OPTIMAL ADAPTIVE OUTPUT REGULATION OF DISCRETE-TIME NONLINEAR STOCHASTIC SYSTEMS*

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Abstract. This paper addresses the output regulation problem associated with a basic class of discrete-time nonlinear stochastic systems with unknown parameters. We allow the controlled plants to exhibit highly nonlinear growth, as well as nonlinearly parameterized structures in the exosystems. Under certain mild conditions, it is shown that with stable exosystems, the closed-loop regulated outputs are asymptotically optimal in the average sense almost surely. Technologically, we derive a near-optimal convergence rate for the nonlinear least squares (NLS) problem when identifying exosystems. In particular, upon degeneration of exosystems to null, we actually extend the existing stabilizability theorems to a more general setting.

Key words. discrete-time, nonlinear least squares, nonlinear systems, output regulation, stochastic adaptive control

MSC codes. 93D15, 93E12, 93E24

1. Introduction.

Output regulation is a fundamental problem in control theory, aiming to control a designated plant to track external references or counteract disturbances, originating from an exosystem. Initially considered for linear deterministic systems, output regulation leverages the widely recognized internal model principle [6, 11]. Over time, the scope of output regulation expanded to include a wide range of nonlinear systems [16, 19, 29] and infinite-dimensional systems [12]. A general framework to address nonlinear output regulation is to transform the original system into an augmented system, comprising the plant and a constructed internal model [18]. This transformation simplifies the problem to the stabilization of the augmented system.

There inherently exist uncertainties in modeling real-world systems, which makes it impractical to precisely solve the nonlinear regulator equations and construct satisfactory internal models. This limitation has spurred interest in adaptive construction methods for internal models. For example, linearly parameterized internal models have been constructed in [33, 34], while [3, 4, 10] have considered the adaptive approach for nonlinear internal models as a system identification problem. But most existing studies primarily concentrate on the deterministic case, despite the prevalence of stochastic modeling in engineering control. Considering that computer simulation requires discrete implementations of continuous systems for testing and analysis, we try to solve the adaptive output regulation problem for a basic class of discrete-time nonlinear stochastic systems.

However, in stochastic frameworks, the existing research on output regulation primarily revolves around linear systems in various settings [15, 30, 35, 37]. These

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methods cannot be used to deal with nonlinear systems. Especially in the discrete-time setting with unknown parameters, addressing the identification of exosystems and the stabilization of augmented systems using traditional adaptive methods, presents significant challenges. These challenges arise due to feedback limitations that occur when the output nonlinearities grow faster than linearities [14, 27, 39]. In fact, establishing the stabilizability of discrete-time nonlinear stochastic systems is pivotal for solving the corresponding adaptive output regulation problem. Nevertheless, this issue has only been addressed in the scalar parameter case [2, 14] or in the Bayesian framework [25, 28]. Moreover, the controllers designed within the Bayesian framework require precise a priori knowledge of the distributions of the noises and unknown parameters, which is practically unrealistic.

This paper studies a basic class of uncertain discrete-time nonlinear multi-input multi-output (MIMO) stochastic systems, and solves the associated optimal adaptive output regulation problem by designing the control law that combines the idea of self-tuning regulator (STR) and a modified nonlinear least squares (NLS) algorithm. The main contributions of this paper are threefold:

- For stable exosystems, we prove that the closed-loop systems asymptotically achieve optimal quadratic performance under some mild conditions. Our results are applicable to highly nonlinear controlled systems and nonlinearly parameterized exosystems. This provides a theoretical foundation for investigating output regulation problems in more general stochastic nonlinear systems.
- For general nonlinearly parameterized stochastic dynamical systems, we propose a new formula regarding the strong convergence rates of the NLS, which only depends on the data generated by systems. This formula has a potential to be applied to a wide spectrum of closed-loop system identification and machine learning problems with nonstationary data.
- As a corollary of our regulation results, we derive a stabilizability theorem that extends the relevant works from Bayesian frameworks and SISO systems in [25, 28] to a more general setting.

The remainder of this paper is organized as follows. Section 2 defines the optimal output regulation problem addressed in this paper. Section 3 investigates the degenerate case where the exosystem is null, and establishes the stabilizability of the plant. Section 4 presents the main results about the presented output regulation problem and the strong convergence rates of the NLS. Section 5 and the Appendices provide proofs of the main results. Section 6 demonstrates the numerical simulation results, and Section 7 concludes the study.

Notation. I_n denotes the n-dimensional identity matrix. For vectors $x_i, i = 1, \ldots, n$, $\mathrm{col}\,(x_1, \ldots, x_n) = (x_1^T, \ldots, x_n^T)^T$. $\|\cdot\|$ denotes the Euclidean norm. $|A|$ denotes the determinant of matrix A. We denote $\sigma\{y_i, 0 \leqslant i \leqslant t\}$ as the σ-field generated by $\{y_i, 0 \leqslant i \leqslant t\}$. I_A is the indicator function of event A. The essential supremum $\text{ess sup}_{x\in\Theta} f(x)$ is denoted as $\inf\{b \in \mathbb{R} : \ell(\{x \in \Theta : f(x) > b\}) = 0\}$, where $\ell(\cdot)$ is the Lebesgue measure. $\mathrm{dist}(x, S)$ is the distance between a point x and a set S, specifically as $\inf_{y\in S} \|x - y\|$. The notation $B(x, \rho)$ represents the open ball of radius ρ around x. If S is a set, $\bar{S}$ denotes its closure, and $\mathrm{diam}(S)$ denotes its diameter $\sup_{x,y\in S} \|x - y\|$. For two random quantities X, Y, we use $X \lesssim Y$ as shorthand for the inequality $X(\omega) \leq CY(\omega)$ for some random number $C > 0$ which depends only on $\omega \in \Omega$. In particular, when X, Y are deterministic, C is a universal constant. For two sequences $\{X_t\}_{t\geq 1}, \{Y_t\}_{t\geq 1}$, we say $X_t \lesssim Y_t$ and $X_t = o(Y_t)$ to denote $\limsup_{t\to\infty} X_t/Y_t < \infty$ and $\limsup_{t\to\infty} X_t/Y_t = 0$, respectively. The minimal eigenvalue of symmetric matrix A is $\lambda_{\min}[A]$. The abbreviations i.o. and a.s. mean "infinitely often" and "almost surely", respectively. $[n]$ denotes the set $\{1, 2, \ldots, n\}$.

2. Problem Formulation and Preliminaries. We aim to study the output regulation problem of discrete-time stochastic systems modeled by

$$y_{t+1} = \theta_1^T \phi(\varphi_t, \zeta_t) + u_t + w_{t+1}, \quad e_{t+1} = y_{t+1} - F\zeta_{t+1}, \quad t \geq 0, \tag{2.1}$$

where y_{t+1}, u_t, w_{t+1} denote the d-dimensional state, input, noise, and $\theta_1 \in \mathbb{R}^{p\times d}$ represents the unknown parameters. Moreover, $\varphi_t = \operatorname{col}(y_t, \ldots, y_{t-m+1})$, and $\phi = (f_1, \ldots, f_p)^T : \mathbb{R}^{dm} \times \mathbb{R}^l \to \mathbb{R}^p$ is a known continuous function. Term $e_t \in \mathbb{R}^d$ denotes the regulated output with matrix $F \in \mathbb{R}^{d\times l}$ given. We assume $\zeta_t \in \mathbb{R}^l$ is an exogenous signal representing the reference input to be tracked and the external disturbance to be rejected, generated by a so-called exosystem:

$$\zeta_{t+1} = g(\theta_2, \zeta_t) + v_{t+1}, \quad t \geq 0, \tag{2.2}$$

where $\theta_2 \in \Theta$ is the unknown parameter vector with set $\Theta \subset \mathbb{R}^q$ compact and convex, $v_{t+1} \in \mathbb{R}^l$ is the noise, and $g = (g_1, \ldots, g_l)^T : \mathbb{R}^q \times \mathbb{R}^l \to \mathbb{R}^l$ is a known three-times continuously differentiable function*. Here, the exosystem adopts a nonlinearly parameterized structure, due to the inherent difficulty of accurately modeling external disturbances in practical applications. For example, complex machine learning models such as deep neural networks may be used for exosystem prediction. Assume

A1 $\{w_t\}$ and $\{v_t\}$ are independent, and both of them are independent and identically distributed (i.i.d.) with $Ew_1 = 0$, $Ev_1 = 0$ and $E\|w_1\|^\gamma, E\|v_1\|^\gamma < \infty$ for some $\gamma > 0$. Moreover, the supports of w_1 and v_1 are $\mathbb{R}^d$ and $\mathbb{R}^l$, respectively.

A2 There are some positive constants L, K, and $\{b_i\}_{i=1}^p$ such that for any $z \in \mathbb{R}^{dm}$, $\zeta \in \mathbb{R}^l$ and $i \in [p]$,

$$|f_i(z, \zeta)| \leq L(\|z\|^{b_i} + 1)(\|\zeta\|^K + 1). \tag{2.3}$$

A3 There exists some $\beta > 0$ such that

$$\sup_{t\geq 1} E\|\zeta_t\|^\beta < \infty \quad \text{and} \quad \sup_{t\geq 1} E\left(\sup_{x\in\Theta}\left\|\frac{\partial g}{\partial \theta_2}(x, \zeta_t)\right\|^\beta\right) < \infty.$$

Remark 2.1. Assumption A3 imposes two moment conditions that in together with Markov's inequality, control the growth rates of both the exosystem and the sensitivity function with respect to the exosystem. By assuming the geometric ergodicity of the exosystem, we can derive the specific values of the exponent β, as explained in Section 4.2.

We define the optimal adaptive output regulation problem for the stochastic system (2.1)–(2.2) as follows.

Problem 2.2. Design a feedback control law $u_t \in \mathcal{F}_t^{y,\zeta} \triangleq \sigma\{y_i, \zeta_i, 0 \leqslant i \leqslant t\}$, $t = 0, 1, \ldots$ such that the trajectory of the closed-loop system starting from any initial state y_0 has the following two properties:

(i) if $\{\zeta_t\}_{t\geq 1}$ is stable†, then the trajectory of the closed-loop system is stable, i.e, $\sup_{t\geq 1} \frac{1}{t}\sum_{i=1}^t \|y_i\|^2 < \infty$ whenever $\sup_{t\geq 1} \frac{1}{t}\sum_{i=1}^t \|\zeta_i\|^2 < \infty$, a.s.,

*We require g to be three-times continuously differentiable because the convergence analysis of parameter estimation of θ_2 in Section 4.1 relies on the third-order differential information of g.

†The definition of stability here is analogous to that of [14, 24, 25]. It can be characterized as a form of mean-square boundedness. For convenience, we will not distinguish between "stable" and mean-square boundedness in this paper.

(ii) the ergodic cost $\limsup_{t\to\infty}\frac{1}{t}\sum_{i=1}^{t}\|e_i\|^2$ is minimized almost surely.

Remark 2.3. (i) The boundedness of exosystems is a standard setting in the study of regulation problems (see [16, 17] for the deterministic framework with bounded exogenous signals). Therefore, in the stochastic framework, we take some kind of stochastic boundedness as an alternative.
(ii) The optimality of the ergodic cost means that there exists a sequence of controllers $\{u_t^*\}_{t\geq 0}$, where for each time $t\geq 0$, $u_t^*\in\mathcal{F}_t^{y,\zeta}$, such that the regulated outputs $\{e_t^*\}$ produced by them satisfies $\limsup_{t\to\infty}\frac{1}{t}\sum_{i=1}^{t}\|e_i^*\|^2\leq\limsup_{t\to\infty}\frac{1}{t}\sum_{i=1}^{t}\|e_i\|^2$ for any $u_t\in\mathcal{F}_t^{y,\zeta}, t\geq 0$, a.s.. This definition of optimality is analogous to the optimality in the classical self-tuning regulator problem [13]. Clearly, achieving optimality necessitates accurate predictions of the exosystem, which requires an estimation of θ_2 in (2.2). The performance metric $\|e_i\|^2$ can be replaced by any quadratic form $e_i^TQe_i$ with $Q=Q^T>0$. The analysis and results in this paper still apply.

Now, let $\Lambda_i=\theta_1^T\phi(\varphi_{i-1},\zeta_{i-1})+u_{i-1}-Fg(\theta_2,\zeta_{i-1})$, $i\geq 1$. By [5, Theorem 2.8] and $u_i\in\mathcal{F}_i^{y,\zeta}$,

$$\sum_{i=1}^{t}\|e_i\|^2=\sum_{i=1}^{t}\|w_i-Fv_i\|^2+(1+o(1))\sum_{i=1}^{t}\|\Lambda_i\|^2,\quad \text{a.s.},$$

which implies

$$\limsup_{t\to\infty}\frac{1}{t}\sum_{i=1}^{t}\|e_i\|^2\geq E\|w_1-Fv_1\|^2,\quad \text{a.s..} \tag{2.4}$$

So, our main objective is to develop a controller law that achieves equality in (2.4), thereby solving Problem 2.2 for system (2.1)–(2.2).

3. Global Stabilizability. We first consider the degenerate case that the exosystem $\zeta_t\equiv 0$. Therefore, set dimension $l=0$ and the system (2.1) reduces to

$$y_{t+1}=\theta_1^T\phi(\varphi_t)+u_t+w_{t+1},\quad e_t=y_t. \tag{3.1}$$

Then Assumption A2 is simplified as
A2' There are some positive constants L, and $\{b_i\}_{i=1}^p$ such that for any $z\in\mathbb{R}^{dm}$,

$$|f_i(z)|\leq L(\|z\|^{b_i}+1),\quad i\in[p].$$

Hence, the solvability of Problem 2.2 is equivalent to the stabilizability and closed-loop optimality of system (3.1).

DEFINITION 3.1. *System* (3.1) *is said to be globally stabilizable, if there exists a feedback control law* $u_t\in\mathcal{F}_t^y\triangleq\sigma\{y_i,0\leqslant i\leqslant t\}$, $t=0,1,\ldots$ *such that* $\forall y_0\in\mathbb{R}^d$,

$$\sup_{t\geq 1}\frac{1}{t}\sum_{i=1}^{t}\|y_i\|^2<\infty,\qquad a.s..$$

The following result is in fact a corollary of Theorem 4.1 stated in Section 4.

