18.338 Project: Are interactions real

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Abstract

The project reviews connections among random matrix eigenvalues, determinantal point processes (DPPs), positions of fermions, and distributions of electron gas. The connection between permanental point processes (PerPPs) and bosons are also introduced. The common features suggest that enforced randomness in one representation give rise to potential energy after transformation. Besides, the transformation (or the symmetry) itself leads to another interaction term, leading to interactions between free particles. We then ask how general this way of forming interactions is, both for physics systems and mathematical structures. In particular, we test if we can use this way (DPP and PerPP) to model the behavior of zeros of random polynomials and vice versa. The conjectured kernel is inspired by Kac-Rice formula. Primary results suggest that our conjecture is not generally true while the trace of the kernel as well as the eigenvectors of the kernel can reflect some information of the original point distribution (or pattern under some unknown interactions).

1 Motivation

There is a trade-off between being abstract and general and being understandable but specific. Abstract mathematics unifies the essence of different objects despite their seemingly unrelated origins, like eigenvalues of random matrices, zeros of random polynomials, and position measurements of fermions. Taking an concrete example helps to use intuitions and past experience to interpret abstract constructions. I was excited to find that the first sampling algorithm of determinantal point processes (DPPs) can be understood as a measuring process of a many-particle mixed state. By making abstractions and connecting to other things, we are able to gain new insights for something we thought we were familiar but actually not. Here, we review how physical entities like fermions and electron gas are related to eigenvalues of random matrices, and try to have a sense about where the interactions between particles may come from. Are the observed interactions just artifacts of statistics and how we thought about them. For some origins that may lead to "interactions", how general can they be.

We first introduce basics about fermions and bosons in Sec. 2, which naturally lead to DPP and permenantal point processes (PerPPs) in Sec. 3. Then, we review eigenvalues of Gaussian orthogonal ensemble (GOE) and Gaussian orthogonal ensemble (GUE) in Sec. 4, and try to explain why they are related to fermions not just from the fact they have the same distributions. From the discussions, we conclude some

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naive ideas and understanding about emergence of "interactions" from enforced randomness and transformation (symmetry). At last, we try to see how far these understandings can go in other scenarios in Sec. 5. We conclude key ideas from reviewing and tests in Sec. 6.

2 Fermions and bosons

We follow [BFB+22] to show the most basic knowledge of physical systems. For simplicity, let's consider a system with N free particles in a one-dimensional quadratic potential well. Free particles means there is no interaction terms in the Hamiltonian but just potential energies for particles and their kinetic energies. In this case, a generic state of the particles would just be a tensor product of states of individual particles:

$$|\psi\rangle = \bigotimes_{i=1}^{N} |\psi_i\rangle := |\psi_1, ..., \psi_N\rangle. \tag{1}$$

However, the particles are indistinguishable, i.e., that permuting particles leaves the state invariant. We can consider exchanging of two particles, which must only give a global phase factor before the wave function. And we know if we exchange the particles back, we get one more phase factor, while everything should be exactly the same as the original one. Therefore, we know that the square of the phase factor is one and conclude the phase factor can only be 1 or -1. If the factor is 1, we call the particles bosons, and if it is -1, we call it fermions.

We first look at fermions. By adding the symmetry, the state for N indistinguishable fermions is

$$|\psi\rangle \propto \sum_{\sigma} \epsilon(\sigma) |\psi_{\sigma(1)}, ..., \psi_{\sigma(N)}\rangle, \tag{2}$$

where σ represents a permutation of $[1, .., N]$ and $\epsilon(\sigma)$ is the signature of σ. We can also say this operation project the tensor product into the anti-symmetric subspace of the Hilbert space. The first observation is that no two fermions can be in the same state, otherwise $|\psi\rangle = 0$ by definition, which is not allowed. This is the famous Pauli's exclusion principle. If we further assume the N-particle system is in its ground state, then we must put the fermions on the energy levels from lowest and occupy the first N levels. Let $\phi_i(x)$ $(i = 0, 1, ...)$ be the eigenstates, or energy levels in position representation of the quadratic potential well, we have the ground state as

$$\psi_0(x_1, ..., x_N) = \frac{1}{\sqrt{N!}} \sum_{\sigma} \epsilon(\sigma) \prod_{i=0}^{N-1} \phi_i(x_{\sigma(i)}), \tag{3}$$

which can be also written as

$$\psi_0(x_1, ..., x_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_0(x_1) & \cdots & \phi_0(x_N) \\ \vdots & \ddots & \vdots \\ \phi_{N-1}(x_1) & \cdots & \phi_{N-1}(x_N) \end{vmatrix}. \tag{4}$$

