Eleven Tools in Feedback Control

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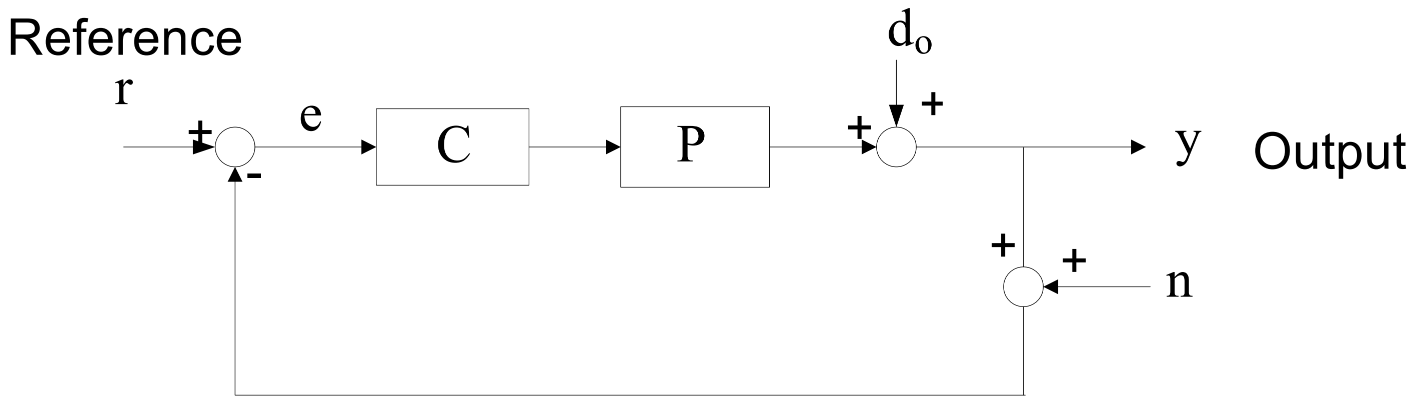
Contents

- Basics: Arithmetic of LTI systems, Goals of feedback, Loop shaping, Tradeoffs
- Fundamental limitations
  - Bandwidth
  - Waterbed
  - Unstable zeros
  - Magnitude-phase relationship
- Practical control engineering
  - Sampling time
  - Delays
  - Time-frequency relationship

#1

Arithmetic of feedback loops



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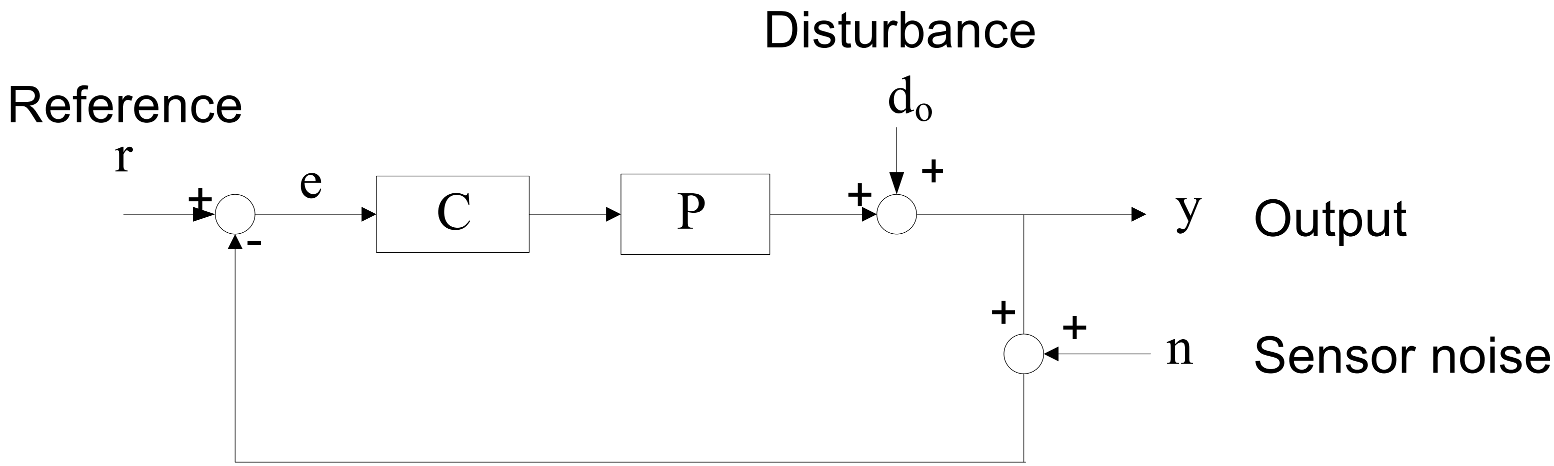

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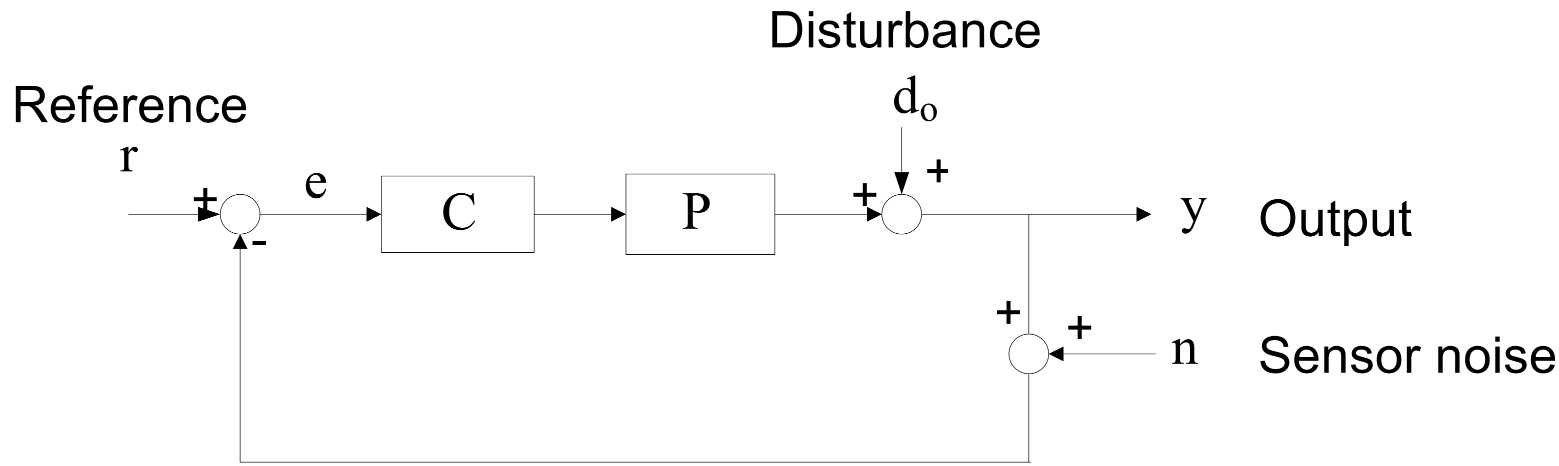
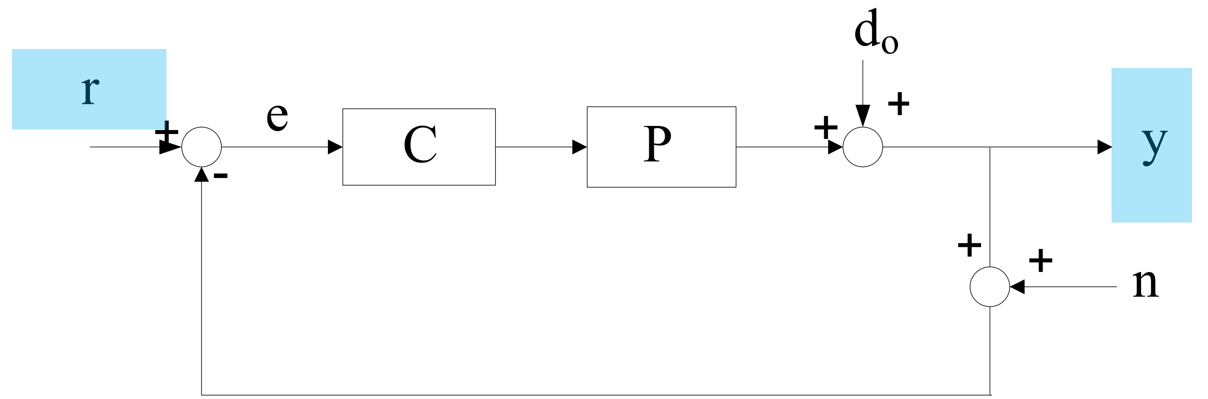
Arithmetic of feedback loops



$$y = \begin{bmatrix} \frac{PC}{1+PC} & \frac{1}{1+PC} & \frac{-PC}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ d_0 \\ n \end{bmatrix} \quad \begin{array}{l} \text{Reference} \\ \text{Disturbance} \\ \text{Sensor noise} \end{array}$$

#2

Goals of feedback



$$y = \begin{bmatrix} \frac{PC}{1+PC} & \frac{1}{1+PC} & \frac{-PC}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ d_0 \\ n \end{bmatrix}$$

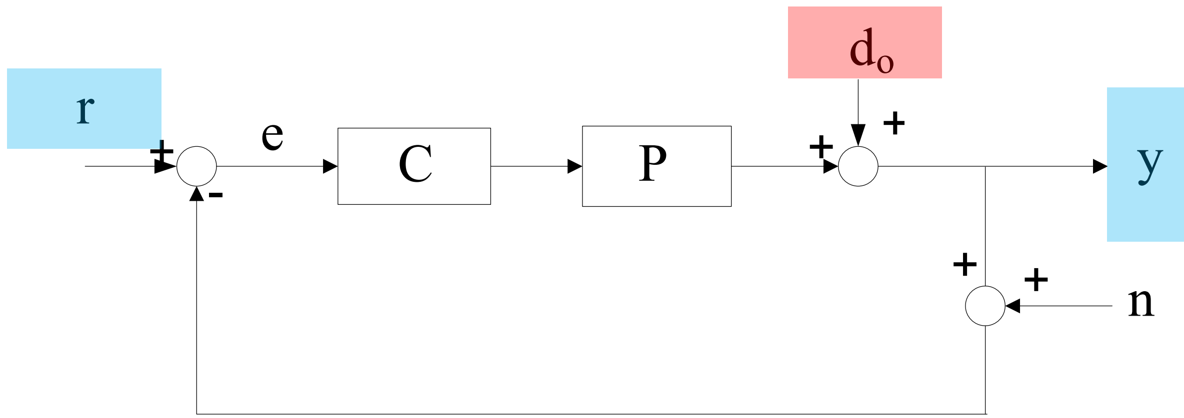
Reference

Desired: ~1

Complementary Sensitivity Function

#2

Goals of feedback



Sensitivity Function

$$y = \begin{bmatrix} \frac{PC}{1+PC} & \frac{1}{1+PC} & \frac{-PC}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ d_0 \\ n \end{bmatrix}$$

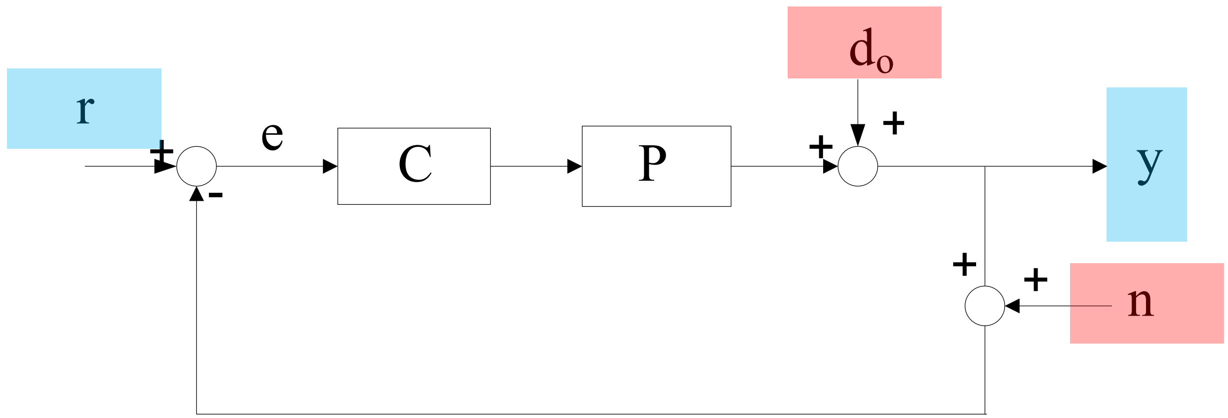
Disturbance

Desired: ~1 ~0

Complementary Sensitivity Function

#2

Goals of feedback



$$y = \begin{bmatrix} \frac{PC}{1+PC} & \frac{1}{1+PC} & \frac{-PC}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ d_0 \\ n \end{bmatrix}$$

r: Reference

d_0: Disturbance

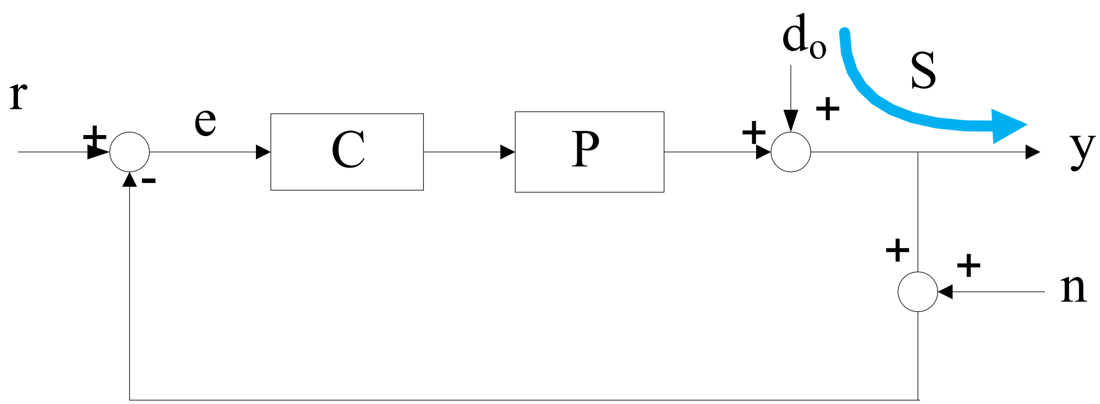
n: Sensor noise

Desired: ~1 ~0 ~0

Can’t do well on both!