THEOREM 3.2. *Under Assumptions A1 and A2' with* $\gamma>2b_1$, *if* $b_1>b_2>\cdots>b_p>0$, $b_1>1$, *and* $\forall x\in(1,b_1)$,

$$P(x)=x^{p+1}-b_1x^p+(b_1-b_2)\,x^{p-1}+\cdots+(b_{p-1}-b_p)x+b_p>0, \tag{3.2}$$

then system (3.1) *is globally stabilizable. Moreover, there exists some* $u_t \in \mathcal{F}_t^y$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t}\sum_{i=1}^{t} \|y_i\|^2 = E\|w_1\|^2, \qquad a.s.. \tag{3.3}$$

Remark 3.3. Apart from its extension to MIMO systems, Theorem 3.2 generalizes [28, Theorem 1] to a non-Bayesian framework, thus covering more general noise distributions and eliminating the need to know the noise variance and parameter expectations, which are prior knowledge for controller design in [28]. It is important to highlight the essential role of the Bayesian framework played in [28, Theorem 1] for the analysis of the closed-loop system. This framework ensures the conditional unbiasedness of the least squares (LS) estimation with respect to the true parameters, i.e., $\theta_{1,t} = E[\theta_1|\mathcal{F}_t^y]$. Consequently, the output y_{t+1} follows a conditional Gaussian distribution with zero mean. But these favorable properties do not hold within the non-Bayesian framework, requiring the development of novel techniques to analyze closed-loop systems.

Remark 3.4. For system (3.1) with $m = d = 1$, if the system function exhibits a specific polynomial growth identified by inequalities $L_1|z|^{b_i} \le |f_i(z)| \le L_2|z|^{b_i},\ i \in [p]$, where L_1 and L_2 are positive constants, the polynomial rule (3.2) has been shown in several frameworks to be a necessary and sufficient condition for the system to be stabilizable, such as Bayesian framework [25] and the deterministic framework [26].

By combining (2.4) and (3.3), we obtain the solvability of Problem 2.2 for $\zeta_t \equiv 0$. In fact, we employ the LS-based self-tuning regulator (LS-STR) in this scenario. However, when $\zeta_t \not\equiv 0$, the optimality of this regulator diminishes.

4. Optimal Output Regulation. We now address Problem 2.2 for general exosystems. Without loss of generality, we suppose $f_1, \dots, f_p$ are linearly independent. Otherwise, we can model the original system with fewer parameters, among which f_i are linearly independent. This will not affect the regulation problem.

For the general case $\zeta_t \not\equiv 0$, a natural idea about the controller design is to involve u_t having the following form[‡],

$$u_t = -\bar\theta_{1,t}^T \phi(\varphi_t, \zeta_t) + Fg(\bar\theta_{2,t}, \zeta_t), \tag{4.1}$$

where $\bar\theta_{2,t} \in \sigma\{\zeta_i, 0 \leqslant i \leqslant t\}$ is an estimator for θ_2, and $\bar\theta_{1,t}$ is the LS estimator for parameter θ_1, recursively defined by

$$\begin{cases} \bar\theta_{1,t+1} = \bar\theta_{1,t} + P_{t+1}\phi_t e_{t+1}^T \\ P_{t+1} = P_t - \left(1 + \phi_t^T P_t \phi_t\right)^{-1} P_t\phi_t\phi_t^T P_t,\ P_0 = I_p \end{cases},$$

where $\phi_t \triangleq \phi(\varphi_t, \zeta_t)$. Under such a controller, the closed-loop system becomes

$$y_{t+1} = (\theta_1 - \bar\theta_{1,t})^T \phi(\varphi_t, \zeta_t) + Fg(\bar\theta_{2,t}, \zeta_t) + w_{t+1}. \tag{4.2}$$

Define set $D \triangleq \{\omega : \sup_{t\ge 1} \frac{1}{t}\sum_{i=1}^{t} \|\zeta_i\|^2 < \infty\}$. We introduce an assumption on $\bar\theta_{2,t}$ such that controller (4.1) can solve Problem 2.2.

[‡]We employ simple additive structures for both the controller u_t and the regulated output e_t in our model settings (2.1)–(2.2), which simplify the controller design. The controller u_t can be divided into two components: the first component is responsible for stabilizing the system, while the second component, $Fg(\bar\theta_{2,t}, \zeta_t)$, predicts ζ_{t+1} and thereby optimize the regulated output e_{t+1}.

A4 For some constant $\varsigma \in (0,1)$,

$$\sum_{i=1}^{t} \|g(\bar{\theta}_{2,i},\zeta_i) - g(\theta_2,\zeta_i)\|^2 \lesssim t^{\varsigma}, \quad \text{a.s. on } D.$$

THEOREM 4.1. *Under Assumptions A1–A4, if* (3.2) *holds and*

$$2K\beta^{-1} + \max\left\{\varsigma b_1, 2b_1\beta^{-1}, 2b_1\gamma^{-1}\right\} < 1, \tag{4.3}$$

then Problem 2.2 is solvable by the feedback control law defined in (4.1)*. Specifically, the closed-loop system satisfies*

$$\limsup_{t\to\infty} \frac{1}{t}\sum_{i=1}^{t} \|e_i\|^2 = E\|w_1 - Fv_1\|^2, \quad \textit{a.s. on } D. \tag{4.4}$$

Proof. The proof is presented in Section 5. ∎

Obviously, the remaining work is to estimate θ_2. By (4.3) in Theorem 4.1, to solve Problem 2.2, we need an estimate $\bar{\theta}_{2,t}$ for the exosystem such that ς in Assumption A4 satisfies $\varsigma b_1 < 1$. According to the polynomial rule (3.2), b_1 can be arbitrarily close to 4, we may expect $\varsigma < \frac{1}{4}$. However, for general exosystems, few algorithms can achieve this condition (Recently, a GS algorithm for general nonlinearly parameterized systems was proposed in [23], with $\varsigma < \frac{1}{2} + \frac{1}{\gamma}$ guaranteed). So, we have to develop an identification algorithm for the exosystem to fulfill (4.3) and Assumption A4.

4.1. Identification of Exosystem. Note that the exosystem (2.2) has a nonlinearly parameterized structure, which makes it natural to use NLS algorithm to estimate the unknown parameters.

4.1.1. Nonlinear Least Square Estimation. Consider the following general stochastic regression:

$$y_{t+1} = g(\vartheta, \psi_t) + \varepsilon_{t+1}, \qquad t \geq 0, \tag{4.5}$$

where $\vartheta = (\vartheta_1, \ldots, \vartheta_q)^T \in \Theta$ is unknown, and there exists a sequence of nondecreasing σ-algebras $\{\mathcal{F}_t\}$ such that $\psi_t, y_t \in \mathcal{F}_t$. The noise sequence $\{\varepsilon_t, \mathcal{F}_t\}$ is a martingale difference sequence.

For the NLS estimator

$$\hat{\vartheta}_t = \operatorname*{argmin}_{x\in\Theta} \sum_{i=1}^{t} \|y_i - g(x,\psi_{i-1})\|^2, \quad t \geq 1, \tag{4.6}$$

a fundamental problem is to derive its convergence rate in the sense of strong consistency. But most research focuses on finding convergence conditions, rather than providing concrete convergence rates (cf. [20, 21, 36]). Particularly, works suited for closed-loop system analysis are even fewer. For example, although [40] provides the strong convergence rates for the NLS, its method requires certain stringent conditions and is not suitable for analyzing the convergence rate of the NLS in closed-loop systems. Besides, [24] employs the implicit function theorem to derive an analytical expression for the NLS and examines its strong convergence rate under the standard Gaussian noises. However, in scenarios involving multiple parameters, this approach cannot guarantee the existence of the implicit function.

Next, we introduce a novel approach to establish the strong convergence rate of NLS. Before presenting the results, we need to introduce some definitions.

The Sobolev space $W^{2,\infty}(\Theta,\mathbb{R}^{l\times q})$ is defined as

$$W^{2,\infty}(\Theta,\mathbb{R}^{l\times q}) \triangleq \left\{u\in L^{\infty}(\Theta,\mathbb{R}^{l\times q}) : D^{\alpha}u\in L^{\infty}(\Theta,\mathbb{R}^{l\times q}),\ \forall|\alpha|\leqslant 2\right\},$$

where

$$\begin{aligned} L^{\infty}(\Theta,\mathbb{R}^{l\times q}) \triangleq \big\{ & u : u \text{ is a measurable function from } \Theta \text{ to } \mathbb{R}^{l\times q},\\ & \|u(x)\|\leq C \text{ for almost every } x\in\Theta \text{ for some } C>0\} \end{aligned}$$

with the norm $\|u\|_{L^{\infty}(\Theta,\mathbb{R}^{l\times q})} = \text{ess sup}_{x\in\Theta}\|u(x)\|$, and $D^{\alpha}u$ denoting the weak α-th partial derivative of u for a multi-index α. We equip $W^{2,\infty}(\Theta,\mathbb{R}^{l\times q})$ with the norm $\|u\|_{2,\infty} = \sum_{0\leq|\alpha|\leq 2}\|D^{\alpha}u\|_{L^{\infty}(\Theta,\mathbb{R}^{l\times q})}$. Let

$$\overline{r}_t \triangleq \sum_{j=0}^{t}\left\|\frac{\partial g}{\partial\vartheta}(x,\psi_j)\right\|_{2,\infty}^{2} \tag{4.7}$$

and for $x=\operatorname{col}(x_{ij})_{i\in[t],j\in[l]}$, $x_{ij}\in\Theta$, define

$$P_{t+1}^{-1}(x) \triangleq I_q + \sum_{j=1}^{l}\sum_{i=1}^{t}\frac{\partial g_j}{\partial\vartheta}(x_{ij},\psi_i)\left(\frac{\partial g_j}{\partial\vartheta}(x_{ij},\psi_i)\right)^{T}. \tag{4.8}$$

We have the following theorem.

THEOREM 4.2. *Assume that there exist constants $\gamma>3$ and $\sigma_\gamma>0$ such that $\sup_{t\geq 1}E\left[\|\varepsilon_{t+1}\|^{\gamma}\mid\mathcal{F}_t\right]<\sigma_\gamma$, a.s., then for any $\epsilon>0$,*

$$\|\hat{\vartheta}_{t+1}-\vartheta\| \lesssim \frac{\overline{r}_t^{\frac{1}{2}}\log^{\frac{1}{2}+\epsilon}\overline{r}_t}{\min_{x\in\Theta^{tl}}\lambda_{\min}[P_{t+1}^{-1}(x)]}, \quad a.s..$$

Proof. The proof is presented in Appendix A. ∎

Remark 4.3. The convergence rate in Theorem 4.2 is similar to that in [24, Proposition 2]. However, the analytical approach employed in [24] depends not only on the implicit function's existence mentioned earlier, but also on the growth rate of the noise, i.e., $\varepsilon_t \lesssim \log t$, a.s.. It also requires a hypothesis similar to the boundedness of the condition number of the data's information matrix [24, assumption (23)]. In contrast, Theorem 4.2 only relies on the moment condition on the noise and does not require any additional assumptions on the data sequence.

4.1.2. A Modified NLS for Exosystem. Now, let us return to the identification of the exosystem (2.2). Set $\nu>0$, for $t=1,2,\ldots$, define events

$$\Xi_t \triangleq \left\{\omega : \|\zeta_t\|^2+\|\zeta_{t-1}\|^2<\nu+\frac{2}{t}\sum_{i=1}^{t-1}\|\zeta_i\|^2\right\}. \tag{4.9}$$

The modified NLS estimator $\bar{\theta}_{2,t}$ is defined by

$$\bar{\theta}_{2,t} \triangleq \operatorname*{argmin}_{x\in\Theta}\sum_{i=1}^{t}\|\zeta_i-g(x,\zeta_{i-1})\|^2 I_{\Xi_{i-1}}, \quad t\geq 1. \tag{4.10}$$

For convenience, we still use the notations $\overline{r}_t$ and $P_{t+1}^{-1}(x)$ defined in (4.7)–(4.8), but with $g(\cdot,\psi_i)$ replaced by $g(\cdot,\zeta_i)I_{\Xi_i}$. By the definitions of $\overline{r}_t$ and Ξ_t, it is not difficult to derive that $\sup_{t\geq 1}\|\zeta_t\|I_{\Xi_t}<\infty$ almost surely on D, and then

$$\overline{r}_t \lesssim t, \quad \text{a.s. on } D. \tag{4.11}$$

According to Theorem 4.2 and (4.11), the convergence rate of $\bar{\theta}_{2,t}$ can be obtained if the growth rate of $\min_{x\in\Theta^{tl}}\lambda_{\min}[P_{t+1}^{-1}(x)]$ is provided. To this end, we introduce a simple algebraic condition from [23]. For $h_j\in[l], j\in[q]$, define

$$\Phi(x,z)\triangleq\left(\frac{\partial g_{h_1}}{\partial\vartheta}(x_1,z_1),\cdots,\frac{\partial g_{h_q}}{\partial\vartheta}(x_q,z_q)\right)^T$$

with $x=\operatorname{col}(x_i)_{i=1}^q, x_i\in\mathbb{R}^q$ and $z=\operatorname{col}(z_i)_{i=1}^q, z_i\in\mathbb{R}^l$. Let $\Phi_k(x,z)$ denote the k-th order leading principal submatrix of $\Phi(x,z), k\in[q]$.

A5 There are q indices $h_j\in[l], j\in[q]$ and some point $z\in\mathbb{R}^{ql}$ such that

$$\operatorname{rank}\Phi_k(x,z)=k \text{ for } k\in[q], \quad \forall x\in\Theta^q.$$

Remark 4.4. In view of the analysis in [23], Assumption A5 plays a crucial role in estimating the growth rate of $\min_{x\in\Theta^{tl}}\lambda_{\min}[P_{t+1}^{-1}(x)]$. Moreover, when g is linearly parameterized, Assumption A5 equals to the linear independence of $\frac{\partial g}{\partial\vartheta_1}(x,\cdot),\cdots,\frac{\partial g}{\partial\vartheta_q}(x,\cdot)$, which is necessary for the identification of system (2.2).

To gain a better understanding of Assumption A5, we propose a condition that is stronger yet more intuitive than Assumption A5.

Definition 4.5. *A sequence of parameterized functions $F_1(x,\cdot),\ldots,F_j(x,\cdot):\mathbb{R}^{n_1}\to\mathbb{R}^{n_2}$ is said to be uniformly linearly independent with respect to $x\in\Theta$, if for all non-zero vector $z=(z_1,\ldots,z_j)^T\in\mathbb{R}^j$, $\inf_{x\in\Theta}\|F_1(x,\cdot)z_1+\cdots+F_j(x,\cdot)z_j\|\not\equiv 0$.*

It is easy to see that Assumption A5 is weaker than the following assumption.