The probability distribution of positions, $\rho_0(x_1, ..., x_N) = |\psi_0(x_1, ..., x_N)|^2$, is then given by

$$\begin{aligned} \rho_0(x_1, ..., x_N) &= \frac{1}{N!} \begin{vmatrix} \phi_0(x_1) & \cdots & \phi_0(x_N) \\ \vdots & \ddots & \vdots \\ \phi_{N-1}(x_1) & \cdots & \phi_{N-1}(x_N) \end{vmatrix}^2 \\ &= \frac{1}{N!} \det[K_N(x_i, x_j)]_{i,j=1}^{N}, \end{aligned} \tag{5}$$

where

$$K_N(x,y) = \sum_{i=0}^{N-1} \phi_i(x)\overline{\phi_j(y)}. \tag{6}$$

The eigenstates ϕ_i are orthonormal. We therefore see that the position distribution of fermions naturally form a projection DPP. Moreover, since the eigenstates for a quadratic potential are Hermite polynomials (up to a factor $e^{-x^2/2}$ and maybe some constant), the distribution here is exactly the same as that of eigenvalues of $N \times N$ GUE matrices. As a historical remark, DPP was defined originally to describe statistics of fermions [Mac75, Mac77, BM03]. Expanding Eq. (5) explicitly, the polynomials naturally yields (determinant of the Vandermonde matrix)

$$\prod_{i \neq j} |x_i - x_j|^2 \tag{7}$$

in the expression, representing emergent repulsion between fermions under position representation.

We next go to see the bosons. Since now exchanging two particles gives $+1$, we just need to go over all the permutations for the tensor products of N particles to get the typical N-particle state

$$|\psi\rangle \propto \sum_{\sigma} |\psi_{\sigma(1)}, ..., \psi_{\sigma(N)}\rangle. \tag{8}$$

An implication of this form of N-particle state is that we can have multiple particles in the same state. In the energy level or occupation number representation, we need to specify the number of particles at each level. For example, let's say there are α_i particles at ϕ_i, and $N = \alpha_0 + \alpha_1 + ... + \alpha_{n-1}$, which means we occupy the first n energy levels by N particles. If we consider the probability distribution of positions, we have something similar to Eq. (5):

$$\rho(x_1, ..., x_N) = \frac{1}{N!\alpha_0! \cdots \alpha_{n-1}!} \left| \text{per} \begin{bmatrix} \{\phi_0(x_1) & \cdots & \phi_0(x_N)\}\alpha_0 \\ \vdots & \ddots & \vdots \\ \{\phi_{n-1}(x_1) & \cdots & \phi_{n-1}(x_N)\}\alpha_{n-1} \end{bmatrix} \right|^2, \tag{9}$$

where the notation $\{\phi_0(x_1) \cdots \phi_0(x_N)\}\alpha_0$ indicates the row $\phi_0(x_1) \cdots \phi_0(x_N)$ is replicated α_0 times. So we can see the matrix has N rows and N columns. The notation per denotes permanent and by definition, it sums up all the permutations. And this is exactly an example of projection PerPP.

3 Determinantal and permenantal point processes

Up to now, we introduced the distribution of a pure state $|\psi\rangle$ for fermions or bosons. In general, we can have mixed states, which classically mixes different pure states rather than make a superposition. The distribution of measurement results of such mixed states form general DPP and PerPP. We first recall the mathematical definition of DPP and PerPP in K-formalism and then explain the sampling algorithm in terms of physics, which shows their fermion or boson natures.

DPP or PerPP depends on the kernel K. Depending on we are sampling from discrete sets or continuous, the kernel is a matrix K_{ij} or a function $K(x,y)$ (can be imagined as a continuous matrix). For DPP, we sample a subset of points, say J, and the probability density that $x_1, ..., x_k$ are in this set is given by

$\det(K(x_i, x_j))_{1\le i,j\le k}$. Similarly, for PerPP, the probability density that $x_1, ..., x_k$ are in the randomly sampled set is given by $\mathrm{per}(K(x_i, x_j))_{1\le i,j\le k}$. Please see [HKPV06] or the lecture notes for rigorous definitions. We can find the eigenfunctions of K and decompose it as

$$K(x, y) = \sum_i \lambda_i \phi_i(x)\overline{\phi_i(y)}. \tag{10}$$

Guaranteed by Theorem 7 of [HKPV06], we can define independent random variables $I_i \sim Bernoulli(\lambda_i)$, and set

$$K_I(x, y) = \sum_i I_i \phi_i(x)\overline{\phi_i(y)}. \tag{11}$$

The distribution of points sampled will be the same if we first sample I_i and form a K_I and then sample from the projection DPP with kernel K_I.

Our first sampling algorithm of DPP follows this equivalence. We first find the eigenvectors and eigenvalues, and use the eigenvalues and Bernoulli distribution to sample which eigenvectors will be used. Using the sampled eigenvectors, we can form a determinant like Eq. (5) which is the distribution of points. The number of eigenvectors sampled will be the number of points being sampled at last, which is also a property of projection DPP. In terms of physics, the eigenvectors of kernel K are the energy levels of a single particle. The first step of sampling I_i is to determine the occupation number of each energy level (for fermions, just 1 or 0), and yields a pure state which is a superposition of the tensor products of eigenstates. And at last, we do projection DPP (which corresponds to the pure state) to sample points, or measure positions from the pure state selected.

