1 PCA

1

```
[ 2.81035710e+02+0.00000000e+00j -5.81067532e-15+0.00000000e+00j
  9.64290269e-01+0.00000000e+00j -3.23092317e-15+0.00000000e+00j
  1.33527898e-18+2.05406773e-17j  1.33527898e-18-2.05406773e-17j]
```

2

By hand:

$$\mathrm{XX}^{\mathrm{T}} = \begin{bmatrix} 112 & 137 \\ 137 & 170 \end{bmatrix}$$

$$\left|XX^{\mathrm{T}} - \lambda I\right| = \begin{bmatrix} 112-\lambda & 137 \\ 137 & 170-\lambda \end{bmatrix}$$

$$(112-\lambda)(170-\lambda) - 137^2 = 0$$

$$\lambda_1 = 0.96429 \quad \lambda_2 = 281.03571$$

$when\ \lambda_1 = 0.96429$:

$$(XX^T - \lambda_1 I)x = \begin{pmatrix} 111.03571 & 137 \\ 137 & 169.03571 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$so, after\ normalization$

$$x_1 = -0.77688 \ \ x_2 = 0.62965$$

$$\xi_1 = \begin{pmatrix} -0.77688 \\ 0.62965 \end{pmatrix}$$

$when\ \lambda_2 = 281.03571$:

$$(XX^T - \lambda_2 I)x = \begin{pmatrix} -169.03571 & 137 \\ 137 & -111.03571 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$so, after\ normalization$

$$x_1 = 0.62965 \ \ x_2 = 0.77688$$

$$\xi_2 = \begin{pmatrix} 0.62965 \\ 0.77688 \end{pmatrix}$$

By code:

```
> 0: array([-0.7768816 , -0.62964671])
> 1: array([ 0.62964671, -0.7768816 ])
```

3

They are the same.

$The\ eigenvalues\ and\ eigenvectors\ of\ X^TX\ are: \lambda_1 v_1 = X^TXv_1$
$Multiply\ matrix\ X\ at\ the\ left: X\lambda_1 v_1 = XX^TXv_1$
$\lambda_1\ is\ a\ number\ and\ thus, \lambda_1\ (Xv_1) = XX^T(Xv_1)$
$The\ eigenvalues\ and\ eigenvectors\ of\ XX^T\ are: \lambda_2 v_2 = XX^Tv_2$
$Let\ Xv_1 = v_2, then\ \lambda_2 Xv_1 = XX^TXv_1$
$After\ compared: (\lambda_1 - \lambda_2)Xv_1 = 0$
$If\ and\ only\ if\ \lambda_1 = 0\ and\ \lambda_2 = 0\ or\ \lambda_1 = \lambda_2\ can\ we\ get\ the\ =.$
$Thus, the\ they\ have\ the\ same\ non-zero\ eigenvalues.$

4

By hand:

$after\ subtract\ mean:$

$$X = \begin{pmatrix} -2 & -1 & -1 & 0 & 1 & 3 \\ -3 & -1 & 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\mathrm{XX^T} = \begin{pmatrix} 16 & 17 \\ 17 & 20 \end{pmatrix}$$

$$\left|\frac{1}{6}XX^{\mathrm{T}} - \lambda I\right| = \begin{bmatrix} \frac{16}{6} - \lambda & \frac{17}{6} \\ \frac{17}{6} & \frac{20}{6} - \lambda \end{bmatrix}$$

$$\left(\frac{16}{6} - \lambda\right)\left(\frac{20}{6} - \lambda\right) - \frac{17^2}{6^2} = 0$$

$$\lambda_1 = 0.147 \quad \lambda_2 = 5.853$$

$when\ \lambda_1 = 0.147:$

$$\xi_1 = \begin{pmatrix} 0.747 \\ -0.665 \end{pmatrix}$$

$when\ \lambda_2 = 5.853:$

$$\xi_2 = \begin{pmatrix} 0.665 \\ 0.747 \end{pmatrix}$$

$1 - dimension\ data:$

$$Y = (0.665 \quad 0.747)\begin{pmatrix} -2 & -1 & -1 & 0 & 1 & 3 \\ -3 & -1 & 0 & 0 & 1 & 3 \end{pmatrix}$$
$$= (-3.571 \quad -1.412 \quad -0.665 \quad 0 \quad 1.412 \quad 4.235)$$

By code:

```
> 0: array([-3.57085518])
> 1: array([-1.41178986])
> 2: array([-0.66451439])
> 3: array([0.])
> 4: array([1.41178986])
> 5: array([4.23536958])
```

2 EM

2.1

No. it can not be solved directly. Because we do not know which coins were flipped after we flipped A. There are some latent data in this problem.
Also, the Eq $\hat{\theta} = argmaxlogP(Y|\theta)$ is difficult to maximize.