#3

Tradeoffs



$$y = \begin{bmatrix} \frac{PC}{1+PC} & \frac{1}{1+PC} & \frac{-PC}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ d_0 \\ n \end{bmatrix}$$

Sensitivity Function: $S \triangleq (I + PC)^{-1}$

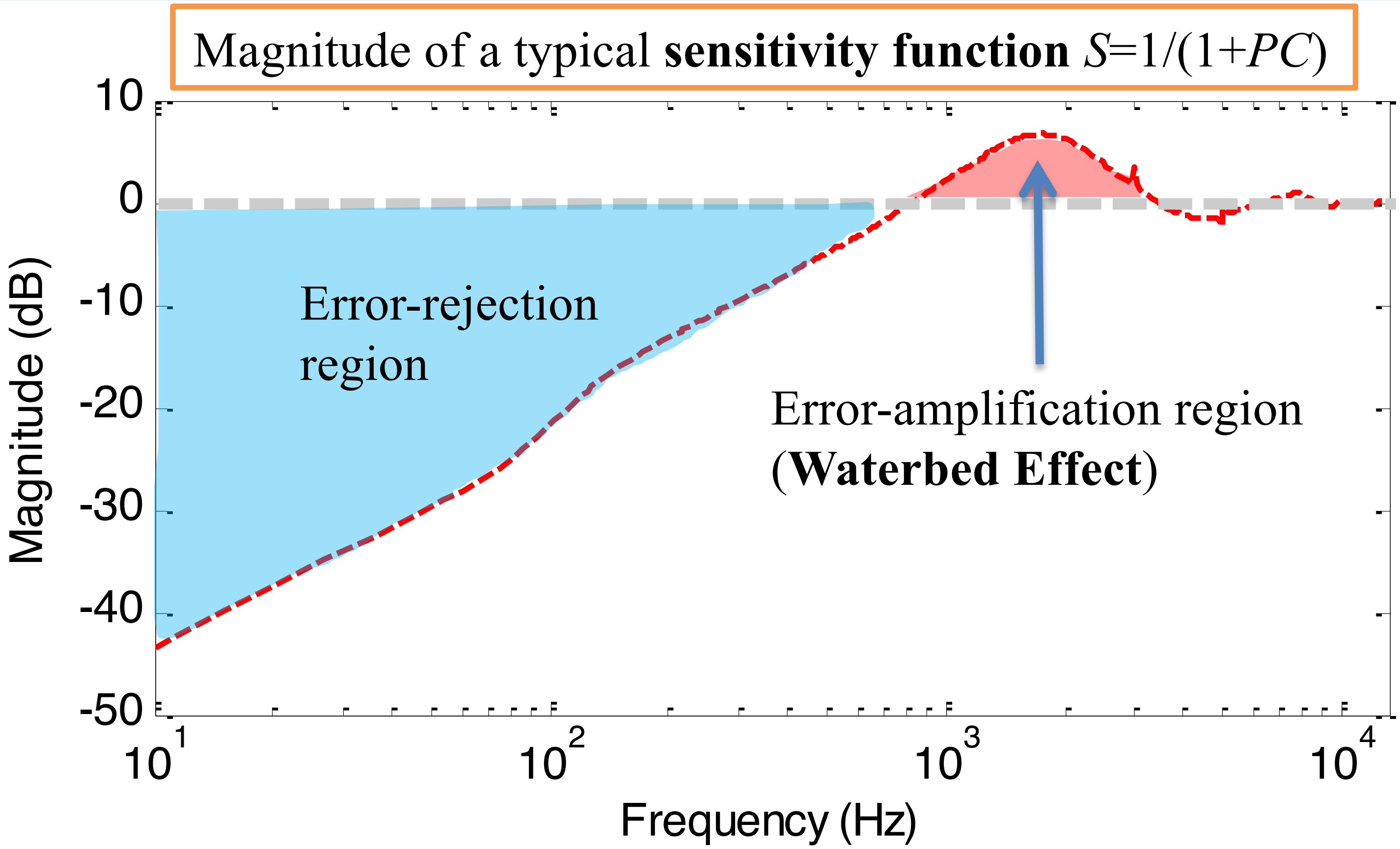
Complementary Sensitivity Function:

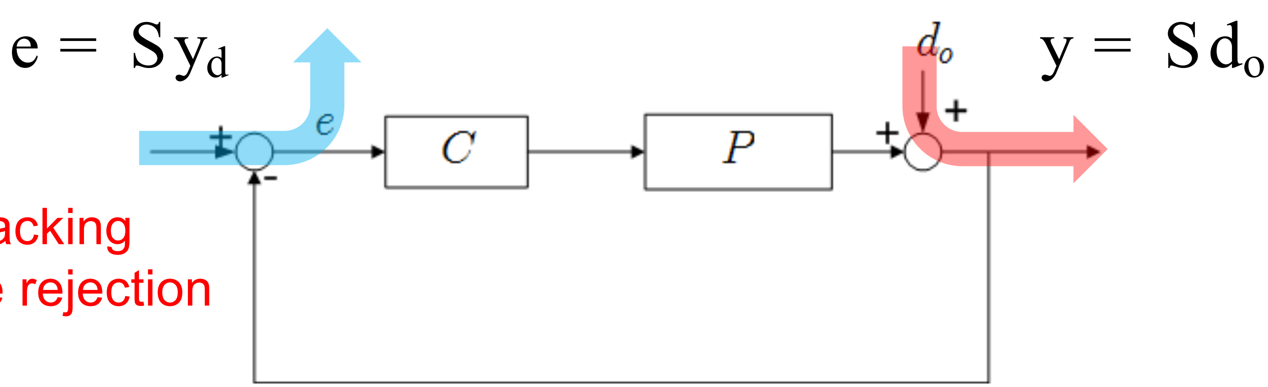
$$T \triangleq PC(I + PC)^{-1}$$

Fundamental Constraint: $S + T = I$

#4

Loop shaping

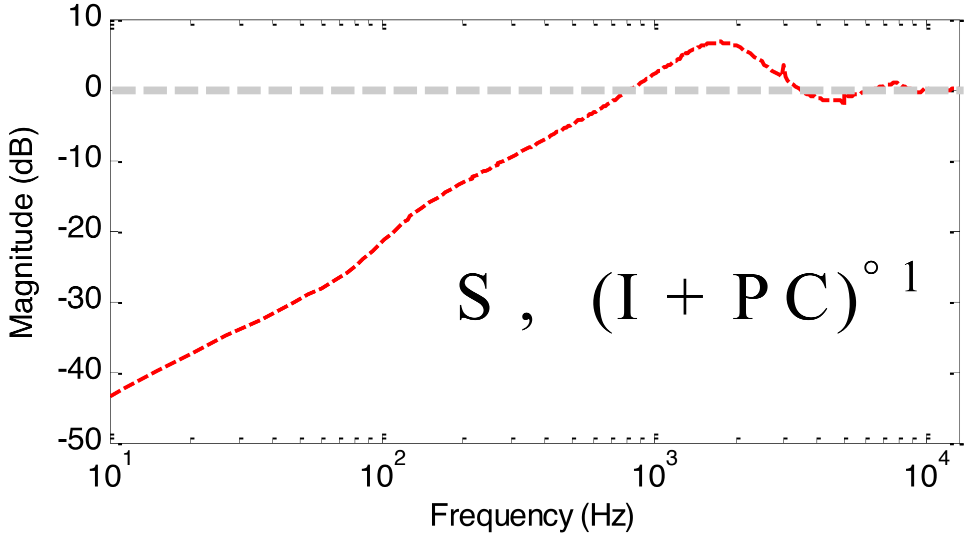




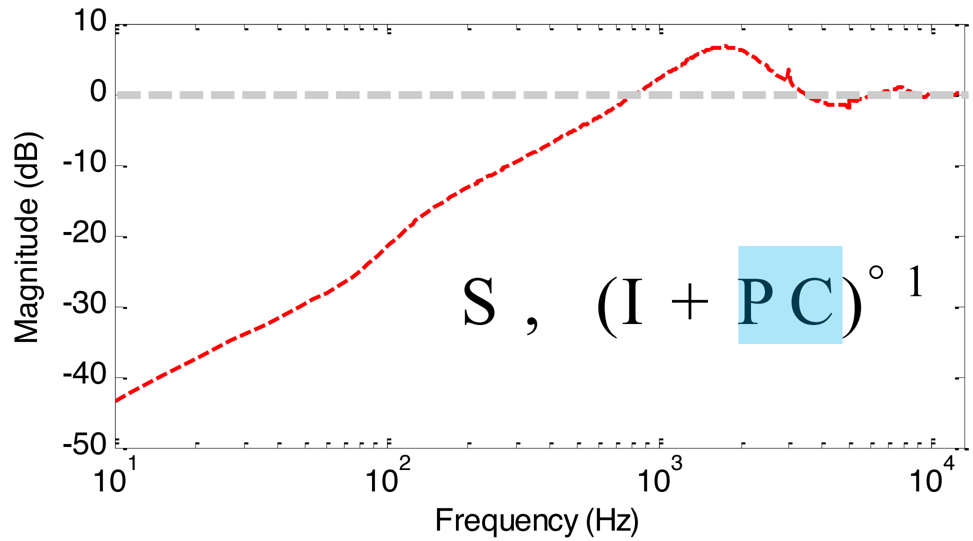
S defines the tracking and disturbance rejection performances

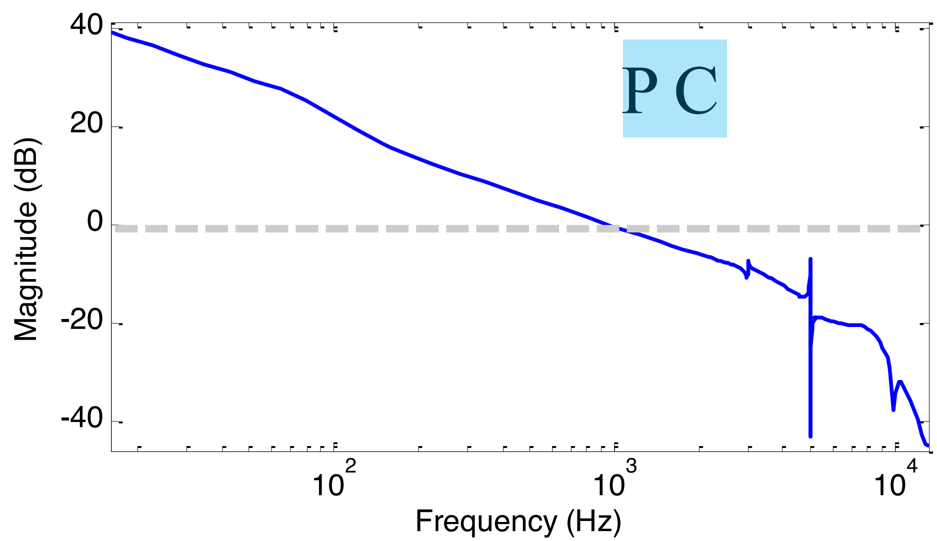
#5

High-gain feedback



High-gain feedback

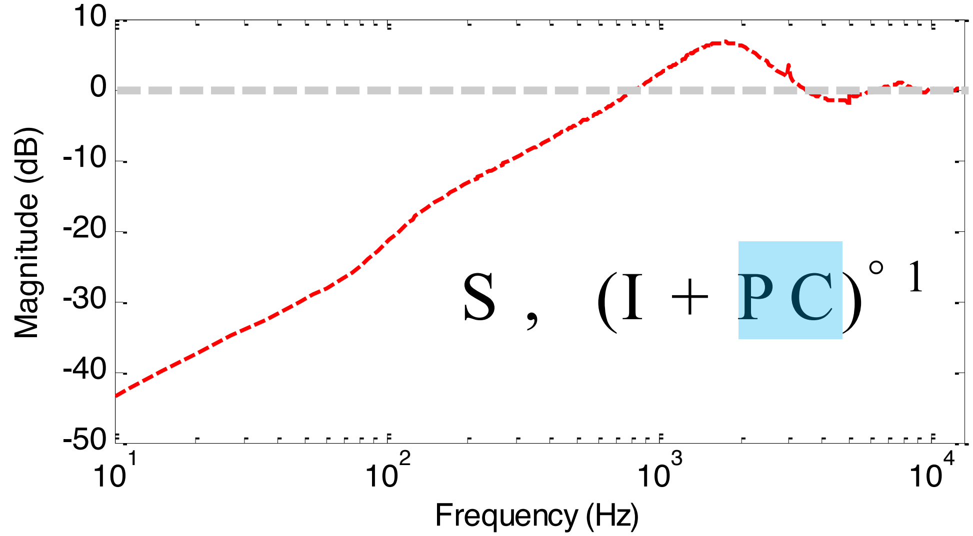




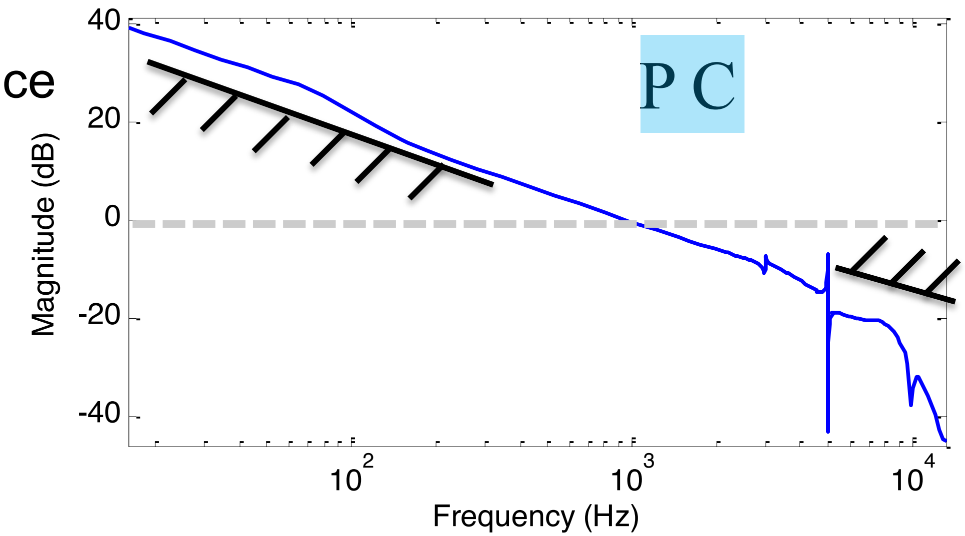
small gain in ***S***
⟷
high gain in ***PC***