As for PerPP, we still have the kernel, and need the eigen-decomposition. Since there can be many particles on the same energy level, the eigenvalues λ_i can be greater than 1 and we use $\alpha_i \sim Geometric\left(\frac{\lambda_i}{\lambda_i+1}\right)$ to sample the occupation number of energy level i. The total number of particles will be $N = \sum_i \alpha_i$. After determining which energy levels have particles and how many particles are on each level, we construct the distribution for $x_1, ..., x_N$ similar to Eq. (9) where we have N rows correspond to all the energy levels selected and how many times they are replicated (number of particles on that level) and N columns correspond to $x_1, ..., x_N$. However, the permanent is much more difficult to calculate than determinant. Even for the pure state Eq. (9) or projection PerPP, as far as we know, there is no elegant efficient way like DPP where we use conditioned probability based on determinant to sample points one by one. In fact, the difficulty of PerPP and calculating the permanent is the key of boson sampling [AA13], which is one of the early cases where people can experimentally show some quantum advantage [SMH+13]. However, some recent works seemed to show that boson sampling is not classically difficult as imagined (increases the problem size needed to establish quantum computational supremacy via Boson Sampling) [CC20]. But we also did not find a simple practical algorithm to do PerPP sampling. The method in page 6 of [JAQK20] seems to just sample from the geometric distribution. And the Markov Chain Monte Carlo method in [NSC+17] is able to output a sample for 30 bosons in half an hour on a standard laptop, which is too slow for us here.

At last, we end this section by pointing out an observation important for later conjectures. Regardless of DPP or PerPP, i.e., no matter we use Bernoulli or geometric distribution to sample the occupation numbers, the expected number at energy level i is given by λ_i. And therefore, the expected number of points sampled is $\mathrm{tr}K$ for both DPP and PerPP. For continuous kernel, $\mathrm{tr}K = \int K(x, x)\mathrm{d}x$ which still is the sum of all eigenvalues.

4 Eigenvalues as physical entities

After the physics and basic understanding of how to use physics to interpret DPP and PerPP, we now review GOE or GUE eigenvalues. It's straightforward to show that GUE eigenvalues have the same distribution as the fermion positions in Sec. 2 and of course, a projection DPP, just by writing down their distributions and we can see they are the same. After reviewing, we want somehow more understanding of why the seemingly unrelated things can have the same distribution.

We use the matrix calculus to understand GOE and GUE eigenvalues. Say $S = Q\Lambda Q'$ is GOE or GUE, we have

$$1 \propto \int e^{-\frac{1}{2}\mathrm{tr}S'S}(\mathrm{d}S)^{\wedge}. \tag{12}$$

To change the variable to eigenvalues, we can know the distribution of eigenvalues. Note that

$$\mathrm{d}S = \mathrm{d}Q\Lambda Q' + Q\mathrm{d}\Lambda Q' + Q\Lambda \mathrm{d}Q', \tag{13}$$

and therefore

$$Q'\mathrm{d}SQ = Q'\mathrm{d}Q\Lambda + \mathrm{d}\Lambda + \Lambda \mathrm{d}Q'Q. \tag{14}$$

The point is that since $Q'Q = I$, so we have $\mathrm{d}Q'Q + Q'\mathrm{d}Q = 0$, and then conclude that $Q'\mathrm{d}Q$ is anti-symmetric. So we can get

$$Q'\mathrm{d}SQ = Q'\mathrm{d}Q\Lambda - \Lambda Q'\mathrm{d}Q + \mathrm{d}\Lambda. \tag{15}$$

It's not hard to show that $(Q'\mathrm{d}SQ)^{\wedge} = (\mathrm{d}S)^{\wedge}$ (imagine $\det AB = \det BA$), so we can get

$$(\mathrm{d}S)^{\wedge} = \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} (Q'\mathrm{d}Q)^{\wedge} (\mathrm{d}\Lambda)^{\wedge}, \tag{16}$$

just by definition expanding the right hand side, where $\beta = 1$ for GOE and $\beta = 2$ for GUE (for GOE we only do production for the upper triangle). Go to the first equation, we can obtain

$$\begin{aligned} \int e^{-\frac{1}{2}\mathrm{tr}S'S}(\mathrm{d}S)^{\wedge} &= \int e^{-\frac{1}{2}\mathrm{tr}\Lambda^2} \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} (Q'\mathrm{d}Q)^{\wedge} (\mathrm{d}\Lambda)^{\wedge} \\ &\propto \int e^{-\frac{1}{2}\sum_i \lambda_i^2} \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} (\mathrm{d}\Lambda)^{\wedge}. \end{aligned} \tag{17}$$

We therefore conclude that the distribution of eigenvalues satisfies

$$\rho(\{\lambda_i\}) \propto e^{-\frac{1}{2}\sum_i \lambda_i^2} \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} = e^{-\frac{1}{2}\sum_i \lambda_i^2 + \sum_{i<j} \log |\lambda_i - \lambda_j|^{\beta}}, \tag{18}$$

where the coefficient depends on Gaussian distributions and $\int (Q'\mathrm{d}Q)^{\wedge}$ which can be understood as surface area of unit high dimensional "sphere".