A5' The sequence of functions $\frac{\partial g}{\partial\vartheta_1}(x,\cdot),\cdots,\frac{\partial g}{\partial\vartheta_q}(x,\cdot)$ are uniformly linearly independent with respect to $x\in\Theta$.

Then, we can derive the strong convergence rate of $\bar{\theta}_{2,t}$ as follows.

Corollary 4.6. *Let Assumptions A1 and A5 (or A5') hold for $\gamma>3$ in exosystem* (2.2). *Then, by appropriately choosing parameter ν in* (4.9), *estimator* (4.10) *satisfies that for any $\epsilon>0$,*

$$\|\bar{\theta}_{2,t}-\theta_2\|^2\lesssim\frac{\log^{1+\epsilon}t}{t}, \quad \textit{a.s. on } D. \tag{4.12}$$

Proof. It suffices to choose ν such that

$$t\lesssim\min_{x\in\Theta^{tl}}\lambda_{\min}[P_{t+1}^{-1}(x)], \quad \text{a.s. on } D.$$

For $\nu>0$, $t\geq 1$, let $\bar{P}_{t+1}^{-1}(x)$ denote the matrix obtained by replacing $\{\Xi_i\}_{i\in[t]}$ with $\{\Omega_i(\nu)\}_{i\in[t]}$ in P_{t+1}^{-1}, $\eta_t(\nu)\triangleq\sum_{i=1}^t I_{\Omega_i(\nu)}$, where $\Omega_i(\nu)\triangleq\left\{\|\zeta_i\|,\|\zeta_{i-1}\|<\frac{1}{2}\sqrt{\nu}\right\}$. Under Assumption A5, if we choose $\nu>\|z\|$, by the proof of [23, Proposition C.1.], it follows that

$$\eta_t(\nu)\lesssim\min_{x\in\Theta^{tl}}\lambda_{\min}[\bar{P}_{t+1}^{-1}(x)], \quad \text{a.s.}. \tag{4.13}$$

By $P_{t+1}^{-1}(x) \geq \bar{P}_{t+1}^{-1}(x)$ and (4.13), it remains to confirm $t \lesssim \eta_t(\nu)$, a.s. on D.

Define $\mathcal{F}_t^{\zeta} = \sigma\{\zeta_0, \ldots, \zeta_t\}$, by [5, Theorem 2.8], for any given $M > 0$ and all sufficiently large t,

$$\sum_{i=1}^{t} I_{\mathbf{V}_i}(I_{\Omega_i(\nu)} - P(\Omega_i(\nu) \mid \mathcal{F}_{i-1}^{\zeta})) = o(t), \quad \text{a.s.,} \tag{4.14}$$

where $\mathbf{V}_i \triangleq \{\|\zeta_{i-1}\|^2 < \nu + 2M\}$. Since the support of v_1 is $\mathbb{R}^l$, we can derive a number $M_\Omega > 0$ (depends on ν and the distribution of v_1) such that

$$P(\Omega_i(\nu) \mid \mathcal{F}_{i-1}^{\zeta}) I_{\mathbf{V}_i} \geq M_\Omega I_{\mathbf{V}_i}.$$

Therefore, by above inequality and (4.14), we have

$$\frac{M_\Omega}{t} \sum_{i=1}^{t} I_{\mathbf{V}_i} \leq \frac{1}{t} \sum_{i=1}^{t} I_{\mathbf{V}_i} P(\Omega_i(\nu) \mid \mathcal{F}_{i-1}^{\zeta}) \lesssim \frac{1}{t} \sum_{i=1}^{t} I_{\mathbf{V}_i} I_{\Omega_i(\nu)} \leq \frac{\eta_t(\nu)}{t}, \quad \text{a.s..} \tag{4.15}$$

Now, denote the set

$$D(C) \triangleq \left\{\omega : \sup_{t \geq 1} \frac{1}{t} \sum_{i=1}^{t} \|\zeta_i\|^2 < C\right\}, \quad C > 0, \tag{4.16}$$

then one has $\liminf_{t\to\infty} \frac{1}{t}\sum_{i=1}^{t} I_{\mathbf{V}_i} \geq \frac{1}{2}$, a.s. on $D\left(\frac{\nu}{2} + M\right)$. This together with (4.15) leads to $t \lesssim \eta_t(\nu)$, a.s. on $D\left(\frac{\nu}{2} + M\right)$. Letting $M \to \infty$, we completes the proof. ∎

Compared to the well-known result for the LS [22], the strong convergence rate of $\bar{\theta}_{2,t}$ in Corollary 4.6 is near-optimal. In fact, the convergence rate in Corollary 4.6 is crucial for ensuring (4.3) and then achieving the optimal output regulation, as discussed in Section 4.2.

4.2. Solutions of the Optimal Output Regulation Problem. We now solve Problem 2.2 in this subsection. By Assumptions A1–A3, we can easily verify that algorithm (4.10) meets (4.3) in Theorem 4.1 under certain mild conditions.

THEOREM 4.7. *Under Assumptions A1–A3 and A5 (or A5'), if* (3.2) *holds and*

$$2K\beta^{-1} + \max\left\{2b_1\beta^{-1}, 2b_1\gamma^{-1}\right\} < 1, \quad \gamma > 3, \tag{4.17}$$

then by appropriately choosing parameter ν *in* (4.9)*, Problem 2.2 is solvable by the feedback control law composed of* (4.1) *and* (4.10)*.*

Proof. Since Assumption A3 holds, by applying *Markov's Inequality* and *Borel-Cantelli-Lemma*, we deduce $\sup_{x\in\Theta} \|\frac{\partial g}{\partial \theta_2}(x, \zeta_t)\| \lesssim t^{1/\beta} \log^{2/\beta} t$, a.s.. Combining this inequality and Corollary 4.6, one has

$$\sum_{i=1}^{t} \|g(\bar{\theta}_{2,i}, \zeta_i) - g(\theta_2, \zeta_i)\|^2 \lesssim t^{2/\beta} \log^{2+4/\beta} t, \quad \text{a.s. on } D. \tag{4.18}$$

Thus, (4.3) is true for $\varsigma = 2/\beta + \epsilon$ with $\epsilon > 0$ sufficiently small. By applying Theorem 4.1, we immediately confirm Theorem 4.7. ∎

Remark 4.8. Condition (4.17) establishes a relationship between the nonlinear growth rate of system (2.1) and the moment conditions of exosystem (2.2). It requires the corresponding exosystem (2.2) to satisfy strong moment conditions when the former exhibits high nonlinearity. Otherwise, the uncertainties introduced by the exosystem during regulation would be amplified, potentially leading to the instability of system (2.1). Furthermore, if we express Assumption A2 using inequality $|f_i(z,\zeta)| \leq L(\|z\|^{b_i} + \|\zeta\|^K + 1)$, then the first inequality in condition (4.17) can be replaced by $\max\left\{2K\beta^{-1}, 2b_1\beta^{-1}, 2b_1\gamma^{-1}\right\} < 1$.

In Theorem 4.7, we do not require the exosystem to satisfy any ergodicity. However, the maximum value of β in Assumption A3 is generally unavailable for exosystem (2.2) in practice, making it difficult to verify (4.17). To address this, we introduce a conveniently verifiable condition that β is absent. Actually, this condition ensures the geometric ergodicity of ζ_t.

A6 There exist a vector norm $\|\cdot\|_v$ on $\mathbb{R}^l$ and three constants $0 < \lambda_1 < 1, \lambda_2, \lambda_3 > 0$ such that for any $\zeta \in \mathbb{R}^l$,

$$\|g(\theta_2,\zeta)\|_v \leq \lambda_1\|\zeta\|_v + \lambda_3 \quad \text{and} \quad \sup_{x\in\Theta}\left\|\frac{\partial g}{\partial \theta_2}(x,\zeta)\right\| \leq \lambda_2\|\zeta\| + \lambda_3. \tag{4.19}$$

Remark 4.9. The first inequality in (4.19) is a commonly used criterion to ensure the ergodicity of ζ_t (see [1] and [41]). Besides, many widely studied nonlinear models that satisfy the first inequality in (4.19) also satisfy the second one, e.g., the bounded AR model, exponential AR model, semi-parametric AR model, generalized linear AR model, ARCH model [1]. In fact, Assumption A6 holds for a wide range of ergodic nonlinear statistical models.

COROLLARY 4.10. *Under Assumptions A1, A2, A5 (or A5') and A6, let* (3.2) *hold and* $\max\{2K + 2b_1, 3\} < \gamma$. *Then by appropriately selecting parameter* ν *in* (4.9), *Problem 2.2 is solvable by the feedback control law composed of* (4.1) *and* (4.10). *Moreover, in the closed-loop system,* $\limsup_{t\to\infty}\frac{1}{t}\sum_{i=1}^{t}\|e_i\|^2 = E\|w_1 - Fv_1\|^2$ *a.s.*.

Proof. In view of [41], we know Assumption A6 implies the geometrical ergodicity of the exosystem. Let μ_0 denote the stationary distribution of $\{\zeta_t\}$. For a given random variable ζ^* with distribution μ_0, we suppose that

$$E\|\zeta^*\|^\gamma < \infty. \tag{4.20}$$

Then, by [31, Theorem 17.0.1], we derive $\lim_{t\to\infty}\frac{1}{t}\sum_{i=1}^{t}\|\zeta_i\|^2 = E\|\zeta^*\|^2 < \infty$, a.s., which implies $P(D) = 1$. In addition, by (4.19) in Assumption A6 and *Minkowski Inequality*, we have the estimation

$$E\|\zeta_t\|_v^\gamma \leq E\|g(\theta_2,\zeta_{t-1})\|_v^\gamma + E\|v_t\|_v^\gamma \leq \lambda_1^\gamma E\|\zeta_{t-1}\|_v^\gamma + \lambda_3^\gamma + E\|v_1\|_v^\gamma. \tag{4.21}$$

Hence it is easy to obtain

$$\sup_{t\geq 1} E\|\zeta_t\|^\gamma \lesssim \sup_{t\geq 1} E\|\zeta_t\|_v^\gamma < \|\zeta_0\|^\gamma + \frac{\lambda_3^\gamma + E\|v_1\|_v^\gamma}{1-\lambda_1^\gamma},$$

which together with Assumption A6 implies that Assumption A3 holds with $\beta = \gamma$. Now, by applying Theorem 4.7, the conclusion of Corollary 4.10 follows.

Next, it suffices to confirm (4.20). Similar to (4.21), we have $E[\|\zeta_t\|_v^\gamma|\zeta_{t-1} = \zeta] \leq \lambda_1^\gamma\|\zeta\|_v^\gamma + \lambda_3^\gamma + E\|v_1\|_v^\gamma$, then there is a closed set $\mathbf{C}$ such that

$$E[\|\zeta_t\|_v^\gamma|\zeta_{t-1} = \zeta] \leq \lambda_1^{\gamma/2}\|\zeta\|_v^\gamma, \quad \forall\zeta \in \mathbb{R}^l \setminus \mathbf{C}, \quad \text{and} \quad \sup_{\zeta\in\mathbf{C}} E[\|\zeta_t\|_v^\gamma|\zeta_{t-1} = \zeta] < \infty.$$

So, we can directly apply [9, Theorem 2] to derive $E\|\zeta\|_v^\gamma < \infty$. Noting the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_v$, (4.20) follows. $\square$

4.3. Linear Parameterization of Exosyetem. Theorem 4.7 solves Problem 2.2 under certain mild conditions, but the nonconvex nature of the optimization problem (4.10) introduces challenges in identifying exosystem (2.2), especially when the dimension q is very large. Solving this optimization problem can be computationally expensive, and finding an exact solution is inherently hard. Therefore, adopting a linearly parameterized model, $g(\theta_2, \cdot) = F(\cdot)\theta_2$, is more practical in exosystem (2.2). This approach is supported by the approximation theorem presented in [32, Theorem 3], which justifies the decomposition $\bar{f}(x, w) \approx \bar{\phi}(x)\bar{a}(w)$ for any vector-valued analytic function $\bar{f}(x, w)$ with a bounded input space. In this decomposition, $\bar{\phi}(x)$ and $\bar{a}(w)$ are two Chebyshev polynomials. Notably, when the dimension of $\bar{f}$ is l, the ranges of $\bar{\phi}$ and $\bar{a}$ are contained in $\mathbb{R}^{l\times h}$ and $\mathbb{R}^h$, respectively. The integer h scales polylogarithmically with the inverse of the desired approximation accuracy.

When $g(\theta_2, \cdot) = \mathbf{F}(\cdot)\theta_2$ for some matrix-valued function $\mathbf{F} = (F_1, \dots, F_q)$, we can take $\bar{\theta}_{2,t}$ to be the recursive LS estimator

$$
\left\{
\begin{array}{l}
\bar{\theta}_{2,t+1} = \bar{\theta}_{2,t} + \bar{P}_{(t+1)l}\bar{\phi}_t(\zeta_{t+1} - F(\zeta_t)\bar{\theta}_{2,t}) \\
\bar{P}_{tl+i} = \bar{P}_{tl+i-1} - \frac{\bar{P}_{tl+i-1}\bar{\phi}_{t,i}\bar{\phi}_{t,i}^T\bar{P}_{tl+i-1}}{1+\bar{\phi}_{t,i}^T\bar{P}_{tl+i-1}\bar{\phi}_{t,i}},\ \bar{P}_0 = I_q,\ i \in [l] \\
\bar{\phi}_t = (\bar{\phi}_{t,1}, \dots, \bar{\phi}_{t,l})I_{\Xi_t} \triangleq \mathbf{F}^T(\zeta_t)I_{\Xi_t}
\end{array}
\right.
\tag{4.22}
$$

in (4.1). Without loss of generality as before, we assume that $F_1, \dots, F_q$ are linearly independent. Now, we present the following theorem.

THEOREM 4.11. *Under Assumptions A1–A3, if* $2K\beta^{-1} + \max\left\{2b_1\beta^{-1}, 2b_1\gamma^{-1}\right\} < 1$ *and* (3.2) *hold, then Problem 2.2 is solvable by the feedback control law composed of* (4.1), (4.10) *and* (4.22).

Proof. By the asymptotic theory of the LS [13] and the proof of Corollary 4.6, it follows that $\bar{\theta}_{2,t}$ defined by (4.22) satisfies $\|\bar{\theta}_{2,t} - \theta_2\| \lesssim \sqrt{\log t/t}$, a.s. on D. Similar to the proof of Theorem 4.7, we obtain (4.18) and Theorem 4.1 applies. $\square$

5. Proof of Theorem 4.1. To facilitate the analysis, let $r_t \triangleq \sum_{i=0}^t \|\phi_i\|^2$. By the definition of P_t, we have $P_t^{-1} = I_p + \sum_{i=0}^{t-1}\phi_i\phi_i^T$. Noting the implicit presence of $\gamma > 2b_1 > 2$ within (4.3), we will proceed with our analysis under $\gamma > 2$.