#5

High-gain feedback



Typical high-gain control for performance at low frequency

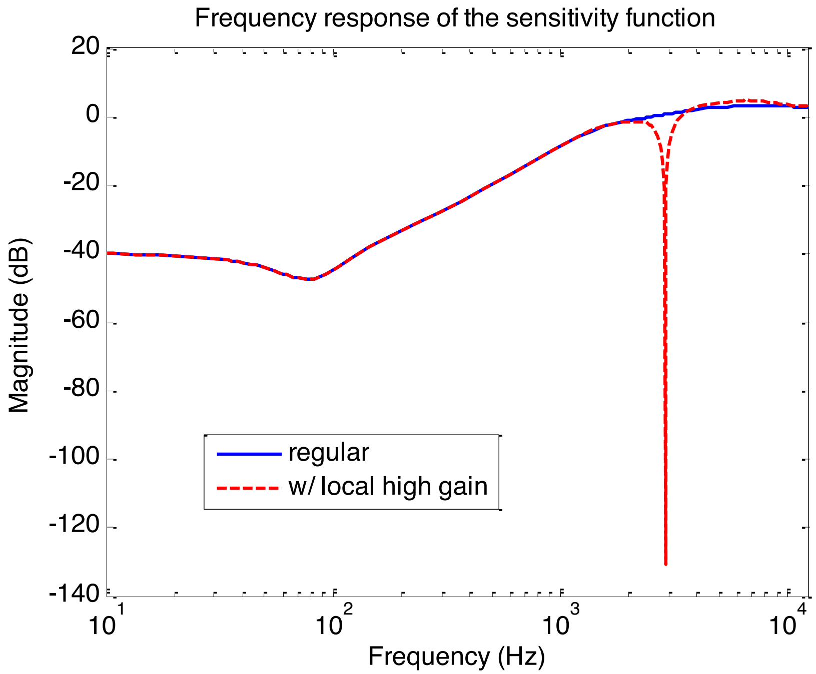


Typical low-gain control for robustness at high frequency

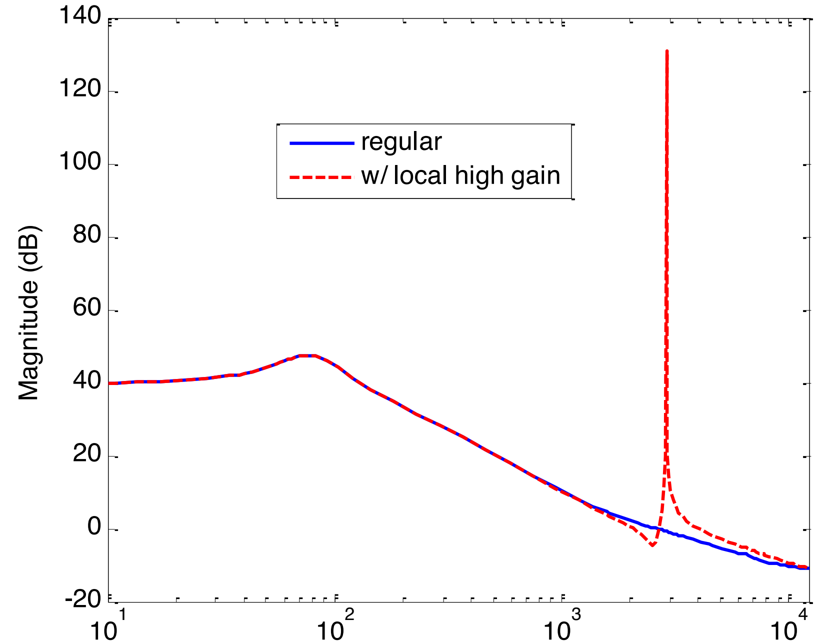
#5

Local high-gain feedback

$$S \triangleq (I + PC)^{-1}$$



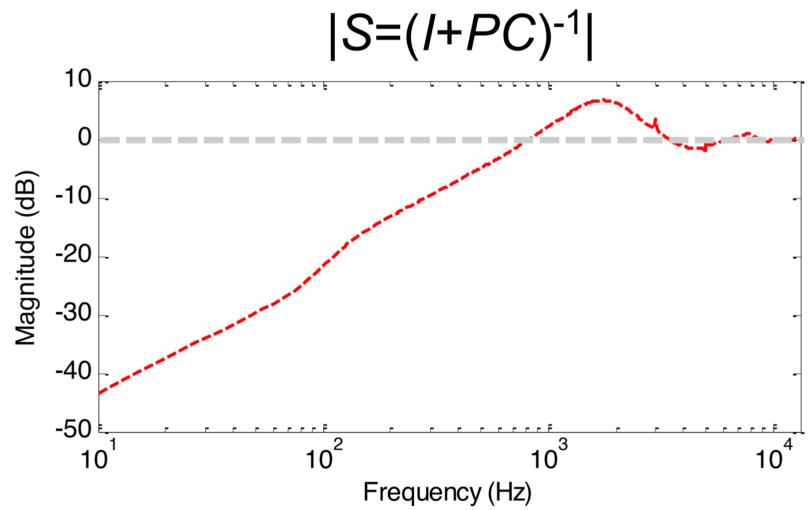
$$PC$$



#6

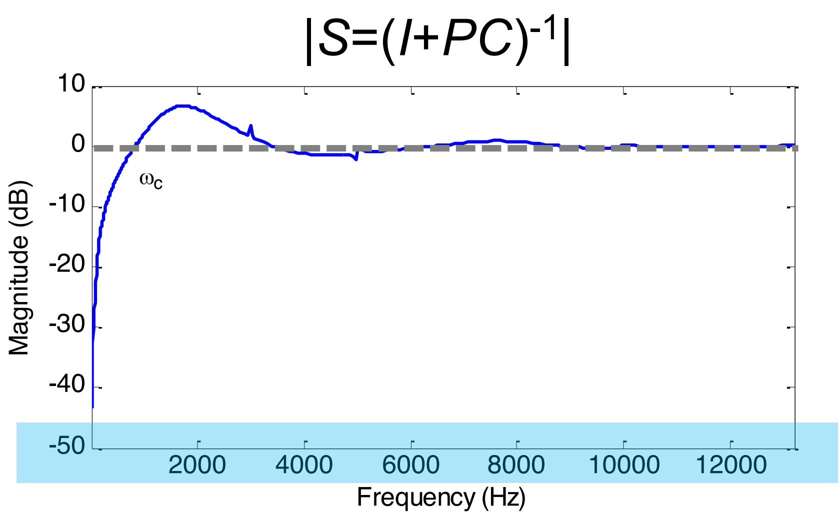
Bode's Integral

Typical feedback design

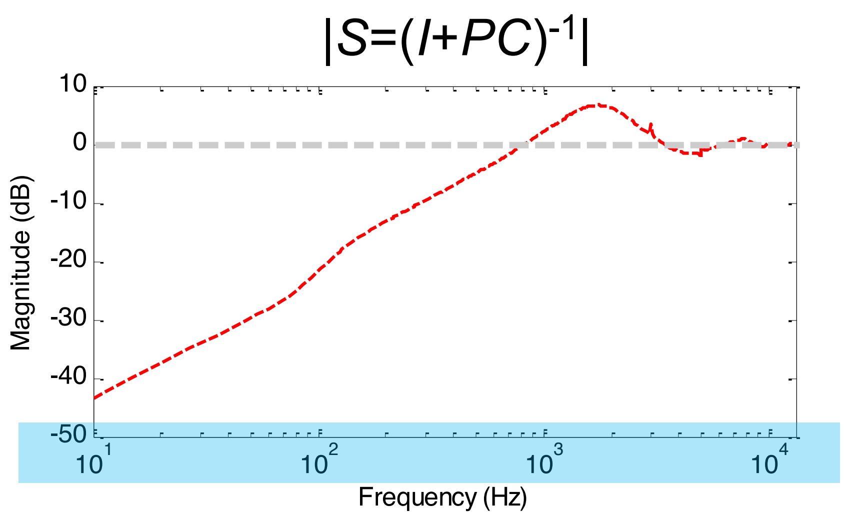


Bode's Integral

x-axis in linear scale

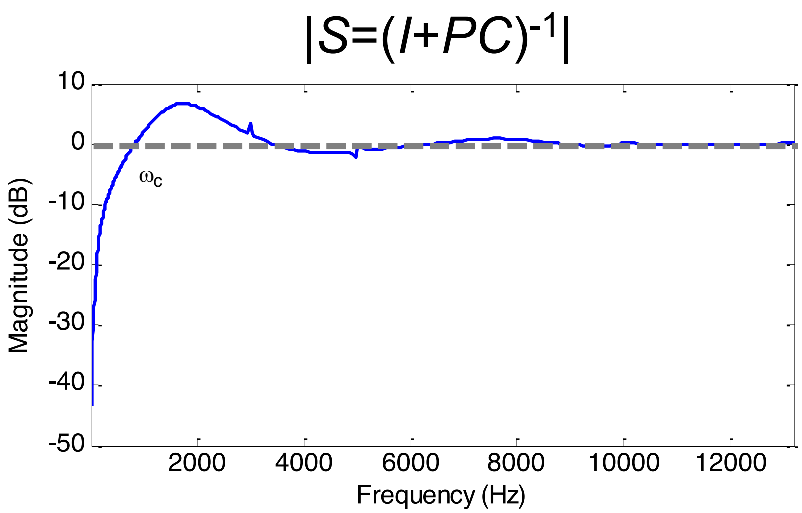


Typical feedback design



#6

Bode's Integral

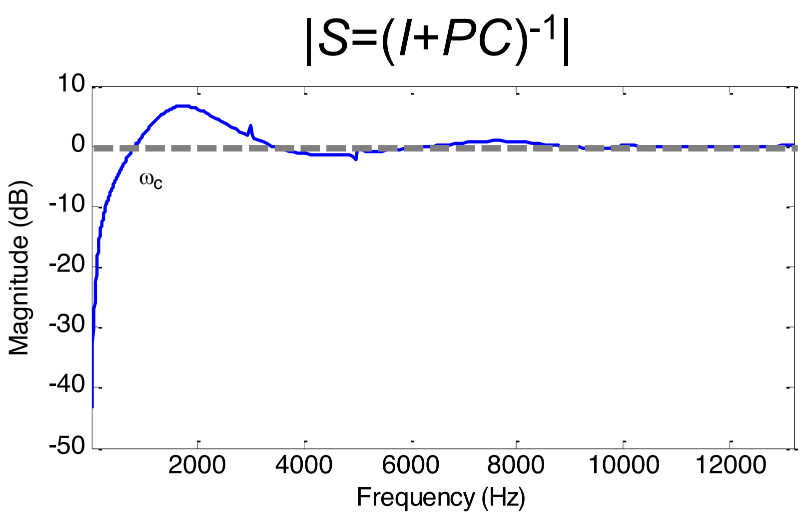


Theorem (basic Bode's Integral): Let $S(s) = 1/(1 + L(s))$. If $L(s)$ and $S(s)$ are both rational and stable. Then

$$\frac{1}{\pi}\int_0^\infty \ln|S(j\omega)|d\omega = -\frac{1}{2}k_s$$

$$k_s = \lim_{s\to\infty} sL(s)$$

Bode's Integral



Theorem (basic Bode's Integral): Let $S(s) = 1/(1 + L(s))$. If $L(s)$ and $S(s)$ are both rational and stable. Then

$$\frac{1}{\pi}\int_0^\infty \ln|S(j\omega)|d\omega = -\frac{1}{2}k_s$$

$$k_s = \lim_{s\to\infty} sL(s)$$

Special case: If the relative degree of $L(s)$ larger than or equal to 2, then

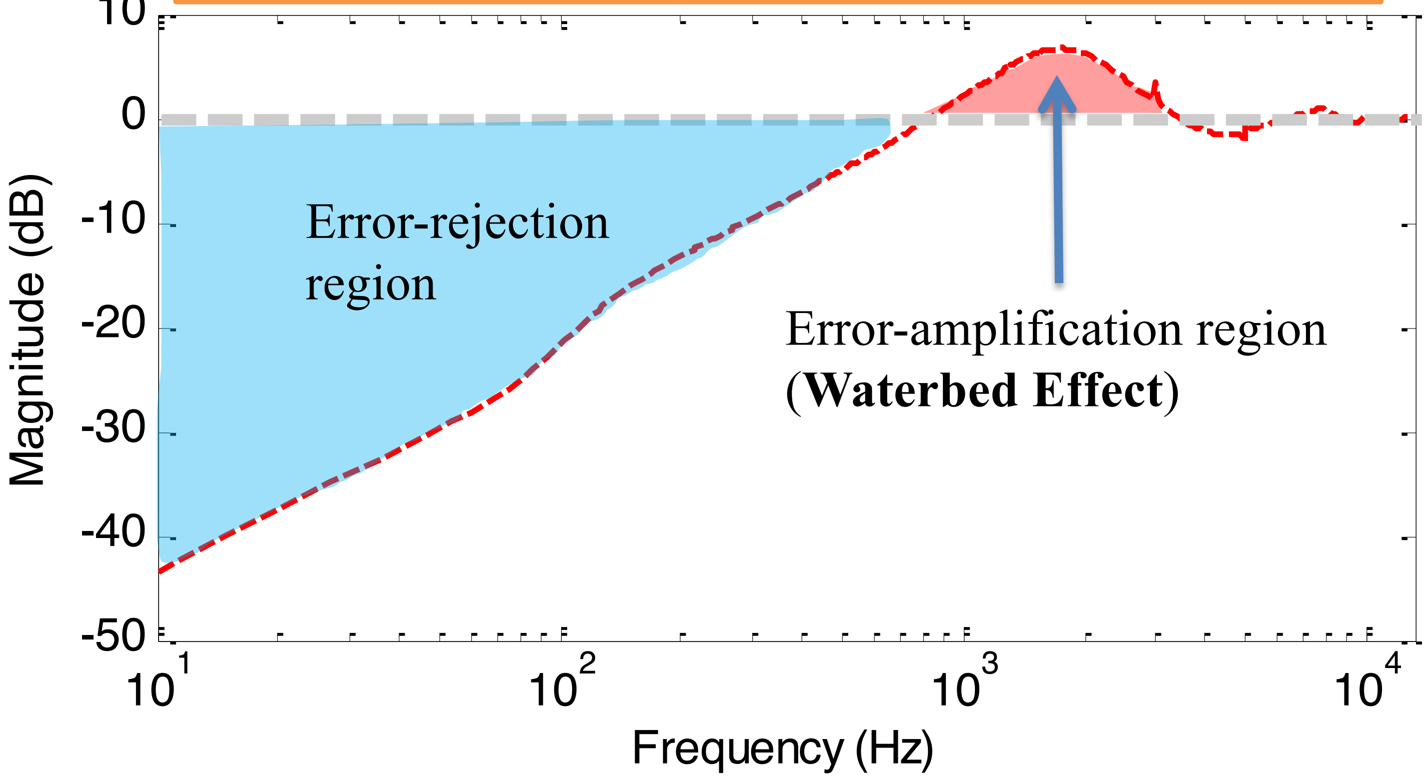
$$\frac{1}{\pi}\int_0^\infty \ln|S(j\omega)|d\omega = 0$$

#6

Bandwidth limitation

Recall:

Magnitude of a typical **sensitivity function** $S=1/(1+PC)$



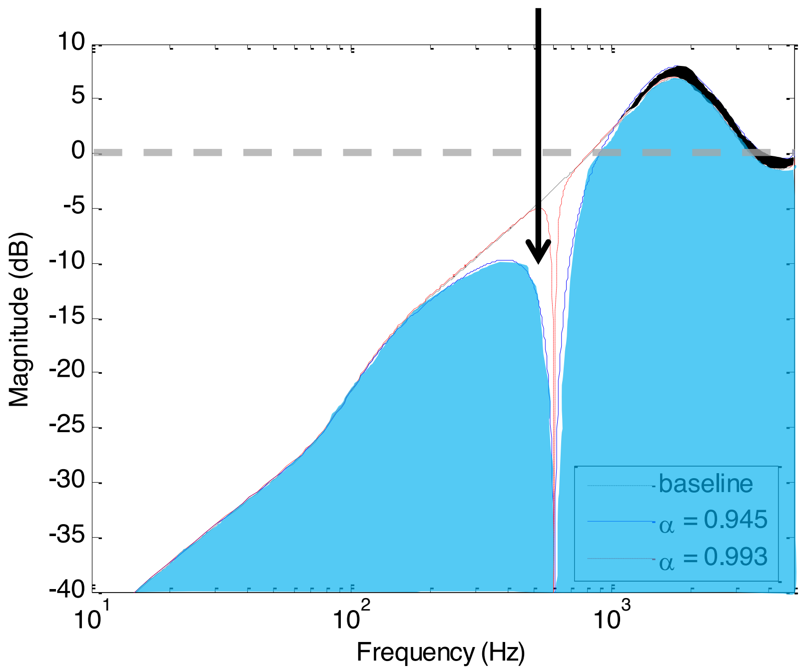
Bode's Integral: $$\frac{1}{\pi}\int_0^\infty \ln|S(j\omega)|\,d\omega = 0$$

Hence it is inevitable to have the error-amplification region.

Waterbed effect: pushing down ***S*** in one region causes amplification in some other region.

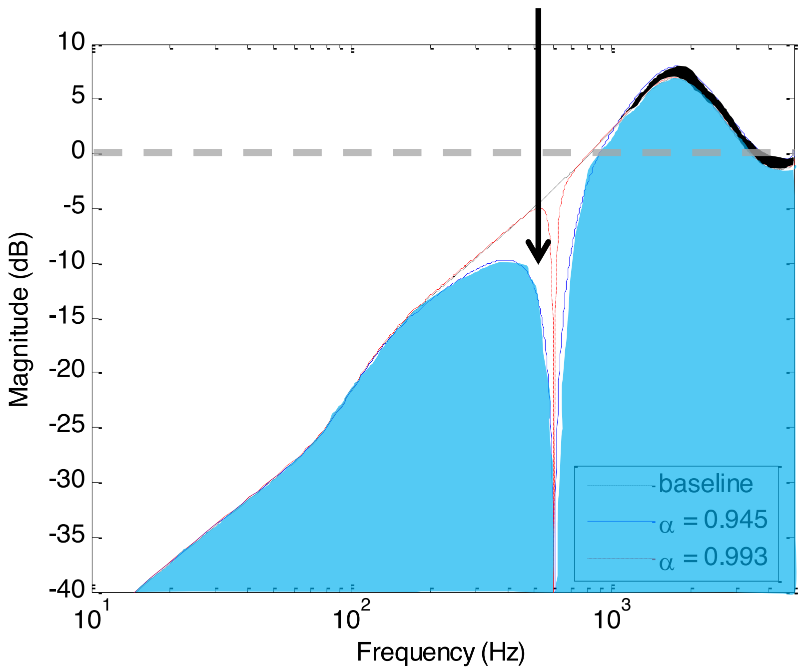
Waterbed Effect

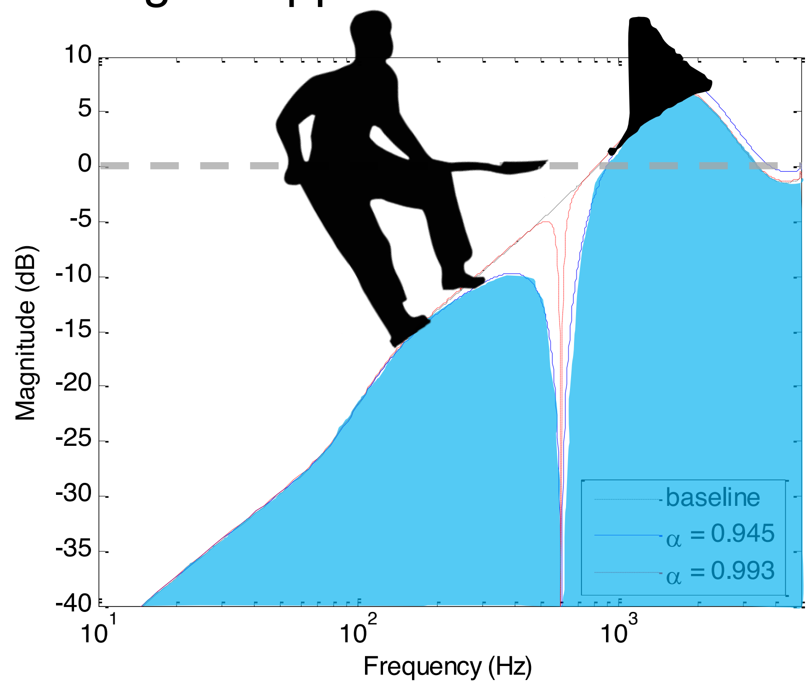
So to achieve this,



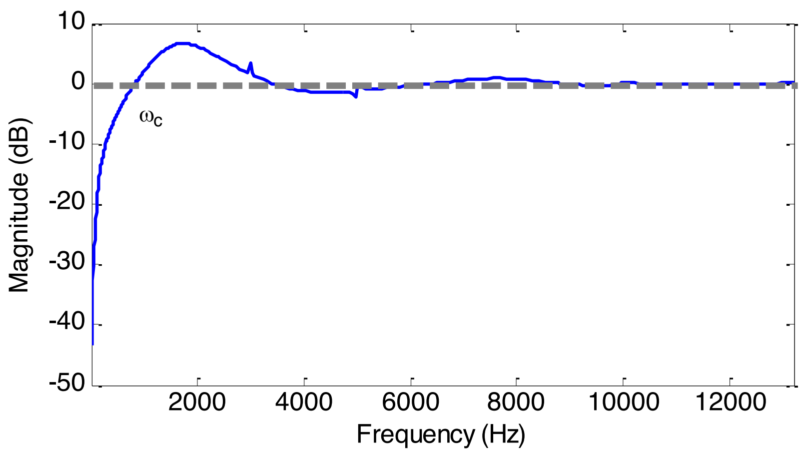
Waterbed Effect

So to achieve this, this might happen…





General Bode's Integral



Theorem (general Bode's Integral): Let $S(s) = 1/(1 + L(s))$. If $S(s)$ is stable and $L(s)$ has unstable poles $\{p_k\}_{k=1}^{q}$. Then

$$\frac{1}{\pi}\int_0^{\infty} \ln|S(j\omega)|\,d\omega = \sum_{k=1}^{q} p_k$$

Proof: complex analysis, analytic functions, Cauchy Integral

#7

Limitations from unstable zeros

- Example: $P = sP_{else}$ ➔ constant inputs can't impact the output

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- More consequences:
  - S *always* has magnitudes larger than one

Limitations from unstable zeros

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- Example: $P = sP_{else}$ ➔ constant inputs can't impact the output
- More consequences:
  - – *S always* has magnitudes larger than one

Proof:

$$P(\omega_o) = 0 \Rightarrow S(\omega_o) = 1/(1 + 0 \times C(\omega_o)) = 1$$

#7

Limitations from unstable zeros

- Example: $P = sP_{else}$ ➔ constant inputs can't impact the output
- More consequences:
  - S *always* has magnitudes larger than one

Proof:

P (æ$_o$) = 0) S (æ$_o$) = 1=(1 + 0 £ C (æ$_o$)) = 1

Closed-loop stability) S(s) is analytic on the right-half complex plane

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Limitations from unstable zeros

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Proof:

P (æo) = 0) S (æo) = 1=(1 + 0 £ C (æo)) = 1

Closed-loop stability) S(s) is analytic on the right-half complex plane

Maximum modulus theorem)

S(j !) > 1 for some !