The derivation is so elegant but everything seems to be classical and has no relation to quantum physics. There is something more fundamental behind both the quantum fermions and the GUE eigenvalues. We focus on the term $\prod_{i<j} |\lambda_i - \lambda_j|^{\beta}$ in Eq. (18). This term suggest negative correlation among eigenvalues, or repulsion between particles. Through the derivation, we know this term comes from the "Jacobian"

due to orthogonal/unitary transformation (change variables), highly depending on the fact $Q'\mathrm{d}Q$ is anti-symmetric, and is irrelevant to the distribution of S. We claim that eigenvalues are entities living in the anti-symmetric subspace which is the reason they have the same behaviors as fermions. To elaborate, we consider a heuristic geometric understanding that $S = Q\Lambda Q'$ is to transform S from Cartesian coordinate to polar coordinate, where Q represents the angle or rotation, and Λ is similar to radius or length. Then, $(\mathrm{d}S)^\wedge$ is the volume element, and we can express that by production surface area element on the sphere, $\prod_{i<j}|\lambda_i-\lambda_j|^\beta(Q'\mathrm{d}Q)^\wedge$, and the radius change perpendicular to the sphere, $(\mathrm{d}\Lambda)^\wedge$. Because of the nature of orthogonal/unitary transformation, we need to project the changes into the tangential space of the sphere, which is anti-symmetric, to get the surface area change. The projection to anti-symmetric subspace leads to $\prod_{i<j}|\lambda_i-\lambda_j|^\beta$. Whenever we consider the measure of eigenvalues, we need to consider this term or a projection to anti-symmetric subspace which is equivalent to say that eigenvalues are living in the anti-symmetric subspace. By definition, fermions are entities living in the anti-symmetric subspace of the multi-particle Hilbert space. Whenever we add one more indistinguishable fermion in, we need to project the tensor product into the anti-symmetric subspace. The anti-symmetry is the origin of intrinsic repulsion for both the fermions and eigenvalues, which is irrelevant to other things but just because they are constrained to have the symmetry.

Another term $e^{-\frac{1}{2}\sum_i\lambda_i^2}$ is easier to understand. For fermions in Sec. 2, we say they are in quadratic well, so naturally there will be a term embody the potential energy. While for eigenvalues, it is interesting that this term comes from the enforced Gaussian distribution of S elements. We therefore conclude a general picture that by enforcing some randomness and symmetry or transformation, the randomness can give some interaction or energy in another representation and the transformation/symmetry independently gives another interaction term.

There are many interesting questions after we get here. I have not fully understood why GUE eigenvalues correspond to many fermions on the ground state of the system. A bigger question is how general this mechanism of "interactions" is. This corresponds to the title "are interactions real." Notice that we can imagine Eq. (18) as Gibbs distribution and the energy is given by

$$H(\{\lambda_i\}) = \frac{1}{2}\sum_i \lambda_i^2 - \sum_{i<j}\log|\lambda_i-\lambda_j|^\beta, \tag{19}$$

where the interaction energy $-\log|\lambda_i-\lambda_j|^\beta$ is exactly the electric potential for two-dimensional electrons. If the electric interactions (even in three dimension) are just artifacts of statistics and the symmetry, it will change the picture a lot. We thought the interactions are fundamental laws, but now it is just some correlation we measured due to the fact the entities are restricted in the anti-symmetric subspace. If this makes sense, we can easily answer some questions like self-energy. Imagine an electron as a ball, if we calculate the electric energy of itself, and take the radius to infinity, the energy will diverge. However, we know that there is no self-energy for the interactions emerging from symmetry (Eq. (18)) and avoid the divergence, as such interactions only shows up when there are multiple particles.

5 Conjecture and test

After knowing that the enforced randomness in one representation give rise to potential energy after transformation, and the transformation (or the symmetry) itself leads to another interaction term. We want to

ask how general this way of forming interactions is, both for physics systems and mathematical structures. In particular, we test if we can use this way (DPP and PerPP) to model the behavior of zeros of random polynomials and vice versa. One reason we test random polynomial is that zeros of polynomials are also eigenvalues of a companion matrix, closely related to something we have known. Another perspective is that we can think there is a transformation between roots and the coefficients of the polynomial (see [BBL96] for example). At last, the roots of random fields (high-dimensional random processes [BLL22, AT07]) which can also be seen as critical points of some other random fields are important and interesting for many subjects across physics [AAv13], machine learning [CHM+15, Ans22], and ecology. It will be great if we can use simple point processes to model the behaviors of these zeros or critical points.