Next, define

$$
\sigma_t^2 \triangleq |P_{t+1}^{-1}|/|P_t^{-1}| = 1 + \phi_t^T P_t \phi_t, \quad \Delta_t \triangleq F(g(\bar{\theta}_{2,t}, \zeta_t) - g(\theta_2, \zeta_t)),
$$

$a_t \triangleq \sigma_t^{-2}$, $\rho_t \triangleq \phi_t^T\left(\theta_1 - \bar{\theta}_{1,t}\right)$, then (4.2) implies

$$
\sum_{i=1}^t \|e_{i+1}\|^2 = \sum_{i=1}^t \|w_{i+1} - Fv_{i+1}\|^2 + (1 + o(1))\sum_{i=1}^t \|\rho_i + \Delta_i\|^2, \quad \text{a.s..}
\tag{5.1}
$$

Under the control law (4.1), it is straightforward to obtain the solvability of Problem 2.2 if the closed-loop system satisfies (4.4). By Assumption A4, (5.1) and noting that $\sum_{i=1}^t \|\rho_i\|^2 \le \sum_{i=1}^t a_i\|\rho_i\|^2 \sup_{k\le t}\sigma_k^2$, it suffices to show

$$
\sum_{i=1}^t a_i\|\rho_i\|^2 = o(t) \quad \text{and} \quad \sup_{t\ge 1}\sigma_t < \infty, \qquad \text{a.s. on } D.
\tag{5.2}
$$

We need some lemmas, where the first one is a modified version of [13, Corollary 3.1].

LEMMA 5.1. *Under Assumption A1,* $\sum_{i=1}^t a_i\|\rho_i\|^2 \lesssim \log r_t \sum_{i=0}^t \|\Delta_i\|^2$, *a.s..*

Proof. Similar to the proof of [13, Lemma 1], we consider the Lyapunov function $V_i = \left(\theta_1 - \bar{\theta}_{1,i}\right)^T P_i^{-1}\left(\theta_1 - \bar{\theta}_{1,i}\right)$. Define $\varpi_{i+1} \triangleq \Delta_i + w_{i+1} - Fv_{i+1},\ i \geq 0$, then by simple calculation,

$$V_{t+1} + \sum_{i=0}^t a_i\|\rho_i\|^2 = V_0 - 2\sum_{i=0}^t a_i\rho_i^T\varpi_{i+1} + \sum_{i=0}^t a_i\phi_i^{\mathrm{T}} P_i\phi_i\|\varpi_{i+1}\|^2. \tag{5.3}$$

In view of [5, Theorem 2.8] and the proof of [13, Lemma 1], we know

$$\sum_{i=0}^t a_i\rho_i^T(w_{i+1} - Fv_{i+1}) \lesssim 1 + o\Big(\sum_{i=0}^t a_i\|\rho_i\|^2\Big), \quad \text{a.s.}$$

and $\sum_{i=0}^t a_i\phi_i^{\mathrm{T}} P_i\phi_i\|w_{i+1} - Fv_{i+1}\|^2 \lesssim \log r_t$, a.s., then (5.3) leads to

$$\begin{aligned}\sum_{i=0}^t a_i\|\rho_i\|^2 &\lesssim \log r_t + \sum_{i=0}^t a_i\phi_i^{\mathrm{T}} P_i\phi_i\|\Delta_i\|^2 + \sum_{i=0}^t a_i\rho_i^T\Delta_i \\ &\lesssim \sum_{i=0}^t \|\Delta_i\|^2 \log r_t + \left(\sum_{i=0}^t a_i\|\rho_i\|^2 \sum_{i=0}^t \|\Delta_i\|^2\right)^{1/2}, \quad \text{a.s.,} \qquad (5.4)\end{aligned}$$

which is exactly Lemma 5.1. The last inequality in (5.4) used $a_i\phi_i^{\mathrm{T}} P_i\phi_i \leq 1$ and *Cauchy-Schwarz Inequality.* ∎

LEMMA 5.2. *Under Assumptions A1–A2 and A4, we have* $\log r_{t+1} \lesssim \log t + \log\sup_{j\leq t}\sigma_j$, *a.s. on* D.

Proof. By (5.1), Lemma 5.1 and Assumption A4,

$$\sum_{i=1}^t \|y_{i+1}\|^2 \lesssim \sum_{i=1}^t \|\zeta_{i+1}\|^2 + \sum_{i=1}^t a_i\|\rho_i\|^2 \sup_{j\leq t}\sigma_j^2 + t \lesssim t^{\varsigma}\log r_t \sup_{j\leq t}\sigma_j^2 + t, \qquad \text{a.s. on } D,$$

which together with Assumption A2 yields that for some positive constant b,

$$\begin{aligned} r_{t+1} &\lesssim \sum_{i=1}^t (\|\zeta_{i+1}\|^{2b} + 1)(\|\varphi_{i+1}\|^{2b} + 1) \lesssim 1 + \left(\sum_{i=1}^t \|y_{i+1}\|^2\right)^b \left(\sum_{i=1}^t \|\zeta_{i+1}\|^2\right)^b \\ &\lesssim t^b(t^{b\varsigma}\log^b r_t)\sup_{j\leq t}\sigma_j^{2b} + t^{2b}, \qquad \text{a.s. on } D.\end{aligned}$$

Taking the logarithm on both sides of the above inequality yields Lemma 5.2. ∎

Note that $\sup_{t\geq 1}\sigma_t < \infty$ and Lemma 5.2 imply $\log r_t \lesssim \log t$, a.s. on D, which together with Lemma 5.1 leads to (5.2), and hence Theorem 4.1 follows. Therefore, our focus is now on establishing the validity of $\sup_{t\geq 1}\sigma_t < \infty$, a.s. on D. This can be directly derived from the two lemmas below.

LEMMA 5.3. *Under Assumptions A1–A4, for any monotonically increasing function* $\psi : \mathbb{N}^+ \to \mathbb{R}_{\geq 0}$ *satisfying* $\lim_{t\to\infty}\psi(t) = \infty$, *if* $P(D\cap D_1) > 0$ *with* $D_1^c = \left\{\left|P_t^{-1}\right| < t^{\psi(t)}, \overline{i.o.}\right\}$, *then* $\sup_{t\geq 1}\sigma_t < \infty$, *a.s. on* $D\cap D_1$.

Proof. Firstly, we can define the same random subscript t_k and matrix Q_k as in [28, Lemma 3]. Let $\mathcal{Q}_k \triangleq \left|Q_{k+1}^{-1}\right| / \left|Q_k^{-1}\right|$, similar to [28, Lemma 3], we have $\mathcal{Q}_{k-1} < \mathcal{Q}_k,\ k \geq 1$, and

$$\sigma_{t-1}^2 \leq \mathcal{Q}_{k-1}, \quad \forall t \in (t_{k-1}, t_k]. \tag{5.5}$$

It suffices to consider the event set $D' \triangleq \{\sup_{k\geq 1} t_k = \infty\} \cap D \cap D_1$ with $P(D') > 0$ and show a contradiction.

Next, by Assumptions A1 and A3, it yields that

$$\|w_t - Fv_t\|^2 \lesssim t^{2/\gamma}\log t \quad \text{and} \quad \|\zeta_t\|^2 \lesssim t^{2/\beta}\log t, \quad \text{a.s..}$$

Therefore, by taking account of (4.2), Lemma 5.1 and Assumption A4,

$$\begin{aligned}
y_t^2 &\lesssim a_{t-1}\|\rho_{t-1}\|^2\sigma_{t-1}^2 + \|w_t - Fv_t\|^2 + \Delta_{t-1}^2 + \zeta_t^2 \\
&\lesssim \sigma_{t-1}^2 t^{\varsigma}\log r_{t-1} + t^{\max\{2/\beta,2/\gamma\}}\log t \\
&\lesssim \sigma_{t-1}^2 t^{\varsigma}\Big(\log\sup_{k\leq t-2}\sigma_k + \log t\Big) + t^{\max\{2/\beta,2/\gamma\}}\log t, \quad \text{a.s. on } D'.
\end{aligned} \tag{5.6}$$

That is, there exists a random number $C_y > 0$ such that for any integer $t \geq 1$,

$$y_t^2 \leq C_y\sigma_{t-1}^2 t^{\max\{\varsigma,2/\beta,2/\gamma\}}\Big(\log\sup_{k\leq t-2}\sigma_k + \log(t+1)\Big), \quad \text{a.s. on } D', \tag{5.7}$$

here we denote $\sigma_{-1} \triangleq 1$.

For $i \geq 1$, we define $\alpha_i(-1)$ as the ith column of the identity matrix I_p and $\alpha_i(j) \triangleq \phi_{t_j} f_i(\varphi_{t_j}, \zeta_{t_j})$, $j \geq 0$. Then, similar to the proof of [28, Lemma 3], we have $|Q_{k+1}^{-1}| = \sum_{(s_1,\ldots,s_p)\in\mathcal{W}(k)}\det(\alpha_1(s_1),\ldots,\alpha_p(s_p))$, where $\mathcal{W}(k)$ is defined as

$$\{(l_1,\ldots,l_p) : l_i \in \{-1,0,\ldots,k\}, i \in [p]; l_i \neq l_{i'} \text{ if } i \neq i',\ l_{i'} \neq -1\}.$$

Further, for given $(l_1,\ldots,l_{p'})$ and k, we use $\mathcal{H}_k^{(l_1,\ldots,l_{p'})}$ to denote set

$$\{(i_1,\ldots,i_k) : i_j \in \{l_1,\ldots,l_{p'}\}, j \in [k]; i_r \neq i_s \text{ if } r \neq s\}.$$

Now, for any $(s_1,\ldots,s_p) \in \mathcal{W}(k)$, $(l_1,\ldots,l_p) \in \mathcal{H}_p^{(1,\ldots,p)}$, if $s_i \neq 1$, by (5.5), (5.7) and Assumption A2, we have

$$\begin{aligned}
&\Big(1 + \|\zeta_{t_{s_i}}\|^K\Big)^{-2}\left|f_{l_i}(y_{t_{s_i}}, \zeta_{t_{s_i}})\right| \cdot \left|f_i(y_{t_{s_i}}, \zeta_{t_{s_i}})\right| \lesssim \max\{1, \max_{j\in[m]}\|y_{t_{s_i}-j+1}\|^{b_{l_i}+b_i}\} \\
&\lesssim \max_{j\in[m]}\Big(\sigma_{t_{s_i}-j}^2 t_{s_i-j+1}^{\varsigma+\frac{2}{\beta}+\frac{2}{\gamma}}\log\sup_{k\leq t_{s_i}-j-1}\sigma_k\Big)^{\frac{b_{l_i}+b_i}{2}} \lesssim \Big(t_{s_i}^{\varsigma+\frac{2}{\beta}+\frac{2}{\gamma}}\mathcal{Q}_{s_i-1}\log\mathcal{Q}_{s_i-1}\Big)^{\frac{b_{l_i}+b_i}{2}},
\end{aligned}$$

where $\mathcal{Q}_{-1} \triangleq I_p$. Then,

$$\begin{aligned}
&\det(\alpha_1(s_1),\ldots,\alpha_p(s_p)) \\
&\leq \sum_{(l_1,\ldots,l_p)\in\mathcal{H}_p^{(1,\ldots,p)}}\ \prod_{i\in[p],s_i\neq-1}\left|f_{l_i}(y_{t_{s_i}}, \zeta_{t_{s_i}})\right| \cdot \left|f_i(y_{t_{s_i}}, \zeta_{t_{s_i}})\right| \\
&\lesssim \Big(1 + \sum_{j=1}^p\|\zeta_{t_{s_j}}\|^K\Big)^{2p}\sum_{(l_1,\ldots,l_p)\in\mathcal{H}_p^{(1,\ldots,p)}}\ \prod_{i\in[p],s_i\neq-1}\Big(t_{s_i}^{\varsigma+\frac{2}{\beta}+\frac{2}{\gamma}}\mathcal{Q}_{s_i-1}\log\mathcal{Q}_{s_i-1}\Big)^{\frac{b_{l_i}+b_i}{2}} \\
&\lesssim \Big(1 + \sum_{j=1}^p\|\zeta_{t_{s_j}}\|^K\Big)^{2p} t_k^{(\varsigma+\frac{2}{\beta}+\frac{2}{\gamma})pb_1}\prod_{i=1}^p\mathcal{Q}_{k-i}^{b_i}\prod_{i=1}^p\log^{b_i}\mathcal{Q}_{s_i-1}, \quad \text{a.s. on } D',
\end{aligned} \tag{5.8}$$

the last inequality in (5.8) used the monotonicity of $\{\mathcal{Q}_i\}_{i\geq 1}$.