#7

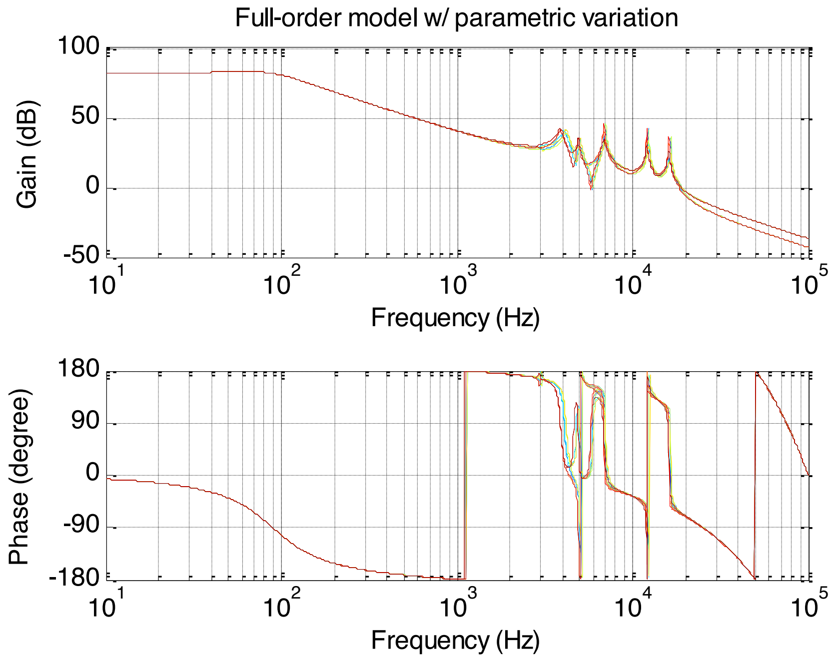
Limitations from unstable zeros

- Example: $P = sP_{else}$ ➔ constant inputs can't impact the output
- More consequences:
  - S *always* has magnitudes larger than one
  - Not able to perform accurate system ID
  - High-gain instability
  - Step responses can have initial undershoot
  - etc

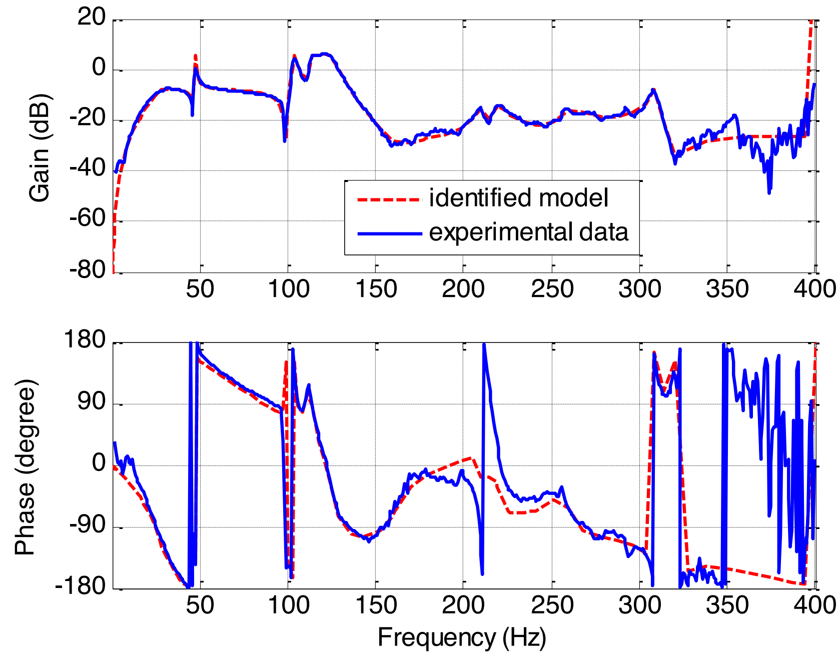
Resonance and anti-resonance

- Typical in mechanical systems.
- Usually identified experimentally.

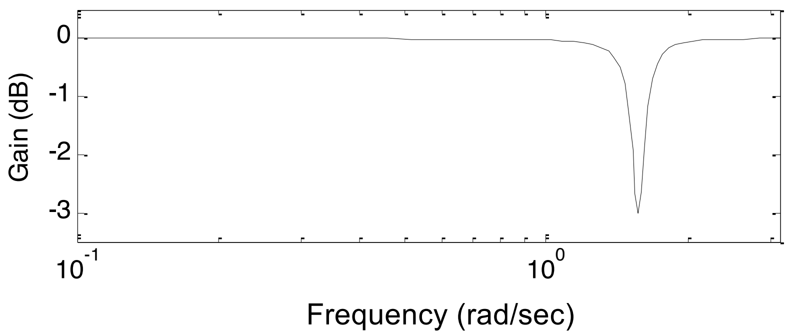
HDD



Active suspension



Notch filters

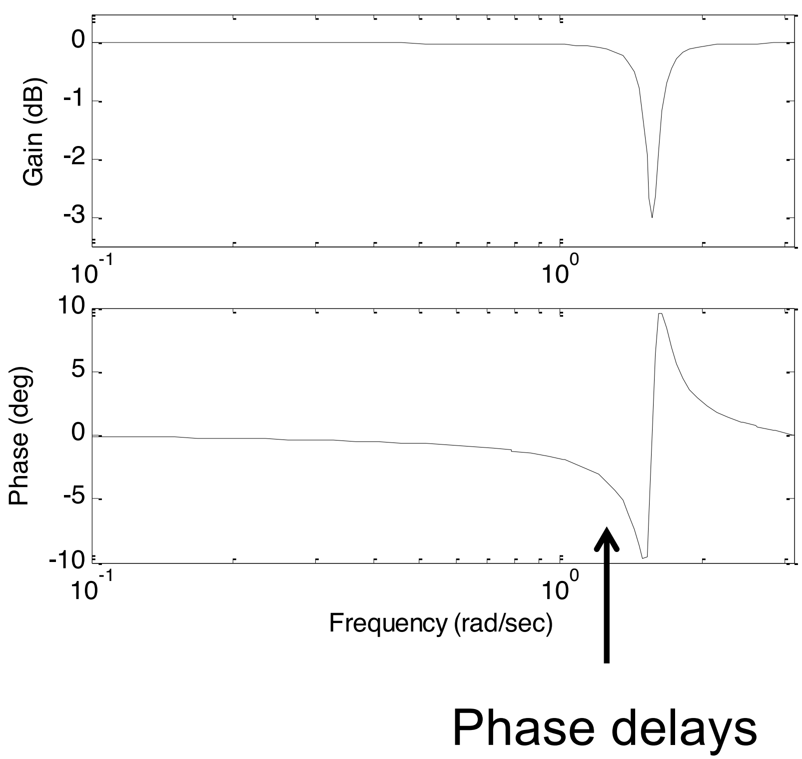


Notch filtering: one common technique to handle resonances

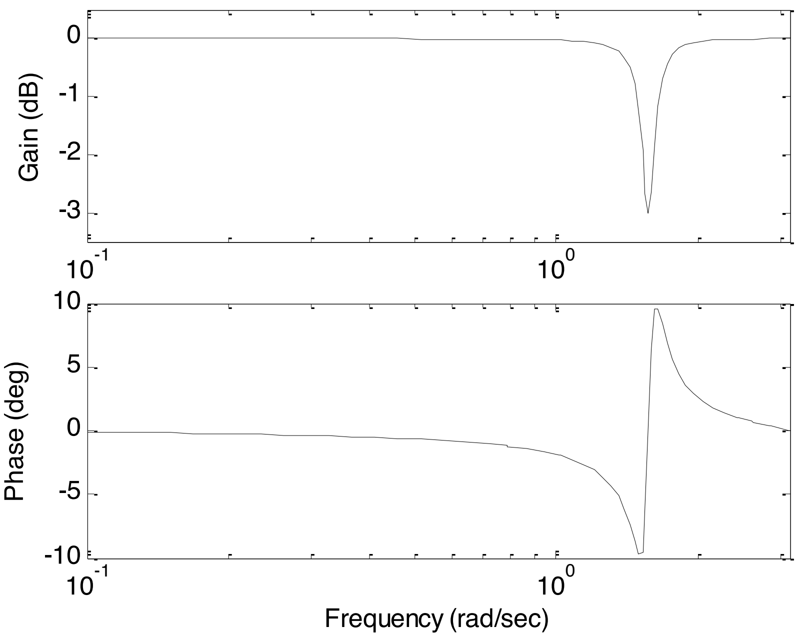
Fundamental constraint in notch filtering: introduces phase delays to the system

#8

Magnitude-phase relationship



Magnitude-phase relationship



Theorem (Bode's Phase Formula): If L is a minimum-phase transfer function, then

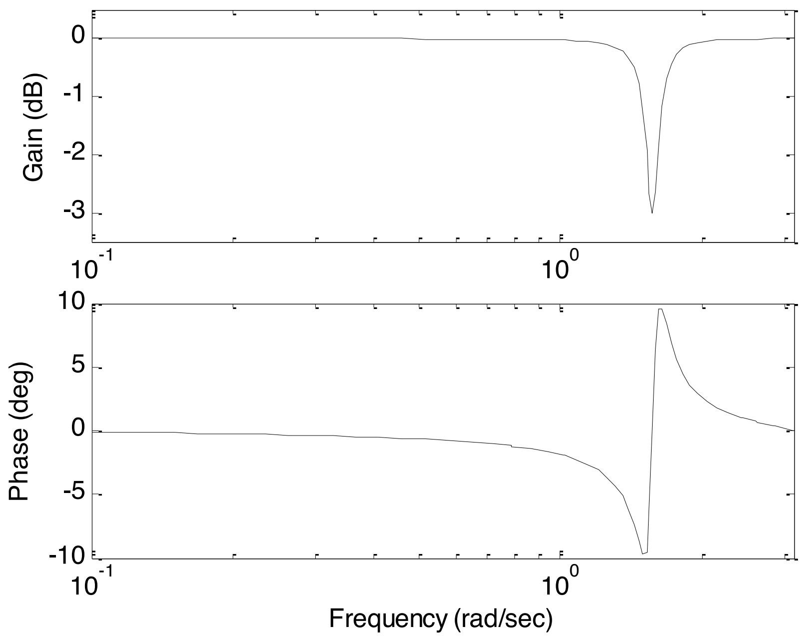
$$\angle L(j\omega) = \int_{-\infty}^{\infty} \frac{d\ln|L(e^{\nu}\omega)|}{d\nu}\psi(\nu)\,d\nu$$

where

$$\psi(\nu) = \frac{1}{\pi}\ln\frac{e^{|\nu|/2} + e^{-|\nu|/2}}{e^{|\nu|/2} - e^{-|\nu|/2}}.$$

#8

Magnitude-phase relationship



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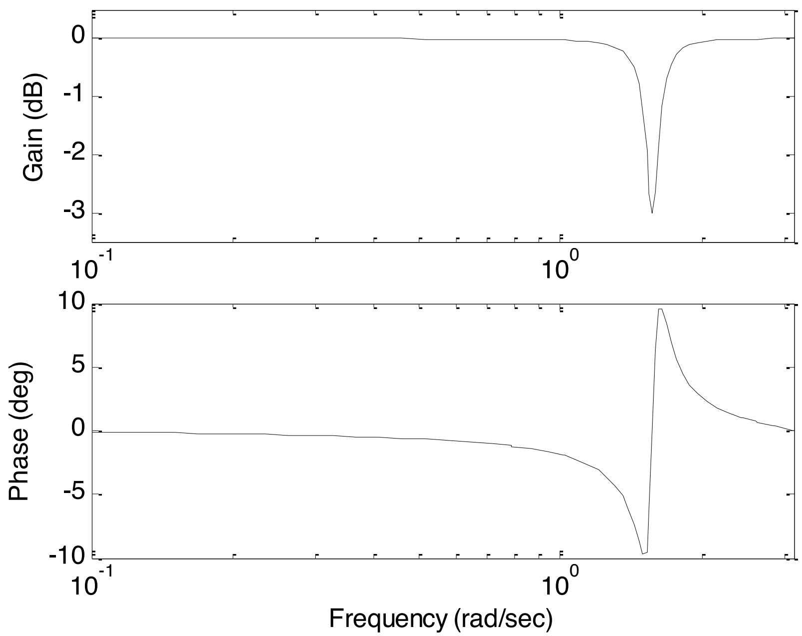
Slope of magnitude response

where

$$\psi(\nu) = \frac{1}{\pi} \ln \frac{e^{|\nu|/2} + e^{-|\nu|/2}}{e^{|\nu|/2} - e^{-|\nu|/2}}.$$

Magnitude-phase relationship



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Slope of magnitude response

where

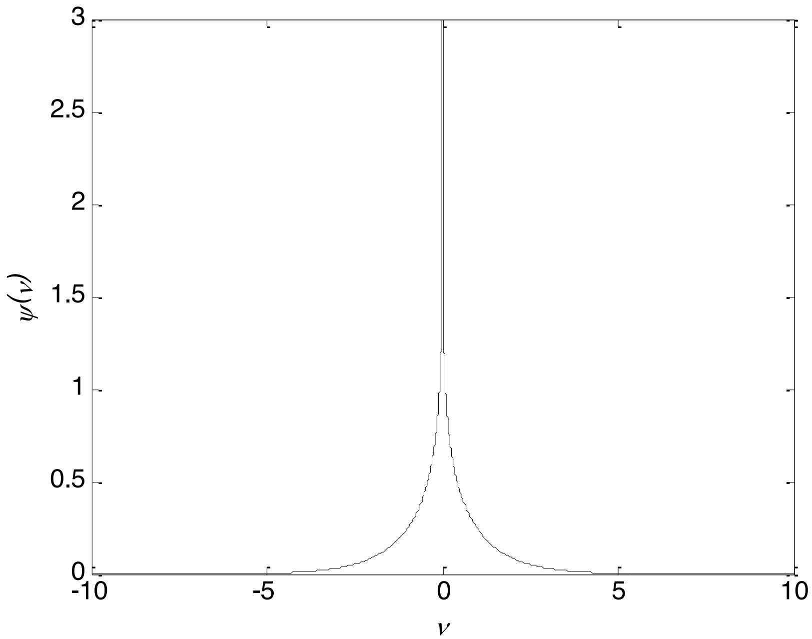
$$W(\nu) = \frac{1}{\pi} \ln \frac{e^{|\nu|/2} + e^{-|\nu|/2}}{e^{|\nu|/2} - e^{-|\nu|/2}}.$$