2.2

$$\mu_j^{i+1} = \frac{\pi^i (p^i)^{y_j} (1-p^i)^{1-y_j}}{\pi^i (p^i)^{y_j} (1-p^i)^{1-y_j} + (1-\pi^i)(q^i)^{y_j}(1-q^i)^{1-y_j}} \Rightarrow$$

$$\mu_j^1 = \frac{\pi^0 (p^0)^{y_j} (1-p^0)^{1-y_j}}{\pi^0 (p^0)^{y_j} (1-p^0)^{1-y_j} + (1-\pi^0)(q^0)^{y_j}(1-q^0)^{1-y_j}}$$

j	1	2	3	4	5	6	7	8	9	10
μ_j^1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5

2.3

$$\pi^{i+1} = \frac{1}{10}\sum_{j=1}^{10} \mu_j^{i+1} \Rightarrow$$

$$\pi^1 = \frac{1}{10}\sum_{j=1}^{10} \mu_j^1$$

$$p^{i+1} = \frac{\sum_{j=1}^{10} \mu_j^{i+1} y_j}{\sum_{j=1}^{10} \mu_j^{i+1}} \Rightarrow$$

$$p^1 = \frac{\sum_{j=1}^{10} \mu_j^1 y_j}{\sum_{j=1}^{10} \mu_j^1}$$

$$q^{i+1} = \frac{\sum_{j=1}^{10} (1-\mu_j^{i+1}) y_j}{\sum_{j=1}^{10} (1-\mu_j^{i+1})} \Rightarrow$$

$$q^1 = \frac{\sum_{j=1}^{10} (1-\mu_j^1) y_j}{\sum_{j=1}^{10} (1-\mu_j^1)}$$

π^1	0.5
p^1	0.6
q^1	0.6

2.4

$$\mu_j^2 = \frac{\pi^1 (p^1)^{y_j} (1-p^1)^{1-y_j}}{\pi^1 (p^1)^{y_j} (1-p^1)^{1-y_j} + (1-\pi^1)(q^1)^{y_j}(1-q^1)^{1-y_j}}$$

j	1	2	3	4	5	6	7	8	9	10
μ_j^2	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5

2.5

$$\pi^2 = \frac{1}{10}\sum_{j=1}^{10} \mu_j^2$$

$$p^2 = \frac{\sum_{j=1}^{10} \mu_j^2 y_j}{\sum_{j=1}^{10} \mu_j^2}$$

$$q^2 = \frac{\sum_{j=1}^{10} (1-\mu_j^2) y_j}{\sum_{j=1}^{10} (1-\mu_j^2)}$$

π^2	0.5
p^2	0.6
q^2	0.6

2.6

$$\mu_j^1 = \frac{\pi^0 (p^0)^{y_j} (1-p^0)^{1-y_j}}{\pi^0 (p^0)^{y_j} (1-p^0)^{1-y_j} + (1-\pi^0)(q^0)^{y_j}(1-q^0)^{1-y_j}}$$

j	1	2	3	4	5	6	7	8	9	10
μ_j^1	0.3636	0.3636	0.4706	0.3636	0.4706	0.4706	0.3636	0.4706	0.3636	0.3636

$$\pi^1 = \frac{1}{10}\sum_{j=1}^{10} \mu_j^1$$

$$p^1 = \frac{\sum_{j=1}^{10} \mu_j^1 y_j}{\sum_{j=1}^{10} \mu_j^1}$$

$$q^1 = \frac{\sum_{j=1}^{10} (1-\mu_j^1) y_j}{\sum_{j=1}^{10} (1-\mu_j^1)}$$

π^1	0.4064
p^1	0.5368
q^1	0.6433

$$\mu_j^2 = \frac{\pi^1 (p^1)^{y_j} (1-p^1)^{1-y_j}}{\pi^1 (p^1)^{y_j} (1-p^1)^{1-y_j} + (1-\pi^1)(q^1)^{y_j}(1-q^1)^{1-y_j}}$$

j	1	2	3	4	5	6	7	8	9	10
μ_j^2	0.3636	0.3636	0.4706	0.3636	0.4706	0.4706	0.3636	0.4706	0.3636	0.3636

$$\pi^2 = \frac{1}{10} \sum_{j=1}^{10} \mu_j^2$$

$$p^2 = \frac{\sum_{j=1}^{10} \mu_j^2 y_j}{\sum_{j=1}^{10} \mu_j^2}$$

$$q^2 = \frac{\sum_{j=1}^{10} (1-\mu_j^2) y_j}{\sum_{j=1}^{10} (1-\mu_j^2)}$$

π^2	0.4064
p^2	0.5368
q^2	0.6433

θ^2 is not the same as that in Question 2.5.

2.7

The results are not the same with different initialization.

$\pi^0 = 0.4, p^0 = 0.6, q^0 = 0.7$:

```
NumberIterations =  1
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631579
NumberIterations =  2
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.536842105263158
NumberIterations =  3
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631581
NumberIterations =  4
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631581
NumberIterations =  5
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631581
NumberIterations =  6
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631581
NumberIterations =  7
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631581
```

```
NumberIterations =  8
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631581
NumberIterations =  9
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631581
NumberIterations =  10
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631581
NumberIterations =  11
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631581
NumberIterations =  12
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631581
NumberIterations =  13
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631581
NumberIterations =  14
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631581
```

```
NumberIterations =  15
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631581
NumberIterations =  16
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631581
NumberIterations =  18
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631581
NumberIterations =  19
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631581
NumberIterations =  20
mu= [[0.36363636 0.36363636 0.47058824 0.36363636 0.47058824 0.47058824
  0.36363636 0.47058824 0.36363636 0.36363636]]
pi= 0.40641711229946526
q= 0.6432432432432431
p= 0.5368421052631581
```

$\pi^0 = 0.5, p^0 = 0.5, q^0 = 0.5$:

```
NumberIterations =  1
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  2
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  3
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  4
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  5
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  6
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  7
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  8
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  9
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  10
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
```

```
NumberIterations =  11
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  12
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  13
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  14
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  15
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  16
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  17
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  18
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  19
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
NumberIterations =  20
mu= [[0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5]]
pi= 0.5
q= 0.6
p= 0.6
```