Note that $\|\zeta_t\|^2 \lesssim t^{2/\beta}\log t \lesssim t^{3/\beta}$ almost surely, we obtain $1+\sum_{j=1}^{p}\|\zeta_{t_{s_j}}\|^K \lesssim t_k^{3K/2\beta}$, a.s., which together with (5.8) leads to

$$\left|Q_{k+1}^{-1}\right| \lesssim (k+2)^p t_k^{\frac{3Kp}{\beta}+(\varsigma+\frac{2}{\beta}+\frac{2}{\gamma})pb_1}\left(\log\left|Q_k^{-1}\right|\right)^{\sum_{i=1}^{p} b_i}\prod_{i=1}^{n}\left(\left|Q_{k+1-i}^{-1}\right|/\left|Q_{k-i}^{-1}\right|\right)^{b_i}.$$

As a consequence, if $\left|Q_{k+1}^{-1}\right| > t_k^{(3pb_1)^{-1}\psi(t_k)}$ for all sufficiently large k, similar to [25, Eq. (31)–(35)], we can derive a contradiction by the definition of $P(x)$ in (3.2). Hence, we immediately deduce that

$$\left|Q_{k+1}^{-1}\right| \le t_k^{(3pb_1)^{-1}\psi(t_k)}, \quad \text{i.o. a.s. on } D'.$$

Similar to (5.8), when k is sufficiently large and satisfies $\left|Q_{k+1}^{-1}\right| \le t_k^{(3pb_1)^{-1}\psi(t_k)}$, for any $t \in (t_k+1, t_{k+1}+1]$,

$$\begin{aligned}
\left|P_t^{-1}\right| \lesssim & \sum_{(s_1,\dots,s_p)\in\mathcal{W}(t-1)}\Big(1+\sum_{j=1}^{p}\|\zeta_{s_j}\|^K\Big)^{2p} \\
& \sum_{(l_1,\dots,l_p)\in\mathcal{H}_p^{(1,\dots,p)}}\prod_{i\in[p],s_i\neq-1}\max_{j\in[m]}\Big(\sigma_{s_i-j}^2 s_i^{\varsigma+\frac{2}{\beta}+\frac{2}{\gamma}}\log\sup_{k\le s_i-j-1}\sigma_k\Big)^{\frac{b_{l_i}+b_i}{2}} \\
& \lesssim t^{3Kp/\beta}\sum_{(s_1,\dots,s_p)\in\mathcal{W}(t-1)}\sum_{(l_1,\dots,l_p)\in\mathcal{H}_p^{(1,\dots,p)}}\prod_{i\in[p],s_i\neq-1}\Big(s_i^{\varsigma+\frac{2}{\beta}+\frac{2}{\gamma}}\mathcal{Q}_k\log\mathcal{Q}_k\Big)^{\frac{b_{l_i}+b_i}{2}} \\
& \lesssim t^{3Kp/\beta}(t+1)^p t^{(\varsigma+\frac{2}{\beta}+\frac{2}{\gamma})pb_1}\left|Q_{k+1}^{-1}\right|^{2pb_1} \\
& \lesssim t^{\frac{3Kp}{\beta}+p+(\varsigma+\frac{2}{\beta}+\frac{2}{\gamma})pb_1}\cdot t_k^{2pb_1(3pb_1)^{-1}\psi(t_k)} = o(t^{\psi(t)}), \quad \text{a.s. on } D'.
\end{aligned}$$

This implies $\left|P_t^{-1}\right| < t^{\psi(t)}$, i.o. a.s. on D', which contradicts to the definition of D_1. ∎

LEMMA 5.4. *Let the parameters in Assumptions A1-A2 and A4 satisfy* (4.3)*, choose* $\psi(t)=\log t$*, then we have* $\sup_{t\ge1}\sigma_t<\infty$*, a.s. on* $D\cap D_1^c$.

Proof. Recall from (4.16) that $D=\lim_{C\to\infty}D(C)$, it suffices to prove that for any $C>1$,

$$\sup_{t\ge1}\sigma_t<\infty, \quad \text{a.s. on } D(C)\cap D_1^c. \tag{5.9}$$

Our core idea is to use induction to prove that for a certain random number t', when $i\ge t'$,

$$\eta_{i-1}(C) > \frac{i-1}{2} \quad \text{and} \quad \sigma_{i-1}^2 < 8mC \quad \text{a.s. on } D(C)\cap D_1^c. \tag{5.10}$$

This induction method will be completed in two steps.

Step 1: Find An Appropriate t'. In fact, we hope t' is sufficiently large and satisfies specific properties. We need to first prove some conclusions to ensure the existence of t', primarily involving the estimation of $\lambda_{\min}[P_{t+1}^{-1}]$, as in (5.11),(5.14).

We utilize the results from [23] as a foundation. For $z=\operatorname{col}(z_j)_{j=1}^p, z_j\in\mathbb{R}^{dm}\times\mathbb{R}^l$. Define a matrix-valued function $\Psi(z)\triangleq\sum_{j=1}^p\phi(z_j)\phi^T(z_j)$. Note that functions $f_1,\dots,f_p$ are linearly independent, by [23, Lemma B.3], there are some $\{\varrho_j\}_{j=1}^p \in$

$\mathbb{R}^{dm} \times \mathbb{R}^l$, and two constants $C_r, C_\lambda > 0$ such that $\min_{z\in\mathcal{D}} \lambda_{\min}[\Psi(z)] > C_\lambda$, where $\mathcal{D} \triangleq \prod_{j=1}^p \mathcal{D}_j$ with $\mathcal{D}_j = \overline{B(\varrho_j, C_r)}$.

Denote $\eta_t(C) \triangleq \sum_{i=m}^t I_{\Omega_{i-m}(C)}$, where

$$\Omega_i(C) \triangleq \bigcap_{j=i}^{i-m+1} \{a_{j-1}\|\rho_{j-1}\|^2 + \sigma_{j-1}^2 + \|\zeta_j\|^2 < 24m^2C\}.$$

By $\min_{z\in\mathcal{D}} \lambda_{\min}[\Psi(z)] > C_\lambda$ and Lemma B.1, for fixed $\epsilon \in (0, \frac{1}{2})$, there is a random number t_1^* such that for any $t > t_1^*$,

$$\lambda_{\min}[P_{t+1}^{-1}] > C_\lambda C_{\mathcal{D}} \eta_t(C) - t^{\frac{1}{2}+\epsilon}, \quad \text{a.s. on } D(C) \cap D_1^c. \tag{5.11}$$

Next, we estimate the growth rate of $\eta_t(C)$. By Assumption A3, Lemma 5.1 and 5.2,

$$\sum_{i=1}^t a_i\|\rho_i\|^2 \lesssim t^\varsigma \log r_t \lesssim t^\varsigma \log \sup_{j\le t-1} \sigma_j + t^\varsigma \log t \lesssim t^\varsigma \log |P_{t-1}^{-1}| + t^\varsigma \log t,$$

there is a random number $C' > 0$ such that

$$\sum_{i=1}^t a_i\|\rho_i\|^2 < C'\Big(t^\varsigma \log \sup_{j\le t-1} \sigma_j + t^\varsigma \log t\Big) \le C'(t^\varsigma \log |P_{t-1}^{-1}| + t^\varsigma \log t) \tag{5.12}$$

for all $t \ge 1$ almost surely. Now, for $j \ge 1$, we define events

$$A_j^1 \triangleq \{a_{j-1}\|\rho_{j-1}\|^2 \ge 8mC\}, \quad A_j^2 \triangleq \{\sigma_{j-1}^2 \ge 8mC\}, \quad A_j^3 \triangleq \{\|\zeta_j\|^2 \ge 8mC\},$$

and let

$$I_{t,C}^k \triangleq \Big\{i : \sum_{j=i}^{i-m+1} I_{A_j^k} > 0, i \in [t]\Big\}, \quad k = 1,2,3, \quad I_{t,C}^4 \triangleq [t] \setminus \cup_{k=1}^3 I_{t,C}^k.$$

If $\left|P_t^{-1}\right| < t^{\psi(t)} = t^{\log t}$ for some sufficiently large t, according to (5.12), we obtain

$$8mC|I_{t,C}^1| \le m\sum_{i=1}^t a_i\|\rho_i\|^2 < 2mC't^\varsigma \log^2 t, \tag{5.13}$$

and

$$(8mC)^{|I_{t,C}^2|} \le \left|P_t^{-1}\right|^m < t^{m\log t}.$$

So, for $C'' \triangleq \frac{C'}{4C} + m\log^{-1}(8mC)$, we have $|I_{t,C}^1| + |I_{t,C}^2| \le C''t^\varsigma \log^2 t$, which implies

$$\eta_t(C) \ge |I_{t,C}^4| - m \ge t - m - C''t^\varsigma \log^2 t - |I_{t,C}^3|.$$

However, for sufficiently large t, it yields that $8mC|I_{t,C}^3| \le m\sum_{i=1}^t \|\zeta_i\|^2 < mCt$, a.s. on $D(C)$, then

$$\eta_t(C) \ge |I_{t,C}^4| - m > \frac{7}{8}t - m - C''t^\varsigma \log^2 t, \quad \text{i.o. a.s. on } D(C) \cap D_1^c. \tag{5.14}$$

Now, denote $C''' \triangleq L^2 m^{2b_1+2} 2^{3b_1+6} C_y^{b_1} C^{b_1} (C_\lambda C_{\mathcal{D}})^{-1}$, by (4.3), there is a random number t_2^* such that $\forall t > t_2^*$,

$$t > (C''' + 16C'')t^{\max\{2K/\beta + b_1\max\{\varsigma, 2/\beta, 2/\gamma\}, 1/2+\epsilon\}} \log^{K+2b_1} t + (8mC)^2. \tag{5.15}$$

According to (5.14), on $D(C) \cap D_1^c$, we can find some sufficiently large t that satisfies

$$|P_t^{-1}| < t^{\log t}, \quad \eta_t(C) > \frac{7}{8}t - m - C''t^\varsigma \log^2 t > t_1^* + t_2^* \quad \text{and} \quad |I_t^4(C)| > \frac{7}{8}t - C''t^\varsigma \log^2 t,$$

and then there is a $t' \in [\frac{7}{8}t - C''t^\varsigma \log^2 t, t]$ such that $t' \in I_{t,C}^4$. This is the t' we are looking for.

Step 2: Induction Process. Let us proceed with the induction of (5.10). Firstly, (5.14) implies

$$\eta_{t'-1}(C) \geq \eta_t(C) - C''t^{\varsigma}\log^2 t - \frac{1}{8}t \geq \frac{6}{8}t - m - 2C''t^{\varsigma}\log^2 t > \frac{t}{2} > \frac{t'-1}{2}. \tag{5.16}$$

Combining (5.16) and $t' \in I^4_{t,C}$, we obtain (5.10) for $i = t'$. Next, suppose that for some $k \geq t'$, (5.10) holds for all $i \in [t', k]$. This means $|I^2_{k,C}| = |I^2_{t',C}|$ and $\sigma^2_{s-1} < 8mC$ for any $s \in [t'-m+1, k]$. Thus, by (5.6), $\forall s \in [k-m+1, k]$,

$$\begin{aligned}
\|y_s\|^2 &\leq C_y\Big(\sigma^2_{s-1}s^{\max\{\varsigma,2/\beta,2/\gamma\}}\Big(\log\sup_{j\leq s-2}\sigma_j + \log s\Big)\Big)\\
&\leq 8C_y mCs^{\max\{\varsigma,2/\beta,2/\gamma\}}\Big(\log\sup_{j\leq t'-1}\sigma_j + \log\sup_{t'\leq j\leq s-2}\sigma_j + \log s\Big)\\
&\leq 4C_y mCs^{\max\{\varsigma,2/\beta,2/\gamma\}}\left(\log|P^{-1}_{t'}| + \log 8mC + 2\log k\right)\\
&\leq 4C_y mCk^{\max\{\varsigma,2/\beta,2/\gamma\}}\left(\log^2 t + \log 8mC + 2\log k\right). \qquad (5.17)
\end{aligned}$$

Notice that $\frac{t}{2} < \frac{7}{8}t - C''t^{\varsigma}\log^2 t \leq t' \leq k$, then (5.17) yields

$$\begin{aligned}
\|\varphi_k\|^2 = \sum_{s=k-m+1}^{k}\|y_s\|^2 &\leq 4C_y m^2Ck^{\max\{\varsigma,2/\beta,2/\gamma\}}\left(\log^2 2k + \log 8mC + 2\log k\right)\\
&< 8C_y m^2Ck^{\max\{\varsigma,2/\beta,2/\gamma\}}\log^2 2k, \quad \text{a.s. on } D(C)\cap D_1^c,
\end{aligned}$$

which gives

$$\begin{aligned}
\sigma_k^2 = 1 + \phi_k^T P_k\phi_k \leq 1 + \frac{\|\phi(\varphi_k,\zeta_k)\|^2}{\lambda_{\min}[P_k^{-1}]} &\overset{(5.11)}{\leq} 1 + \frac{L^2m(\|\zeta_k\|^K+1)^2(\|\varphi_k\|^{b_1}+1)^2}{C_\lambda C_{\mathcal{D}}\eta_{k-1}(C) - (k-1)^{\frac{1}{2}+\epsilon}}\\
&\overset{(5.15)}{\leq} 1 + \frac{4L^2m(2(k^{2K/\beta}\log^K k)+2)}{C_\lambda C_D(k-1)}\cdot(2(8C_y m^2Ck^{\max\{\varsigma,2/\beta,2/\gamma\}}\log^2 2k)^{b_1}+2)\\
&\overset{(5.15)}{<} 1 + \frac{C'''k^{2K/\beta+b_1\max\{\varsigma,2/\beta,2/\gamma\}}\log^{K+2b_1}k}{k-1} \overset{(5.15)}{<} 8mC, \quad \text{a.s. on } D(C)\cap D_1^c.
\end{aligned}$$

Moreover, according to (5.12), $|P_t^{-1}| < t^{\log t}$ and $2k \geq 2t' > t$, it turns out that

$$\begin{aligned}
&\sum_{i=1}^{k}a_i\|\rho_i\|^2 < C'k^{\varsigma}\Big(\log\sup_{j\leq k-1}\sigma_j + \log k\Big) \leq C'k^{\varsigma}\Big(\log\sup_{j\leq t'-1}\sigma_j + \log\sup_{t'\leq j\leq k-1}\sigma_j + \log k\Big)\\
&\leq C'k^{\varsigma}\Big(\frac{1}{2}\log|P_t^{-1}| + \frac{1}{2}\log 8mC + \log k\Big) < C'k^{\varsigma}\Big(\frac{1}{2}\log^2 2k + \frac{1}{2}\log 8mC + \log k\Big)\\
&< 2C'k^{\varsigma}\log^2 2k.
\end{aligned}$$

Similar to (5.13), we derive $|I^1_{k,C}| < C''k^{\varsigma}\log 2k\log k$, which together with

$$|I^2_{k,C}| = |I^2_{t',C}| \leq |I^2_{t,C}| \leq C''t^{\varsigma}\log^2 t \leq C''(2k)^{\varsigma}\log^2 2k$$

implies

$$\begin{aligned}
\eta_k(C) &\geq k - m - |I^1_{k,C}| - |I^2_{k,C}| - |I^3_{k,C}|\\
&\geq k - m - C''k^{\varsigma}\log 2k\log k - C''(2k)^{\varsigma}\log^2 2k - \frac{k}{8} > \frac{k}{2}.
\end{aligned}$$

Hence (5.10) holds for $i = k+1$, and the induction is completed. So (5.9) holds as desired. ∎

6. Simulations. Consider a forced stochastic Van der Pol oscillator with an unknown parameter described by the following equations:

$$\left\{ \begin{array}{l} \dot{p}_1(t) = p_2(t) \\ \dot{p}_2(t) = -p_1(t) + \theta_1 \left(1 - p_1^2(t)\right) p_2(t) + u(t) + w(t) \end{array} \right. , \tag{6.1}$$

where θ_1 represents the unknown parameter, and w denotes a white noise process. By discretizing equation (6.1) using Euler's method, we obtain the following discrete-time nonlinear stochastic system:

$$y_{t+1} = 2y_t - (1 + \Delta^2)y_{t-1} + \theta_1 \Delta(1 - y_t^2)(y_t - y_{t-1}) + \Delta^2 u_t + \Delta^2 w_{t+1}. \tag{6.2}$$

In (6.2), Δ represents the sampling time interval, and we define $u_t = u(t\Delta)$, $w_t = w(t\Delta)$, and $y_t = p_2(t\Delta)$. In this section, we simulate system (6.2) from $t = 0$ to $t = 2000$ using a sampling time interval of $\Delta = 0.1$. The initial output is set as $y_0 = 1$, and $\theta_1 = 0.1$. The noise sequence $\{\Delta^2 w_t\}_{t=1}^{2000}$ follows an i.i.d. distribution of $N(0, 1)$.