Approximately an impulse at 0

Magnitude-phase relationship

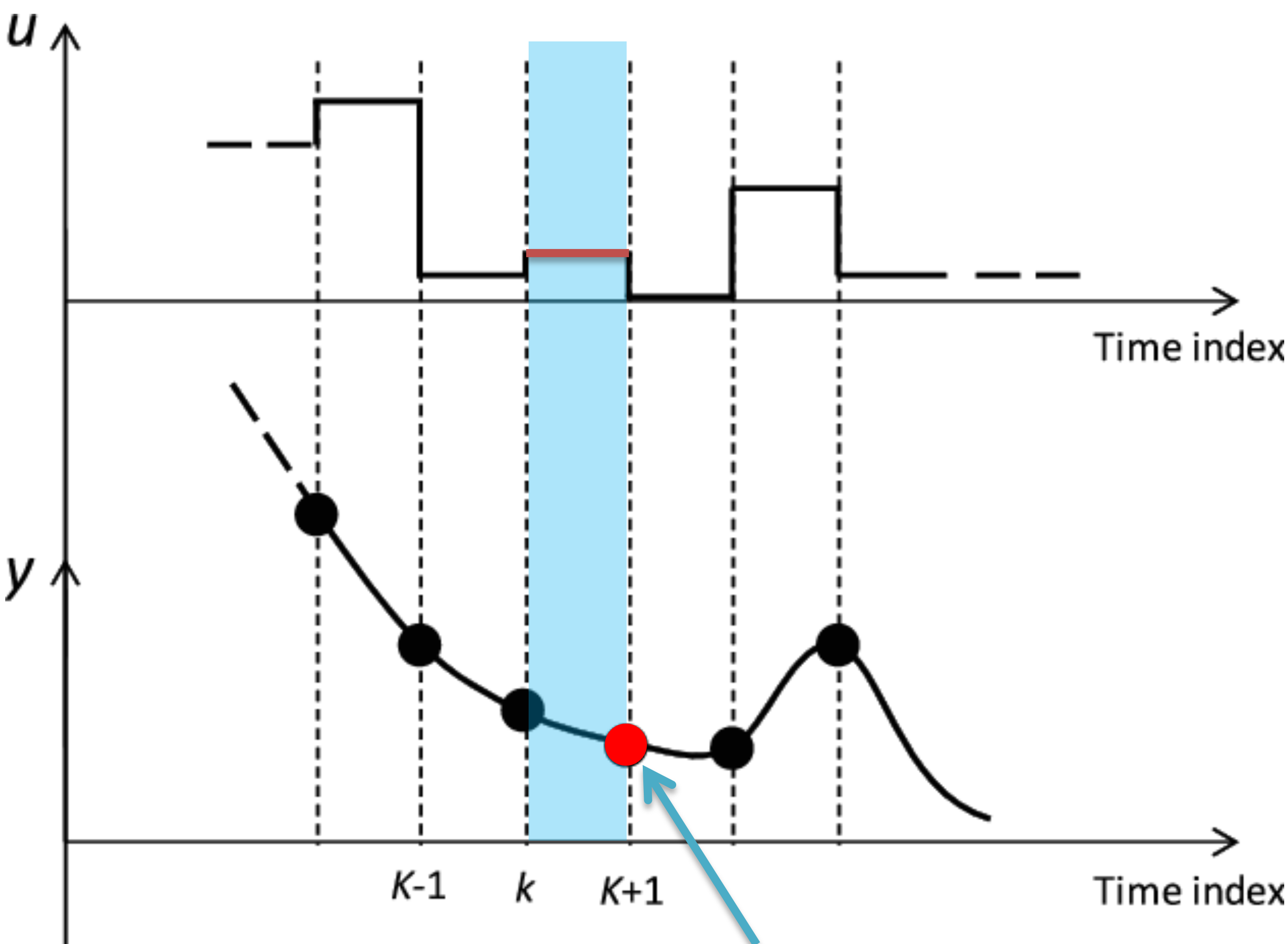
Theorem (Bode's Phase Formula): If L is a minimum-phase transfer function, then

$$\angle L(j\omega) = \int_{-\infty}^{\infty} \frac{d \ln |L(e^{\nu} j\omega)|}{d\nu} \psi(\nu)\, d\nu$$



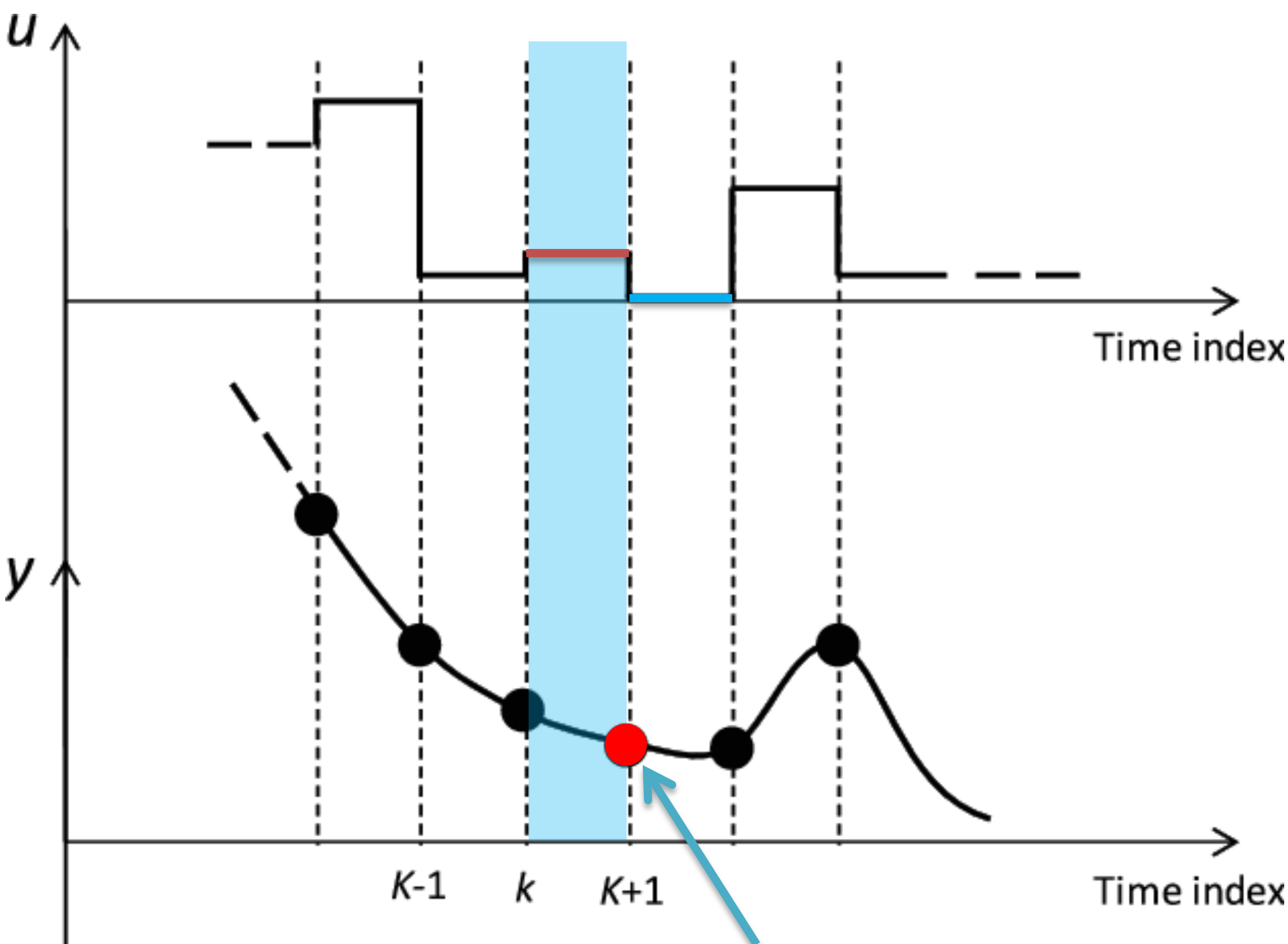
$$\psi(\nu) = \frac{1}{\pi} \ln \frac{e^{j\nu j/2} + e^{-j\nu j/2}}{e^{j\nu j/2} - e^{-j\nu j/2}}.$$

Discrete-time plant delay



$$y(k+1) = f(u(k); u(k-1); \ldots)$$

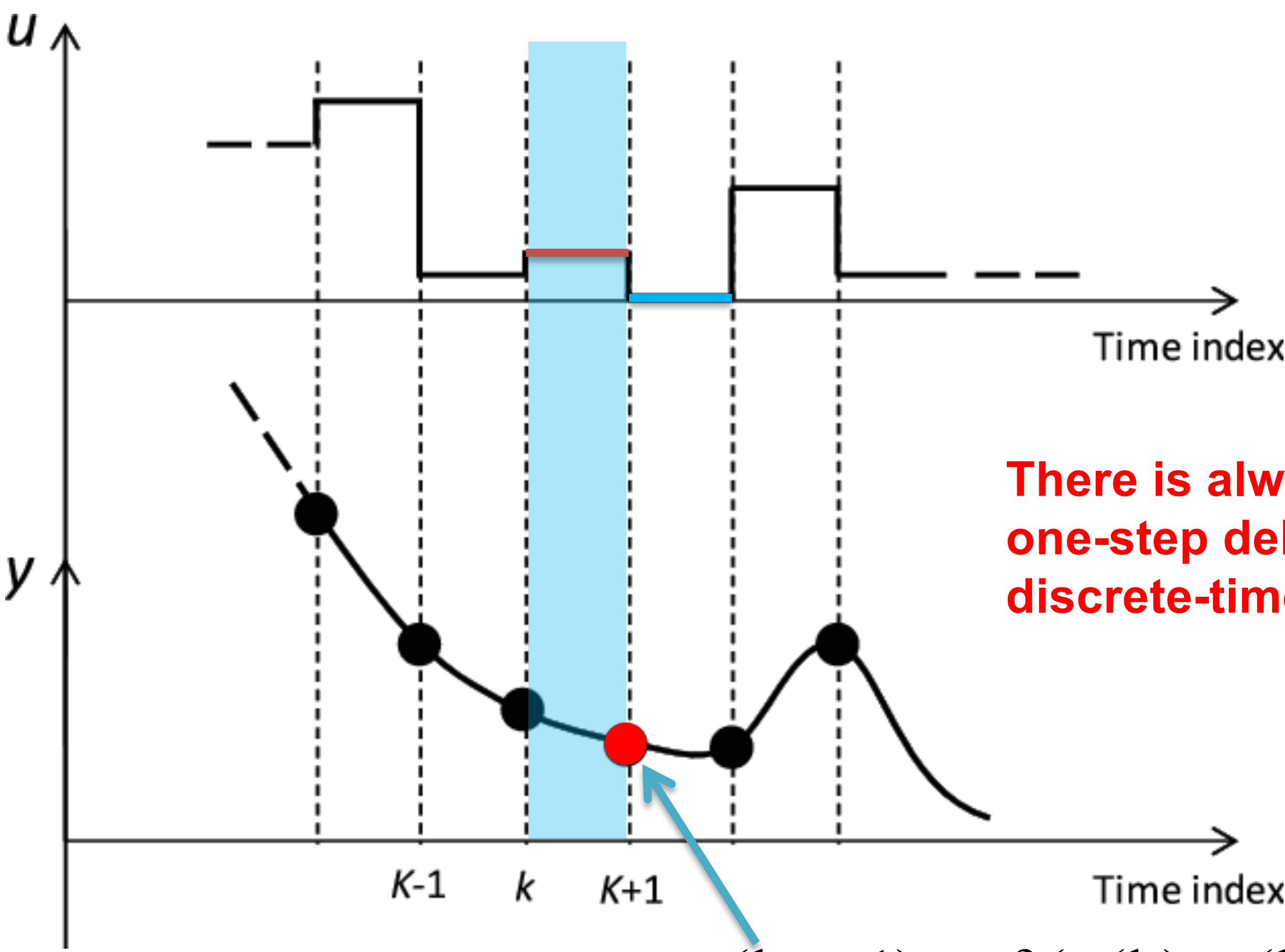
Discrete-time plant delay



$$y(k+1) = f(u(k), u(k-1), \ldots)$$

$$y(k+1) \neq f(u(k+1), \ldots)$$

Discrete-time plant delay

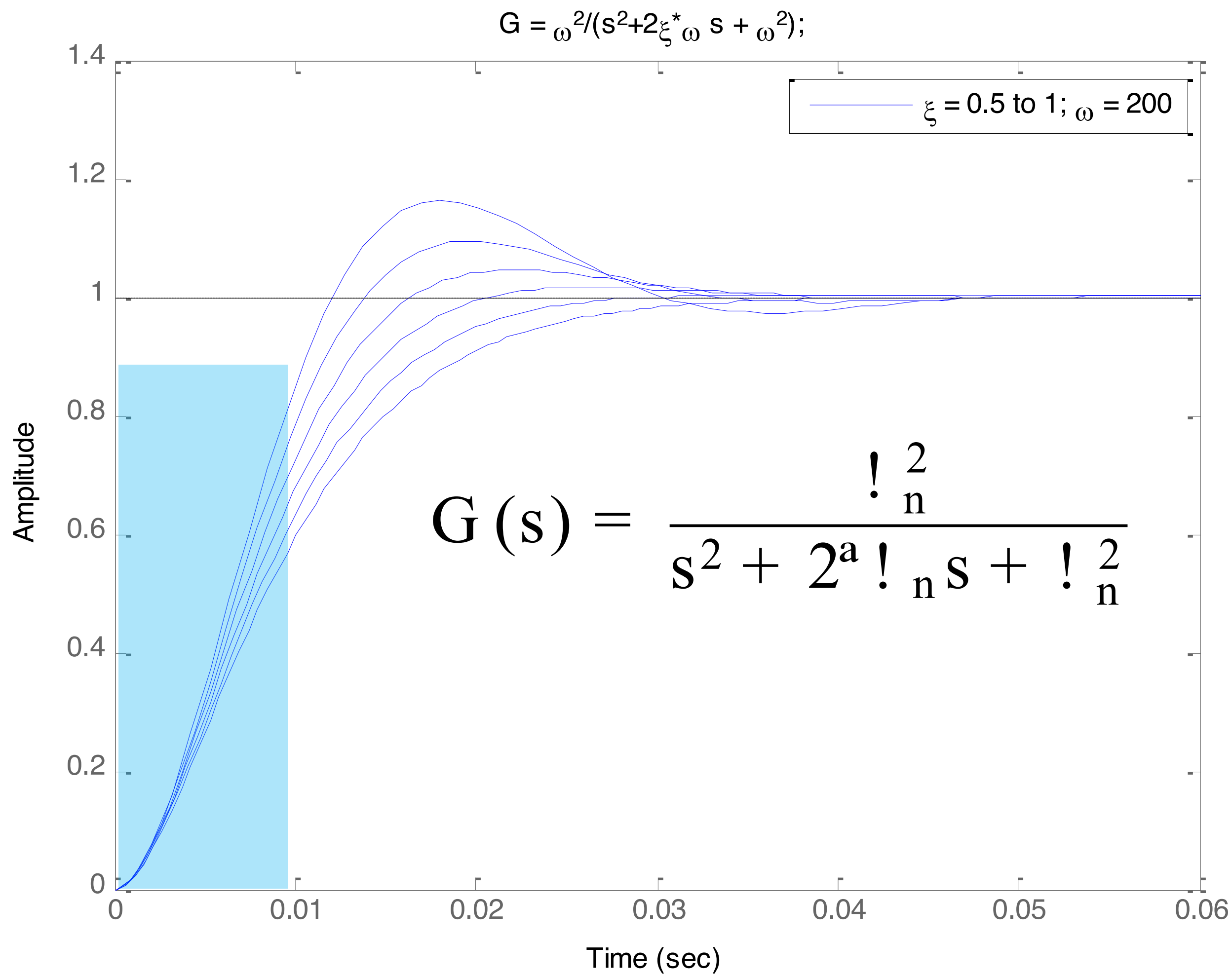


There is always at least one-step delay in discrete-time systems!