The bridge I found is though the number of points. As mentioned, for both DPP and PerPP, the expected number of points sampled is $\mathrm{tr}K$ where K is the kernel. In general, there is also a fixed formula for the expected number of zeros or critical points, i.e., Kac-Rice formula [Kac43, Ric44]. Kac was interested in exactly the same problem, i.e., roots of random polynomials [Kac43], and got

$$\mathbf{E}\{|\mathrm{Val}^X_{[0,T]}(0)|\} = \int_0^T \mathbf{E}\{|X'(t)||X(t)=0\}\mathbf{P}\{X(t)=0\}\mathrm{d}t, \tag{20}$$

where $X(t)$ is a random polynomial and $\mathrm{Val}^X_{[0,T]}(0)$ is the set of its zeros in $[0,T]$. So, in the regime of random polynomials, there must be a connection between DPP and Kac-Rice formula. It is intuitive that repulsion between zeros limit the expected number of zeros. Rice focused on noisy signals (random process) instead [Ric44]. For a random field, say f, which can be regarded as high-dimensional random processes, we have [AT07]

$$\mathbf{E}\{|\mathrm{Crt}^f|\} = \int \mathbf{E}\{|\det(\nabla^2 f(x))||\nabla f(x)=0\}\mathbf{P}\{\nabla f(x)=0\}\mathrm{d}x, \tag{21}$$

where Crt^f is the set of critical points of f. Obviously, the Kac-Rice formula remains similar regardless of whether we are studying random polynomials. The intuition for Kac-Rice formula is simple, which is also related to change of variable. The number of zeros of $X(t)$ is just the integral $\int \delta(X)\mathrm{d}X$ and can be converted to $\int \delta(X(t))|X'(t)|\mathrm{d}t$ introducing the derivative $|X'(t)|$. Later, we just need to remember that when calculating the expectation that both X and X' are random variables given t.

We followed [Boj19] for detailed and completed review of how Kac-Rice formula is applied to random polynomials. In fact, for any random function f that is a weighted sum of smooth function where the weighted coefficients are zero mean Gaussian, we all can define

$$K(x,y) = \frac{1}{\pi}\sqrt{\frac{\partial^2}{\partial x \partial y}\log \boldsymbol{E}(f(x)f(y))}, \tag{22}$$

and have the following

$$\mathbf{E}\{|\mathrm{Val}^f_{[0,T]}(0)|\} = \int_0^T K(t,t)\mathrm{d}t = \mathrm{tr}K \text{ (in } [0,T]). \tag{23}$$

The same result can also be obtained via the elegant geometric argument [EK95]. We therefore conjecture that $K(x,y)$ defined above can give certain DPP or PerPP or mixture of the two and can reproduce the distribution of zeros.

We use numerical experiments to test the idea. All the codes can be found on github.com/liuyz0/RandPoly, containing test-i.ipynb written in Julia ($i = 1, ..., 5$ which will be explained as follows). We first play

with Kac polynomials. By writing the polynomials as

$$P_n(t) = \sum_{k=0}^{n} a_k t^k, \tag{24}$$

Kac polynomials means a_k are i.i.d. standard normal. If we let a_k be complex and study the complex zeros, [PV05, FI18] has proved the complex zeros are exactly DPP and the kernel is square of what we defined, Eq. (22). The square is reasonable as it is complex and has tow variables now. We test the real zeros with real coefficients. The kernel can be calculated explicitly as

$$K(x,y) = \frac{1}{\pi}\sqrt{\frac{1}{(1-xy)^2} - \frac{(n+1)^2(xy)^n}{(1-(xy)^{n+1})^2}}. \tag{25}$$

Interestingly, for Kac polynomials, when $n \to \infty$, $K(x,y) \propto \boldsymbol{E}(f(x)f(y))$. In test-1.ipynb, we solved many Kac polynomials and collect the zeros, we also use the kernel calculated to construct DPP and sample. The results suggest the zeros agree well with DPP sampling (Fig. 1, **a** and **b**). What is also interesting is that after determining the occupation number of each energy level, if we do not do projection DPP but sample one point from a eigenfunction with non-zero occupation number, we still get very similar results (Fig. 1, **c**). This suggest that the eigenfunctions already encode a lot of information, and correlations between particles are less important in some cases.

We then tested more polynomials. The coefficients (a_k in Eq. (24)) of Kostlan-Shub-Smale (KSS) polynomials satisfy $\mathrm{Var}(a_k) = \binom{n}{k}$. The kernel can also be calculated explicitly as

$$K(x,y) = \frac{1}{\pi}\sqrt{\frac{n}{(1+xy)^2}}. \tag{26}$$

The zeros of the random polynomials are quite concentrated (Fig. 2**a**). It turns out that the kernel K cannot give neither DPP nor PerPP as it has negative eigenvalues. We check the eigenvectors and eigenvalues, and found that the eigenvalues are quite symmetric regarding 0, and eigenvectors whose eigenvalues have similar absolute values are similar. We then think about some kind of particle and antiparticle. We sample the number of particles (geometric distribution) on the eigenstate based on the absolute value of the eigenvalues and then do measurements of the eigenstates repeating as many times as the number of particles on that state. Sampled points will be classified to be particles or antiparticles based on the signs of eigenvalues. The particles have the histogram (Fig. 2**b**) similar to antiparticles (Fig. 2**c**). And both particles and antiparticles have similar shaped distribution as the zeros in Fig. 2**a**. If we eliminate a particles and antiparticles if they appear closely, we get the only particles left and giving another histogram (Fig. 2**d**), which is not good and we may not draw further conclusions.

