overline{\nu} + \frac{1}{r\rho z} = 0. \tag{23}$$

Clearly, this solution of $\overline{\nu}$ will only be a function of z. Since the eigenvalue density can be now expressed as

$$f_{AB'}(x,y) = \frac{1}{\pi}\partial_{z^*}\overline{\nu}(z,z^*), \tag{24}$$

the solution with $\overline{\chi}=0$ corresponds to zero density in a region. Now, if $\overline{\chi}\neq 0$, it is true we will get a solution of $\overline{\nu}$ which depends on z^*. But the calculation seems to be too complicated. I cannot find a simple connection between what we have in Eq. (21) and the results in [2].

We then try to identify the boundary where the eigenvalue density vanishes. To simplify the calculation, we introduce

$$\begin{aligned} \phi &= r^2\overline{\chi}^2 \geq 0, \\ \psi &= 1 - r\overline{\nu}\rho. \end{aligned} \tag{25}$$

The first equation in Eq. (21) gives when $\overline{\chi}\neq 0$:

$$|z\psi|^2 - 2\rho\mathrm{Re}(z\psi) + \rho^2 - r \leq 0. \tag{26}$$

The inequality is equality when we take $\overline{\chi}=0$, that is, when the first equation in Eq. (21) has two degenerate solutions $\overline{\chi}=0$. To get the boundary, we only need this degenerate case. Therefore, at the boundary, the above equation gives

$$\left(\frac{\mathrm{Re}(z\psi)-\rho}{\sqrt{r}}\right)^2 + \left(\frac{\mathrm{Im}(z\psi)}{\sqrt{r}}\right)^2 = 1, \tag{27}$$

which shows that $z\psi$ is on a circle. At the boundary, where we have set $\overline{\chi}=0$, the second equation in Eq. (21) gives

$$\frac{1-\psi}{r\rho} = \frac{\psi}{z\psi-\rho}. \tag{28}$$

We can make use of that fact $z\psi$ is on a circle, $z\psi-\rho=\sqrt{r}e^{\mathrm{i}\theta}$, to simplify the above equation and obtain

$$z - \rho(1+r) = \sqrt{r}e^{\mathrm{i}\theta} + \rho^2\sqrt{r}e^{-\mathrm{i}\theta}. \tag{29}$$

Clearly, the boundary shown above is an ellipse. Let $z = x+\mathrm{i}y$, we rewrite the above equation as

$$\left(\frac{x-\rho(1+r)}{\sqrt{r}(1+\rho^2)}\right)^2 + \left(\frac{y}{\sqrt{r}(1-\rho^2)}\right)^2 = 1. \tag{30}$$

If we rescale the system back which puts σ_A and σ_B back, the boundary on the complex plane is given by

$$\left(\frac{x/\sigma_A\sigma_B-\rho(1+r)}{\sqrt{r}(1+\rho^2)}\right)^2 + \left(\frac{y/\sigma_A\sigma_B}{\sqrt{r}(1-\rho^2)}\right)^2 = 1. \tag{31}$$

It is clear that inside the ellipse, eigenvalue density is nonzero while it is zero outside the ellipse. The eigenvalue with minimum real part lies on the real axis:

$$x_{\min} = (1-\rho\sqrt{r})(\rho-\sqrt{r})\sigma_A\sigma_B. \tag{32}$$

By definition, $\rho\leq 1$ is the correlation and $r=N/M\leq 1$, so $(1-\rho\sqrt{r})\geq 0$. If we want eigenvalues to have non-positive real part which corresponds to instabilities, we need

$$\rho - \sqrt{r} < 0. \tag{33}$$

In other words, the stability criterion is given by

$$\rho \geq \sqrt{r}. \tag{34}$$

II. MORE CORRELATION

[1] W. Cui, J. W. Rocks, and P. Mehta, The perturbative resolvent method: spectral densities of random matrix ensembles via perturbation theory (2020), arXiv:2012.00663.
[2] Vinayak and L. Benet, Phys. Rev. E **90**, 042109 (2014).