$$y(k+1) = f(u(k); u(k-1); \ldots)$$

$$y(k+1) \neq f(u(k+1); \ldots)$$

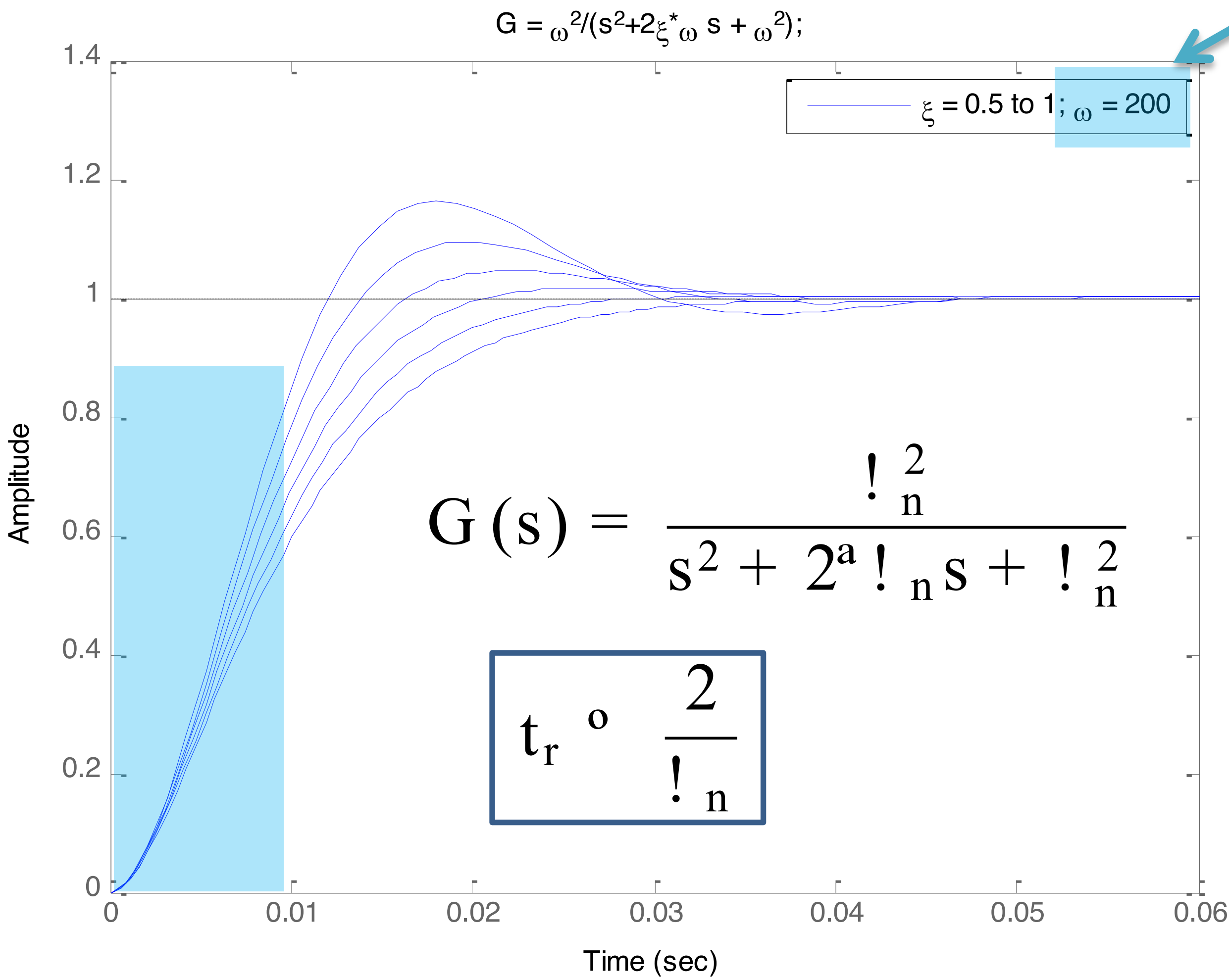
#10 Estimate rise time from “bandwidth”



$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

#10 Estimate rise time from "bandwidth"

$\omega_n = 200 \text{ rad/sec}$

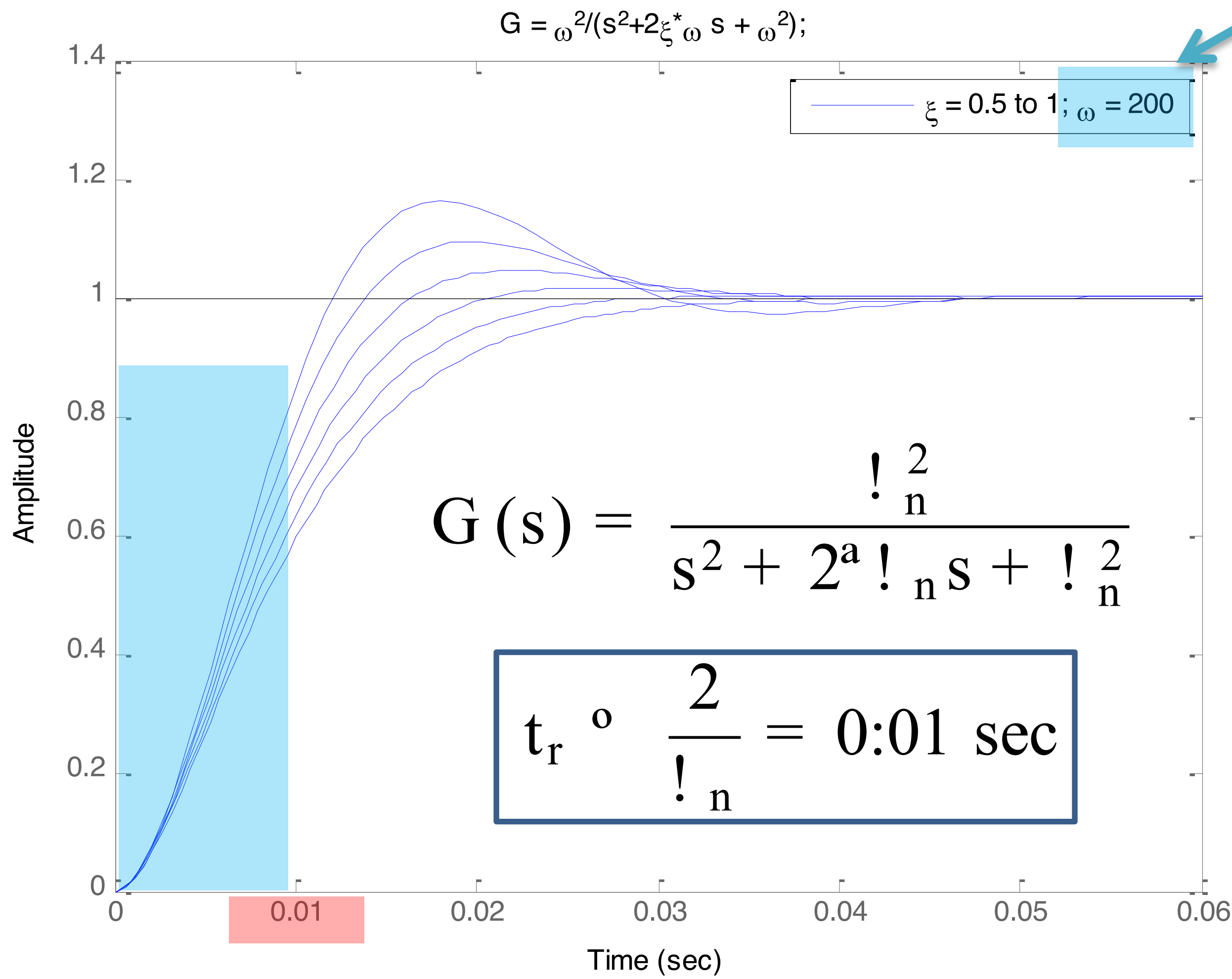


$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\boxed{t_r \approx \frac{2}{\omega_n}}$$

#10 Estimate rise time from “bandwidth”

$\omega_n = 200$ rad/sec



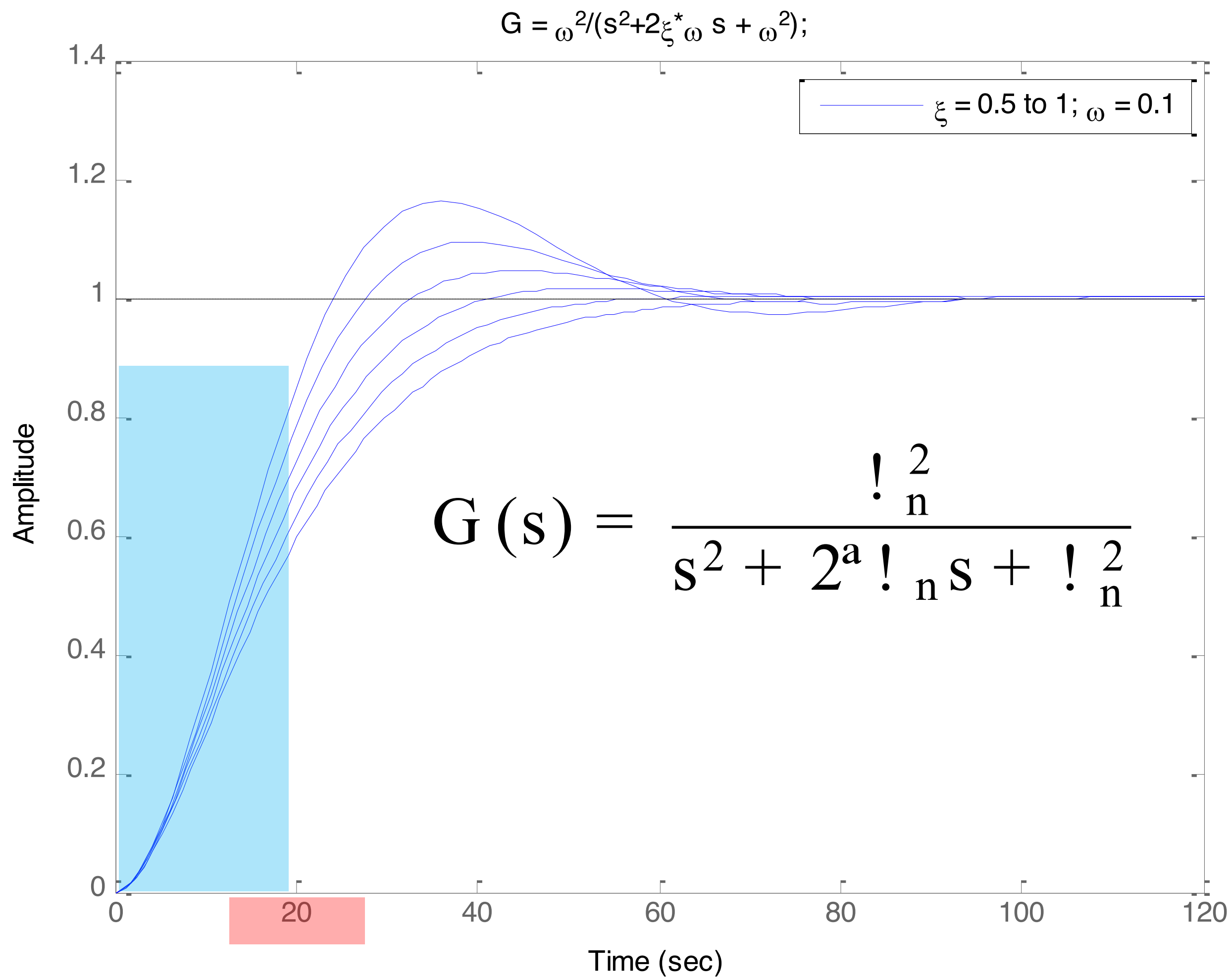
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\boxed{t_r \approx \frac{2}{\omega_n} = 0.01 \text{ sec}}$$

$t_r \approx 0.01$ sec

#10

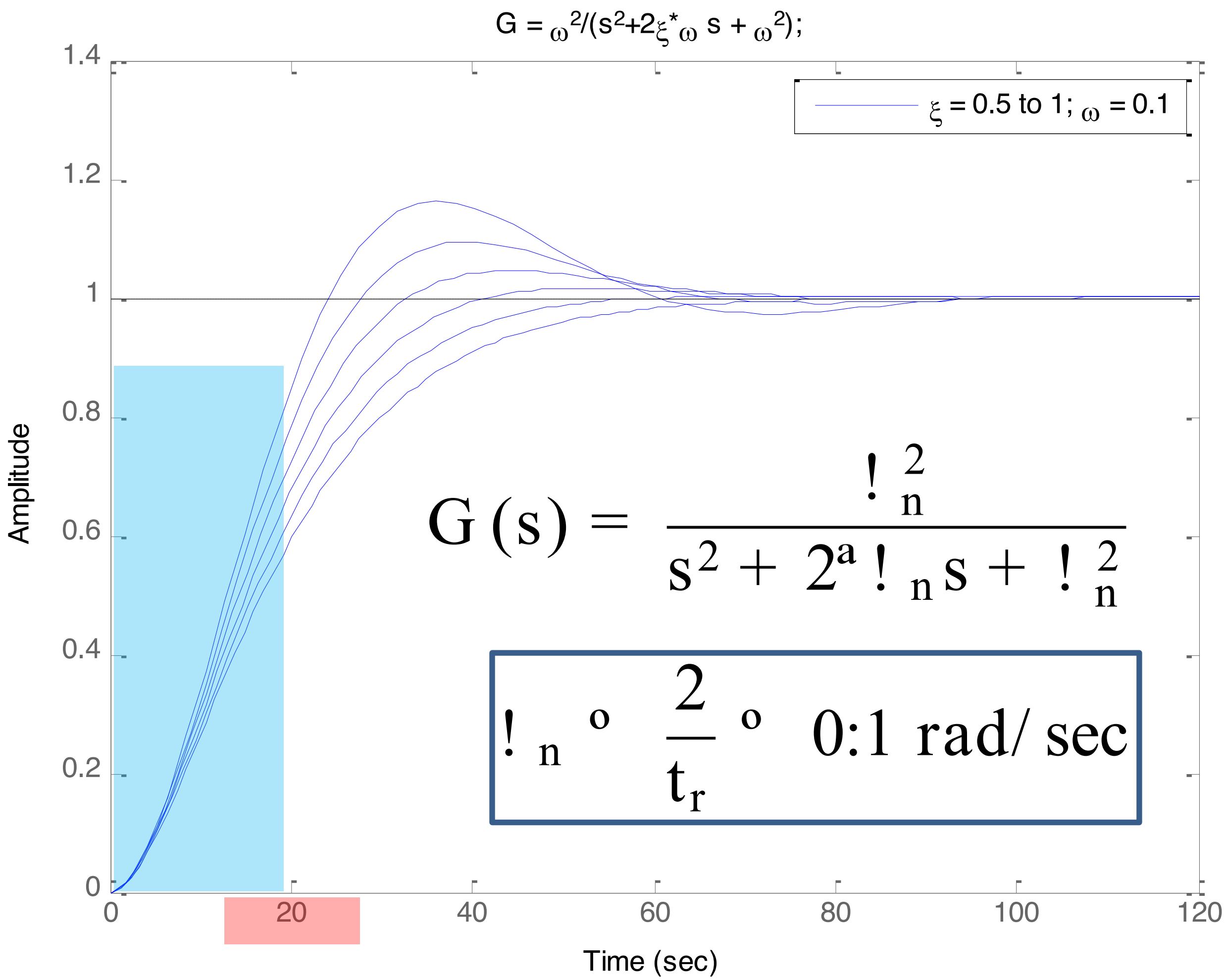
Estimate "bandwidth" from rise time



$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$t_r \approx 20$ sec

Estimate "bandwidth" from rise time



$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

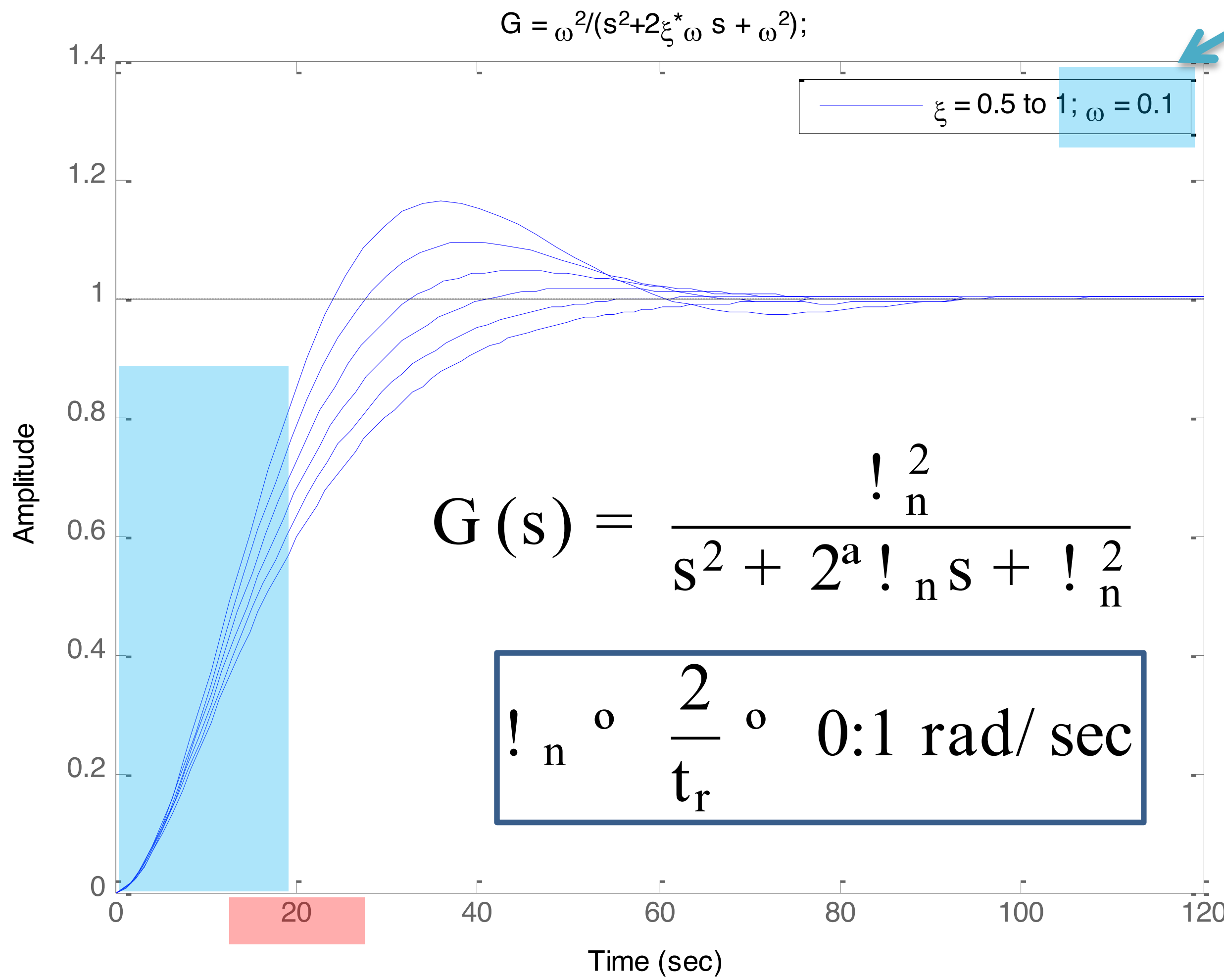
$$\boxed{\omega_n \approx \frac{2}{t_r} \approx 0.1\ \text{rad/sec}}$$