Up to now, we already know that simple DPP or PerPP may not able to describe all behaviors of zeros, or our conjecture about kernel is not correct. But we also do not have more conjectures. To test if the conjectured kernel (Eq. (22)) may be correct for some general cases, we use DPP to construct random polynomials and then calculate the kernel and see if it can give the original DPP. We tested GUE eigenvalues. We can sample a set of GUE eigenvalues and then use the eigenvalues as zeros to construct a polynomial f. By discretization of the space, $f(x)$ is numerically stored as a vector and $f(x)f(y)$ is a matrix. By sample a lot of GUE, we get a lot of $f(x)f(y)$ matrices and can estimate $\boldsymbol{E}(f(x)f(y))$. And later operations like taking derivatives can all be done by finite difference. Anyway, we use numerical methods to get the kernel as a matrix. Unfortunately, this kernel still has negative eigenvalues and cannot give the original DPP. However,

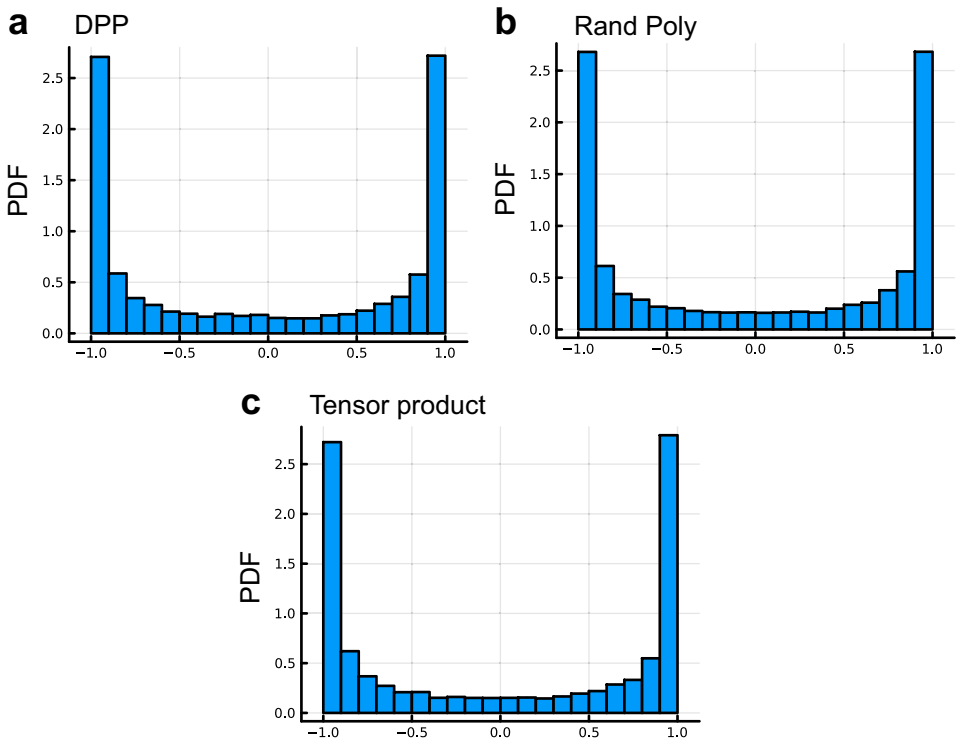

Figure 1: Histograms of points. We tested degrees $n = 100, 200, 300$, and present $n = 200$ in this figure. **a** is sampled from the conjectured kernel with DPP sampling rules. **b** draws the solved zeros from Kac random polynomials. **a** is sampled from the conjectured kernel not doing projection DPP after sampling occupation numbers but sample a point from each sampled eigenvectors independently. **a** and **b** are obtained from test-1.ipynb, while **c** is obtained from test-2.ipynb.