To generate the exosystem (2.2), we employ the additive autoregressive (AAR) model [8, Example 8.13] with $\zeta_t = (Y_t, Y_{t-1})^T$ and $g(\theta_2, \zeta_t) = (g(\theta_2, \zeta_t), Y_t)^T$. Here $g^*(\theta_2, \zeta_t)$ is defined as:

$$g^*(\theta_2, \zeta_t) = \frac{\theta_2(1)Y_{t-1}}{1 + 0.8Y_{t-1}^2} + \frac{\exp\{\theta_2(2)(Y_{t-2} - 2)\}}{1 + \exp\{\theta_2(2)(Y_{t-2} - 2)\}}$$

with $\theta_2 = (\theta_2(1), \theta_2(2))^T = (4, 3)^T$. For the simulations, we let $Y_0 = Y_{-1} = 1$, $\{v_t\}_{t=1}^{2000}$ be a sequence of i.i.d. standard normal random variables, independent of $\{w_t\}_{t=1}^{2000}$.

Now, let $F = (0, 1)$ in (2.1), which corresponds to the control goal of driving the output y_t in (6.2) to the reference trajectory Y_t. Based on the control strategy given by (4.1) and (4.10), we can design the controller as follows:

$$\Delta^2 u_t = -2y_t + (1 + \Delta^2)y_{t-1} - \bar{\theta}_{1,t}^T \Delta(1 - y_t^2)(y_t - y_{t-1}) + g\left(\bar{\theta}_{2,t}, \zeta_t\right).$$

Here, ν and Θ in (4.10) are set to 1 and $[0, 5] \times [0, 5]$, respectively.

It is straightforward to verify that the models (2.1)-(2.2) in the above setup satisfy the assumptions in Corollary 4.10. According to Corollary 4.10, we conclude that the upper limit of $\frac{1}{t}\sum_{i=1}^t e_i^2 = \frac{1}{t}\sum_{i=1}^t (y_i - Y_i)^2$ is equal to 2, aligning with the simulation results shown in FIG. 1. Furthermore, FIG. 2 displays the estimation errors for parameters θ_1 and θ_2. Despite the relatively less stable convergence of $\bar{\theta}_{2,t}$ compared to $\bar{\theta}_{1,t}$, both estimations tend to zero, aligning with our theoretical results.

7. Conclusions. This paper investigates the optimal adaptive output regulation problem for a class of MIMO discrete-time nonlinear stochastic systems. We show the solvability of the optimal output regulation problem under mild conditions. This study serves as a preliminary step for further exploring output regulation problems for nonlinear stochastic systems in more general settings, e.g., there may be nonlinear coupling among the control input, the plant, and the regulated output. Moreover, this study develops new parameter identification algorithms, stabilizability theorems, and related analytical methods, laying the foundation for future studies of a broader range of output regulation and stabilization problems.

Appendix A. Proof of Theorem 4.2. First, we provide some definitions. For any $a > 0$, $B(a) \triangleq \{(x_1, \ldots, x_q)^T \in \Theta : \max_{i \in [1,q]} |x_i - \vartheta_i| \geq a\}$, Define

$$S_t(x) \triangleq \sum_{k=1}^t \|y_k - g(x, \psi_{k-1})\|^2, \quad D_t(x) \triangleq \sum_{k=1}^t \|d_k(x)\|^2,$$

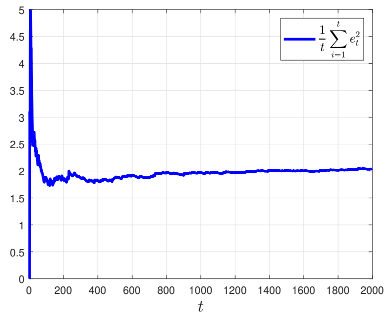

FIG. 1. *Simulation results on the time evolution of tracking performance.*

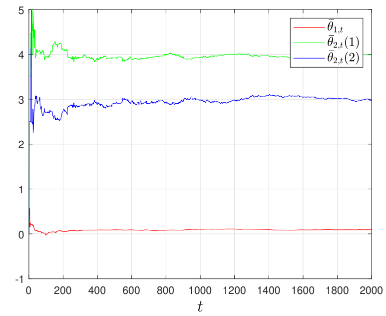

FIG. 2. *Simulation results on the time evolution of parameter estimations.*

where $d_k(x) \triangleq g(x, \psi_{k-1}) - g(\vartheta, \psi_{k-1})$. Denote $d_k(x) = (d_{k,1}(x), \ldots, d_{k,l}(x))^T$ and $\varepsilon_k = (\varepsilon_{k,1}, \ldots, \varepsilon_{k,l})^T$.

It is straightforward to observe that Theorem 4.2 is a corollary of the following lemma.

LEMMA A.1. *Let $\{h_t, t \geq 0\}$ be a positive process adapted to filtration $\{\mathcal{F}_t, t \geq 0\}$, and satisfy $h_t < \frac{1}{2\sqrt{q}} dist(\vartheta, \partial\Theta)$, assume that $\lim_{t\to\infty} \overline{r}_t = \infty$, and for some $\epsilon > 0$,*

$$\text{(A.1)} \qquad \overline{r}_{t-1}^{\frac{1}{2}} \log^{\frac{1}{2}+\epsilon} \overline{r}_{t-1} \lesssim \inf_{x \in B(h_{t-1})} \frac{D_t(x)}{\|x - \vartheta\|}, \quad a.s..$$

Then $\|\hat{\vartheta}_t - \vartheta\| \lesssim h_{t-1}$, a.s..

Proof. Through an analysis similar to that of [38, Lemma 1] , it is easy to see $\|\hat{\vartheta}_t - \vartheta\| \lesssim h_{t-1}$, a.s. if

$$\text{(A.2)} \qquad \liminf_{t\to\infty} \inf_{x\in B(h_{t-1})} (S_t(x) - S_t(\vartheta)) > 0, \quad \text{a.s.},$$

So it suffices to prove (A.2). Note that $S_t(x) - S_t(\vartheta) = D_t(x) - 2\sum_{k=1}^t d_k^T(x)\varepsilon_k$, since (A.1) shows that $\liminf_{t\to\infty} \inf_{x\in B(h_{t-1})} D_t(x) = \infty$, then (A.2) will follow provided by

$$\text{(A.3)} \qquad \limsup_{t\to\infty} \sup_{x\in B(h_{t-1})} \frac{|\sum_{k=1}^t d_{k,i}(x)\varepsilon_{k,i}|}{D_t(x)} = 0, \quad \text{a.s.}, \quad i \in [l].$$

Now, in order to simplify the proof, we only prove (A.3) for $i = 1$. By the remainder of multivariate Taylor polynomials, $d_{k,1}(x) = e_{k,1}^T(x) \cdot (x - \vartheta)$, where

$$\text{(A.4)} \qquad e_{k,1}(x) \triangleq \int_0^1 (1-s)\frac{\partial^2 g_1}{\partial \vartheta^2}(\vartheta + s(x-\vartheta), \psi_{k-1})(x-\vartheta)ds + \frac{\partial g_1}{\partial \vartheta}(\vartheta, \psi_{k-1}).$$

Finally, in view of (A.1) and Lemma A.3, it is easy to see the assertion holds by inequality

$$\frac{|\sum_{k=1}^t d_{k,1}(x)\varepsilon_{k,1}|}{D_t(x)} \leq \frac{\|\sum_{k=1}^t e_{k,1}(x)\varepsilon_{k,1}\|}{\overline{r}_{t-1}^{\frac{1}{2}} \log^{\frac{1}{2}+\epsilon} \overline{r}_{t-1}} \cdot \frac{\overline{r}_{t-1}^{\frac{1}{2}} \log^{\frac{1}{2}+\epsilon} \overline{r}_{t-1}}{D_t(x)/\|x-\vartheta\|}. \qquad \square$$

LEMMA A.2. *For continuous function $\mu : \Theta \to \mathbb{R}^q$, denote*

$$\begin{cases} c(\mu,\delta) \triangleq \sup_{y,z\in\Theta,\|y-z\|\leqslant\delta} \|\mu(y)-\mu(z)\| \\ C_n(\mu,\delta) \triangleq \sup_{y,z\in\Theta\cap\mathbb{D}_n,\|y-z\|\leqslant\delta} \|\mu(y)-\mu(z)\| \end{cases}, \quad \delta>0,\ n=1,2,\ldots,$$

where $\mathbb{D}_n \triangleq \mathbb{Z}^q/2^n$. For any integer $n \geq 1$, we have

$$c(\mu,2^{-n}) \leq 6q\sum_{r\geqslant n} C_r(\mu,2^{-r}). \tag{A.5}$$

Proof. Actually, for any $y,z\in\Theta\cap\mathbb{D}_n$, there are $j'<\sqrt{q}2^n\|y-z\|+1$ and $\mathbf{z}_1,\ldots,\mathbf{z}_{j'}\in\Theta\cap\mathbb{D}_n$ such that $\mathbf{z}_1=y$, $\mathbf{z}_{j'}=z$, $\|\mathbf{z}_{j+1}-\mathbf{z}_j\|=2^{-n}$, $j\in[j'-1]$. So

$$\begin{aligned} \|\mu(y)-\mu(z)\| &\leq \sum_{i=1}^{j'-1}\|\mu(z_i)-\mu(z_{i+1})\| \leq (j'-1)C_n(\mu,2^{-n}) \\ &\leq \sqrt{q}2^n\|y-z\|C_n(\mu,2^{-n}). \end{aligned} \tag{A.6}$$

Now, for any fixed $y,z\in\Theta$ with $\|y-z\|\leq 2^{-n}$, define $y(r)\triangleq \operatorname{argmin}_{x\in\Theta\cap\mathbb{D}_r}\|x-y\|$, $z(r)\triangleq \operatorname{argmin}_{x\in\Theta\cap\mathbb{D}_r}\|x-z\|$, we have $\|y-y(r)\|\leq\sqrt{q}2^{-r}$, $\|z-z(r)\|\leq\sqrt{q}2^{-r}$, it follows that for any $r\geq 1$,

$$\|y(r)-y(r+1)\|,\|z(r)-z(r+1)\| \leq \sqrt{q}2^{-r}+\sqrt{q}2^{-r-1}=3\sqrt{q}2^{-r-1}. \tag{A.7}$$

Note that $\mathbb{D}_r\subset\mathbb{D}_{r+1}$, by (A.6) and (A.7) we deduce

$$\begin{aligned} &q^{-1/2}(\|\mu(y(r))-\mu(y(r+1))\|+\|\mu(z(r))-\mu(z(r+1))\|) \\ &\leq 2^{r+1}(\|y(r)-y(r+1)\|+\|z(r)-z(r+1)\|)C_{r+1}(\mu,2^{-(r+1)}) \\ &\leq 6\sqrt{q}C_{r+1}(\mu,2^{-(r+1)}), \end{aligned} \tag{A.8}$$

and

$$\begin{aligned} &\|\mu(y(n))-\mu(z(n))\| \leq \sqrt{q}2^n\|y(n)-z(n)\|C_n(\mu,2^{-n}) \\ &\leq \sqrt{q}2^n(\|y(n)-y\|+\|z(n)-z\|+\|y-z\|)C_n(\mu,2^{-n}) \leq 3qC_n(\mu,2^{-n}). \end{aligned} \tag{A.9}$$

Combine the continuity of μ, (A.8) and (A.9), it holds that

$$\begin{aligned} \|\mu(y)-\mu(z)\| \leq \lim_{r\to\infty}\sum_{j=n}^{r}&(\|\mu(y(j))-\mu(y(j+1))\| \\ &+\|\mu(z(j))-\mu(z(j+1))\|)+\|\mu(y(n))-\mu(z(n))\| \\ \leq{}& 6q\sum_{r\geqslant n}C_r(\mu,2^{-r}), \end{aligned}$$

which implies (A.5). ∎

LEMMA A.3. *For $e_{k,1}(x)$ defined in (A.4), we have*

$$\limsup_{t\to\infty}\sup_{x\in\Theta}\|\sum_{k=1}^{t}e_{k,1}(x)\varepsilon_{k,1}\| = o(\overline{r}_{t-1}^{\frac{1}{2}}\log^{\frac{1}{2}+\epsilon}\overline{r}_{t-1}), \quad a.s., \quad \forall\epsilon>0. \tag{A.10}$$

Proof. Define a martingale $M_t(x) \triangleq \sum_{k=1}^t e_{k,1}(x)\varepsilon_{k,1}$ adapted to $\{\mathcal{F}_t\}_{t\geq 1}$ for any $x \in \Theta$. Let $m_t(x) \triangleq M_t(x) \cdot \overline{r}_{t-1}^{-\frac{1}{2}} \log^{-\frac{1}{2}-\epsilon} \overline{r}_{t-1}$, we assert that (A.10) holds if

$$\lim_{\delta\to 0} \limsup_{t\to\infty} c(m_t, \delta) = 0, \quad \text{a.s..} \tag{A.11}$$

For any $\delta \in (0, \frac{1}{2}\text{dist}(\vartheta, \partial\Theta))$, fix this δ. For finite set $\mathbf{L} = \frac{\delta}{\sqrt{q}}\mathbb{Z}^q \cap \Theta$, $\forall z \in \Theta$, there is $y \in \mathbf{L} \cap \Theta$ such that $\|y - z\| \leq \delta$. Now, we can use the finite set $\mathbf{L} \cap \Theta$ to approximate Θ, one has

$$\sup_{x\in\Theta} \|m_t(x)\| \leq \sup_{x\in\mathbf{L}\cap\Theta} \|m_t(x)\| + c(m_t, \delta) \leq \sup_{x\in\mathbf{L}} \|m_t(x)\| + c(m_t, \delta). \tag{A.12}$$

By [5, Theorem 2.8] and noting that $\sum_{k=1}^t \|e_{k,1}(x)\|^2 \lesssim \overline{r}_{t-1}$, we have $\limsup_{t\to\infty} \sup_{x\in\mathbf{L}} \|m_t(x)\| = 0$ a.s., thus by taking the limit superior on both sides of (A.12) with respect to t and letting $\delta \to 0$, (A.10) is true provided (A.11) holds.