$$t_r \approx 20\ \text{sec}$$

#10

Estimate "bandwidth" from rise time

$\omega_n = 0.1 \text{ rad/sec}$



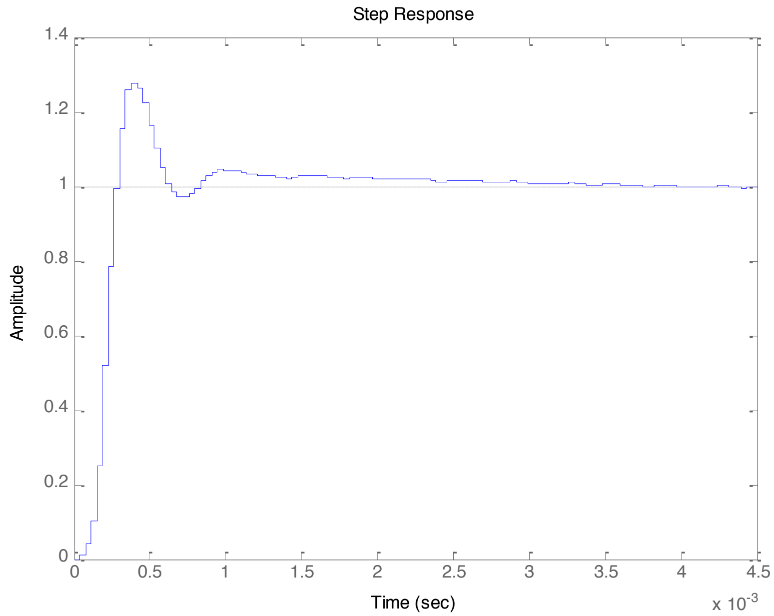
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\boxed{\omega_n \approx \frac{2}{t_r} \approx 0.1 \text{ rad/sec}}$$

$t_r \approx 20 \text{ sec}$

#10 Bandwidth and rise time: practical application

Step response of a high-order closed-loop system



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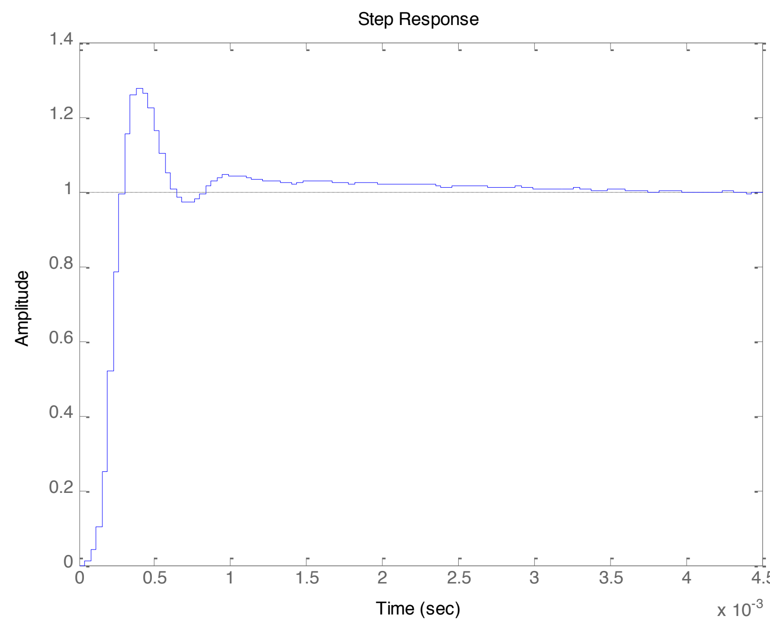
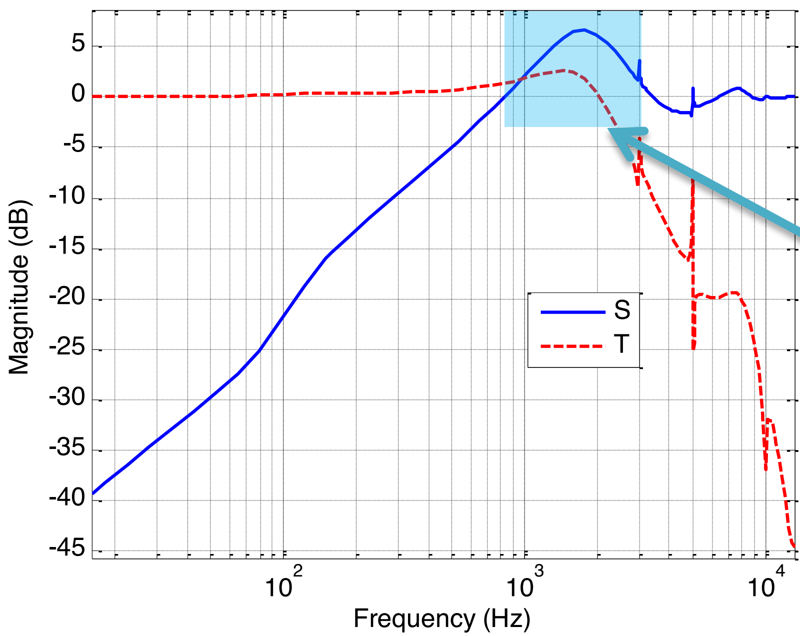


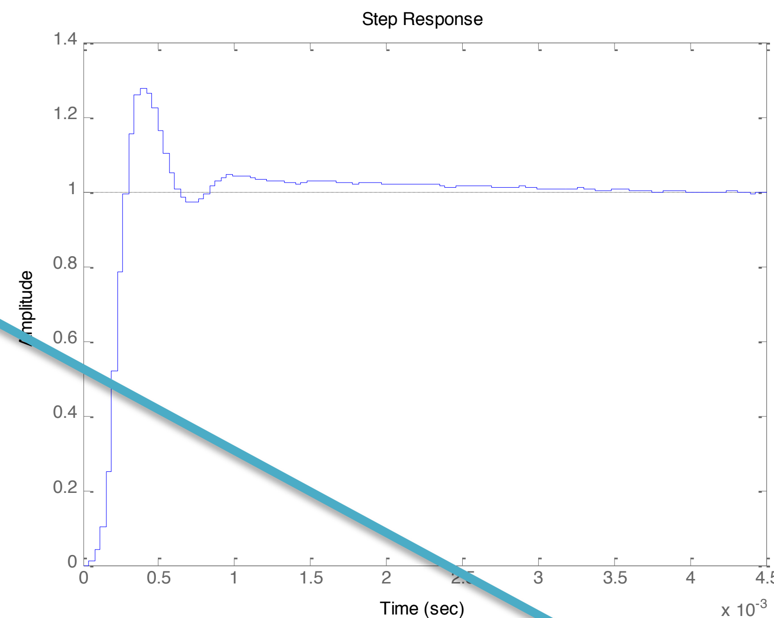
$$\text{Bandwidth} \approx \frac{2}{0.25 \times 10^{-3} \times 2\pi} = 1273 \text{ Hz}$$

#10 Bandwidth and rise time: practical application

Actual system frequency response



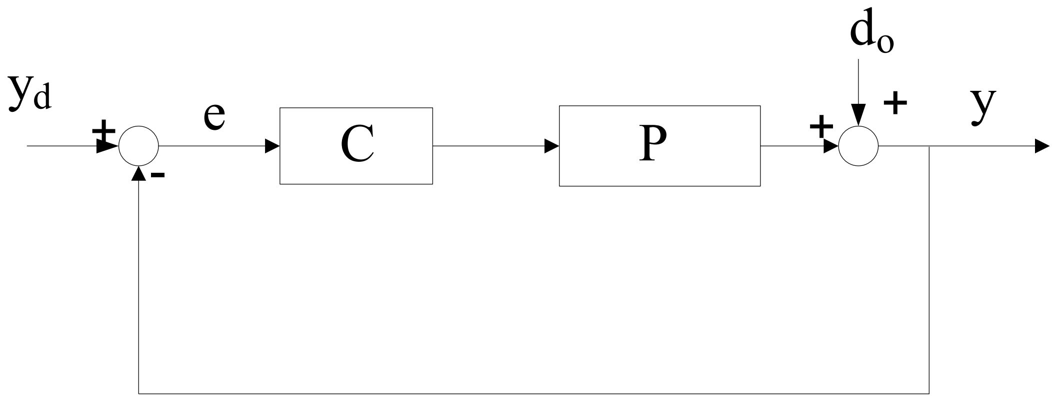
Step response of a high-order closed-loop system



$$\text{Bandwidth} \approx \frac{2}{0.25\times 10^{-3}\times 2\pi} = 1273\ \text{Hz}$$

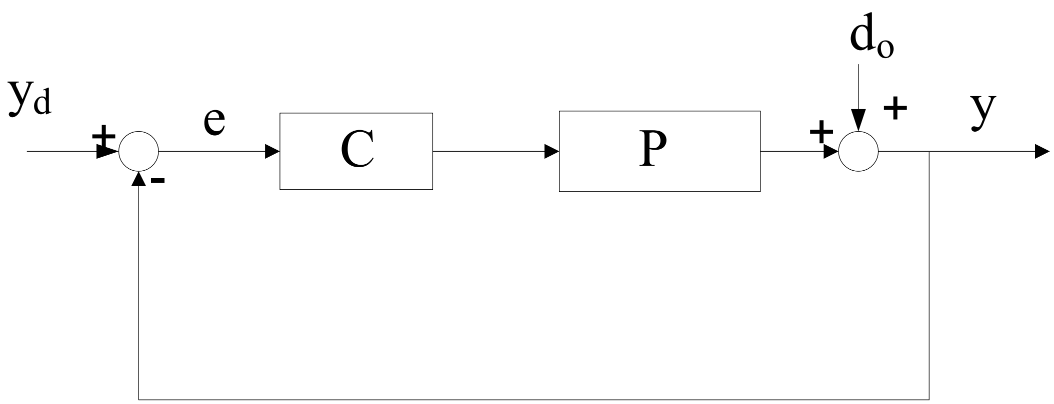
#11

Sampling-time selection

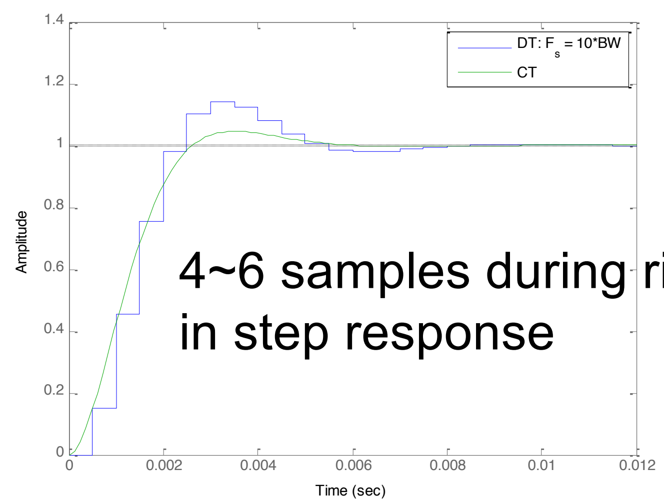


- Rule of thumb:
  - Sampling frequency 10 ~ 20 bandwidth (in Hz)

Sampling-time selection



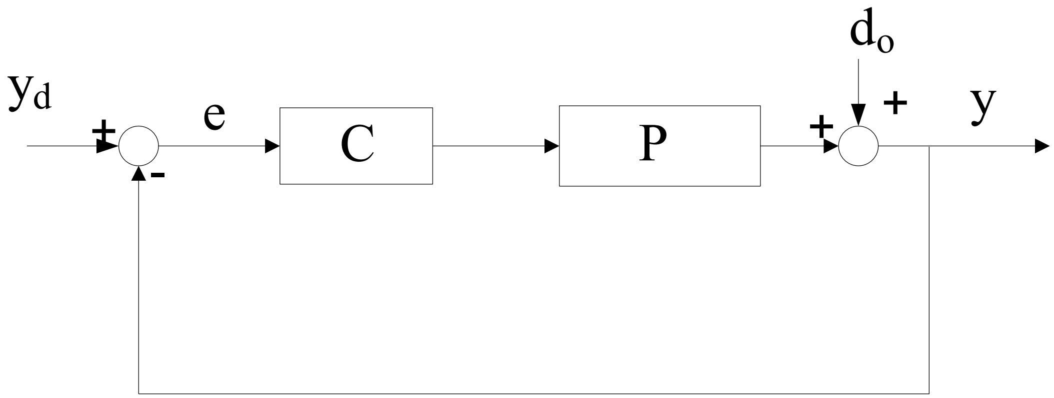
- Rule of thumb:
  - Sampling frequency 10 ~ 20 bandwidth (in Hz)



4~6 samples during rise time
in step response

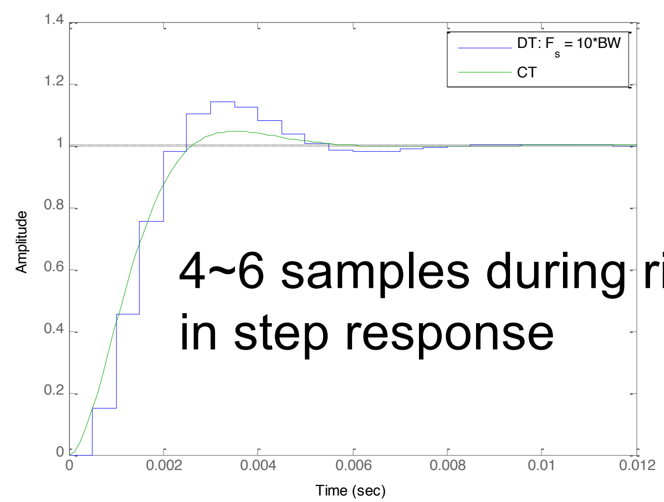
#11

Sampling-time selection



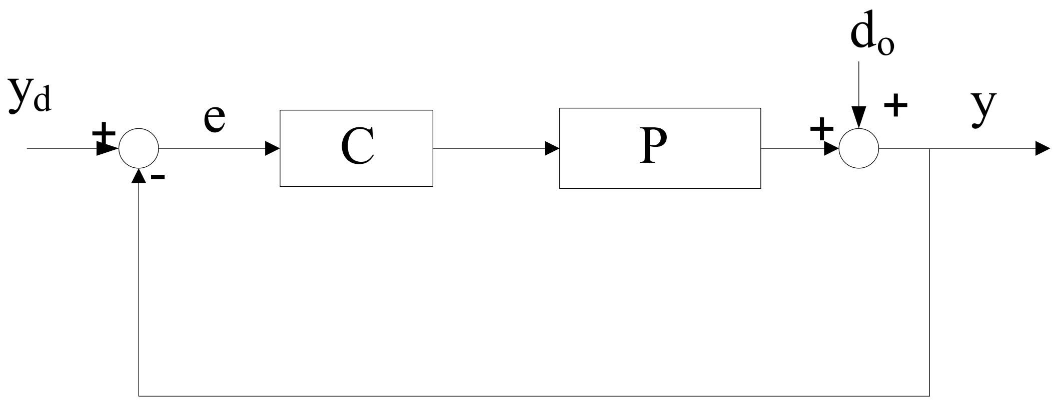
Intuition: 20 = the number of letters in "sampling frequencies"