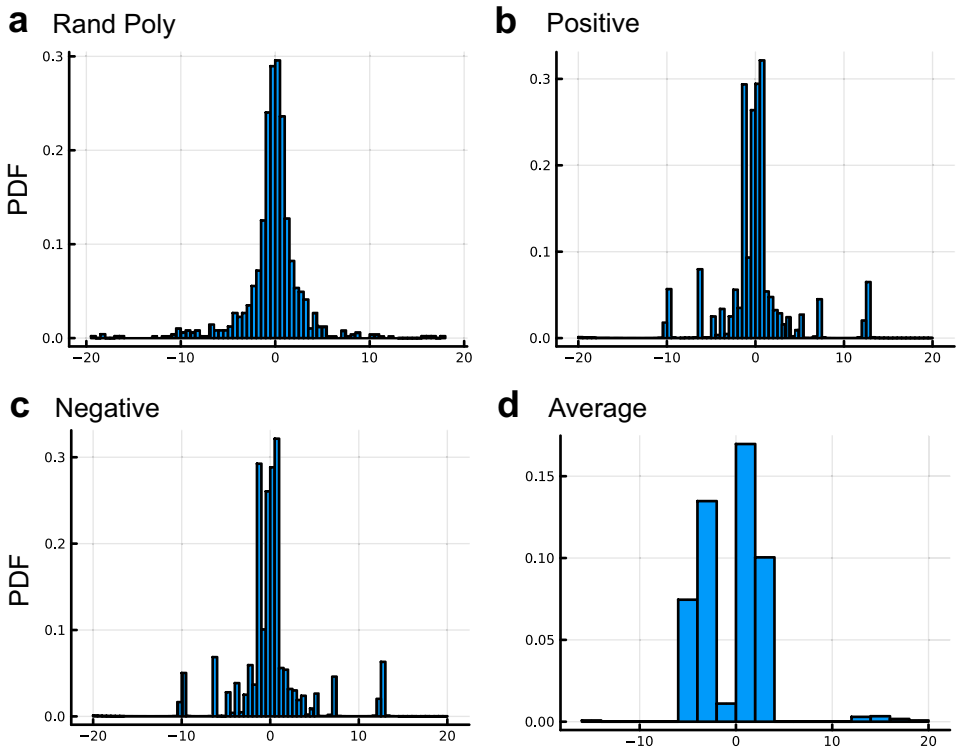

Figure 2: Histograms of points. **a** is the histogram of zeros from degree 100 KSS random polynomials. **b** draws the particles sampled from the kernel. **c** draws the antiparticles sampled from the kernel. And **d** draws the left particles after annihilation due to meeting antiparticles. Figures are obtained from test-3.ipynb.

we find that the eigenvector with the largest eigenvalue show some patterns of the underlying DPP. When there are n peaks (corresponding to n zeros) in DPP distributions (Fig. 3, **a** or **b**), there will be $n-1$ peaks in the distribution given by the eigenvector (Fig. 3, **c** or **d**). And although the tr$K$ formula does not reflect the number of zeros exactly, we have tr$K$ nearly linear with respect to number of zeros (Fig. 3**e**).

Rethinking about the formula Eq. (22) and its assumption that coefficients should be mean zero (the expectation of the polynomial at every point is 0), we guess more generally Eq. (22) is related to how many times the random field go back to its own mean value. Therefore, we define a new kernel

$$K(x,y) = \frac{1}{\pi}\sqrt{\frac{\partial^2}{\partial x \partial y}\log \boldsymbol{E}[(f(x)-\mathbf{E}f(x))(f(y)-\mathbf{E}f(y))]}, \tag{27}$$

and want to see if this can better approximate zeros or the number of zeros. It turns out this modification does do something! When there are n peaks (corresponding to n zeros) in DPP distributions (Fig. 4, **a** or **b**), there will be n peaks in the distribution given by the eigenvector (Fig. 4, **c** or **d**).

6 Conclusion

In conclusion, this project delves into the intricate interplay among random matrix eigenvalues, determinantal point processes (DPPs), fermion positions, and electron gas distributions, while also exploring connections between permanental point processes (PerPPs) and bosons. The identification of common features points to the intriguing concept that enforced randomness in one representation can give rise to potential energy upon transformation. Moreover, the transformation or symmetry itself introduces an additional interaction term, fostering interactions among initially free particles. The overarching question posed by the study revolves around the generalizability of this method for forming interactions, applicable to both physical systems and mathematical structures.

A notable aspect of the investigation involves testing the viability of using DPPs and PerPPs to model the behavior of zeros of random polynomials and vice versa. Despite the initial conjecture inspired by the Kac-Rice formula, primary results suggest that this conjecture is not universally true. Nevertheless, we find the intriguing insight that the trace of the conjectured kernel and the eigenvectors of the conjectured kernel can convey valuable information about the original point distribution or pattern, even in the presence of unknown interactions.

While the initial conjecture may not hold in all cases, the findings contribute to a deeper understanding of the nuanced ways in which randomness, transformations, and symmetries give rise to interactions, offering an avenue for further exploration and refinement in both physics and mathematics.