Next, we are devoted to proving (A.11). Before do this, we provide some definitions. Firstly, define

$$\overline{r}_t^{(k)} \triangleq \sum_{j=0}^t \left\| D^{k-1} \frac{\partial g}{\partial\vartheta}(x, \psi_j) \right\|^2_{L^\infty(\Theta, \mathbb{R}^{q\times l})}, \quad k = 1, 2, 3, \tag{A.13}$$

then $\overline{r}_t = \overline{r}_t^{(1)} + \overline{r}_t^{(2)} + \overline{r}_t^{(3)}$. Moreover, define events

$$A_k \triangleq \{|\varepsilon_{k,1}| \leq \overline{r}_{k-1}^{\frac{1}{2}} / (\overline{r}_{k-1}^{(3)} - \overline{r}_{k-2}^{(3)} + \overline{r}_{k-1}^{(2)} - \overline{r}_{k-2}^{(2)})^{1/2}\},$$

and

$$\overline{\varepsilon}_{k,1} \triangleq \varepsilon_{k,1} I_{A_k} - E[\varepsilon_{k,1} I_{A_k} | \mathcal{F}_{k-1}], \quad \underline{\varepsilon}_{k,1} \triangleq \varepsilon_{k,1} - \overline{\varepsilon}_{k,1}, \quad k \geq 1.$$

Accordingly, denote $\overline{M}_t(x) \triangleq \sum_{k=1}^t e_{k,1}(x)\overline{\varepsilon}_{k,1}$, and let

$$\overline{m}_t(x) \triangleq \overline{M}_t(x) \cdot \overline{r}_{t-1}^{-\frac{1}{2}} \log^{-\frac{1}{2}-\epsilon} \overline{r}_{t-1}, \quad \underline{m}_t(x) \triangleq m_t(x) - \overline{m}_t(x).$$

Next, we prove

$$\lim_{n\to\infty} \limsup_{t\to\infty} c(\overline{m}_t, 2^{-n}) = 0, \quad \text{a.s.} \tag{A.14}$$

by estimating $C_r(\overline{m}_t, 2^{-r})$. Fix some $\lambda > 1$, define sequence $t_j \triangleq \inf\{t; t \geq 1,\ \overline{r}_t \geq \lambda^j\}$. For any fixed $y, z \in \Theta$, define $N_{t,j}(y, z) \triangleq \overline{M}_{\min\{t,t_j\}}(y) - \overline{M}_{\min\{t,t_j\}}(z)$, then for each j, $\{N_{t,j}(y, z)\}_{t\geq 1}$ is family of martingales, adapted to filtration $\{\mathcal{F}_t\}_{t\geq 1}$. We have the following inequality about the increments of $N_{t,j}(y, z)$,

$$\begin{aligned}
&\|\Delta N_{t+1,j}(y, z)\| = \|N_{t+1,j}(y, z) - N_{t,j}(y, z)\| \\
&= I_{\{\overline{r}_t < \lambda^j\}} \|(\overline{M}_{t+1}(y) - \overline{M}_t(y)) - (\overline{M}_{t+1}(z) - \overline{M}_t(z))\| \\
&= I_{\{\overline{r}_t < \lambda^j\}} \left\| \int_0^1 (1-s) K_s(y) ds - \int_0^1 (1-s) K_s(z) ds \right\| \cdot |\overline{\varepsilon}_{t+1,1}|, \qquad \text{(A.15)}
\end{aligned}$$

where the map $K_s(\cdot) : \mathbb{R}^q \to \mathbb{R}^q$ satisfying $K_s(x) = \frac{\partial^2 g_1}{\partial\vartheta^2}(\vartheta + s(x - \vartheta), \psi_t)(x - \vartheta)$, $\forall s \in [0, 1]$. By generalization of mean value theorem,

$$\|K_s(y) - K_s(z)\| \leq \sup_{0<\eta<1} \|J_{K_s}(z + \eta(y - z))\| \cdot \|y - z\|,$$

where J_{K_s} is the Jacobian matrix of K_s. Observe that

$$J_{K_s}(x) = s(x-\vartheta)^T \frac{\partial}{\partial\vartheta}\left(\frac{\partial^2 g_1}{\partial\vartheta^2}\right)(\vartheta + s(x-\vartheta), \psi_t) + \frac{\partial^2 g_1}{\partial\vartheta^2}(\vartheta + s(x-\vartheta), \psi_t),$$

then by applying basic inequality scaling and noting (A.13), one has

$$\sup_{0<\eta<1} \|J_{K_s}(z+\eta(y-z))\|^2 \le \frac{(1+s)^2}{2}\Upsilon^2(\overline{r}_t^{(3)} - \overline{r}_{t-1}^{(3)}) + \overline{r}_t^{(2)} - \overline{r}_{t-1}^{(2)}. \tag{A.16}$$

where $\Upsilon \triangleq 2+2\mathrm{dist}(\vartheta, \partial\Theta)$. Thus, in view of (A.15), (A.16), and $|\overline{\varepsilon}_{t+1,1}| \le 2\overline{r}_t^{\frac{1}{2}}/(\overline{r}_t^{(3)} - \overline{r}_{t-1}^{(3)} + \overline{r}_t^{(2)} - \overline{r}_{t-1}^{(2)})^{1/2}$, the martingale $\{N_{t,j}(y,z)\}_{t\ge 1}$ has bounded increments

$$\|\Delta N_{t+1,j}(y,z)\| \le I_{\{\overline{r}_{\tau+t}<\lambda^j\}}\Upsilon \overline{r}_t^{\frac{1}{2}}\|y-z\| \le \lambda^{\frac{j}{2}}\Upsilon\|y-z\| \triangleq Q_{1,j}(y,z). \tag{A.17}$$

Further, denote $\sigma^2 \triangleq 1+\sigma_\gamma$, then $E[\overline{\varepsilon}_{k,1}^2|\mathcal{F}_{k-1}] \le 1+\sup_{k\ge 1} E[\|\varepsilon_k\|^\gamma|\mathcal{F}_{k-1}] \le \sigma^2,\ k \ge 1$, a.s.. Then, as long as $t_j > t$, by (A.15),(A.16) and simple integral calculations, it follows that

$$\begin{aligned}
&\sum_{k=1}^{t+1} E[\|\Delta N_{k,j}(y,z)\|^2|\mathcal{F}_{\tau+k-1}] \le \sum_{k=1}^{t+1} E[\|\Delta(\overline{M}_k(y) - \overline{M}_k(z))\|^2|\mathcal{F}_{k-1}] \\
&\le \sigma^2\Upsilon^2\|y-z\|^2 \sum_{k=1}^{t+1}\left(\overline{r}_{k-1}^{(3)} - \overline{r}_{k-2}^{(3)} + \overline{r}_{k-1}^{(2)} - \overline{r}_{k-2}^{(2)}\right) \\
&= \sigma^2\Upsilon^2\|y-z\|^2\left(\overline{r}_t^{(3)} + \overline{r}_t^{(2)}\right) \le \sigma^2\Upsilon^2\|y-z\|^2\lambda^j \triangleq Q_{2,j}(y,z).
\end{aligned}$$

Denote events

$$\begin{cases} \mathcal{A}_{t,j} \triangleq \{\|N_{t+1,j}(y,z)\| \ge Q_{1,j}^{-1}(y,z)Q_{2,j}(y,z) + L_j(y,z)\} \\ \mathcal{B}_{t,j} \triangleq \{\sum_{k=1}^{t+1} E[\|\Delta N_{k,j}(y,z)\|^2|\mathcal{F}_{k-1}] \le Q_{2,j}(y,z)\} \end{cases},$$

where $L_j(y,z) \triangleq \|y-z\|^{\frac{1}{2}}\lambda^{\frac{j}{2}}\log\log\lambda^j$. By [7, Theorem 1.2A],

$$\begin{aligned}
P(\cup_{t\ge 0}(\mathcal{A}_{t,j}\cap\mathcal{B}_{t,j})) &\le 2q\cdot\exp\left\{-\frac{(Q_{1,j}^{-1}(y,z)Q_{2,j}(y,z) + L_j(y,z))^2}{2(2Q_{2,j}(y,z) + L_j(y,z)Q_{1,j}(y,z))}\right\} \\
&\le 2q\cdot\exp\left\{-\frac{L_j(y,z)}{2Q_{1,j}(y,z)}\right\}.
\end{aligned} \tag{A.18}$$

Note that $\cup_{k=0}^{t_j-1}\mathcal{A}_{k,j} \subset \cup_{k=0}^{t_j-1}(\mathcal{A}_{k,j}\cap\mathcal{B}_{k,j}) \subset \cup_{t\ge 0}(\mathcal{A}_{t,j}\cap\mathcal{B}_{t,j})$, which together with (A.18) leads to

$$\begin{aligned}
&P\Big(\sup_{k\in(t_{j-1},t_j]}\|\overline{m}_k(y) - \overline{m}_k(z)\| \ge \frac{Q_{1,j}^{-1}(y,z)Q_{2,j}(y,z) + L_j(y,z)}{\lambda^{(j-1)/2}((j-1)\log\lambda)^{\frac{1}{2}+\epsilon}}\Big) \\
&\le P\Big(\sup_{k\in(t_{j-1},t_j]}\|\overline{M}_k(y) - \overline{M}_k(z)\| \ge Q_{1,j}^{-1}(y,z)Q_{2,j}(y,z) + L_j(x,y)\Big) \\
&\le P\Big(\sup_{k\in[1,t_j]}\|N_{k,j}(y,z)\| \ge Q_{1,j}^{-1}(y,z)Q_{2,j}(y,z) + L_j(y,z)\Big) \\
&= P\Big(\cup_{k=0}^{t_j-1}\mathcal{A}_j\Big) \le P(\cup_{t\ge 0}(\mathcal{A}_{t,j}\cap\mathcal{B}_{t,j})) \le 2q\cdot\exp\left\{-\frac{L_j(y,z)}{2Q_{1,j}(y,z)}\right\}.
\end{aligned} \tag{A.19}$$

On the other hand,

$$\begin{aligned}
&\sup_{\substack{y,z\in\Theta\cap\mathbb{D}_n,\\ \|y-z\|\leqslant 2^{-n}}} \frac{\|\overline{m}_k(y)-\overline{m}_k(z)\|}{Q_{1,j}^{-1}(y,z)Q_{2,j}(y,z)+L_j(y,z)} \\
&\geq \sup_{\substack{y,z\in\Theta\cap\mathbb{D}_n,\\ \|y-z\|\leqslant 2^{-n}}} \frac{\|\overline{m}_k(y)-\overline{m}_k(z)\|}{\sigma^2\Upsilon\lambda^{\frac{j}{2}}\|y-z\|+\|y-z\|^{\frac{1}{2}}\lambda^{\frac{j}{2}}\log\log\lambda^j} \\
&\geq C_n(\overline{m}_k,2^{-n})\Big(\sigma^2\Upsilon\lambda^{\frac{j}{2}}2^{-n}+2^{-\frac{n}{2}}\lambda^{\frac{j}{2}}\log\log\lambda^j\Big)^{-1}.
\end{aligned}\tag{A.20}$$

Observe that for some constant $C>0$ depends on Θ and q, there are at most $C2^{nq}$ different (y,z) with $\|y-z\|\leq 2^{-n}$ in $\Theta\cap\mathbb{D}_n$, as a consequence of (A.19) and (A.20),

$$\begin{aligned}
&P\Big(\sup_{k\in(t_{j-1},t_j]} C_n(\overline{m}_k,2^{-n})\geq \frac{\sigma^2\Upsilon\lambda^{\frac{j}{2}}2^{-n}+2^{-\frac{n}{2}}\lambda^{\frac{j}{2}}\log\log\lambda^j}{\lambda^{\frac{j-1}{2}}((j-1)\log\lambda)^{\frac{1}{2}+\epsilon}}\Big) \\
&\leq \sum_{y,z\in\Theta\cap\mathbb{D}_n,\|y-z\|\leqslant 2^{-n}} 2q\cdot\exp\Big\{-\frac{L_j(y,z)}{2Q_{1,j}(y,z)}\Big\} \\
&\leq 2Cq\cdot 2^{nq}\exp\Big\{-(2\Upsilon)^{-1}2^{\frac{n}{2}}\log\log\lambda^j\Big\}\lesssim 2^{nq}(j\log\lambda)^{-\frac{1}{2\Upsilon}2^{\frac{n}{2}}}.
\end{aligned}\tag{A.21}$$

According to (A.21) and

$$\frac{\sigma^2\Upsilon\lambda^{\frac{j}{2}}2^{-n}+2^{-\frac{n}{2}}\lambda^{\frac{j}{2}}\log\log\lambda^j}{\lambda^{\frac{j-1}{2}}((j-1)\log\lambda)^{\frac{1}{2}+\epsilon}}=\frac{\sigma^2\Upsilon 2^{-n}+2^{-\frac{n}{2}}\log\log\lambda^j}{\lambda^{-\frac{1}{2}}((j-1)\log\lambda)^{\frac{1}{2}+\epsilon}}\lesssim 2^{-\frac{n}{2}},$$

there is some constant $C'>0$ such that

$$P\Big(\sup_{k\in(t_{j-1},t_j]} C_n(\overline{m}_k,2^{-n})\geq C'2^{-\frac{n}{2}}\Big)\lesssim 2^{nq}(j\log\lambda)^{-\frac{1}{2\Upsilon}2^{\frac{n}{2}}}.\tag{A.22}$$

Now, define events

$$\mathcal{B}_j^n\triangleq\bigcup_{r\geq n}\Big\{\sup_{k\in(t_{j-1},t_j]} C_r(\overline{m}_k,2^{-r})\geq C'2^{-\frac{r}{2}}\Big\},\quad \forall n,j\geq 1.$$

There is j^* such that $\forall j\geq j^*$, $2^q(j\log\lambda)^{-\frac{1}{8\Upsilon}}<1$. Then, by (A.22), $\forall j\geq j^*$,

$$P(\mathcal{B}_j^n)\lesssim\sum_{r=n}^{\infty}\Big((j\log\lambda)^{-\frac{1}{8\Upsilon}}\Big)^{2^{\frac{r}{2}+2}-r}\lesssim\sum_{r=n}^{\infty}(j\log\lambda)^{-\frac{r}{8\Upsilon}}\lesssim(j\log\lambda)^{-\frac{n}{8\Upsilon}}.$$

This means that for $n>8\Upsilon$, $\sum_{j=1}^{\infty}P(\mathcal{B}_j^n)<\infty$. Thus, by *Borel-Cantelli-Lemma*, $P(\limsup_{j\to\infty}\mathcal{B}_j^n)=0$, which implies that for any $r>8\Upsilon$ and sufficiently large j,

$$\sup_{k\geq t_{j-1}} C_r(\overline{m}_k,2^{-r})<C'2^{-\frac{r}{2}},\quad \text{a.s..}\tag{A.23}$$

So, for any $n>8\Upsilon$, by combining (A.5) and (A.23),

$$\limsup_{t\to\infty} c(\overline{m}_t,2^{-n})\lesssim\sum_{r\geqslant n}\limsup_{t\to\infty} C_r(\overline{m}_t,2^{-r})\lesssim 2^{-\frac{n}{2}},\quad \text{a.s.,}$$

which confirms (A.14).