- Rule of thumb:
  - Sampling frequency ∪ 10 ~ 20 bandwidth (in Hz)



4~6 samples during rise time in step response

#11 Sampling-time selection: example



Example:

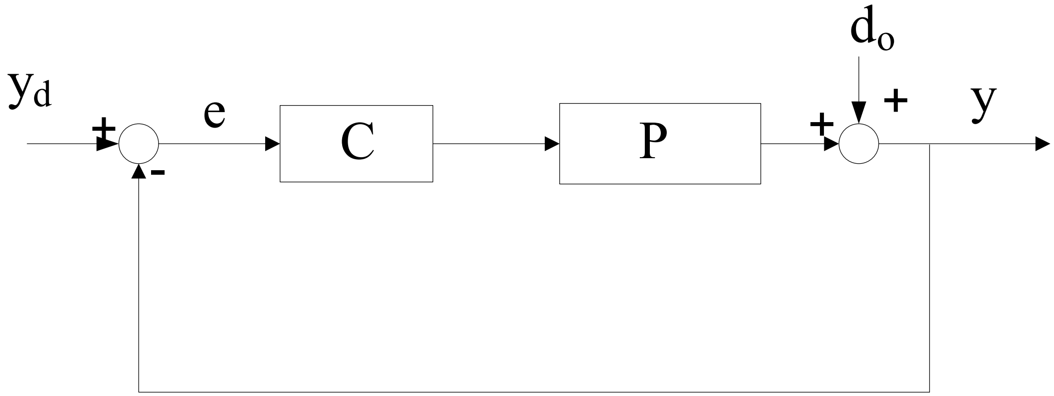
$$P = k$$

$$C = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}\frac{1}{k}$$

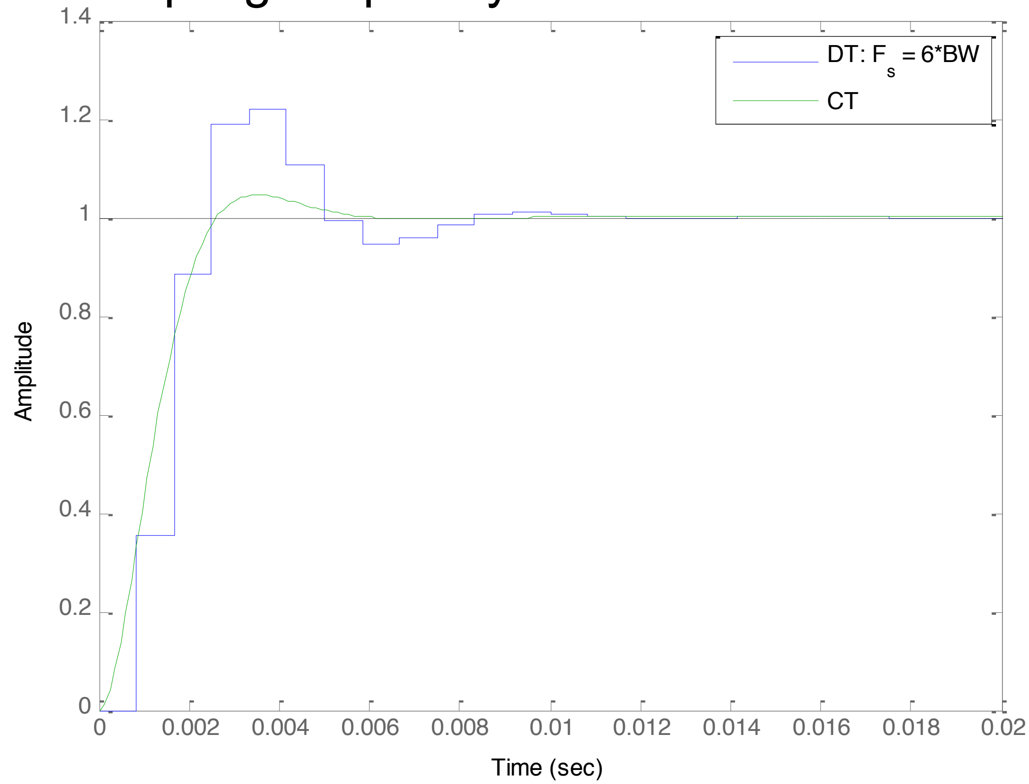
$$S = \frac{1}{1 + PC} = \frac{s^2 + 2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$T = 1 - S = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

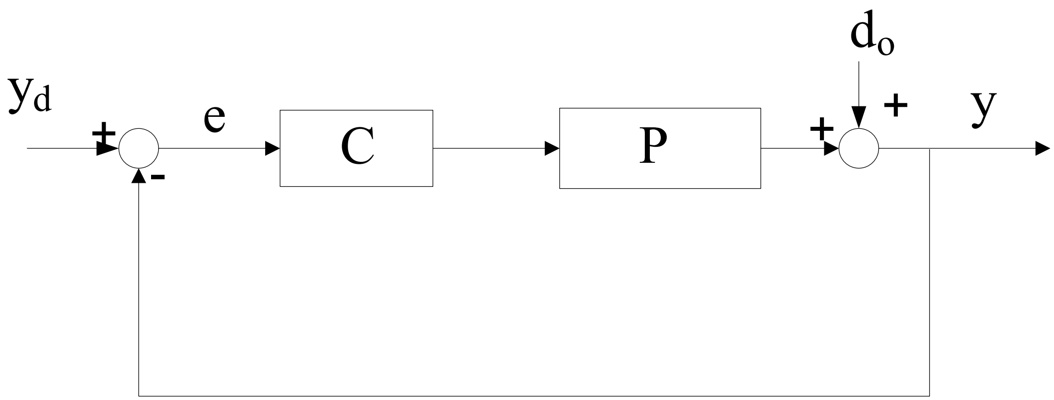
#11 Sampling-time selection: example

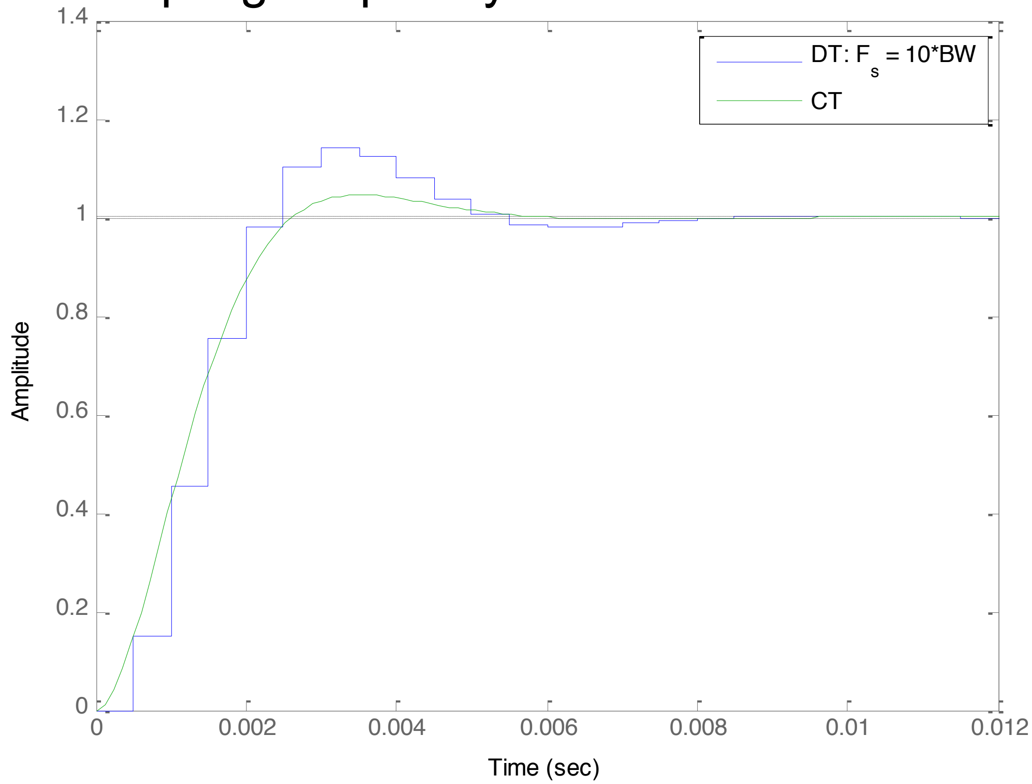


Sampling frequency = 6 x bandwidth

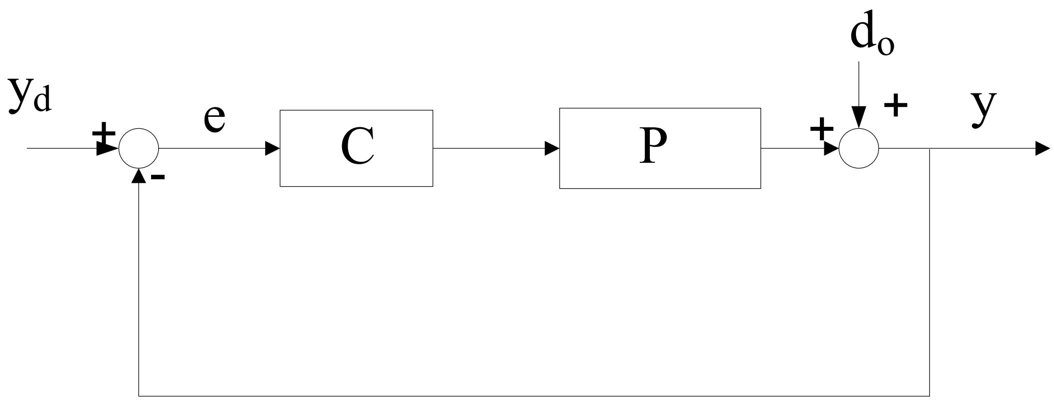


#11 Sampling-time selection: example

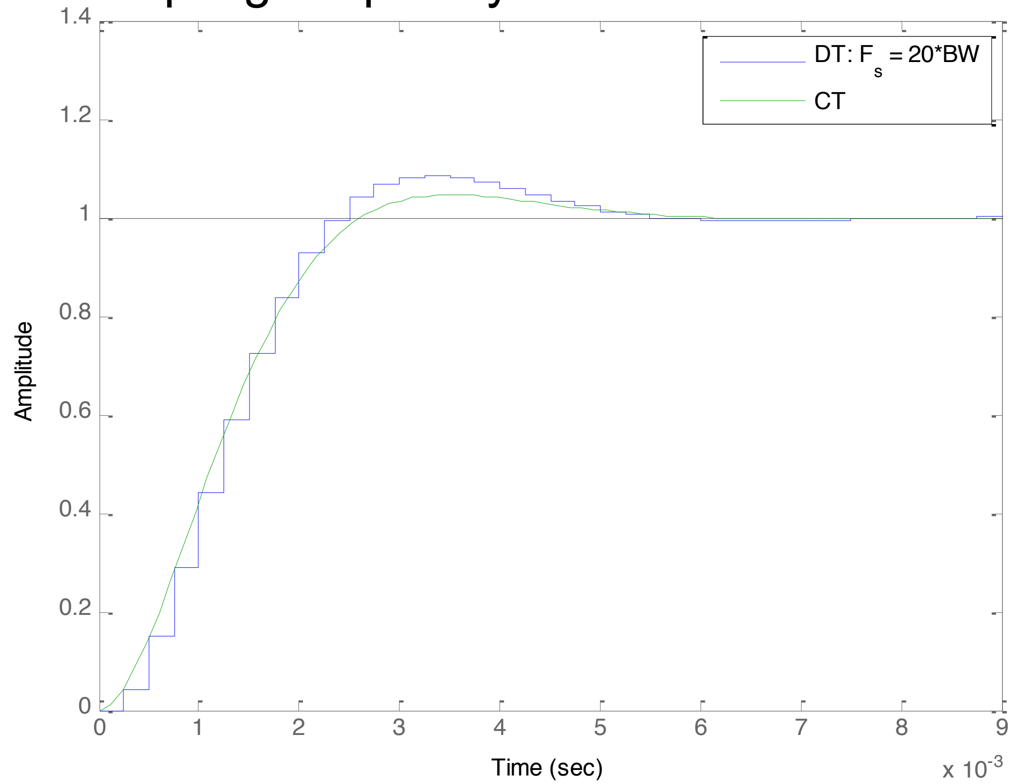




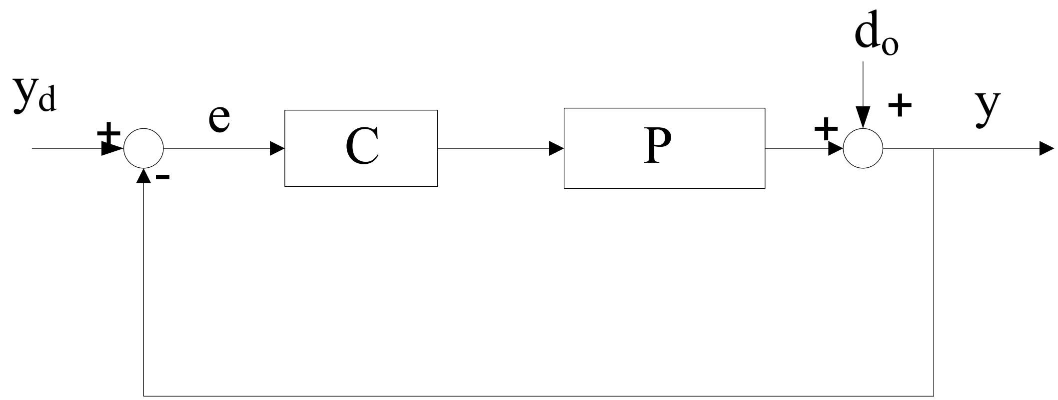
#11 Sampling-time selection: example



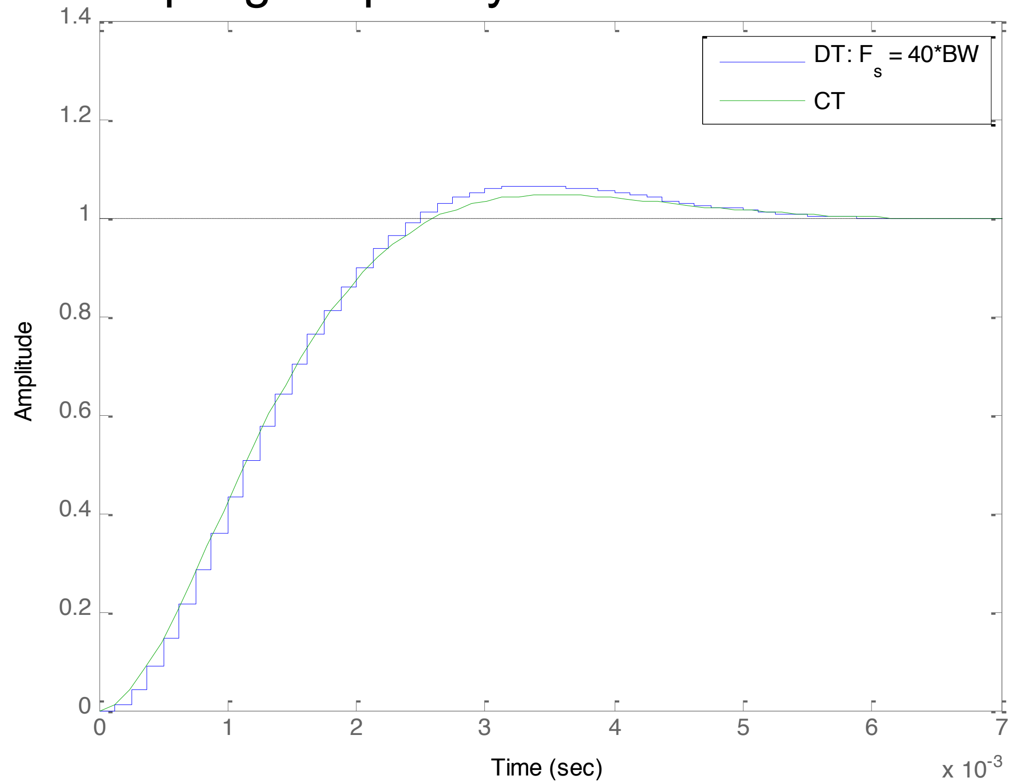
Sampling frequency = 20 x bandwidth



#11 Sampling-time selection: example



Sampling frequency = 40 x bandwidth



Related active research field

- Flexible loop shaping
- Vibration rejection and motion control
- MIMO loop shaping
- Delay compensation
- Adaptive control
- Nonlinear control and breaking the waterbed effect