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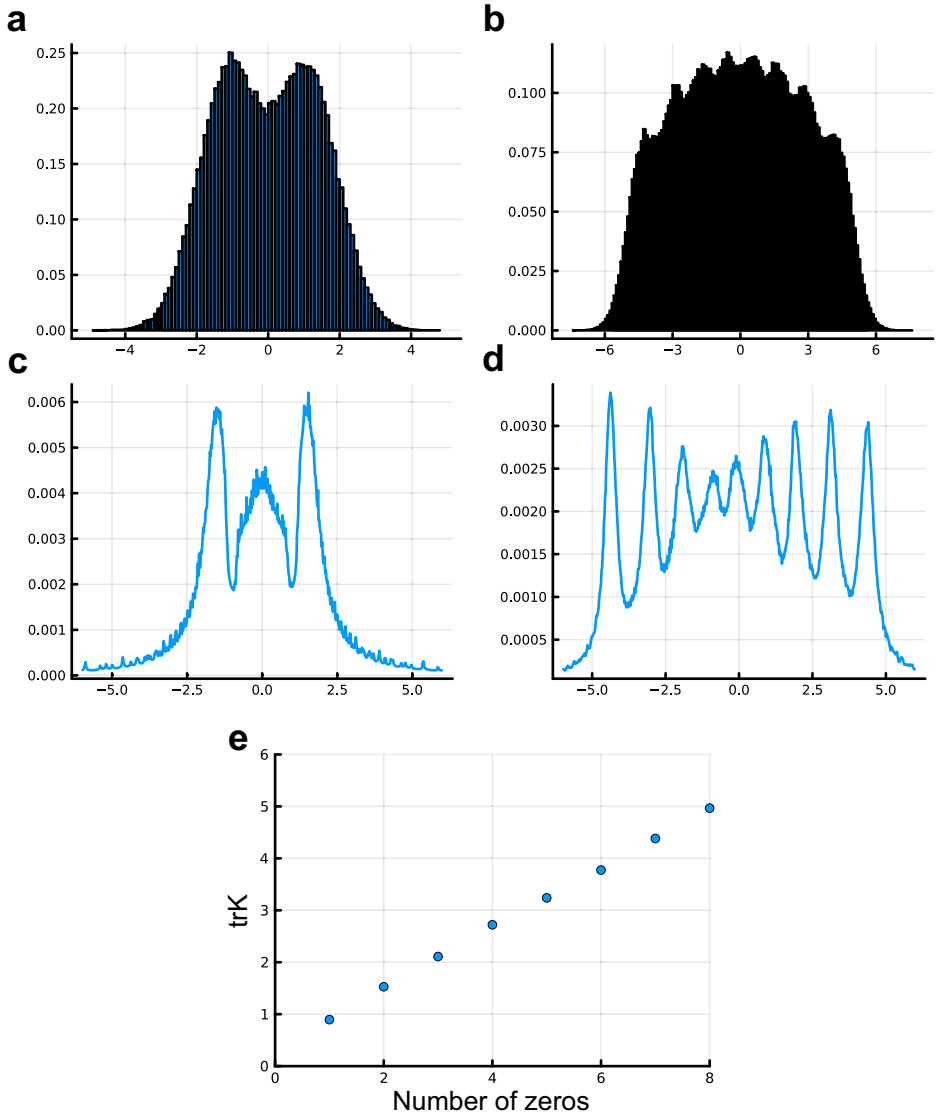

Figure 3: GUE eigenvalue distributions (**a** and **b**) and the distribution of corresponding conjectured kernel (Eq. (22)) eigenvectors (**c** and **d**). We tested $n \times n$ GUE from $n = 1$ to $n = 10$ and the pattern is always true. **a** and **c** show the case $n = 2$, while **b** and **d** show the case $n = 8$. **e** plots the relation between $\mathrm{tr}K$ and the real number of particles or zeros or just n. Figures are obtained from test-4.ipynb.

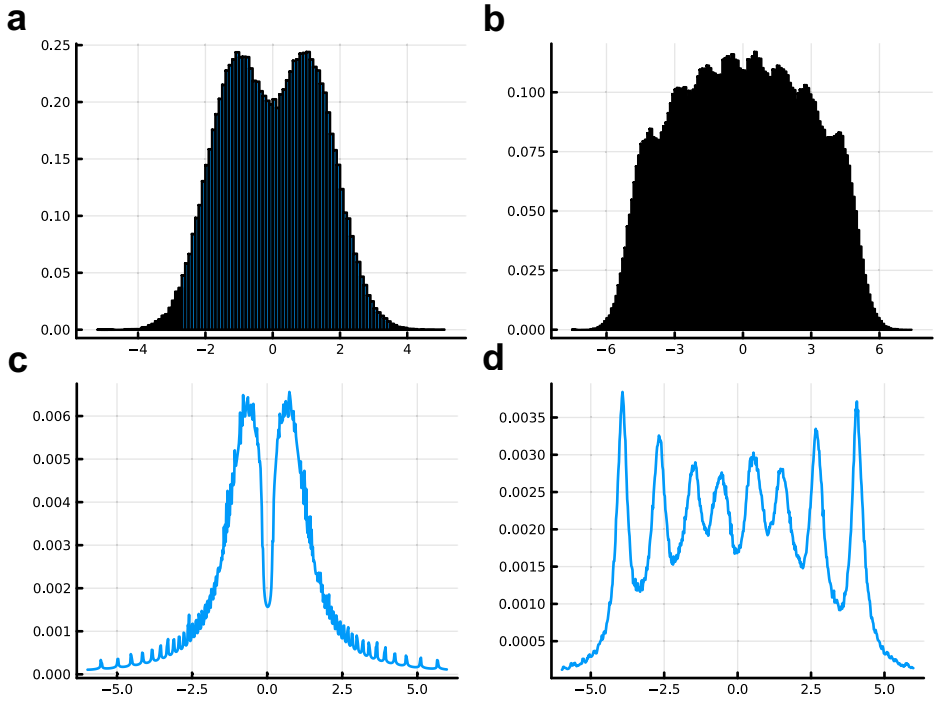

Figure 4: GUE eigenvalue distributions (**a** and **b**) and the distribution of corresponding conjectured kernel (Eq. (27)) eigenvectors (**c** and **d**). We tested $n \times n$ GUE from $n = 1$ to $n = 10$ and the pattern is always true. **a** and **c** show the case $n = 2$, while **b** and **d** show the case $n = 8$. Figures are obtained from test-4.ipynb.

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