Now, to prove (A.11), it remains to show

$$\lim_{\delta\to 0}\limsup_{t\to\infty} c(\underline{m}_t,\delta)=0,\quad \text{a.s..}\tag{A.24}$$

For any $y, z \in \Theta$, similar to (A.17), we have

$$\|e_{k,1}(y) - e_{k,1}(z)\| \lesssim (\bar{r}^{(3)}_{k-1} - \bar{r}^{(3)}_{k-2} + \bar{r}^{(2)}_{k-1} - \bar{r}^{(2)}_{k-2})^{1/2}\|y - z\| \lesssim (\bar{r}_{k-1} - \bar{r}_{k-2})^{1/2}\|y - z\|,$$

then

$$\|\underline{m}_t(y) - \underline{m}_t(z)\| \leq \bar{r}_{t-1}^{-\frac{1}{2}} \log^{-\frac{1}{2}-\epsilon} \bar{r}_{t-1} \sum_{k=1}^{t} \|e_{k,1}(y) - e_{k,1}(z)\| \cdot |\underline{\varepsilon}_{k,1}|$$

$$\lesssim \|y - z\| \cdot \frac{\sum_{k=1}^{t} (\bar{r}_{k-1} - \bar{r}_{k-2})^{1/2} \cdot |\underline{\varepsilon}_{k,1}|}{\bar{r}_{t-1}^{\frac{1}{2}} \log^{\frac{1}{2}+\epsilon} \bar{r}_{t-1}} \triangleq \|y - z\| \cdot \chi_t. \tag{A.25}$$

Define

$$\chi_t^* \triangleq \bar{r}_{t-1}^{-\frac{1}{2}} \log^{-\frac{1}{2}-\epsilon} \bar{r}_{t-1} \sum_{k=1}^{t} (\bar{r}_{k-1} - \bar{r}_{k-2})^{1/2} E[|\underline{\varepsilon}_{k,1}||\mathcal{F}_{k-1}],$$

by $\sup_{k\geq 1} E[\mathrm{var}(|\underline{\varepsilon}_{k,1}|)|\mathcal{F}_{k-1}] < \infty$, we deduce

$$\sum_{k=1}^{t} (\bar{r}_{k-1} - \bar{r}_{k-2})^{1/2} (|\underline{\varepsilon}_{k,1}| - E[|\underline{\varepsilon}_{k,1}||\mathcal{F}_{k-1}]) \lesssim \bar{r}_{t-1}^{\frac{1}{2}} \log^{\frac{1}{2}+\epsilon} \bar{r}_{t-1}, \quad \text{a.s.,}$$

then $\chi_t - \chi_t^*$ converges almost surely. Moreover, observe that

$$E[|\underline{\varepsilon}_{k,1}||\mathcal{F}_{k-1}] \leq (\bar{r}_{k-1}^{-\frac{1}{2}} (\bar{r}_{k-1} - \bar{r}_{k-2})^{\frac{1}{2}})^{\gamma-1} E[|\varepsilon_{k,1}|^{\gamma}|\mathcal{F}_{k-1}],$$

then

$$\chi_t^* \leq \sup_{k\geq 1} E[|\varepsilon_{k,1}|^{\gamma}|\mathcal{F}_{k-1}] \sum_{k=1}^{t} \frac{(\bar{r}_{k-1} - \bar{r}_{k-2})^{\frac{\gamma}{2}}}{\bar{r}_{k-1}^{\frac{1}{2}(\gamma-1)}} \bar{r}_{t-1}^{-\frac{1}{2}} \log^{-\frac{1}{2}-\epsilon} \bar{r}_{t-1}$$

$$\lesssim \bar{r}_{t-1}^{-\frac{1}{2}} \log^{-\frac{1}{2}-\epsilon} \bar{r}_{t-1} \sum_{k=1}^{t} \frac{\bar{r}_{k-1} - \bar{r}_{k-2}}{\bar{r}_{k-1}^{\frac{1}{2}(\gamma-1)}} \sup_{k\in[t]} (\bar{r}_{k-1} - \bar{r}_{k-2})^{\frac{\gamma-2}{2}}$$

$$\lesssim \sum_{k=1}^{t} \int_{\bar{r}_{k-2}}^{\bar{r}_{k-1}} \frac{1}{x^{\frac{1}{2}(\gamma-1)}} dx \cdot \bar{r}_{t-1}^{\frac{\gamma}{2}-\frac{3}{2}} \log^{-\frac{1}{2}-\epsilon} \bar{r}_{t-1} \lesssim \frac{\log^{-\frac{1}{2}-\epsilon} \bar{r}_{t-1}}{\gamma - 3} < \infty.$$

This implies $\lim_{t\to\infty} \chi_t = \chi_\infty$ almost surely, which together with (A.25) gives $\limsup_{t\to\infty} c(\underline{m}_t, \delta) \leq \delta \cdot \limsup_{t\to\infty} \chi_t = \delta \cdot \chi_\infty$, a.s., then (A.24) follows. ∎

Appendix B. A Technical Lemma.

Lemma B.1. *Under the conditions of Lemma 5.4, for all sufficiently large t,*

$$\min_{j\in[p]} \sum_{i=m}^{t} I_{\{col(\varphi_i,\zeta_i)\in\mathcal{D}_j\}} \geq C_{\mathcal{D}} \sum_{i=m}^{t} I_{\Omega_{i-m}(C)} - o(t^{\frac{1}{2}+\epsilon}), \quad a.s., \quad \forall \epsilon > 0,$$

for some constant $C_{\mathcal{D}} > 0$.

Proof. Recall the definitions in the proof of Lemma 5.4. For any fixed $j \in [p]$, there exist corresponding hypercubes $\{D_{j,k}\}_{k\in[m+1]}$ such that $\times_{k=1}^{m+1} \overline{D_{j,k}} \subset \mathcal{D}_j$, where $D_{j,k} \subset \mathbb{R}^d$ for $k \in [m]$, and $D_{j,m+1} \subset \mathbb{R}^l$. For $k \in [m]$, define events

$$\Omega_{i,k} \triangleq \{\mathrm{col}(y_{i-m+k}, \zeta_{i-m+k}) \in \overline{D_{j,m-k+1}} \times \overline{D_{j,m+1}}\}$$

and

$$\overline{\Omega}_{i,k} \triangleq \{a_{i-m+k}\|\rho_{i-m+k}\|^2 + \sigma^2_{i-m+k} + \|\zeta_{i-m+k+1}\|^2 < 24m^2C\}.$$

Subsequently, we introduce

$$\mathcal{I}_{i,0} = \cap_{s=-m+1}^{0}\overline{\Omega}_{i,s} \quad \text{and} \quad \mathcal{I}_{i,k} = \Omega_{i,k} \cap (\cap_{s=1}^{k-1}(\Omega_{i,s} \cap \overline{\Omega}_{i,s})) \cap \mathcal{I}_{i,0}, \quad k \in [m].$$

Now, for a new sequence of σ-algebra $\mathcal{F}'_t \triangleq \sigma\left\{y_i, \zeta_i,\ 0 \leqslant i \leqslant t\right\},\ t = 0, 1, \ldots,$ one has

$$\begin{aligned}
&\sum_{i=m}^{t}\left(I_{\mathcal{I}_{i,k}} - P(\Omega_{i,k}|\mathcal{F}'_{i-m+k-1})\prod_{s=1}^{k-1} I_{\Omega_{i,s}} \prod_{s=-m+1}^{k-1} I_{\overline{\Omega}_{i,s}}\right) \\
\text{(B.1)} \qquad &\leq \sum_{i=m}^{t}\left(I_{\Omega_{i,k}} - P(\Omega_{i,k}|\mathcal{F}'_{i-m+k-1})\right) = o(t^{\frac{1}{2}+\epsilon}), \quad \text{a.s.}, \quad \forall \epsilon > 0, \quad k \in [m].
\end{aligned}$$

Moreover, by Assumption A1 and the boundedness of $\bar{\theta}_{2,i-m+k-1}$,

$$\begin{aligned}
&P(\Omega_{i,k}|\mathcal{F}'_{i-m+k-1}) \prod_{s=-m+1}^{k-1} I_{\overline{\Omega}_{i,s}} \\
\geq & P(w_{i-m+k} \in \overline{D_{j,m-k+1}} - \rho_{i-m+k-1} - Fg(\bar{\theta}_{2,i-m+k-1}, \zeta_{i-m+k-1})|\mathcal{F}'_{i-m+k-1}) \\
&\cdot P(v_{i-m+k} \in \overline{D_{j,m+1}} - g(\theta_2, \zeta_{i-m+k-1})|\mathcal{F}'_{i-m+k-1}) \prod_{s=-m+1}^{k-1} I_{\overline{\Omega}_{i,s}} \\
\text{(B.2)} \qquad \geq & c^* \prod_{s=-m+1}^{k-1} I_{\overline{\Omega}_{i,s}},
\end{aligned}$$

where $c^* > 0$ is a constant. Then, by combining (B.1) and (B.2),

$$\begin{aligned}
\sum_{i=m}^{t} I_{\mathcal{I}_{i,k}} \geq & \sum_{i=m}^{t} P(\Omega_{i,k}|\mathcal{F}'_{i-m+k-1}) \prod_{s=1}^{k-1} I_{\Omega_{i,s}} \prod_{s=-m+1}^{k-1} I_{\overline{\Omega}_{i,s}} - o(t^{\frac{1}{2}+\epsilon}) \\
\geq & c^* \sum_{i=m}^{t} \prod_{s=-m+1}^{k-1} I_{\overline{\Omega}_{i,s}} \prod_{s=1}^{k-1} I_{\Omega_{i,s}} - o(t^{\frac{1}{2}+\epsilon}) \\
\text{(B.3)} \qquad \triangleq & c^* \sum_{i=m}^{t} I_{\mathcal{I}'_{i,k}} - o(t^{\frac{1}{2}+\epsilon}), \quad \text{a.s.},
\end{aligned}$$

where

$$\mathcal{I}'_{i,k} \triangleq (\cap_{s=1}^{k-1}(\Omega_{i,s} \cap \overline{\Omega}_{i,s})) \cap (\cap_{s=-m+1}^{0}\overline{\Omega}_{i,s}).$$

Next, similar to (B.1), we derive that

$$\begin{aligned}
&\sum_{i=m}^{t}\left(I_{\mathcal{I}'_{i,k}} - P(\Omega_{i,k-1}|\mathcal{F}'_{i-m+k-2}) I_{\mathcal{I}_{i,k-1}}\right) \\
\leq & \sum_{i=m}^{t}(I_{\Omega_{i,k-1}} - P(\Omega_{i,k-1}|\mathcal{F}'_{i-m+k-2})) I_{\mathcal{I}_{i,k-1}} = o(t^{1/2+\epsilon}), \quad \text{a.s.}, \quad \forall \epsilon > 0, \quad k \in [m].
\end{aligned}$$

Combining this result with (B.3) yields

$$\text{(B.4)} \qquad \sum_{i=m}^{t} I_{\mathcal{I}_{i,k}} \geq c^* \sum_{i=m}^{t} P(\Omega_{i,k-1}|\mathcal{F}'_{i-m+k-2}) I_{\mathcal{I}_{i,k-1}} - o(t^{1/2+\epsilon}), \quad \text{a.s..}$$

Similar to (B.2), we derive that for $k \in [m]$, $P(\Omega_{i,k-1}|\mathcal{F}'_{i-m+k-2})I_{\mathcal{I}_{i,k-1}} \geq c^* I_{\mathcal{I}_{i,k-1}}$, which together with (B.4) leads to

$$\sum_{i=m}^{t} I_{\mathcal{I}_{i,k}} \geq (c^*)^2 \sum_{i=m}^{t} I_{\mathcal{I}_{i,k-1}} - o(t^{\frac{1}{2}+\epsilon}), \quad \text{a.s.}, \quad k \in [m].$$

From this, we can deduce

$$\begin{aligned}\sum_{i=m}^{t} I_{\mathcal{I}_{i,m}} \geq & (c^*)^2 \sum_{i=m}^{t} I_{\mathcal{I}_{i,m-1}} - o(t^{\frac{1}{2}+\epsilon}) \geq (c^*)^4 \sum_{i=m}^{t} I_{\mathcal{I}_{i,m-2}} - o(t^{\frac{1}{2}+\epsilon}) \\ & \geq \cdots \geq (c^*)^{2m} \sum_{i=m}^{t} I_{\mathcal{I}_{i,0}} - o(t^{\frac{1}{2}+\epsilon}), \quad \text{a.s.}, \quad \forall \epsilon > 0. \end{aligned} \tag{B.5}$$

Note that $\sum_{i=m}^{t} I_{\{\mathrm{col}(\varphi_i,\varsigma_i)\in\mathcal{D}_j\}} \geq \sum_{i=m}^{t} I_{\mathcal{I}_{i,m}}$, then (B.5) imply

$$\sum_{i=m}^{t} I_{\{\mathrm{col}(\varphi_i,\varsigma_i)\in\mathcal{D}_j\}} \geq (c^*)^{2m} \sum_{i=m}^{t} I_{\mathcal{I}_{i,0}} - o(t^{\frac{1}{2}+\epsilon}), \quad \text{a.s.}, \quad \forall \epsilon > 0. \tag{B.6}$$ □

Since for any $j \in [p]$, there exists a constant c^* such that the above inequality (B.6) holds, we conclude that Lemma B.1 is true.

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